Part III

Data Structures

Abstract Data Type

An abstract data type (ADT) is defined by an interface of operations or methods that can be performed and that have a defined behavior.

The data types in this lecture all operate on objects that are represented by a [key, value] pair.

- The key comes from a totally ordered set, and we assume that there is an efficient comparison function.
- The value can be anything; it usually carries satellite information important for the application that uses the ADT.

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  Requires $\mathrm{key}[S.\,\mathrm{maximum}()] \le \mathrm{key}[S'.\,\mathrm{minimum}()]$.

- $\boldsymbol{S.\,\mathrm{decrease\text{-}key}(x, k)}$: Replace $\mathrm{key}[x]$ by $k \le \mathrm{key}[x]$.

Examples of ADTs

Stack:

- $S.\mathbf{push}(x)$: Insert an element.
- $S.\mathbf{pop}()$: Return the element from S that was inserted most recently; delete it from S.
- $S.\mathbf{empty}()$: Tell if S contains any object.

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Queue:

- $\boldsymbol{S}.\,\mathbf{enqueue}(\boldsymbol{x})$: Insert an element.
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Priority-Queue:

- $S.\mathrm{insert}(x)$: Insert an element.
- $S.\mathrm{delete\text{-}min}()$: Return the element with lowest key-value; delete it from S.

7 Dictionary

Dictionary:

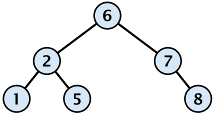
- $\boldsymbol{S.\,\mathrm{insert}(x)}$: Insert an element x.
- $\boldsymbol{S.\,\mathrm{delete}(x)}$: Delete the element pointed to by x.
- $\boldsymbol{S.\,\mathrm{search}(k)}$: Return a pointer to an element e with $\mathrm{key}[e] = k$ in S if it exists; otherwise return null.

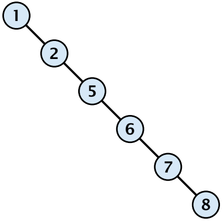
7.1 Binary Search Trees

An (internal) binary search tree stores the elements in a binary tree. Each tree-node corresponds to an element. All elements in the left sub-tree of a node v have a smaller key-value than $\text{key}[v]$ and elements in the right sub-tree have a larger-key value. We assume that all key-values are different.

(External Search Trees store objects only at leaf-vertices)

Examples:





7.1 Binary Search Trees

We consider the following operations on binary search trees. Note that this is a super-set of the dictionary-operations.

- $T.\textbf{insert}(x)$
- $T.\textbf{delete}(x)$
- $T.\textbf{search}(k)$
- $T.\textbf{successor}(x)$
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- $T.\textbf{minimum}()$
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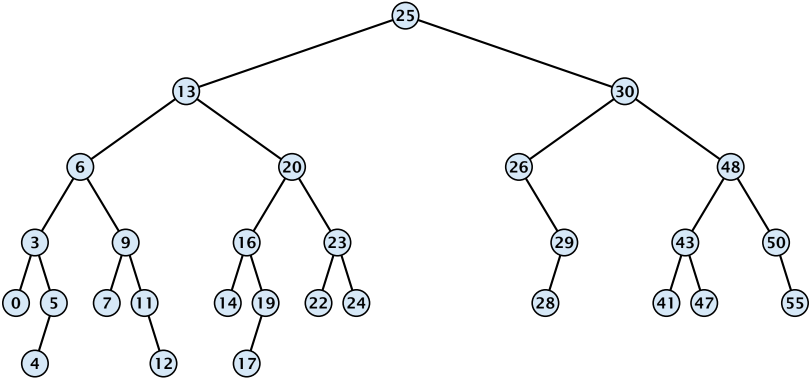
Binary Search Trees: Searching



Algorithm 1 TreeSearch(x, k)

if $x = \text{null}$ **or** $k = \text{key}[x]$ **return** x

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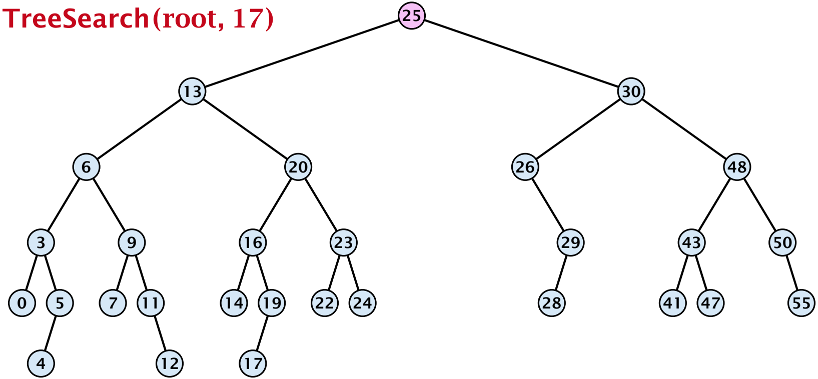

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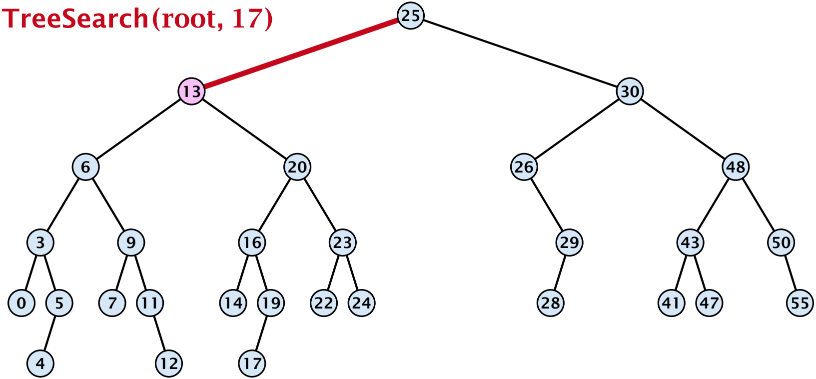

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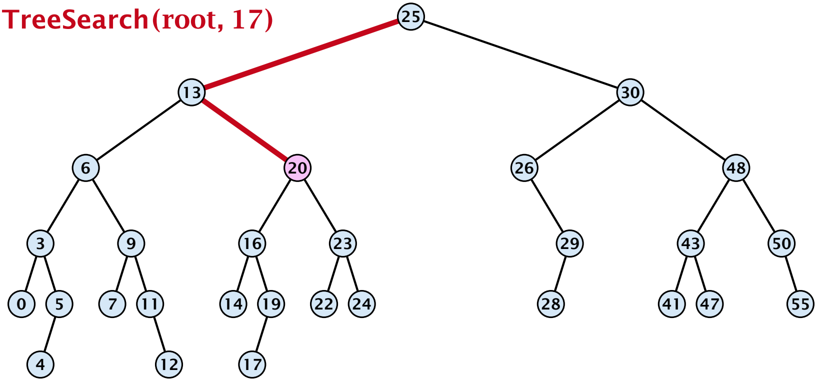

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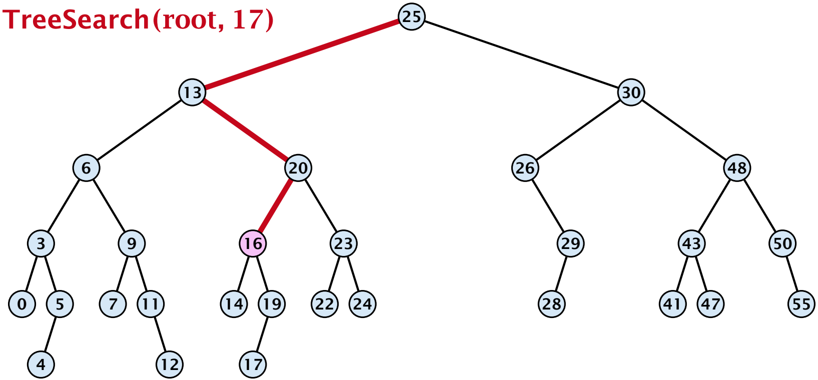

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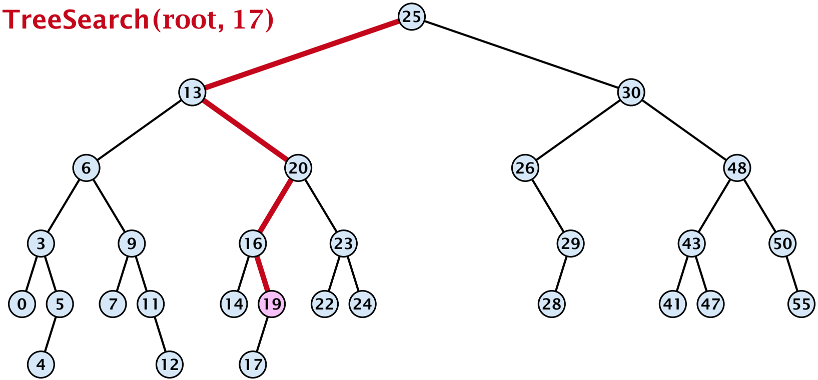

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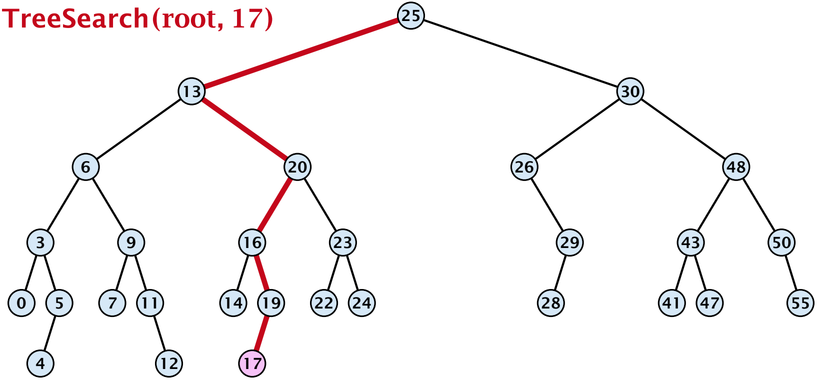

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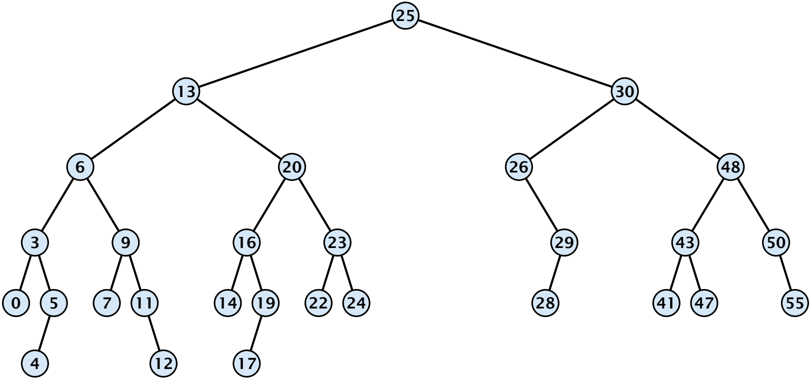

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Binary Search Trees: Searching

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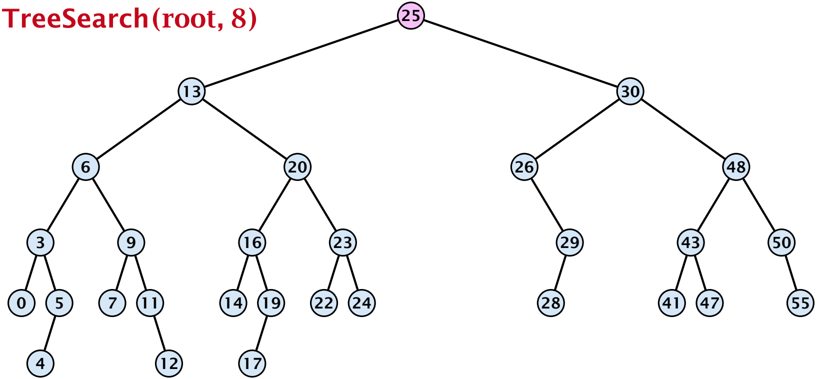

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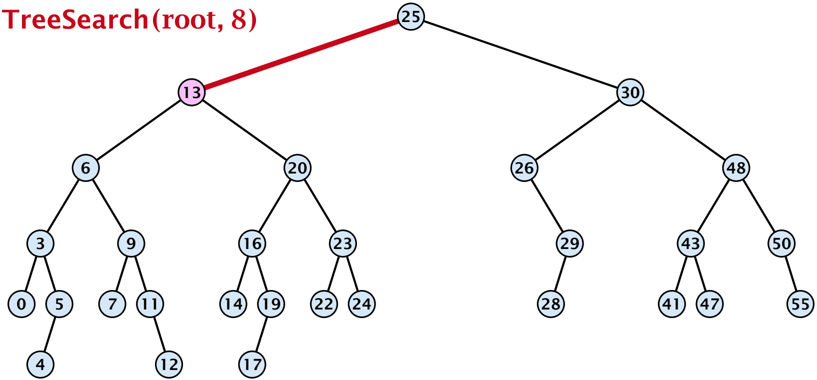

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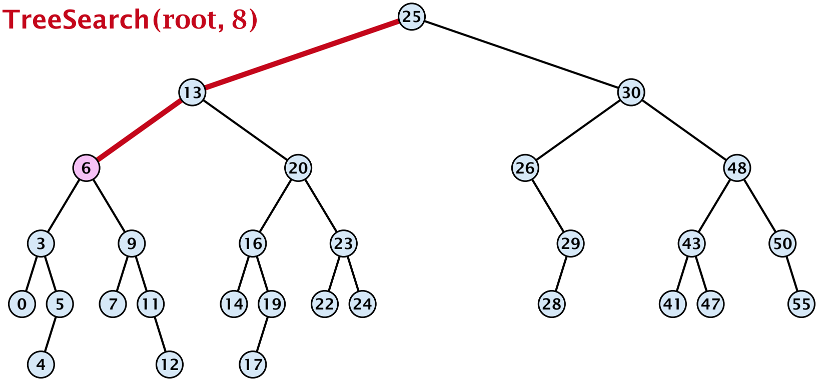

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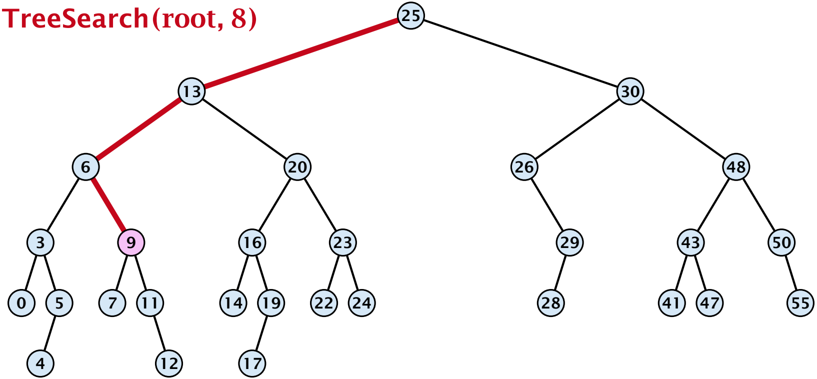

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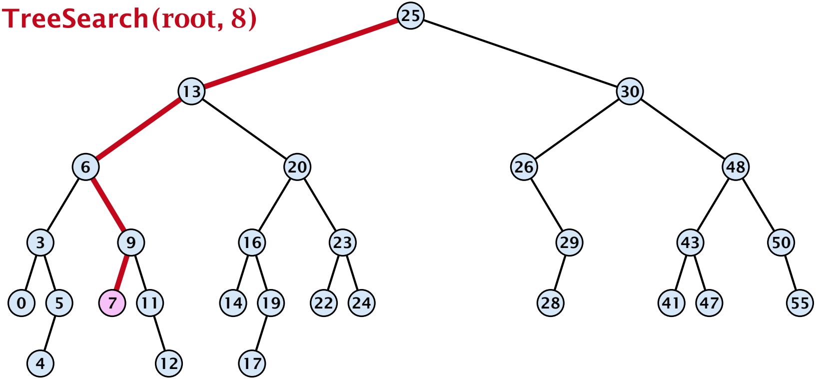

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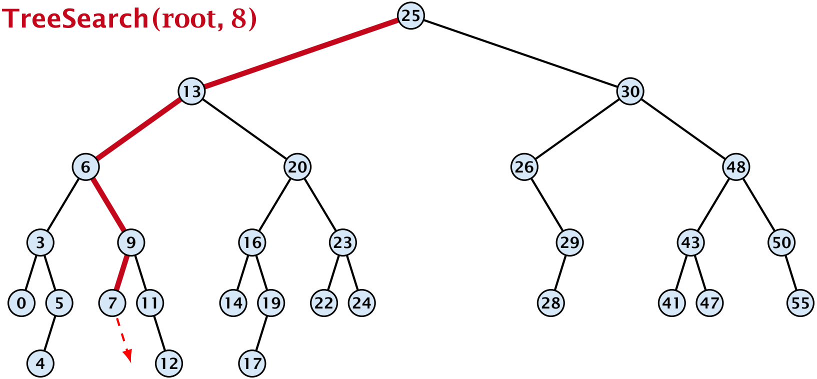

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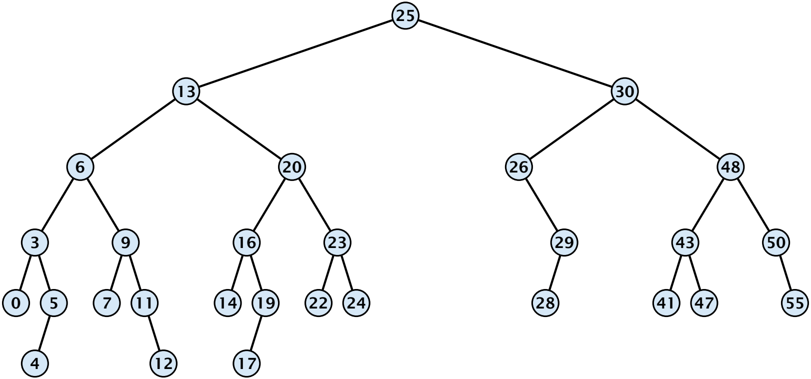

Algorithm 2 TreeMin(x)

1: **if** x = null **or** left[x] = null **return** x

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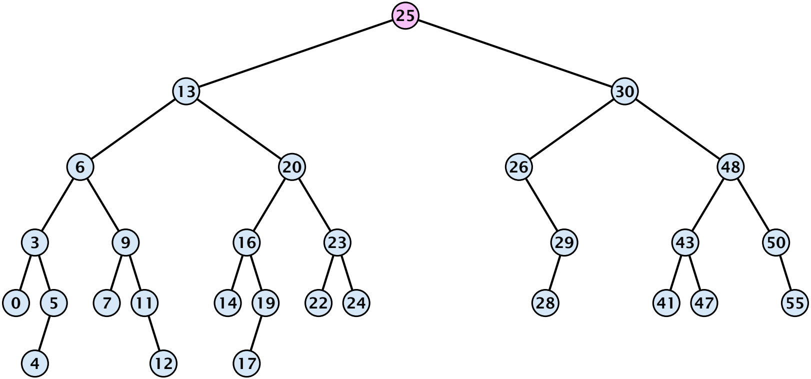
Binary Search Trees: Minimum



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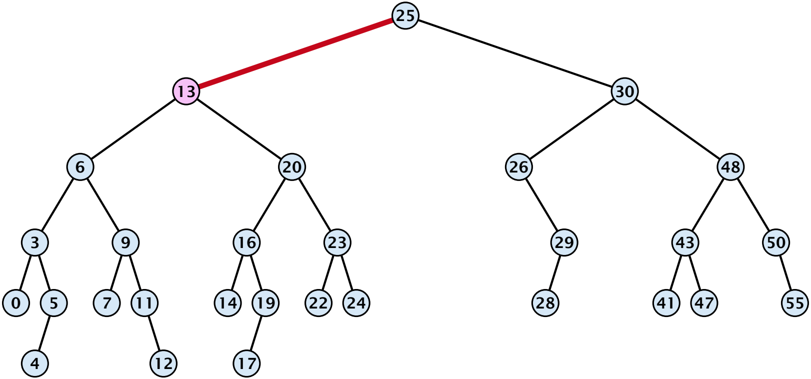

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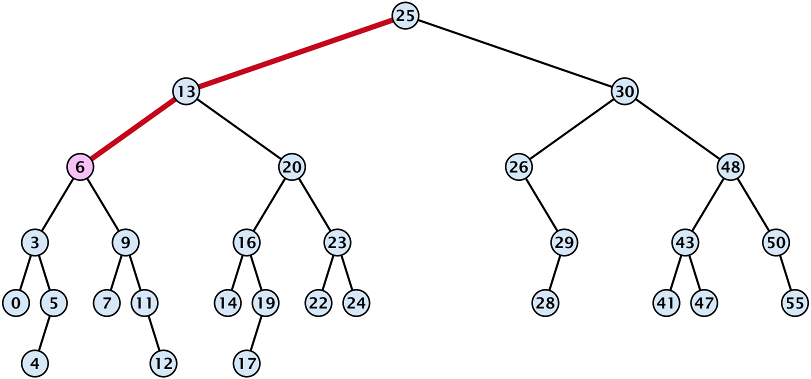

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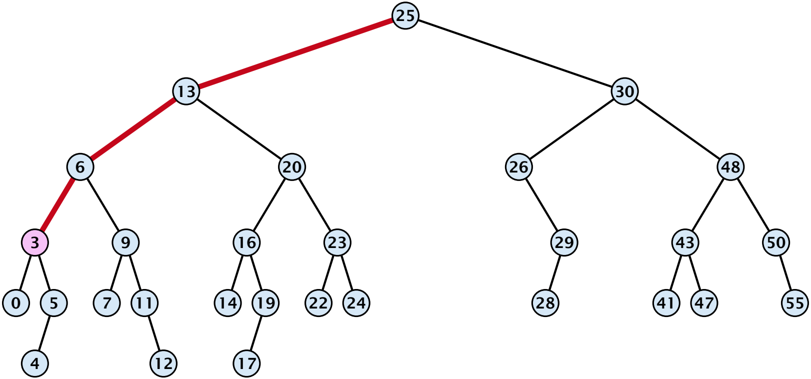

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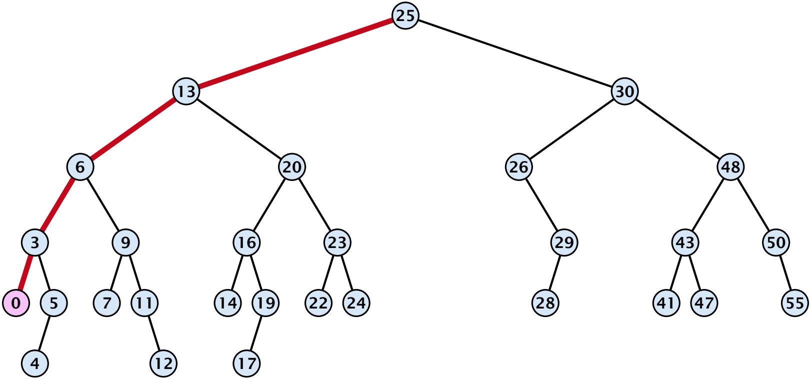

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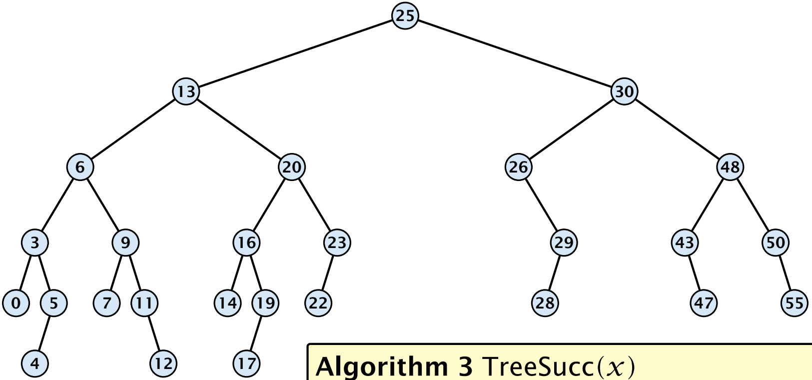
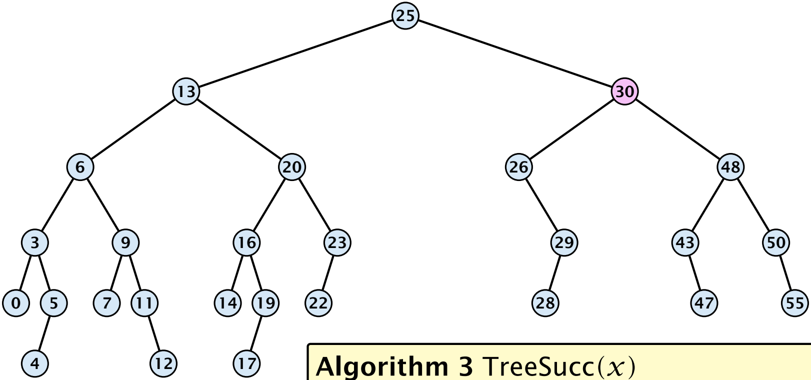
Binary Search Trees: Successor



Algorithm 3 TreeSucc(x)

1: **if** right[x] $\neq$ null **return** TreeMin(right[x])
2: $y \leftarrow$ parent[x]
3: **while** $y \neq$ null **and** $x =$ right[y] **do**
4: $\quad x \leftarrow y$; $y \leftarrow$ parent[x]
5: **return** y;

Binary Search Trees: Successor



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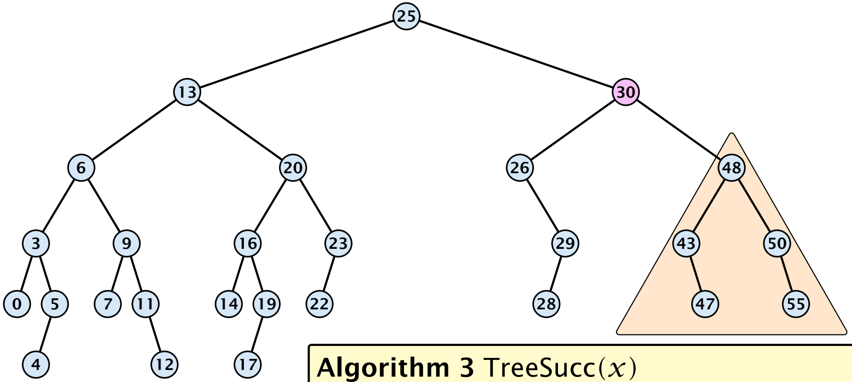

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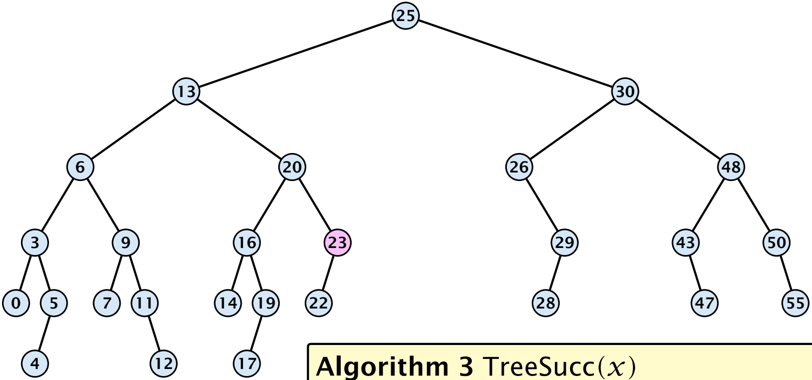

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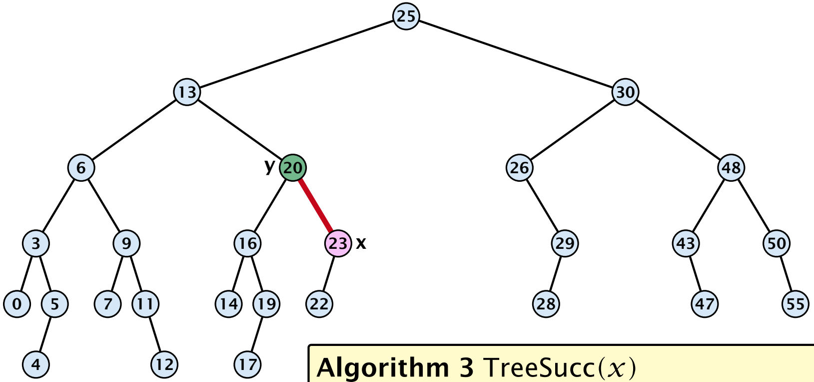

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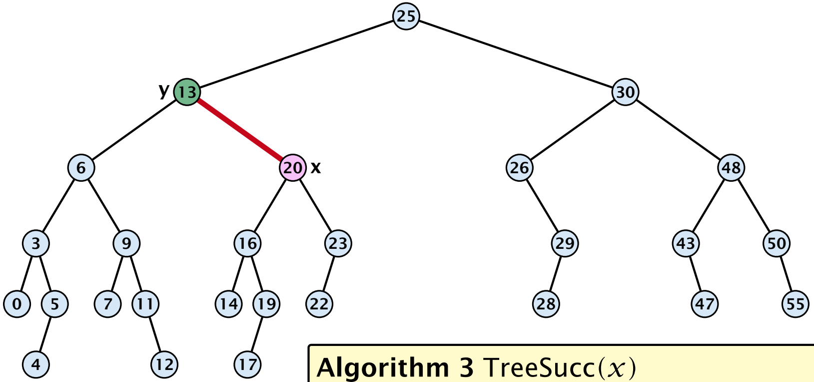

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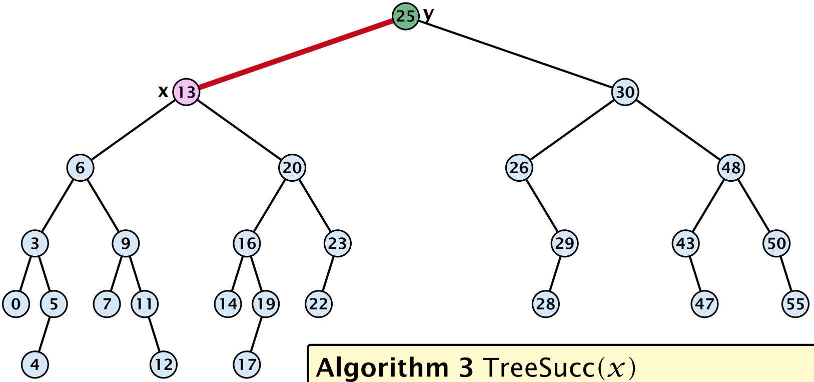

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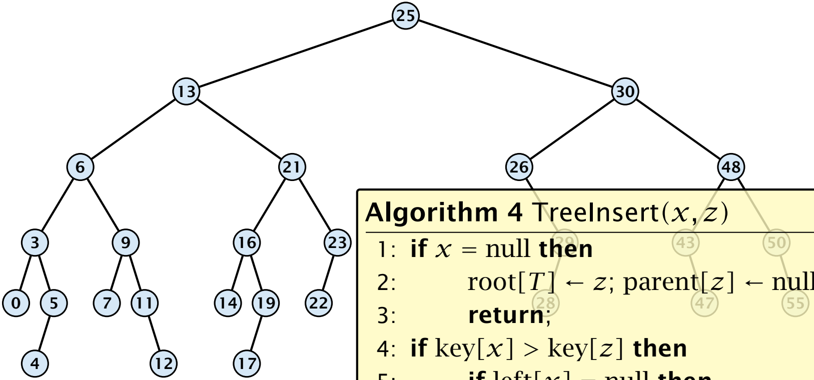
return y;

Binary Search Trees: Insert



Algorithm 4 TreeInsert(x, z)

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if x = null then
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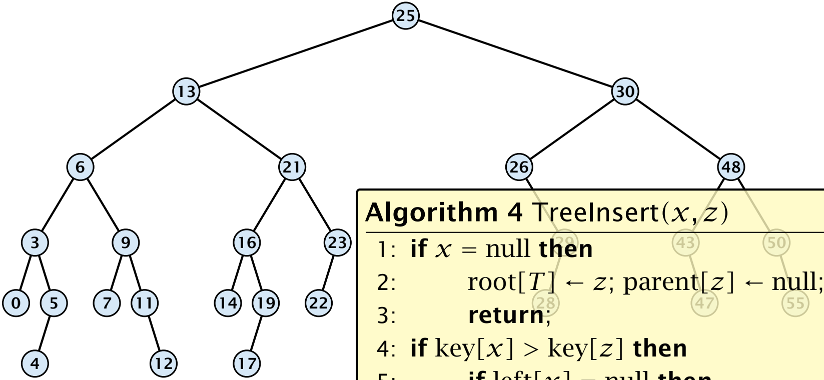
Insert element **not** in the tree.



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if $x = \text{null}$ **then**
    $\text{root}[T] \leftarrow z$; $\text{parent}[z] \leftarrow \text{null}$;
    return;
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    if $\text{left}[x] = \text{null}$ **then**
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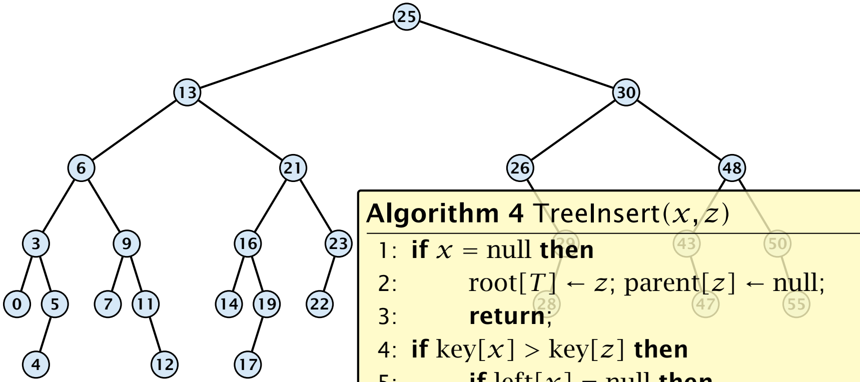

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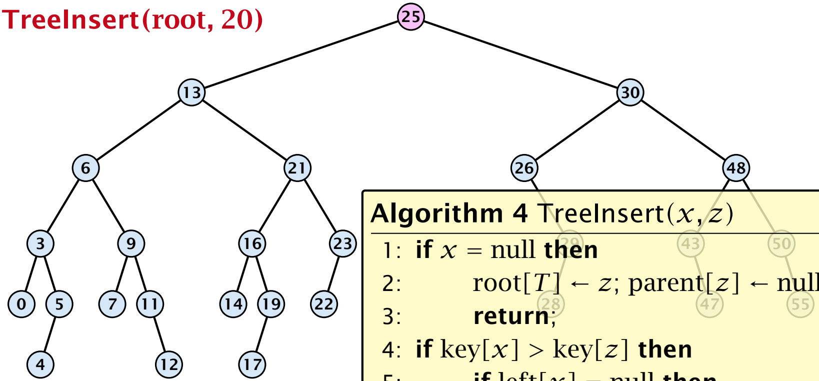

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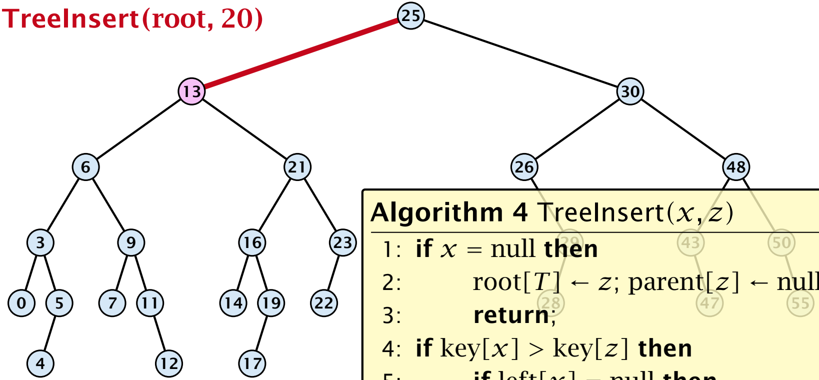

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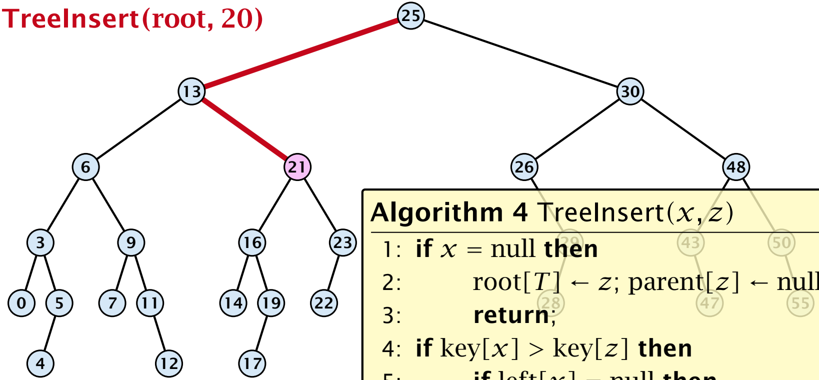

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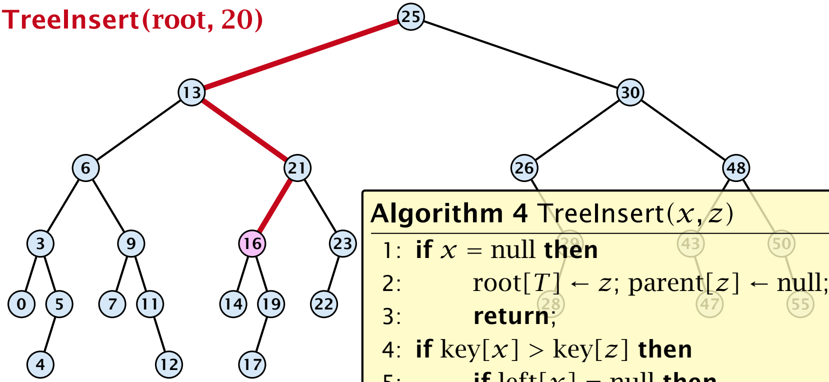

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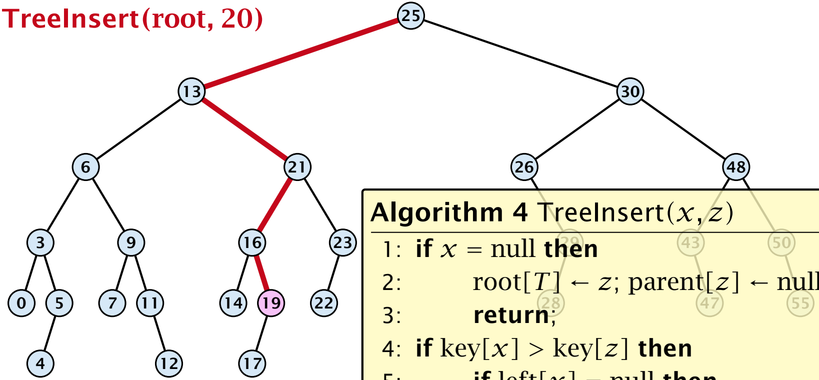

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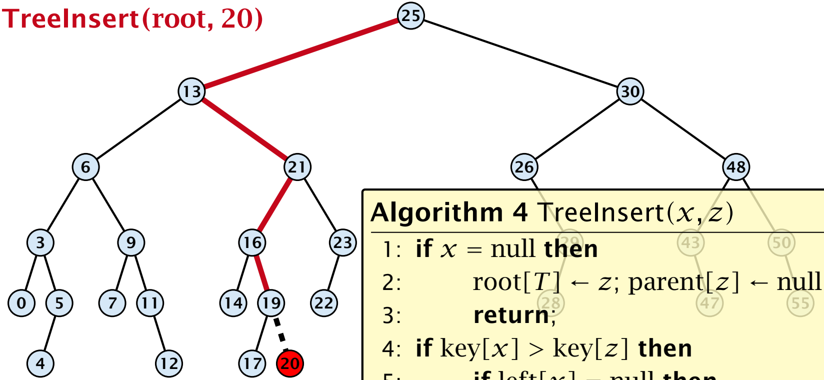

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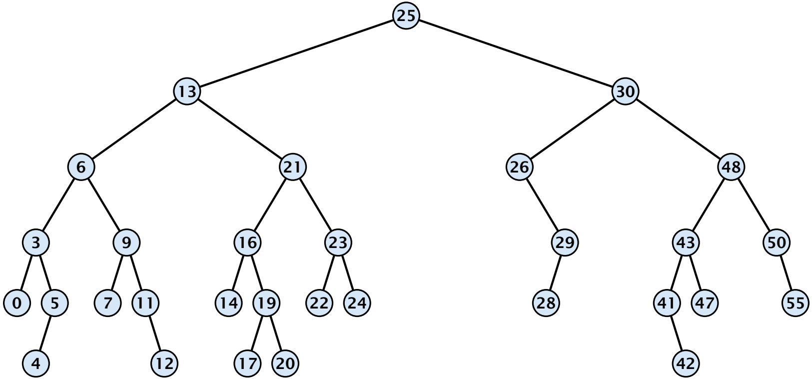

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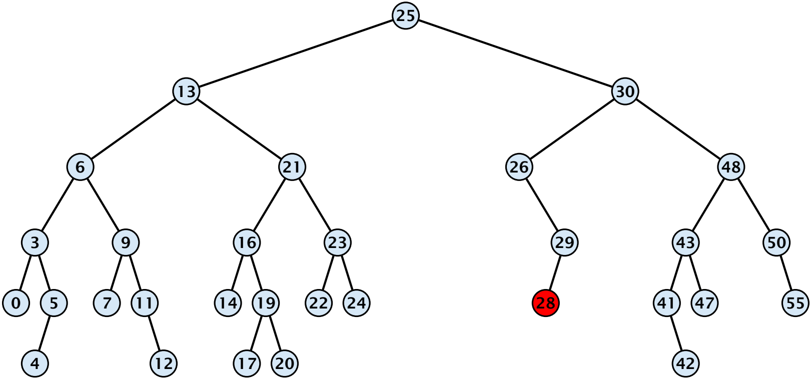

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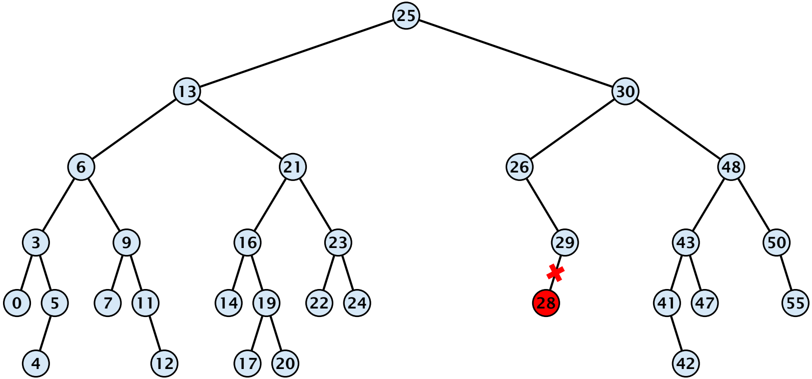

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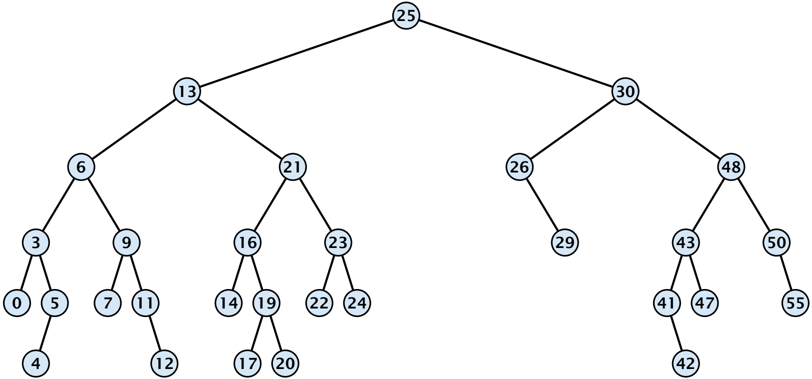

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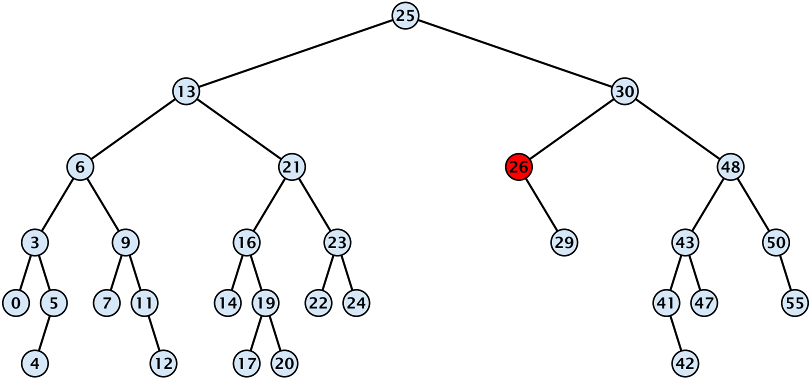

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Element has exactly one child

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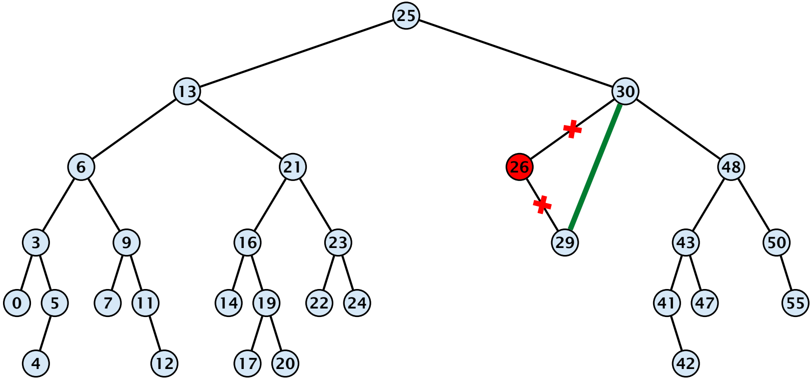

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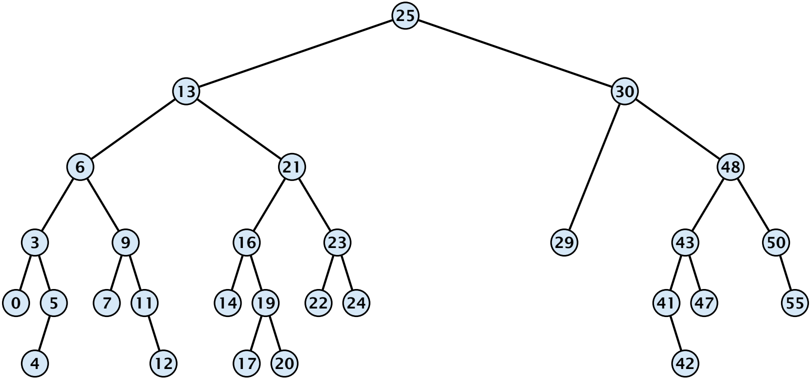

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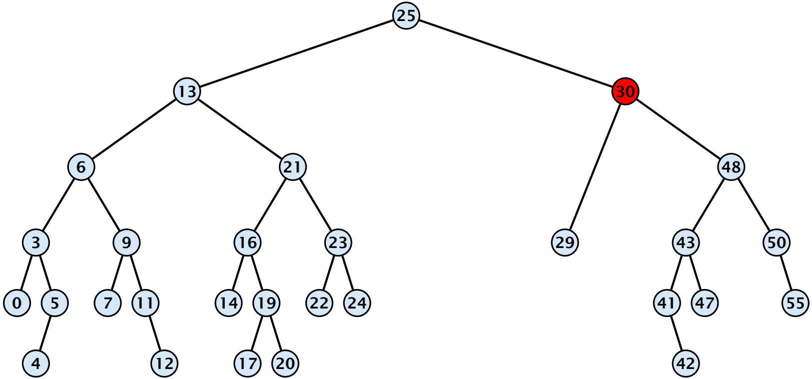

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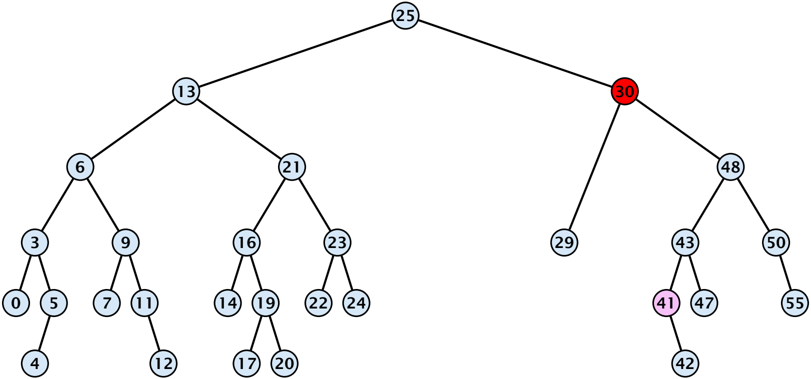

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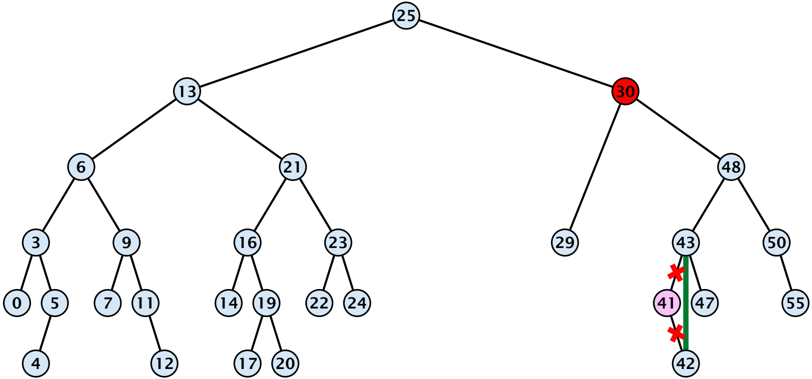

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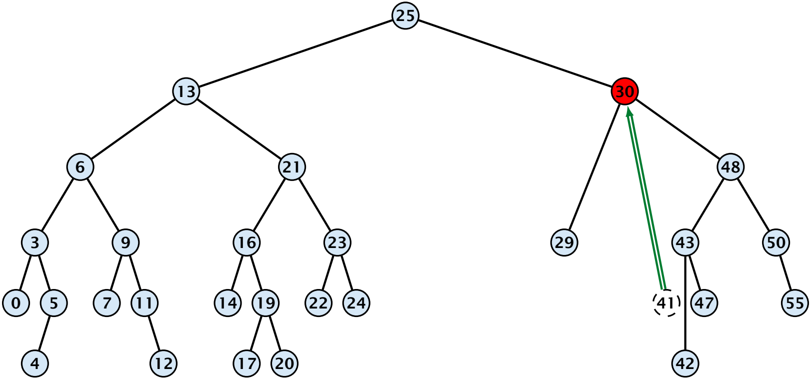

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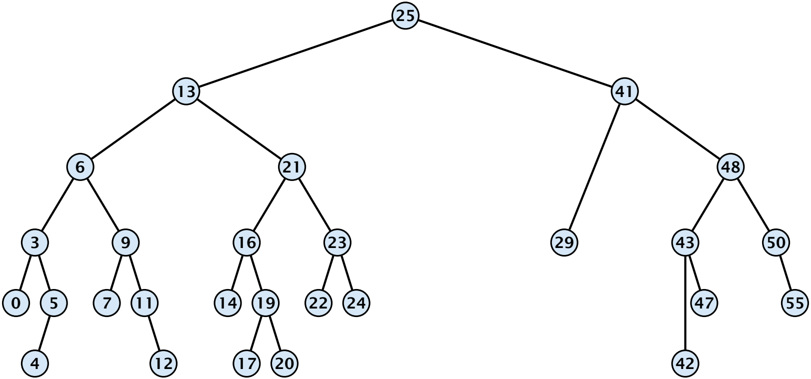

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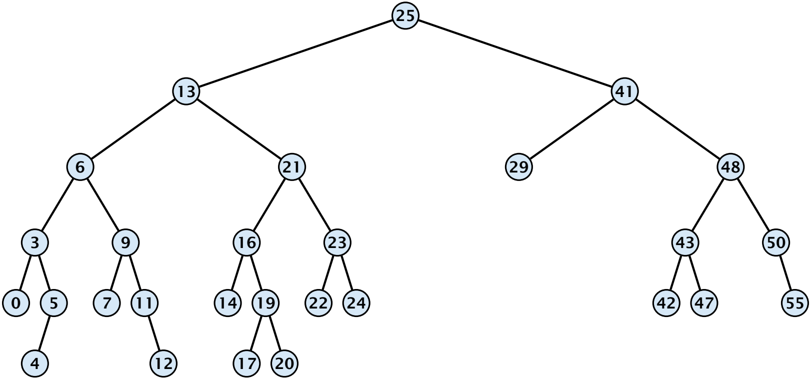

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Binary Search Trees: Delete

Algorithm 9 TreeDelete(z)

if left[z] = null **or** right[z] = null
    then $y \leftarrow z$ **else** $y \leftarrow$ TreeSucc(z); — select y to splice out
if left[y] $\neq$ null
    then $x \leftarrow$ left[y] **else** $x \leftarrow$ right[y]; — x is child of y (or null)
if $x \neq$ null **then** parent[x] $\leftarrow$ parent[y]; — parent[x] is correct
if parent[y] = null **then**
    root[T] $\leftarrow x$
else
    if y = left[parent[y]] **then**
        left[parent[y]] $\leftarrow x$
    else
        right[parent[y]] $\leftarrow x$
if $y \neq z$ **then** copy y-data to z

(lines 6–12: fix pointer to x)

Balanced Binary Search Trees

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With each insert- and delete-operation perform local adjustments to guarantee a height of $\mathcal{O}(\log n)$.

