WS 2024/25

Efficient Algorithms

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https://www.moodle.tum.de/course/view.php?id=100478

Winter Term 2024/25

Part I

Organizational Matters

Part I

Organizational Matters

- Modul: IN2003
- Name: “Efficient Algorithms and Data Structures”
  “Effiziente Algorithmen und Datenstrukturen”
- ECTS: 8 Credit points
- Lectures:
  - 4 SWS
    Mon 10:00–12:00 (Room Interim2)
    Fri 10:00–12:00 (Room Interim2)
- Webpage:
  `https://www.moodle.tum.de/course/view.php?id=100478`

- Required knowledge:
  - IN0001, IN0003
    "Introduction to Informatics 1/2"
    "Einführung in die Informatik 1/2"
  - IN0007
    "Fundamentals of Algorithms and Data Structures"
    "Grundlagen: Algorithmen und Datenstrukturen" (GAD)
  - IN0011
    "Basic Theoretic Informatics"
    "Einführung in die Theoretische Informatik" (THEO)
  - IN0015
    "Discrete Structures"
    "Diskrete Strukturen" (DS)
  - IN0018
    "Discrete Probability Theory"
    "Diskrete Wahrscheinlichkeitstheorie" (DWT)

The Lecturer

- Harald Räcke
- Email: raecke@in.tum.de
- Room: 03.09.044
- Office hours: (by appointment)

Tutorials

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- omar.abdelwanis@in.tum.de
- Room: 03.09.042
- Office hours: (by appointment)

1 Contents

- Foundations
    - Machine models
    - Efficiency measures
    - Asymptotic notation
    - Recursion

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    - Search trees
    - Hashing
    - Priority queues
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- Cuts/Flows
- Matchings

2 Literatur

- Alfred V. Aho, John E. Hopcroft, Jeffrey D. Ullman:
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  Addison-Wesley Publishing Company: Reading (MA), 1974
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- Donald E. Knuth:
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Part II

Foundations

3 Goals

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- Learn how to analyze and judge the efficiency of algorithms.
- Learn how to design efficient algorithms.

4 Modelling Issues

What do you measure?

- Memory requirement

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- Implementing and testing on representative inputs
  - How do you choose your inputs?
  - May be very time-consuming.
  - Very reliable results if done correctly.
  - Results only hold for a specific machine and for a specific set of inputs.

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  - May be very time-consuming.
  - Very reliable results if done correctly.
  - Results only hold for a specific machine and for a specific set of inputs.

- Theoretical analysis in a specific model of computation.
  - Gives asymptotic bounds like “this algorithm always runs in time $\mathcal{O}(n^2)$”.
  - Typically focuses on the worst case.
  - Can give lower bounds like “any comparison-based sorting algorithm needs at least $\Omega(n \log n)$ comparisons in the worst case”.

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The theoretical bounds are usually given by a function $f : \mathbb{N} \to \mathbb{N}$ that maps the input length to the running time (or storage space, comparisons, multiplications, program size etc.).

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Example 1

Suppose n numbers from the interval $\{1, \dots, N\}$ have to be sorted. In this case we usually say that the input length is n instead of e.g. $n \log N$, which would be the number of bits required to encode the input.

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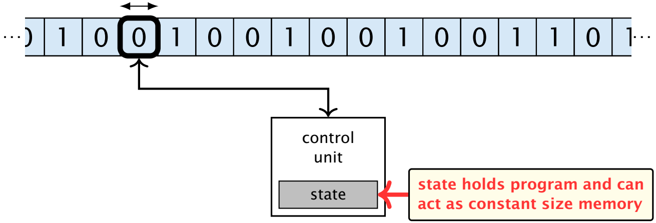
Version 2. is often easier, but focusing on one type of operation makes it more difficult to obtain meaningful results.

Turing Machine

- Very simple model of computation.



state holds program and can act as constant size memory

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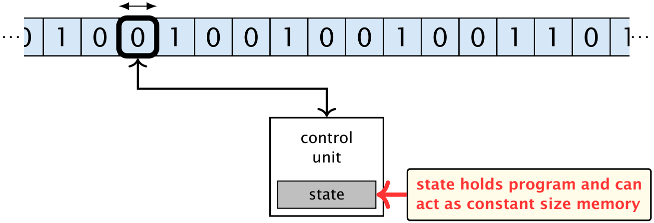

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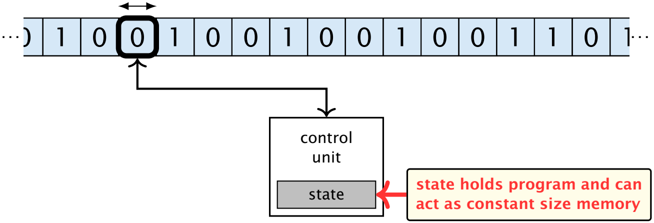

