

## 7.7 Hashing

### Dictionary:

- ▶  **$S.insert(x)$** : Insert an element  $x$ .
- ▶  **$S.delete(x)$** : Delete the element pointed to by  $x$ .
- ▶  **$S.search(k)$** : Return a pointer to an element  $e$  with  $key[e] = k$  in  $S$  if it exists; otherwise return **null**.

So far we have implemented the search for a key by carefully choosing split-elements.

Then the memory location of an object  $x$  with key  $k$  is determined by successively comparing  $k$  to split-elements.

**Hashing** tries to **directly** compute the memory location from the given key. The goal is to have constant search time.

## 7.7 Hashing

### Definitions:

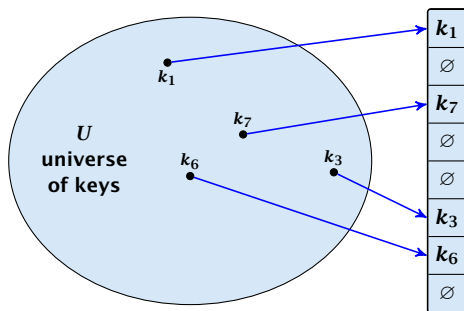
- ▶ Universe  $U$  of keys, e.g.,  $U \subseteq \mathbb{N}_0$ .  $U$  very large.
- ▶ Set  $S \subseteq U$  of keys,  $|S| = m \leq |U|$ .
- ▶ Array  $T[0, \dots, n - 1]$  hash-table.
- ▶ Hash function  $h : U \rightarrow [0, \dots, n - 1]$ .

### The hash-function $h$ should fulfill:

- ▶ Fast to evaluate.
- ▶ Small storage requirement.
- ▶ Good distribution of elements over the whole table.

# Direct Addressing

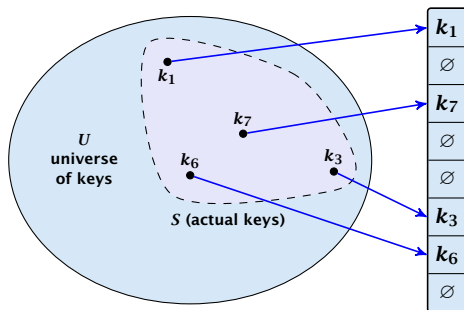
Ideally the hash function maps **all** keys to different memory locations.



This special case is known as **Direct Addressing**. It is usually very unrealistic as the universe of keys typically is quite large, and in particular larger than the available memory.

# Perfect Hashing

Suppose that we **know** the set  $S$  of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



Such a hash function  $h$  is called a **perfect hash function** for set  $S$ .

# Collisions

If we do not know the keys in advance, the best we can hope for is that the hash function distributes keys evenly across the table.

## Problem: Collisions

Usually the universe  $U$  is much larger than the table-size  $n$ .

Hence, there may be two elements  $k_1, k_2$  from the set  $S$  that map to the same memory location (i.e.,  $h(k_1) = h(k_2)$ ). This is called a **collision**.

# Collisions

Typically, collisions do not appear once the size of the set  $S$  of actual keys gets close to  $n$ , but already when  $|S| \geq \omega(\sqrt{n})$ .

## Lemma 1

*The probability of having a collision when hashing  $m$  elements into a table of size  $n$  under **uniform hashing** is at least*

$$1 - e^{-\frac{m(m-1)}{2n}} \approx 1 - e^{-\frac{m^2}{2n}} .$$

## Uniform hashing:

Choose a hash function uniformly at random from all functions  $f : U \rightarrow [0, \dots, n-1]$ .

# Collisions

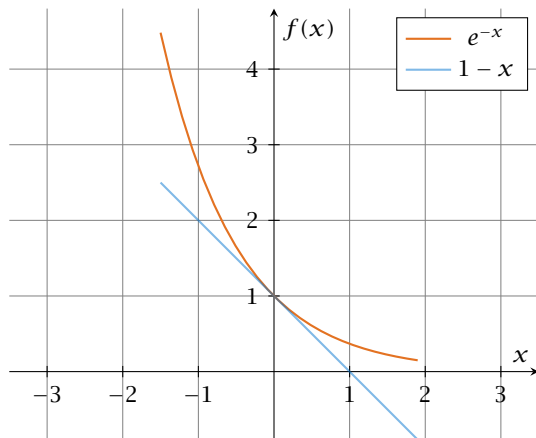
## Proof.

Let  $A_{m,n}$  denote the event that inserting  $m$  keys into a table of size  $n$  does **not** generate a collision. Then

$$\begin{aligned}\Pr[A_{m,n}] &= \prod_{\ell=1}^m \frac{n - \ell + 1}{n} = \prod_{j=0}^{m-1} \left(1 - \frac{j}{n}\right) \\ &\leq \prod_{j=0}^{m-1} e^{-j/n} = e^{-\sum_{j=0}^{m-1} \frac{j}{n}} = e^{-\frac{m(m-1)}{2n}}.\end{aligned}$$

Here the first equality follows since the  $\ell$ -th element that is hashed has a probability of  $\frac{n-\ell+1}{n}$  to not generate a collision under the condition that the previous elements did not induce collisions. □

# Collisions



The inequality  $1 - x \leq e^{-x}$  is derived by stopping the Taylor-expansion of  $e^{-x}$  after the second term.



# Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

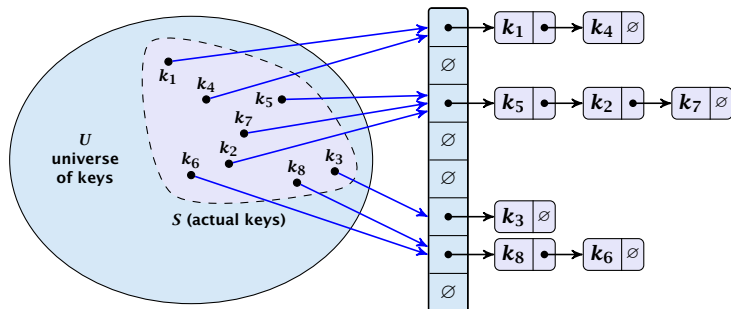
- ▶ **open addressing**, aka. closed hashing
- ▶ **hashing with chaining**, aka. closed addressing, open hashing.

There are applications e.g. computer chess where you do not resolve collisions at all.

# Hashing with Chaining

Arrange elements that map to the same position in a linear list.

- ▶ Access: compute  $h(x)$  and search list for  $\text{key}[x]$ .
- ▶ Insert: insert at the front of the list.



# Hashing with Chaining

Let  $A$  denote a strategy for resolving collisions. We use the following notation:

- ▶  $A^+$  denotes the average time for a **successful** search when using  $A$ ;
- ▶  $A^-$  denotes the average time for an **unsuccessful** search when using  $A$ ;
- ▶ We parameterize the complexity results in terms of  $\alpha := \frac{m}{n}$ , the so-called **fill factor** of the hash-table.