AVL-trees, Red-black trees, Scapegoat trees, 2-3 trees, B-trees, AA trees, Treaps

similar: SPLAY trees.

7.2 Red Black Trees

Definition 12

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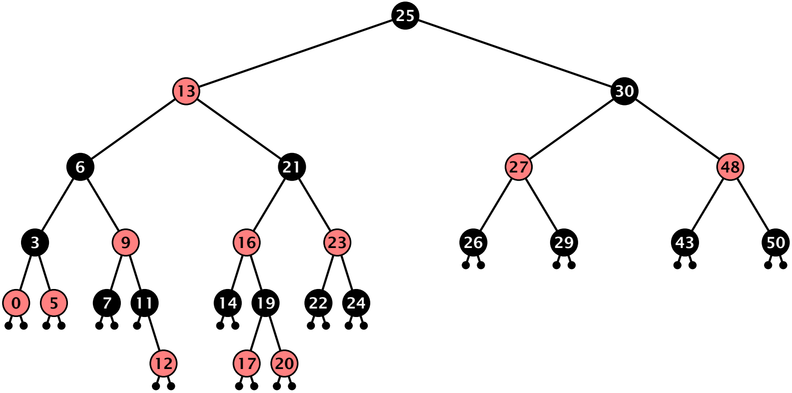
Red Black Trees: Example



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A red-black tree with n *internal nodes has height at most* $\mathcal{O}(\log n)$.

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The black height $\mathrm{bh}(v)$ of a node v in a red black tree is the number of black nodes on a path from v to a leaf vertex (not counting v).

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The black height $\mathrm{bh}(v)$ of a node v in a red black tree is the number of black nodes on a path from v to a leaf vertex (not counting v).

We first show:

Lemma 15
A sub-tree of black height $\mathrm{bh}(v)$ in a red black tree contains at least $2^{\mathrm{bh}(v)} - 1$ internal vertices.

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- The sub-tree rooted at v contains $0 = 2^{\text{bh}(v)} - 1$ inner vertices.

7.2 Red Black Trees

Proof (cont.)

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induction step

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- Then T_v contains at least $2(2^{\text{bh}(v)-1} - 1) + 1 \geq 2^{\text{bh}(v)} - 1$ vertices.

□

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Hence, $h \leq 2\log(n+1) = \mathcal{O}(\log n)$. □

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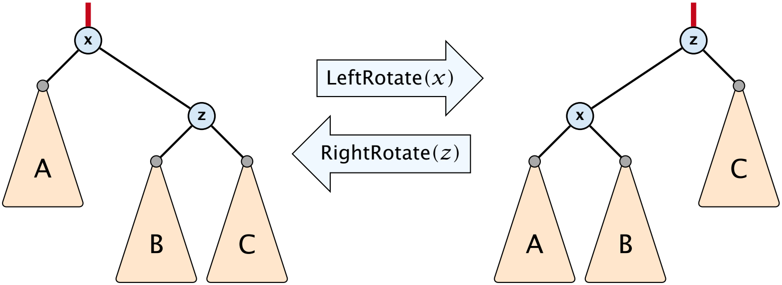
7.2 Red Black Trees

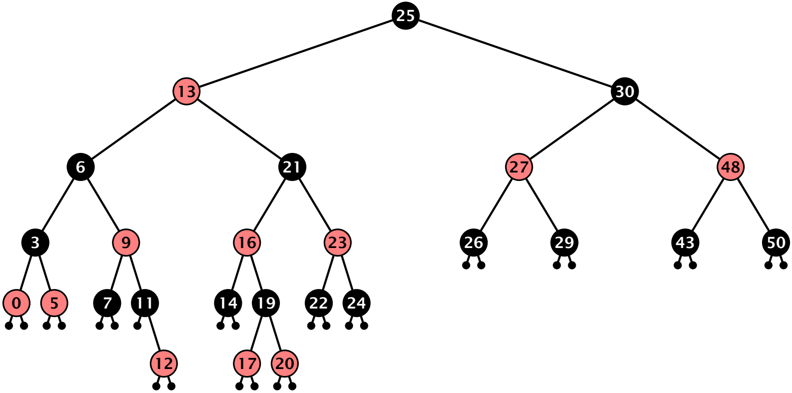
We need to adapt the insert and delete operations so that the red black properties are maintained.

Rotations

The properties will be maintained through rotations:



Red Black Trees: Insert

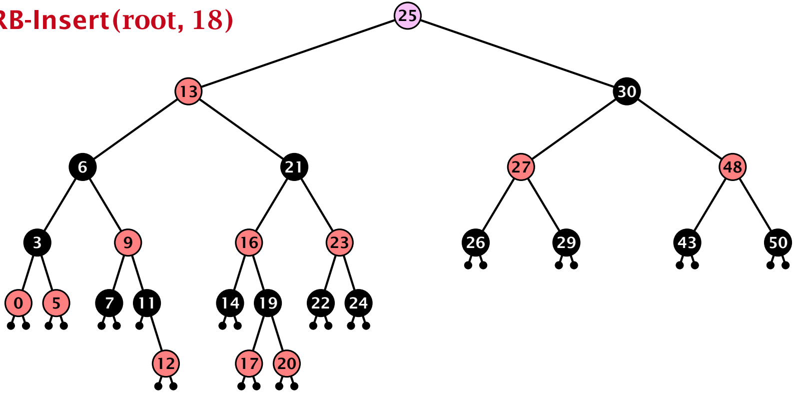


Insert:

- first make a normal insert into a binary search tree
- then fix red-black properties

Red Black Trees: Insert

RB-Insert(root, 18)



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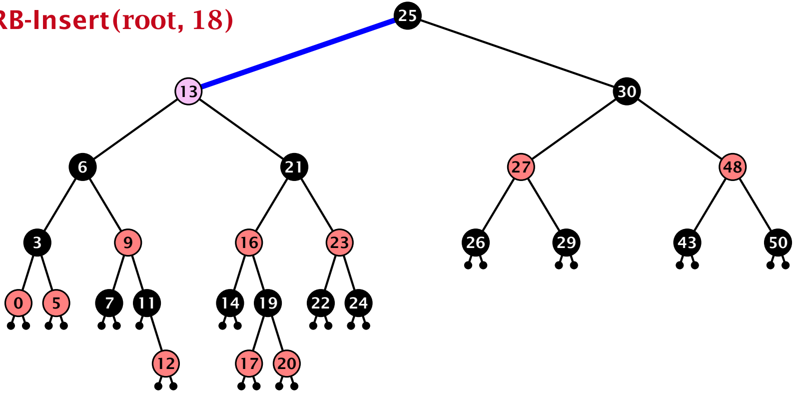

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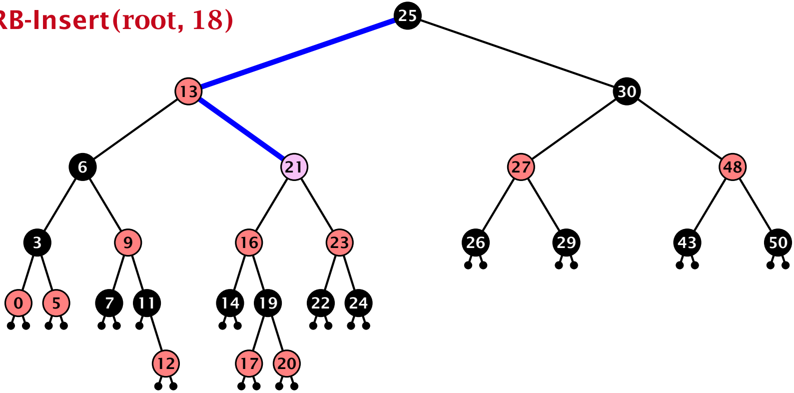

Insert:

- first make a normal insert into a binary search tree
- then fix red-black properties

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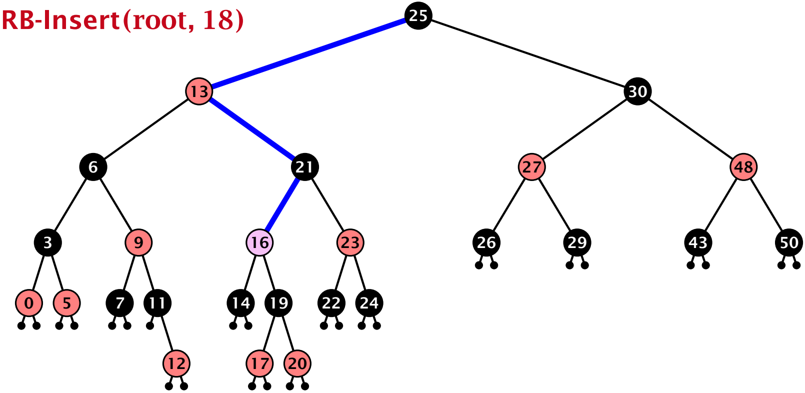

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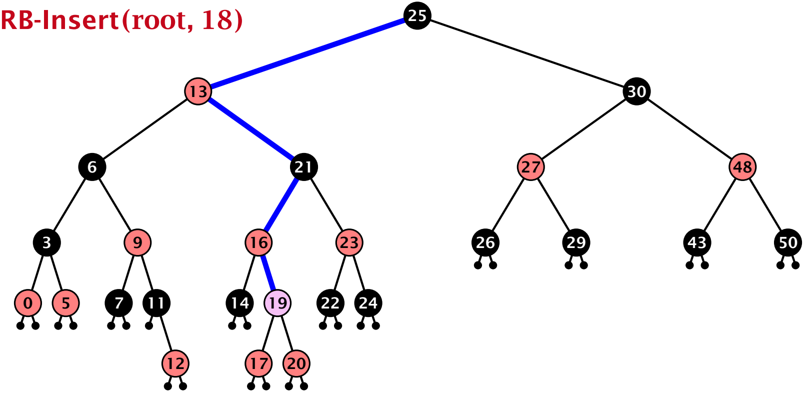

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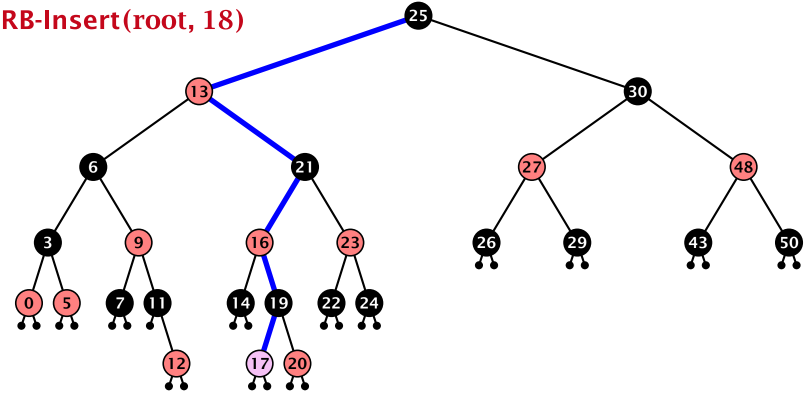

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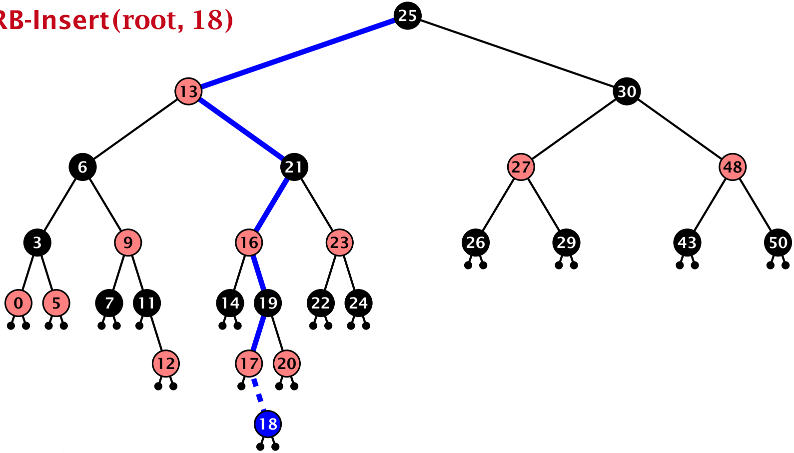

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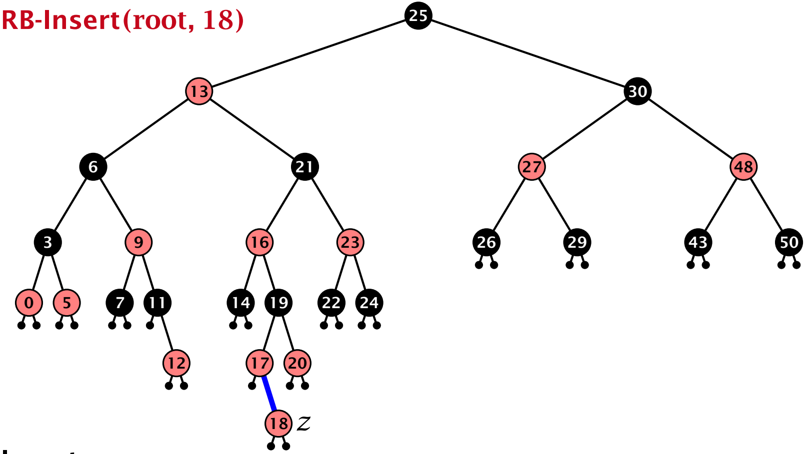

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Red Black Trees: Insert

Invariant of the fix-up algorithm:

- z is a red node

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    (most important case)

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- the only violation of red-black properties occurs at z and $\text{parent}[z]$
  - either both of them are red (most important case)
  - or the parent does not exist (violation since root must be black)

If z has a parent but no grand-parent we could simply color the parent/root black; however this case never happens.

Red Black Trees: Insert

Algorithm 10 InsertFix(z)

```
while parent[z] ≠ null and col[parent[z]] = red do
    if parent[z] = left[gp[z]] then
        uncle ← right[grandparent[z]]
        if col[uncle] = red then
            col[p[z]] ← black; col[u] ← black;
            col[gp[z]] ← red; z ← grandparent[z];
        else
            if z = right[parent[z]] then
                z ← p[z]; LeftRotate(z);
            col[p[z]] ← black; col[gp[z]] ← red;
            RightRotate(gp[z]);
    else same as then-clause but right and left exchanged
col(root[T]) ← black;
```

Red Black Trees: Insert

Algorithm 10 InsertFix(z)

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while parent[z] ≠ null and col[parent[z]] = red do
    if parent[z] = left[gp[z]] then        z in left subtree of grandparent
        uncle ← right[grandparent[z]]
        if col[uncle] = red then
            col[p[z]] ← black; col[u] ← black;
            col[gp[z]] ← red; z ← grandparent[z];
        else
            if z = right[parent[z]] then
                z ← p[z]; LeftRotate(z);
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            col[gp[z]] ← red; z ← grandparent[z];
        else                                        Case 2: uncle black
            if z = right[parent[z]] then
                z ← p[z]; LeftRotate(z);
            col[p[z]] ← black; col[gp[z]] ← red;
            RightRotate(gp[z]);
    else same as then-clause but right and left exchanged
col(root[T]) ← black;
```

Red Black Trees: Insert

Algorithm 10 InsertFix(z)

while parent[z] $\neq$ null **and** col[parent[z]] = red **do**
    if parent[z] = left[gp[z]] **then**
        uncle $\leftarrow$ right[grandparent[z]]
        if col[*uncle*] = red **then**
            col[p[z]] $\leftarrow$ black; col[u] $\leftarrow$ black;
            col[gp[z]] $\leftarrow$ red; $z \leftarrow$ grandparent[z];
        else
            if z = right[parent[z]] **then** — 2a: z right child
                $z \leftarrow$ p[z]; LeftRotate(z);
            col[p[z]] $\leftarrow$ black; col[gp[z]] $\leftarrow$ red;
            RightRotate(gp[z]);
    else same as then-clause but right and left exchanged
col(root[T]) $\leftarrow$ black;

Red Black Trees: Insert

Algorithm 10 InsertFix(z)

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        else
            if z = right[parent[z]] then
                z ← p[z]; LeftRotate(z);
            col[p[z]] ← black; col[gp[z]] ← red;   2b: z left child
            RightRotate(gp[z]);
    else same as then-clause but right and left exchanged
col(root[T]) ← black;
```
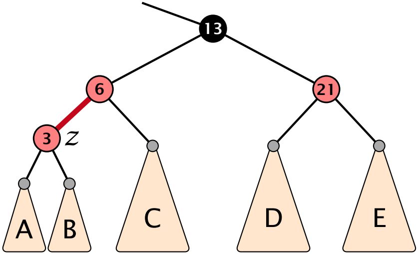

Case 1: Red Uncle



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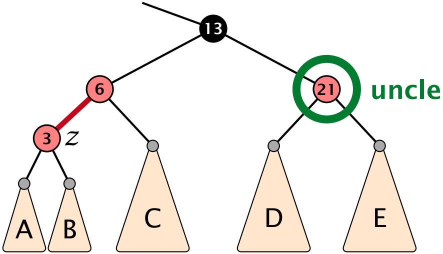

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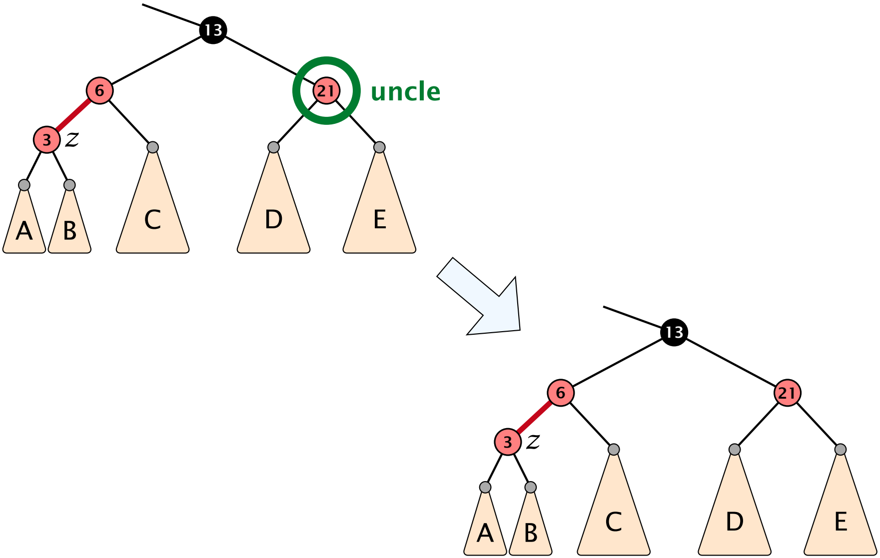

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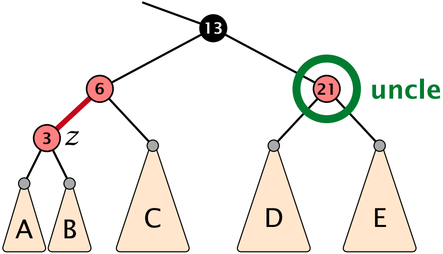
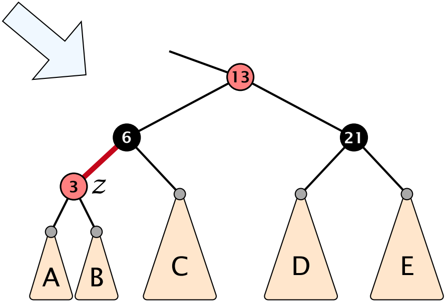
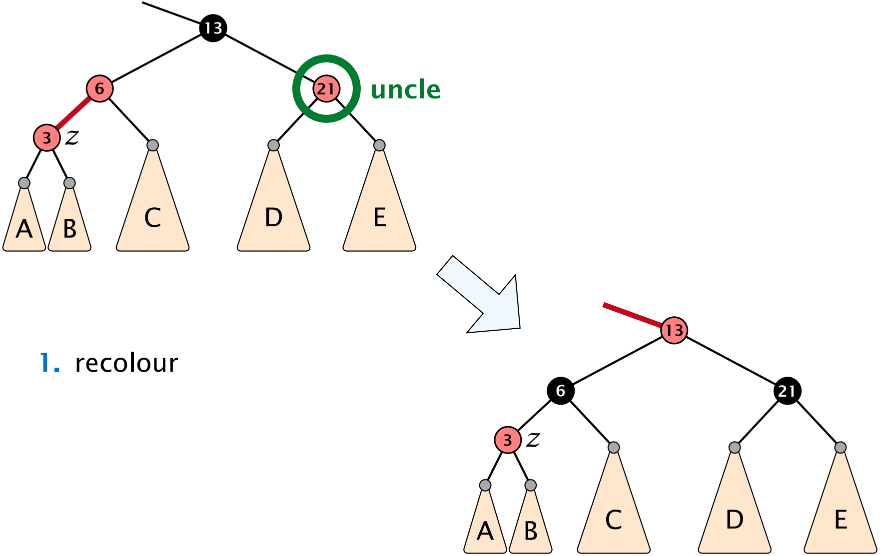




1. recolour

Case 1: Red Uncle



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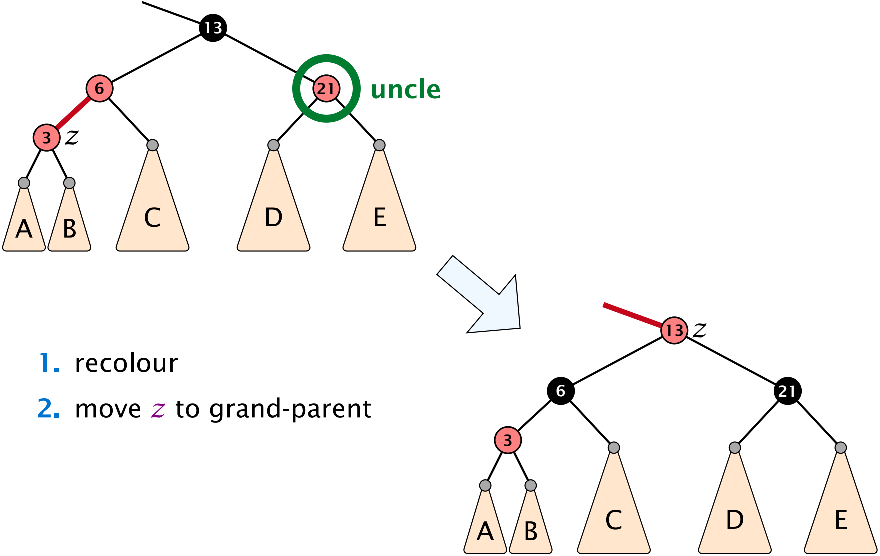
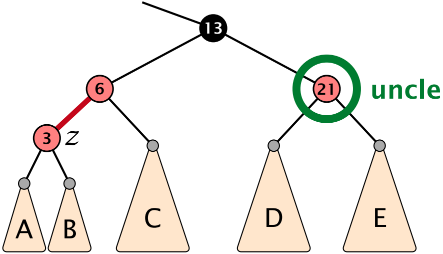
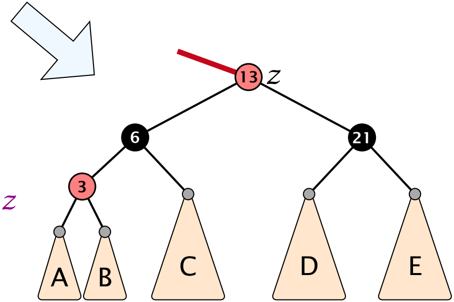


1. recolour
2. move z to grand-parent

Case 1: Red Uncle





1. recolour
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3. invariant is fulfilled for new z

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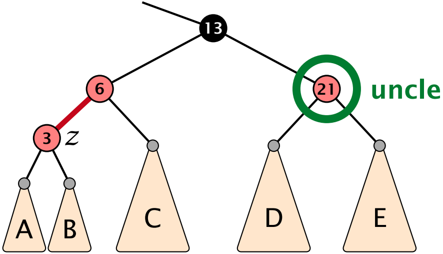
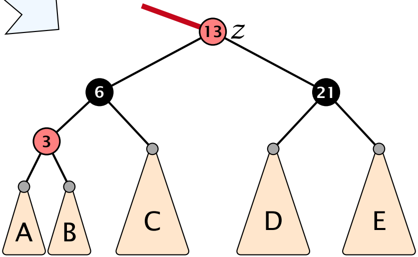
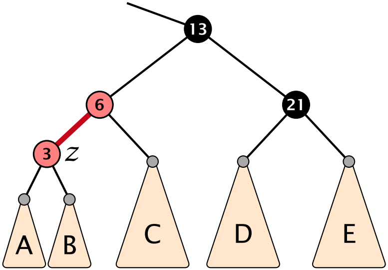




1. recolour
2. move z to grand-parent
3. invariant is fulfilled for new z
4. you made progress

Case 2b: Black uncle and z is left child



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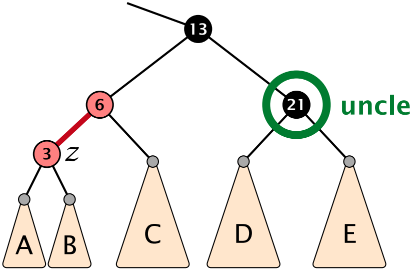
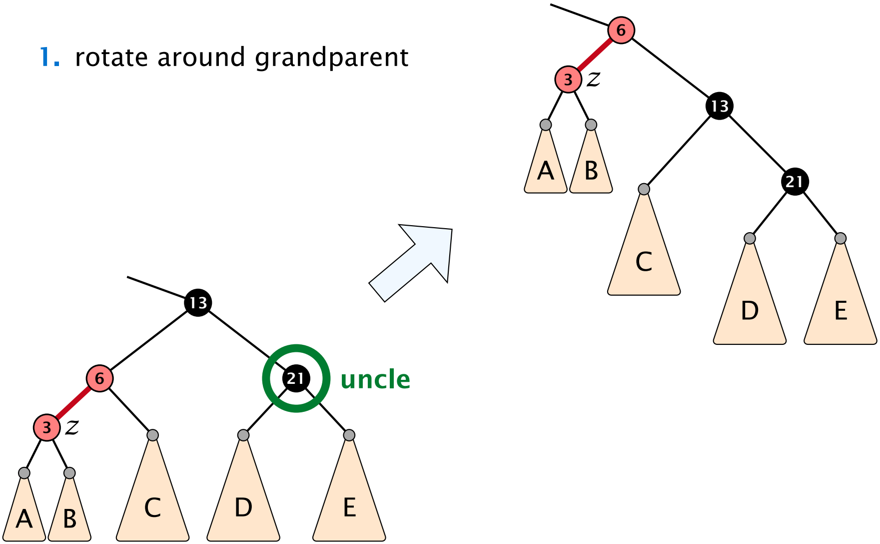


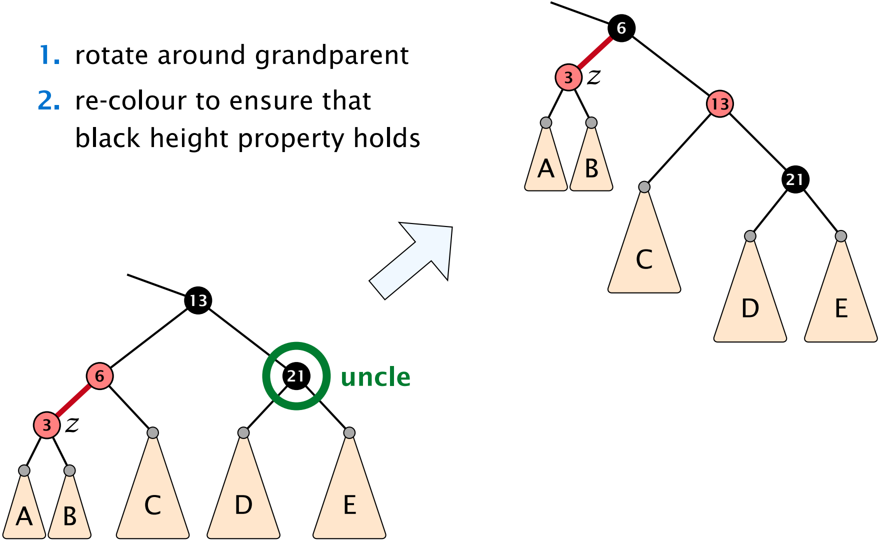
Case 2b: Black uncle and z is left child

1. rotate around grandparent



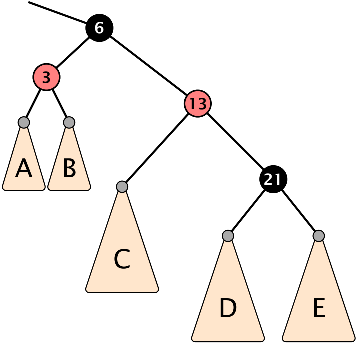
Case 2b: Black uncle and z is left child

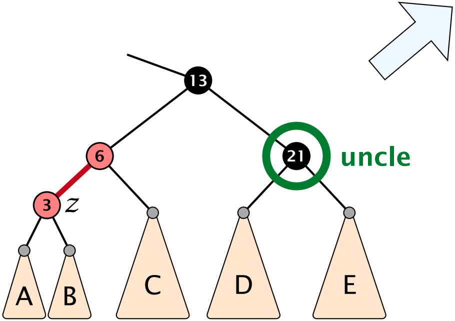
1. rotate around grandparent
2. re-colour to ensure that black height property holds



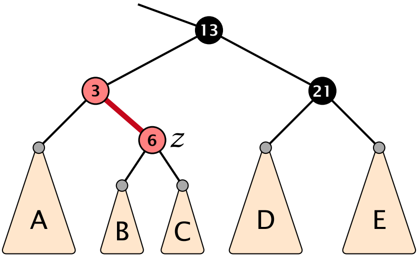
Case 2b: Black uncle and z is left child

1. rotate around grandparent
2. re-colour to ensure that black height property holds
3. you have a red black tree





Case 2a: Black uncle and z is right child



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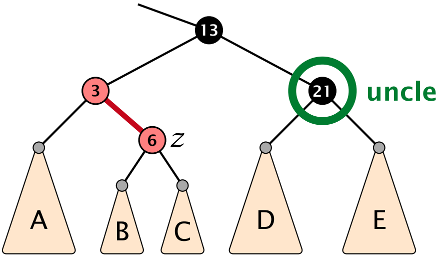

Case 2a: Black uncle and z is right child

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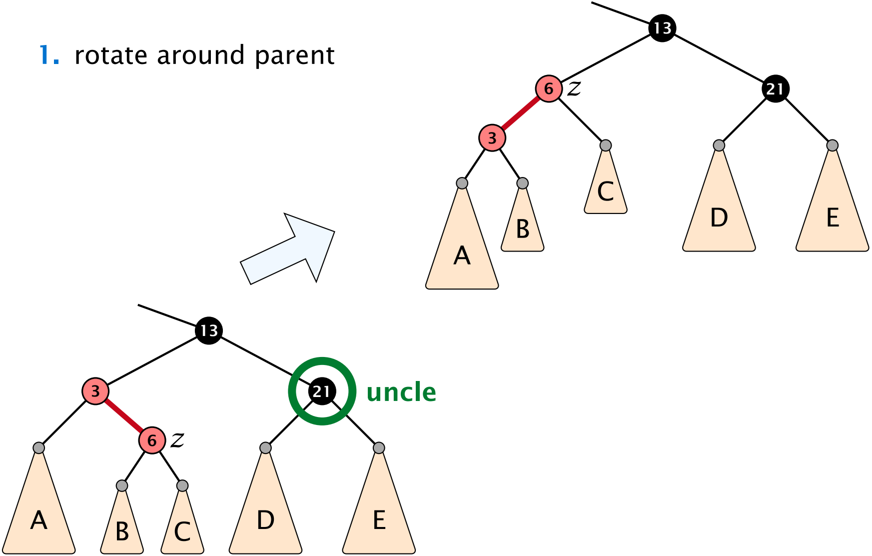
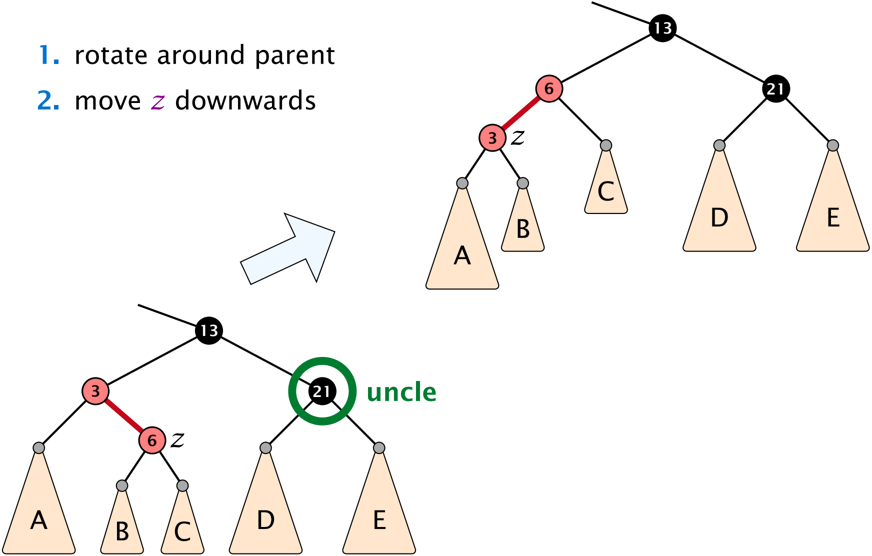


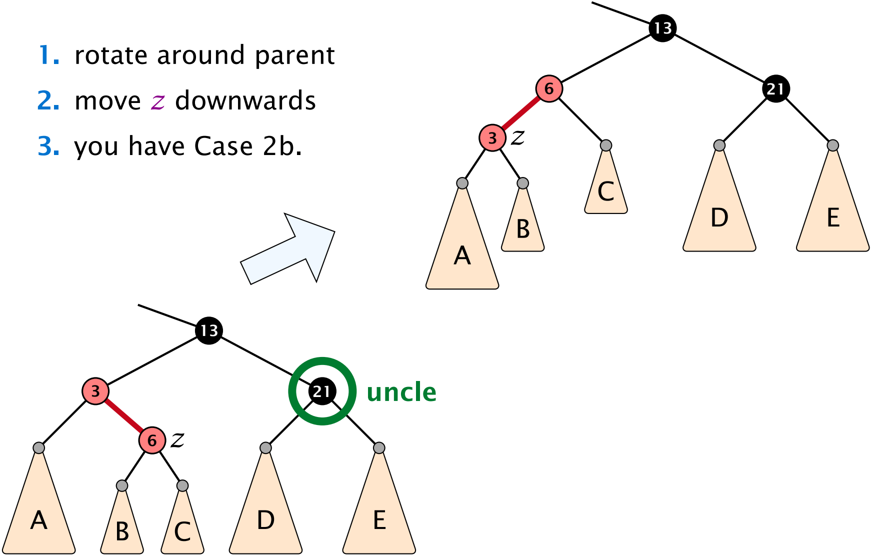
Case 2a: Black uncle and z is right child

1. rotate around parent
2. move z downwards



Case 2a: Black uncle and z is right child

1. rotate around parent
2. move z downwards
3. you have Case 2b.



Red Black Trees: Insert

Running time:

- Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.

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- Case 2a $\rightarrow$ Case 2b $\rightarrow$ red-black tree
- Case 2b $\rightarrow$ red-black tree

Performing Case 1 at most $\mathcal{O}(\log n)$ times and every other case at most once, we get a red-black tree. Hence $\mathcal{O}(\log n)$ re-colorings and at most 2 rotations.

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- Parent and child of x were red; two adjacent red vertices.

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- Parent and child of x were red; two adjacent red vertices.
- If you delete the root, the root may now be red.

Red Black Trees: Delete

First do a standard delete.

If the spliced out node x was red everything is fine.

If it was black there may be the following problems.

- Parent and child of x were red; two adjacent red vertices.
- If you delete the root, the root may now be red.
- Every path from an ancestor of x to a descendant leaf of x changes the number of black nodes. Black height property might be violated.

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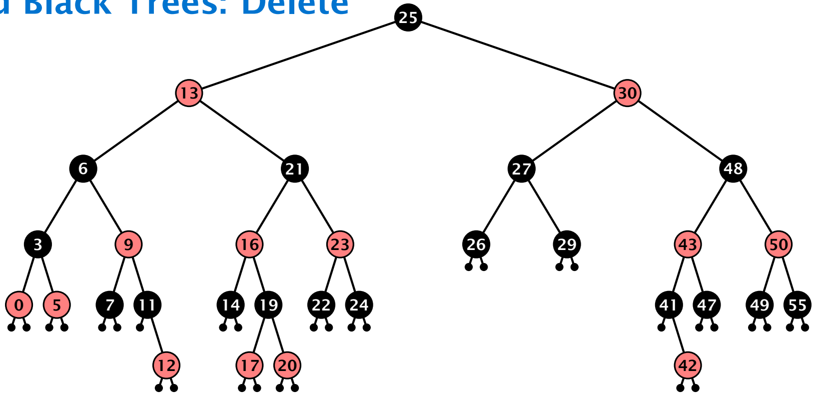

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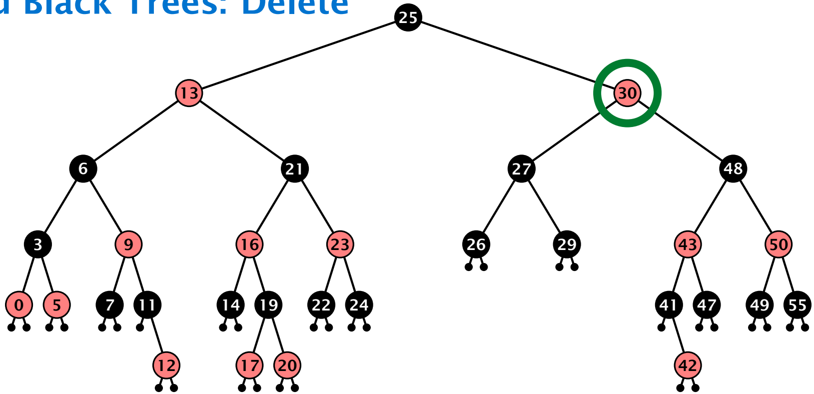

Case 3:

Element has two children

- do normal delete
- when replacing content by content of successor, don't change color of node

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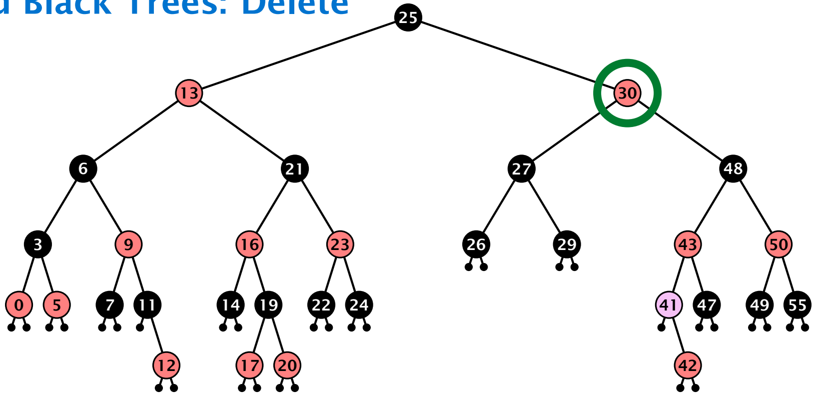

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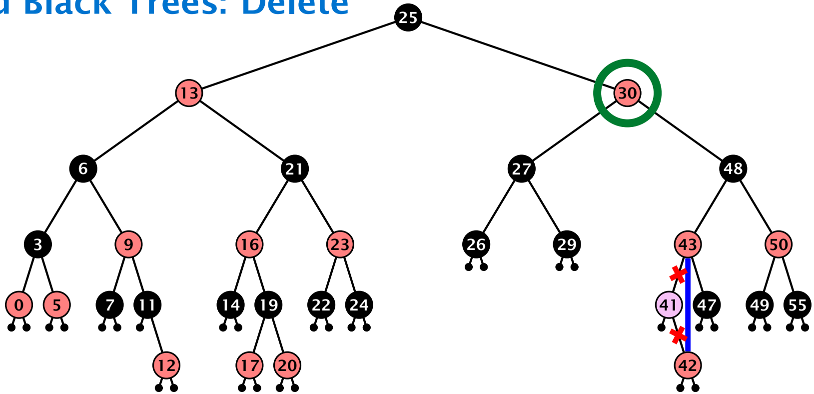

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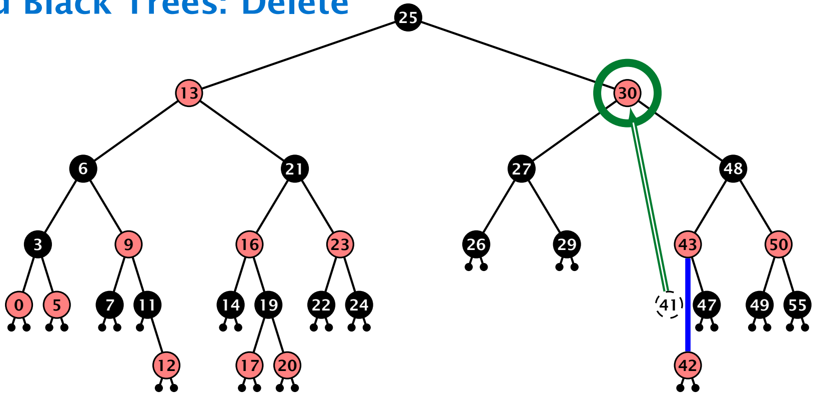

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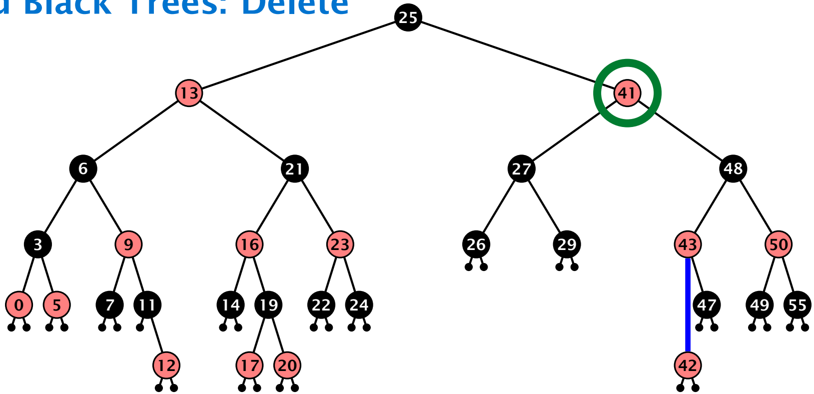

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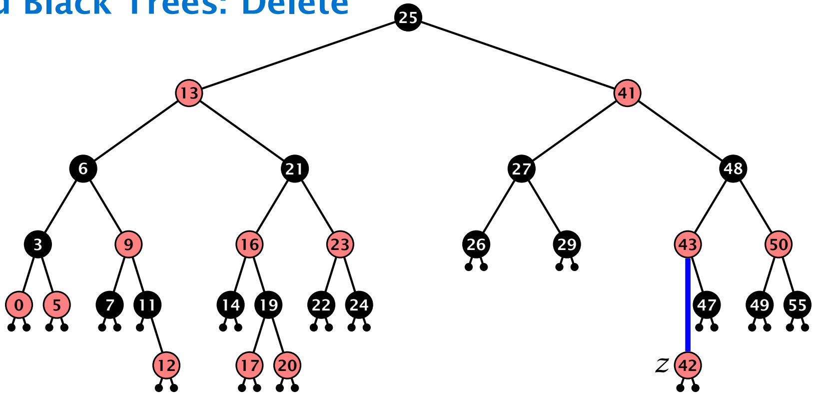
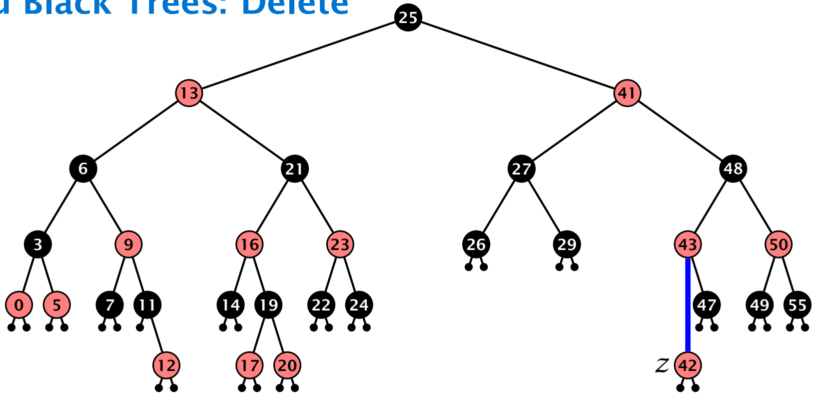


Delete:

- deleting black node messes up black-height property

Red Black Trees: Delete



Delete:

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- if z is red, we can simply color it black and everything is fine

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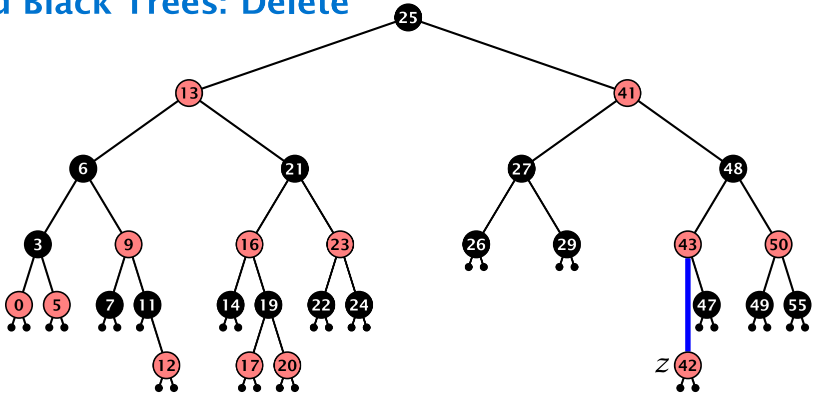


Delete:

- deleting black node messes up black-height property
- if z is red, we can simply color it black and everything is fine
- the problem is if z is black (e.g. a dummy-leaf); we call a fix-up procedure to fix the problem.

Red Black Trees: Delete

Invariant of the fix-up algorithm

- the node z is black

Red Black Trees: Delete

Invariant of the fix-up algorithm

- the node z is black
- if we “assign” a fake black unit to the edge from z to its parent then the black-height property is fulfilled

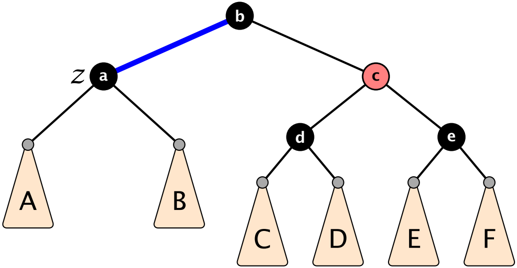
Red Black Trees: Delete

Invariant of the fix-up algorithm

- the node z is black
- if we "assign" a fake black unit to the edge from z to its parent then the black-height property is fulfilled

Goal: make rotations in such a way that you at some point can remove the fake black unit from the edge.

