

7.7 Hashing

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Hashing tries to **directly** compute the memory location from the given key. The goal is to have constant search time.

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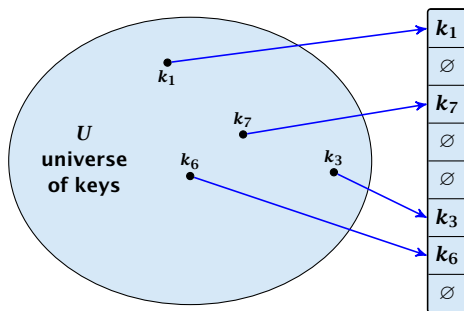
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The hash-function h should fulfill:

- ▶ Fast to evaluate.
- ▶ Small storage requirement.
- ▶ Good distribution of elements over the whole table.

Direct Addressing

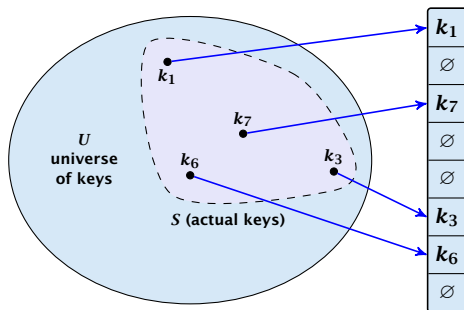
Ideally the hash function maps **all** keys to different memory locations.



This special case is known as **Direct Addressing**. It is usually very unrealistic as the universe of keys typically is quite large, and in particular larger than the available memory.

Perfect Hashing

Suppose that we **know** the set S of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



Such a hash function h is called a **perfect hash function** for set S .

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Hence, there may be two elements k_1, k_2 from the set S that map to the same memory location (i.e., $h(k_1) = h(k_2)$). This is called a **collision**.

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Lemma 1

The probability of having a collision when hashing m elements into a table of size n under uniform hashing is at least

$$1 - e^{-\frac{m(m-1)}{2n}} \approx 1 - e^{-\frac{m^2}{2n}} .$$

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Uniform hashing:

Choose a hash function uniformly at random from all functions $f : U \rightarrow [0, \dots, n-1]$.

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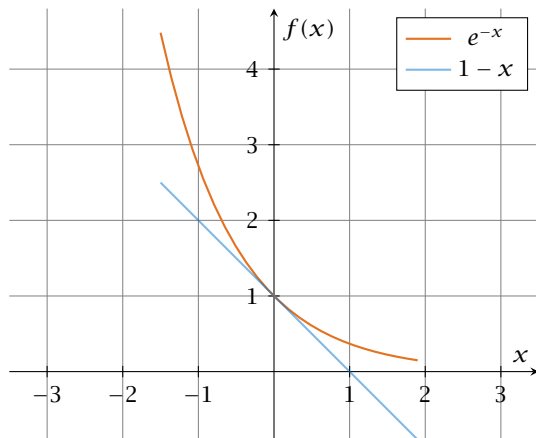
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Here the first equality follows since the ℓ -th element that is hashed has a probability of $\frac{n-\ell+1}{n}$ to not generate a collision under the condition that the previous elements did not induce collisions. □

Collisions



The inequality $1 - x \leq e^{-x}$ is derived by stopping the Taylor-expansion of e^{-x} after the second term.

Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

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- ▶ **hashing with chaining**, aka. closed addressing, open hashing.

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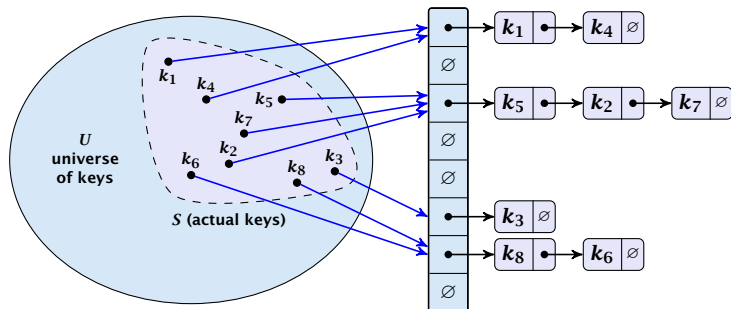
- ▶ **open addressing**, aka. closed hashing
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There are applications e.g. computer chess where you do not resolve collisions at all.

Hashing with Chaining

Arrange elements that map to the same position in a linear list.

- ▶ Access: compute $h(x)$ and search list for $\text{key}[x]$.
- ▶ Insert: insert at the front of the list.



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We assume **uniform hashing** for the following analysis.

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$$A^- = 1 + \alpha .$$

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Hence, the expected cost for a successful search is $A^+ \leq 1 + \frac{\alpha}{2}$.

Hashing with Chaining

Disadvantages:

- ▶ pointers increase memory requirements
- ▶ pointers may lead to bad cache efficiency

Advantages:

- ▶ no à priori limit on the number of elements
- ▶ deletion can be implemented efficiently
- ▶ by using balanced trees instead of linked list one can also obtain worst-case guarantees.

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Insert(x): Search until you find an empty slot; insert your element there. If your search reaches $h(k, n - 1)$, and this slot is non-empty then your table is full.

Open Addressing

Choices for $h(k, j)$:

- ▶ **Linear probing:**

$$h(k, i) = h(k) + i \bmod n$$

(sometimes: $h(k, i) = h(k) + ci \bmod n$).

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For quadratic probing and double hashing one has to ensure that the search covers all positions in the table (i.e., for double hashing $h_2(k)$ must be relatively prime to n (**teilerfremd**); for quadratic probing c_1 and c_2 have to be chosen carefully).

Linear Probing

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Lemma 2

Let L be the method of linear probing for resolving collisions:

$$L^+ \approx \frac{1}{2} \left(1 + \frac{1}{1 - \alpha} \right)$$

$$L^- \approx \frac{1}{2} \left(1 + \frac{1}{(1 - \alpha)^2} \right)$$

Quadratic Probing

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Lemma 3

Let Q be the method of quadratic probing for resolving collisions:

$$Q^+ \approx 1 + \ln\left(\frac{1}{1-\alpha}\right) - \frac{\alpha}{2}$$

$$Q^- \approx \frac{1}{1-\alpha} + \ln\left(\frac{1}{1-\alpha}\right) - \alpha$$

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Lemma 4

Let D be the method of double hashing for resolving collisions:

$$D^+ \approx \frac{1}{\alpha} \ln \left(\frac{1}{1 - \alpha} \right)$$

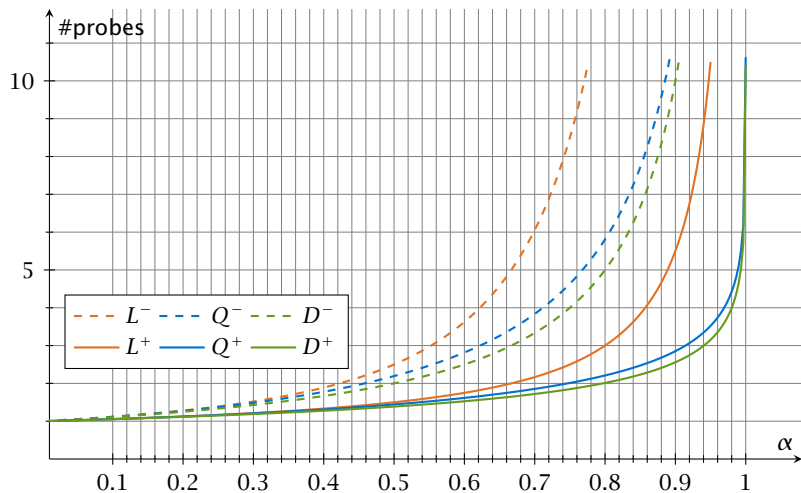
$$D^- \approx \frac{1}{1 - \alpha}$$

Open Addressing

Some values:

α	<i>Linear Probing</i>		<i>Quadratic Probing</i>		<i>Double Hashing</i>	
	L^+	L^-	Q^+	Q^-	D^+	D^-
0.5	1.5	2.5	1.44	2.19	1.39	2
0.9	5.5	50.5	2.85	11.40	2.55	10
0.95	10.5	200.5	3.52	22.05	3.15	20

Open Addressing



Analysis of Idealized Open Address Hashing

We analyze the time for a search in a very idealized Open Addressing scheme.