We assume **uniform hashing** for the following analysis.

# Hashing with Chaining

The time required for an unsuccessful search is 1 plus the length of the list that is examined. The average length of a list is  $\alpha = \frac{m}{n}$ . Hence, if  $A$  is the collision resolving strategy “Hashing with Chaining” we have

$$A^- = 1 + \alpha .$$

## Hashing with Chaining

For a successful search observe that we do **not** choose a list at random, but we consider a random key  $k$  in the hash-table and ask for the search-time for  $k$ .

This is 1 plus the number of elements that lie before  $k$  in  $k$ 's list.

Let  $k_\ell$  denote the  $\ell$ -th key inserted into the table.

Let for two keys  $k_i$  and  $k_j$ ,  $X_{ij}$  denote the indicator variable for the event that  $k_i$  and  $k_j$  hash to the same position. Clearly,  $\Pr[X_{ij} = 1] = 1/n$  for uniform hashing.

The expected successful search cost is

$$\mathbb{E} \left[ \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{\substack{j=i+1 \\ \text{cost for key } k_j}}^m X_{ij} \right) \right]$$

keys before  $k_i$

# Hashing with Chaining

$$\begin{aligned} E \left[ \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m X_{ij} \right) \right] &= \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m E[X_{ij}] \right) \\ &= \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m \frac{1}{n} \right) \\ &= 1 + \frac{1}{mn} \sum_{i=1}^m (m - i) \\ &= 1 + \frac{1}{mn} \left( m^2 - \frac{m(m+1)}{2} \right) \\ &= 1 + \frac{m-1}{2n} = 1 + \frac{\alpha}{2} - \frac{\alpha}{2m} . \end{aligned}$$

Hence, the expected cost for a successful search is  $A^+ \leq 1 + \frac{\alpha}{2}$ .

# Hashing with Chaining

## Disadvantages:

- ▶ pointers increase memory requirements
- ▶ pointers may lead to bad cache efficiency

## Advantages:

- ▶ no à priori limit on the number of elements
- ▶ deletion can be implemented efficiently
- ▶ by using balanced trees instead of linked list one can also obtain worst-case guarantees.

# Open Addressing

All objects are stored in the table itself.

Define a function  $h(k, j)$  that determines the table-position to be examined in the  $j$ -th step. The values  $h(k, 0), \dots, h(k, n - 1)$  must form a permutation of  $0, \dots, n - 1$ .

**Search( $k$ ):** Try position  $h(k, 0)$ ; if it is empty your search fails; otw. continue with  $h(k, 1), h(k, 2), \dots$

**Insert( $x$ ):** Search until you find an empty slot; insert your element there. If your search reaches  $h(k, n - 1)$ , and this slot is non-empty then your table is full.



# Open Addressing

Choices for  $h(k, j)$ :

▶ **Linear probing:**

$$h(k, i) = h(k) + i \pmod n$$

(sometimes:  $h(k, i) = h(k) + ci \pmod n$ ).

▶ **Quadratic probing:**

$$h(k, i) = h(k) + c_1i + c_2i^2 \pmod n.$$

▶ **Double hashing:**

$$h(k, i) = h_1(k) + ih_2(k) \pmod n.$$

For quadratic probing and double hashing one has to ensure that the search covers all positions in the table (i.e., for double hashing  $h_2(k)$  must be relatively prime to  $n$  (**teilerfremd**); for quadratic probing  $c_1$  and  $c_2$  have to be chosen carefully).

# Linear Probing

- ▶ Advantage: **Cache-efficiency**. The new probe position is very likely to be in the cache.
- ▶ Disadvantage: **Primary clustering**. Long sequences of occupied table-positions get longer as they have a larger probability to be hit. Furthermore, they can merge forming larger sequences.

## Lemma 2

Let  $L$  be the method of linear probing for resolving collisions:

$$L^+ \approx \frac{1}{2} \left( 1 + \frac{1}{1 - \alpha} \right)$$

$$L^- \approx \frac{1}{2} \left( 1 + \frac{1}{(1 - \alpha)^2} \right)$$

# Quadratic Probing

- ▶ Not as cache-efficient as Linear Probing.
- ▶ **Secondary clustering**: caused by the fact that all keys mapped to the same position have the same probe sequence.

## Lemma 3

Let  $Q$  be the method of quadratic probing for resolving collisions:

$$Q^+ \approx 1 + \ln\left(\frac{1}{1-\alpha}\right) - \frac{\alpha}{2}$$

$$Q^- \approx \frac{1}{1-\alpha} + \ln\left(\frac{1}{1-\alpha}\right) - \alpha$$

# Double Hashing

- ▶ Any probe into the hash-table usually creates a cache-miss.

## Lemma 4

Let  $D$  be the method of double hashing for resolving collisions:

$$D^+ \approx \frac{1}{\alpha} \ln \left( \frac{1}{1 - \alpha} \right)$$

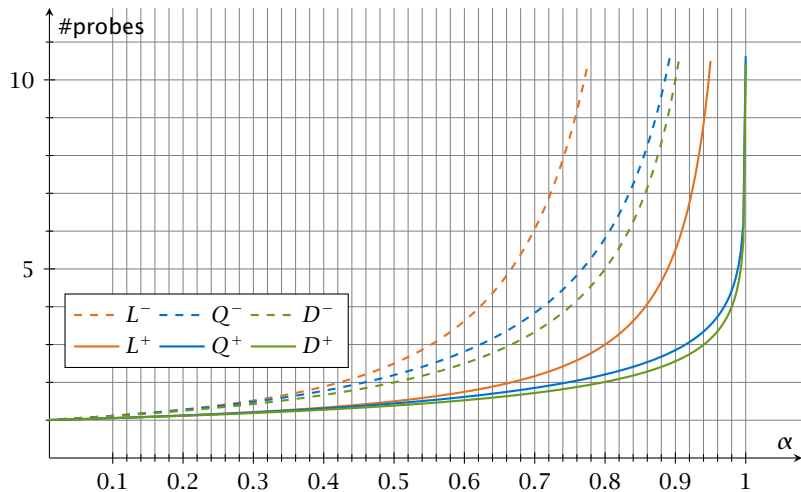
$$D^- \approx \frac{1}{1 - \alpha}$$

# Open Addressing

Some values:

$\alpha$	<i>Linear Probing</i>		<i>Quadratic Probing</i>		<i>Double Hashing</i>	
	$L^+$	$L^-$	$Q^+$	$Q^-$	$D^+$	$D^-$
0.5	1.5	2.5	1.44	2.19	1.39	2
0.9	5.5	50.5	2.85	11.40	2.55	10
0.95	10.5	200.5	3.52	22.05	3.15	20

# Open Addressing



# Analysis of Idealized Open Address Hashing

We analyze the time for a search in a very idealized Open Addressing scheme.