Case 1: Sibling of z is red



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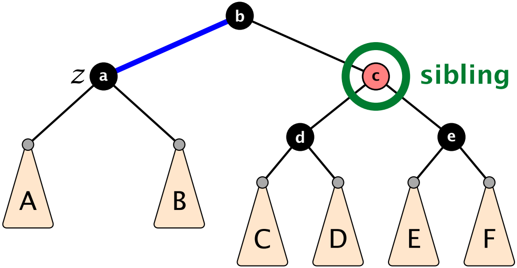

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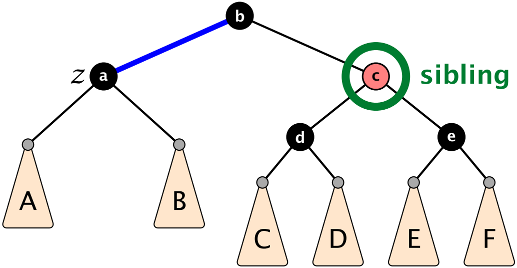
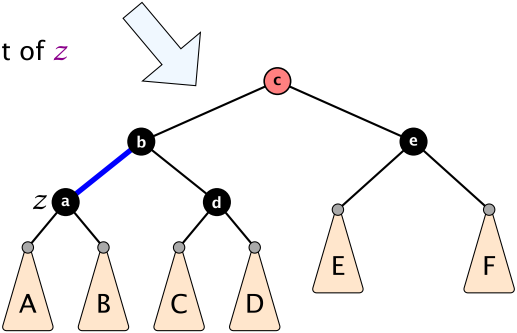


1. left-rotate around parent of z



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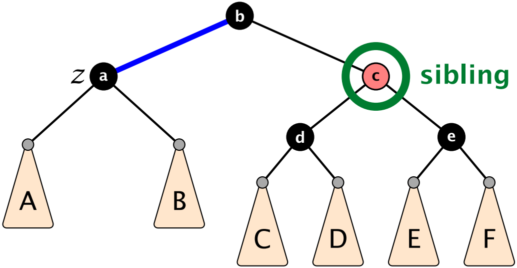
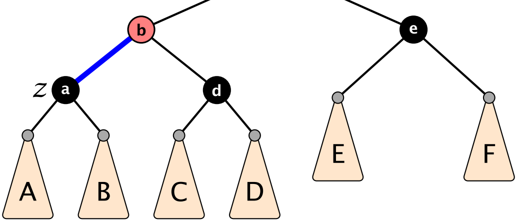
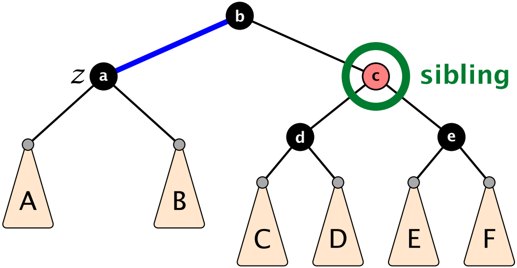


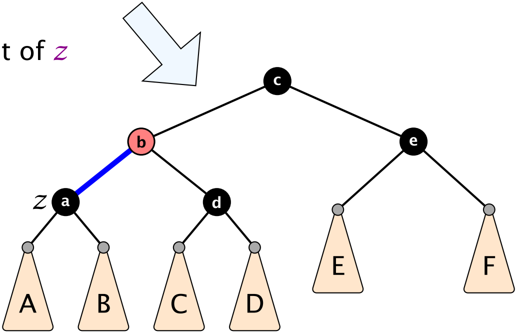
1. left-rotate around parent of z
2. recolor nodes b and c



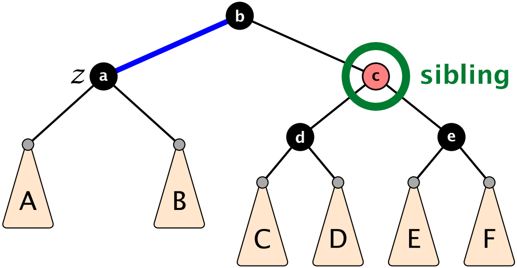
Case 1: Sibling of z is red



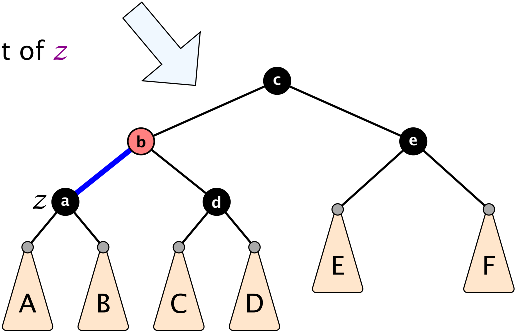
1. left-rotate around parent of z
2. recolor nodes b and c
3. the new sibling is black (and parent of z is red)



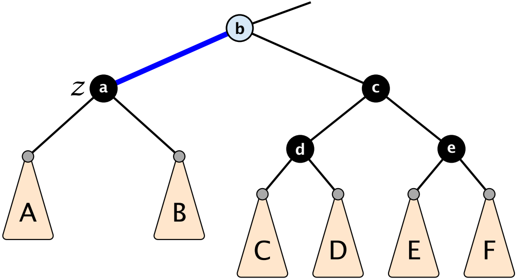
Case 1: Sibling of z is red



1. left-rotate around parent of z
2. recolor nodes b and c
3. the new sibling is black (and parent of z is red)
4. Case 2 (special), or Case 3, or Case 4



Case 2: Sibling is black with two black children



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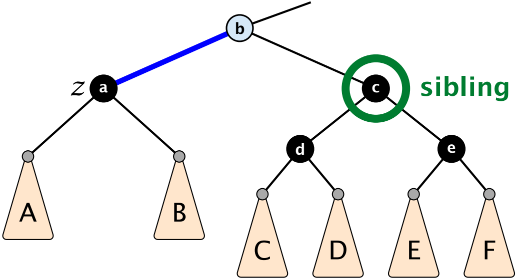

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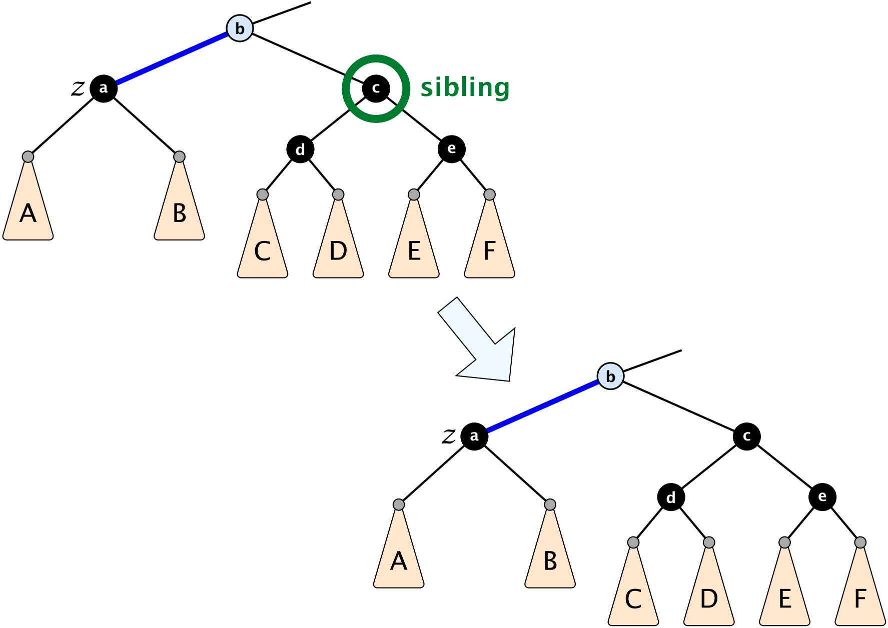

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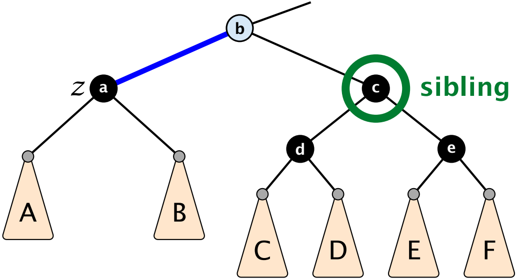
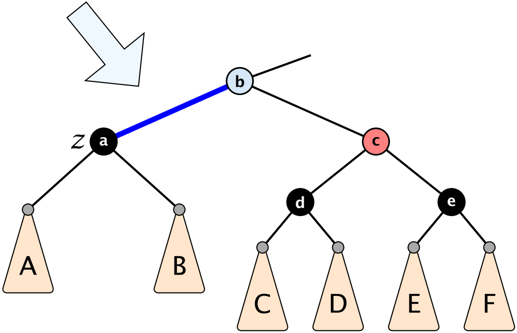


1. re-color node c



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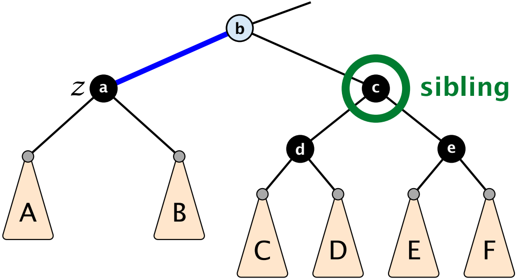
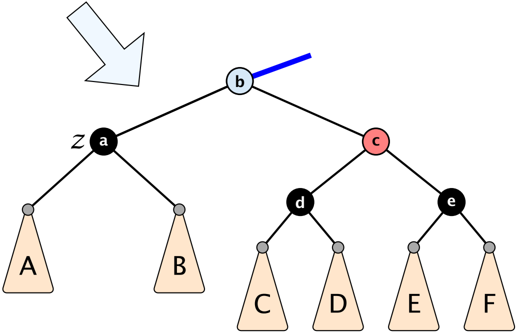
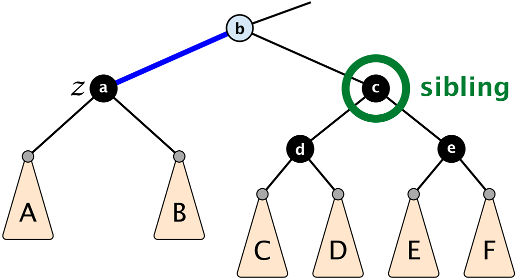


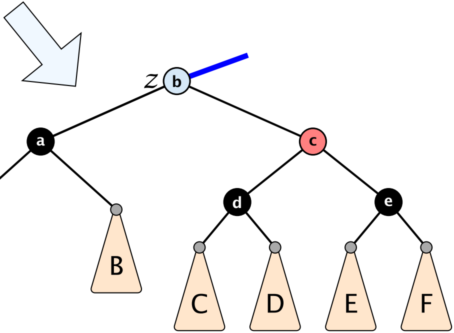
1. re-color node c
2. move fake black unit upwards



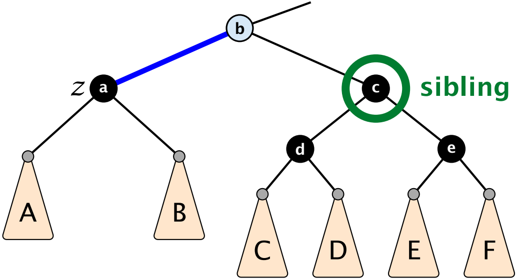
Case 2: Sibling is black with two black children



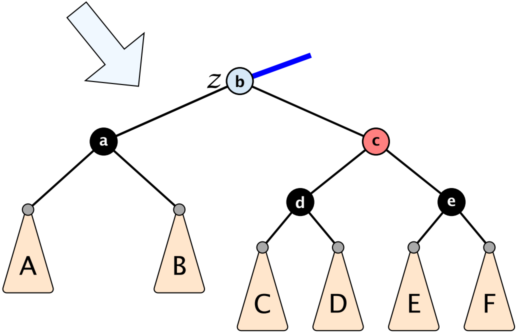
1. re-color node c
2. move fake black unit upwards
3. move z upwards



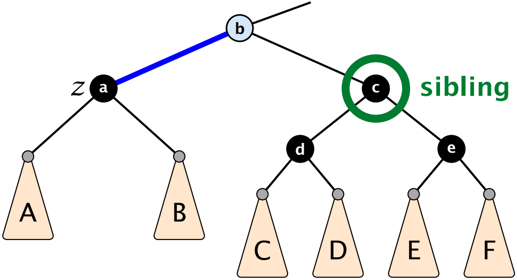
Case 2: Sibling is black with two black children

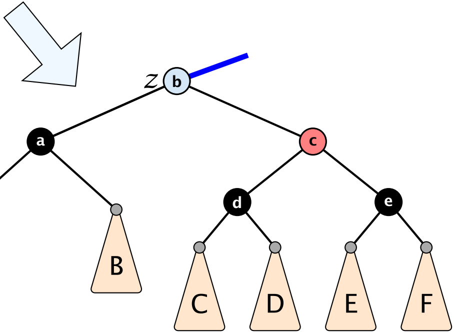


1. re-color node c
2. move fake black unit upwards
3. move z upwards
4. we made progress



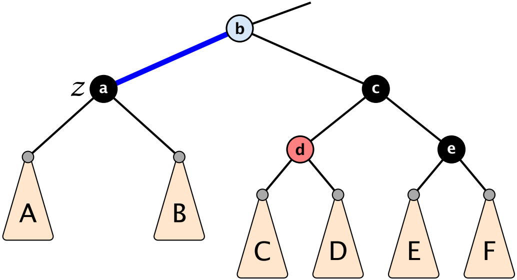
Case 2: Sibling is black with two black children



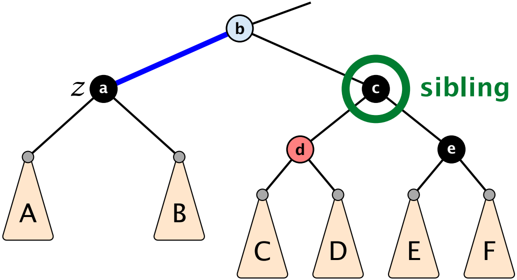


1. re-color node c
2. move fake black unit upwards
3. move z upwards
4. we made progress
5. if b is red we color it black and are done

Case 3: Sibling black with one black child to the right

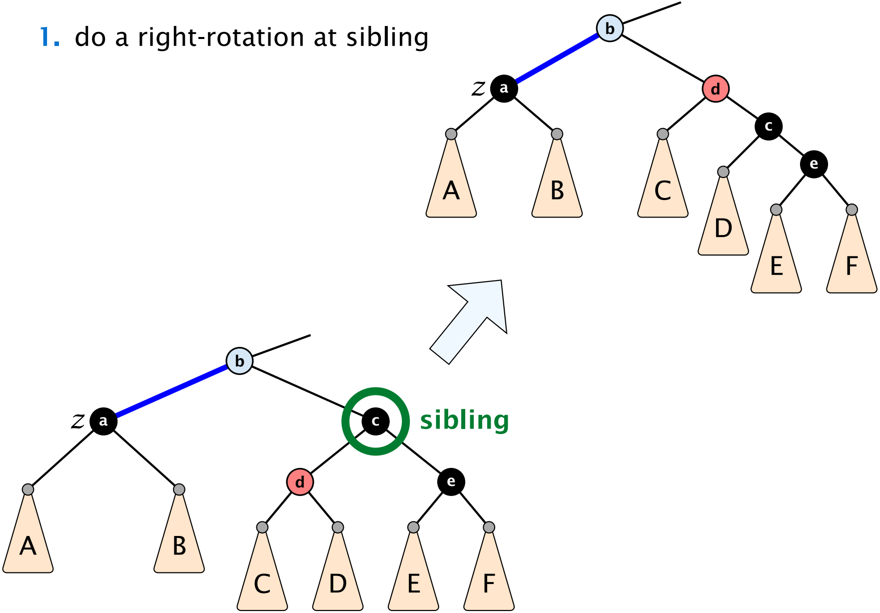


Case 3: Sibling black with one black child to the right



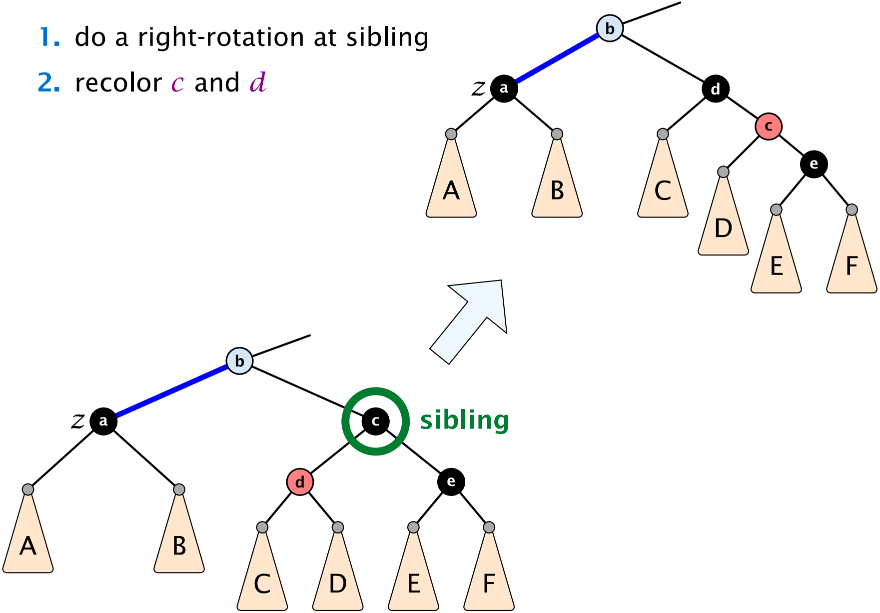
Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling



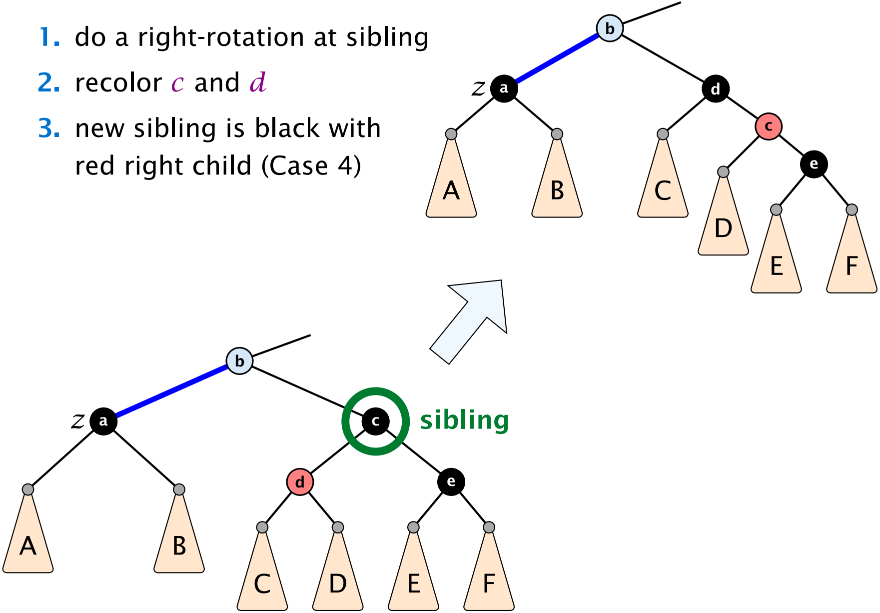
Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling
2. recolor c and d

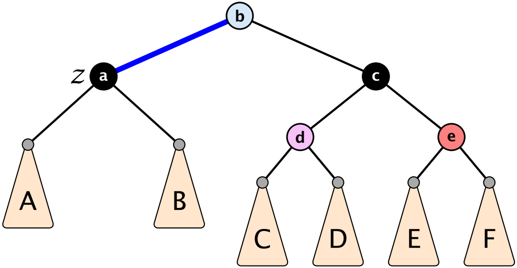


Case 3: Sibling black with one black child to the right

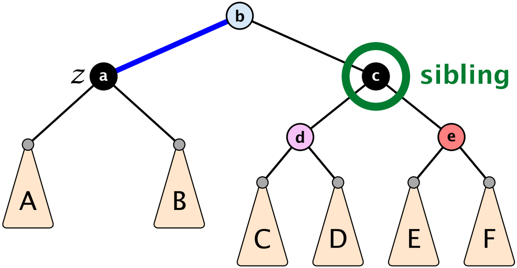
1. do a right-rotation at sibling
2. recolor c and d
3. new sibling is black with red right child (Case 4)



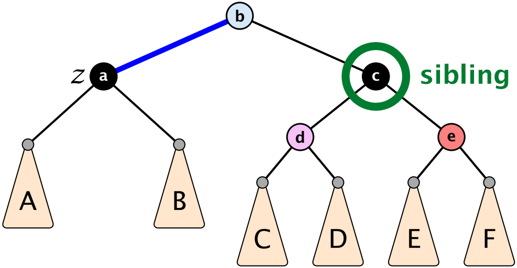
Case 4: Sibling is black with red right child



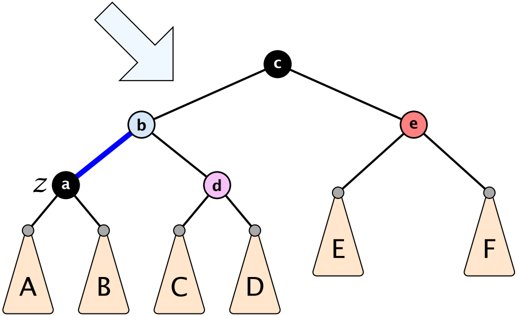
Case 4: Sibling is black with red right child



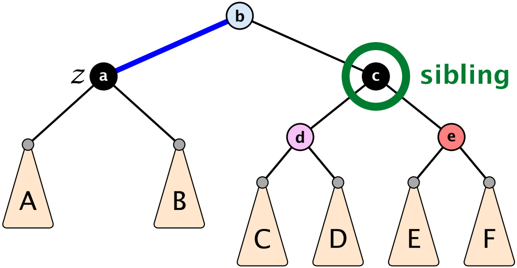
Case 4: Sibling is black with red right child



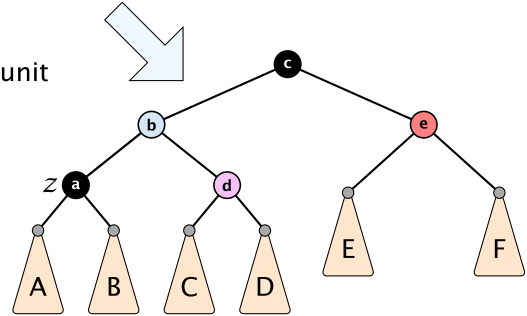
1. left-rotate around b



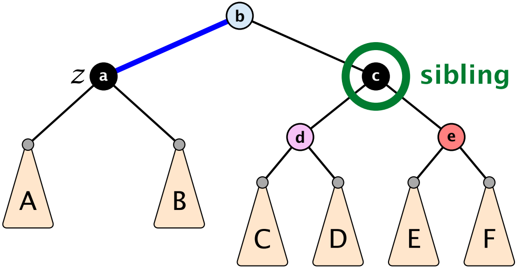
Case 4: Sibling is black with red right child



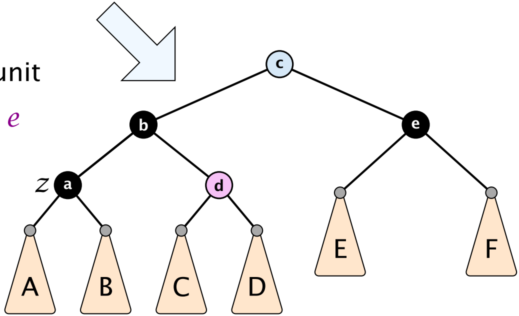
1. left-rotate around b
2. remove the fake black unit



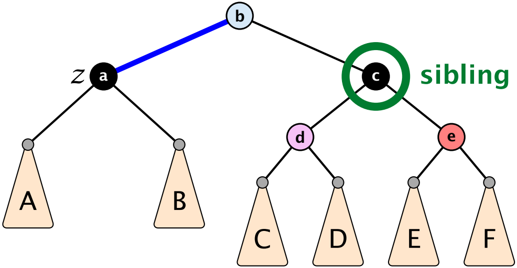
Case 4: Sibling is black with red right child



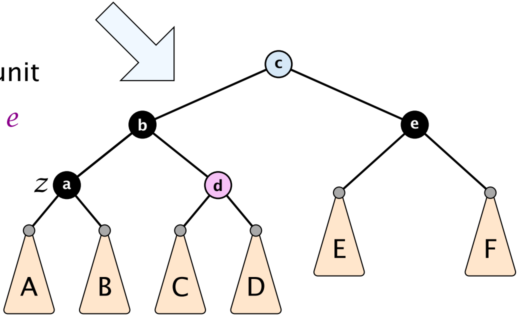
1. left-rotate around b
2. remove the fake black unit
3. recolor nodes b, c, and e



Case 4: Sibling is black with red right child



1. left-rotate around b
2. remove the fake black unit
3. recolor nodes b, c, and e
4. you have a valid red black tree



Running time:

- only Case 2 can repeat; but only h many steps, where h is the height of the tree

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- Case 1 → Case 2 (special) → red black tree
  Case 1 → Case 3 → Case 4 → red black tree
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Running time:

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Running time:

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- Case 1 → Case 2 (special) → red black tree
  Case 1 → Case 3 → Case 4 → red black tree
  Case 1 → Case 4 → red black tree
- Case 3 → Case 4 → red black tree
- Case 4 → red black tree

Performing Case 2 at most $\mathcal{O}(\log n)$ times and every other step at most once, we get a red black tree. Hence, $\mathcal{O}(\log n)$ re-colorings and at most 3 rotations.

Splay Trees

Disadvantage of balanced search trees:

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- worst case; no advantage for easy inputs

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Splay Trees:

+ after access, an element is moved to the root; splay(x)
  repeated accesses are faster
- only amortized guarantee
- read-operations change the tree

Splay Trees

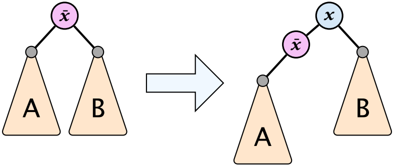
find(x)

- search for x according to a search tree
- let $\bar{x}$ be last element on search-path
- splay($\bar{x}$)

Splay Trees

insert(x)

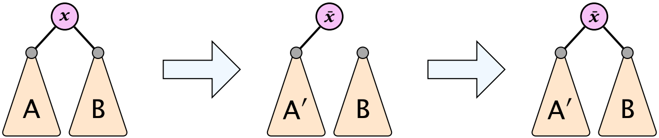
- search for x; $\bar{x}$ is last visited element during search (successer or predecessor of x)
- splay($\bar{x}$) moves $\bar{x}$ to the root
- insert x as new root



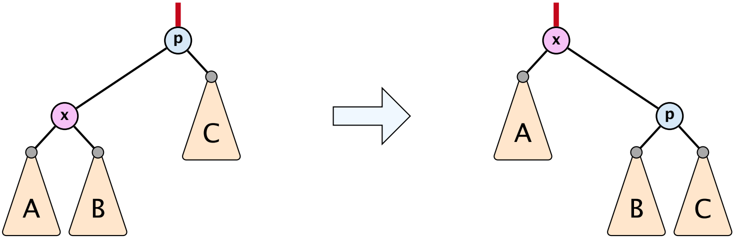
Splay Trees

delete(x)

- search for x; splay(x); remove x
- search largest element $\bar{x}$ in A
- splay($\bar{x}$) (on subtree A)
- connect root of B as right child of $\bar{x}$



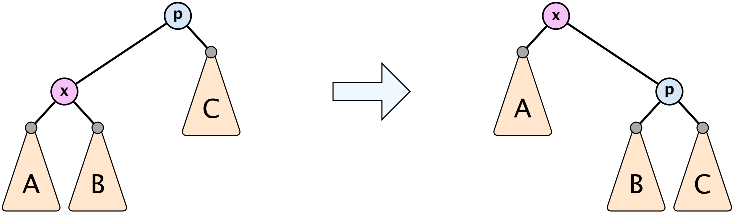
Move to Root



How to bring element to root?

- one (bad) option: moveToRoot(x)
- iteratively do rotation around parent of x until x is root
- if x is left child do right rotation otw. left rotation

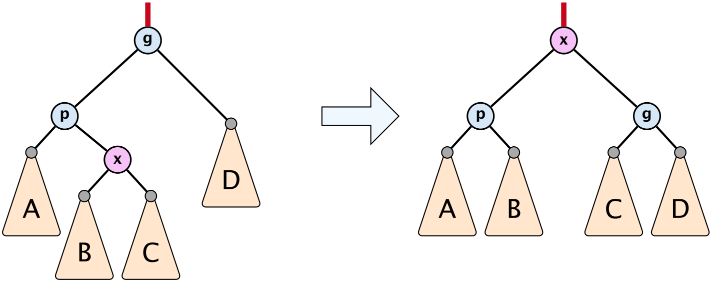
Splay: Zig Case



better option splay(x):

- zig case: if x is child of root do left rotation or right rotation around parent

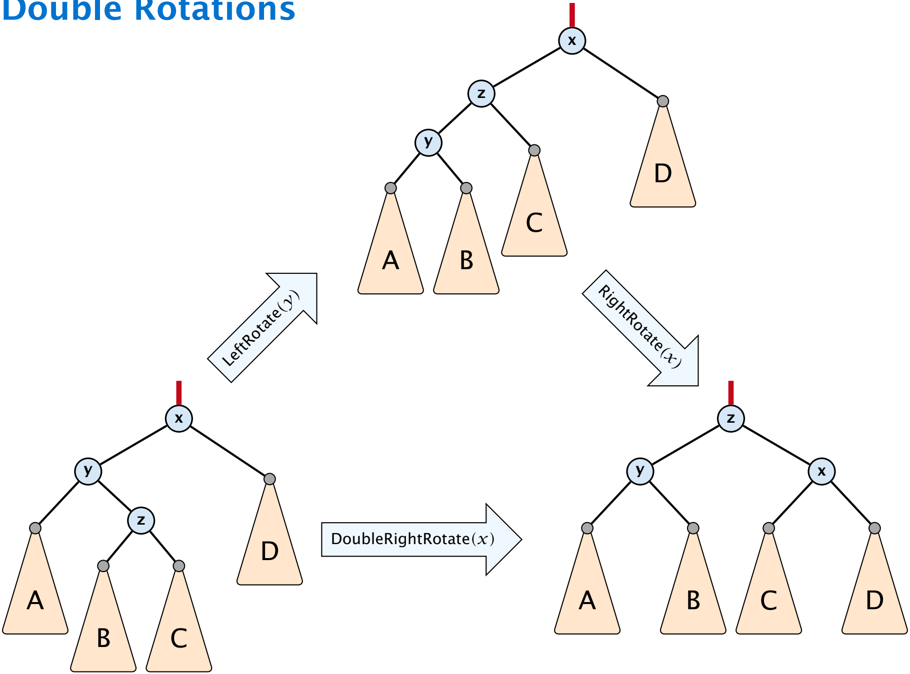
Splay: Zigzag Case



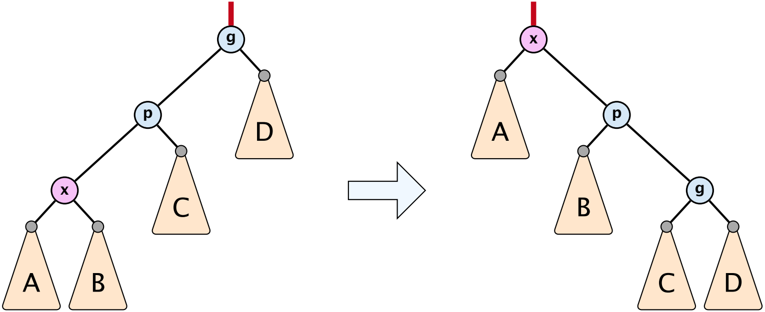
better option splay(x):

- zigzag case: if x is right child and parent of x is left child (or x left child parent of x right child)
- do double right rotation around grand-parent (resp. double left rotation)

Double Rotations



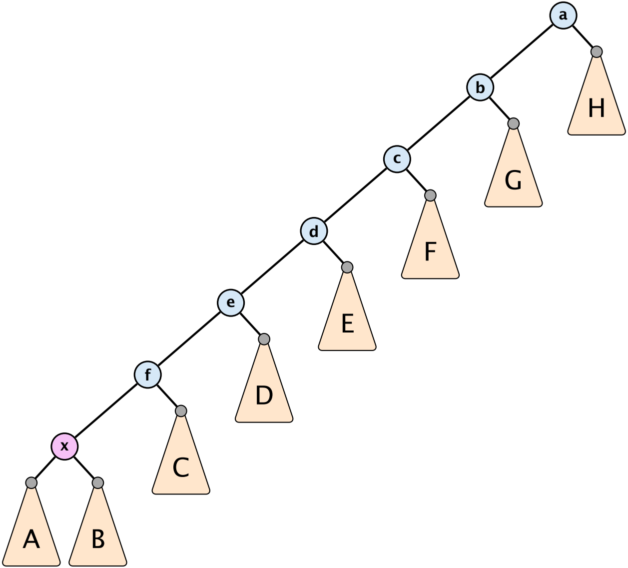
Splay: Zigzig Case



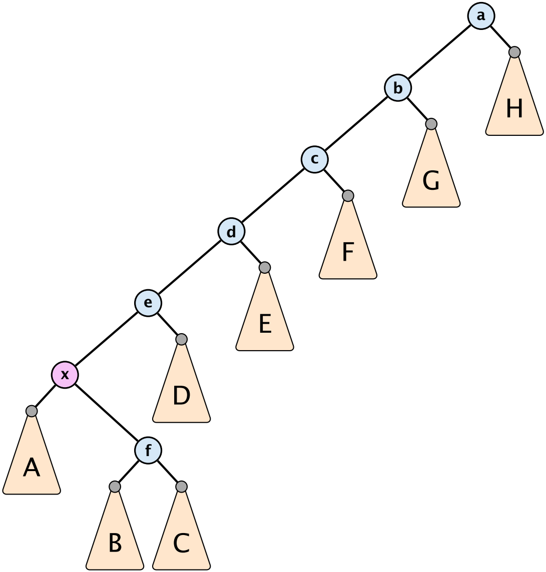
better option splay(x):

- zigzig case: if x is left child and parent of x is left child (or x right child, parent of x right child)
- do right roation around grand-parent followed by right rotation around parent (resp. left rotations)

Splay vs. Move to Root



Splay vs. Move to Root



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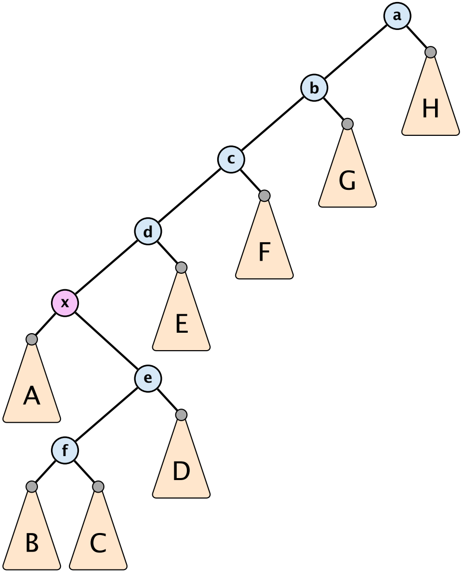

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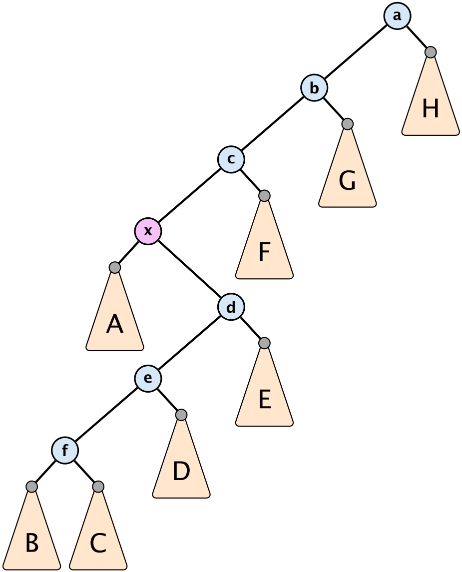
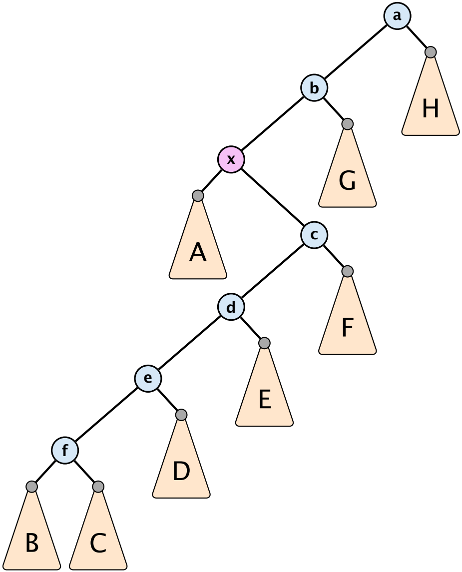
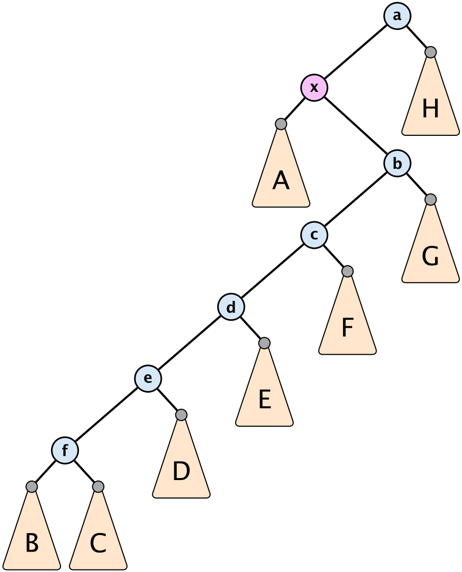


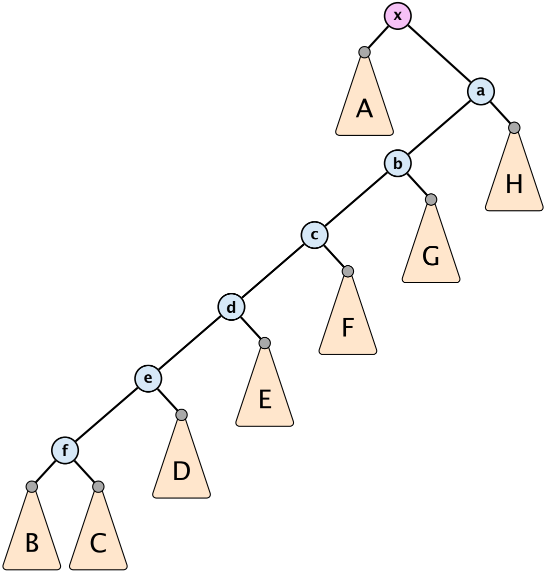
Splay vs. Move to Root



Splay vs. Move to Root



Splay vs. Move to Root



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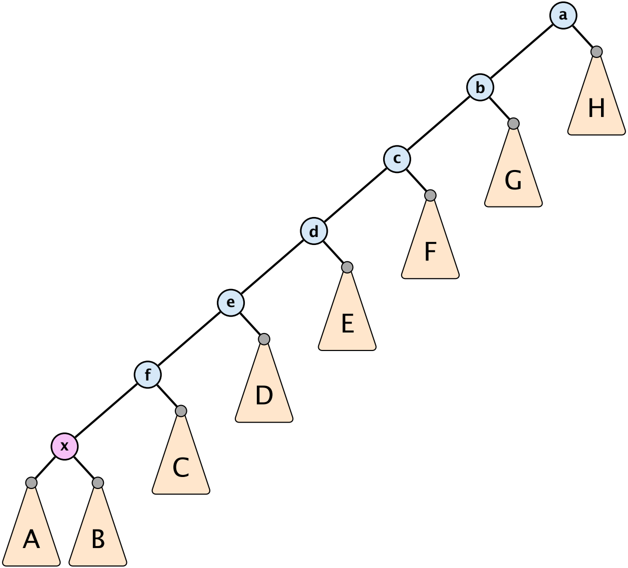

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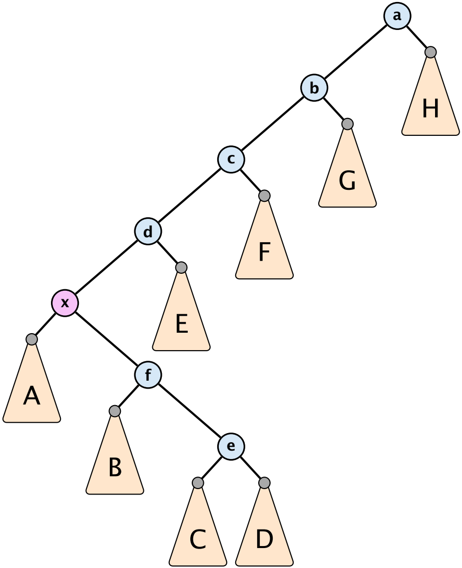

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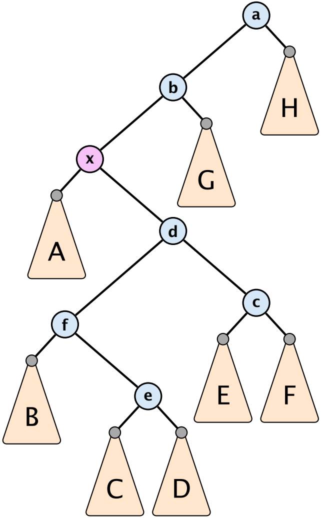
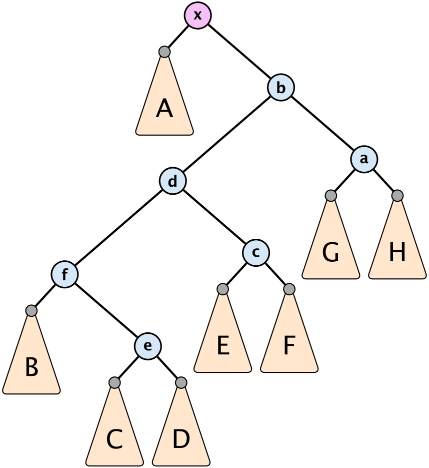


Splay vs. Move to Root



Static Optimality

Suppose we have a sequence of m find-operations. find(x) appears h_x times in this sequence.

The cost of a static search tree T is:

$$\text{cost}(T) = m + \sum_x h_x \,\text{depth}_T(x)$$

The total cost for processing the sequence on a splay-tree is $\mathcal{O}(\text{cost}(T_{\text{min}}))$, where T_{min} is an optimal static search tree.

Dynamic Optimality

Let S be a sequence with m find-operations.

Let A be a data-structure based on a search tree:

- the cost for accessing element x is $1 + \text{depth}(x)$;
- after accessing x the tree may be re-arranged through rotations;

Conjecture:
A splay tree that only contains elements from S has cost $\mathcal{O}(\text{cost}(A, S))$, for processing S.

Lemma 16

Splay Trees have an amortized running time of $\mathcal{O}(\log n)$ *for all operations.*

Amortized Analysis

Definition 17

A data structure with operations $\mathrm{op}_1(), \dots, \mathrm{op}_k()$ has amortized running times $t_1, \dots, t_k$ for these operations if the following holds.

Suppose you are given a sequence of operations (starting with an empty data-structure) that operate on at most n elements, and let k_i denote the number of occurences of $\mathrm{op}_i()$ within this sequence. Then the actual running time must be at most $\sum_i k_i \cdot t_i(n)$.

Potential Method

Introduce a potential for the data structure.

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- Show that $\Phi(D_i) \geq \Phi(D_0)$.

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Then

$$\sum_{i=1}^{k} c_i$$

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$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) \ .$$

- Show that $\Phi(D_i) \geq \Phi(D_0)$.

Then

$$\sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} c_i + \Phi(D_k) - \Phi(D_0)$$

Potential Method

Introduce a potential for the data structure.

- $\Phi(D_i)$ is the potential after the i-th operation.
- Amortized cost of the i-th operation is

$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) \ .$$

- Show that $\Phi(D_i) \geq \Phi(D_0)$.

Then

$$\sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} c_i + \Phi(D_k) - \Phi(D_0) = \sum_{i=1}^{k} \hat{c}_i$$

This means the amortized costs can be used to derive a bound on the total cost.

Example: Stack

Stack

- $S.\mathbf{push}()$
- $S.\mathbf{pop}()$
- $S.\mathbf{multipop}(k)$: removes k items from the stack. If the stack currently contains less than k items it empties the stack.
- The user has to ensure that pop and multipop do not generate an underflow.

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- The user has to ensure that pop and multipop do not generate an underflow.

Actual cost:

- $S.\mathbf{push}()$: cost 1.
- $S.\mathbf{pop}()$: cost 1.
- $S.\mathbf{multipop}(k)$: cost $\min\{\text{size}, k\} = k$.

Example: Stack

Use potential function $\Phi(S) =$ number of elements on the stack.

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Amortized cost:

- $S.\mathbf{push()}$: cost

$$\hat{C}_{\text{push}} = C_{\text{push}} + \Delta\Phi = 1 + 1 \leq 2 \enspace .$$

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$$\hat{C}_{\mathrm{pop}} = C_{\mathrm{pop}} + \Delta\Phi = 1 - 1 \le 0\ .$$

- $\boldsymbol{S.\,\mathrm{multipop}(k)}$: cost

$$\hat{C}_{\mathrm{mp}} = C_{\mathrm{mp}} + \Delta\Phi = \min\{\mathrm{size}, k\} - \min\{\mathrm{size}, k\} \le 0\ .$$

Example: Binary Counter

Incrementing a binary counter:

Consider a computational model where each bit-operation costs one time-unit.

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Actual cost:

- Changing bit from 0 to 1: cost 1.
- Changing bit from 1 to 0: cost 1.
- Increment: cost is $k + 1$, where k is the number of consecutive ones in the least significant bit-positions (e.g, 001101 has $k = 1$).

Example: Binary Counter

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Choose potential function $\Phi(x) = k$, where k denotes the number of ones in the binary representation of x.

Amortized cost:

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Amortized cost:

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- Changing bit from 1 to 0:

$$\hat{C}_{1\to 0} = C_{1\to 0} + \Delta\Phi = 1 - 1 \le 0 \ .$$

- Increment: Let k denotes the number of consecutive ones in the least significant bit-positions. An increment involves k $(1 \to 0)$-operations, and one $(0 \to 1)$-operation.

  Hence, the amortized cost is $k\hat{C}_{1\to 0} + \hat{C}_{0\to 1} \le 2$.

Splay Trees

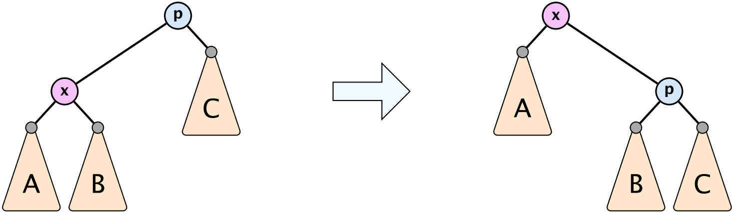
potential function for splay trees:

- size $\mathrm{s}(x) = |T_x|$
- rank $\mathrm{r}(x) = \log_2(s(x))$
- $\Phi(T) = \sum_{v \in T} r(v)$

amortized cost = real cost + potential change

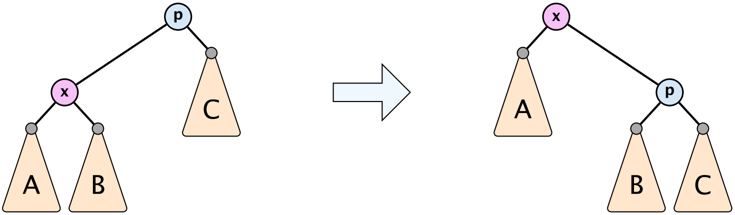
The cost is essentially the cost of the splay-operation, which is 1 plus the number of rotations.

