4 Modelling Issues

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- Implementing and testing on representative inputs
  - How do you choose your inputs?
  - May be very time-consuming.
  - Very reliable results if done correctly.
  - Results only hold for a specific machine and for a specific set of inputs.

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  - Results only hold for a specific machine and for a specific set of inputs.

- Theoretical analysis in a specific model of computation.
  - Gives asymptotic bounds like “this algorithm always runs in time $\mathcal{O}(n^2)$”.
  - Typically focuses on the worst case.
  - Can give lower bounds like “any comparison-based sorting algorithm needs at least $\Omega(n \log n)$ comparisons in the worst case”.

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Example 1

Suppose n numbers from the interval $\{1, \ldots, N\}$ have to be sorted. In this case we usually say that the input length is n instead of e.g. $n \log N$, which would be the number of bits required to encode the input.

Model of Computation

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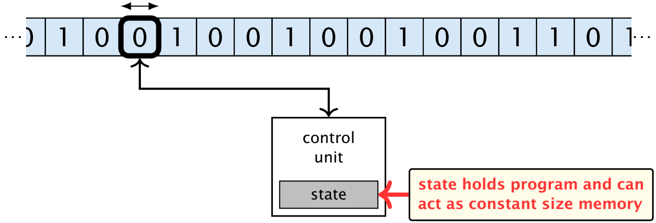
Version 2. is often easier, but focusing on one type of operation makes it more difficult to obtain meaningful results.

Turing Machine

- Very simple model of computation.



Turing Machine

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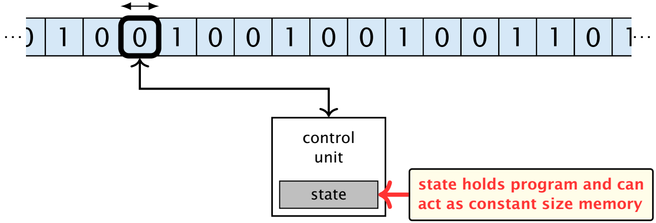

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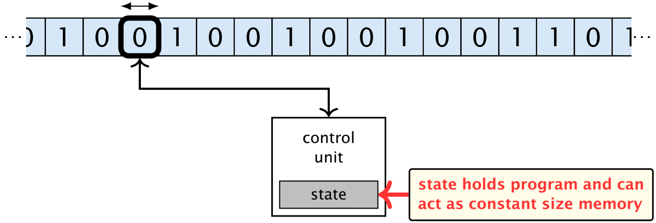

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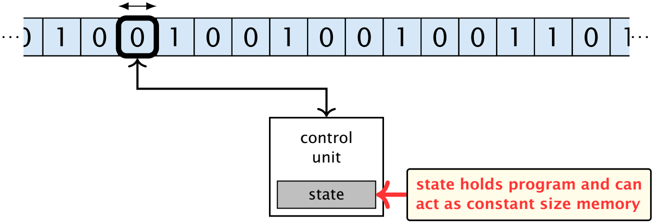

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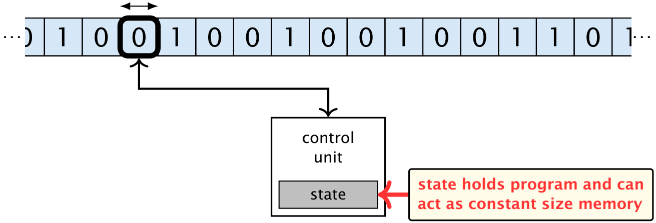
$\Rightarrow$ **Not a good model for developing efficient algorithms.**



Random Access Machine (RAM)

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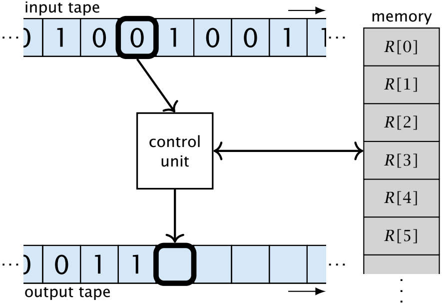

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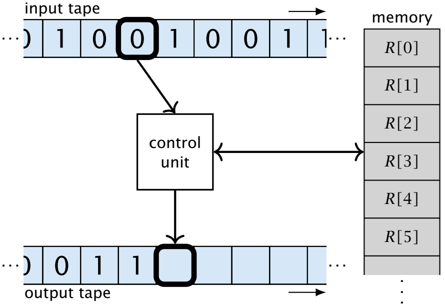

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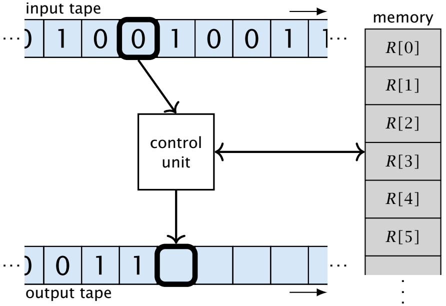

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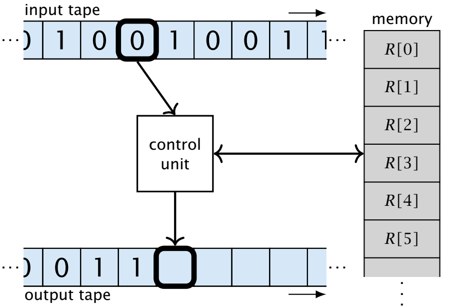

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The jump-directives are very close to the jump-instructions contained in the assembler language of real machines.

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  - $R[i]$ `:=` $R[j]$ `+` $R[k]$`;`
    $R[i]$ `:=` `-`$R[k]$`;`

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Model of Computation

- uniform cost model
  Every operation takes time 1.

The latter model is quite realistic as the word-size of a standard computer that handles a problem of size n must be at least $\log_2 n$ as otherwise the computer could either not store the problem instance or not address all its memory.

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Bounded word RAM model: cost is uniform but the largest value stored in a register may not exceed 2^w, where usually $w = \log_2 n$.

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Example 2

Algorithm 1 RepeatedSquaring(n)

1: $r \leftarrow 2$;
2: **for** $i = 1 \rightarrow n$ **do**
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- best-case complexity:

$$C_{bc}(n) := \min\{C(x) \mid |x| = n\}$$

  Usually easy to analyze, but not very meaningful.

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  more general: probability measure μ

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- amortized complexity:
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- randomized complexity:
  The algorithm may use random bits. Expected running time (over all possible choices of random bits) for a fixed input x. Then take the worst-case over all x with $|x| = n$.

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