- ▶ The probe sequence  $h(k, 0), h(k, 1), h(k, 2), \dots$  is equally likely to be any permutation of  $\langle 0, 1, \dots, n - 1 \rangle$ .

# Analysis of Idealized Open Address Hashing

Let  $X$  denote a random variable describing the number of probes in an **unsuccessful** search.

Let  $A_i$  denote the event that the  $i$ -th probe **occurs** and is to a non-empty slot.

$$\begin{aligned}\Pr[A_1 \cap A_2 \cap \dots \cap A_{i-1}] \\ &= \Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1 \cap A_2] \cdot \\ &\quad \dots \cdot \Pr[A_{i-1} \mid A_1 \cap \dots \cap A_{i-2}]\end{aligned}$$

$$\begin{aligned}\Pr[X \geq i] &= \frac{m}{n} \cdot \frac{m-1}{n-1} \cdot \frac{m-2}{n-2} \cdot \dots \cdot \frac{m-i+2}{n-i+2} \\ &\leq \left(\frac{m}{n}\right)^{i-1} = \alpha^{i-1} .\end{aligned}$$



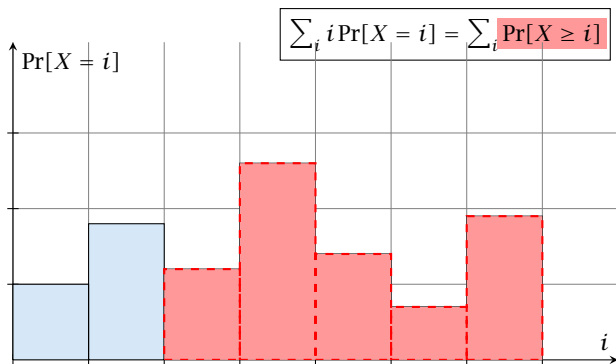
# Analysis of Idealized Open Address Hashing

$$E[X] = \sum_{i=1}^{\infty} \Pr[X \geq i] \leq \sum_{i=1}^{\infty} \alpha^{i-1} = \sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha} .$$

$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \dots$$

# Analysis of Idealized Open Address Hashing

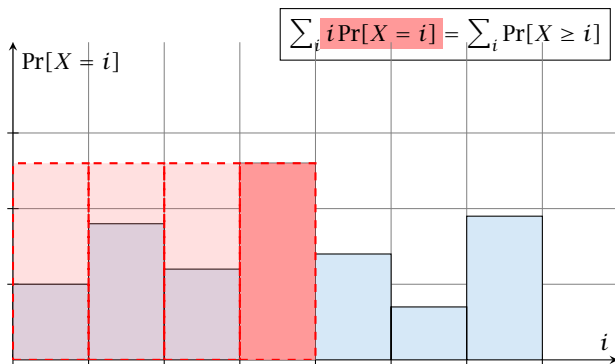
$i = 3$



The  $j$ -th rectangle appears in both sums  $j$  times. ( $j$  times in the first due to multiplication with  $j$ ; and  $j$  times in the second for summands  $i = 1, 2, \dots, j$ )

# Analysis of Idealized Open Address Hashing

$i = 4$



The  $j$ -th rectangle appears in both sums  $j$  times. ( $j$  times in the first due to multiplication with  $j$ ; and  $j$  times in the second for summands  $i = 1, 2, \dots, j$ )

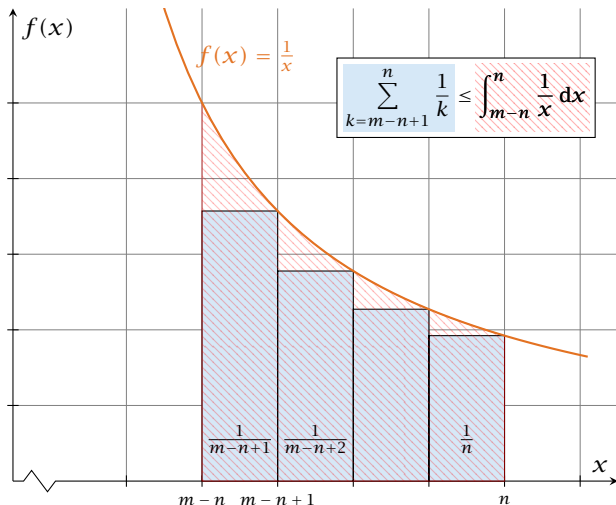
# Analysis of Idealized Open Address Hashing

The number of probes in a **successful** search for  $k$  is equal to the number of probes made in an unsuccessful search for  $k$  at the time that  $k$  is inserted.

Let  $k$  be the  $i + 1$ -st element. The expected time for a search for  $k$  is at most  $\frac{1}{1-i/n} = \frac{n}{n-i}$ .

$$\begin{aligned} \frac{1}{m} \sum_{i=0}^{m-1} \frac{n}{n-i} &= \frac{n}{m} \sum_{i=0}^{m-1} \frac{1}{n-i} = \frac{1}{\alpha} \sum_{k=n-m+1}^n \frac{1}{k} \\ &\leq \frac{1}{\alpha} \int_{n-m}^n \frac{1}{x} dx = \frac{1}{\alpha} \ln \frac{n}{n-m} = \frac{1}{\alpha} \ln \frac{1}{1-\alpha} . \end{aligned}$$

# Analysis of Idealized Open Address Hashing



## How do we delete in a hash-table?

- ▶ For hashing with chaining this is not a problem. Simply search for the key, and delete the item in the corresponding list.
- ▶ For open addressing this is difficult.

# Deletions in Hashtables

- ▶ Simply removing a key might interrupt the probe sequence of other keys which then cannot be found anymore.
- ▶ One can delete an element by replacing it with a **deleted**-marker.
  - ▶ During an insertion if a **deleted**-marker is encountered an element can be inserted there.
  - ▶ During a search a **deleted**-marker must not be used to terminate the probe sequence.
- ▶ The table could fill up with **deleted**-markers leading to bad performance.
- ▶ If a table contains many deleted-markers (linear fraction of the keys) one can rehash the whole table and amortize the cost for this rehash against the cost for the deletions.

# Deletions for Linear Probing

- ▶ For Linear Probing one can delete elements without using **deletion**-markers.
- ▶ Upon a deletion elements that are further down in the probe-sequence may be moved to guarantee that they are still found during a search.