Splay: Zig Case



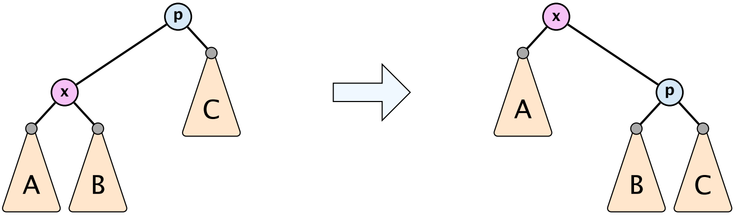
$\Delta\Phi =$

Splay: Zig Case



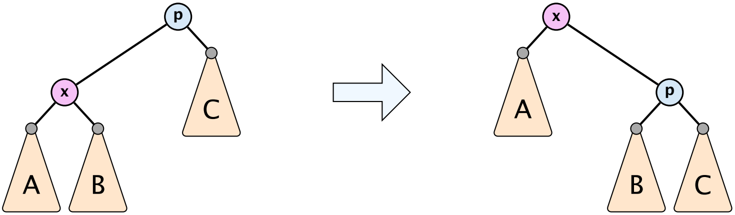
$$\Delta\Phi = r'(x) + r'(p) - r(x) - r(p)$$

Splay: Zig Case



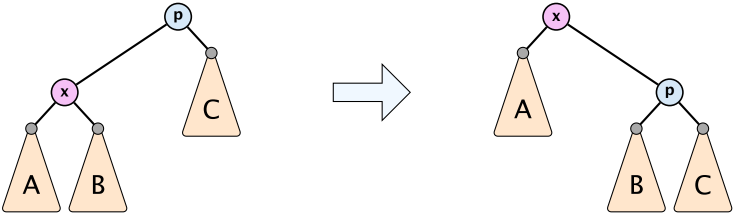
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p) \\ &= r'(p) - r(x)\end{aligned}$$

Splay: Zig Case



$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p) \\ &= r'(p) - r(x) \\ &\leq r'(x) - r(x)\end{aligned}$$

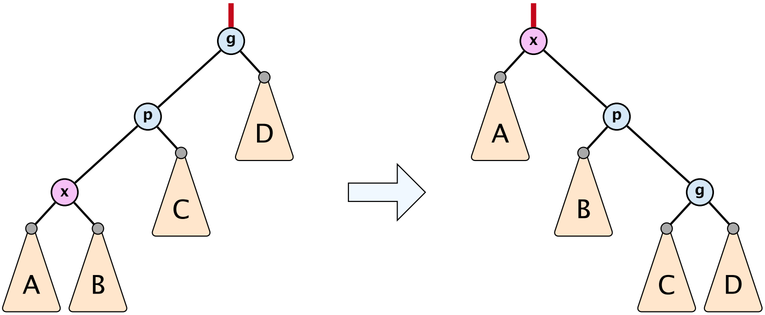
Splay: Zig Case



$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p)\\ &= r'(p) - r(x)\\ &\le r'(x) - r(x)\end{aligned}$$

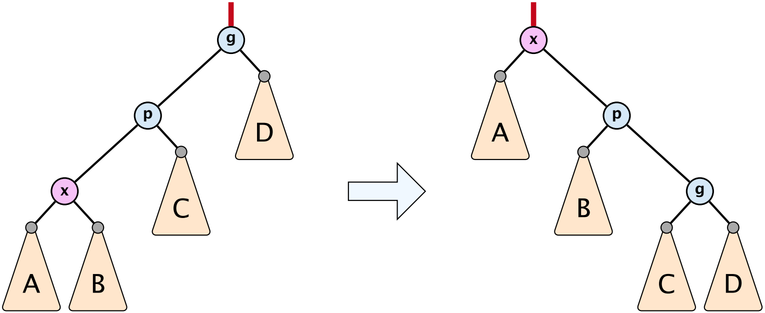
$$\mathrm{cost}_{\mathrm{zig}} \le 1 + 3(r'(x) - r(x))$$

Splay: Zigzig Case



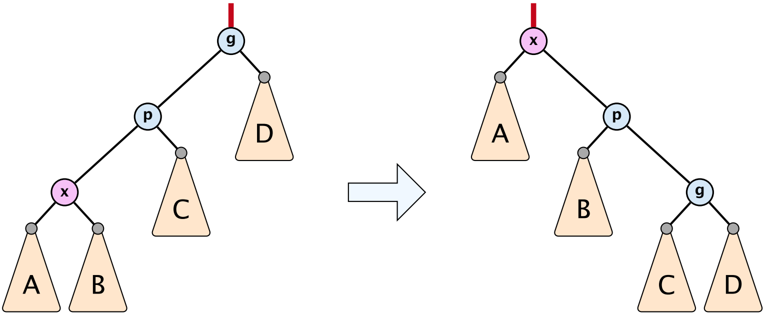
$\Delta\Phi =$

Splay: Zigzig Case



$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$

Splay: Zigzig Case



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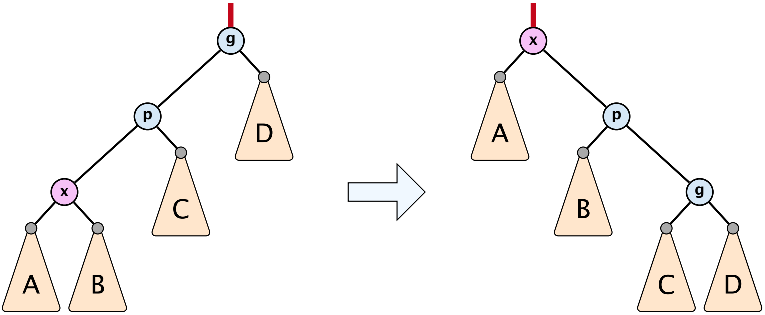

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$$= r'(p) + r'(g) - r(x) - r(p)$$
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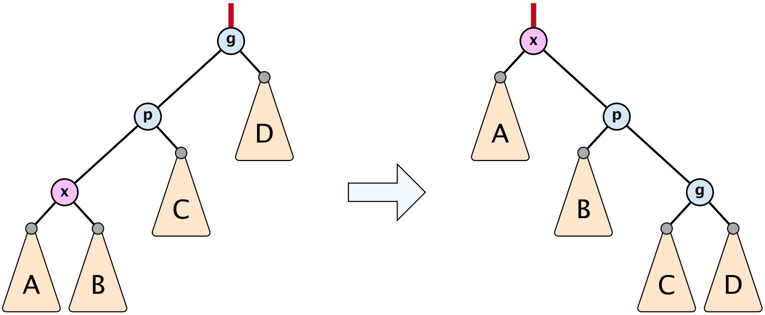

$$\begin{aligned}
\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\
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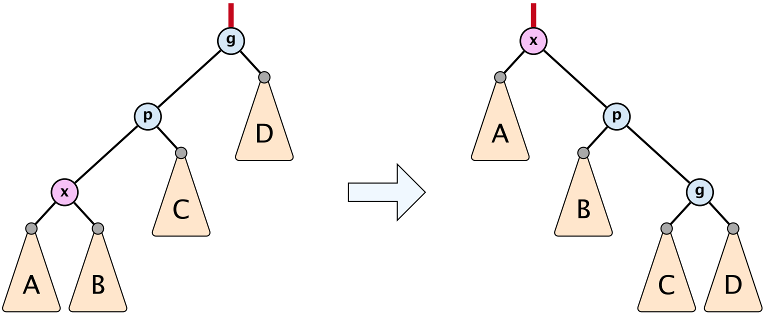
&= r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x)
\end{aligned}$$

Splay: Zigzig Case



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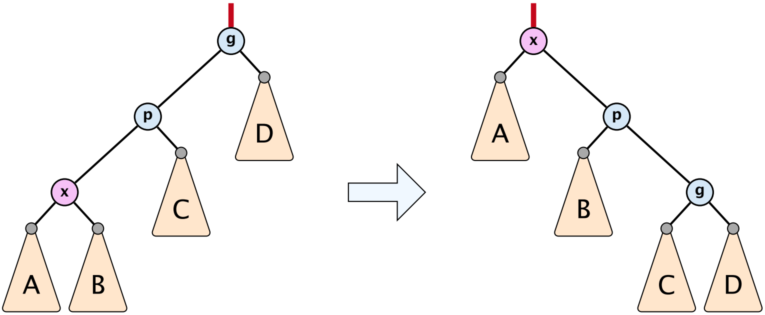
&= -2r'(x) + r'(g) + r(x) + 3(r'(x) - r(x))
\end{aligned}
$$

Splay: Zigzig Case



$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$
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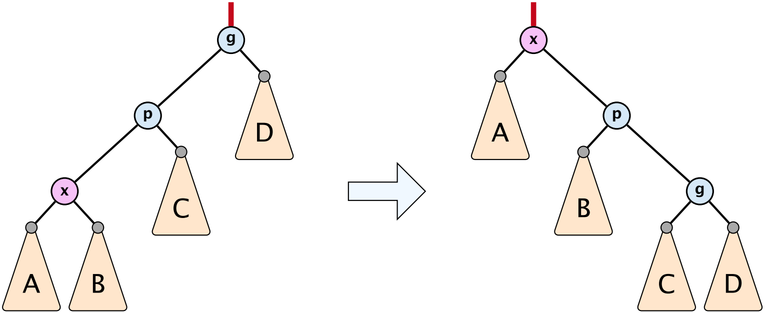

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\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\
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\end{aligned}$$

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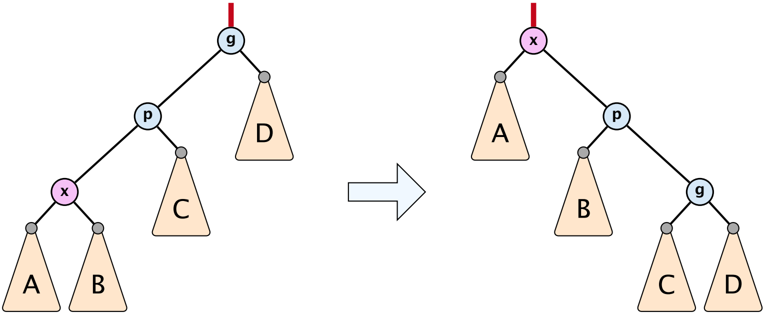

$$\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)$$

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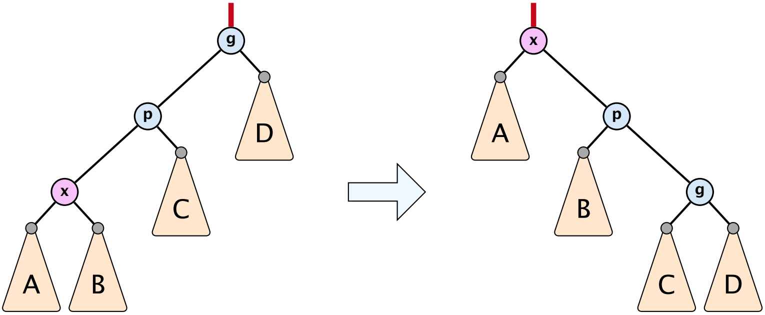

$$\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)$$
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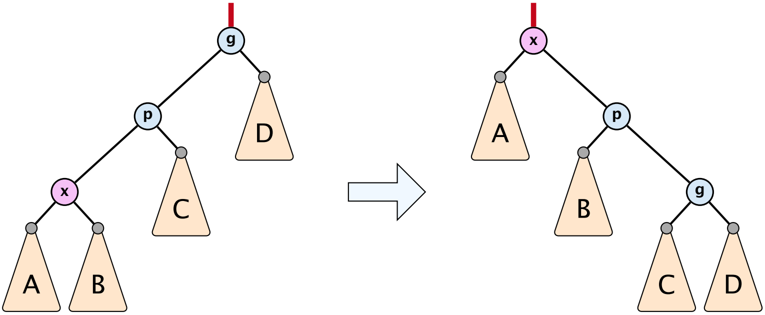
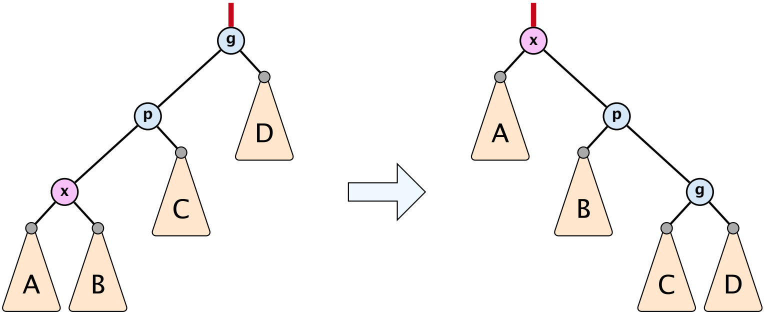


$$\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)$$

$$= \frac{1}{2}\Big(\log(s(x)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

$$= \frac{1}{2}\log\Big(\frac{s(x)}{s'(x)}\Big) + \frac{1}{2}\log\Big(\frac{s'(g)}{s'(x)}\Big)$$

Splay: Zigzig Case



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$$\leq \log\Big(\frac{1}{2}\frac{s(x)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\Big)$$

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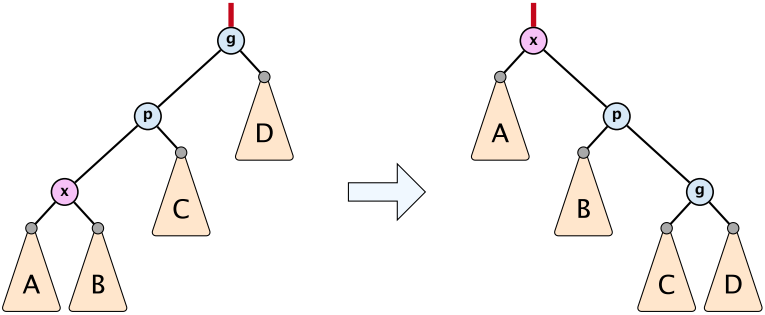

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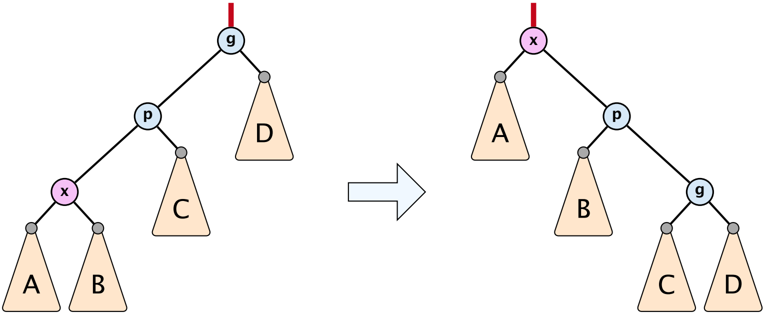

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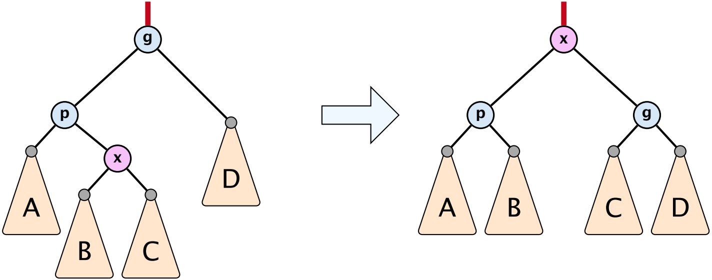
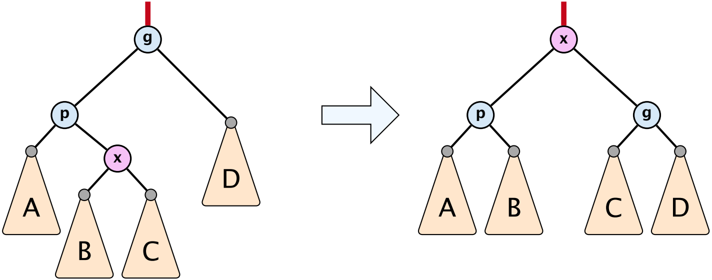
$$\leq \log\Big(\frac{1}{2}\frac{s(x)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\Big) \leq \log\Big(\frac{1}{2}\Big) = -1$$

Splay: Zigzag Case



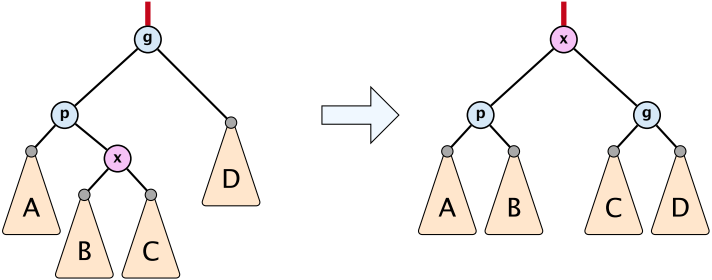
$\Delta\Phi =$

Splay: Zigzag Case



$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$

Splay: Zigzag Case



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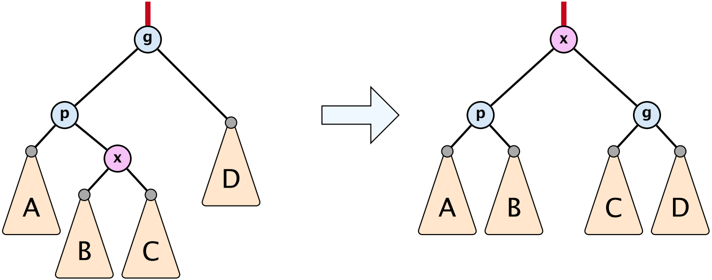
Splay: Zigzag Case



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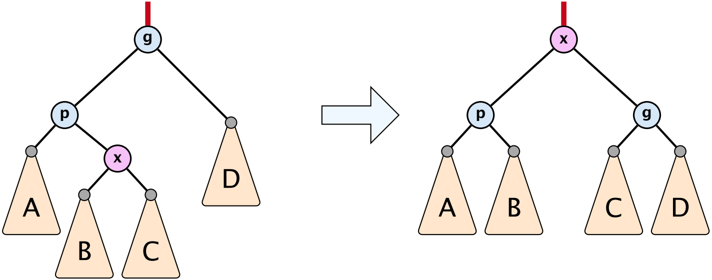

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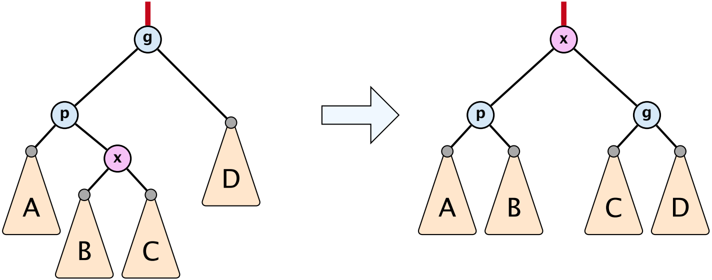
\begin{aligned}
\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\
&= r'(p) + r'(g) - r(x) - r(p) \\
&\leq r'(p) + r'(g) - r(x) - r(x) \\
&= r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x)
\end{aligned}
$$

Splay: Zigzag Case



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&= r'(p) + r'(g) - r(x) - r(p) \\
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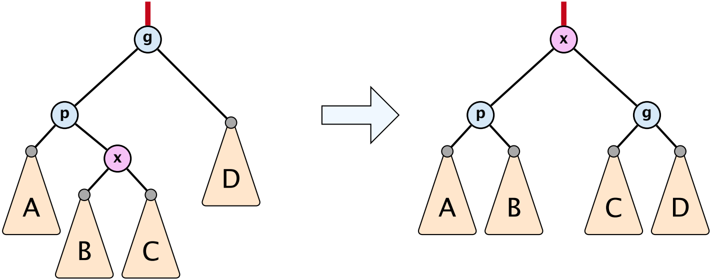
&= r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x) \\
&\leq -2 + 2(r'(x) - r(x))
\end{aligned}$$

Splay: Zigzag Case



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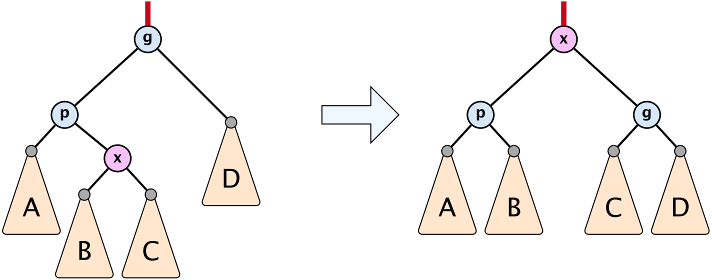
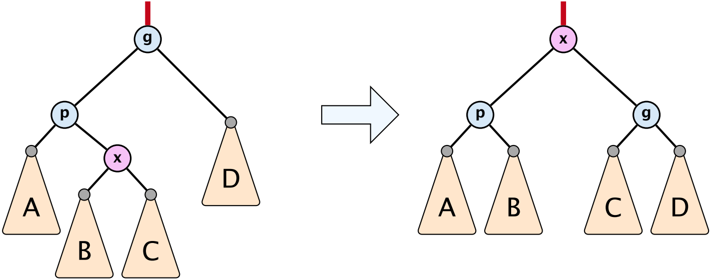
$$\leq -2 + 2(r'(x) - r(x)) \quad \Rightarrow \text{cost}_{\text{zigzag}} \leq 3(r'(x) - r(x))$$

Splay: Zigzag Case



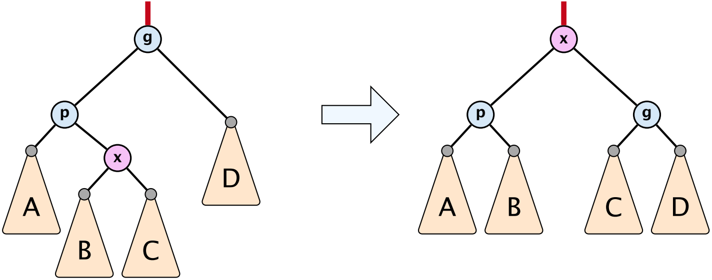
$$\frac{1}{2}\Big(r'(p) + r'(g) - 2r'(x)\Big)$$

Splay: Zigzag Case



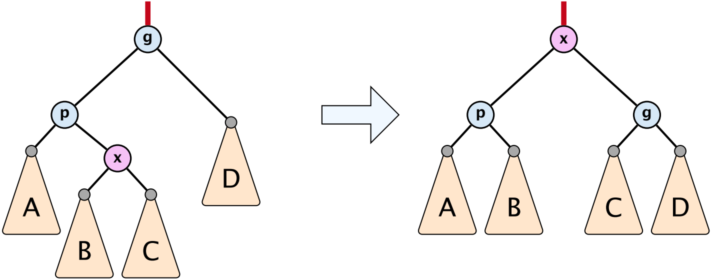
$$\frac{1}{2}\Big(r'(p) + r'(g) - 2r'(x)\Big)$$
$$= \frac{1}{2}\Big(\log(s'(p)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

Splay: Zigzag Case



$$\frac{1}{2}\Big(r'(p) + r'(g) - 2r'(x)\Big)$$
$$= \frac{1}{2}\Big(\log(s'(p)) + \log(s'(g)) - 2\log(s'(x))\Big)$$
$$\leq \log\Big(\frac{1}{2}\frac{s'(p)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\Big)$$

Splay: Zigzag Case



$$\frac{1}{2}\Big(r'(p) + r'(g) - 2r'(x)\Big)$$

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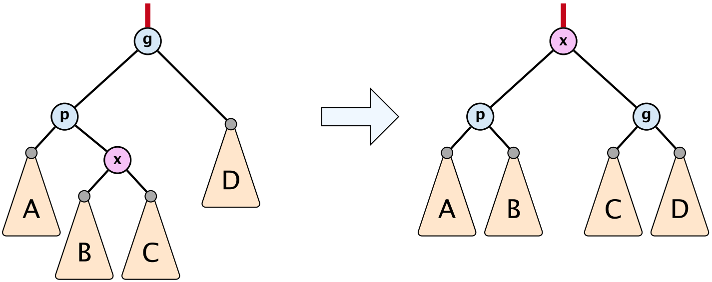
Splay: Zigzag Case



$$\frac{1}{2}\Big(r'(p) + r'(g) - 2r'(x)\Big)$$

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$$\leq \log\left(\frac{1}{2}\frac{s'(p)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\right) \leq \log\left(\frac{1}{2}\right) = -1$$

Amortized cost of the whole splay operation:

$$\leq 1 + 1 + \sum_{\text{steps } t} 3(r_t(x) - r_{t-1}(x))$$

$$= 2 + 3(r(\text{root}) - r_0(x))$$

$$\leq \mathcal{O}(\log n)$$

7.4 Augmenting Data Structures

Suppose you want to develop a data structure with:

- **Insert(x)**: insert element x.
- **Search(k)**: search for element with key k.
- **Delete(x)**: delete element referenced by pointer x.
- **find-by-rank(ℓ)**: return the ℓ-th element; return "error" if the data-structure contains less than ℓ elements.

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Augment an existing data-structure instead of developing a new one.

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4. develop the new operations

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Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

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2. We store in each node v the size of the sub-tree rooted at v.
3. We need to be able to update the size-field in each node without asymptotically affecting the running time of insert, delete, and search. We come back to this step later...

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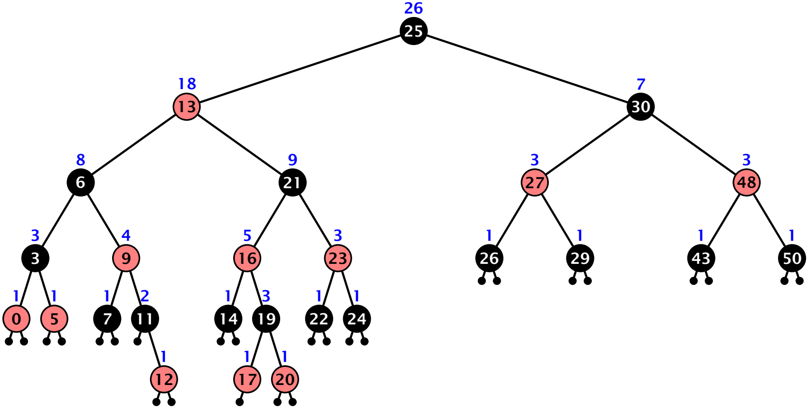
Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

4. How does find-by-rank work?
   Find-by-rank(k) := Select(root, k) with

Algorithm 1 Select(x, i)

if $x = \text{null}$ **then return** error
if $\text{left}[x] \neq \text{null}$ **then** $r \leftarrow \text{left}[x].\text{size} + 1$ **else** $r \leftarrow 1$
if $i = r$ **then return** x
if $i < r$ **then**
    return Select$(\text{left}[x], i)$
else
    return Select$(\text{right}[x], i - r)$

Select(x, i)



Find-by-rank:

- decide whether you have to proceed into the left or right sub-tree
- adjust the rank that you are searching for if you go right

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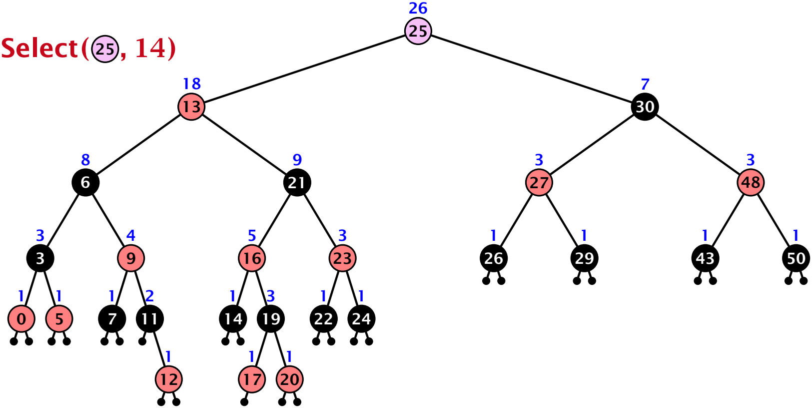

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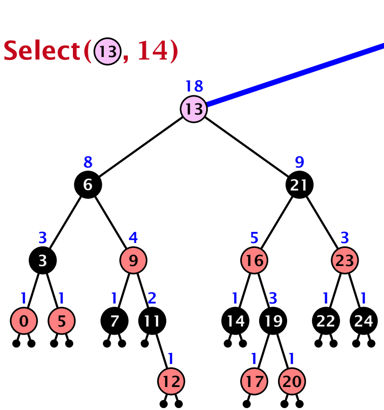

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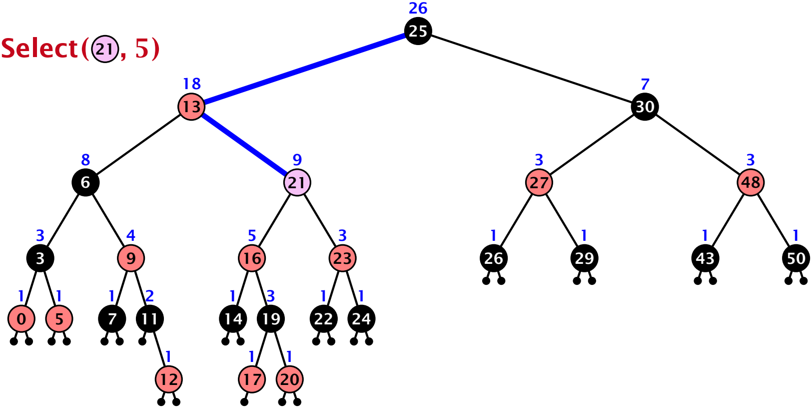

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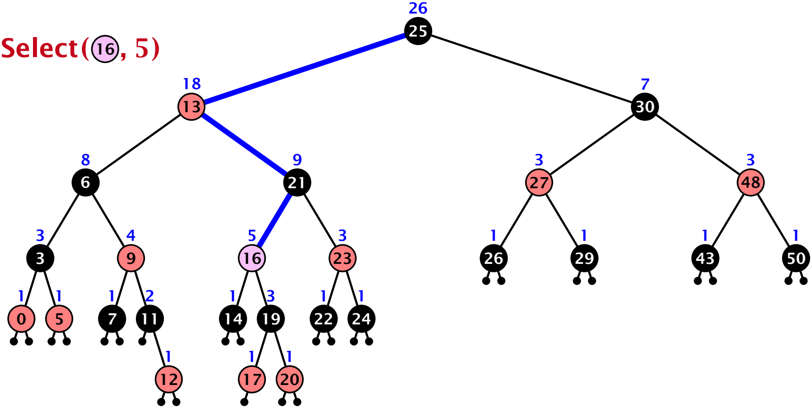

Find-by-rank:

- decide whether you have to proceed into the left or right sub-tree
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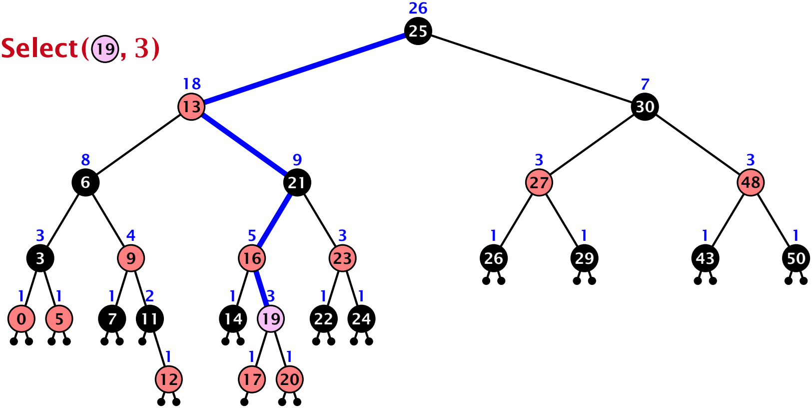
Select(x, i)

Select(⑲, 3)



Find-by-rank:

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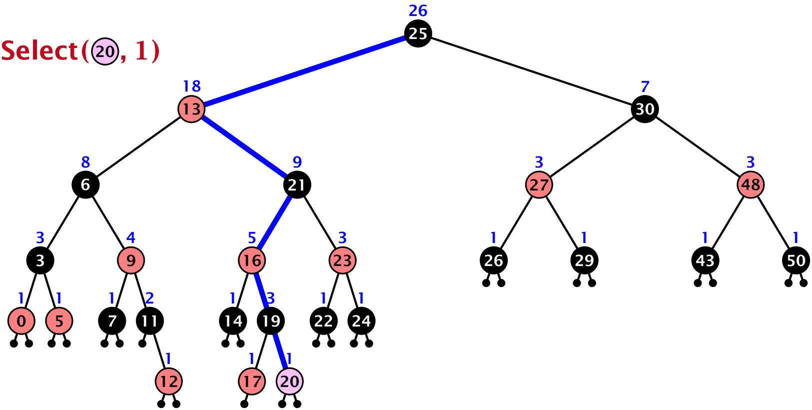
Select(x, i)

Select(20, 1)



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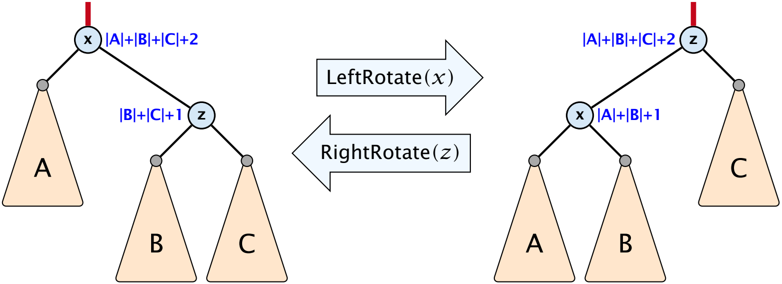
Insert(x): When going down the search path increase the size field for each visited node. Maintain the size field during rotations.

Delete(x): Directly after splicing out a node traverse the path from the spliced out node upwards, and decrease the size counter on every node on this path. Maintain the size field during rotations.

Rotations

The only operation during the fix-up procedure that alters the tree and requires an update of the size-field:



The nodes x and z are the only nodes changing their size-fields.

The new size-fields can be computed locally from the size-fields of the children.

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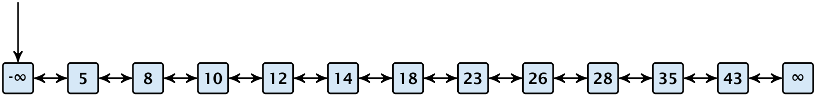

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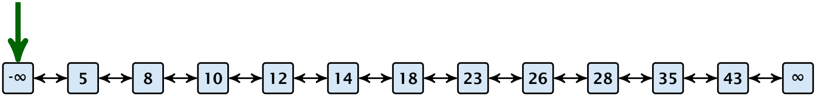

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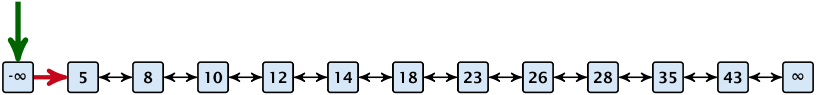

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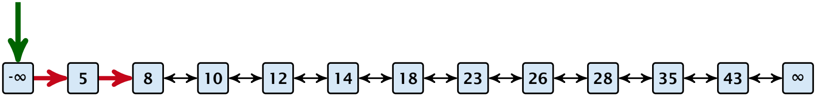

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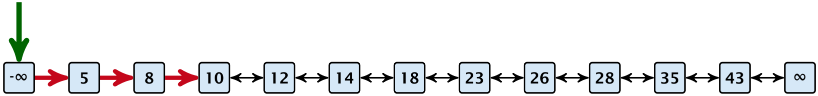

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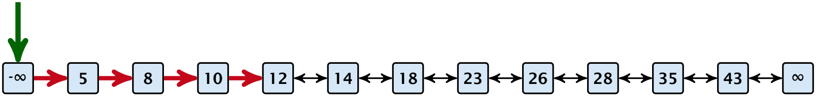

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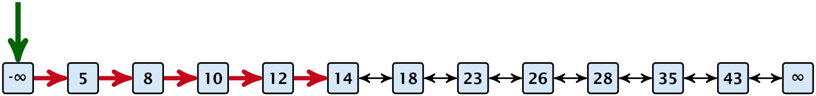

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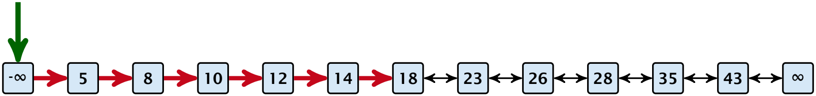

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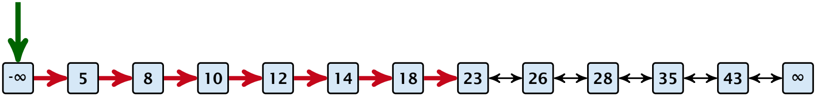

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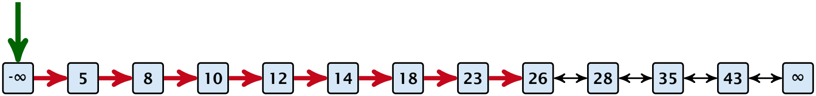

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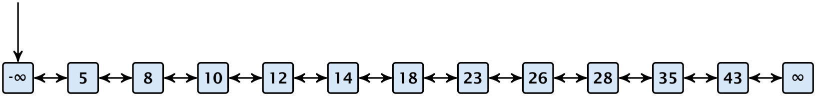

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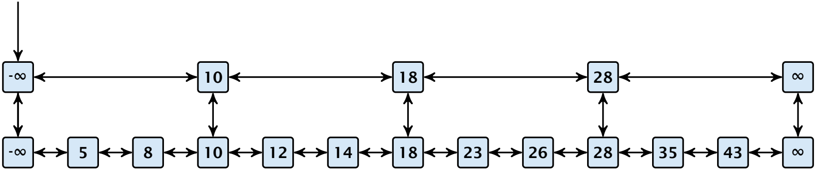

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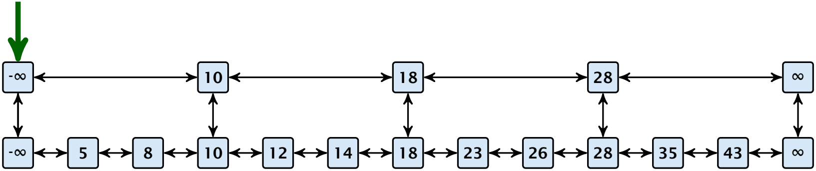

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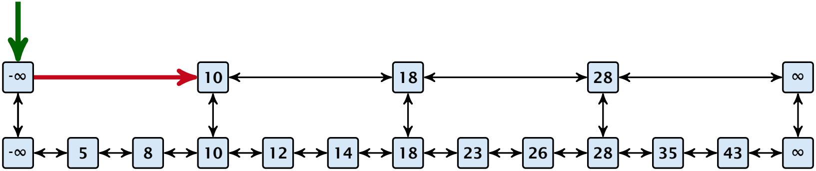

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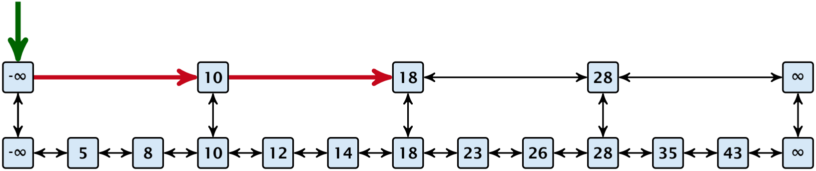

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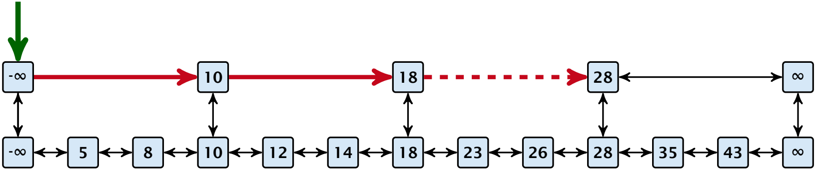

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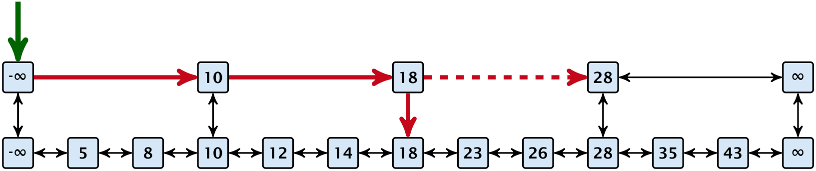

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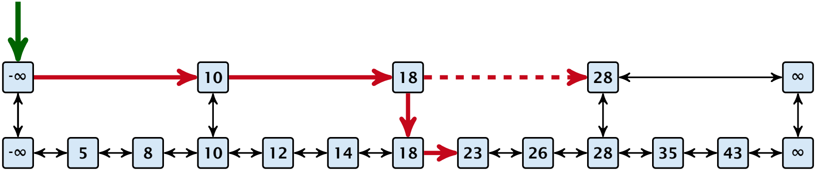

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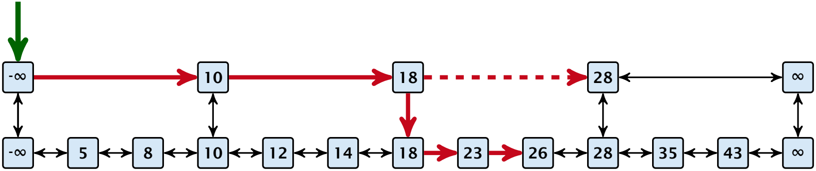

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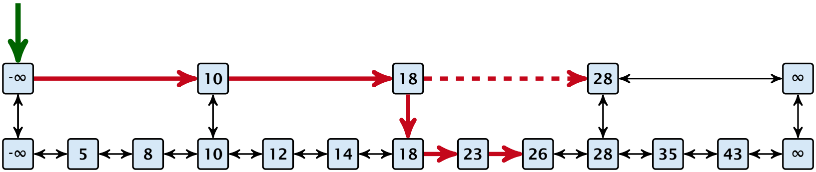

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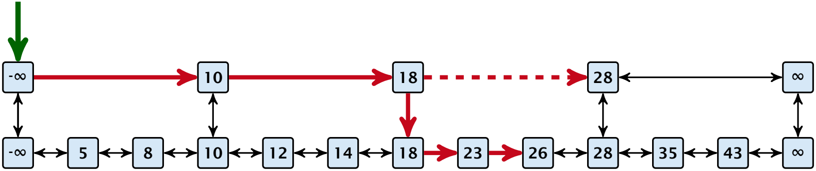

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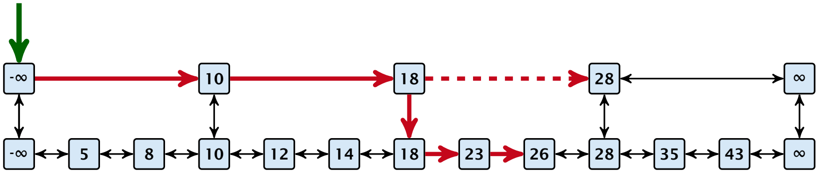

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Choose $|L_1| = \sqrt{n}$. Then search time $\Theta(\sqrt{n})$.

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$$\begin{aligned} r^{-k}n + kr &= \left(n^{\frac{1}{k+1}}\right)^{-k} n + kn^{\frac{1}{k+1}} \\ &= n^{1-\frac{k}{k+1}} + kn^{\frac{1}{k+1}} \end{aligned}$$

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$$\begin{aligned} r^{-k}n + kr &= \left(n^{\frac{1}{k+1}}\right)^{-k} n + kn^{\frac{1}{k+1}} \\ &= n^{1-\frac{k}{k+1}} + kn^{\frac{1}{k+1}} \\ &= (k+1)n^{\frac{1}{k+1}} \ . \end{aligned}$$

7.5 Skip Lists

Choose ratios between list-lengths evenly, i.e., $\frac{|L_{i-1}|}{|L_i|} = r$, and, hence, $L_k \approx r^{-k}n$.

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Choosing $k = \Theta(\log n)$ gives a logarithmic running time.

7.5 Skip Lists

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- If we want that in L_i we always skip over roughly the same number of elements in L_{i-1} an insert or delete may require a lot of re-organisation.

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Use randomization instead!

7.5 Skip Lists

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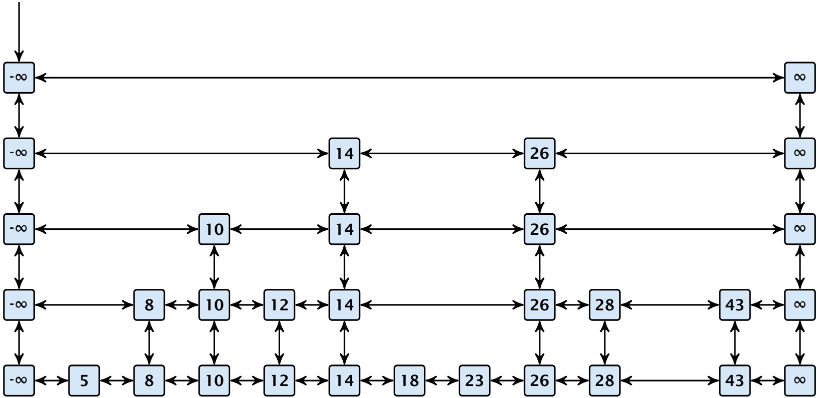
The time for both operations is dominated by the search time.

7.5 Skip Lists

Insert (35):



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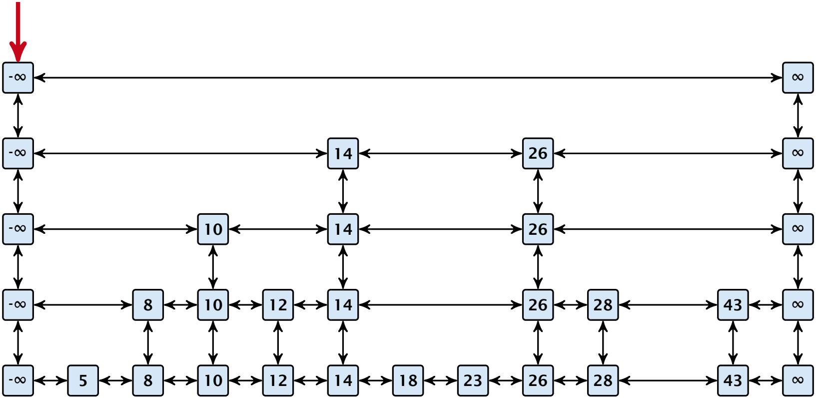

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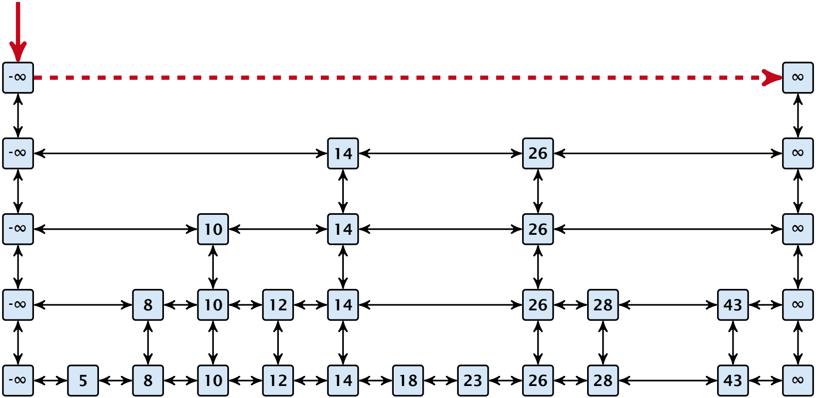

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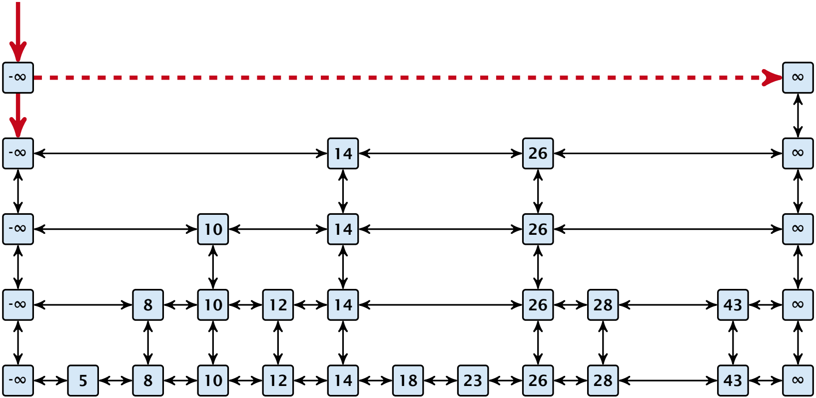

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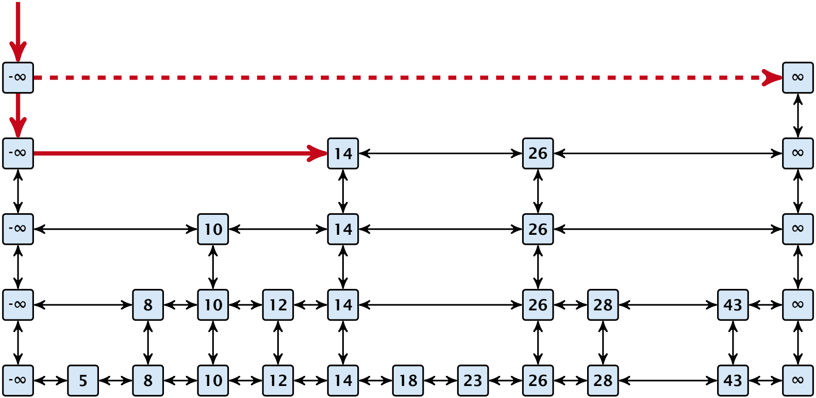

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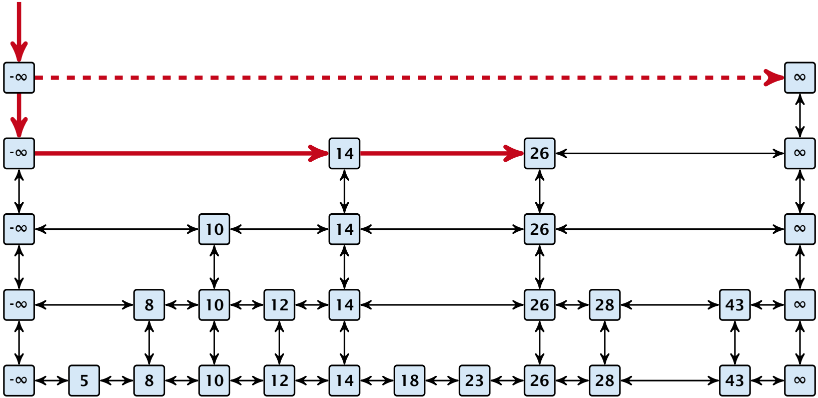

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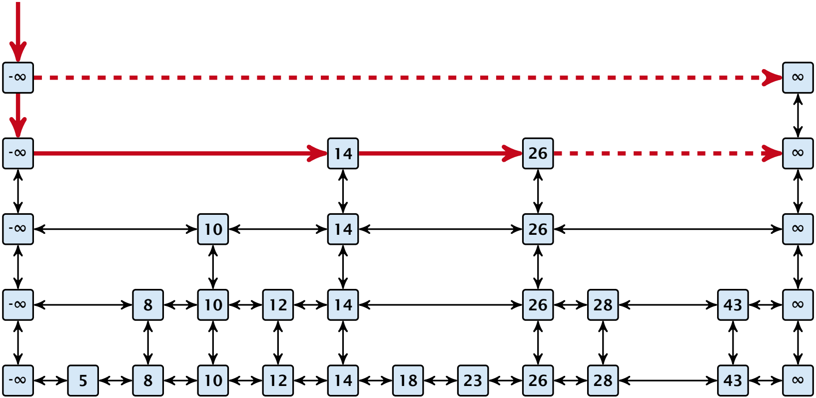

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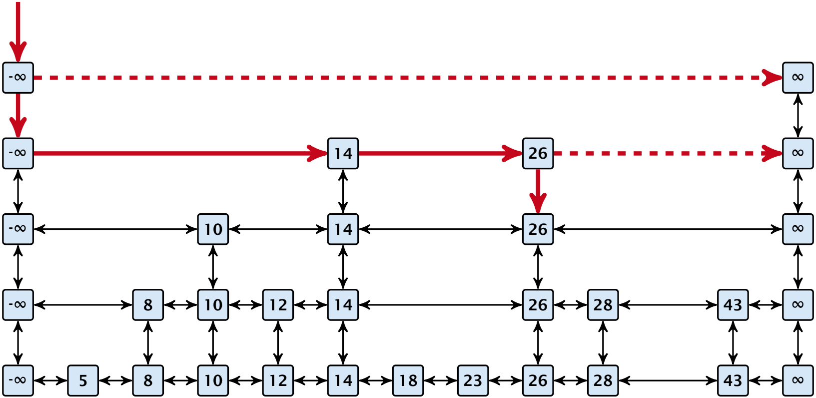

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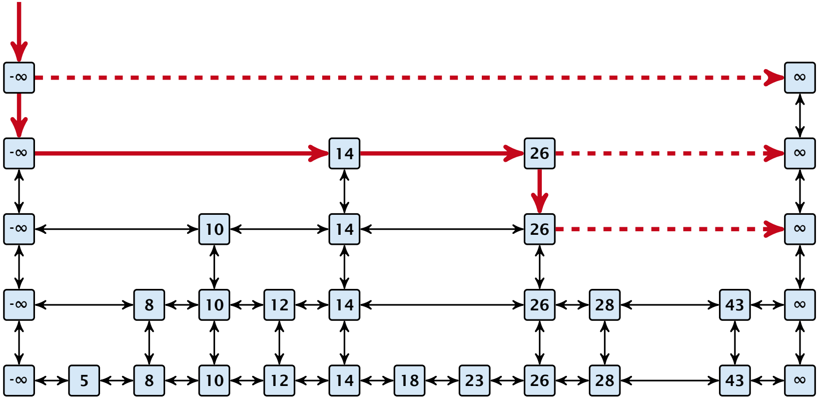

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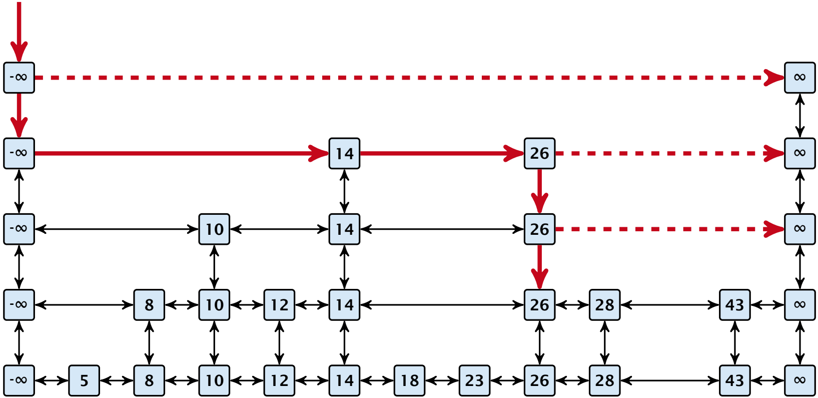

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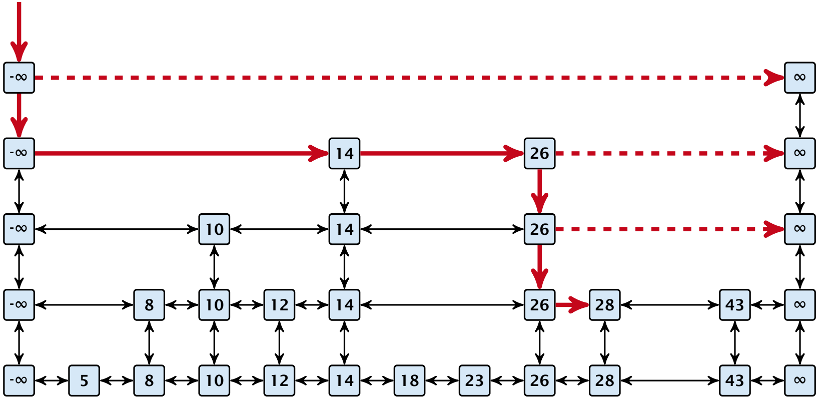

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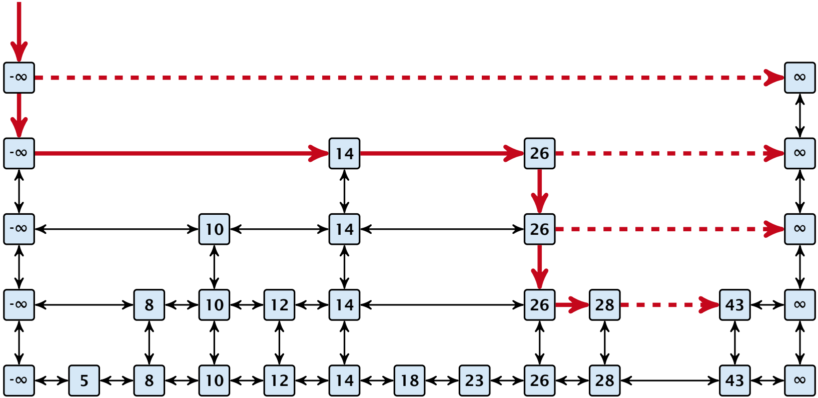

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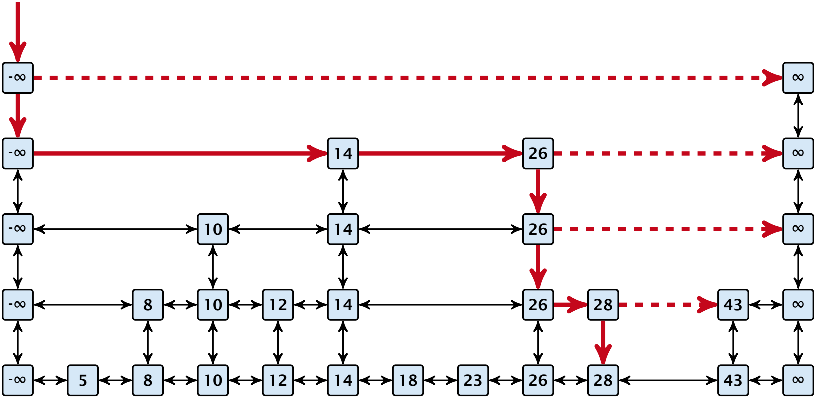

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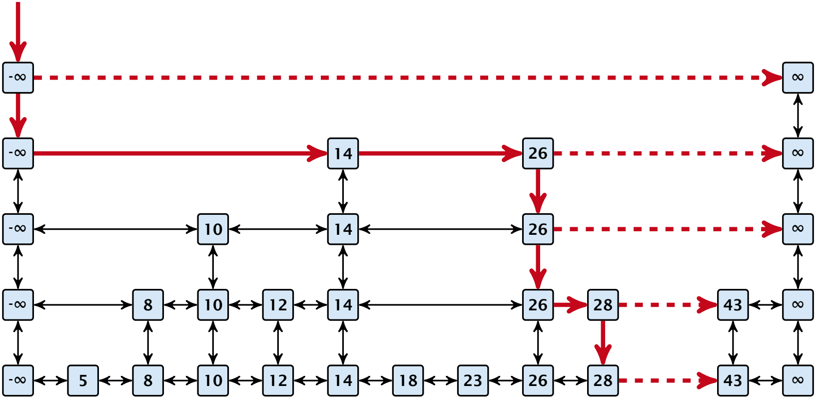

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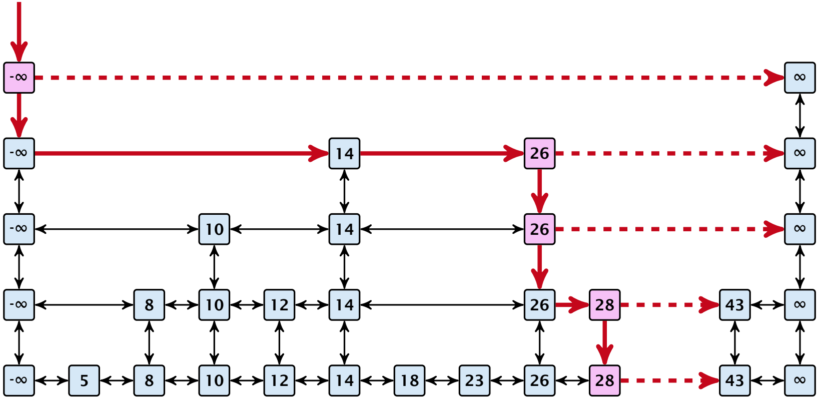

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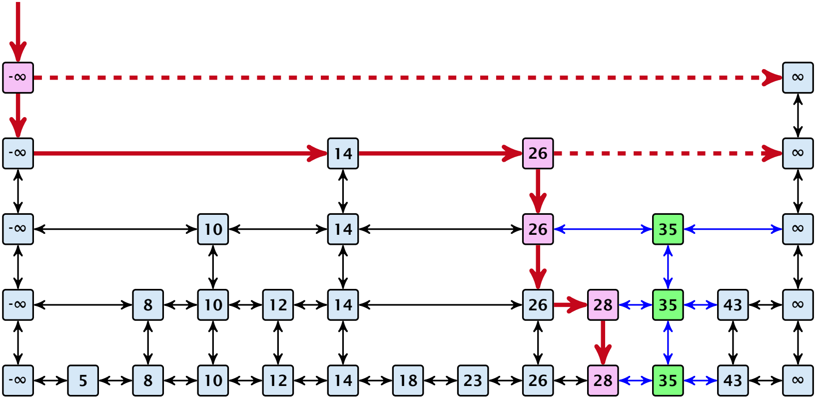

High Probability

Definition 18 (High Probability)

We say a **randomized** algorithm has running time $\mathcal{O}(\log n)$ with high probability if for any constant α the running time is at most $\mathcal{O}(\log n)$ with probability at least $1 - \frac{1}{n^\alpha}$.