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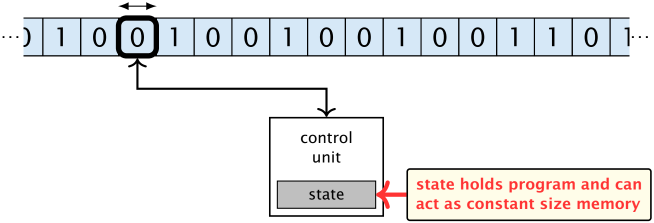

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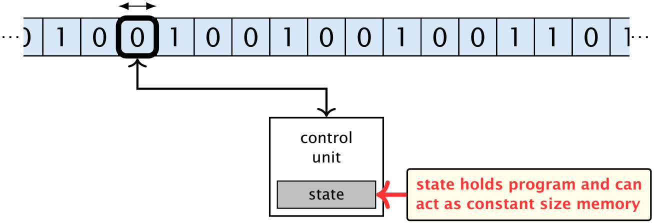
$\Longrightarrow$ **Not a good model for developing efficient algorithms.**



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Random Access Machine (RAM)

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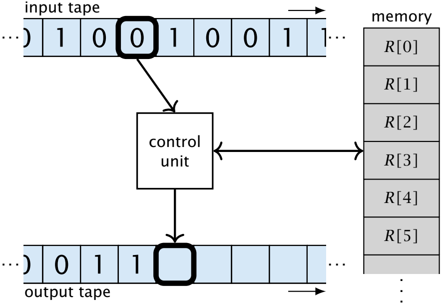

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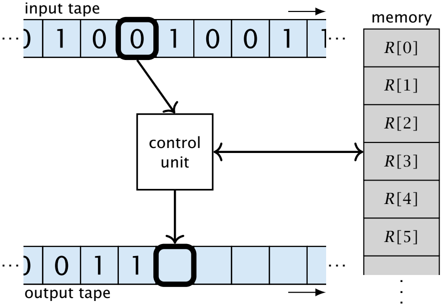

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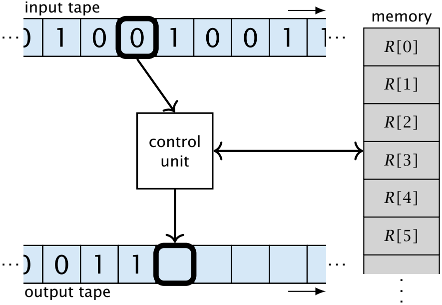

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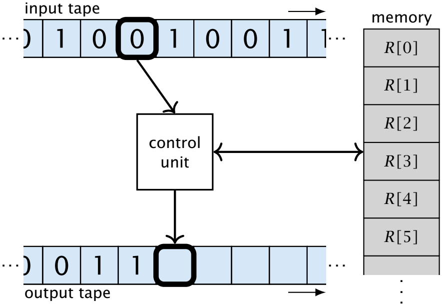

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  - $R[R[i]] := R[j]$
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  - $R[i]$ `:=` $R[j]$ `+` $R[k]$;
    $R[i]$ `:=` `-`$R[k]$;

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Model of Computation

- uniform cost model
  Every operation takes time 1.

The latter model is quite realistic as the word-size of a standard computer that handles a problem of size n must be at least $\log_2 n$ as otherwise the computer could either not store the problem instance or not address all its memory.

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Bounded word RAM model: cost is uniform but the largest value stored in a register may not exceed 2^w, where usually $w = \log_2 n$.

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4 Modelling Issues

Example 2

Algorithm 1 RepeatedSquaring(n)

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r ← 2;
for i = 1 → n do
        r ← r^2
return r
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There are different types of complexity bounds:

- best-case complexity:

$$C_{\mathrm{bc}}(n) := \min\{C(x) \mid |x| = n\}$$

Usually easy to analyze, but not very meaningful.

μ is a probability distribution over inputs of length n.

$C(x)$	cost of instance x
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I_n	set of instances of length n

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  more general: probability measure μ

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  The average cost of data structure operations over a worst case sequence of operations.
- randomized complexity:
  The algorithm may use random bits. Expected running time (over all possible choices of random bits) for a fixed input x. Then take the worst-case over all x with $|x| = n$.

$C(x)$	cost of instance x
$\|x\|$	input length of instance x
I_n	set of instances of length n

μ is a probability distribution over inputs of length n.