- ▶ The probe sequence $h(k, 0), h(k, 1), h(k, 2), \dots$ is equally likely to be any permutation of $\langle 0, 1, \dots, n - 1 \rangle$.

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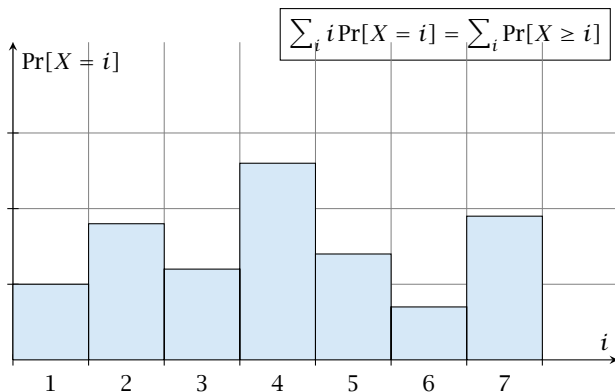
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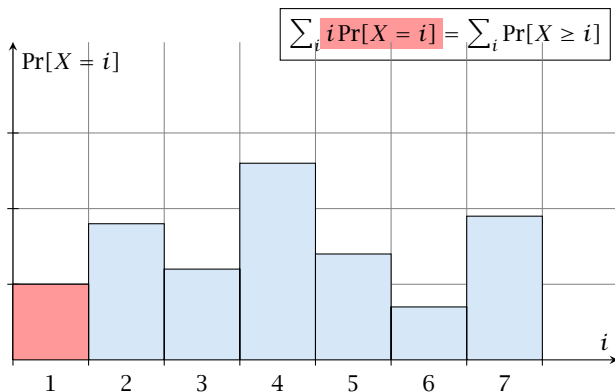
$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \dots$$

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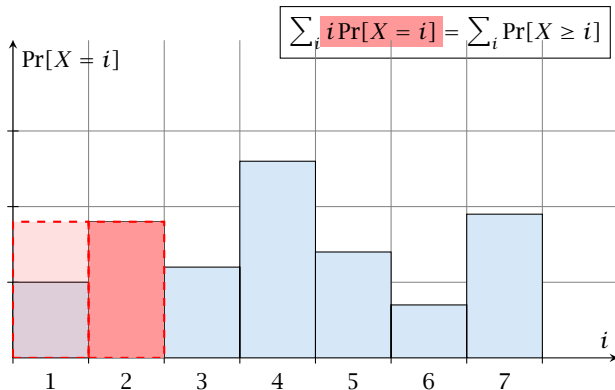
Analysis of Idealized Open Address Hashing

$i = 1$



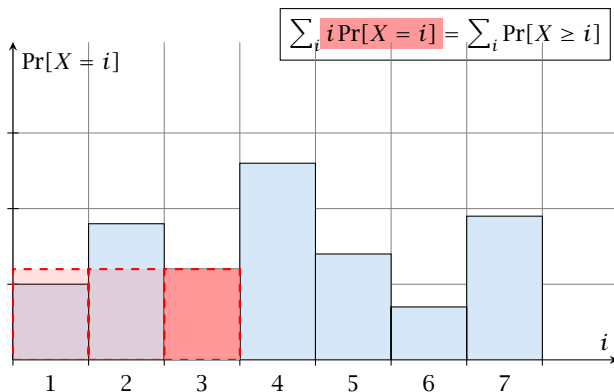
Analysis of Idealized Open Address Hashing

$i = 2$



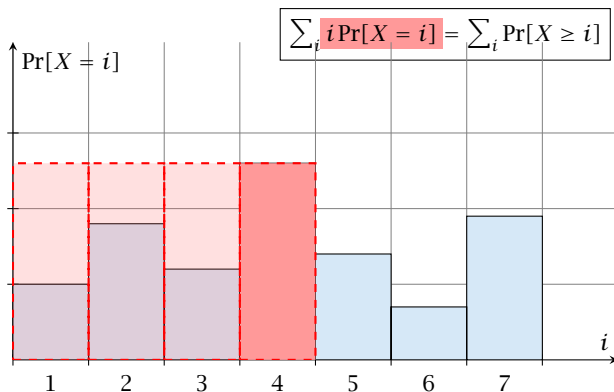
Analysis of Idealized Open Address Hashing

$i = 3$



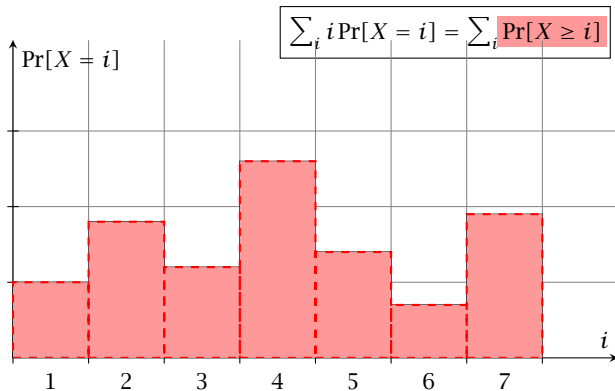
Analysis of Idealized Open Address Hashing

$i = 4$



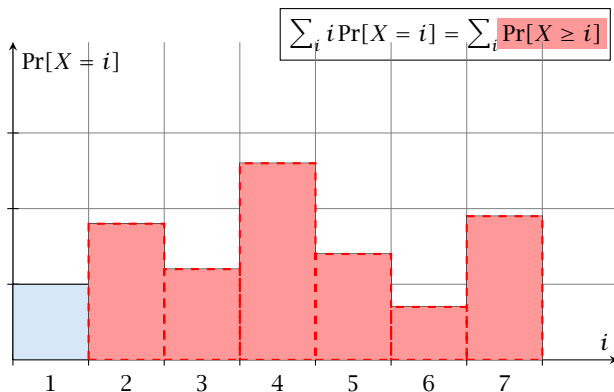
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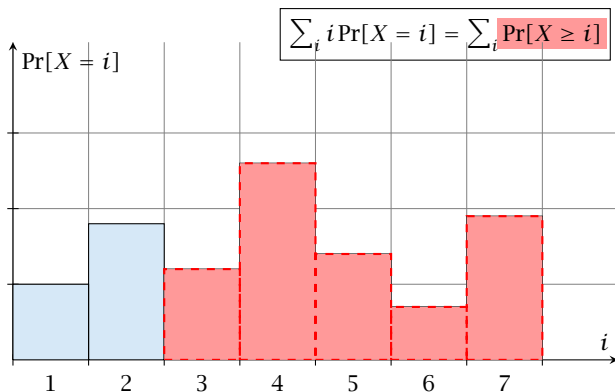
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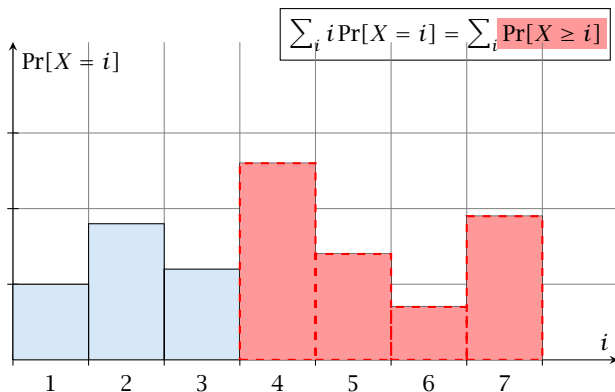
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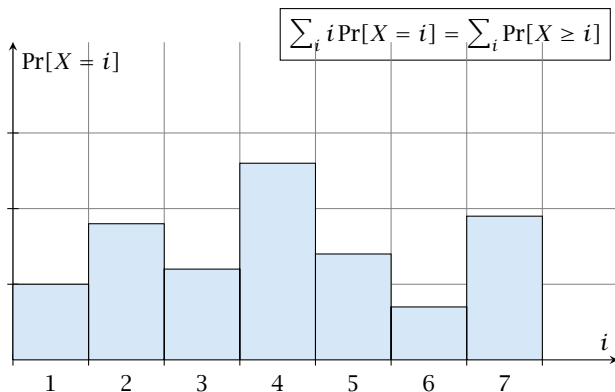