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Here the $\mathcal{O}$-notation hides a constant that may depend on α.

High Probability

Suppose there are polynomially many events $E_1, E_2, \ldots, E_\ell$, $\ell = n^c$ each holding with high probability (e.g. E_i may be the event that the i-th search in a skip list takes time at most $\mathcal{O}(\log n)$).

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Then the probability that all E_i hold is at least

$$\Pr[E_1 \wedge \cdots \wedge E_\ell]$$

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This means $E_1 \wedge \cdots \wedge E_\ell$ holds with high probability.

7.5 Skip Lists

Lemma 19

A search (and, hence, also insert and delete) in a skip list with n elements takes time $\mathcal{O}(\log n)$ with high probability (w. h. p.).

7.5 Skip Lists

Backward analysis:

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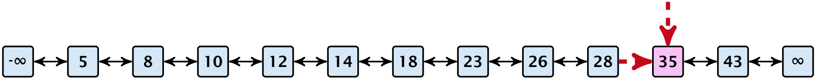

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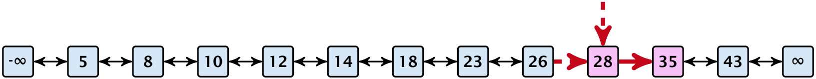

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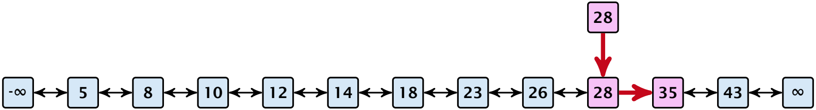

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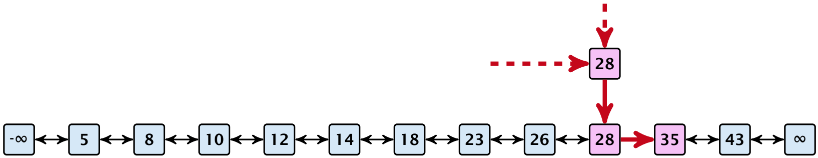

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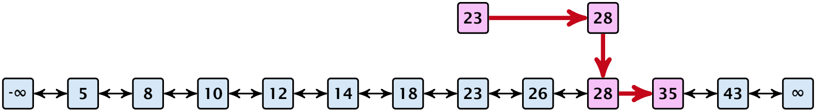

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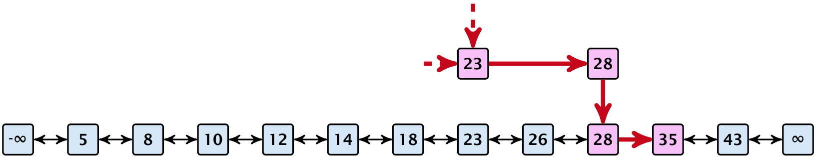

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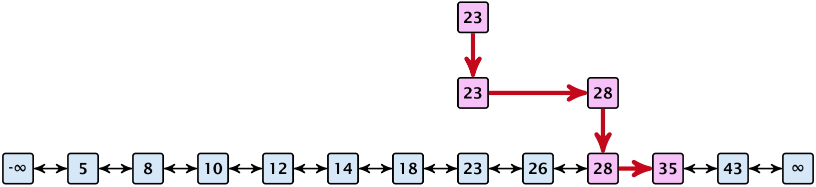

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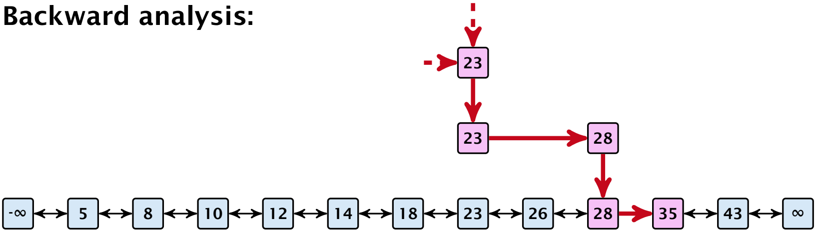

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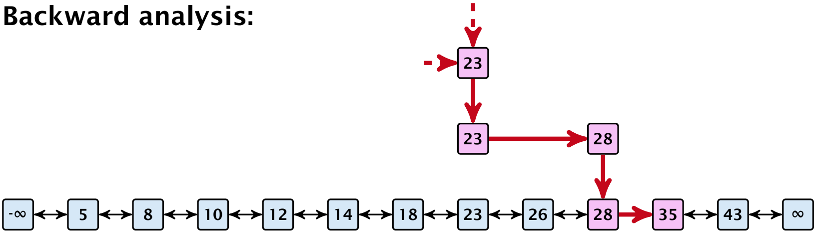

At each point the path goes up with probability $1/2$ and left with probability $1/2$.

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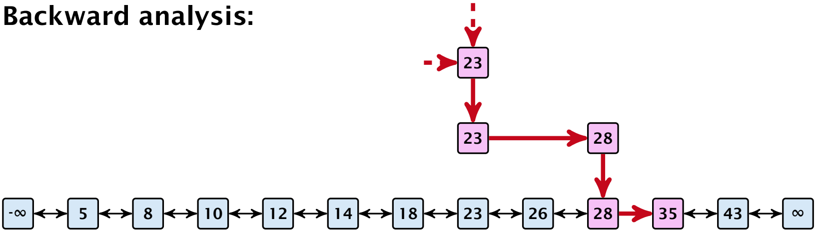

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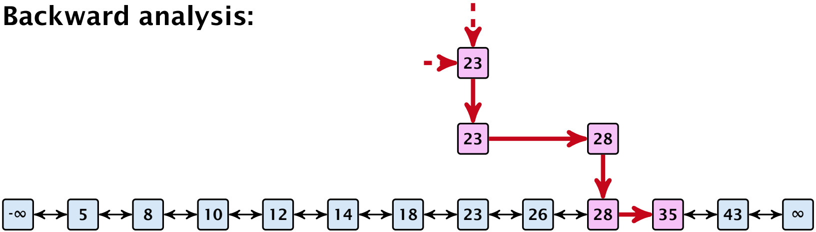

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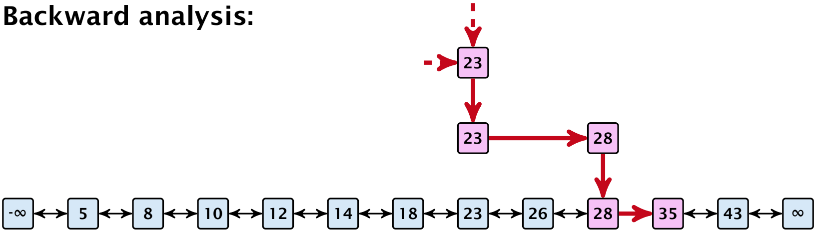

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We show that w.h.p:

- A "long" search path must also go very high.
- There are no elements in high lists.

From this it follows that w.h.p. there are no long paths.

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Estimation for Binomial Coefficients

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

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In particular, this means that during the construction in the backward analysis we see at most k heads (i.e., coin flips that tell you to go up) in z trials.

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$$\Pr[E_{z,k}]$$

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$$\leq \binom{z}{k} 2^{-(z-k)} \leq \left(\frac{ez}{k}\right)^k 2^{-(z-k)} \leq \left(\frac{2ez}{k}\right)^k 2^{-z}$$

choosing $k = \gamma \log n$ with $\gamma \geq 1$ and $z = (\beta + \alpha)\gamma \log n$

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So far we fixed $k = \gamma \log n$, $\gamma \geq 1$, and $z = 7\alpha\gamma \log n$, $\alpha \geq 1$.

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$$\Pr[\text{search requires } z \text{ steps}]$$

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Hence,

$$\Pr[\text{search requires } z \text{ steps}] \leq \Pr[E_{z,k}] + \Pr[A_{k+1}]$$

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Hence,

$$\begin{aligned}\Pr[\text{search requires } z \text{ steps}] &\leq \Pr[E_{z,k}] + \Pr[A_{k+1}] \\ &\leq n^{-\alpha} + n^{-(\gamma-1)}\end{aligned}$$

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Let A_{k+1} denote the event that the list L_{k+1} is non-empty. Then

$$\Pr[A_{k+1}] \leq n2^{-(k+1)} \leq n^{-(\gamma-1)} \ .$$

For the search to take at least $z = 7\alpha\gamma \log n$ steps either the event $E_{z,k}$ or the event A_{k+1} must hold.
Hence,

$$\begin{aligned}\Pr[\text{search requires } z \text{ steps}] &\leq \Pr[E_{z,k}] + \Pr[A_{k+1}] \\ &\leq n^{-\alpha} + n^{-(\gamma-1)}\end{aligned}$$

This means, the search requires at most z steps, w. h. p.

7.6 van Emde Boas Trees

Dynamic Set Data Structure S:

- $S.\text{insert}(x)$
- $S.\text{delete}(x)$
- $S.\text{search}(x)$
- $S.\text{min}()$
- $S.\text{max}()$
- $S.\text{succ}(x)$
- $S.\text{pred}(x)$

7.6 van Emde Boas Trees

For this chapter we ignore the problem of storing satellite data:

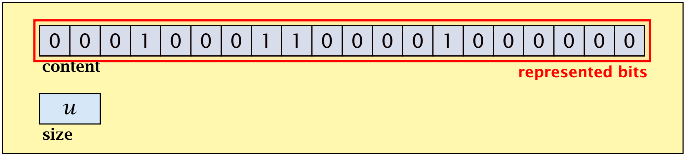
- $S.\mathbf{insert}(x)$: Inserts x into S.
- $S.\mathbf{delete}(x)$: Deletes x from S. Usually assumes that $x \in S$.
- $S.\mathbf{member}(x)$: Returns 1 if $x \in S$ and 0 otw.
- $S.\mathbf{min}()$: Returns the value of the minimum element in S.
- $S.\mathbf{max}()$: Returns the value of the maximum element in S.
- $S.\mathbf{succ}(x)$: Returns successor of x in S. Returns null if x is maximum or larger than any element in S. Note that x needs not to be in S.
- $S.\mathbf{pred}(x)$: Returns the predecessor of x in S. Returns null if x is minimum or smaller than any element in S. Note that x needs not to be in S.

7.6 van Emde Boas Trees

Can we improve the existing algorithms when the keys are from a restricted set?

In the following we assume that the keys are from $\{0, 1, \ldots, u - 1\}$, where u denotes the size of the universe.

Implementation 1: Array



one array of u bits

Use an array that encodes the indicator function of the dynamic set.

Implementation 1: Array

Algorithm 1 array.insert(x)

1: content[x] $\leftarrow 1$;

Algorithm 2 array.delete(x)

1: content[x] $\leftarrow 0$;

Algorithm 3 array.member(x)

1: **return** content[x];

- Note that we assume that x is valid, i.e., it falls within the array boundaries.
- Obviously(?) the running time is constant.

Implementation 1: Array

Algorithm 4 array.max()

```
for (i = size − 1; i ≥ 0; i−−) do
    if content[i] = 1 then return i;
return null;
```

Implementation 1: Array

Algorithm 4 array.max()

for $(i = \text{size} - 1;\ i \geq 0;\ i\text{--})$ **do**
  if $\text{content}[i] = 1$ **then return** i;
return null;

Algorithm 5 array.min()

for $(i = 0;\ i < \text{size};\ i\text{++})$ **do**
  if $\text{content}[i] = 1$ **then return** i;
return null;

Implementation 1: Array

Algorithm 4 array.max()

```
for (i = size − 1; i ≥ 0; i−−) do
    if content[i] = 1 then return i;
return null;
```

Algorithm 5 array.min()

```
for (i = 0; i < size; i++) do
    if content[i] = 1 then return i;
return null;
```

- Running time is $\mathcal{O}(u)$ in the worst case.

Implementation 1: Array

Algorithm 6 array.succ(x)

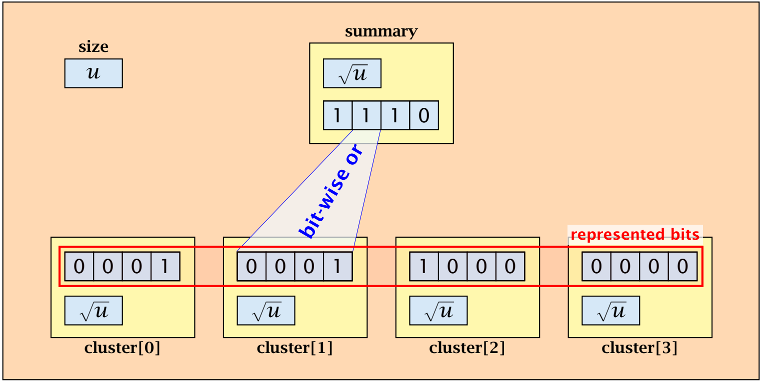
for ($i = x + 1$; $i <$ size; i++) **do**
    if content[i] $= 1$ **then return** i;
return null;

Algorithm 7 array.pred(x)

for ($i = x - 1$; $i \geq 0$; i--) **do**
    if content[i] $= 1$ **then return** i;
return null;

- Running time is $\mathcal{O}(u)$ in the worst case.

Implementation 2: Summary Array



- $\sqrt{u}$ cluster-arrays of $\sqrt{u}$ bits.
- One summary-array of $\sqrt{u}$ bits. The i-th bit in the summary array stores the bit-wise or of the bits in the i-th cluster.

Implementation 2: Summary Array

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The bit for a key x is contained in cluster number $\left\lfloor \frac{x}{\sqrt{u}} \right\rfloor$.

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Within the cluster-array the bit is at position $x \bmod \sqrt{u}$.

For simplicity we assume that $u = 2^{2k}$ for some $k \geq 1$. Then we can compute the cluster-number for an entry x as $\mathrm{high}(x)$ (the upper half of the dual representation of x) and the position of x within its cluster as $\mathrm{low}(x)$ (the lower half of the dual representation).

Implementation 2: Summary Array

Algorithm 8 member(x)

1: **return** cluster[high(x)]. member(low(x));

Implementation 2: Summary Array

Algorithm 8 member(x)

return cluster[high(x)].member(low(x));

Algorithm 9 insert(x)

cluster[high(x)].insert(low(x));
summary.insert(high(x));

Implementation 2: Summary Array

Algorithm 8 member(x)

return cluster[high(x)]. member(low(x));

Algorithm 9 insert(x)

cluster[high(x)]. insert(low(x));
summary . insert(high(x));

- The running times are constant, because the corresponding array-functions have constant running times.

Implementation 2: Summary Array

Algorithm 10 delete(x)

```
cluster[high(x)].delete(low(x));
if cluster[high(x)].min() = null then
    summary.delete(high(x));
```

Implementation 2: Summary Array

Algorithm 10 delete(x)

```
1: cluster[high(x)].delete(low(x));
2: if cluster[high(x)].min() = null then
3:     summary.delete(high(x));
```

- The running time is dominated by the cost of a minimum computation on an array of size $\sqrt{u}$. Hence, $\mathcal{O}(\sqrt{u})$.

Implementation 2: Summary Array

Algorithm 11 max()

1: $\mathit{maxcluster} \leftarrow \text{summary.max}()$;
2: **if** $\mathit{maxcluster} = \text{null}$ **return** null;
3: $\mathit{offs} \leftarrow \text{cluster}[\mathit{maxcluster}].\text{max}()$
4: **return** $\mathit{maxcluster} \circ \mathit{offs}$;

Implementation 2: Summary Array

Algorithm 11 max()

```
maxcluster ← summary.max();
if maxcluster = null return null;
offs ← cluster[maxcluster].max()
return maxcluster ∘ offs;
```

Algorithm 12 min()

```
mincluster ← summary.min();
if mincluster = null return null;
offs ← cluster[mincluster].min();
return mincluster ∘ offs;
```

Implementation 2: Summary Array

Algorithm 11 max()

```
maxcluster ← summary.max();
if maxcluster = null return null;
offs ← cluster[maxcluster].max()
return maxcluster ∘ offs;
```

Algorithm 12 min()

```
mincluster ← summary.min();
if mincluster = null return null;
offs ← cluster[mincluster].min();
return mincluster ∘ offs;
```

The operator $\circ$ stands for the concatenation of two bitstrings.
This means if $x = 0111_2$ and $y = 0001_2$ then $x \circ y = 01110001_2$.

- Running time is roughly $2\sqrt{u} = \mathcal{O}(\sqrt{u})$ in the worst case.

Implementation 2: Summary Array

Algorithm 13 succ(x)

```
m ← cluster[high(x)].succ(low(x))
if m ≠ null then return high(x) ∘ m;
succcluster ← summary.succ(high(x));
if succcluster ≠ null then
        offs ← cluster[succcluster].min();
        return succcluster ∘ offs;
return null;
```

Implementation 2: Summary Array

Algorithm 13 succ(x)

```
m ← cluster[high(x)].succ(low(x))
if m ≠ null then return high(x) ∘ m;
succcluster ← summary.succ(high(x));
if succcluster ≠ null then
    offs ← cluster[succcluster].min();
    return succcluster ∘ offs;
return null;
```

- Running time is roughly $3\sqrt{u} = \mathcal{O}(\sqrt{u})$ in the worst case.

Implementation 2: Summary Array

Algorithm 14 pred(x)

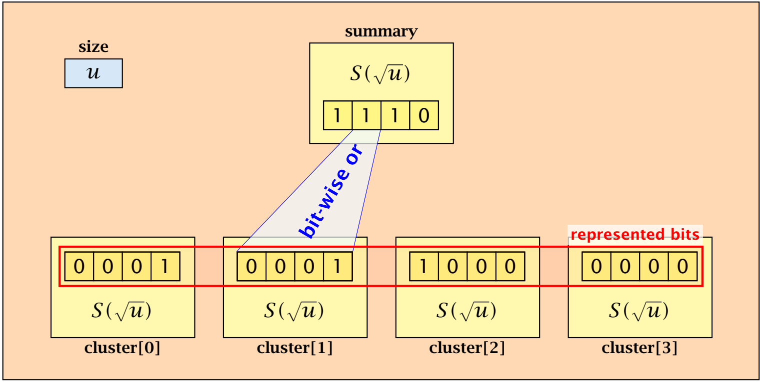
```
m ← cluster[high(x)].pred(low(x))
if m ≠ null then return high(x) ∘ m;
predcluster ← summary.pred(high(x));
if predcluster ≠ null then
    offs ← cluster[predcluster].max();
    return predcluster ∘ offs;
return null;
```

- Running time is roughly $3\sqrt{u} = \mathcal{O}(\sqrt{u})$ in the worst case.

Implementation 3: Recursion

Instead of using sub-arrays, we build a recursive data-structure.

$S(u)$ is a dynamic set data-structure representing u bits:



Implementation 3: Recursion

We assume that $u = 2^{2^k}$ for some k.

The data-structure $S(2)$ is defined as an array of 2-bits (end of the recursion).

Implementation 3: Recursion

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The code from Implementation 2 can be used unchanged. We only need to redo the analysis of the running time.

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Note that in the code we do not need to specifically address the non-recursive case. This is achieved by the fact that an $S(4)$ will contain $S(2)$'s as sub-datastructures, which are arrays. Hence, a call like $\text{cluster}[1].\text{min}()$ from within the data-structure $S(4)$ is not a recursive call as it will call the function $\text{array}.\text{min}()$.

Implementation 3: Recursion

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Note that in the code we do not need to specifically address the non-recursive case. This is achieved by the fact that an $S(4)$ will contain $S(2)$'s as sub-datastructures, which are arrays. Hence, a call like $\text{cluster}[1].\text{min}()$ from within the data-structure $S(4)$ is not a recursive call as it will call the function $\text{array}.\text{min}()$.

This means that the non-recursive case is been dealt with while initializing the data-structure.

Implementation 3: Recursion

Algorithm 15 member(x)

1: **return** cluster[high(x)]. member(low(x));

- $T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1$.

Implementation 3: Recursion

Algorithm 16 insert(x)

```
1: cluster[high(x)].insert(low(x));
2: summary.insert(high(x));
```

- $T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1$.

Implementation 3: Recursion

Algorithm 17 delete(x)

1: cluster[high(x)]. delete(low(x));
2: **if** cluster[high(x)]. min() = null **then**
3: summary . delete(high(x));

- $T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1$.

Implementation 3: Recursion

Algorithm 18 min()

```
mincluster ← summary.min();
if mincluster = null return null;
offs ← cluster[mincluster].min();
return mincluster ∘ offs;
```

- $T_{\min}(u) = 2T_{\min}(\sqrt{u}) + 1$.

Implementation 3: Recursion

Algorithm 19 succ(x)

1: $m \leftarrow \text{cluster}[\text{high}(\text{x})].\text{succ}(\text{low}(x))$
2: **if** $m \neq \text{null}$ **then return** $\text{high}(x) \circ m$;
3: $\mathit{succcluster} \leftarrow \text{summary}.\text{succ}(\text{high}(x))$;
4: **if** $\mathit{succcluster} \neq \text{null}$ **then**
5: $\quad \mathit{offs} \leftarrow \text{cluster}[\mathit{succcluster}].\text{min}()$;
6: $\quad$ **return** $\mathit{succcluster} \circ \mathit{offs}$;
7: **return** null;

- $T_{\text{succ}}(u) = 2T_{\text{succ}}(\sqrt{u}) + T_{\text{min}}(\sqrt{u}) + 1$.

Implementation 3: Recursion

$$T_{\mathrm{mem}}(u) = T_{\mathrm{mem}}(\sqrt{u}) + 1:$$

Implementation 3: Recursion

$\boldsymbol{T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1}$**:**

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^\ell)$.

Implementation 3: Recursion

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Implementation 3: Recursion

$\boldsymbol{T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1}$:

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^\ell)$. Then

$$\begin{aligned} X(\ell) = T_{\text{mem}}(2^\ell) = T_{\text{mem}}(u) &= T_{\text{mem}}(\sqrt{u}) + 1 \\ &= T_{\text{mem}}(2^{\frac{\ell}{2}}) + 1 \end{aligned}$$

Implementation 3: Recursion

$$T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1:$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^{\ell})$.Then

$$\begin{aligned} X(\ell) = T_{\text{mem}}(2^{\ell}) &= T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1 \\ &= T_{\text{mem}}(2^{\frac{\ell}{2}}) + 1 = X(\tfrac{\ell}{2}) + 1 \enspace . \end{aligned}$$

Implementation 3: Recursion

$\boldsymbol{T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1}$**:**

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^\ell)$.Then

$$X(\ell) = T_{\text{mem}}(2^\ell) = T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1$$
$$= T_{\text{mem}}(2^{\frac{\ell}{2}}) + 1 = X(\tfrac{\ell}{2}) + 1 \enspace .$$

Using Master theorem gives $X(\ell) = \mathcal{O}(\log \ell)$, and hence $T_{\text{mem}}(u) = \mathcal{O}(\log\log u)$.

Implementation 3: Recursion

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$$X(\ell) = T_{\text{ins}}(2^\ell) = T_{\text{ins}}(u)$$

Implementation 3: Recursion

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Implementation 3: Recursion

$$\boldsymbol{T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.}$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$. Then

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Using Master theorem gives $X(\ell) = \mathcal{O}(\ell)$, and hence $T_{\text{ins}}(u) = \mathcal{O}(\log u)$.

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The same holds for $T_{\max}(u)$ and $T_{\min}(u)$.

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$$T_{\mathrm{del}}(u) = 2T_{\mathrm{del}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1 \leq 2T_{\mathrm{del}}(\sqrt{u}) + c\log(u).$$

Set $\ell := \log u$ and $X(\ell) := T_{\mathrm{del}}(2^{\ell})$.

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$$X(\ell)$$

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$$X(\ell) = T_{\text{del}}(2^\ell)$$

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Set $\ell := \log u$ and $X(\ell) := T_{\text{del}}(2^\ell)$. Then

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$$\mathbf{T_{del}(u) = 2T_{del}(\sqrt{u}) + T_{min}(\sqrt{u}) + 1 \leq 2T_{del}(\sqrt{u}) + c\log(u).}$$

Set $\ell := \log u$ and $X(\ell) := T_{del}(2^\ell)$. Then

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Using Master theorem gives $X(\ell) = \Theta(\ell \log \ell)$, and hence $T_{del}(u) = \mathcal{O}(\log u \log\log u)$.

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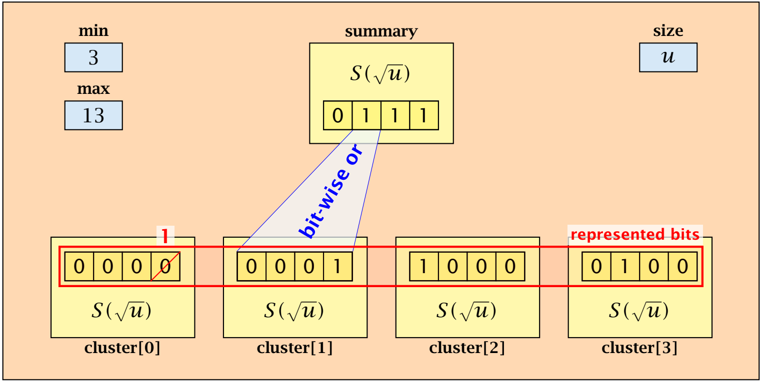
Using Master theorem gives $X(\ell) = \Theta(\ell \log \ell)$, and hence $T_{\text{del}}(u) = \mathcal{O}(\log u \log\log u)$.

The same holds for $T_{\text{pred}}(u)$ and $T_{\text{succ}}(u)$.

Implementation 4: van Emde Boas Trees



- The bit referenced by min is not set within sub-datastructures.
- The bit referenced by max is set within sub-datastructures (if $\max \neq \min$).

Implementation 4: van Emde Boas Trees

Advantages of having max/min pointers:

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- Recursive calls for min and max are constant time.

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- Recursive calls for $\min$ and $\max$ are constant time.
- $\min = \text{null}$ means that the data-structure is empty.
- $\min = \max \neq \text{null}$ means that the data-structure contains exactly one element.

Implementation 4: van Emde Boas Trees

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- Recursive calls for min and max are constant time.
- $\text{min} = \text{null}$ means that the data-structure is empty.
- $\text{min} = \text{max} \neq \text{null}$ means that the data-structure contains exactly one element.
- We can insert into an empty datastructure in constant time by only setting $\text{min} = \text{max} = x$.

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Advantages of having max/min pointers:

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- $\text{min} = \text{max} \neq \text{null}$ means that the data-structure contains exactly one element.
- We can insert into an empty datastructure in constant time by only setting $\text{min} = \text{max} = x$.
- We can delete from a data-structure that just contains one element in constant time by setting $\text{min} = \text{max} = \text{null}$.

Implementation 4: van Emde Boas Trees

Algorithm 20 max()

1: **return** max;

Algorithm 21 min()

1: **return** min;

- Constant time.

Implementation 4: van Emde Boas Trees

Algorithm 22 member(x)

```
if x = min then return 1; // TRUE
return cluster[high(x)].member(low(x));
```

- $T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1 \Longrightarrow T(u) = \mathcal{O}(\log\log u)$.

Implementation 4: van Emde Boas Trees

Algorithm 23 succ(x)

```
if min ≠ null ∧ x < min then return min;
maxincluster ← cluster[high(x)].max();
if maxincluster ≠ null ∧ low(x) < maxincluster then
    offs ← cluster[high(x)].succ(low(x));
    return high(x) ∘ offs;
else
    succcluster ← summary.succ(high(x));
    if succcluster = null then return null;
    offs ← cluster[succcluster].min();
    return succcluster ∘ offs;
```

- $T_{\text{succ}}(u) = T_{\text{succ}}(\sqrt{u}) + 1 \implies T_{\text{succ}}(u) = \mathcal{O}(\log\log u)$.

Implementation 4: van Emde Boas Trees

Algorithm 35 insert(x)

```
if min = null then
    min = x; max = x;
else
    if x < min then exchange x and min;
    if x > max then max = x;
    if cluster[high(x)].min = null; then
        summary.insert(high(x));
        cluster[high(x)].insert(low(x));
    else
        cluster[high(x)].insert(low(x));
```

- $T_{\text{ins}}(u) = T_{\text{ins}}(\sqrt{u}) + 1 \Longrightarrow T_{\text{ins}}(u) = \mathcal{O}(\log \log u)$.

Implementation 4: van Emde Boas Trees

Note that the recusive call in Line 8 takes constant time as the if-condition in Line 6 ensures that we are inserting in an empty sub-tree.

The only non-constant recursive calls are the call in Line 7 and in Line 10. These are mutually exclusive, i.e., only one of these calls will actually occur.

From this we get that $T_{\text{ins}}(u) = T_{\text{ins}}(\sqrt{u}) + 1$.

Implementation 4: van Emde Boas Trees

- **Assumes that x is contained in the structure.**

Algorithm 36 delete(x)

```
if min = max then
    min = max = null;
else
    if x = min then
        firstcluster ← summary.min();
        offs ← cluster[firstcluster].min();
        x ← firstcluster ∘ offs;
        min ← x;
    cluster[high(x)].delete(low(x));
```

continued...

Implementation 4: van Emde Boas Trees

- **Assumes that x is contained in the structure.**

Algorithm 36 delete(x)

```
if min = max then
    min = max = null;
else
    if x = min then                                  find new minimum
        firstcluster ← summary.min();
        offs ← cluster[firstcluster].min();
        x ← firstcluster ∘ offs;
        min ← x;
    cluster[high(x)].delete(low(x));
```

continued...

Implementation 4: van Emde Boas Trees

- **Assumes that x is contained in the structure.**

Algorithm 36 delete(x)

```
if min = max then
    min = max = null;
else
    if x = min then
        firstcluster ← summary.min();
        offs ← cluster[firstcluster].min();
        x ← firstcluster ∘ offs;
        min ← x;
    cluster[high(x)].delete(low(x));                delete
```

continued...

Implementation 4: van Emde Boas Trees

Algorithm 36 delete(x)

...continued

```
if cluster[high(x)].min() = null then
    summary.delete(high(x));
    if x = max then
        summax ← summary.max();
        if summax = null then max ← min;
        else
            offs ← cluster[summax].max();
            max ← summax ∘ offs
else
    if x = max then
        offs ← cluster[high(x)].max();
        max ← high(x) ∘ offs;
```

Implementation 4: van Emde Boas Trees

Algorithm 36 delete(x)

...continued — fix maximum

```
if cluster[high(x)].min() = null then
    summary.delete(high(x));
    if x = max then
        summax ← summary.max();
        if summax = null then max ← min;
        else
            offs ← cluster[summax].max();
            max ← summax ∘ offs
else
    if x = max then
        offs ← cluster[high(x)].max();
        max ← high(x) ∘ offs;
```

Implementation 4: van Emde Boas Trees

Note that only one of the possible recusive calls in Line 9 and Line 11 in the deletion-algorithm may take non-constant time.

To see this observe that the call in Line 11 only occurs if the cluster where x was deleted is now empty. But this means that the call in Line 9 deleted the last element in $\mathrm{cluster}[\mathrm{high}(x)]$. Such a call only takes constant time.

Hence, we get a recurrence of the form

$$T_{\mathrm{del}}(u) = T_{\mathrm{del}}(\sqrt{u}) + c \ .$$

This gives $T_{\mathrm{del}}(u) = \mathcal{O}(\log\log u)$.

7.6 van Emde Boas Trees

Space requirements:

- The space requirement fulfills the recurrence

$$S(u) = (\sqrt{u} + 1)S(\sqrt{u}) + \mathcal{O}(\sqrt{u}) \enspace .$$

- Note that we cannot solve this recurrence by the Master theorem as the branching factor is not constant.
- One can show by induction that the space requirement is $S(u) = \mathcal{O}(u)$. Exercise.

- Let the “real” recurrence relation be

$$S(k^2) = (k+1)S(k) + c_1 \cdot k;\ S(4) = c_2$$

- Replacing $S(k)$ by $R(k) := S(k)/c_2$ gives the recurrence

$$R(k^2) = (k+1)R(k) + ck;\ R(4) = 1$$

where $c = c_1/c_2 < 1$.

- Now, we show $R(k^2) \le k^2 - 2$ for $k^2 \ge 4$.
  - Obviously, this holds for $k^2 = 4$.
  - For $k^2 > 4$ we have

$$\begin{aligned} R(k^2) &= (1+k)R(k) + ck \\ &\le (1+k)(k-2) + k \le k^2 - 2 \end{aligned}$$

  - This shows that $R(k)$ and, hence, $S(k)$ grows linearly.

7.7 Hashing

Dictionary:

- $\boldsymbol{S}.\mathbf{insert}(\boldsymbol{x})$: Insert an element x.
- $\boldsymbol{S}.\mathbf{delete}(\boldsymbol{x})$: Delete the element pointed to by x.
- $\boldsymbol{S}.\mathbf{search}(\boldsymbol{k})$: Return a pointer to an element e with $\mathrm{key}[e] = k$ in S if it exists; otherwise return null.

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So far we have implemented the search for a key by carefully choosing split-elements.

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So far we have implemented the search for a key by carefully choosing split-elements.

Then the memory location of an object x with key k is determined by successively comparing k to split-elements.

Hashing tries to directly compute the memory location from the given key. The goal is to have constant search time.

7.7 Hashing

Definitions:

- Universe U of keys, e.g., $U \subseteq \mathbb{N}_0$. U very large.

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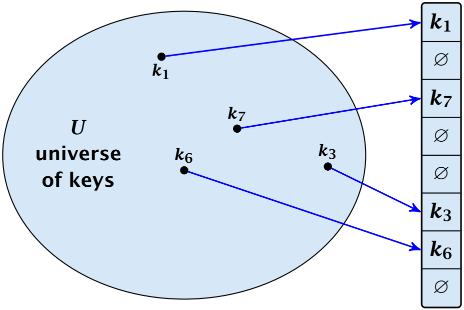
The hash-function h should fulfill:

- Fast to evaluate.
- Small storage requirement.
- Good distribution of elements over the whole table.

Direct Addressing

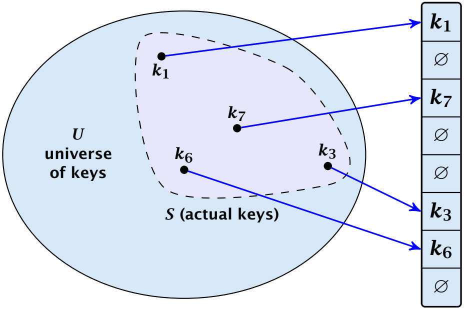
Ideally the hash function maps all keys to different memory locations.



This special case is known as Direct Addressing. It is usually very unrealistic as the universe of keys typically is quite large, and in particular larger than the available memory.

Perfect Hashing

Suppose that we know the set S of actual keys (no insert/no delete). Then we may want to design a simple hash-function that maps all these keys to different memory locations.



Such a hash function h is called a perfect hash function for set S.

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If we do not know the keys in advance, the best we can hope for is that the hash function distributes keys evenly across the table.

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Hence, there may be two elements k_1, k_2 from the set S that map to the same memory location (i.e., $h(k_1) = h(k_2)$). This is called a collision.

Collisions

Typically, collisions do not appear once the size of the set S of actual keys gets close to n, but already when $|S| \geq \omega(\sqrt{n})$.

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Lemma 20

The probability of having a collision when hashing m elements into a table of size n under uniform hashing is at least

$$1 - e^{-\frac{m(m-1)}{2n}} \approx 1 - e^{-\frac{m^2}{2n}} \enspace .$$

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Uniform hashing:

Choose a hash function uniformly at random from all functions $f : U \to [0, \dots, n-1]$.

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Proof.

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$$\le \prod_{j=0}^{m-1} e^{-j/n} = e^{-\sum_{j=0}^{m-1}\frac{j}{n}} = e^{-\frac{m(m-1)}{2n}} \quad .$$

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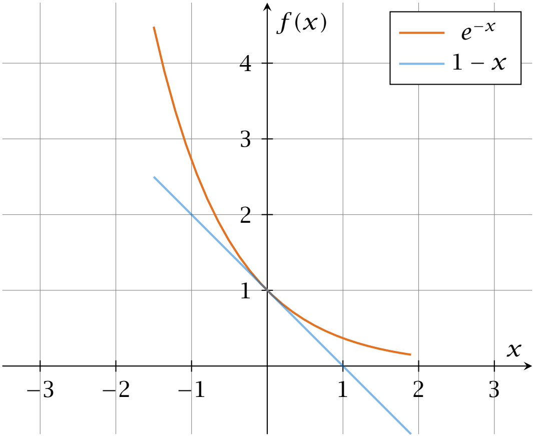
$$\Pr[A_{m,n}] = \prod_{\ell=1}^{m} \frac{n-\ell+1}{n} = \prod_{j=0}^{m-1} \Big(1 - \frac{j}{n}\Big)$$
$$\le \prod_{j=0}^{m-1} e^{-j/n} = e^{-\sum_{j=0}^{m-1} \frac{j}{n}} = e^{-\frac{m(m-1)}{2n}} \enspace .$$

Here the first equality follows since the ℓ-th element that is hashed has a probability of $\frac{n-\ell+1}{n}$ to not generate a collision under the condition that the previous elements did not induce collisions. □

Collisions



The inequality $1 - x \leq e^{-x}$ is derived by stopping the Taylor-expansion of e^{-x} after the second term.

Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

- **open addressing**, aka. closed hashing
- **hashing with chaining**, aka. closed addressing, open hashing.

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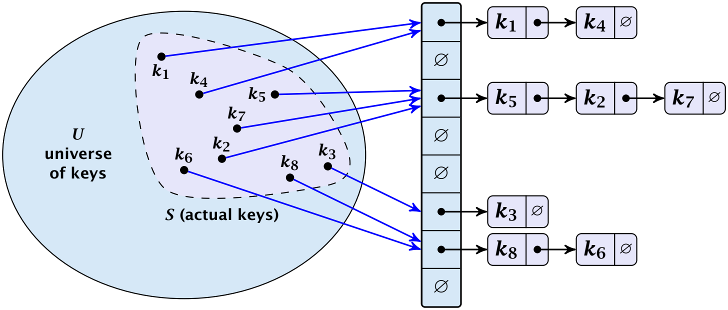
There are applications e.g. computer chess where you do not resolve collisions at all.

Hashing with Chaining

Arrange elements that map to the same position in a linear list.

- Access: compute $h(x)$ and search list for $\mathrm{key}[x]$.
- Insert: insert at the front of the list.



Hashing with Chaining

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We assume uniform hashing for the following analysis.

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$$A^{-} = 1 + \alpha \ .$$

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For a successful search observe that we do **not** choose a list at random, but we consider a random key k in the hash-table and ask for the search-time for k.

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The expected successful search cost is

$$\mathrm{E}\Big[\frac{1}{m}\sum_{i=1}^{m}\underbrace{\Big(1+\sum_{j=i+1}^{m} X_{ij}\Big)}_{\text{cost for key } k_i}\Big]$$

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$$=\frac{1}{m}\sum_{i=1}^{m}\left(1+\sum_{j=i+1}^{m}\frac{1}{n}\right)$$

$$=1+\frac{1}{mn}\sum_{i=1}^{m}(m-i)$$

Hashing with Chaining

$$\begin{aligned}
\mathrm{E}\left[\frac{1}{m}\sum_{i=1}^{m}\left(1+\sum_{j=i+1}^{m}X_{ij}\right)\right] &= \frac{1}{m}\sum_{i=1}^{m}\left(1+\sum_{j=i+1}^{m}\mathrm{E}\left[X_{ij}\right]\right)\\
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&= 1+\frac{1}{mn}\sum_{i=1}^{m}(m-i)\\
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&= 1+\frac{m-1}{2n} = 1+\frac{\alpha}{2}-\frac{\alpha}{2m}\ .
\end{aligned}$$

Hence, the expected cost for a successful search is $A^{+} \leq 1 + \frac{\alpha}{2}$.

Hashing with Chaining

Disadvantages:

- pointers increase memory requirements
- pointers may lead to bad cache efficiency

Advantages:

- no à priori limit on the number of elements
- deletion can be implemented efficiently
- by using balanced trees instead of linked list one can also obtain worst-case guarantees.

Open Addressing

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All objects are stored in the table itself.

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Define a function $h(k,j)$ that determines the table-position to be examined in the j-th step. The values $h(k,0)$,...,$h(k,n-1)$ must form a permutation of $0, \ldots, n-1$.

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Search(*k*): Try position $h(k, 0)$; if it is empty your search fails; otw. continue with $h(k, 1)$, $h(k, 2)$, ...

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Search(k): Try position $h(k,0)$; if it is empty your search fails; otw. continue with $h(k,1)$, $h(k,2)$, ...

Insert(x): Search until you find an empty slot; insert your element there. If your search reaches $h(k,n-1)$, and this slot is non-empty then your table is full.

Open Addressing

Choices for $h(k, j)$:

- Linear probing:
  $h(k, i) = h(k) + i \mod n$
  (sometimes: $h(k, i) = h(k) + ci \mod n$).

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- Double hashing:
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For quadratic probing and double hashing one has to ensure that the search covers all positions in the table (i.e., for double hashing $h_2(k)$ must be relatively prime to n (teilerfremd); for quadratic probing c_1 and c_2 have to be chosen carefully).

Linear Probing

- Advantage: Cache-efficiency. The new probe position is very likely to be in the cache.

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Lemma 21

Let L be the method of linear probing for resolving collisions:

$$L^+ \approx \frac{1}{2}\left(1 + \frac{1}{1-\alpha}\right)$$

$$L^- \approx \frac{1}{2}\left(1 + \frac{1}{(1-\alpha)^2}\right)$$

Quadratic Probing

- Not as cache-efficient as Linear Probing.
- Secondary clustering: caused by the fact that all keys mapped to the same position have the same probe sequence.

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Lemma 22

Let Q be the method of quadratic probing for resolving collisions:

$$Q^+ \approx 1 + \ln\left(\frac{1}{1-\alpha}\right) - \frac{\alpha}{2}$$

$$Q^- \approx \frac{1}{1-\alpha} + \ln\left(\frac{1}{1-\alpha}\right) - \alpha$$

Double Hashing

- Any probe into the hash-table usually creates a cache-miss.

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Lemma 23
Let D be the method of double hashing for resolving collisions:

$$D^+ \approx \frac{1}{\alpha}\ln\left(\frac{1}{1-\alpha}\right)$$

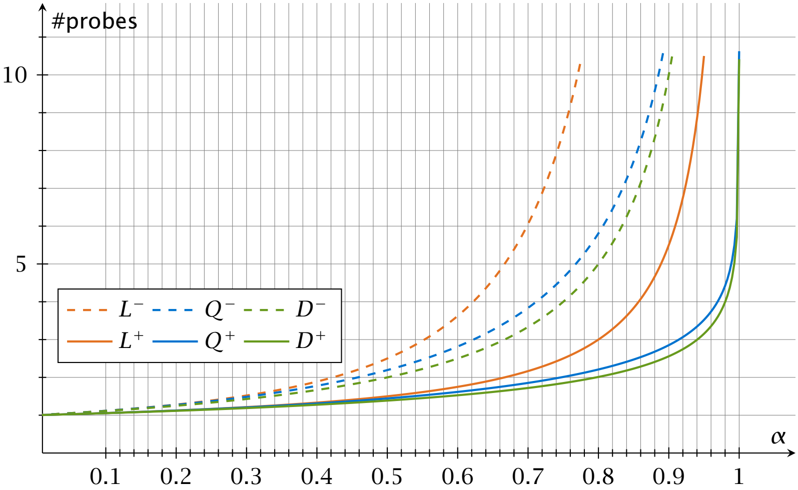
$$D^- \approx \frac{1}{1-\alpha}$$

Open Addressing

Some values:

α	Linear Probing		Quadratic Probing		Double Hashing	
	L^+	L^-	Q^+	Q^-	D^+	D^-
0.5	1.5	2.5	1.44	2.19	1.39	2
0.9	5.5	50.5	2.85	11.40	2.55	10
0.95	10.5	200.5	3.52	22.05	3.15	20

Open Addressing



Analysis of Idealized Open Address Hashing

We analyze the time for a search in a very idealized Open Addressing scheme.

- The probe sequence $h(k,0), h(k,1), h(k,2), \ldots$ is equally likely to be any permutation of $\langle 0, 1, \ldots, n-1 \rangle$.

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Let X denote a random variable describing the number of probes in an unsuccessful search.