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- Running time should be expressed by simple functions.

Asymptotic Notation

Formal Definition

Let f, g denote functions from $\mathbb{N}$ to $\mathbb{R}^+$.

- $\mathcal{O}(f) = \{g \mid \exists c > 0\ \exists n_0 \in \mathbb{N}_0\ \forall n \geq n_0 \colon [g(n) \leq c \cdot f(n)]\}$
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- $g \in \mathcal{O}(f)$: $\quad 0 \leq \lim\limits_{n\to\infty} \dfrac{g(n)}{f(n)} < \infty$

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Abuse of notation

1. People write $f = \mathcal{O}(g)$, when they mean $f \in \mathcal{O}(g)$. This is **not** an equality (how could a function be equal to a set of functions).

2. In this context $f(n)$ does **not** mean the function f evaluated at n, but instead it is a shorthand for the function itself (leaving out domain and codomain and only giving the rule of correspondence of the function).

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4. People write $\mathcal{O}(f(n)) = \mathcal{O}(g(n))$, when they mean $\mathcal{O}(f(n)) \subseteq \mathcal{O}(g(n))$. Again this is not an equality.

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Asymptotic Notation in Equations

How do we interpret an expression like:

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Note that $\Theta(n)$ is on the right hand side, otw. this interpretation is wrong.

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How do we interpret an expression like:

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Regardless of how we choose the anonymous function $f(n) \in \mathcal{O}(n)$ there is an anonymous function $g(n) \in \Theta(n^2)$ that makes the expression true.

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How do we interpret an expression like:

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Careful!

Asymptotic Notation in Equations

The $\Theta(i)$-symbol on the left represents one anonymous function $f : \mathbb{N} \to \mathbb{R}^+$, and then $\sum_i f(i)$ is computed.

How do we interpret an expression like:

$$\sum_{i=1}^{n} \Theta(i) = \Theta(n^2)$$

Careful!

"It is understood" that every occurence of an $\mathcal{O}$-symbol (or $\Theta, \Omega, o, \omega$) on the left represents one anonymous function.

Hence, the left side is **not** equal to

$$\Theta(1) + \Theta(2) + \cdots + \Theta(n-1) + \Theta(n)$$

$\Theta(1) + \Theta(2) + \cdots + \Theta(n-1) + \Theta(n)$ does not really have a reasonable interpretation.

Asymptotic Notation in Equations

We can view an expression containing asymptotic notation as generating a set:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n)$$

represents

$$\Big\{f : \mathbb{N} \to \mathbb{R}^+ \mid f(n) = n^2 \cdot g(n) + h(n)$$

$$\text{with } g(n) \in \mathcal{O}(n) \text{ and } h(n) \in \mathcal{O}(\log n)\Big\}$$

Recall that according to the previous slide e.g. the expressions $\sum_{i=1}^{n} \mathcal{O}(i)$ and $\sum_{i=1}^{n/2} \mathcal{O}(i) + \sum_{i=n/2+1}^{n} \mathcal{O}(i)$ generate different sets.

Asymptotic Notation in Equations

Then an asymptotic equation can be interpreted as containement btw. two sets:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) = \Theta(n^2)$$

represents

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) \subseteq \Theta(n^2)$$

Note that the equation does not hold.

Asymptotic Notation

Lemma 3

Let f, g be functions with the property $\exists n_0 > 0\, \forall n \geq n_0 : f(n) > 0$ (the same for g). Then

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The expressions also hold for Ω*. Note that this means that* $f(n) + g(n) \in \Theta(\max\{f(n), g(n)\})$*.*