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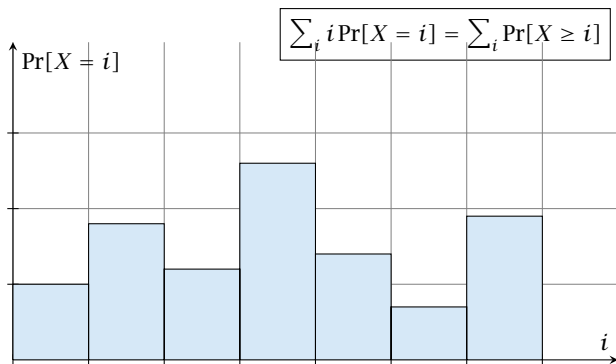
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Analysis of Idealized Open Address Hashing



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The j -th rectangle appears in both sums j times. (j times in the first due to multiplication with j ; and j times in the second for summands $i = 1, 2, \dots, j$)

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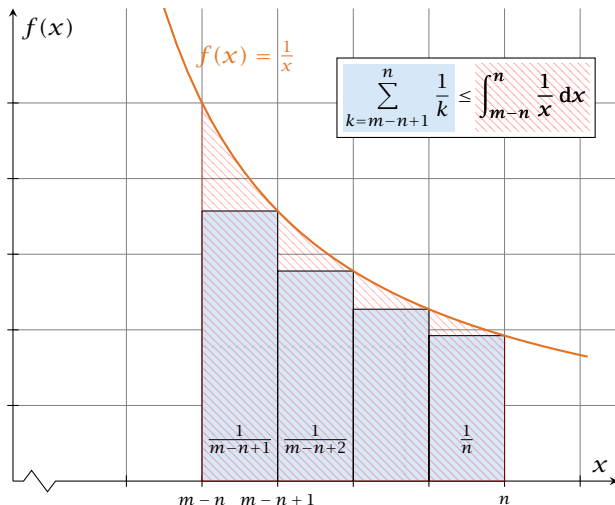
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- ▶ For open addressing this is difficult.

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- ▶ The table could fill up with **deleted**-markers leading to bad performance.
- ▶ If a table contains many deleted-markers (linear fraction of the keys) one can rehash the whole table and amortize the cost for this rehash against the cost for the deletions.

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- ▶ Upon a deletion elements that are further down in the probe-sequence may be moved to guarantee that they are still found during a search.