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$$\Pr[A_1 \cap A_2 \cap \cdots \cap A_{i-1}]$$

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$$\Pr[X \geq i]$$

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$$\Pr[X \geq i] = \frac{m}{n} \cdot \frac{m-1}{n-1} \cdot \frac{m-2}{n-2} \cdot \ldots \cdot \frac{m-i+2}{n-i+2}$$

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Analysis of Idealized Open Address Hashing

$E[X]$

Analysis of Idealized Open Address Hashing

$$\mathrm{E}[X] = \sum_{i=1}^{\infty} \Pr[X \geq i]$$

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Analysis of Idealized Open Address Hashing

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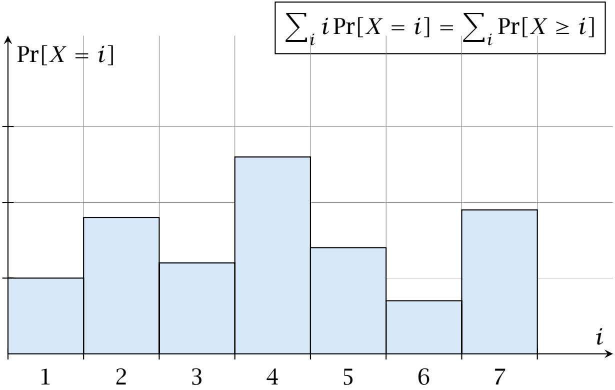
Analysis of Idealized Open Address Hashing

$$\mathrm{E}[X] = \sum_{i=1}^{\infty} \Pr[X \geq i] \leq \sum_{i=1}^{\infty} \alpha^{i-1} = \sum_{i=0}^{\infty} \alpha^{i} = \frac{1}{1-\alpha} \ .$$

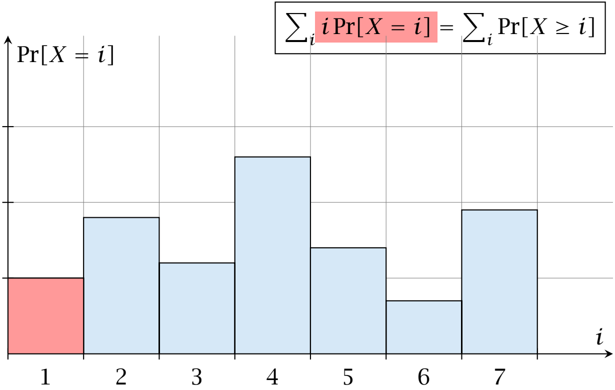
$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \dots$$

Analysis of Idealized Open Address Hashing



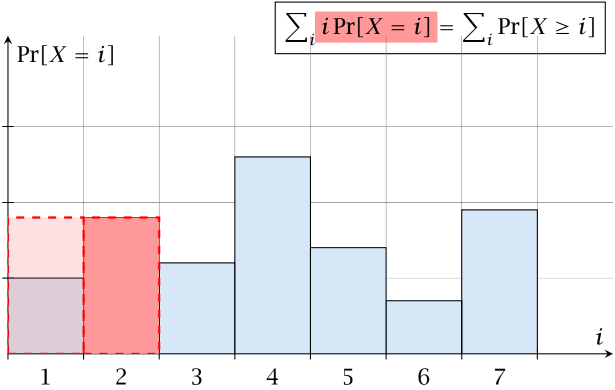
Analysis of Idealized Open Address Hashing

$i = 1$



Analysis of Idealized Open Address Hashing

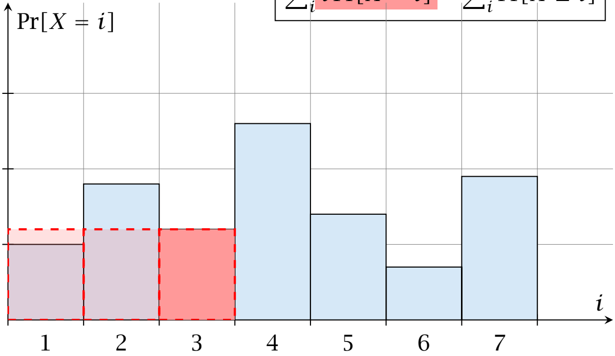
$i = 2$



Analysis of Idealized Open Address Hashing

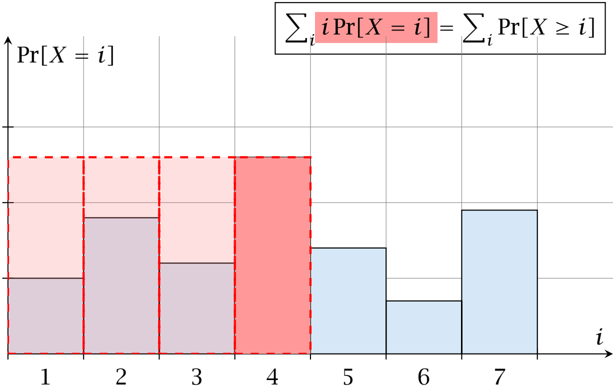
$i = 3$

$\sum_i i \Pr[X = i] = \sum_i \Pr[X \geq i]$



Analysis of Idealized Open Address Hashing

$i = 4$



Analysis of Idealized Open Address Hashing

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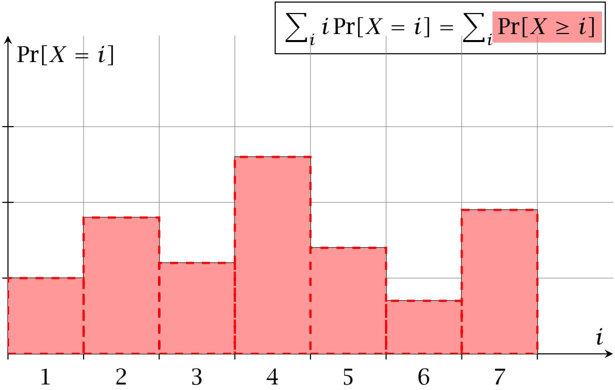

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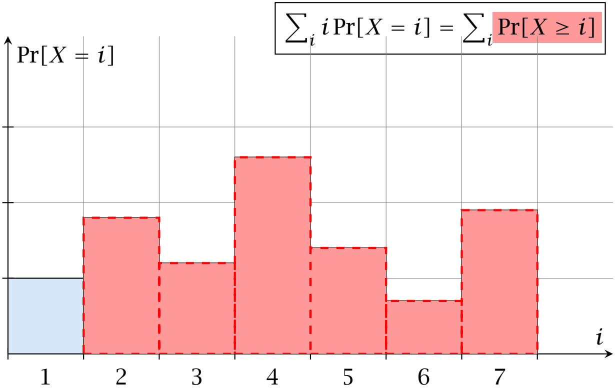

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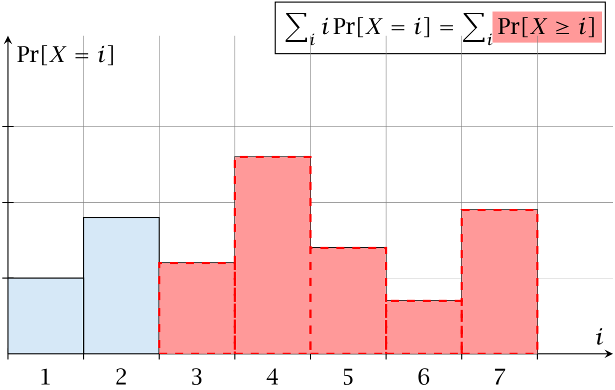

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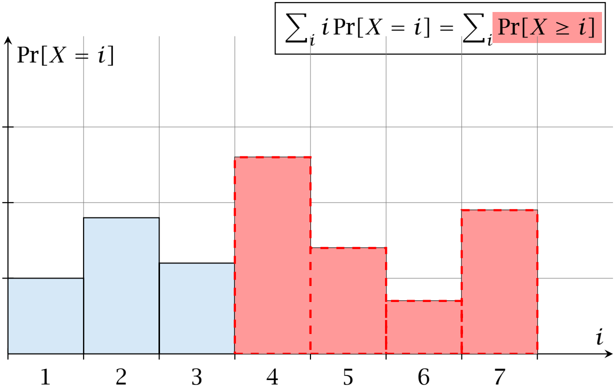

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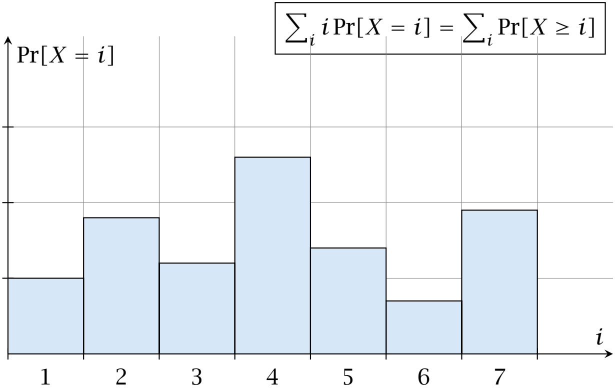

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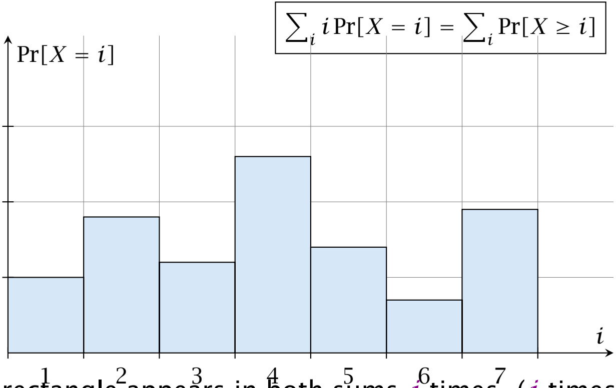

The j-th rectangle appears in both sums j times. (j times in the first due to multiplication with j; and j times in the second for summands $i = 1, 2, \ldots, j$)

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$$\frac{1}{m}\sum_{i=0}^{m-1}\frac{n}{n-i}$$

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$$\frac{1}{m}\sum_{i=0}^{m-1}\frac{n}{n-i} = \frac{n}{m}\sum_{i=0}^{m-1}\frac{1}{n-i}$$

Analysis of Idealized Open Address Hashing

The number of probes in a successful search for k is equal to the number of probes made in an unsuccessful search for k at the time that k is inserted.

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$$\frac{1}{m}\sum_{i=0}^{m-1}\frac{n}{n-i} = \frac{n}{m}\sum_{i=0}^{m-1}\frac{1}{n-i} = \frac{1}{\alpha}\sum_{k=n-m+1}^{n}\frac{1}{k}$$

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$$\leq \frac{1}{\alpha}\int_{n-m}^{n}\frac{1}{x}\mathrm{d}x$$

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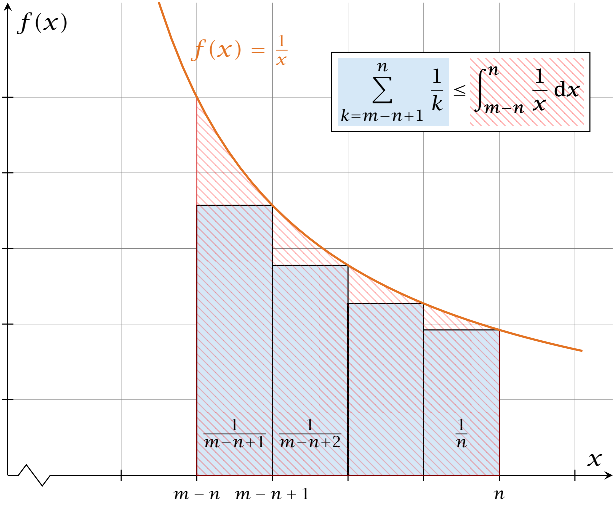
$$\leq \frac{1}{\alpha}\int_{n-m}^{n}\frac{1}{x}\mathrm{d}x = \frac{1}{\alpha}\ln\frac{n}{n-m} = \frac{1}{\alpha}\ln\frac{1}{1-\alpha} \; .$$

Analysis of Idealized Open Address Hashing



Deletions in Hashtables

How do we delete in a hash-table?

- For hashing with chaining this is not a problem. Simply search for the key, and delete the item in the corresponding list.

Deletions in Hashtables

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- For hashing with chaining this is not a problem. Simply search for the key, and delete the item in the corresponding list.
- For open addressing this is difficult.

Deletions in Hashtables

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  - During an insertion if a deleted-marker is encountered an element can be inserted there.

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  - During a search a deleted-marker must not be used to terminate the probe sequence.

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- The table could fill up with deleted-markers leading to bad performance.

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- One can delete an element by replacing it with a deleted-marker.
  - During an insertion if a deleted-marker is encountered an element can be inserted there.
  - During a search a deleted-marker must not be used to terminate the probe sequence.
- The table could fill up with deleted-markers leading to bad performance.
- If a table contains many deleted-markers (linear fraction of the keys) one can rehash the whole table and amortize the cost for this rehash against the cost for the deletions.

Deletions for Linear Probing

- For Linear Probing one can delete elements without using deletion-markers.

Deletions for Linear Probing

- For Linear Probing one can delete elements without using deletion-markers.
- Upon a deletion elements that are further down in the probe-sequence may be moved to guarantee that they are still found during a search.

Deletions for Linear Probing

Algorithm 37 delete(p)

```
1: T[p] ← null
2: p ← succ(p)
3: while T[p] ≠ null do
4:     y ← T[p]
5:     T[p] ← null
6:     p ← succ(p)
7:     insert(y)
```

p is the index into the table-cell that contains the object to be deleted.

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Pointers into the hash-table become invalid.

Universal Hashing

Universal Hashing

Regardless, of the choice of hash-function there is always an input (a set of keys) that has a very poor worst-case behaviour.

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Therefore, so far we assumed that the hash-function is random so that regardless of the input the average case behaviour is good.

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However, the assumption of uniform hashing that h is chosen randomly from all functions $f : U \to [0, \dots, n-1]$ is clearly unrealistic as there are $n^{|U|}$ such functions. Even writing down such a function would take $|U| \log n$ bits.

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Universal hashing tries to define a set $\mathcal{H}$ of functions that is much smaller but still leads to good average case behaviour when selecting a hash-function uniformly at random from $\mathcal{H}$.

Universal Hashing

Definition 24
A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called universal if for all $u_1, u_2 \in U$ with $u_1 \neq u_2$

$$\Pr[h(u_1) = h(u_2)] \leq \frac{1}{n} \ ,$$

where the probability is w. r. t. the choice of a random hash-function from set $\mathcal{H}$.

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where the probability is w. r. t. the choice of a random hash-function from set $\mathcal{H}$.

Note that this means that the probability of a collision between two arbitrary elements is at most $\frac{1}{n}$.

Universal Hashing

Definition 25

A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0,\ldots,n-1\}$ is called 2-independent (pairwise independent) if the following two conditions hold

- For any key $u \in U$, and $t \in \{0,\ldots,n-1\}$ $\Pr[h(u)=t] = \frac{1}{n}$, i.e., a key is distributed uniformly within the hash-table.
- For all $u_1, u_2 \in U$ with $u_1 \neq u_2$, and for any two hash-positions t_1, t_2:

$$\Pr[h(u_1)=t_1 \wedge h(u_2)=t_2] \le \frac{1}{n^2} .$$

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This requirement clearly implies a universal hash-function.

Universal Hashing

Definition 26

A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0,\ldots,n-1\}$ is called k-independent if for any choice of $\ell \le k$ distinct keys $u_1,\ldots,u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions $t_1,\ldots,t_\ell$:

$$\Pr[h(u_1) = t_1 \wedge \cdots \wedge h(u_\ell) = t_\ell] \le \frac{1}{n^\ell} \ ,$$

where the probability is w. r. t. the choice of a random hash-function from set $\mathcal{H}$.

Universal Hashing

Definition 27

A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0,\dots,n-1\}$ is called (μ,k)-independent if for any choice of $\ell \le k$ distinct keys $u_1,\dots,u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions $t_1,\dots,t_\ell$:

$$\Pr[h(u_1) = t_1 \wedge \dots \wedge h(u_\ell) = t_\ell] \le \frac{\mu}{n^\ell} \ ,$$

where the probability is w. r. t. the choice of a random hash-function from set $\mathcal{H}$.

Universal Hashing

Universal Hashing

Let $U := \{0, \ldots, p-1\}$ for a prime p. Let $\mathbb{Z}_p := \{0, \ldots, p-1\}$, and let $\mathbb{Z}_p^* := \{1, \ldots, p-1\}$ denote the set of invertible elements in $\mathbb{Z}_p$.

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Define

$$h_{a,b}(x) := (ax + b \bmod p) \bmod n$$

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Define

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Lemma 28

The class

$$\mathcal{H} = \{h_{a,b} \mid a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$$

is a universal class of hash-functions from U to $\{0, \dots, n-1\}$.

Universal Hashing

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Proof.

Let $x, y \in U$ be two distinct keys. We have to show that the probability of a collision is only $1/n$.

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  If $x \neq y$ then $(x - y) \not\equiv 0 \pmod{p}$.

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  Multiplying with $a \not\equiv 0 \pmod{p}$ gives

$$a(x - y) \not\equiv 0 \pmod{p}$$

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  where we use that $\mathbb{Z}_p$ is a field (Körper) and, hence, has no zero divisors (nullteilerfrei).

Universal Hashing

- The hash-function does not generate collisions before the $\pmod{n}$-operation. Furthermore, every choice (a, b) is mapped to a different pair (t_x, t_y) with $t_x := ax + b$ and $t_y := ay + b$.

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$$
\begin{aligned}
t_x &\equiv ax + b \pmod{p} \\
t_y &\equiv ay + b \pmod{p}
\end{aligned}
$$

$$
\begin{aligned}
t_x - t_y &\equiv a(x - y) \pmod{p} \\
t_y &\equiv ay + b \pmod{p}
\end{aligned}
$$

Universal Hashing

- The hash-function does not generate collisions before the $(\bmod\ n)$-operation. Furthermore, every choice (a, b) is mapped to a different pair (t_x, t_y) with $t_x := ax + b$ and $t_y := ay + b$.

  This holds because we can compute a and b when given t_x and t_y:

$$t_x \equiv ax + b \pmod{p}$$
$$t_y \equiv ay + b \pmod{p}$$

$$t_x - t_y \equiv a(x - y) \pmod{p}$$
$$t_y \equiv ay + b \pmod{p}$$

$$a \equiv (t_x - t_y)(x - y)^{-1} \pmod{p}$$
$$b \equiv t_y - ay \pmod{p}$$

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There is a one-to-one correspondence between hash-functions (pairs (a,b), $a \neq 0$) and pairs (t_x,t_y), $t_x \neq t_y$.

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Fix a value t_x. There are $p-1$ possible values for choosing t_y.

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What happens when we do the $\bmod n$ operation?

Fix a value t_x. There are $p - 1$ possible values for choosing t_y.

From the range $0, \ldots, p - 1$ the values $t_x, t_x + n, t_x + 2n, \ldots$ map to t_x after the modulo-operation. These are at most $\lceil p/n \rceil$ values.

Universal Hashing

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As $t_y \neq t_x$ there are

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possibilities for choosing t_y such that the final hash-value creates a collision.

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possibilities for choosing t_y such that the final hash-value creates a collision.

This happens with probability at most $\frac{1}{n}$.

Universal Hashing

Universal Hashing

It is also possible to show that $\mathcal{H}$ is an (almost) pairwise independent class of hash-functions.

$$\Pr_{t_x \neq t_y \in \mathbb{Z}_p^2}\left[\begin{array}{c} t_x \bmod n = h_1 \\ \wedge \\ t_y \bmod n = h_2 \end{array}\right]$$

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$$\frac{\left\lfloor\frac{p}{n}\right\rfloor^2}{p(p-1)} \le \Pr_{t_x \neq t_y \in \mathbb{Z}_p^2}\left[\begin{array}{c} t_x \bmod n = h_1 \\ \wedge \\ t_y \bmod n = h_2 \end{array}\right] \le \frac{\left\lceil\frac{p}{n}\right\rceil^2}{p(p-1)}$$

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Note that the middle is the probability that $h(x) = h_1$ and $h(y) = h_2$. The total number of choices for (t_x, t_y) is $p(p-1)$. The number of choices for t_x (t_y) such that $t_x \bmod n = h_1$ ($t_y \bmod n = h_2$) lies between $\lfloor \frac{p}{n} \rfloor$ and $\lceil \frac{p}{n} \rceil$.

Universal Hashing

Definition 29

Let $d \in \mathbb{N}$; $q \geq (d+1)n$ be a prime; and let $\bar{a} \in \{0, \ldots, q-1\}^{d+1}$. Define for $x \in \{0, \ldots, q-1\}$

$$h_{\bar{a}}(x) := \Big(\sum_{i=0}^{d} a_i x^i \bmod q \Big) \bmod n \ .$$

Let $\mathcal{H}_n^d := \{h_{\bar{a}} \mid \bar{a} \in \{0, \ldots, q-1\}^{d+1}\}$. The class $\mathcal{H}_n^d$ is $(e, d+1)$-independent.

Note that in the previous case we had $d = 1$ and chose $a_d \neq 0$.

Universal Hashing

Universal Hashing

For the coefficients $\bar{a} \in \{0, \dots, q-1\}^{d+1}$ let $f_{\bar{a}}$ denote the polynomial

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$$f_{\bar{a}}(x) = \Big(\sum_{i=0}^{d} a_i x^i\Big) \bmod q$$

The polynomial is defined by $d+1$ distinct points.

Universal Hashing

Universal Hashing

Fix $\ell \leq d+1$; let $x_1, \ldots, x_\ell \in \{0, \ldots, q-1\}$ be keys, and let $t_1, \ldots, t_\ell$ denote the corresponding hash-function values.

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Let $A^\ell = \{h_{\bar{a}} \in \mathcal{H} \mid h_{\bar{a}}(x_i) = t_i \text{ for all } i \in \{1,\dots,\ell\}\}$

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Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

$$f_{\bar{a}}(x_i) \in \underbrace{\{t_i + \alpha \cdot n \mid \alpha \in \{0, \dots, \lceil \tfrac{q}{n} \rceil - 1\}\}}_{=: B_i}$$

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In order to obtain the cardinality of A^ℓ we choose our polynomial by fixing $d+1$ points.

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In order to obtain the cardinality of A^ℓ we choose our polynomial by fixing $d+1$ points.

We first fix the values for inputs $x_1,\dots,x_\ell$.
We have

$$|B_1| \cdot \ldots \cdot |B_\ell|$$

possibilities to do this (so that $h_{\bar{a}}(x_i) = t_i$).

Universal Hashing

Now, we choose $d - \ell + 1$ other inputs and choose their value arbitrarily. We have $q^{d-\ell+1}$ possibilities to do this.

Universal Hashing

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Therefore we have

$$|B_1| \cdot \ldots \cdot |B_\ell| \cdot q^{d-\ell+1} \leq \lceil \frac{q}{n} \rceil^\ell \cdot q^{d-\ell+1}$$

possibilities to choose $\bar{a}$ such that $h_{\bar{a}} \in A_\ell$.

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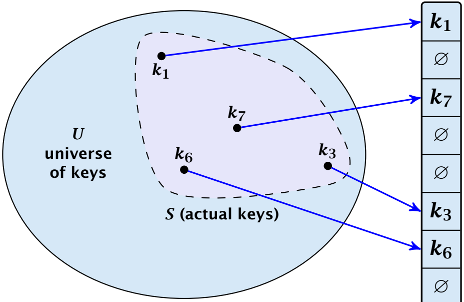
This shows that the $\mathcal{H}$ is $(e, d+1)$-universal.

The last step followed from $q \ge (d+1)n$, and $\ell \le d+1$.

Perfect Hashing

Suppose that we **know** the set S of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



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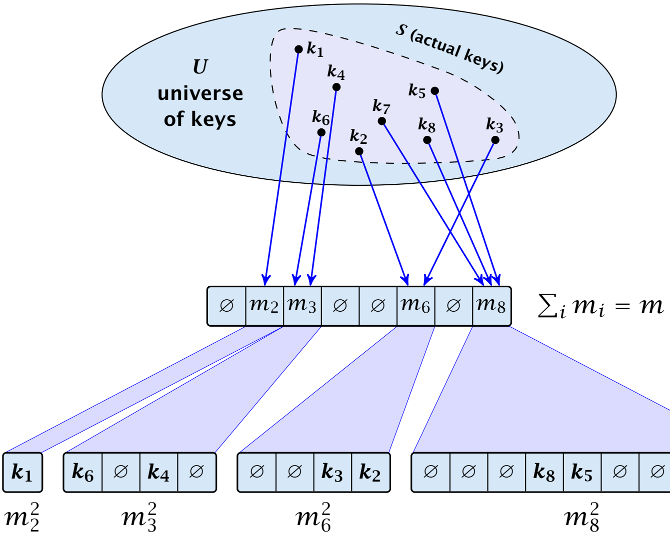
Let m_j denote the number of items that are hashed to the j-th bucket. For each bucket we choose a second hash-function that maps the elements of the bucket into a table of size m_j^2. The second function can be chosen such that all elements are mapped to different locations.

Perfect Hashing



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$$
= 2\binom{m}{2}\frac{1}{m} + m = 2m - 1 \ .
$$

Perfect Hashing

We need only $\mathcal{O}(m)$ time to construct a hash-function h with $\sum_j m_j^2 = \mathcal{O}(4m)$, because with probability at least $1/2$ a random function from a universal family will have this property.

Then we construct a hash-table h_j for every bucket. This takes expected time $\mathcal{O}(m_j)$ for every bucket. A random function h_j is collision-free with probability at least $1/2$. We need $\mathcal{O}(m_j)$ to test this.

We only need that the hash-functions are chosen from a universal family!!!

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Goal:
Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

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- A search clearly takes constant time if the above constraint is met.

Cuckoo Hashing

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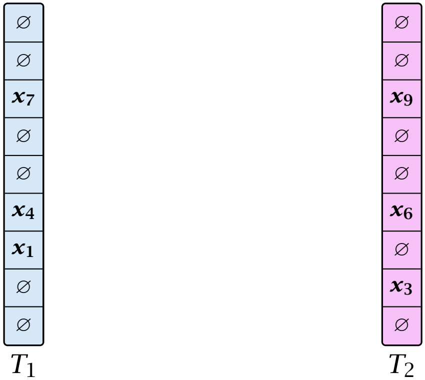

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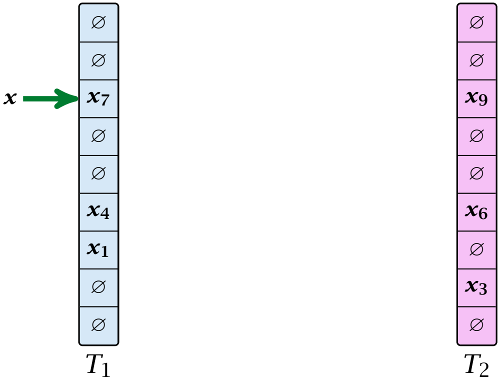

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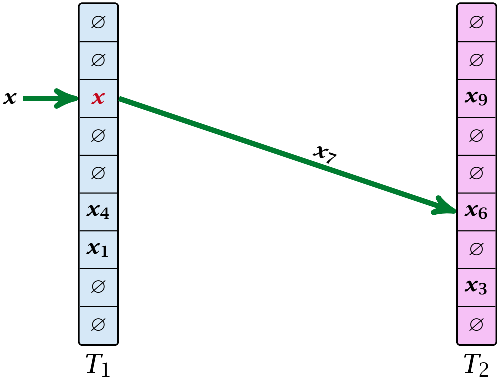

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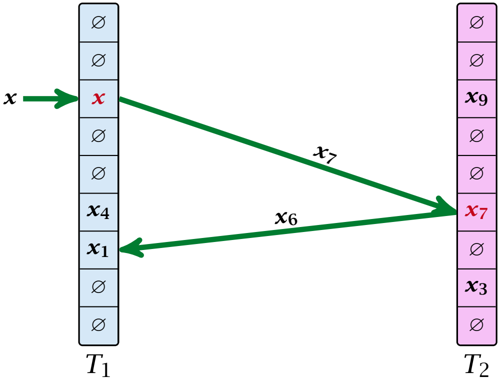

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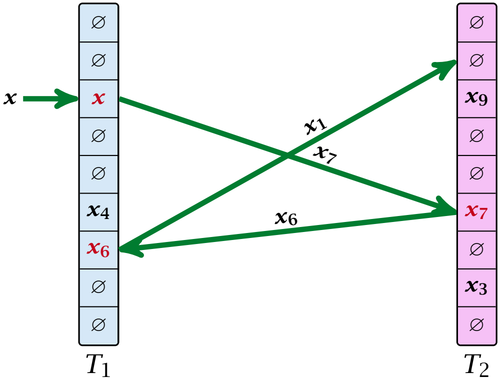

Cuckoo Hashing

Algorithm 38 Cuckoo-Insert(x)

if $T_1[h_1(x)] = x \vee T_2[h_2(x)] = x$ **then return**
$\text{steps} \leftarrow 1$
while $\text{steps} \le \text{maxsteps}$ **do**
    exchange x and $T_1[h_1(x)]$
    if $x = \text{null}$ **then return**
    exchange x and $T_2[h_2(x)]$
    if $x = \text{null}$ **then return**
    $\text{steps} \leftarrow \text{steps} + 1$
rehash() // change hash-functions; rehash everything
Cuckoo-Insert(x)

Cuckoo Hashing

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- We call a sequence of iterations through the while-loop without the termination condition becoming true a phase of the algorithm.
- We say a phase is successful if it is not terminated by the $\mathrm{maxstep}$-condition, but the while loop is left because $x = \mathrm{null}$.

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We first analyze the probability that we end-up in an infinite loop (that is then terminated after $\mathrm{maxsteps}$ steps).

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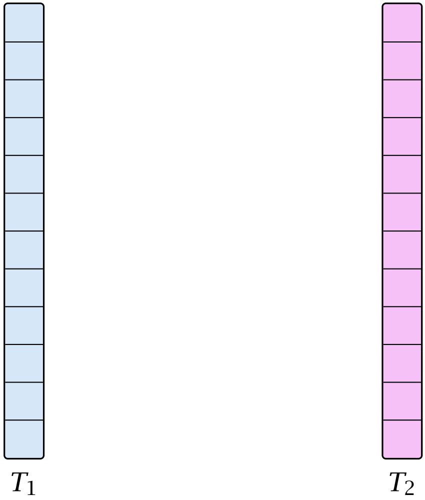
Formally what is the probability to enter an infinite loop that touches s different keys?

Cuckoo Hashing: Insert



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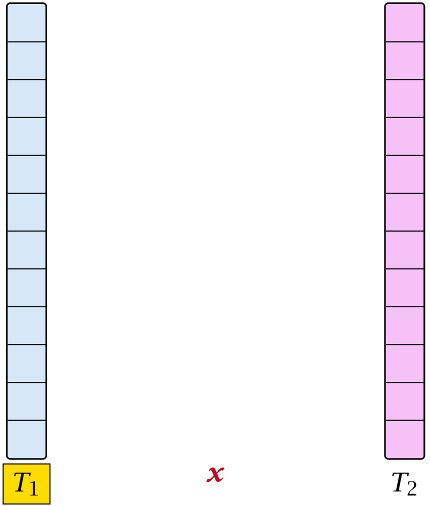

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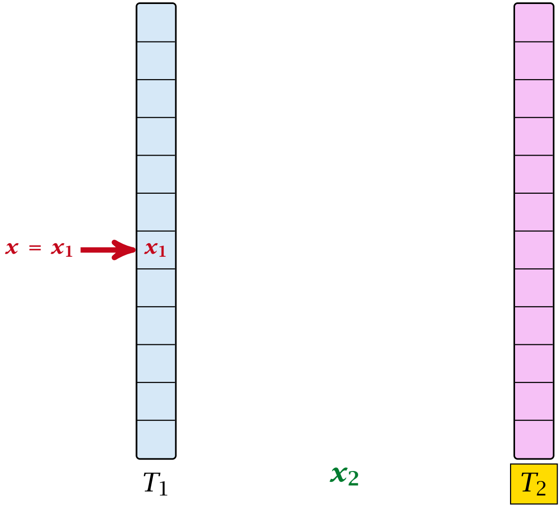

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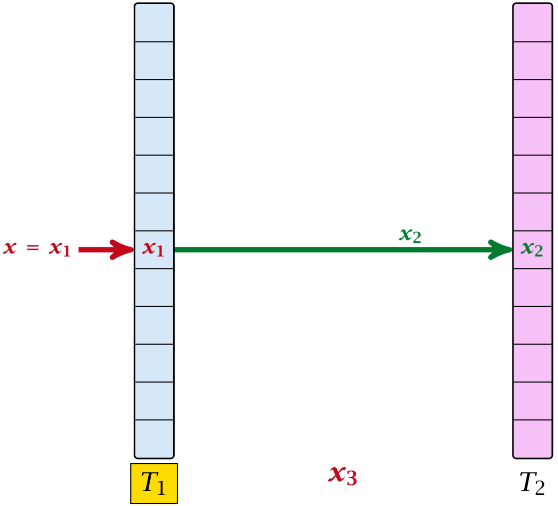

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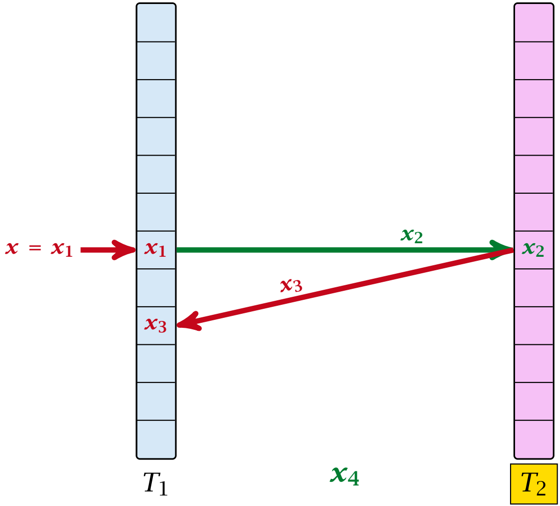

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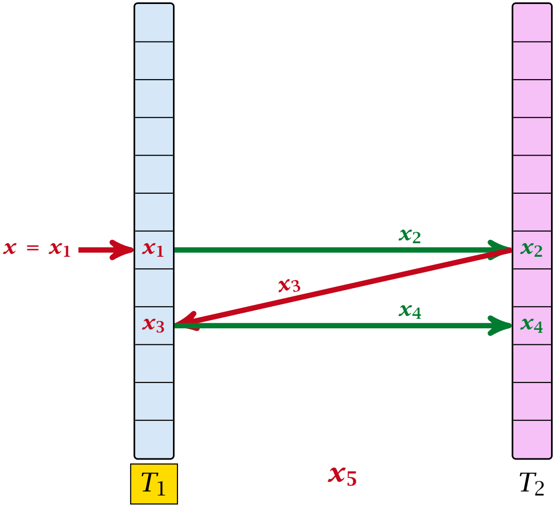

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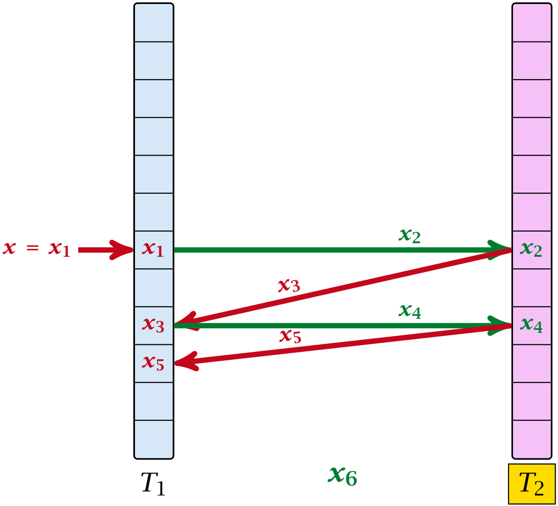

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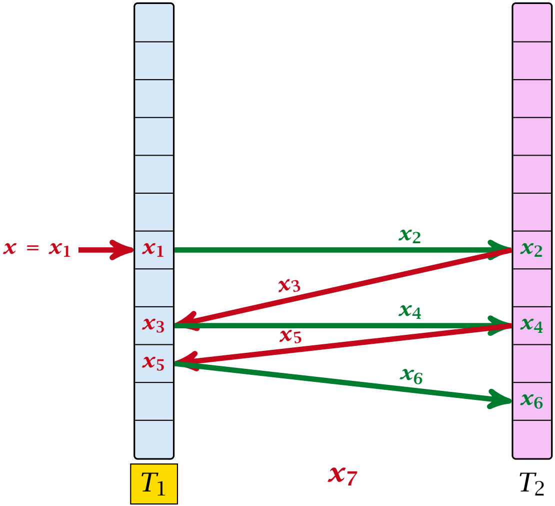

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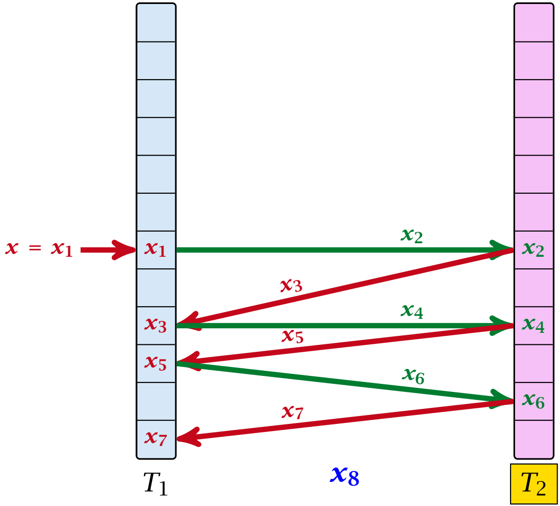

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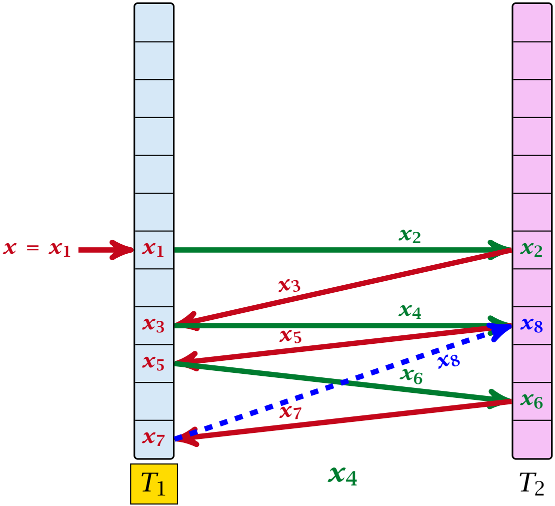

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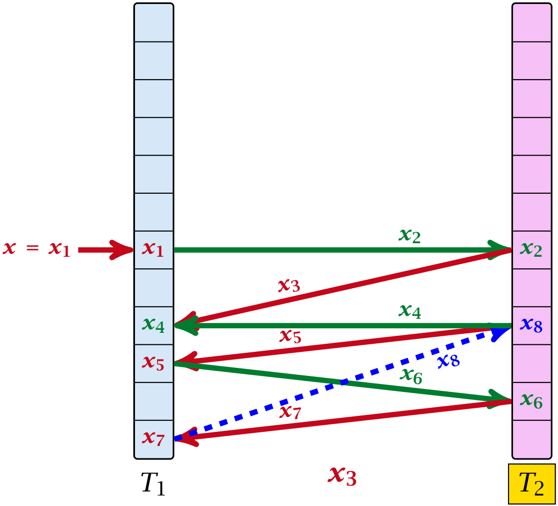

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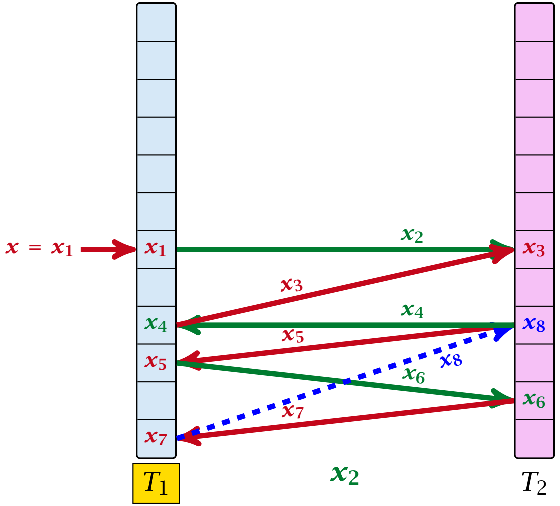

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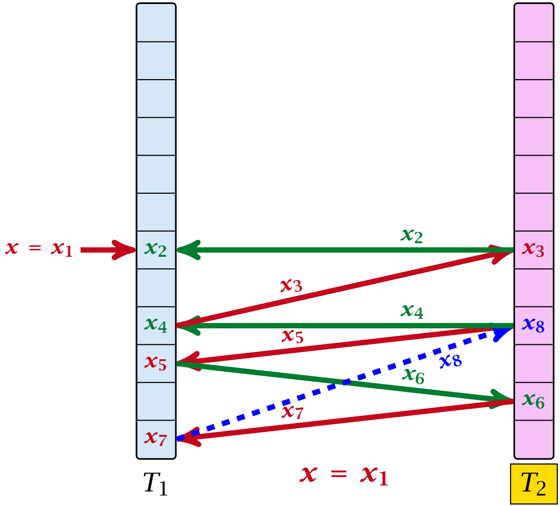

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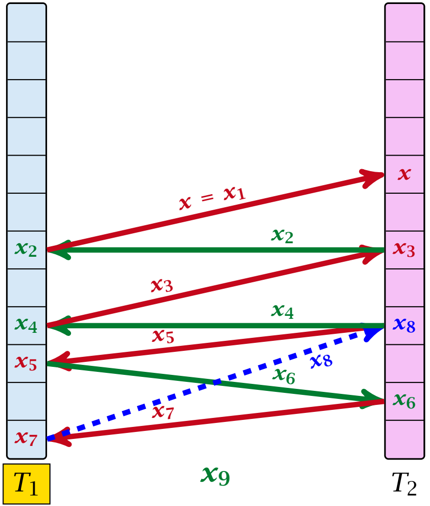

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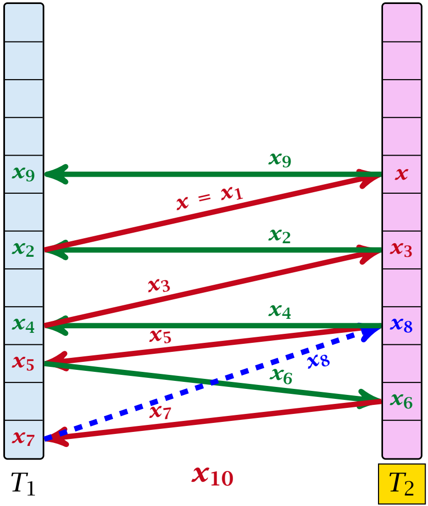

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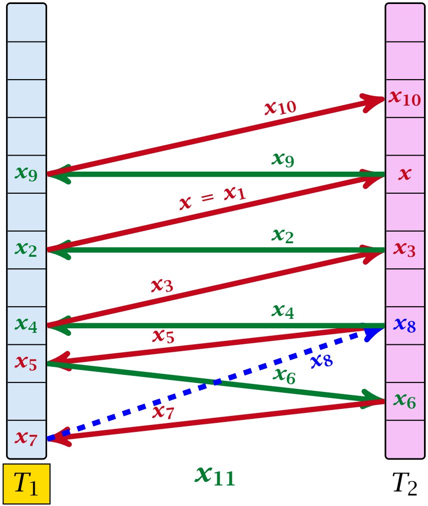

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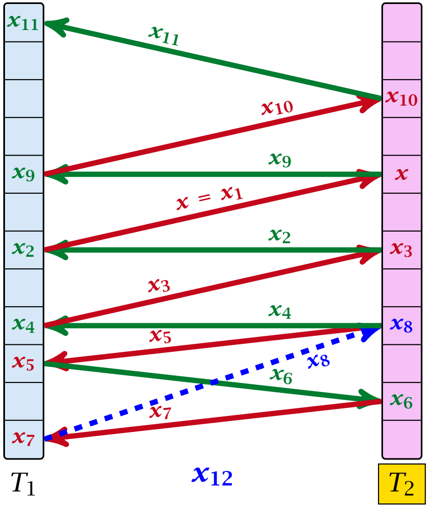

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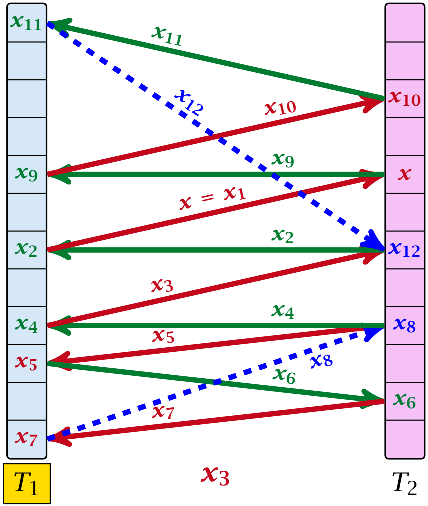

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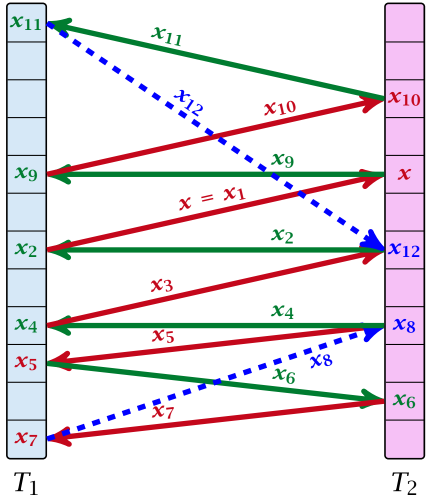

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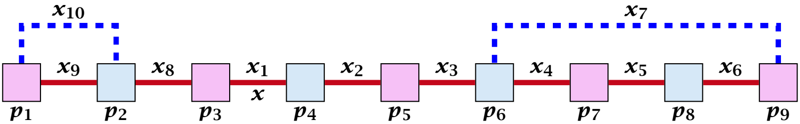
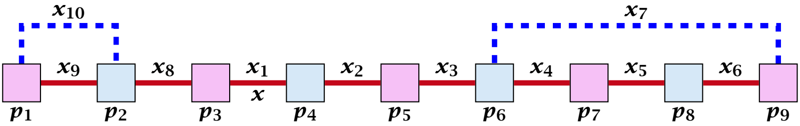


A cycle-structure of size s is defined by

Cuckoo Hashing



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- $s - 1$ different cells (alternating btw. cells from T_1 and T_2).

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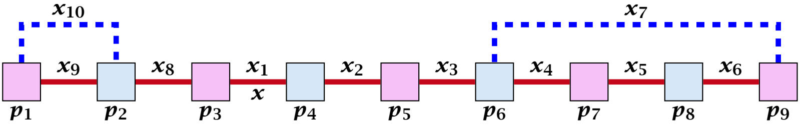

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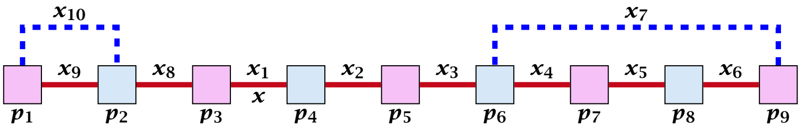

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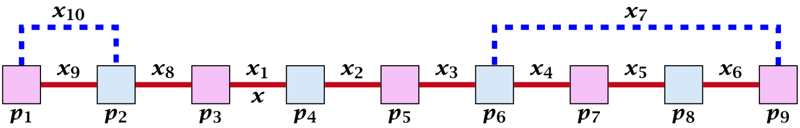

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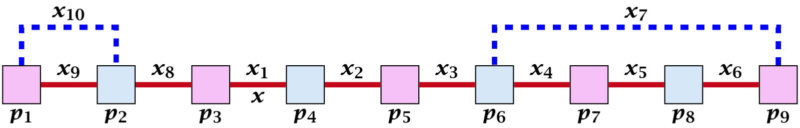

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- The leftmost cell is “linked forward” to some cell on the right.
- The rightmost cell is “linked backward” to a cell on the left.
- One link represents key x; this is where the counting starts.

Cuckoo Hashing

A cycle-structure is active if for every key x_ℓ (linking a cell p_i from T_1 and a cell p_j from T_2) we have

$$h_1(x_\ell) = p_i \qquad \text{and} \qquad h_2(x_\ell) = p_j$$

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Observation:
If during a phase the insert-procedure runs into a cycle there must exist an active cycle structure of size $s \geq 3$.

Cuckoo Hashing

What is the probability that all keys in a cycle-structure of size s correctly map into their T_1-cell?

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This probability is at most $\frac{\mu}{n^s}$ since h_1 is a (μ, s)-independent hash-function.

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This probability is at most $\frac{\mu}{n^s}$ since h_2 is a (μ, s)-independent hash-function.

These events are independent.

Cuckoo Hashing

The probability that a given cycle-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

Cuckoo Hashing

The probability that a given cycle-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

What is the probability that there exists an active cycle structure of size s?

Cuckoo Hashing

The number of cycle-structures of size s is at most

$$s^3 \cdot n^{s-1} \cdot m^{s-1} \ .$$

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- There are n^{s-1} possibilities to choose the cells.

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} = \frac{\mu^2}{nm} \sum_{s=3}^{\infty} s^3 \left(\frac{m}{n}\right)^s$$

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Here we used the fact that $(1+\epsilon)m \leq n$.

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Here we used the fact that $(1+\epsilon)m \leq n$.

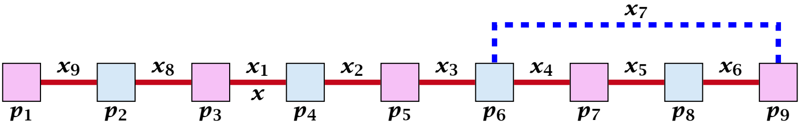
Hence,

$$\Pr[\text{cycle}] = \mathcal{O}\left(\frac{1}{m^2}\right) .$$

Cuckoo Hashing

Now, we analyze the probability that a phase is not successful without running into a closed cycle.

Cuckoo Hashing



Sequence of visited keys:

$x = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_3, x_2, x_1 = x, x_8, x_9, \dots$

Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with x in the order that they are visited during the phase.

Cuckoo Hashing

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Lemma 30

If the sequence is of length p then there exists a sub-sequence of at least $\frac{p+2}{3}$ keys starting with x of distinct keys.

Cuckoo Hashing

Proof.

Let i be the number of keys (including x) that we see before the first repeated key. Let j denote the total number of distinct keys.

The sequence is of the form:

$$x = x_1 \to x_2 \to \cdots \to x_i \to x_r \to x_{r-1} \to \cdots \to x_1 \to x_{i+1} \to \cdots \to x_j$$

As $r \leq i - 1$ the length p of the sequence is

$$p = i + r + (j - i) \leq i + j - 1 \ .$$

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As $r \leq i - 1$ the length p of the sequence is

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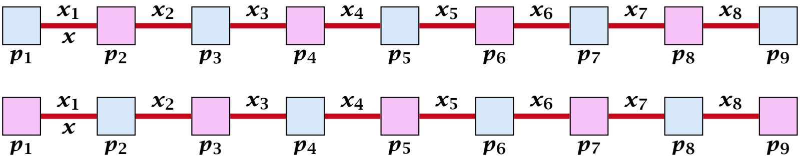
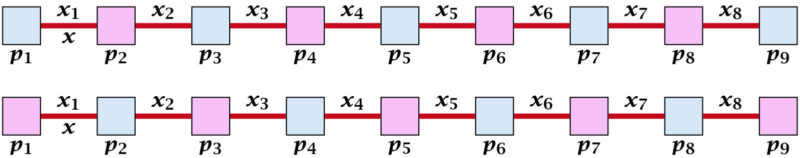
Either sub-sequence $x_1 \to x_2 \to \cdots \to x_i$ or sub-sequence $x_1 \to x_{i+1} \to \cdots \to x_j$ has at least $\frac{p+2}{3}$ elements. □

Cuckoo Hashing



A path-structure of size s is defined by

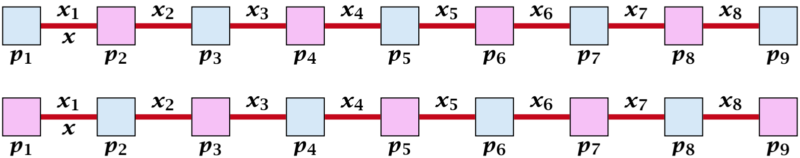
Cuckoo Hashing



A path-structure of size s is defined by

- $s+1$ different cells (alternating btw. cells from T_1 and T_2).

Cuckoo Hashing



A path-structure of size s is defined by

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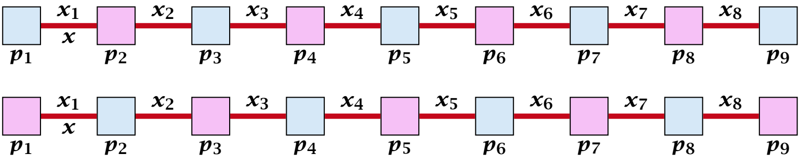

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- $s + 1$ different cells (alternating btw. cells from T_1 and T_2).
- s distinct keys $x = x_1, x_2, \ldots, x_s$, linking the cells.
- The leftmost cell is either from T_1 or T_2.

Cuckoo Hashing

A path-structure is active if for every key x_ℓ (linking a cell p_i from T_1 and a cell p_j from T_2) we have

$$h_1(x_\ell) = p_i \qquad \text{and} \qquad h_2(x_\ell) = p_j$$

Observation:
If a phase takes at least t steps without running into a cycle there must exist an active path-structure of size $(2t+2)/3$.

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$$\leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{(2t+2)/3-1} = 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{(2t-1)/3} .$$

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This gives $\text{maxsteps} = \Theta(\log m)$.

Cuckoo Hashing

So far we estimated

$$\Pr[\text{cycle}] \leq \mathcal{O}\Big(\frac{1}{m^2}\Big)$$

and

$$\Pr[\text{unsuccessful} \mid \text{no cycle}] \leq \mathcal{O}\Big(\frac{1}{m^2}\Big)$$

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for a suitable constant $c > 0$.

Cuckoo Hashing

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This means the expected cost for a successful phase is constant (even after accounting for the cost of the incomplete step that finishes the phase).

Cuckoo Hashing

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A phase that is not successful induces cost for doing a complete rehash (this dominates the cost for the steps in the phase).

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The probability that a phase is not successful is $q = \mathcal{O}(1/m^2)$ (probability $\mathcal{O}(1/m^2)$ of running into a cycle and probability $\mathcal{O}(1/m^2)$ of reaching maxsteps without running into a cycle).

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Therefore the expected cost for re-hashes is $\mathcal{O}(m) \cdot \mathcal{O}(p) = \mathcal{O}(1)$.

Formal Proof

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Therefore, it is sufficient to have $(\mu, \Theta(\log m))$-independent hash-functions.

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- Therefore we can amortize the rehash cost after a change in table-size against the cost for insertions and deletions.

Cuckoo Hashing

Lemma 31

Cuckoo Hashing has an expected constant insert-time and a worst-case constant search-time.

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Cuckoo Hashing has an expected constant insert-time and a worst-case constant search-time.

Note that the above lemma only holds if the fill-factor (number of keys/total number of hash-table slots) is at most $\frac{1}{2(1+\epsilon)}$.

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- **boolean** $\boldsymbol{S}.\,\mathbf{is\text{-}empty}()$: Returns true if the data-structure is empty and false otherwise.

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- **boolean** $\boldsymbol{S}.\,\textbf{is-empty}()$: Returns true if the data-structure is empty and false otherwise.

Sometimes we also have

- $\boldsymbol{S}.\,\textbf{merge}(\boldsymbol{S'})$: $S := S \cup S'$; $S' := \emptyset$.

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- $\textbf{handle } S.\textbf{insert}(x)$: Adds element x to the data-structure, and returns a handle to the object for future reference.
- $S.\textbf{delete}(h)$: Deletes element specified through handle h.
- $S.\textbf{decrease-key}(h, k)$: Decreases the key of the element specified by handle h to k. Assumes that the key is at least k before the operation.