## Deletions for Linear Probing

### Algorithm 37 delete( $p$ )

```
1:  $T[p] \leftarrow \text{null}$ 
2:  $p \leftarrow \text{succ}(p)$ 
3: while  $T[p] \neq \text{null}$  do
4:      $y \leftarrow T[p]$ 
5:      $T[p] \leftarrow \text{null}$ 
6:      $p \leftarrow \text{succ}(p)$ 
7:     insert( $y$ )
```

$p$  is the index into the table-cell that contains the object to be deleted.

Pointers into the hash-table become invalid.

# Universal Hashing

Regardless, of the choice of hash-function there is always an input (a set of keys) that has a very poor worst-case behaviour.

Therefore, so far we assumed that the hash-function is random so that regardless of the input the average case behaviour is good.

However, the assumption of uniform hashing that  $h$  is chosen randomly from all functions  $f : U \rightarrow [0, \dots, n - 1]$  is clearly unrealistic as there are  $n^{|U|}$  such functions. Even writing down such a function would take  $|U| \log n$  bits.

Universal hashing tries to define a set  $\mathcal{H}$  of functions that is much smaller but still leads to good average case behaviour when selecting a hash-function uniformly at random from  $\mathcal{H}$ .

# Universal Hashing

## Definition 5

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called **universal** if for all  $u_1, u_2 \in U$  with  $u_1 \neq u_2$

$$\Pr[h(u_1) = h(u_2)] \leq \frac{1}{n} ,$$

where the probability is w. r. t. the choice of a random hash-function from set  $\mathcal{H}$ .

Note that this means that the probability of a collision between two arbitrary elements is at most  $\frac{1}{n}$ .

# Universal Hashing

## Definition 6

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called **2-independent** (pairwise independent) if the following two conditions hold

- ▶ For any key  $u \in U$ , and  $t \in \{0, \dots, n-1\}$   $\Pr[h(u) = t] = \frac{1}{n}$ , i.e., a key is distributed uniformly within the hash-table.
- ▶ For all  $u_1, u_2 \in U$  with  $u_1 \neq u_2$ , and for any two hash-positions  $t_1, t_2$ :

$$\Pr[h(u_1) = t_1 \wedge h(u_2) = t_2] \leq \frac{1}{n^2} .$$

This requirement clearly implies a universal hash-function.

# Universal Hashing

## Definition 7

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called  **$k$ -independent** if for any choice of  $\ell \leq k$  distinct keys  $u_1, \dots, u_\ell \in U$ , and for any set of  $\ell$  not necessarily distinct hash-positions  $t_1, \dots, t_\ell$ :

$$\Pr[h(u_1) = t_1 \wedge \dots \wedge h(u_\ell) = t_\ell] \leq \frac{1}{n^\ell} ,$$

where the probability is w. r. t. the choice of a random hash-function from set  $\mathcal{H}$ .

## Definition 8

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called  $(\mu, k)$ -independent if for any choice of  $\ell \leq k$  distinct keys  $u_1, \dots, u_\ell \in U$ , and for any set of  $\ell$  not necessarily distinct hash-positions  $t_1, \dots, t_\ell$ :

$$\Pr[h(u_1) = t_1 \wedge \dots \wedge h(u_\ell) = t_\ell] \leq \frac{\mu}{n^\ell},$$

where the probability is w. r. t. the choice of a random hash-function from set  $\mathcal{H}$ .

# Universal Hashing

Let  $U := \{0, \dots, p-1\}$  for a prime  $p$ . Let  $\mathbb{Z}_p := \{0, \dots, p-1\}$ , and let  $\mathbb{Z}_p^* := \{1, \dots, p-1\}$  denote the set of invertible elements in  $\mathbb{Z}_p$ .

Define

$$h_{a,b}(x) := (ax + b \bmod p) \bmod n$$

## Lemma 9

*The class*

$$\mathcal{H} = \{h_{a,b} \mid a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$$

*is a universal class of hash-functions from  $U$  to  $\{0, \dots, n-1\}$ .*

# Universal Hashing

## Proof.

Let  $x, y \in U$  be two distinct keys. We have to show that the probability of a collision is only  $1/n$ .

►  $ax + b \not\equiv ay + b \pmod{p}$

If  $x \neq y$  then  $(x - y) \not\equiv 0 \pmod{p}$ .

Multiplying with  $a \not\equiv 0 \pmod{p}$  gives

$$a(x - y) \not\equiv 0 \pmod{p}$$

where we use that  $\mathbb{Z}_p$  is a field (**Körper**) and, hence, has no zero divisors (**nullteilerfrei**).



## Universal Hashing

- ▶ The hash-function does not generate collisions before the  $(\text{mod } n)$ -operation. Furthermore, every choice  $(a, b)$  is mapped to a different pair  $(t_x, t_y)$  with  $t_x := ax + b$  and  $t_y := ay + b$ .

This holds because we can compute  $a$  and  $b$  when given  $t_x$  and  $t_y$ :

$$t_x \equiv ax + b \pmod{p}$$

$$t_y \equiv ay + b \pmod{p}$$

$$t_x - t_y \equiv a(x - y) \pmod{p}$$

$$t_y \equiv ay + b \pmod{p}$$

$$a \equiv (t_x - t_y)(x - y)^{-1} \pmod{p}$$

$$b \equiv t_y - ay \pmod{p}$$

# Universal Hashing

There is a one-to-one correspondence between hash-functions (pairs  $(a, b)$ ,  $a \neq 0$ ) and pairs  $(t_x, t_y)$ ,  $t_x \neq t_y$ .

Therefore, we can view the first step (before the  $\text{mod } n$ -operation) as choosing a pair  $(t_x, t_y)$ ,  $t_x \neq t_y$  uniformly at random.

What happens when we do the  $\text{mod } n$  operation?

Fix a value  $t_x$ . There are  $p - 1$  possible values for choosing  $t_y$ .

From the range  $0, \dots, p - 1$  the values  $t_x, t_x + n, t_x + 2n, \dots$  map to  $t_x$  after the modulo-operation. These are at most  $\lceil p/n \rceil$  values.

# Universal Hashing

As  $t_y \neq t_x$  there are

$$\left\lceil \frac{p}{n} \right\rceil - 1 \leq \frac{p}{n} + \frac{n-1}{n} - 1 \leq \frac{p-1}{n}$$

possibilities for choosing  $t_y$  such that the final hash-value creates a collision.

This happens with probability at most  $\frac{1}{n}$ .

# Universal Hashing

It is also possible to show that  $\mathcal{H}$  is an (almost) pairwise independent class of hash-functions.

$$\frac{\lfloor \frac{p}{n} \rfloor^2}{p(p-1)} \leq \Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[ \begin{array}{l} t_x \bmod n = h_1 \\ t_y \bmod n = h_2 \end{array} \right] \leq \frac{\lceil \frac{p}{n} \rceil^2}{p(p-1)}$$

Note that the middle is the probability that  $h(x) = h_1$  and  $h(y) = h_2$ . The total number of choices for  $(t_x, t_y)$  is  $p(p-1)$ . The number of choices for  $t_x$  ( $t_y$ ) such that  $t_x \bmod n = h_1$  ( $t_y \bmod n = h_2$ ) lies between  $\lfloor \frac{p}{n} \rfloor$  and  $\lceil \frac{p}{n} \rceil$ .