Deletions for Linear Probing

Algorithm 37 delete(p)

```
1:  $T[p] \leftarrow \text{null}$ 
2:  $p \leftarrow \text{succ}(p)$ 
3: while  $T[p] \neq \text{null}$  do
4:    $y \leftarrow T[p]$ 
5:    $T[p] \leftarrow \text{null}$ 
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7:    $\text{insert}(y)$ 
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p is the index into the table-cell that contains the object to be deleted.

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Pointers into the hash-table become invalid.

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However, the assumption of uniform hashing that h is chosen randomly from all functions $f : U \rightarrow [0, \dots, n - 1]$ is clearly unrealistic as there are $n^{|U|}$ such functions. Even writing down such a function would take $|U| \log n$ bits.

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Universal hashing tries to define a set \mathcal{H} of functions that is much smaller but still leads to good average case behaviour when selecting a hash-function uniformly at random from \mathcal{H} .

Universal Hashing

Definition 5

A class \mathcal{H} of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called **universal** if for all $u_1, u_2 \in U$ with $u_1 \neq u_2$

$$\Pr[h(u_1) = h(u_2)] \leq \frac{1}{n} ,$$

where the probability is w. r. t. the choice of a random hash-function from set \mathcal{H} .

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Note that this means that the probability of a collision between two arbitrary elements is at most $\frac{1}{n}$.

Universal Hashing

Definition 6

A class \mathcal{H} of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called **2-independent** (pairwise independent) if the following two conditions hold

- ▶ For any key $u \in U$, and $t \in \{0, \dots, n-1\}$ $\Pr[h(u) = t] = \frac{1}{n}$,
i.e., a key is distributed uniformly within the hash-table.
- ▶ For all $u_1, u_2 \in U$ with $u_1 \neq u_2$, and for any two hash-positions t_1, t_2 :

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This requirement clearly implies a universal hash-function.

Universal Hashing

Definition 7

A class \mathcal{H} of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called **k -independent** if for any choice of $\ell \leq k$ distinct keys $u_1, \dots, u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions t_1, \dots, t_ℓ :

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where the probability is w. r. t. the choice of a random hash-function from set \mathcal{H} .

Definition 8

A class \mathcal{H} of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called (μ, k) -independent if for any choice of $\ell \leq k$ distinct keys $u_1, \dots, u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions t_1, \dots, t_ℓ :

$$\Pr[h(u_1) = t_1 \wedge \dots \wedge h(u_\ell) = t_\ell] \leq \frac{\mu}{n^\ell},$$

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Let $U := \{0, \dots, p - 1\}$ for a prime p . Let $\mathbb{Z}_p := \{0, \dots, p - 1\}$, and let $\mathbb{Z}_p^* := \{1, \dots, p - 1\}$ denote the set of invertible elements in \mathbb{Z}_p .

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Lemma 9

The class

$$\mathcal{H} = \{h_{a,b} \mid a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$$

is a universal class of hash-functions from U to $\{0, \dots, n-1\}$.

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where we use that \mathbb{Z}_p is a field (**Körper**) and, hence, has no zero divisors (**nullteilerfrei**).

Universal Hashing

- ▶ The hash-function does not generate collisions before the $(\text{mod } n)$ -operation. Furthermore, every choice (a, b) is mapped to a different pair (t_x, t_y) with $t_x := ax + b$ and $t_y := ay + b$.

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$$t_x - t_y \equiv a(x - y) \pmod{p}$$

$$t_y \equiv ay + b \pmod{p}$$

Universal Hashing

- ▶ The hash-function does not generate collisions before the $(\text{mod } n)$ -operation. Furthermore, every choice (a, b) is mapped to a different pair (t_x, t_y) with $t_x := ax + b$ and $t_y := ay + b$.

This holds because we can compute a and b when given t_x and t_y :

$$t_x \equiv ax + b \pmod{p}$$

$$t_y \equiv ay + b \pmod{p}$$

$$t_x - t_y \equiv a(x - y) \pmod{p}$$

$$t_y \equiv ay + b \pmod{p}$$

$$a \equiv (t_x - t_y)(x - y)^{-1} \pmod{p}$$

$$b \equiv t_y - ay \pmod{p}$$

Universal Hashing

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What happens when we do the $\text{mod } n$ operation?

Fix a value t_x . There are $p - 1$ possible values for choosing t_y .

From the range $0, \dots, p - 1$ the values $t_x, t_x + n, t_x + 2n, \dots$ map to t_x after the modulo-operation. These are at most $\lceil p/n \rceil$ values.

Universal Hashing

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$$\left[\frac{p}{n} \right] - 1$$

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This happens with probability at most $\frac{1}{n}$.

Universal Hashing

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It is also possible to show that \mathcal{H} is an (almost) pairwise independent class of hash-functions.

$$\Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[\begin{array}{l} t_x \bmod n = h_1 \\ t_y \bmod n = h_2 \end{array} \right]$$

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$$\frac{\left\lfloor \frac{p}{n} \right\rfloor^2}{p(p-1)} \leq \Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[\begin{array}{l} t_x \bmod n = h_1 \\ t_y \bmod n = h_2 \end{array} \right] \leq \frac{\left\lfloor \frac{p}{n} \right\rfloor^2}{p(p-1)}$$