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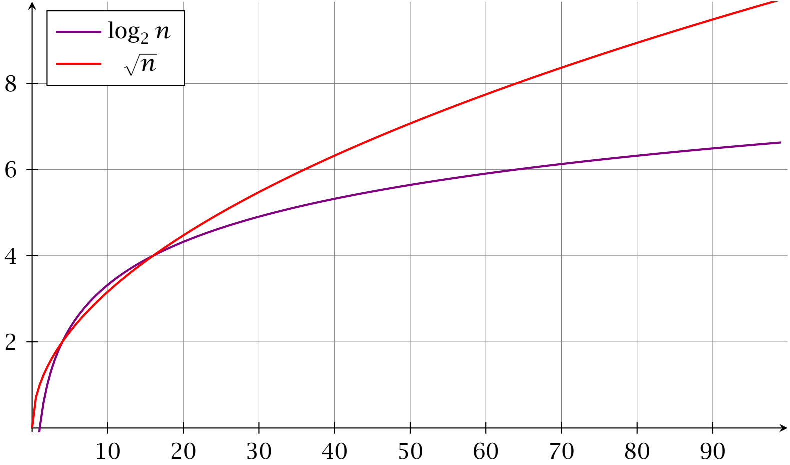
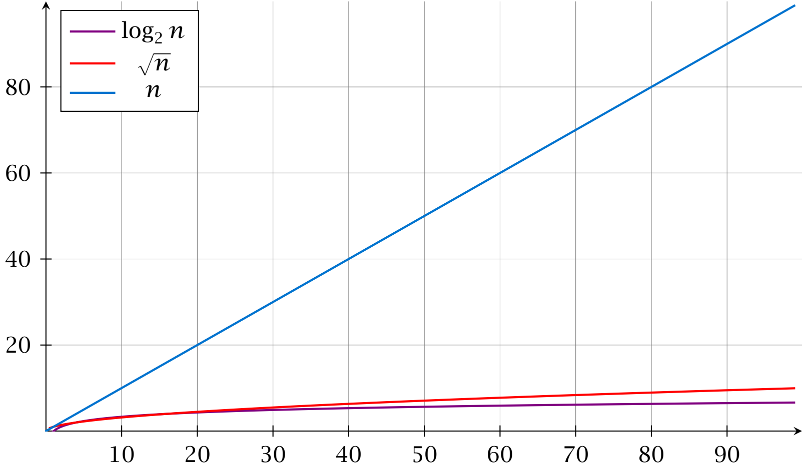
- For any constants a, b we have $\log_a n = \Theta(\log_b n)$. Therefore, we will usually ignore the base of a logarithm within asymptotic notation.
- In general $\log n = \log_2 n$, i.e., we use 2 as the default base for the logarithm.

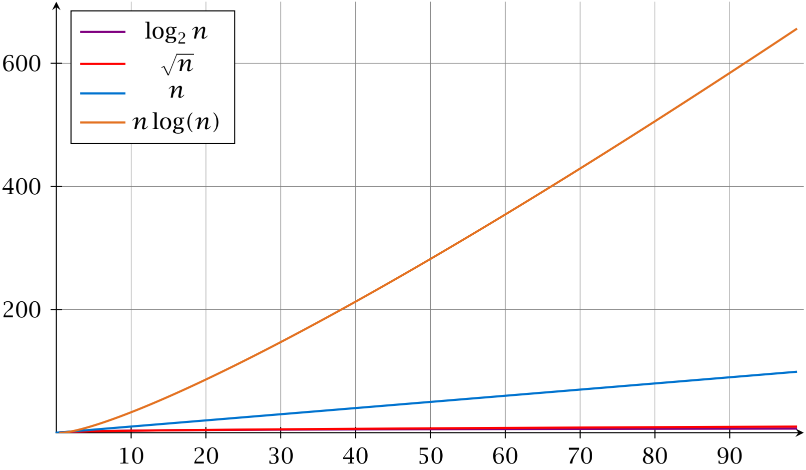
Funktionen



Funktionen



Funktionen



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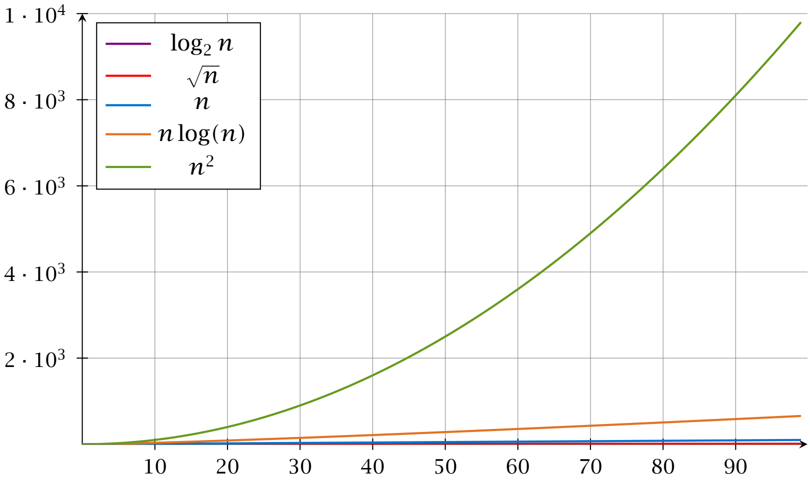
$1 \cdot 10^4$
$8 \cdot 10^3$
$6 \cdot 10^3$
$4 \cdot 10^3$
$2 \cdot 10^3$
10
20
30
40
50
60
70
80
90
$\log_2 n$
$\sqrt{n}$
n
$n \log(n)$
n^2