# Universal Hashing

## Definition 10

Let  $d \in \mathbb{N}$ ;  $q \geq (d + 1)n$  be a prime; and let  $\bar{a} \in \{0, \dots, q - 1\}^{d+1}$ . Define for  $x \in \{0, \dots, q - 1\}$

$$h_{\bar{a}}(x) := \left( \sum_{i=0}^d a_i x^i \bmod q \right) \bmod n .$$

Let  $\mathcal{H}_n^d := \{h_{\bar{a}} \mid \bar{a} \in \{0, \dots, q - 1\}^{d+1}\}$ . The class  $\mathcal{H}_n^d$  is  $(e, d + 1)$ -independent.

Note that in the previous case we had  $d = 1$  and chose  $a_d \neq 0$ .

# Universal Hashing

For the coefficients  $\bar{a} \in \{0, \dots, q-1\}^{d+1}$  let  $f_{\bar{a}}$  denote the polynomial

$$f_{\bar{a}}(x) = \left( \sum_{i=0}^d a_i x^i \right) \bmod q$$

The polynomial is defined by  $d+1$  distinct points.

# Universal Hashing

Fix  $\ell \leq d + 1$ ; let  $x_1, \dots, x_\ell \in \{0, \dots, q - 1\}$  be keys, and let  $t_1, \dots, t_\ell$  denote the corresponding hash-function values.

Let  $A^\ell = \{h_{\bar{a}} \in \mathcal{H} \mid h_{\bar{a}}(x_i) = t_i \text{ for all } i \in \{1, \dots, \ell\}\}$

Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

$$f_{\bar{a}}(x_i) \in \underbrace{\{t_i + \alpha \cdot n \mid \alpha \in \{0, \dots, \lceil \frac{q}{n} \rceil - 1\}\}}_{=: B_i}$$

In order to obtain the cardinality of  $A^\ell$  we choose our polynomial by fixing  $d + 1$  points.

We first fix the values for inputs  $x_1, \dots, x_\ell$ .

We have

$$|B_1| \cdot \dots \cdot |B_\ell|$$

possibilities to do this (so that  $h_{\bar{a}}(x_i) = t_i$ ).

- $A^\ell$  denotes the set of hash-functions such that every  $x_i$  hits its pre-defined position  $t_i$ .
- $B_i$  is the set of positions that  $f_{\bar{a}}$  can hit so that  $h_{\bar{a}}$  still hits  $t_i$ .

# Universal Hashing

Now, we choose  $d - \ell + 1$  other inputs and choose their value arbitrarily. We have  $q^{d-\ell+1}$  possibilities to do this.

Therefore we have

$$|B_1| \cdot \dots \cdot |B_\ell| \cdot q^{d-\ell+1} \leq \left\lceil \frac{q}{n} \right\rceil^\ell \cdot q^{d-\ell+1}$$

possibilities to choose  $\bar{a}$  such that  $h_{\bar{a}} \in A_\ell$ .



# Universal Hashing

Therefore the probability of choosing  $h_{\bar{a}}$  from  $A_\ell$  is only

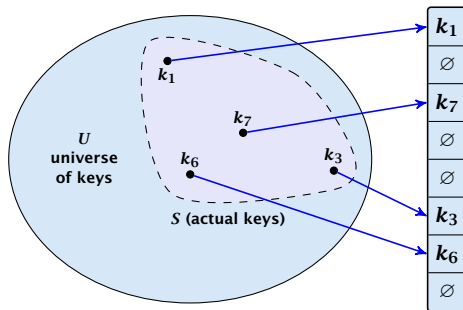
$$\begin{aligned}\frac{\lceil \frac{q}{n} \rceil^\ell \cdot q^{d-\ell+1}}{q^{d+1}} &\leq \frac{(\frac{q+n}{n})^\ell}{q^\ell} \leq \left(\frac{q+n}{q}\right)^\ell \cdot \frac{1}{n^\ell} \\ &\leq \left(1 + \frac{1}{\ell}\right)^\ell \cdot \frac{1}{n^\ell} \leq \frac{e}{n^\ell}.\end{aligned}$$

This shows that the  $\mathcal{H}$  is  $(e, d+1)$ -universal.

The last step followed from  $q \geq (d+1)n$ , and  $\ell \leq d+1$ .

# Perfect Hashing

Suppose that we **know** the set  $S$  of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



# Perfect Hashing

Let  $m = |S|$ . We could simply choose the hash-table size very large so that we don't get any collisions.

Using a universal hash-function the expected number of collisions is

$$E[\#\text{Collisions}] = \binom{m}{2} \cdot \frac{1}{n}.$$

If we choose  $n = m^2$  the **expected number** of collisions is strictly less than  $\frac{1}{2}$ .

Can we get an upper bound on the **probability of having collisions**?

The probability of having **1** or more collisions can be at most  $\frac{1}{2}$  as otherwise the expectation would be larger than  $\frac{1}{2}$ .

# Perfect Hashing

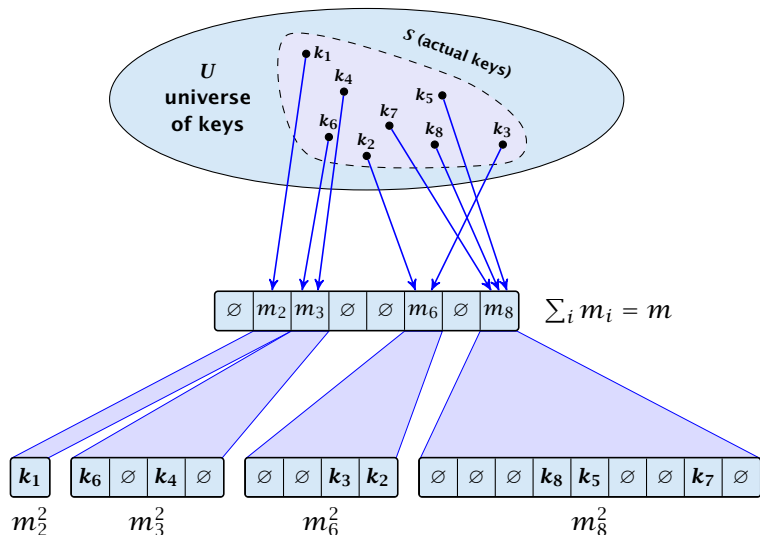
We can find such a hash-function by a few trials.

However, a hash-table size of  $n = m^2$  is very very high.