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Note that the middle is the probability that $h(x) = h_1$ and $h(y) = h_2$. The total number of choices for (t_x, t_y) is $p(p-1)$. The number of choices for t_x (t_y) such that $t_x \bmod n = h_1$ ($t_y \bmod n = h_2$) lies between $\lfloor \frac{p}{n} \rfloor$ and $\lceil \frac{p}{n} \rceil$.

Universal Hashing

Definition 10

Let $d \in \mathbb{N}$; $q \geq (d + 1)n$ be a prime; and let $\bar{a} \in \{0, \dots, q - 1\}^{d+1}$. Define for $x \in \{0, \dots, q - 1\}$

$$h_{\bar{a}}(x) := \left(\sum_{i=0}^d a_i x^i \bmod q \right) \bmod n .$$

Let $\mathcal{H}_n^d := \{h_{\bar{a}} \mid \bar{a} \in \{0, \dots, q - 1\}^{d+1}\}$. The class \mathcal{H}_n^d is $(e, d + 1)$ -independent.

Note that in the previous case we had $d = 1$ and chose $a_d \neq 0$.

Universal Hashing

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For the coefficients $\bar{a} \in \{0, \dots, q-1\}^{d+1}$ let $f_{\bar{a}}$ denote the polynomial

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The polynomial is defined by $d+1$ distinct points.

Universal Hashing

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Fix $\ell \leq d + 1$; let $x_1, \dots, x_\ell \in \{0, \dots, q - 1\}$ be keys, and let t_1, \dots, t_ℓ denote the corresponding hash-function values.

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Let $A^\ell = \{h_{\bar{a}} \in \mathcal{H} \mid h_{\bar{a}}(x_i) = t_i \text{ for all } i \in \{1, \dots, \ell\}\}$

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Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

$$f_{\bar{a}}(x_i) \in \underbrace{\{t_i + \alpha \cdot n \mid \alpha \in \{0, \dots, \lceil \frac{q}{n} \rceil - 1\}\}}_{=: B_i}$$

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We have

$$|B_1| \cdot \dots \cdot |B_\ell|$$

possibilities to do this (so that $h_{\bar{a}}(x_i) = t_i$).

Universal Hashing

Now, we choose $d - \ell + 1$ other inputs and choose their value arbitrarily. We have $q^{d-\ell+1}$ possibilities to do this.

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Therefore we have

$$|B_1| \cdot \dots \cdot |B_\ell| \cdot q^{d-\ell+1} \leq \left\lceil \frac{q}{n} \right\rceil^\ell \cdot q^{d-\ell+1}$$

possibilities to choose \bar{a} such that $h_{\bar{a}} \in A_\ell$.

Universal Hashing

Therefore the probability of choosing $h_{\bar{a}}$ from A_ℓ is only

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$$\frac{\left[\frac{q}{n}\right]^\ell \cdot q^{d-\ell+1}}{q^{d+1}} \leq \frac{\left(\frac{q+n}{n}\right)^\ell}{q^\ell} \leq \left(\frac{q+n}{q}\right)^\ell \cdot \frac{1}{n^\ell}$$

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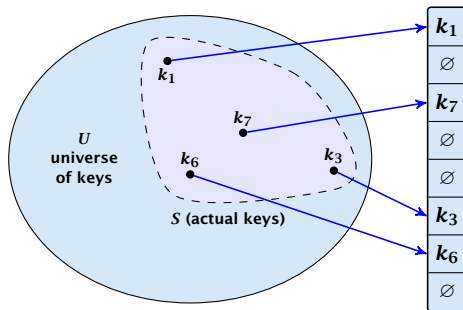
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This shows that the \mathcal{H} is $(e, d+1)$ -universal.

The last step followed from $q \geq (d+1)n$, and $\ell \leq d+1$.

Perfect Hashing

Suppose that we **know** the set S of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



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The probability of having **1** or more collisions can be at most $\frac{1}{2}$ as otherwise the expectation would be larger than $\frac{1}{2}$.

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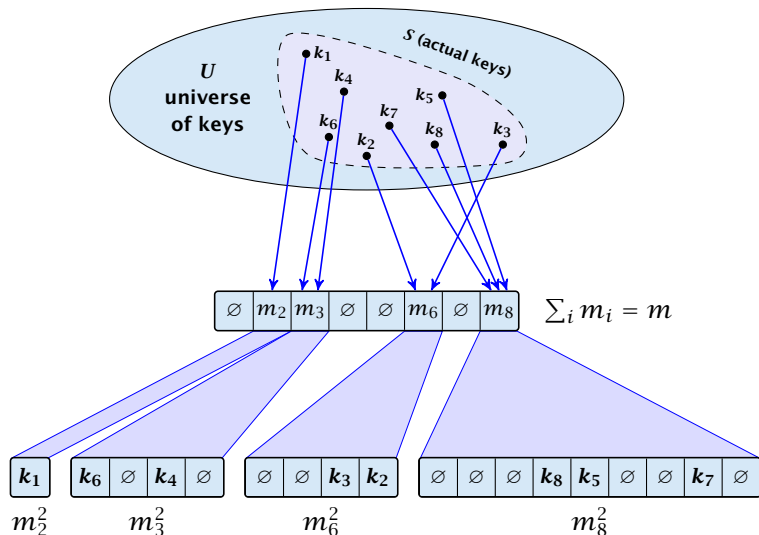
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Let m_j denote the number of items that are hashed to the j -th bucket. For each bucket we choose a second hash-function that maps the elements of the bucket into a table of size m_j^2 . The second function can be chosen such that all elements are mapped to different locations.

Perfect Hashing



Perfect Hashing

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The total memory that is required by all hash-tables is $\mathcal{O}(\sum_j m_j^2)$.
Note that m_j is a random variable.

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The first expectation is simply the expected number of collisions, for the first level. Since we use universal hashing we have

$$= 2 \binom{m}{2} \frac{1}{m} + m = 2m - 1 .$$

Perfect Hashing

We need only $\mathcal{O}(m)$ time to construct a hash-function h with $\sum_j m_j^2 = \mathcal{O}(4m)$, because with probability at least $1/2$ a random function from a universal family will have this property.

Then we construct a hash-table h_j for every bucket. This takes expected time $\mathcal{O}(m_j)$ for every bucket. A random function h_j is collision-free with probability at least $1/2$. We need $\mathcal{O}(m_j)$ to test this.

We only need that the hash-functions are chosen from a universal family!!!

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Goal:

Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

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- ▶ An object x is either stored at location $T_1[h_1(x)]$ or $T_2[h_2(x)]$.

Cuckoo Hashing

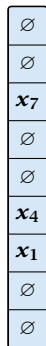
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- ▶ An object x is either stored at location $T_1[h_1(x)]$ or $T_2[h_2(x)]$.
- ▶ A search clearly takes constant time if the above constraint is met.

Cuckoo Hashing

Insert:



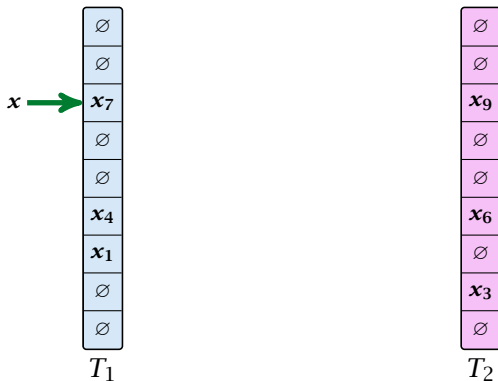
T_1



T_2

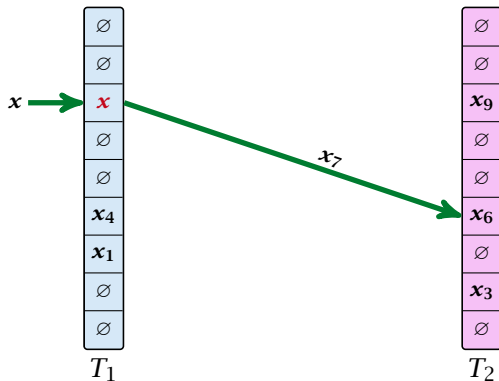
Cuckoo Hashing

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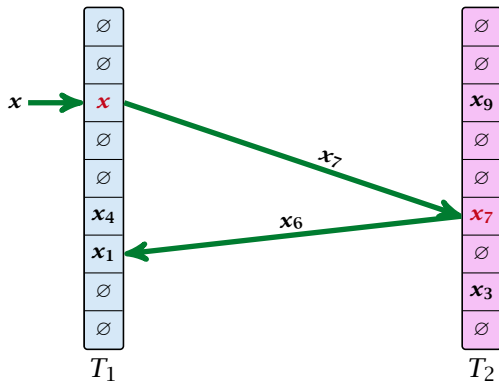
Cuckoo Hashing

Insert:



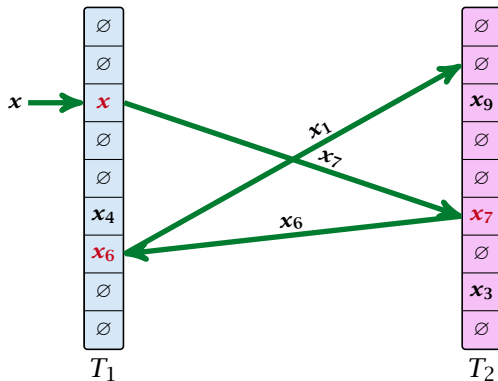
Cuckoo Hashing

Insert:



Cuckoo Hashing

Insert:



Algorithm 38 Cuckoo-Insert(x)