Dijkstra's Shortest Path Algorithm

Algorithm 39 Shortest-Path($G = (V, E, d), s \in V$)

Input: weighted graph $G = (V, E, d)$; start vertex s;
Output: key-field of every node contains distance from s;
S.build(); // build empty priority queue
for all $v \in V \setminus \{s\}$ **do**
    $v.\text{key} \leftarrow \infty$;
    $h_v \leftarrow S$.insert(v);
$s.\text{key} \leftarrow 0$; S.insert(s);
while S.is-empty() $=$ false **do**
    $v \leftarrow S$.delete-min();
    for all $x \in V$ s.t. $(v, x) \in E$ **do**
        if $x.\text{key} > v.\text{key} + d(v, x)$ **then**
            S.decrease-key(h_x, $v.\text{key} + d(v, x)$);
            $x.\text{key} \leftarrow v.\text{key} + d(v, x)$;

Prim's Minimum Spanning Tree Algorithm

Algorithm 40 Prim-MST$(G = (V, E, d), s \in V)$

Input: weighted graph $G = (V, E, d)$; start vertex s;
Output: pred-fields encode MST;
S.build(); // build empty priority queue
for all $v \in V \setminus \{s\}$ **do**
    $v.\mathrm{key} \leftarrow \infty$;
    $h_v \leftarrow S.\mathrm{insert}(v)$;
$s.\mathrm{key} \leftarrow 0$; $S.\mathrm{insert}(s)$;
while S.is-empty$() = \mathrm{false}$ **do**
    $v \leftarrow S$.delete-min$()$;
    for all $x \in V$ s.t. $\{v, x\} \in E$ **do**
        if $x.\mathrm{key} > d(v, x)$ **then**
            S.decrease-key$(h_x, d(v, x))$;
            $x.\mathrm{key} \leftarrow d(v, x)$;
            $x.\mathrm{pred} \leftarrow v$;

Analysis of Dijkstra and Prim

Both algorithms require:

- 1 build() operation
- $|V|$ insert() operations
- $|V|$ delete-min() operations
- $|V|$ is-empty() operations
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How good a running time can we obtain?

8 Priority Queues

Operation	*Binary Heap*	*BST*	*Binomial Heap*	*Fibonacci Heap**
build	n	$n \log n$	$n \log n$	n
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n^{**}$	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	n	$n \log n$	$\log n$	1

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delete-min	$\log n$	$\log n$	$\log n$	$\log n$
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Note that most applications use **build()** only to create an empty heap which then costs time 1.

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The standard version of binary heaps is not addressable, and hence does not support a delete operation.

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Note that most applications use **build()** only to create an empty heap which then costs time 1.

The standard version of binary heaps is not addressable, and hence does not support a delete operation.

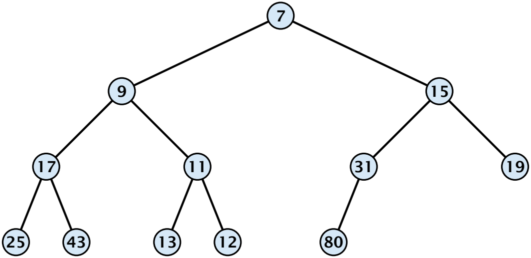
Fibonacci heaps only give an amortized guarantee.

8 Priority Queues

Using Binary Heaps, Prim and Dijkstra run in time $\mathcal{O}((|V| + |E|) \log |V|)$.

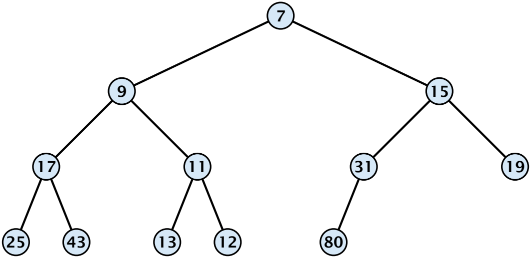
Using Fibonacci Heaps, Prim and Dijkstra run in time $\mathcal{O}(|V| \log |V| + |E|)$.

8.1 Binary Heaps



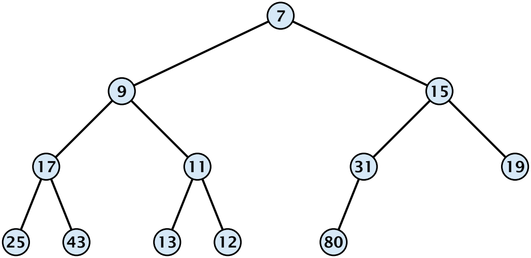
8.1 Binary Heaps

- Nearly complete binary tree; only the last level is not full, and this one is filled from left to right.



8.1 Binary Heaps

- Nearly complete binary tree; only the last level is not full, and this one is filled from left to right.
- Heap property: A node's key is not larger than the key of one of its children.



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- **minimum()**: return the root-element. Time $\mathcal{O}(1)$.
- **is-empty()**: check whether root-pointer is null. Time $\mathcal{O}(1)$.

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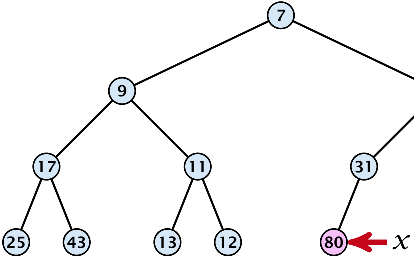
Maintain a pointer to the last element x.



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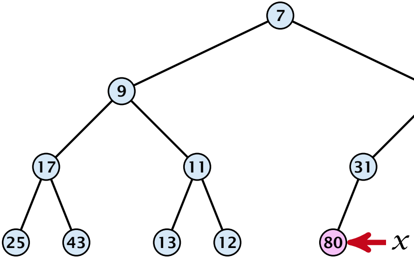
Maintain a pointer to the last element x.

- We can compute the predecessor of x (last element when x is deleted) in time $\mathcal{O}(\log n)$.



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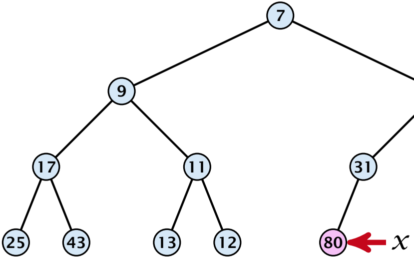
  go up until the last edge used was a right edge.
  go left; go right until you reach a leaf



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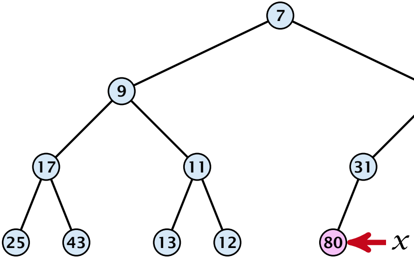
  if you hit the root on the way up, go to the rightmost element



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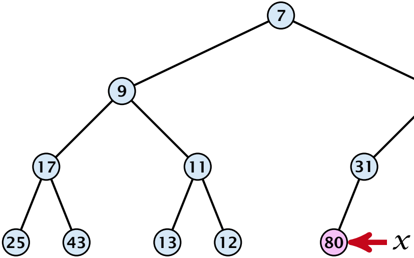

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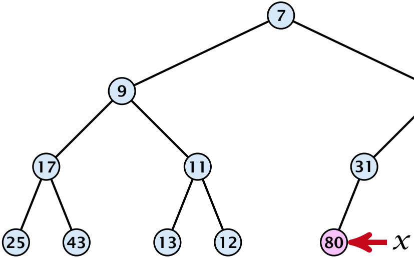

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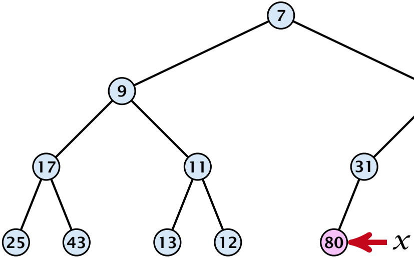

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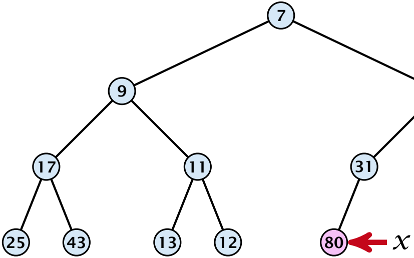
  if you hit the root on the way up, go to the leftmost element;
  insert a new element as a left child;



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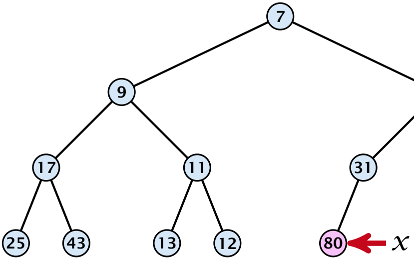

Insert

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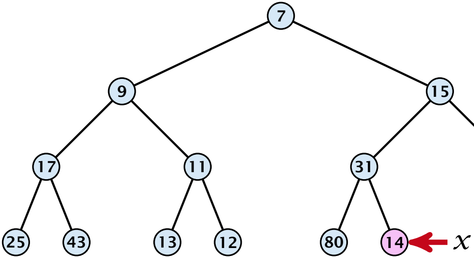
2. Exchange with parent until heap property is fulfilled.



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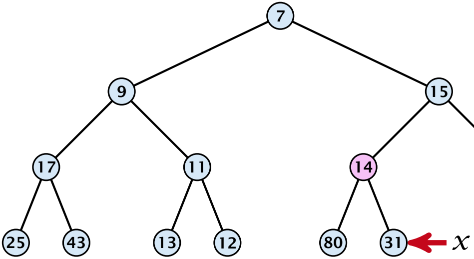

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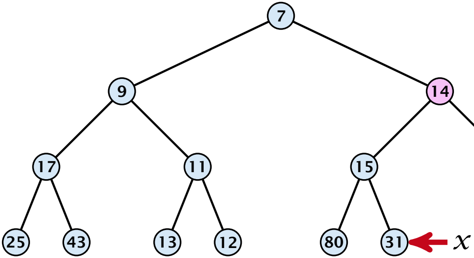

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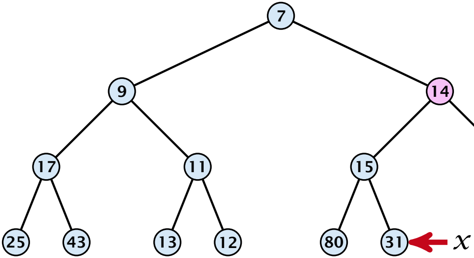
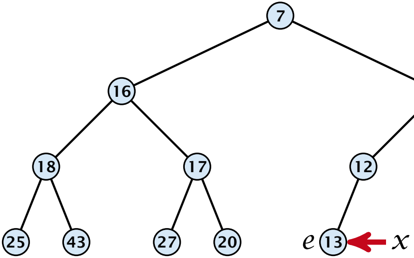


Note that an exchange can either be done by moving the data or by changing pointers. The latter method leads to an addressable priority queue.

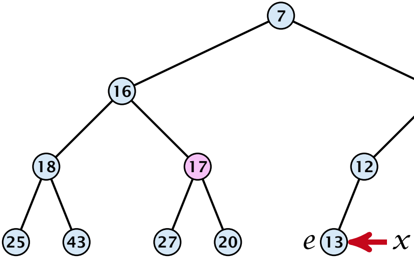
Delete

1. Exchange the element to be deleted with the element e pointed to by x.



Delete

1. Exchange the element to be deleted with the element e pointed to by x.
2. Restore the heap-property for the element e.



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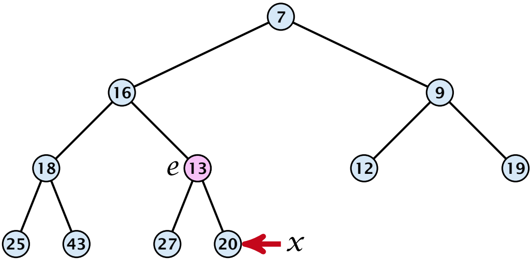

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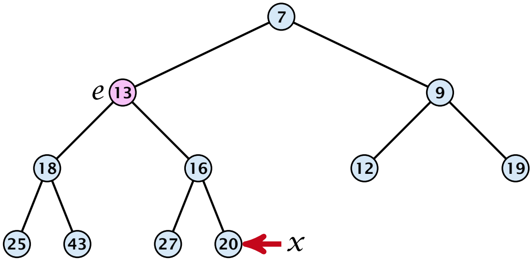

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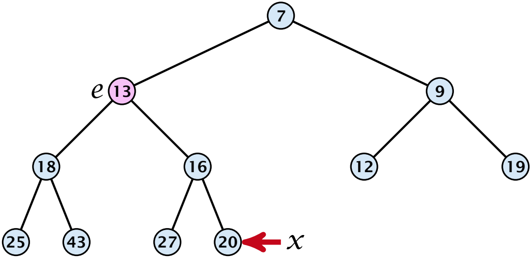

At its new position e may either travel up or down in the tree (but not both directions).

Binary Heaps

Operations:

- **minimum()**: return the root-element. Time $\mathcal{O}(1)$.
- **is-empty()**: check whether root-pointer is null. Time $\mathcal{O}(1)$.
- **insert(k)**: insert at successor of x and bubble up. Time $\mathcal{O}(\log n)$.
- **delete(h)**: swap with x and bubble up or sift-down. Time $\mathcal{O}(\log n)$.

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- **delete(h)**: Swap with x and bubble up or sift-down. Time $\mathcal{O}(\log n)$.
- **build($x_1, \dots, x_n$)**: Insert elements arbitrarily; then do sift-down operations starting with the lowest layer in the tree. Time $\mathcal{O}(n)$.

Binary Heaps

Binary Heaps

The standard implementation of binary heaps is via arrays. Let $A[0,\ldots,n-1]$ be an array

- The parent of i-th element is at position $\lfloor\frac{i-1}{2}\rfloor$.
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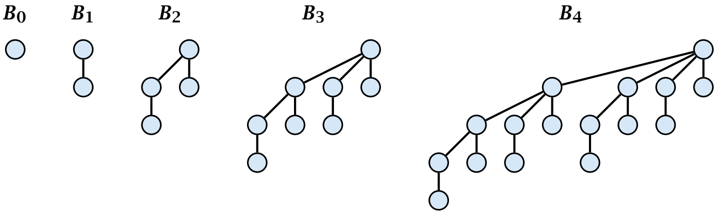
Finding the successor of x is much easier than in the description on the previous slide. Simply increase or decrease x.

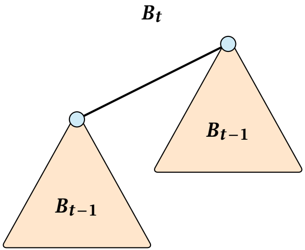
The resulting binary heap is not addressable. The elements don't maintain their positions and therefore there are no stable handles.

8.2 Binomial Heaps

Operation	*Binary Heap*	*BST*	*Binomial Heap*	*Fibonacci Heap**
build	n	$n \log n$	$n \log n$	n
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n^{**}$	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	n	$n \log n$	$\log n$	1

Binomial Trees





Binomial Trees

Properties of Binomial Trees

- B_k has 2^k nodes.

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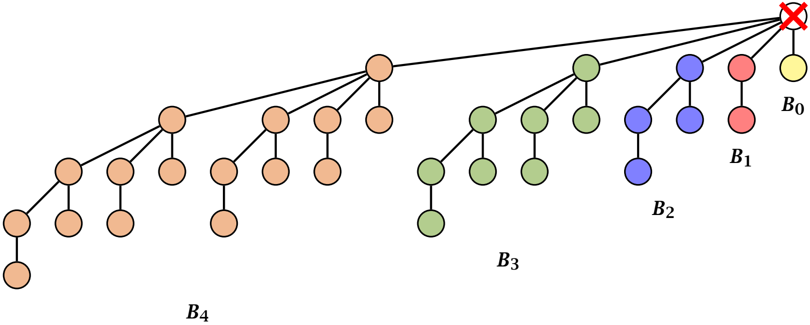
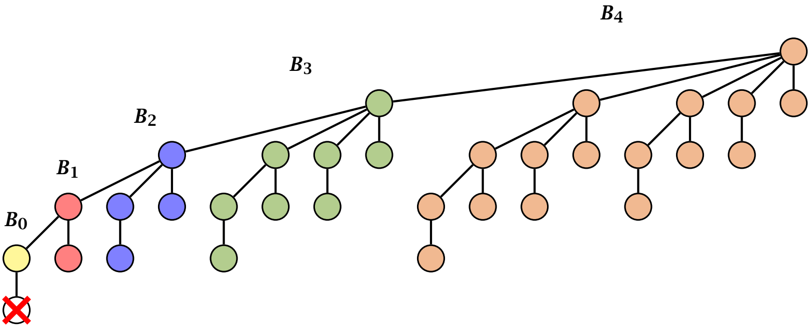
- B_k has height k.
- The root of B_k has degree k.
- B_k has $\binom{k}{\ell}$ nodes on level ℓ.
- Deleting the root of B_k gives trees $B_0, B_1, \ldots, B_{k-1}$.

Binomial Trees



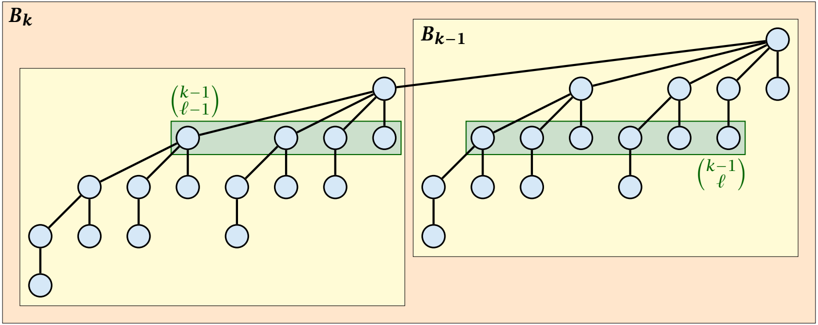
Deleting the root of B_5 leaves sub-trees B_4, B_3, B_2, B_1, and B_0.

Binomial Trees



Deleting the leaf furthest from the root (in B_5) leaves a path that connects the roots of sub-trees B_4, B_3, B_2, B_1, and B_0.

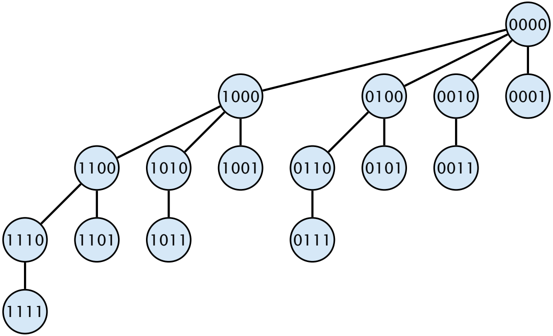
Binomial Trees



The number of nodes on level ℓ in tree B_k is therefore

$$\binom{k-1}{\ell-1} + \binom{k-1}{\ell} = \binom{k}{\ell}$$

Binomial Trees



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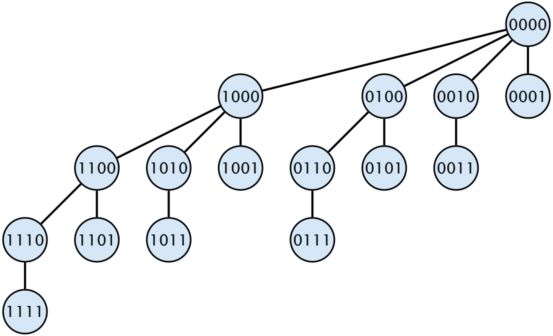
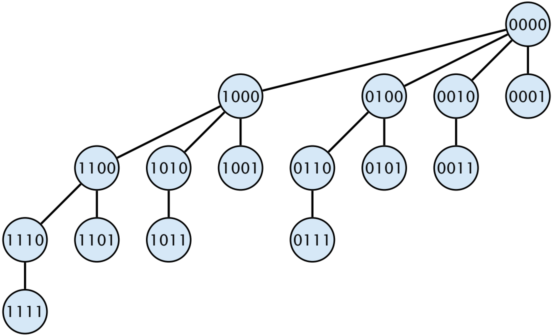


The binomial tree B_k is a sub-graph of the hypercube H_k.

Binomial Trees



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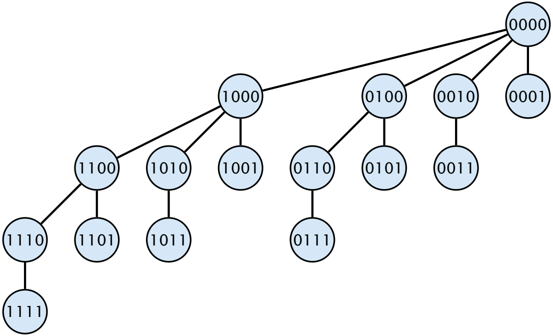
The parent of a node with label $b_k, \dots, b_1$ is obtained by setting the least significant 1-bit to 0.

Binomial Trees



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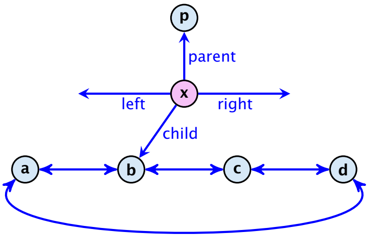
The ℓ-th level contains nodes that have ℓ 1's in their label.

8.2 Binomial Heaps

How do we implement trees with non-constant degree?

- The children of a node are arranged in a circular linked list.



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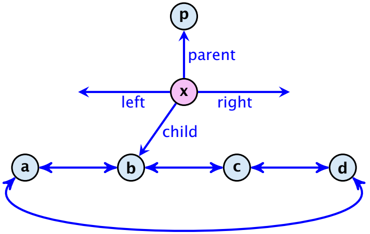

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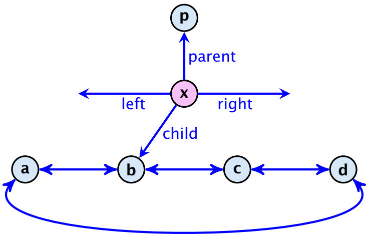

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How do we implement trees with non-constant degree?

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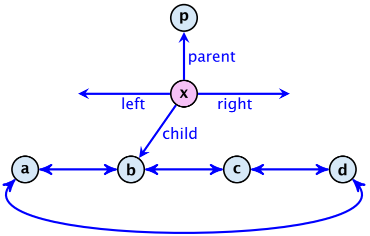
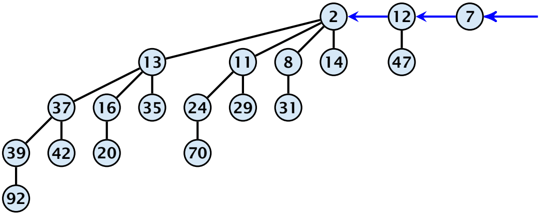
- A child-pointer points to an arbitrary node within the list.
- A parent-pointer points to the parent node.
- Pointers $x.\text{left}$ and $x.\text{right}$ point to the left and right sibling of x (if x does not have siblings then $x.\text{left} = x.\text{right} = x$).



8.2 Binomial Heaps

- Given a pointer to a node x we can splice out the sub-tree rooted at x in constant time.
- We can add a child-tree T to a node x in constant time if we are given a pointer to x and a pointer to the root of T.

Binomial Heap



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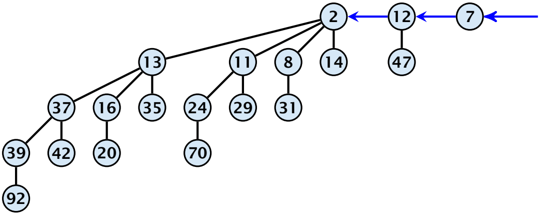
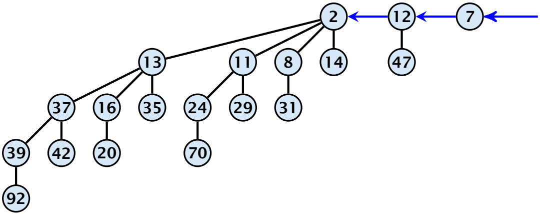


In a binomial heap the keys are arranged in a collection of binomial trees.

Binomial Heap



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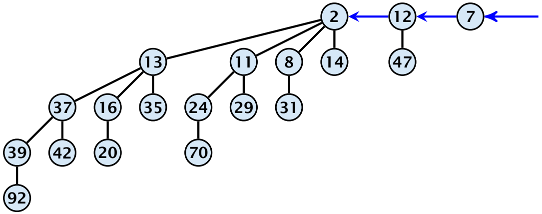
Binomial Heap



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There is at most one tree for every dimension/order. For example the above heap contains trees B_0, B_1, and B_4.

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Given the number n of keys to be stored in a binomial heap we can deduce the binomial trees that will be contained in the collection.

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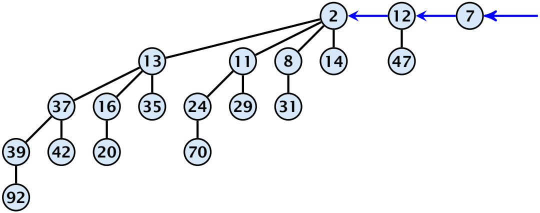
Then $n = \sum_i 2^{k_i}$ must hold. But since the k_i are all distinct this means that the k_i define the non-zero bit-positions in the binary representation of n.

Binomial Heap

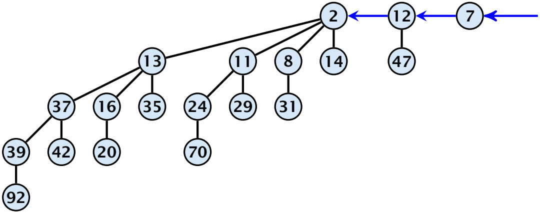
Properties of a heap with n keys:



Binomial Heap

Properties of a heap with n keys:

- Let $n = b_d b_{d-1}, \ldots, b_0$ denote binary representation of n.



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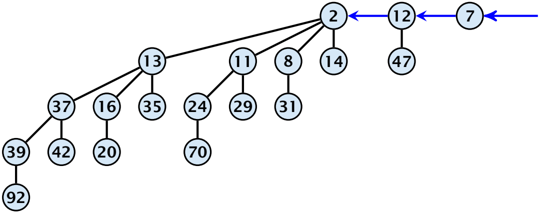

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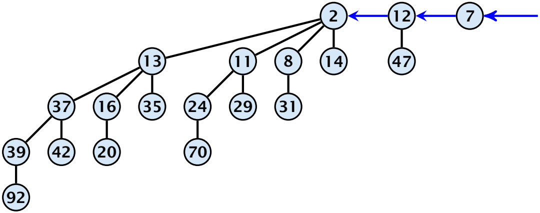

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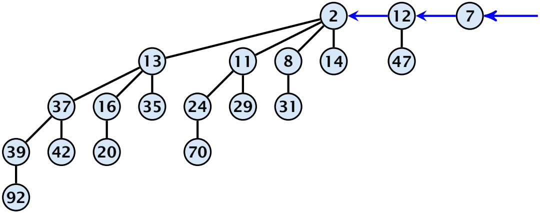

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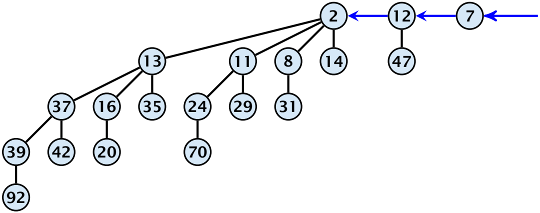

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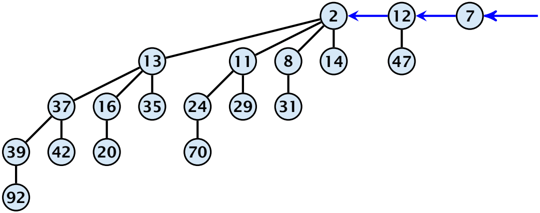
- Hence, at most $\lfloor \log n \rfloor + 1$ trees.
- The minimum must be contained in one of the roots.
- The height of the largest tree is at most $\lfloor \log n \rfloor$.
- The trees are stored in a single-linked list; ordered by dimension/size.



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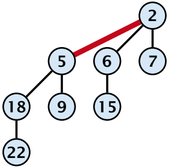

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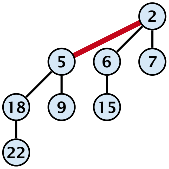
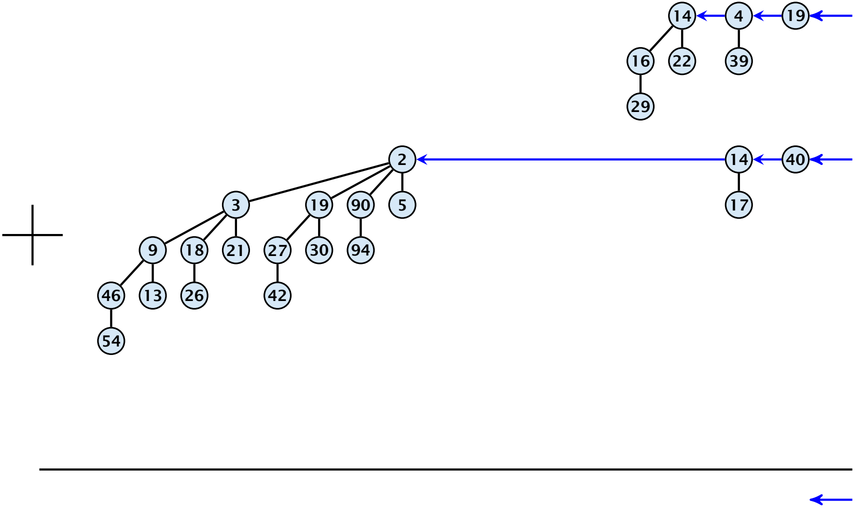


For more trees the technique is analogous to binary addition.



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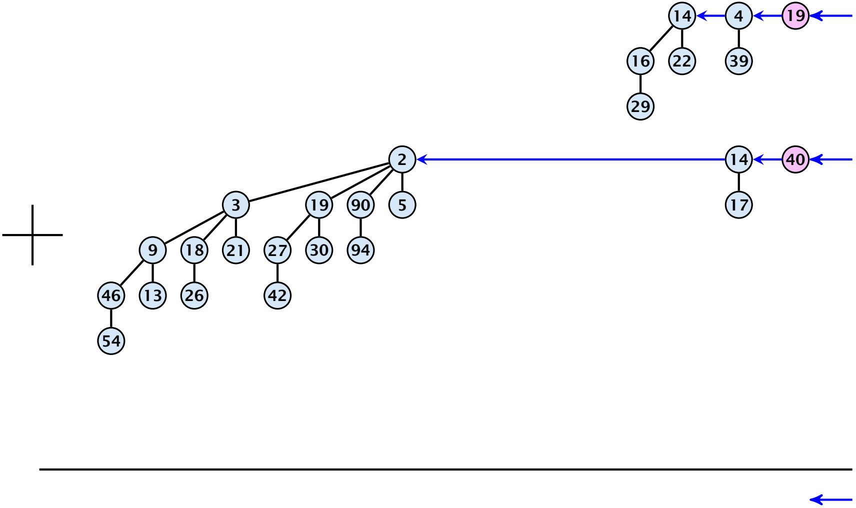
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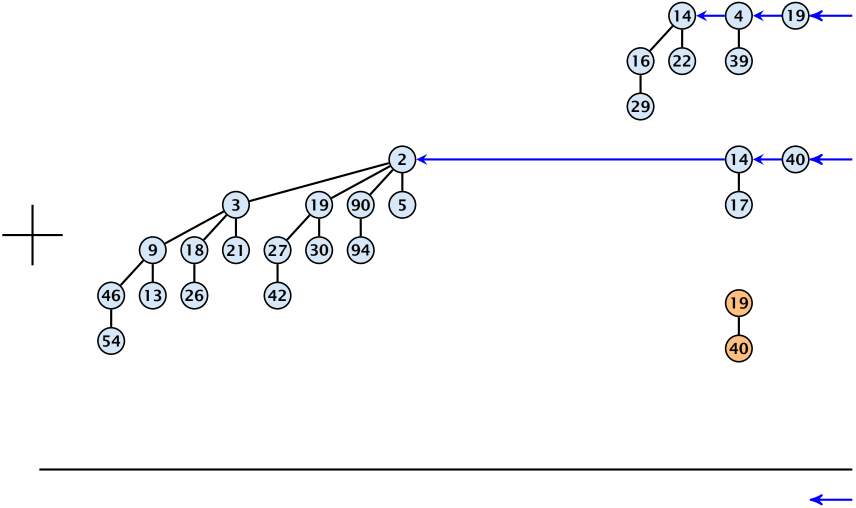

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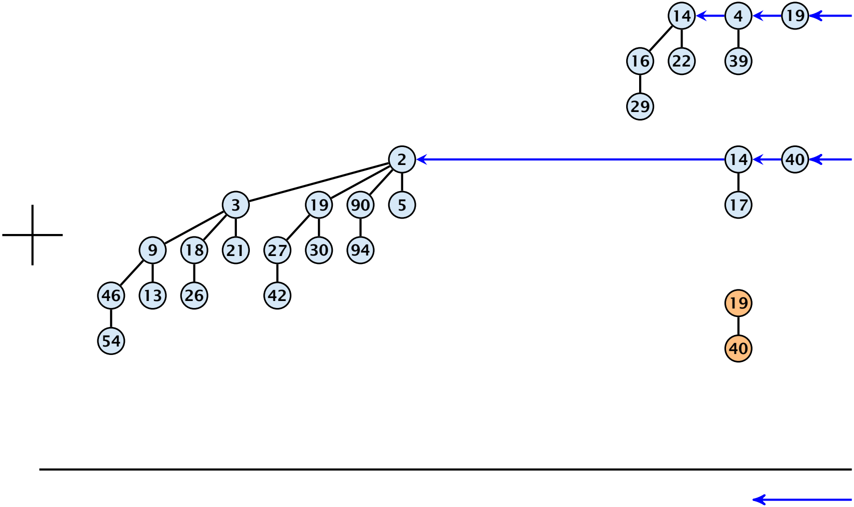

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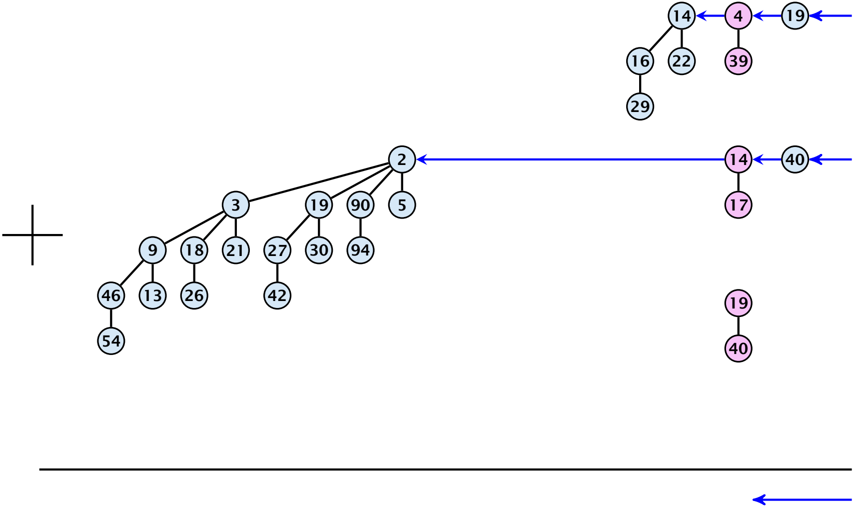

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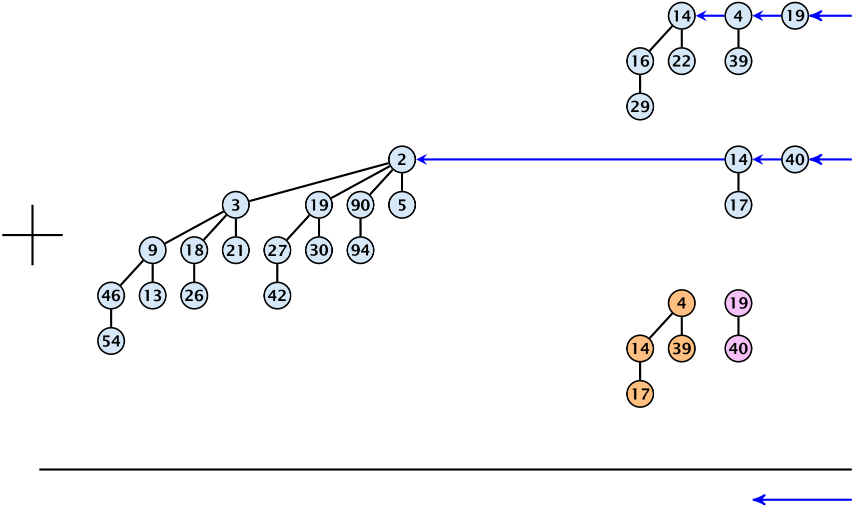
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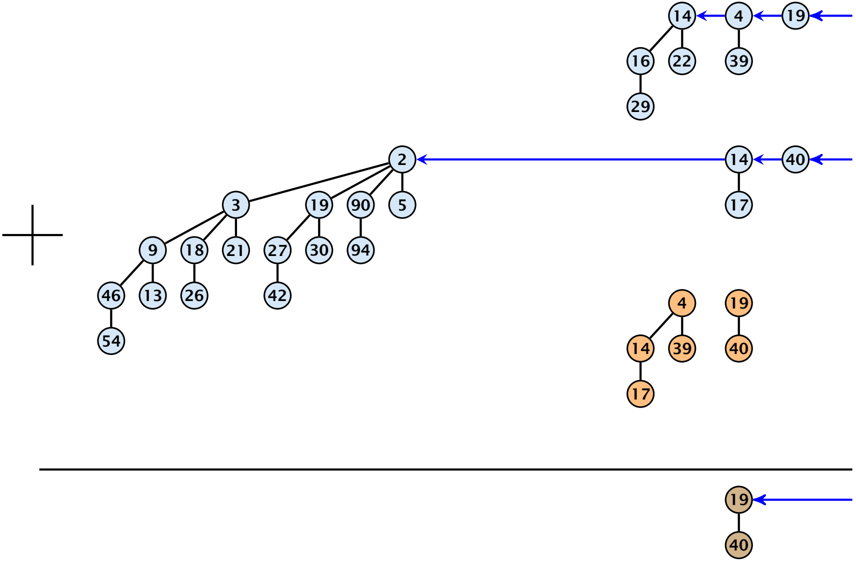
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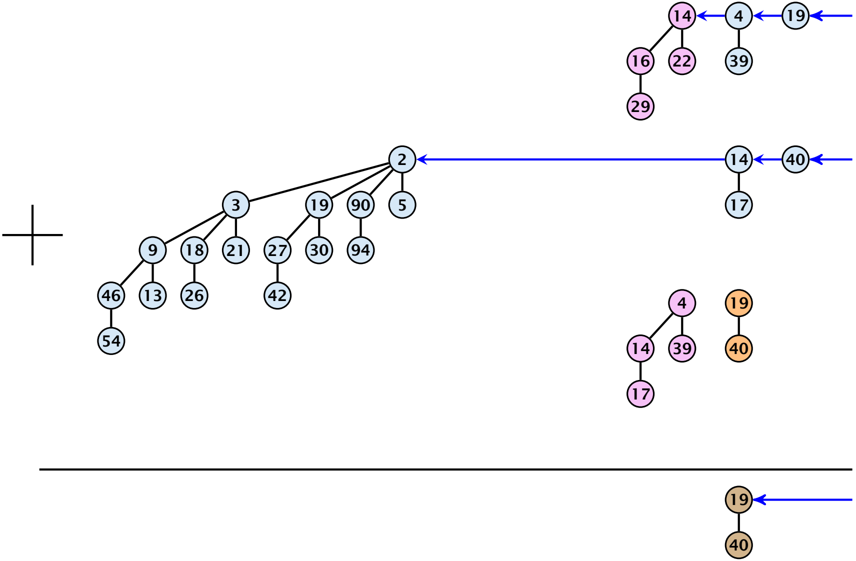
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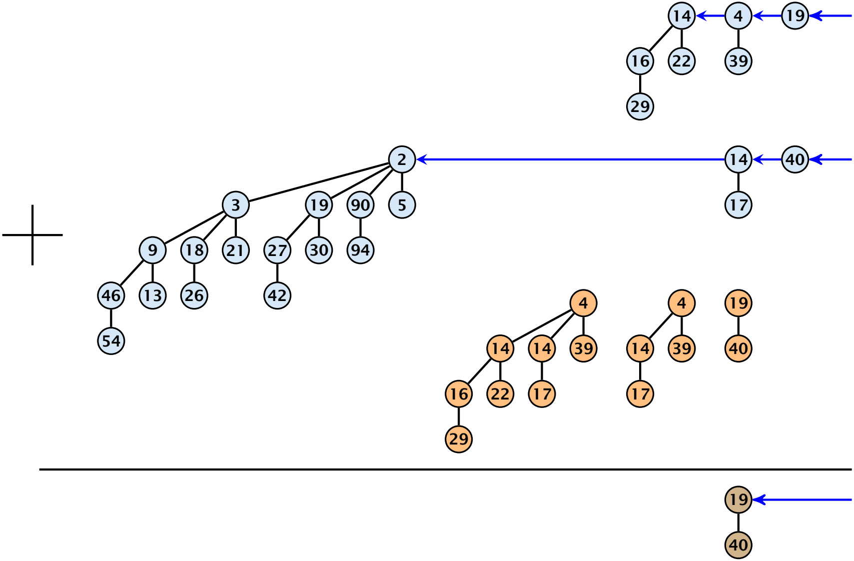
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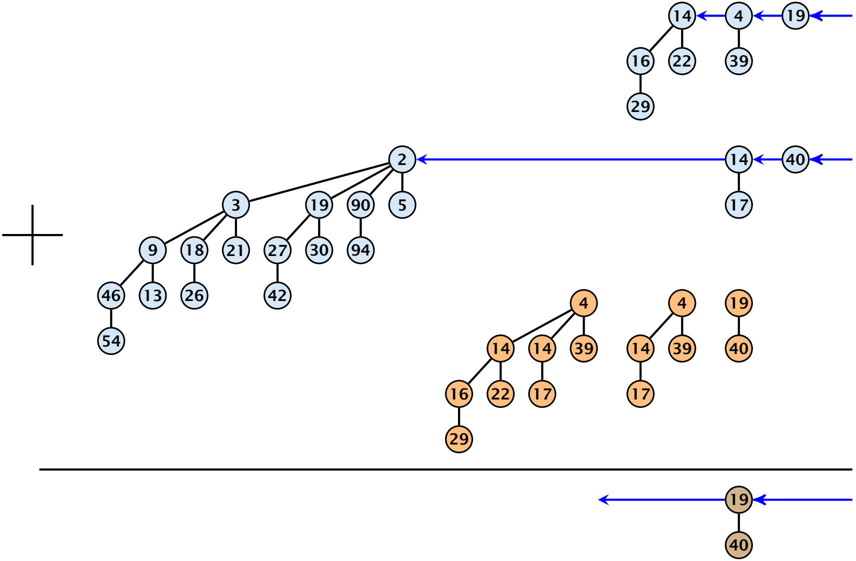

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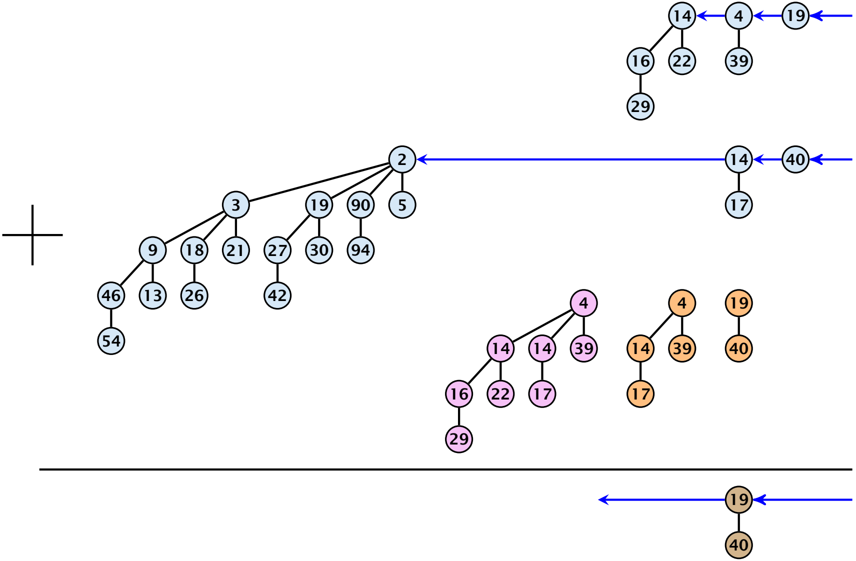

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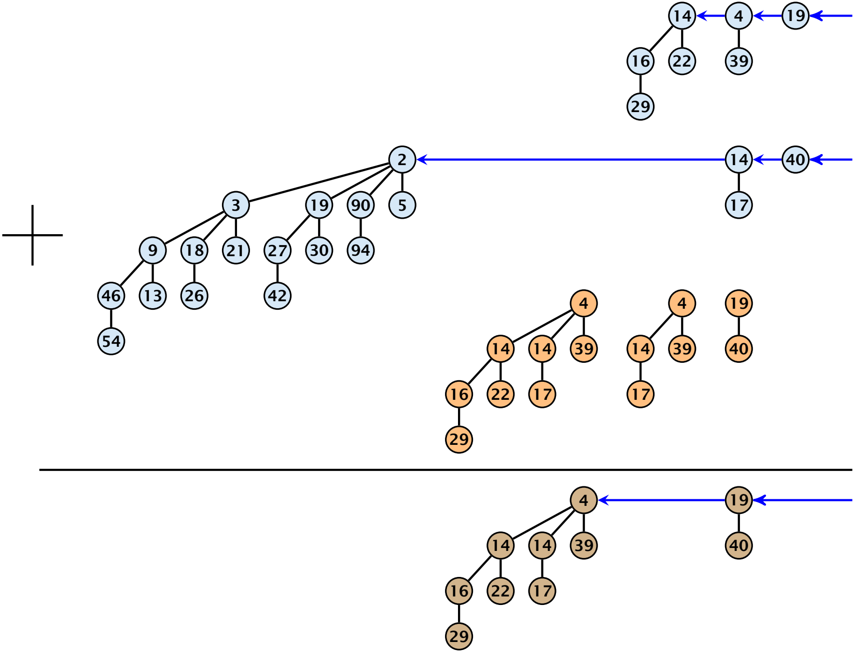
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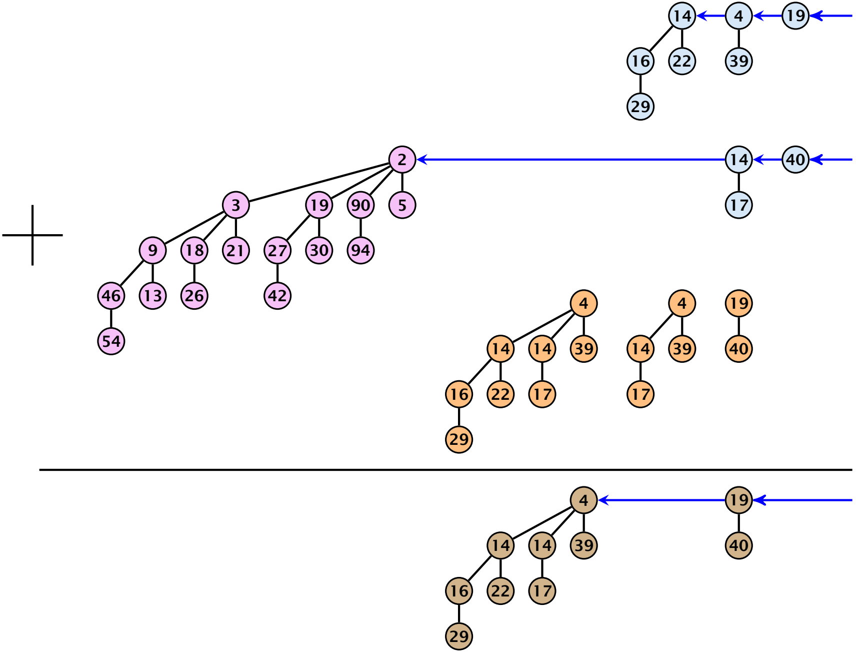

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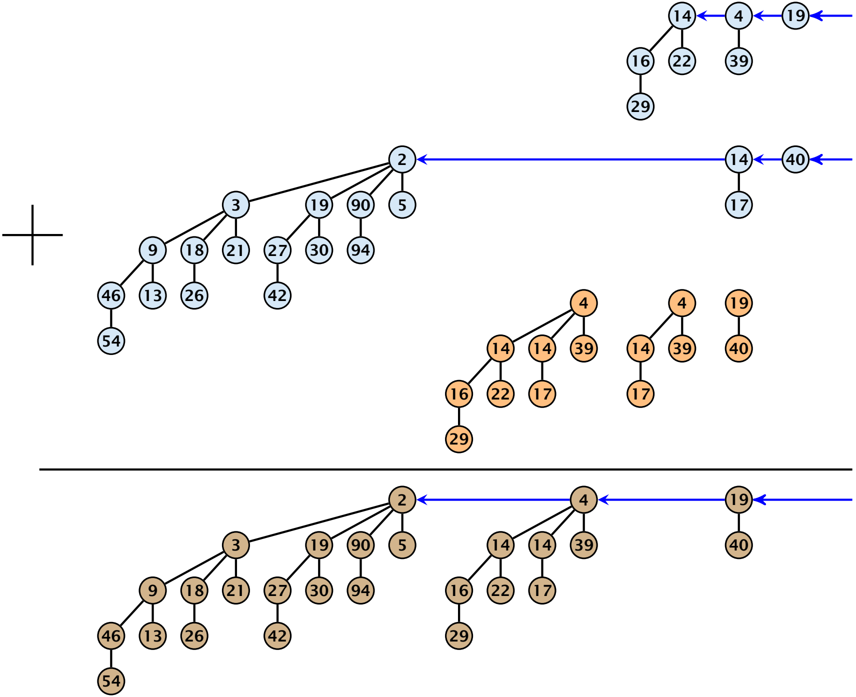
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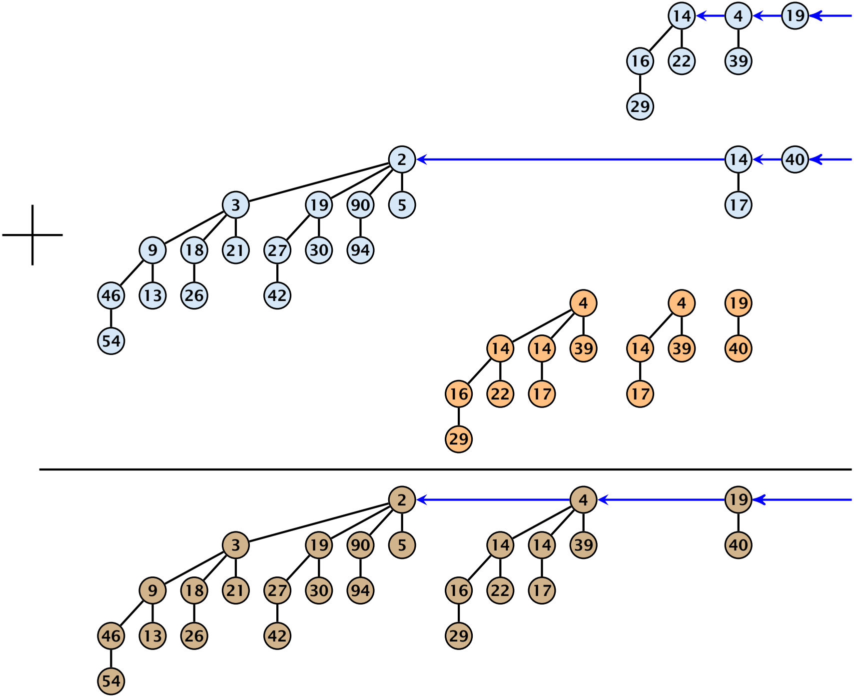
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8.2 Binomial Heaps

S_1.**merge**(S_2):

- Analogous to binary addition.