We construct a two-level scheme. We first use a hash-function that maps elements from  $S$  to  $m$  buckets.

Let  $m_j$  denote the number of items that are hashed to the  $j$ -th bucket. For each bucket we choose a second hash-function that maps the elements of the bucket into a table of size  $m_j^2$ . The second function can be chosen such that all elements are mapped to different locations.

# Perfect Hashing



# Perfect Hashing

The total memory that is required by all hash-tables is  $\mathcal{O}(\sum_j m_j^2)$ . Note that  $m_j$  is a random variable.

$$\begin{aligned} \mathbb{E} \left[ \sum_j m_j^2 \right] &= \mathbb{E} \left[ 2 \sum_j \binom{m_j}{2} + \sum_j m_j \right] \\ &= 2 \mathbb{E} \left[ \sum_j \binom{m_j}{2} \right] + \mathbb{E} \left[ \sum_j m_j \right] \end{aligned}$$

The first expectation is simply the expected number of collisions, for the first level. Since we use universal hashing we have

$$= 2 \binom{m}{2} \frac{1}{m} + m = 2m - 1 .$$

# Perfect Hashing

We need only  $\mathcal{O}(m)$  time to construct a hash-function  $h$  with  $\sum_j m_j^2 = \mathcal{O}(4m)$ , because with probability at least  $1/2$  a random function from a universal family will have this property.

Then we construct a hash-table  $h_j$  for every bucket. This takes expected time  $\mathcal{O}(m_j)$  for every bucket. A random function  $h_j$  is collision-free with probability at least  $1/2$ . We need  $\mathcal{O}(m_j)$  to test this.

We only need that the hash-functions are chosen from a universal family!!!

# Cuckoo Hashing

## Goal:

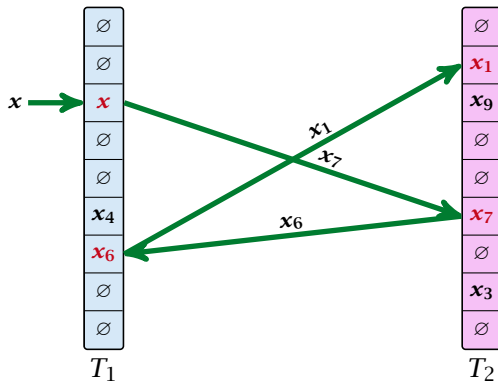
Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

- ▶ Two hash-tables  $T_1[0, \dots, n - 1]$  and  $T_2[0, \dots, n - 1]$ , with hash-functions  $h_1$ , and  $h_2$ .
- ▶ An object  $x$  is either stored at location  $T_1[h_1(x)]$  or  $T_2[h_2(x)]$ .
- ▶ A search clearly takes constant time if the above constraint is met.



# Cuckoo Hashing

Insert:



## Algorithm 38 Cuckoo-Insert( $x$ )

```
1: if  $T_1[h_1(x)] = x \vee T_2[h_2(x)] = x$  then return  
2: steps  $\leftarrow$  1  
3: while steps  $\leq$  maxsteps do  
4:     exchange  $x$  and  $T_1[h_1(x)]$   
5:     if  $x = \text{null}$  then return  
6:     exchange  $x$  and  $T_2[h_2(x)]$   
7:     if  $x = \text{null}$  then return  
8:     steps  $\leftarrow$  steps + 1  
9: rehash() // change hash-functions; rehash everything  
10: Cuckoo-Insert( $x$ )
```

# Cuckoo Hashing

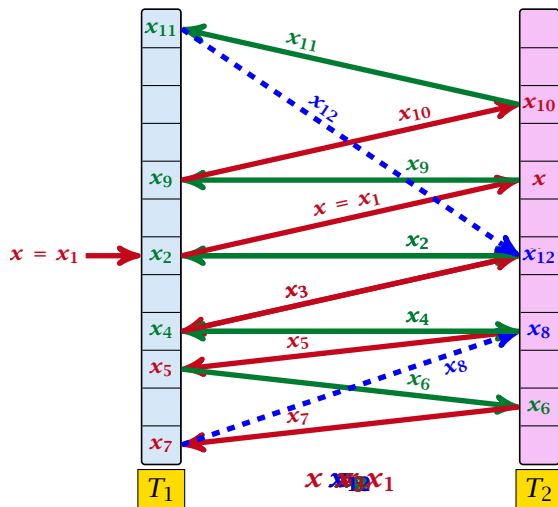
- ▶ We call one iteration through the while-loop a **step** of the algorithm.
- ▶ We call a sequence of iterations through the while-loop without the termination condition becoming true a **phase** of the algorithm.
- ▶ We say a phase is **successful** if it is not terminated by the **maxstep**-condition, but the while loop is left because  $x = \text{null}$ .

## What is the expected time for an insert-operation?

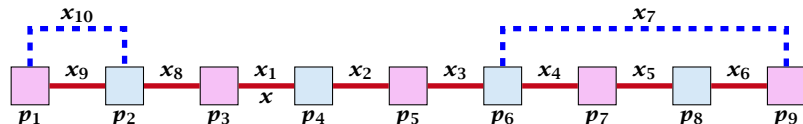
We first analyze the probability that we end-up in an infinite loop (that is then terminated after  $\text{maxsteps}$  steps).

Formally what is the probability to enter an infinite loop that touches  $s$  different keys?

# Cuckoo Hashing: Insert



# Cuckoo Hashing



A **cycle-structure of size  $s$**  is defined by

- ▶  $s - 1$  different cells (alternating btw. cells from  $T_1$  and  $T_2$ ).
- ▶  $s$  distinct keys  $x = x_1, x_2, \dots, x_s$ , linking the cells.
- ▶ The leftmost cell is “linked forward” to some cell on the right.
- ▶ The rightmost cell is “linked backward” to a cell on the left.
- ▶ One link represents key  $x$ ; this is where the counting starts.

# Cuckoo Hashing

A cycle-structure is **active** if for every key  $x_\ell$  (linking a cell  $p_i$  from  $T_1$  and a cell  $p_j$  from  $T_2$ ) we have

$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

## Observation:

If during a phase the insert-procedure runs into a cycle there must exist an active cycle structure of size  $s \geq 3$ .

# Cuckoo Hashing

What is the probability that all keys in a cycle-structure of size  $s$  correctly map into their  $T_1$ -cell?

This probability is at most  $\frac{\mu}{n^s}$  since  $h_1$  is a  $(\mu, s)$ -independent hash-function.

What is the probability that all keys in the cycle-structure of size  $s$  correctly map into their  $T_2$ -cell?

This probability is at most  $\frac{\mu}{n^s}$  since  $h_2$  is a  $(\mu, s)$ -independent hash-function.

These events are independent.



# Cuckoo Hashing

The probability that a given cycle-structure of size  $s$  is active is at most  $\frac{\mu^2}{n^{2s}}$ .