```
1: if  $T_1[h_1(x)] = x \vee T_2[h_2(x)] = x$  then return  
2: steps  $\leftarrow$  1  
3: while steps  $\leq$  maxsteps do  
4:     exchange  $x$  and  $T_1[h_1(x)]$   
5:     if  $x = \text{null}$  then return  
6:     exchange  $x$  and  $T_2[h_2(x)]$   
7:     if  $x = \text{null}$  then return  
8:     steps  $\leftarrow$  steps + 1  
9: rehash() // change hash-functions; rehash everything  
10: Cuckoo-Insert( $x$ )
```

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- ▶ We call a sequence of iterations through the while-loop without the termination condition becoming true a **phase** of the algorithm.
- ▶ We say a phase is **successful** if it is not terminated by the **maxstep**-condition, but the while loop is left because $x = \text{null}$.

Cuckoo Hashing

What is the expected time for an insert-operation?

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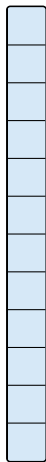
We first analyze the probability that we end-up in an infinite loop (that is then terminated after **maxsteps** steps).

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Formally what is the probability to enter an infinite loop that touches s different keys?

Cuckoo Hashing: Insert

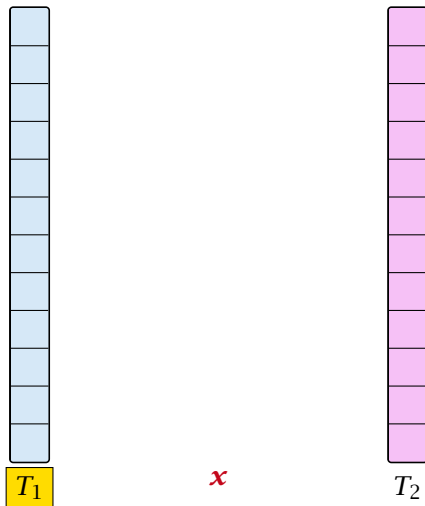


T_1

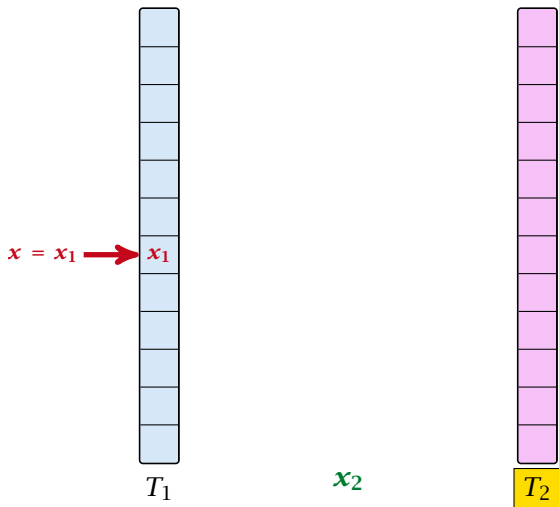


T_2

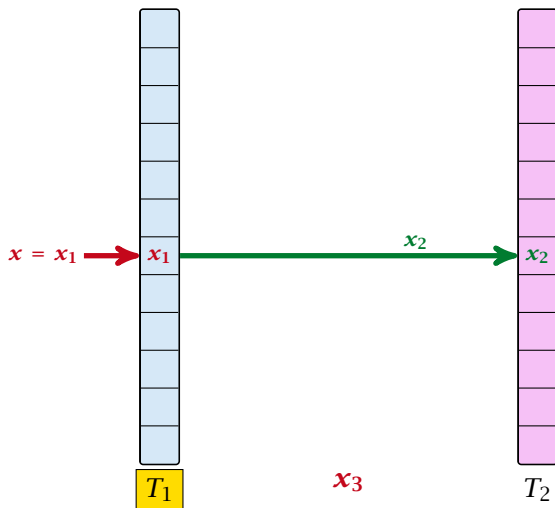
Cuckoo Hashing: Insert



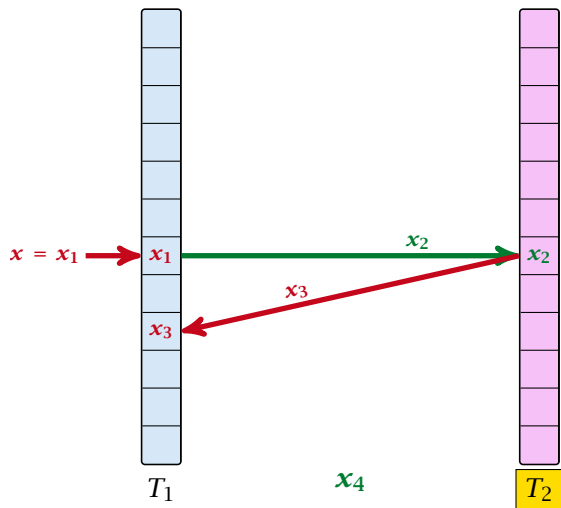
Cuckoo Hashing: Insert



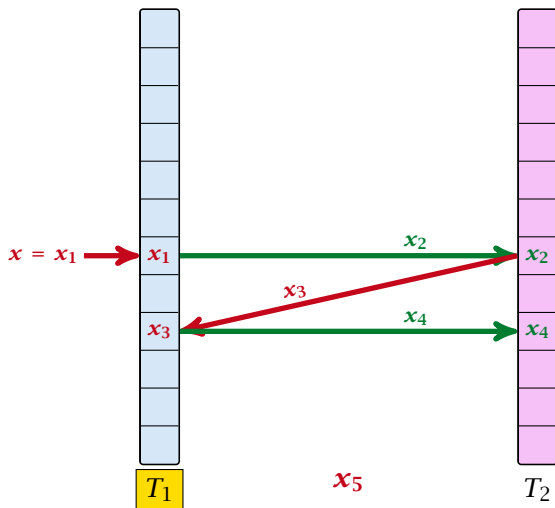
Cuckoo Hashing: Insert



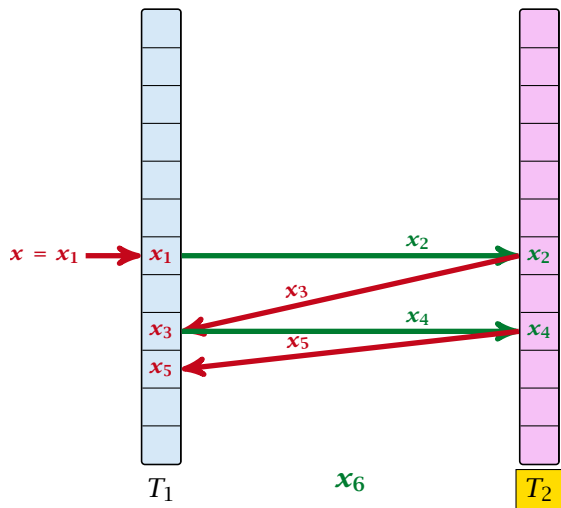
Cuckoo Hashing: Insert



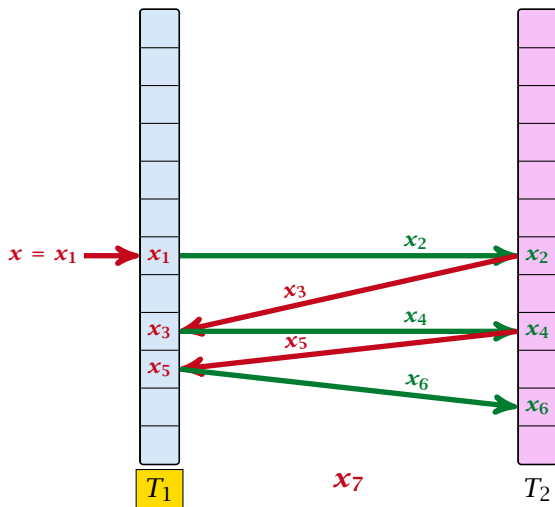
Cuckoo Hashing: Insert



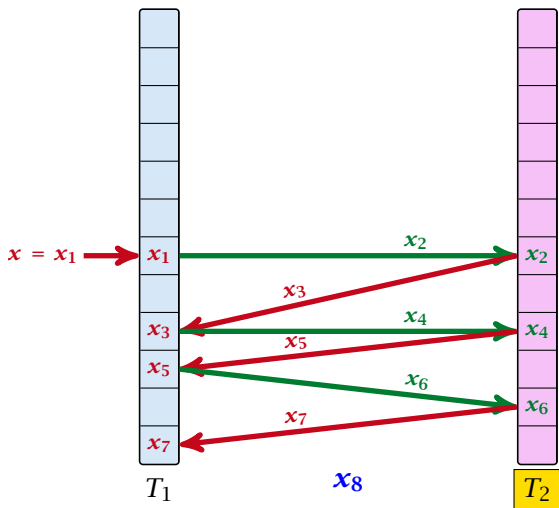
Cuckoo Hashing: Insert



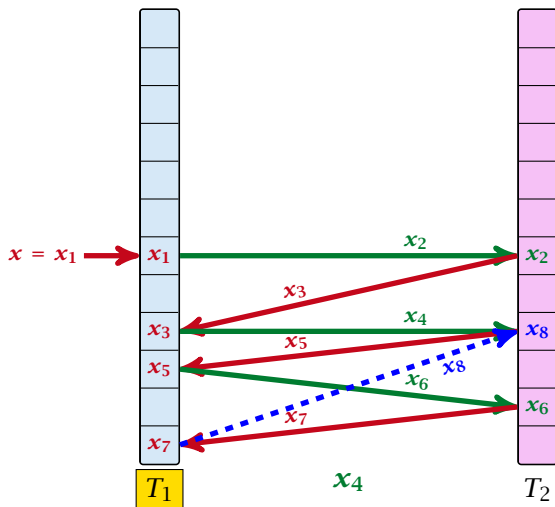
Cuckoo Hashing: Insert



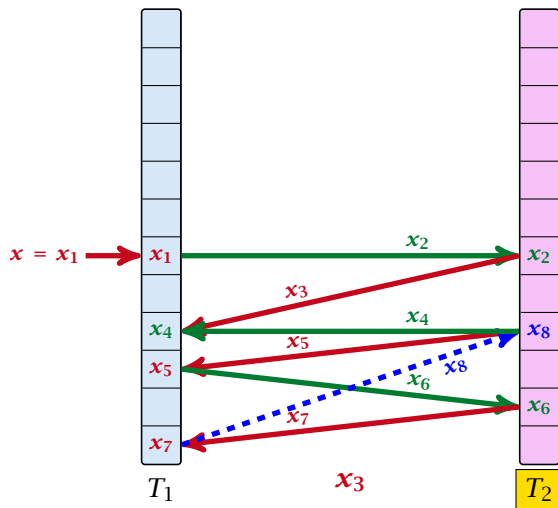
Cuckoo Hashing: Insert



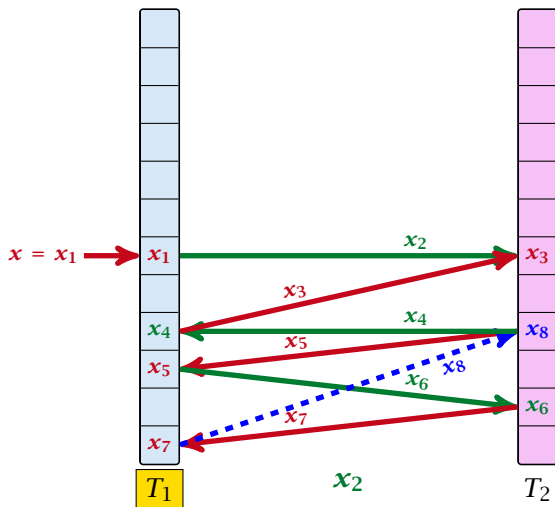
Cuckoo Hashing: Insert



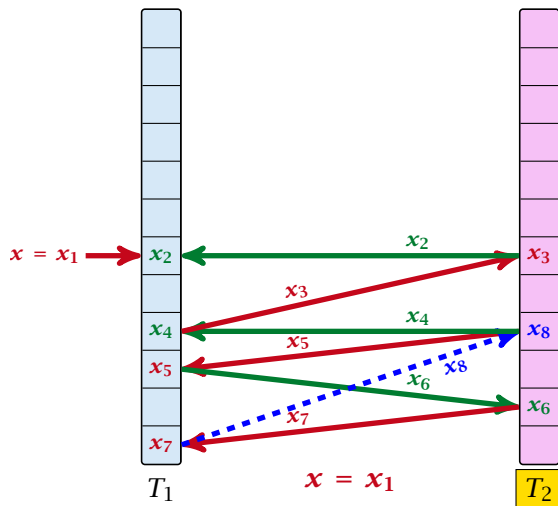
Cuckoo Hashing: Insert



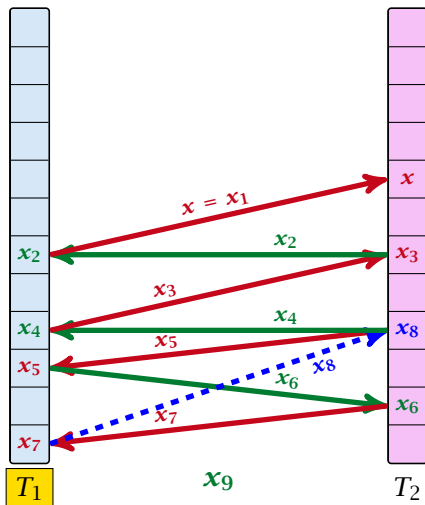
Cuckoo Hashing: Insert



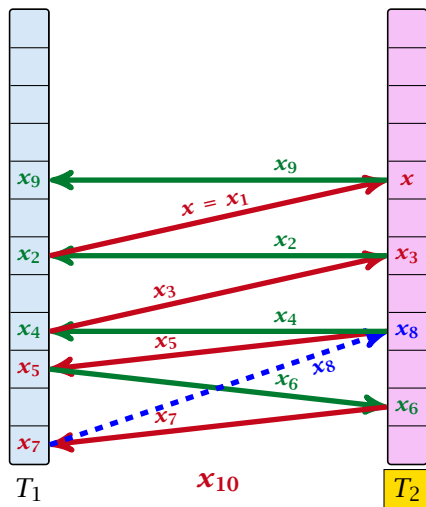
Cuckoo Hashing: Insert



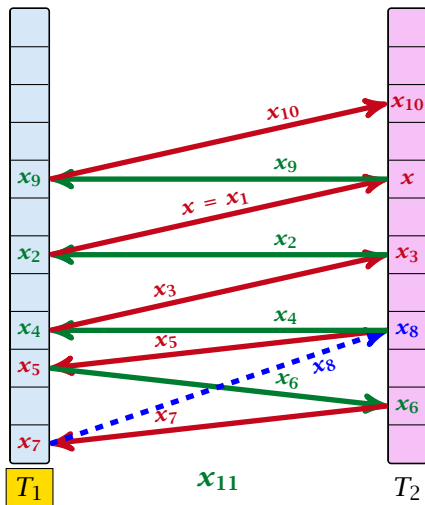
Cuckoo Hashing: Insert



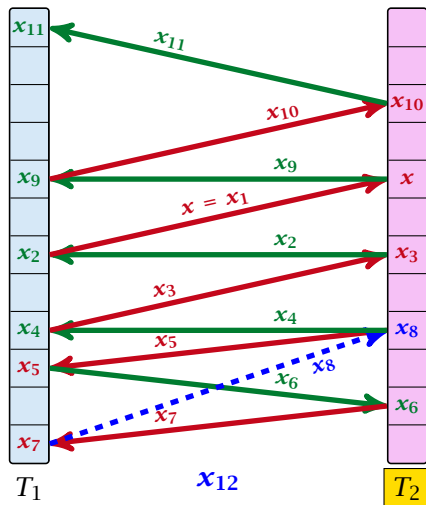
Cuckoo Hashing: Insert



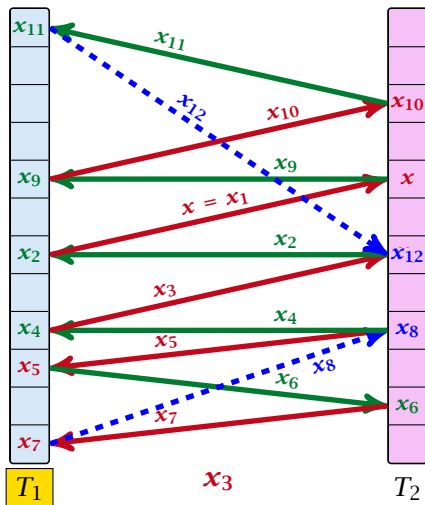
Cuckoo Hashing: Insert



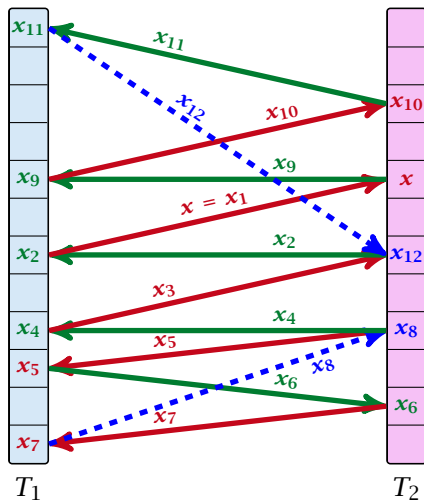
Cuckoo Hashing: Insert



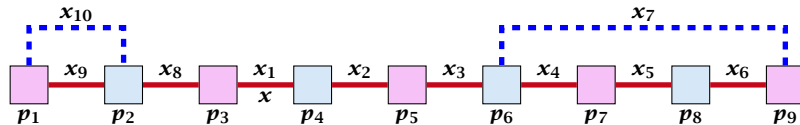
Cuckoo Hashing: Insert



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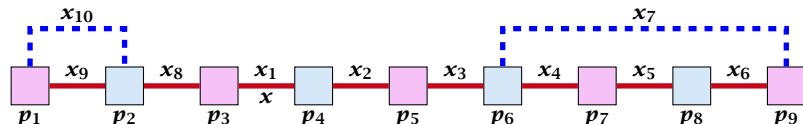


Cuckoo Hashing



A cycle-structure of size s is defined by

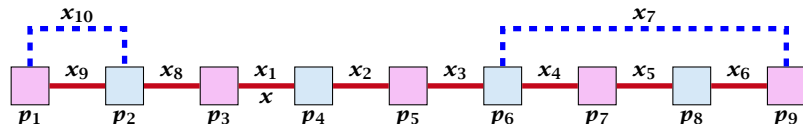
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- ▶ $s - 1$ different cells (alternating btw. cells from T_1 and T_2).

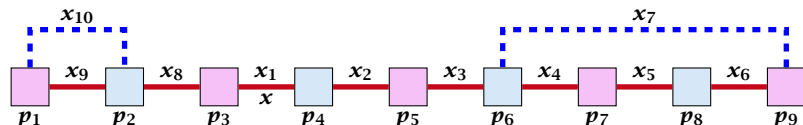
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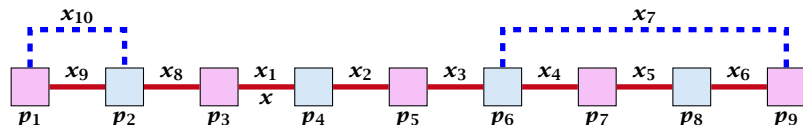