8.2 Binomial Heaps

S_1.merge(S_2):

- Analogous to binary addition.
- Time is proportional to the number of trees in both heaps.

8.2 Binomial Heaps

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- Time is proportional to the number of trees in both heaps.
- Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

All other operations can be reduced to $\text{merge}()$.

$S.\text{insert}(x)$:

- Create a new heap S' that contains just the element x.

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- Execute $S.\text{merge}(S')$.

8.2 Binomial Heaps

All other operations can be reduced to merge().

$S.\text{insert}(x)$:

- Create a new heap S' that contains just the element x.
- Execute $S.\text{merge}(S')$.
- Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

$S.$ **minimum()**:

- Find the minimum key-value among all roots.
- Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

S. delete-min():

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- Compute $S.\mathrm{merge}(S')$.

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- Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

$S.$ **decrease-key(handle** h**):**

8.2 Binomial Heaps

S. decrease-key(handle h):

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8.2 Binomial Heaps

S. decrease-key(handle h):

- Decrease the key of the element pointed to by h.
- Bubble the element up in the tree until the heap property is fulfilled.
- Time: $\mathcal{O}(\log n)$ since the trees have height $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

$S.$ **delete(handle h):**

8.2 Binomial Heaps

S. delete(handle h):

- Execute $S.\text{decrease-key}(h, -\infty)$.

8.2 Binomial Heaps

$S.\text{delete}(\text{handle } h)$:

- Execute $S.\text{decrease-key}(h, -\infty)$.
- Execute $S.\text{delete-min}()$.

8.2 Binomial Heaps

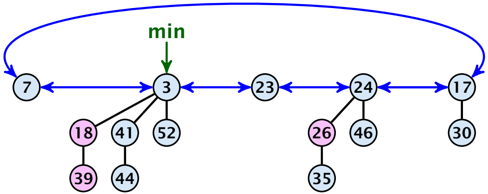
$S.$ delete(handle h):

- Execute $S.\text{decrease-key}(h, -\infty)$.
- Execute $S.\text{delete-min}()$.
- Time: $\mathcal{O}(\log n)$.

8.3 Fibonacci Heaps

Collection of trees that fulfill the heap property.

Structure is much more relaxed than binomial heaps.



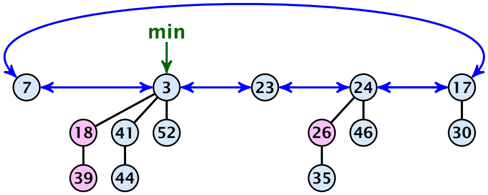
Additional implementation details:

- Every node x stores its degree in a field $x.\mathrm{degree}$. Note that this can be updated in constant time when adding a child to x.
- Every node stores a boolean value $x.\mathrm{marked}$ that specifies whether x is marked or not.

8.3 Fibonacci Heaps

The potential function:

- $t(S)$ denotes the number of trees in the heap.
- $m(S)$ denotes the number of marked nodes.
- We use the potential function $\Phi(S) = t(S) + 2m(S)$.



The potential is $\Phi(S) = 5 + 2 \cdot 3 = 11$.

8.3 Fibonacci Heaps

We assume that one unit of potential can pay for a constant amount of work, where the constant is chosen “big enough” (to take care of the constants that occur).

To make this more explicit we use c to denote the amount of work that a unit of potential can pay for.

8.3 Fibonacci Heaps

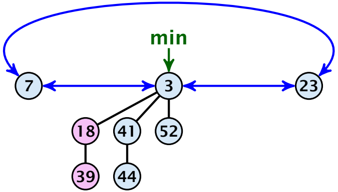
S. minimum()

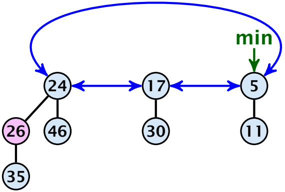
- Access through the min-pointer.
- Actual cost $\mathcal{O}(1)$.
- No change in potential.
- Amortized cost $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

S. merge(S')

- Merge the root lists.
- Adjust the min-pointer

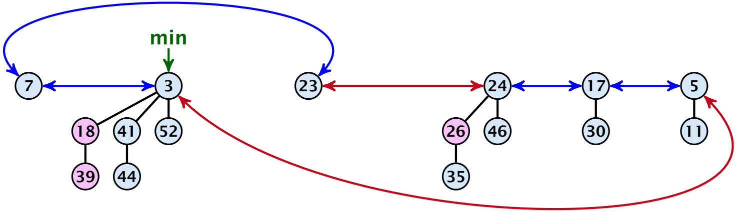




8.3 Fibonacci Heaps

$S.\mathrm{merge}(S')$

- Merge the root lists.
- Adjust the min-pointer



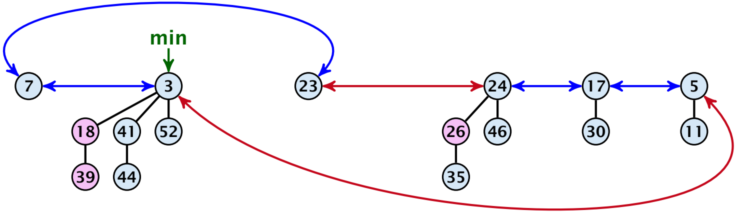
Running time:

- Actual cost $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

$S.\mathrm{merge}(S')$

- Merge the root lists.
- Adjust the min-pointer



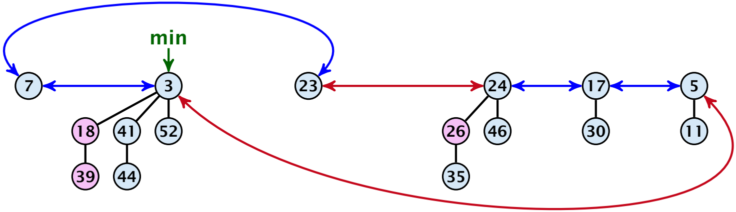
Running time:

- Actual cost $\mathcal{O}(1)$.
- No change in potential.

8.3 Fibonacci Heaps

$S.\,\textbf{merge}(S')$

- Merge the root lists.
- Adjust the min-pointer



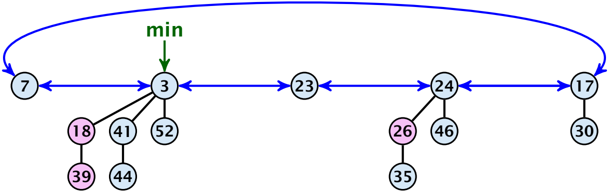
Running time:

- Actual cost $\mathcal{O}(1)$.
- No change in potential.
- Hence, amortized cost is $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

$S.\,\mathbf{insert}(x)$

- Create a new tree containing x.
- Insert x into the root-list.
- Update min-pointer, if necessary.



8.3 Fibonacci Heaps

S. insert(x)

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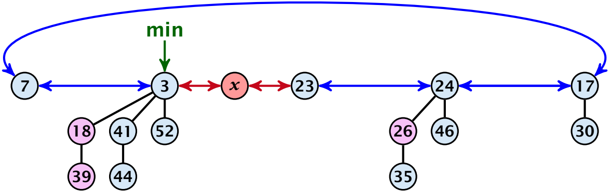

8.3 Fibonacci Heaps

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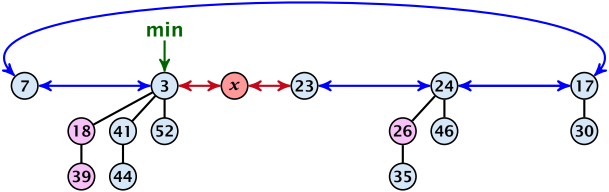
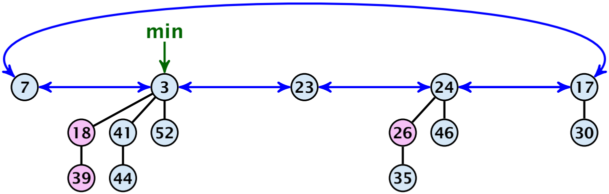


Running time:

- Actual cost $\mathcal{O}(1)$.
- Change in potential is $+1$.
- Amortized cost is $c + \mathcal{O}(1) = \mathcal{O}(1)$.

8.3 Fibonacci Heaps

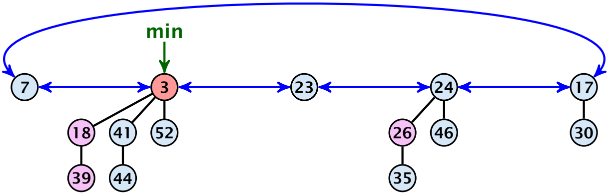
$S.$ delete-min(x)



8.3 Fibonacci Heaps

$S.$ **delete-min**(x)

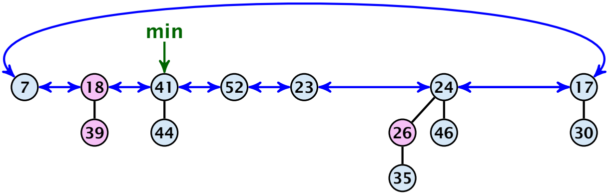
- Delete minimum; add child-trees to heap;
  time: $D(\min) \cdot \mathcal{O}(1)$.



8.3 Fibonacci Heaps

$S.$ **delete-min**(x)

- Delete minimum; add child-trees to heap; time: $D(\min) \cdot \mathcal{O}(1)$.
- Update min-pointer; time: $(t + D(\min)) \cdot \mathcal{O}(1)$.



8.3 Fibonacci Heaps

***S*. delete-min(*x*)**

- Delete minimum; add child-trees to heap; time: $D(\min) \cdot \mathcal{O}(1)$.
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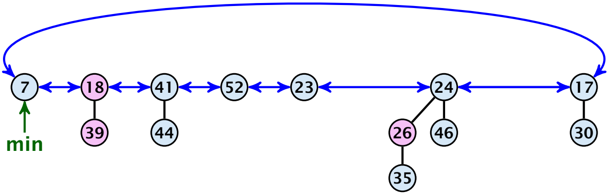

8.3 Fibonacci Heaps

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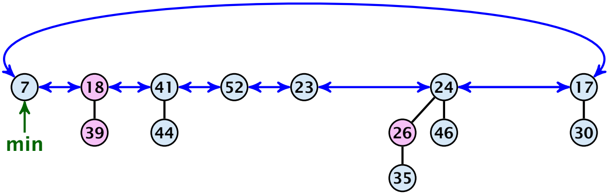

- Consolidate root-list so that no roots have the same degree. Time $t \cdot \mathcal{O}(1)$ (see next slide).

8.3 Fibonacci Heaps

Consolidate:

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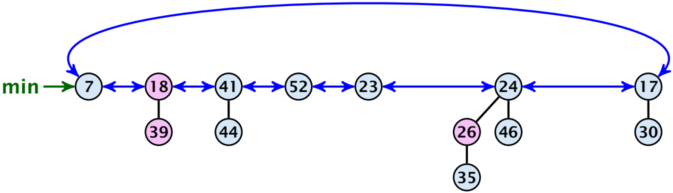

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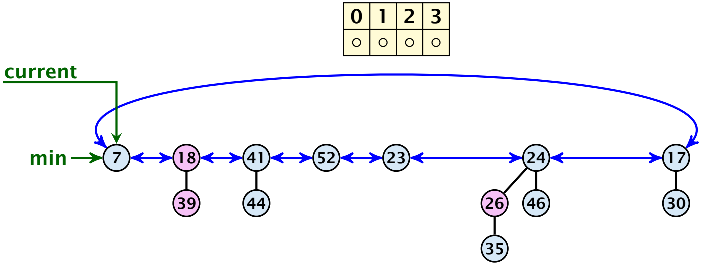

8.3 Fibonacci Heaps

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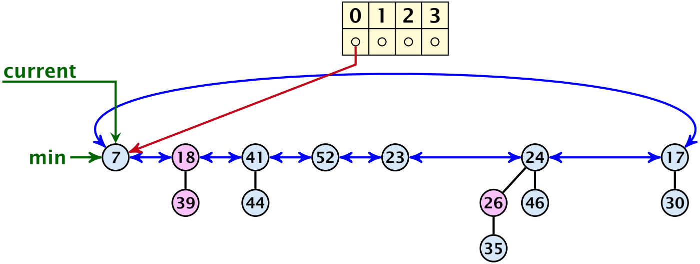

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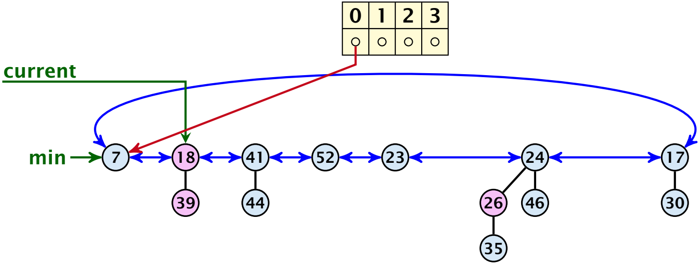

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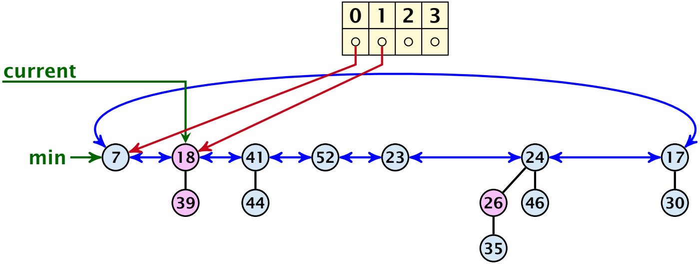

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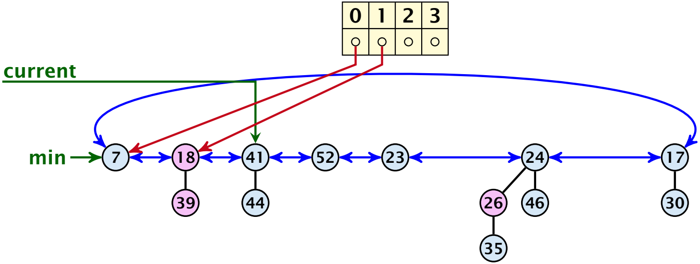

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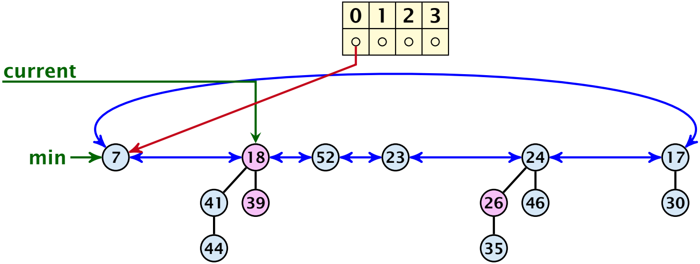

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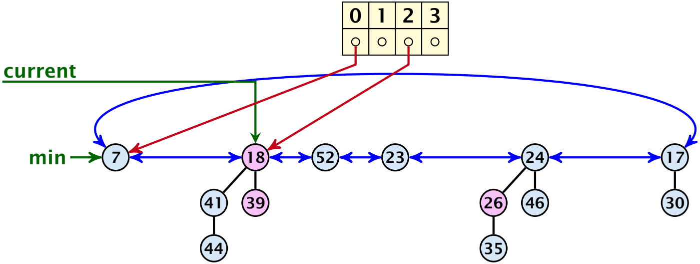

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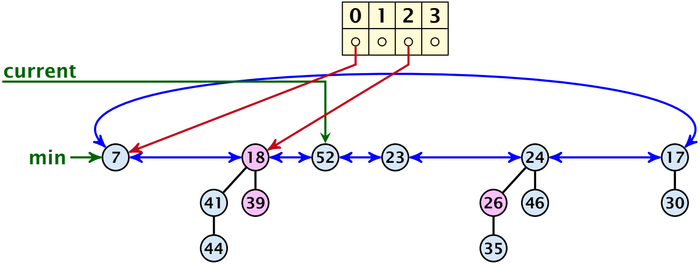

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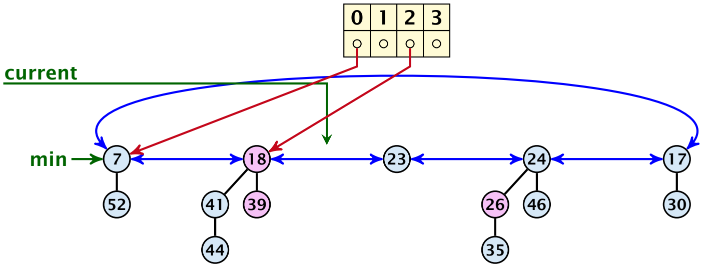

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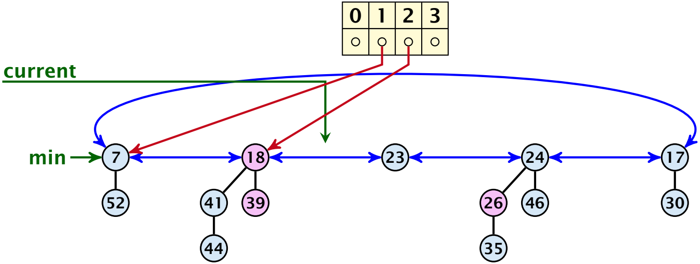

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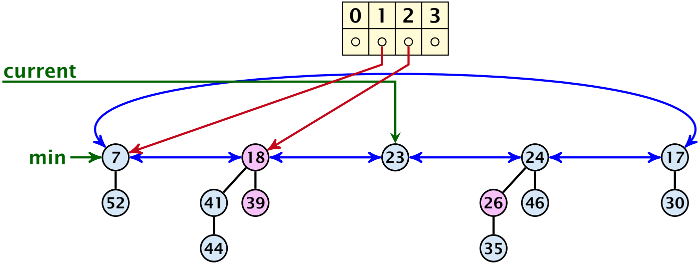

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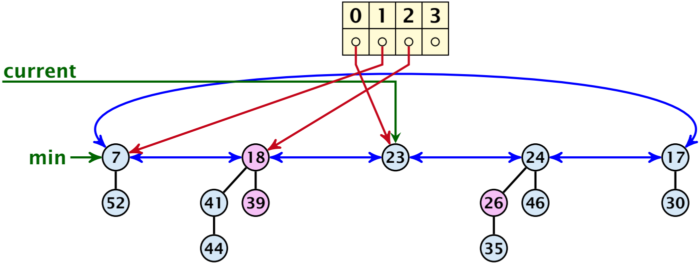

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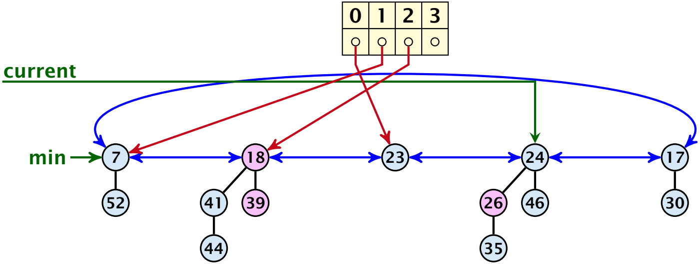

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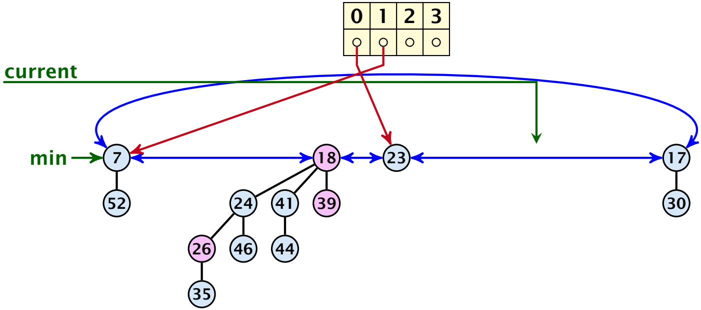

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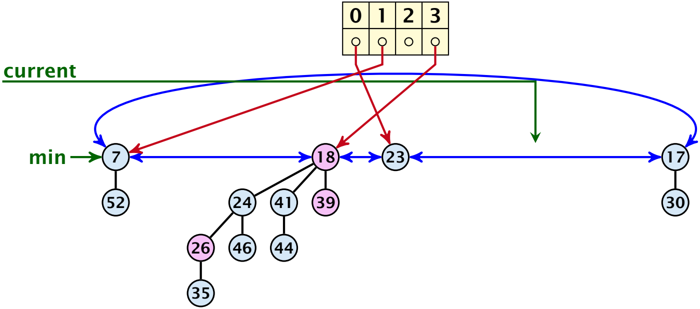
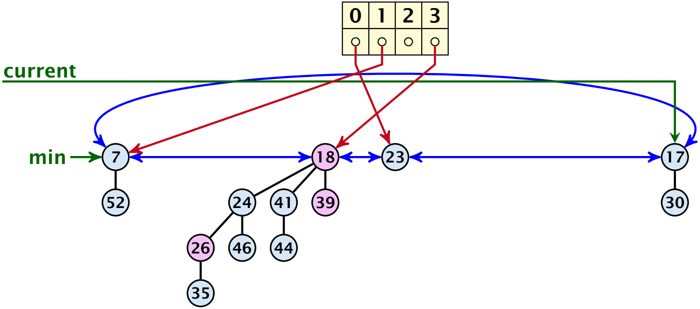


Consolidate:



8.3 Fibonacci Heaps

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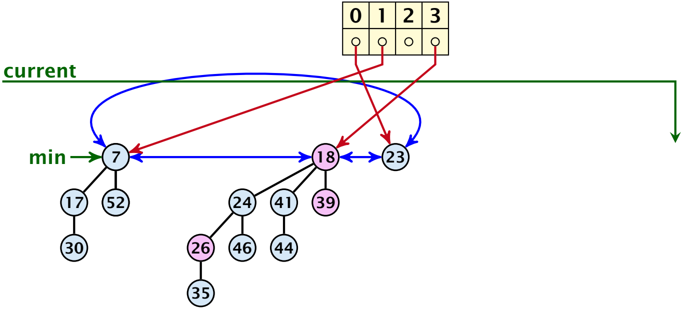

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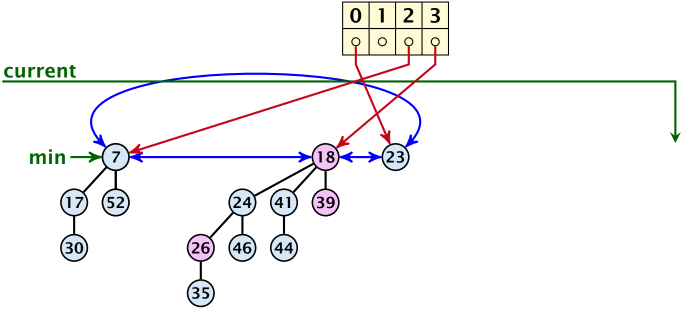

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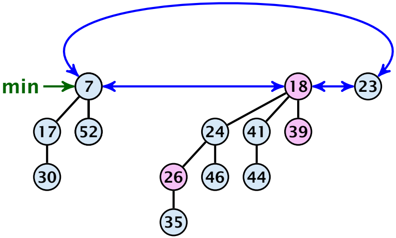

8.3 Fibonacci Heaps

Actual cost for delete-min()

- At most $D_n + t$ elements in root-list before consolidate.

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$$\begin{aligned} & c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1) \\ & \qquad \le (c_1 + c)D_n + (c_1 - c)t + c \end{aligned}$$

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for $c \ge c_1$.

8.3 Fibonacci Heaps

If the input trees of the consolidation procedure are binomial trees (for example only singleton vertices) then the output will be a set of distinct binomial trees, and, hence, the Fibonacci heap will be (more or less) a Binomial heap right after the consolidation.

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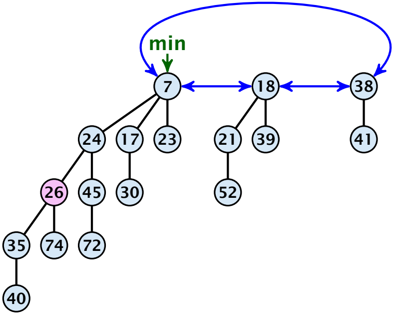
If we do not have delete or decrease-key operations then $D_n \le \log n$.

Fibonacci Heaps: decrease-key(handle h, v)



Case 1: decrease-key does not violate heap-property

- Just decrease the key-value of element referenced by h. Nothing else to do.

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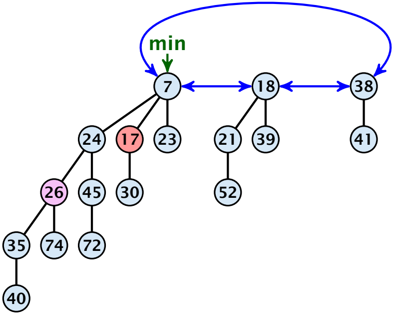

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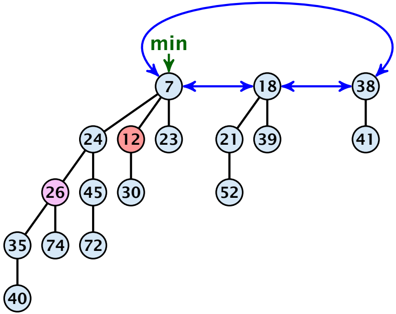

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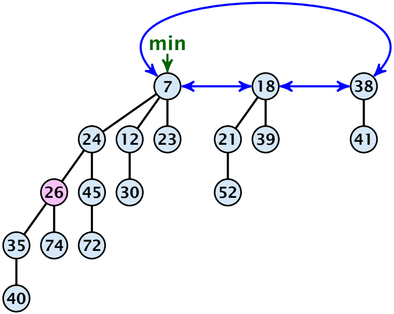

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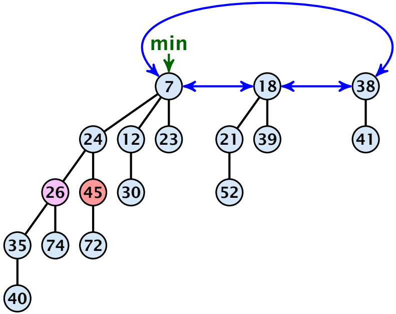

Case 2: heap-property is violated, but parent is not marked

- Decrease key-value of element x reference by h.
- If the heap-property is violated, cut the parent edge of x, and make x into a root.
- Adjust min-pointers, if necessary.
- Mark the (previous) parent of x (unless it's a root).

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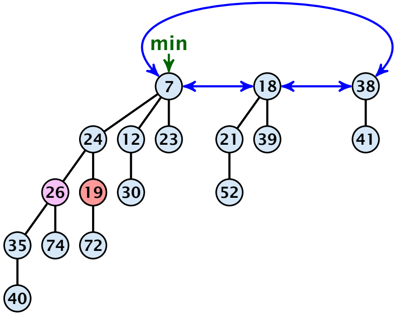

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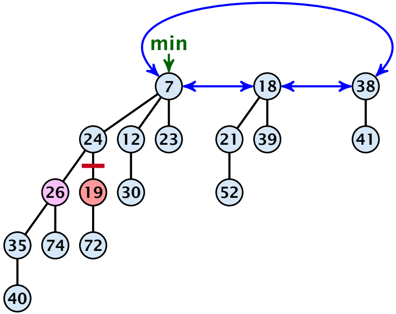

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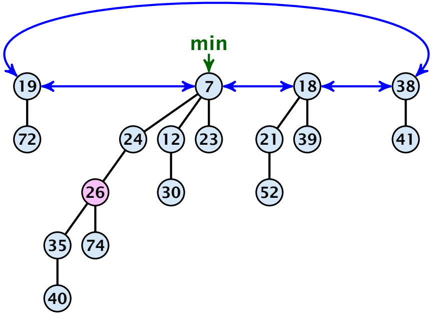

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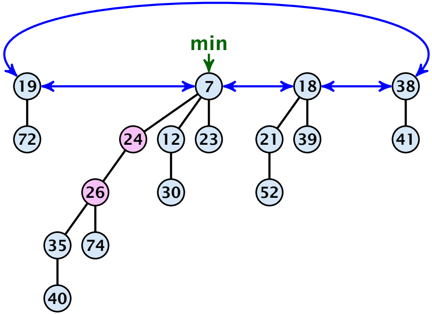

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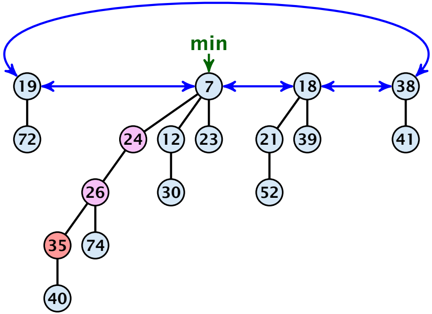

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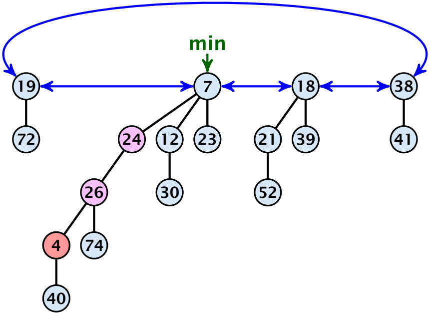

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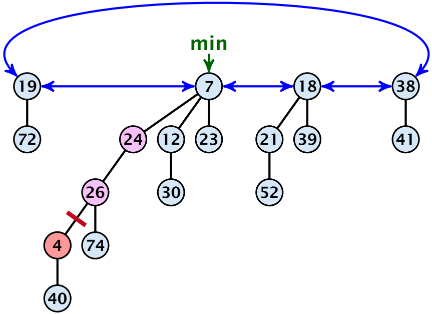

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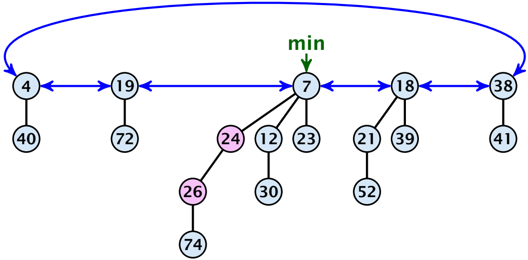

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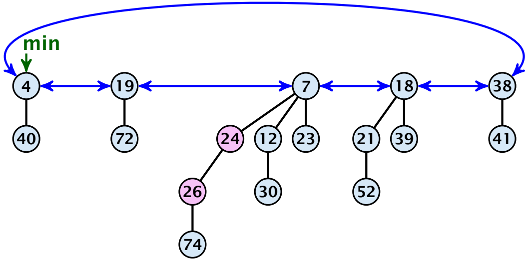

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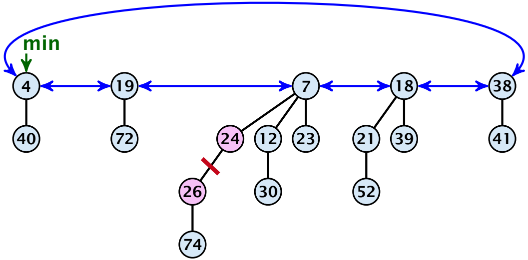

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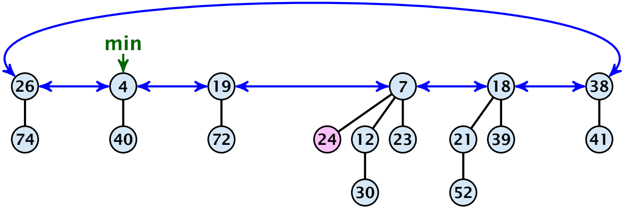

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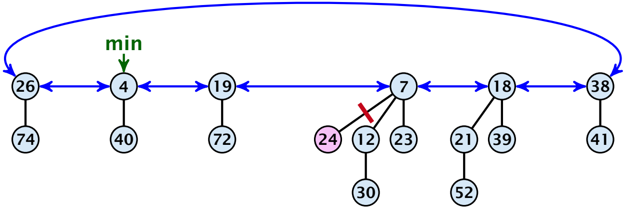

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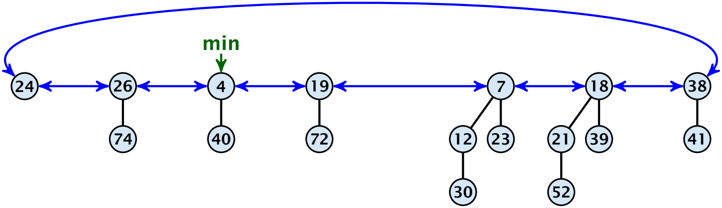

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- Decrease key-value of element x reference by h.
- Cut the parent edge of x, and make x into a root.
- Adjust min-pointers, if necessary.
- Execute the following:

$p \leftarrow \text{parent}[x]$;
while (p is marked)
  $pp \leftarrow \text{parent}[p]$;
  cut of p; make it into a root; unmark it;
  $p \leftarrow pp$;
if p is unmarked and not a root mark it;

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

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- Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2.

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- $t' = t + \ell$, as every cut creates one new root.

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Amortized cost:

- $t' = t + \ell$, as every cut creates one new root.
- $m' \leq m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.

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- $\Delta\Phi \leq \ell + 2(-\ell + 2) = 4 - \ell$

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  $$c_2(\ell+1) + c(4-\ell) \leq (c_2 - c)\ell + 4c + c_2 = \mathcal{O}(1)\,,$$

  if $c \geq c_2$.

Delete node

$H.\mathbf{delete}(x)$:

- decrease value of x to $-\infty$.
- delete-min.

Amortized cost: $\mathcal{O}(D_n)$

- $\mathcal{O}(1)$ for decrease-key.
- $\mathcal{O}(D_n)$ for delete-min.

8.3 Fibonacci Heaps

Lemma 32
Let x be a node with degree k and let $y_1, \ldots, y_k$ denote the children of x in the order that they were linked to x. Then

$$\text{degree}(y_i) \geq \begin{cases} 0 & \textit{if } i = 1 \\ i - 2 & \textit{if } i > 1 \end{cases}$$

8.3 Fibonacci Heaps

Proof

- When y_i was linked to x, at least $y_1, \ldots, y_{i-1}$ were already linked to x.

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- When y_i was linked to x, at least $y_1, \ldots, y_{i-1}$ were already linked to x.
- Hence, at this time $\text{degree}(x) \geq i - 1$, and therefore also $\text{degree}(y_i) \geq i - 1$ as the algorithm links nodes of equal degree only.

8.3 Fibonacci Heaps

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8.3 Fibonacci Heaps

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- Therefore, $\text{degree}(y_i) \geq i - 2$.

8.3 Fibonacci Heaps

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$$\begin{aligned} s_k &= 2 + \sum_{i=2}^{k} \mathrm{size}(y_i) \\ &\geq 2 + \sum_{i=2}^{k} s_{i-2} \\ &= 2 + \sum_{i=0}^{k-2} s_i \end{aligned}$$

8.3 Fibonacci Heaps

Definition 33

Consider the following non-standard Fibonacci type sequence:

$$F_k = \begin{cases} 1 & \text{if } k = 0 \\ 2 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

Facts:

1. $F_k \geq \phi^k$.
2. For $k \geq 2$: $F_k = 2 + \sum_{i=0}^{k-2} F_i$.

The above facts can be easily proved by induction. From this it follows that $s_k \geq F_k \geq \phi^k$, which gives that the maximum degree in a Fibonacci heap is logarithmic.

k=0: $1 = F_0 \geq \Phi^0 = 1$

k=1: $2 = F_1 \geq \Phi^1 \approx 1.61$

k-2,k-1→ k: $F_k = F_{k-1} + F_{k-2} \geq \Phi^{k-1} + \Phi^{k-2} = \Phi^{k-2}\overbrace{(\Phi+1)}^{\Phi^2} = \Phi^k$

k=2: $3 = F_2 = 2 + 1 = 2 + F_0$

k-1→ k: $F_k = F_{k-1} + F_{k-2} = 2 + \sum_{i=0}^{k-3} F_i + F_{k-2} = 2 + \sum_{i=0}^{k-2} F_i$

9 Union Find

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- **$\mathcal{P}.\text{find}(x)$:** Given a handle for an element x; find the set that contains x. Returns a representative/identifier for this set.
- **$\mathcal{P}.\text{union}(x, y)$:** Given two elements x, and y that are currently in sets S_x and S_y, respectively, the function replaces S_x and S_y by $S_x \cup S_y$ and returns an identifier for the new set.

9 Union Find

Applications:

- Keep track of the connected components of a dynamic graph that changes due to insertion of nodes and edges.

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- Kruskals Minimum Spanning Tree Algorithm

Algorithm 1 Kruskal-MST$(G = (V, E), w)$

$A \leftarrow \emptyset$;
for all $v \in V$ **do**
    $v.\text{set} \leftarrow \mathcal{P}.\text{makeset}(v.\text{label})$
sort edges in non-decreasing order of weight w
for all $(u, v) \in E$ in non-decreasing order **do**
    if $\mathcal{P}.\text{find}(u.\text{set}) \neq \mathcal{P}.\text{find}(v.\text{set})$ **then**
        $A \leftarrow A \cup \{(u, v)\}$
        $\mathcal{P}.\text{union}(u.\text{set}, v.\text{set})$

List Implementation

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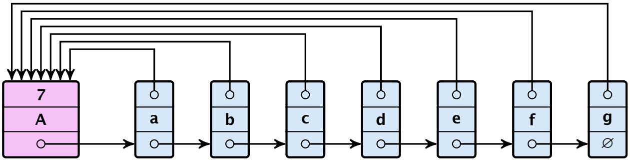

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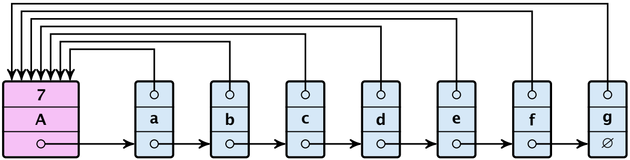

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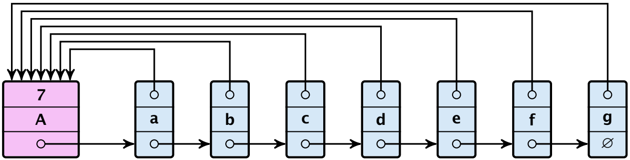

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- Time: $\min\{|S_x|, |S_y|\}$.

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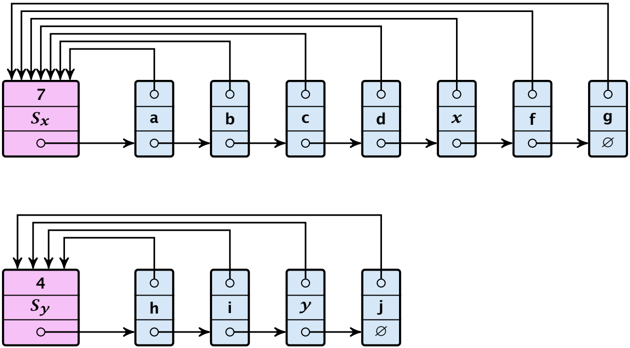

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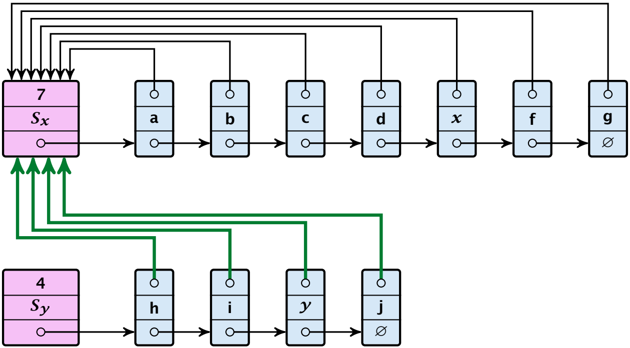

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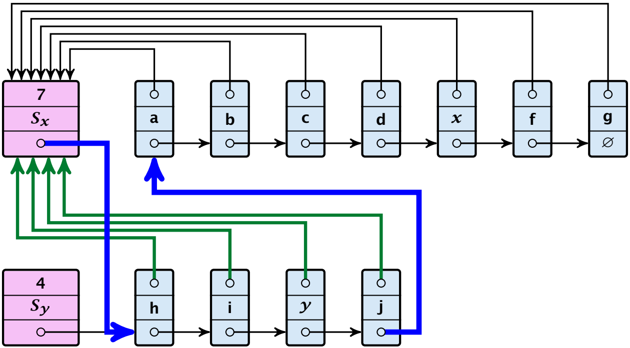

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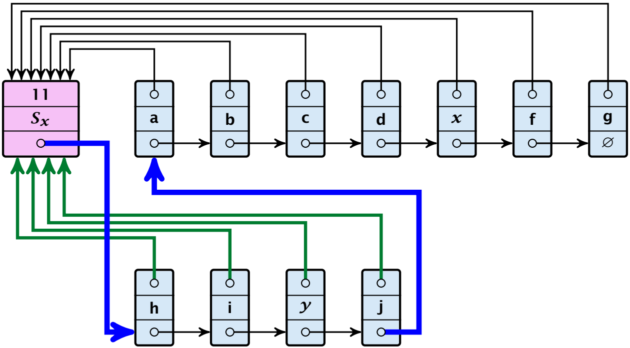

List Implementation

Running times:

- $\text{find}(x)$: constant
- $\text{makeset}(x)$: constant
- $\text{union}(x, y)$: $\mathcal{O}(n)$, where n denotes the number of elements contained in the set system.

List Implementation

Lemma 34

The list implementation for the ADT union find fulfills the following amortized time bounds:

- $\text{find}(x)$: $\mathcal{O}(1)$.
- $\text{makeset}(x)$: $\mathcal{O}(\log n)$.
- $\text{union}(x, y)$: $\mathcal{O}(1)$.

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- If we can find a charging scheme that guarantees that balances always stay positive the amortized time bounds are proven.

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- Later operations charge the account but the balance never drops below zero.

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- Charge c to every element in set S_x.

List Implementation

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Proof.

Whenever an element x is charged the number of elements in x's set doubles. This can happen at most $\lfloor \log n \rfloor$ times. □

Implementation via Trees

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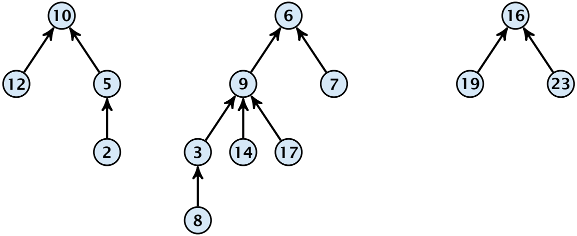

Set system $\{2,5,10,12\}$, $\{3,6,7,8,9,14,17\}$, $\{16,19,23\}$.

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- Start at element x in the tree. Go upwards until you reach the root.
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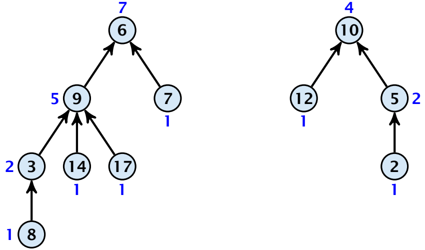

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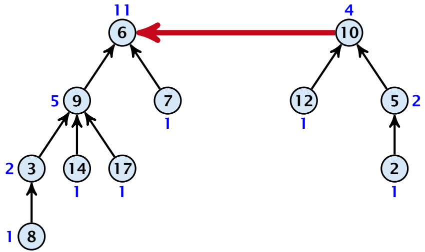

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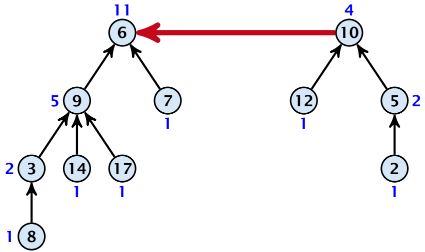

- Time: constant for $\text{link}(a, b)$ plus two find-operations.

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The running time (non-amortized!!!) for $\text{find}(x)$ *is* $\mathcal{O}(\log n)$.

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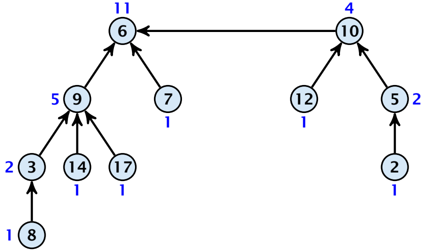

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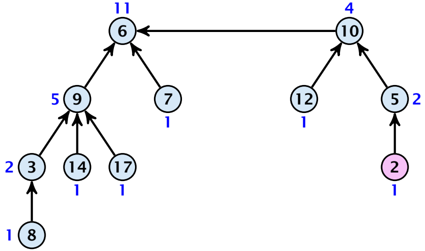

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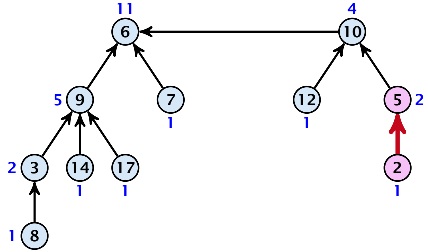

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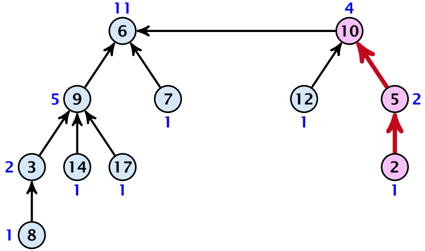

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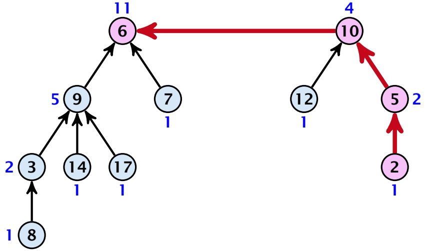

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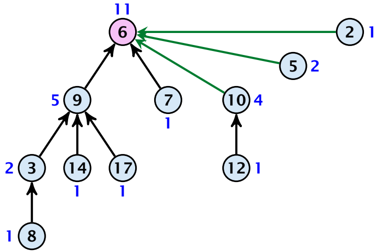
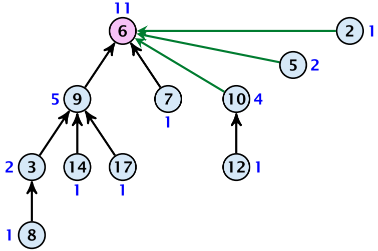


Path Compression

find(x):

- Go upward until you find the root.
- Re-attach all visited nodes as children of the root.
- Speeds up successive find-operations.



- Note that the size-fields now only give an upper bound on the size of a sub-tree.

Path Compression

Asymptotically the cost for a find-operation does not increase due to the path compression heuristic.

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However, for a worst-case analysis there is no improvement on the running time. It can still happen that a find-operation takes time $\mathcal{O}(\log n)$.

Amortized Analysis

Definitions:

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- $\text{size}(v) :=$ the number of nodes that were in the sub-tree rooted at v when v became the child of another node (or the number of nodes if v is the root).

  Note that this is the same as the size of v's subtree in the case that there are no find-operations.

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Lemma 37
The rank of a parent must be strictly larger than the rank of a child.

Amortized Analysis

Lemma 38
There are at most $n/2^s$ *nodes of rank* s*.*

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Proof.

- Let's say a node v **sees** node x if v is in x's sub-tree at the time that x becomes a child.

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Amortized Analysis

Lemma 38

There are at most $n/2^s$ nodes of rank s.

Proof.

- Let's say a node v sees node x if v is in x's sub-tree at the time that x becomes a child.
- A node v sees at most one node of rank s during the running time of the algorithm.

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- This holds because the rank-sequence of the roots of the different trees that contain v during the running time of the algorithm is a strictly increasing sequence.

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- A node v sees at most one node of rank s during the running time of the algorithm.
- This holds because the rank-sequence of the roots of the different trees that contain v during the running time of the algorithm is a strictly increasing sequence.
- Hence, every node *sees* at most one rank s node, but every rank s node is seen by at least 2^s different nodes. □

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Theorem 39
Union find with path compression fulfills the following amortized running times:

- $\text{makeset}(x) : \mathcal{O}(\log^*(n))$
- $\text{find}(x) : \mathcal{O}(\log^*(n))$
- $\text{union}(x, y) : \mathcal{O}(\log^*(n))$

Amortized Analysis

In the following we assume $n \geq 2$.

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- The maximum non-empty rank group is $\log^*(\lfloor \log n \rfloor) \leq \log^*(n) - 1$ (which holds for $n \geq 2$).

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- The maximum non-empty rank group is $\log^*(\lfloor \log n \rfloor) \leq \log^*(n) - 1$ (which holds for $n \geq 2$).
- Hence, the total number of rank-groups is at most $\log^* n$.

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- create an account for every find-operation

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- If $\text{parent}[v]$ is the root we charge the cost to the find-account.
- If the group-number of $\text{rank}(v)$ is the same as that of $\text{rank}(\text{parent}[v])$ (before starting path compression) we charge the cost to the node-account of v.

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- If the group-number of $\text{rank}(v)$ is the same as that of $\text{rank}(\text{parent}[v])$ (before starting path compression) we charge the cost to the node-account of v.
- Otherwise we charge the cost to the find-account.

Amortized Analysis

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- After a node v is charged its parent-edge is re-assigned. The rank of the parent strictly increases.
- After some charges to v the parent will be in a larger rank-group. $\Longrightarrow$ v will never be charged again.
- The total charge made to a node in rank-group g is at most $\mathrm{tow}(g) - \mathrm{tow}(g-1) - 1 \le \mathrm{tow}(g)$.

Amortized Analysis

What is the total charge made to nodes?

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- The total charge is at most

$$\sum_g n(g) \cdot \mathrm{tow}(g) \ ,$$

where $n(g)$ is the number of nodes in group g.

Amortized Analysis

For $g \geq 1$ we have

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Hence,

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$$\sum_g n(g)\,\mathrm{tow}(g) \leq n(0)\,\mathrm{tow}(0) + \sum_{g\geq 1} n(g)\,\mathrm{tow}(g)$$

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Hence,

$$\sum_g n(g)\,\mathrm{tow}(g) \leq n(0)\,\mathrm{tow}(0) + \sum_{g \geq 1} n(g)\,\mathrm{tow}(g) \leq n \log^*(n)$$

Amortized Analysis

Without loss of generality we can assume that all makeset-operations occur at the start.

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This means if we inflate the cost of makeset to $\log^* n$ and add this to the node account of v then the balances of all node accounts will sum up to a positive value (this is sufficient to obtain an amortized bound).

Amortized Analysis

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The analysis is not tight. In fact it has been shown that the amortized time for the union-find data structure with path compression is $\mathcal{O}(\alpha(m,n))$, where $\alpha(m,n)$ is the inverse Ackermann function which grows a lot lot slower than $\log^* n$. (Here, we consider the average running time of m operations on at most n elements).

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There is also a lower bound of $\Omega(\alpha(m,n))$.

Amortized Analysis

$$A(x,y) = \begin{cases} y+1 & \text{if } x = 0 \\ A(x-1,1) & \text{if } y = 0 \\ A(x-1, A(x, y-1)) & \text{otw.} \end{cases}$$

$$\alpha(m,n) = \min\{i \geq 1 : A(i, \lfloor m/n \rfloor) \geq \log n\}$$

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$$\alpha(m,n) = \min\{i \geq 1 : A(i, \lfloor m/n \rfloor) \geq \log n\}$$

- $A(0,y) = y+1$
- $A(1,y) = y+2$
- $A(2,y) = 2y+3$
- $A(3,y) = 2^{y+3} - 3$
- $A(4,y) = \underbrace{2^{2^{2^2}}}_{y+3 \text{ times}} - 3$