What is the probability that **there exists** an active cycle structure of size  $s$ ?

# Cuckoo Hashing

The number of cycle-structures of size  $s$  is at most

$$s^3 \cdot n^{s-1} \cdot m^{s-1} .$$

- ▶ There are at most  $s^2$  possibilities where to attach the forward and backward links.
- ▶ There are at most  $s$  possibilities to choose where to place key  $x$ .
- ▶ There are  $m^{s-1}$  possibilities to choose the keys apart from  $x$ .
- ▶ There are  $n^{s-1}$  possibilities to choose the cells.

# Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\begin{aligned} \sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} &= \frac{\mu^2}{nm} \sum_{s=3}^{\infty} s^3 \left(\frac{m}{n}\right)^s \\ &\leq \frac{\mu^2}{m^2} \sum_{s=3}^{\infty} s^3 \left(\frac{1}{1+\epsilon}\right)^s \leq \mathcal{O}\left(\frac{1}{m^2}\right). \end{aligned}$$

Here we used the fact that  $(1 + \epsilon)m \leq n$ .

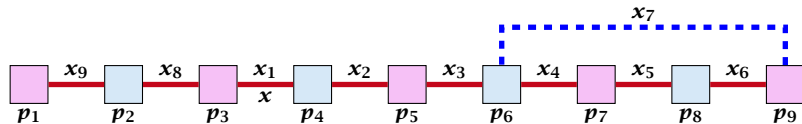
Hence,

$$\Pr[\text{cycle}] = \mathcal{O}\left(\frac{1}{m^2}\right).$$

# Cuckoo Hashing

Now, we analyze the probability that a phase is not successful without running into a closed cycle.

# Cuckoo Hashing



Sequence of visited keys:

$x = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_3, x_2, x_1 = x, x_8, x_9, \dots$

# Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with  $x$  in the order that they are visited during the phase.

## Lemma 11

*If the sequence is of length  $p$  then there exists a sub-sequence of at least  $\frac{p+2}{3}$  keys starting with  $x$  of *distinct* keys.*

# Cuckoo Hashing

Taking  $x_1 \rightarrow \dots \rightarrow x_i$  twice, and  $x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$  once gives  $2i + (j - i + 1) = i + j + 1 \geq p + 2$  keys. Hence, one of the sequences contains at least  $(p + 2)/3$  keys.

## Proof.

Let  $i$  be the number of keys (including  $x$ ) that we see before the first repeated key. Let  $j$  denote the total number of distinct keys.

The sequence is of the form:

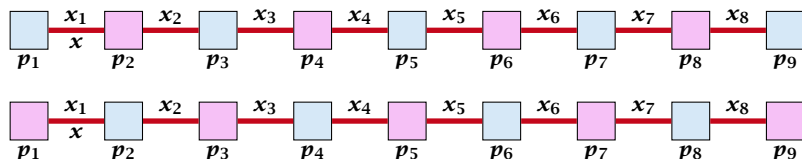
$$x = x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i \rightarrow x_r \rightarrow x_{r-1} \rightarrow \dots \rightarrow x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$$

As  $r \leq i - 1$  the length  $p$  of the sequence is

$$p = i + r + (j - i) \leq i + j - 1 .$$

Either sub-sequence  $x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i$  or sub-sequence  $x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$  has at least  $\frac{p+2}{3}$  elements. □

# Cuckoo Hashing



A path-structure of size  $s$  is defined by

- ▶  $s + 1$  different cells (alternating btw. cells from  $T_1$  and  $T_2$ ).
- ▶  $s$  distinct keys  $x = x_1, x_2, \dots, x_s$ , linking the cells.
- ▶ The leftmost cell is either from  $T_1$  or  $T_2$ .



# Cuckoo Hashing

A path-structure is **active** if for every key  $x_\ell$  (linking a cell  $p_i$  from  $T_1$  and a cell  $p_j$  from  $T_2$ ) we have

$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

## Observation:

If a phase takes at least  $t$  steps without running into a cycle there must exist an active path-structure of size  $(2t + 2)/3$ .

Note that we count **complete** steps. A search that touches  $2t$  or  $2t + 1$  keys takes  $t$  steps.

# Cuckoo Hashing

The probability that a given path-structure of size  $s$  is active is at most  $\frac{\mu^2}{n^{2s}}$ .

The probability that there exists an active path-structure of size  $s$  is at most

$$2 \cdot n^{s+1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} \\ \leq 2\mu^2 \left(\frac{m}{n}\right)^{s-1} \leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{s-1}$$

Plugging in  $s = (2t + 2)/3$  gives

$$\leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{(2t+2)/3-1} = 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{(2t-1)/3}.$$

# Cuckoo Hashing

We choose  $\text{maxsteps} \geq 3\ell/2 + 1/2$ . Then the probability that a phase terminates unsuccessfully without running into a cycle is at most

$$\begin{aligned} & \Pr[\text{unsuccessful} \mid \text{no cycle}] \\ & \leq \Pr[\exists \text{ active path-structure of size at least } \frac{2\text{maxsteps}+2}{3}] \\ & \leq \Pr[\exists \text{ active path-structure of size at least } \ell + 1] \\ & \leq \Pr[\exists \text{ active path-structure of size exactly } \ell + 1] \\ & \leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^\ell \leq \frac{1}{m^2} \end{aligned}$$

by choosing  $\ell \geq \log\left(\frac{1}{2\mu^2 m^2}\right) / \log\left(\frac{1}{1+\epsilon}\right) = \log(2\mu^2 m^2) / \log(1+\epsilon)$

This gives  $\text{maxsteps} = \Theta(\log m)$ .

Note that the existence of a path structure of size larger than  $s$  implies the existence of a path structure of size exactly  $s$ .

# Cuckoo Hashing

So far we estimated

$$\Pr[\text{cycle}] \leq \mathcal{O}\left(\frac{1}{m^2}\right)$$

and

$$\Pr[\text{unsuccessful} \mid \text{no cycle}] \leq \mathcal{O}\left(\frac{1}{m^2}\right)$$

Observe that

$$\begin{aligned}\Pr[\text{successful}] &= \Pr[\text{no cycle}] - \Pr[\text{unsuccessful} \mid \text{no cycle}] \\ &\geq c \cdot \Pr[\text{no cycle}]\end{aligned}$$

for a suitable constant  $c > 0$ .

This is a very weak (and trivial) statement but still sufficient for our asymptotic analysis.