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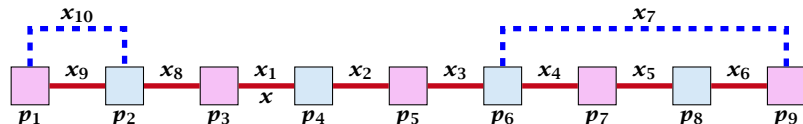
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- ▶ The rightmost cell is “linked backward” to a cell on the left.

Cuckoo Hashing



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- ▶ $s - 1$ different cells (alternating btw. cells from T_1 and T_2).
- ▶ s distinct keys $x = x_1, x_2, \dots, x_s$, linking the cells.
- ▶ The leftmost cell is “linked forward” to some cell on the right.
- ▶ The rightmost cell is “linked backward” to a cell on the left.
- ▶ One link represents key x ; this is where the counting starts.

Cuckoo Hashing

A cycle-structure is **active** if for every key x_ℓ (linking a cell p_i from T_1 and a cell p_j from T_2) we have

$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

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Observation:

If during a phase the insert-procedure runs into a cycle there must exist an active cycle structure of size $s \geq 3$.

Cuckoo Hashing

What is the probability that all keys in a cycle-structure of size s correctly map into their T_1 -cell?

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These events are independent.

Cuckoo Hashing

The probability that a given cycle-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

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What is the probability that **there exists** an active cycle structure of size s ?

Cuckoo Hashing

The number of cycle-structures of size s is at most

$$s^3 \cdot n^{s-1} \cdot m^{s-1} .$$

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- ▶ There are at most s possibilities to choose where to place key x .
- ▶ There are m^{s-1} possibilities to choose the keys apart from x .
- ▶ There are n^{s-1} possibilities to choose the cells.

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} = \frac{\mu^2}{nm} \sum_{s=3}^{\infty} s^3 \left(\frac{m}{n}\right)^s$$

Cuckoo Hashing

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Here we used the fact that $(1 + \epsilon)m \leq n$.

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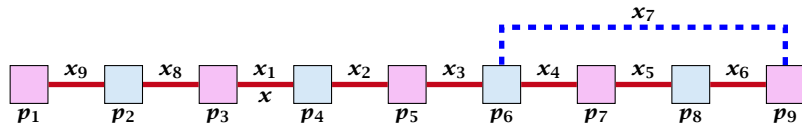
Hence,

$$\Pr[\text{cycle}] = \mathcal{O}\left(\frac{1}{m^2}\right).$$

Cuckoo Hashing

Now, we analyze the probability that a phase is not successful without running into a closed cycle.

Cuckoo Hashing



Sequence of visited keys:

$x = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_3, x_2, x_1 = x, x_8, x_9, \dots$

Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with x in the order that they are visited during the phase.

Cuckoo Hashing

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Lemma 11

*If the sequence is of length p then there exists a sub-sequence of at least $\frac{p+2}{3}$ keys starting with x of *distinct* keys.*

Cuckoo Hashing

Proof.

Let i be the number of keys (including x) that we see before the first repeated key. Let j denote the total number of distinct keys.

The sequence is of the form:

$$x = x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i \rightarrow x_r \rightarrow x_{r-1} \rightarrow \dots \rightarrow x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$$

As $r \leq i - 1$ the length p of the sequence is

$$p = i + r + (j - i) \leq i + j - 1 .$$

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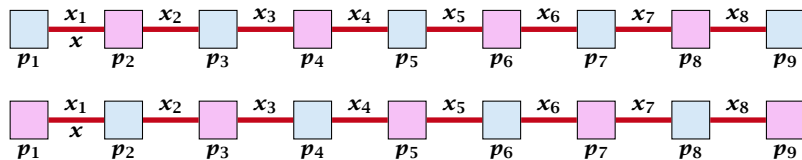
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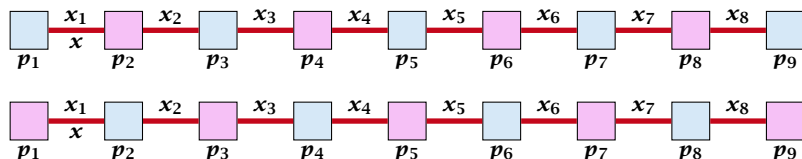
Either sub-sequence $x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i$ or sub-sequence $x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$ has at least $\frac{p+2}{3}$ elements. □

Cuckoo Hashing



A path-structure of size s is defined by

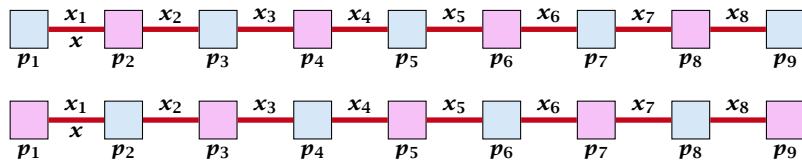
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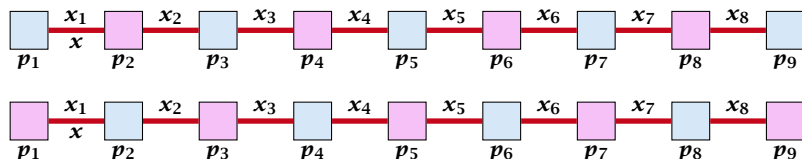
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$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

Observation:

If a phase takes at least t steps without running into a cycle there must exist an active path-structure of size $(2t + 2)/3$.

Cuckoo Hashing

The probability that a given path-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

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by choosing $\ell \geq \log \left(\frac{1}{2\mu^2 m^2} \right) / \log \left(\frac{1}{1+\epsilon} \right) = \log(2\mu^2 m^2) / \log(1+\epsilon)$

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This gives $\text{maxsteps} = \Theta(\log m)$.

Cuckoo Hashing

So far we estimated

$$\Pr[\text{cycle}] \leq \mathcal{O}\left(\frac{1}{m^2}\right)$$

and

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This means the expected cost for a successful phase is constant (even after accounting for the cost of the incomplete step that finishes the phase).

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Therefore the expected cost for re-hashes is $\mathcal{O}(m) \cdot \mathcal{O}(p) = \mathcal{O}(1)$.

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Therefore, it is sufficient to have $(\mu, \Theta(\log m))$ -independent hash-functions.

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- ▶ Therefore we can amortize the rehash cost after a change in table-size against the cost for insertions and deletions.

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Cuckoo Hashing has an expected constant insert-time and a worst-case constant search-time.

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Note that the above lemma only holds if the fill-factor (number of keys/total number of hash-table slots) is at most $\frac{1}{2(1+\epsilon)}$.

The $1/(2(1+\epsilon))$ fill-factor comes from the fact that the total hash-table is of size $2n$ (because we have two tables of size n); moreover $m \leq (1+\epsilon)n$.