# Cuckoo Hashing

The expected number of complete steps in the **successful phase** of an insert operation is:

$$\begin{aligned} & E[\text{number of steps} \mid \text{phase successful}] \\ &= \sum_{t \geq 1} \Pr[\text{search takes at least } t \text{ steps} \mid \text{phase successful}] \end{aligned}$$

We have

$$\begin{aligned} & \Pr[\text{search at least } t \text{ steps} \mid \text{successful}] \\ &= \Pr[\text{search at least } t \text{ steps} \wedge \text{successful}] / \Pr[\text{successful}] \\ &\leq \frac{1}{c} \Pr[\text{search at least } t \text{ steps} \wedge \text{successful}] / \Pr[\text{no cycle}] \\ &\leq \frac{1}{c} \Pr[\text{search at least } t \text{ steps} \wedge \text{no cycle}] / \Pr[\text{no cycle}] \\ &= \frac{1}{c} \Pr[\text{search at least } t \text{ steps} \mid \text{no cycle}] . \end{aligned}$$

$$\Pr[A \mid B] = \frac{\Pr[A \wedge B]}{\Pr[B]}$$

# Cuckoo Hashing

Hence,

$E[\text{number of steps} \mid \text{phase successful}]$

$$\begin{aligned} &\leq \frac{1}{c} \sum_{t \geq 1} \Pr[\text{search at least } t \text{ steps} \mid \text{no cycle}] \\ &\leq \frac{1}{c} \sum_{t \geq 1} 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{(2t-1)/3} = \frac{1}{c} \sum_{t \geq 0} 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{(2(t+1)-1)/3} \\ &= \frac{2\mu^2}{c(1+\epsilon)^{1/3}} \sum_{t \geq 0} \left(\frac{1}{(1+\epsilon)^{2/3}}\right)^t = \mathcal{O}(1) . \end{aligned}$$

This means the expected cost for a successful phase is constant (even after accounting for the cost of the incomplete step that finishes the phase).

# Cuckoo Hashing

A phase that is not successful induces cost for doing a complete rehash (this dominates the cost for the steps in the phase).

The probability that a phase is not successful is  $q = \mathcal{O}(1/m^2)$  (probability  $\mathcal{O}(1/m^2)$  of running into a cycle and probability  $\mathcal{O}(1/m^2)$  of reaching `maxsteps` without running into a cycle).

A rehash try requires  $m$  insertions and takes expected constant time per insertion. It fails with probability  $p := \mathcal{O}(1/m)$ .

The expected number of unsuccessful rehashes is

$$\sum_{i \geq 1} p^i = \frac{1}{1-p} - 1 = \frac{p}{1-p} = \mathcal{O}(p).$$

Therefore the expected cost for re-hashes is  $\mathcal{O}(m) \cdot \mathcal{O}(p) = \mathcal{O}(1)$ .

## Formal Proof

Let  $Y_i$  denote the event that the  $i$ -th rehash occurs and does not lead to a valid configuration (i.e., one of the  $m + 1$  insertions fails):

$$\Pr[Y_i | Z_i] \leq (m + 1) \cdot \mathcal{O}(1/m^2) \leq \mathcal{O}(1/m) =: p .$$

Let  $Z_i$  denote the event that the  $i$ -th rehash occurs:

The 0-th (re)hash is the initial configuration when doing the insert.

$$\Pr[Z_i] \leq \prod_{j=0}^{i-1} \Pr[Y_h | Z_j] \leq p^i$$

Let  $X_i^s$ ,  $s \in \{1, \dots, m + 1\}$  denote the cost for inserting the  $s$ -th element during the  $i$ -th rehash (assuming  $i$ -th rehash occurs):

$$\begin{aligned} \mathbb{E}[X_i^s] &= \mathbb{E}[\text{steps} \mid \text{phase successful}] \cdot \Pr[\text{phase successful}] \\ &\quad + \text{maxsteps} \cdot \Pr[\text{not successful}] = \mathcal{O}(1) . \end{aligned}$$



The expected cost for all rehashes is

$$\mathbb{E} \left[ \sum_i \sum_s Z_i X_i^s \right]$$

Note that  $Z_i$  is independent of  $X_j^s$ ,  $j \geq i$  (however, it is not independent of  $X_j^s$ ,  $j < i$ ). Hence,

$$\begin{aligned} \mathbb{E} \left[ \sum_i \sum_s Z_i X_i^s \right] &= \sum_i \sum_s \mathbb{E}[Z_i] \cdot \mathbb{E}[X_i^s] \\ &\leq \mathcal{O}(m) \cdot \sum_i p^i \\ &\leq \mathcal{O}(m) \cdot \frac{p}{1-p} \\ &= \mathcal{O}(1) . \end{aligned}$$

## What kind of hash-functions do we need?

Since  $\text{maxsteps}$  is  $\Theta(\log m)$  the largest size of a path-structure or cycle-structure contains just  $\Theta(\log m)$  different keys.

Therefore, it is sufficient to have  $(\mu, \Theta(\log m))$ -independent hash-functions.

# Cuckoo Hashing

How do we make sure that  $n \geq (1 + \epsilon)m$ ?

- ▶ Let  $\alpha := 1/(1 + \epsilon)$ .
- ▶ Keep track of the number of elements in the table. When  $m \geq \alpha n$  we double  $n$  and do a complete re-hash (**table-expand**).
- ▶ Whenever  $m$  drops below  $\alpha n/4$  we divide  $n$  by 2 and do a rehash (**table-shrink**).
- ▶ Note that right after a change in table-size we have  $m = \alpha n/2$ . In order for a table-expand to occur at least  $\alpha n/2$  insertions are required. Similar, for a table-shrink at least  $\alpha n/4$  deletions must occur.
- ▶ Therefore we can amortize the rehash cost after a change in table-size against the cost for insertions and deletions.

# Cuckoo Hashing

## Lemma 12

*Cuckoo Hashing has an expected constant insert-time and a worst-case constant search-time.*

Note that the above lemma only holds if the fill-factor (number of keys/total number of hash-table slots) is at most  $\frac{1}{2(1+\epsilon)}$ .

The  $1/(2(1+\epsilon))$  fill-factor comes from the fact that the total hash-table is of size  $2n$  (because we have two tables of size  $n$ ); moreover  $m \leq (1+\epsilon)n$ .

## Bibliography

- [MS08] Kurt Mehlhorn, Peter Sanders:  
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- [CLRS90] Thomas H. Cormen, Charles E. Leiserson, Ron L. Rivest, Clifford Stein:  
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MIT Press and McGraw-Hill, 2009

Chapter 4 of [MS08] contains a detailed description about Hashing with Linear Probing and Hashing with Chaining. Also the Perfect Hashing scheme can be found there.

The analysis of Hashing with Chaining under the assumption of uniform hashing can be found in Chapter 11.2 of [CLRS90]. Chapter 11.3.3 describes Universal Hashing. Collision resolution with Open Addressing is described in Chapter 11.4. Chapter 11.5 describes the Perfect Hashing scheme.

Reference for Cuckoo Hashing???