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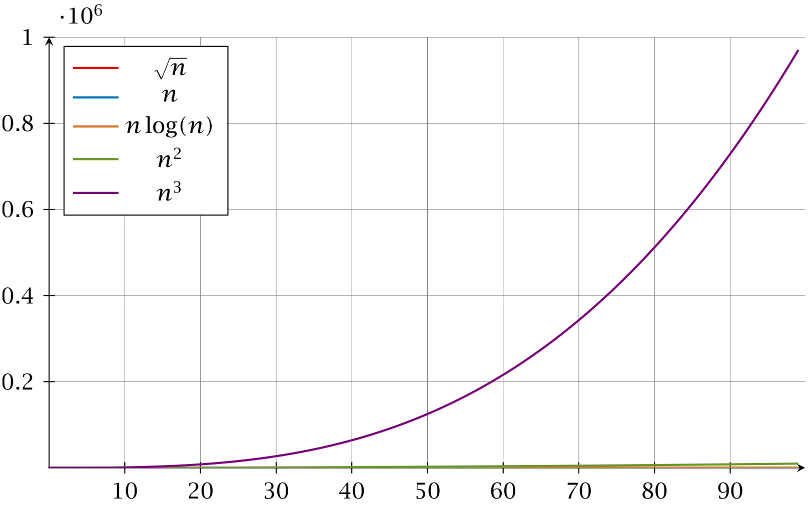

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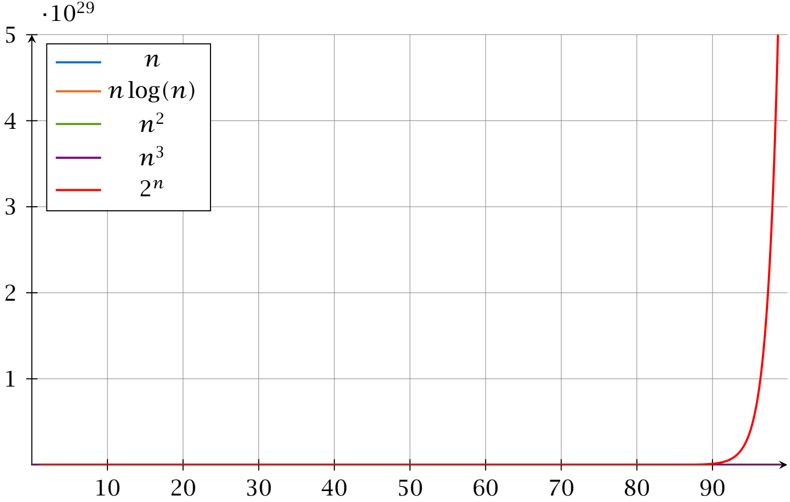

Laufzeiten

Funktion	***Eingabelänge n***							
$f(n)$	**10**	**10^2**	**10^3**	**10^4**	**10^5**	**10^6**	**10^7**	**10^8**
$\log n$	33ns	66ns	0.1μs	0.1μs	0.2μs	0.2μs	0.2μs	0.3μs
$\sqrt{n}$	32ns	0.1μs	0.3μs	1μs	3.1μs	10μs	31μs	0.1ms
n	100ns	1μs	10μs	0.1ms	1ms	10ms	0.1s	1s
$n \log n$	0.3μs	6.6μs	0.1ms	1.3ms	16ms	0.2s	2.3s	27s
$n^{3/2}$	0.3μs	10μs	0.3ms	10ms	0.3s	10s	5.2min	2.7h
n^2	1μs	0.1ms	10ms	1s	1.7min	2.8h	11d	3.2y
n^3	10μs	10ms	10s	2.8h	115d	317y	$3.2 \cdot 10^5$y	
1.1^n	26ns	0.1ms	$7.8 \cdot 10^{25}$y					
2^n	10μs	$4 \cdot 10^{14}$y						
$n!$	36ms	$3 \cdot 10^{142}$y						

1 Operation = 10ns; 100MHz

Alter des Universums: ca. $13.8 \cdot 10^9$y

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In general asymptotic classification of running times is a good measure for comparing algorithms:

- If the running time analysis is tight and actually occurs in practise (i.e., the asymptotic bound is not a purely theoretical worst-case bound), then the algorithm that has better asymptotic running time will always outperform a weaker algorithm for large enough values of n.

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  Clearly $f = o(g)$. However, as long as $\log n \leq 1000$ Algorithm B will be more efficient.

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Let f, g denote functions from $\mathbb{N}^d$ to $\mathbb{R}_0^+$.

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  (set of functions that asymptotically grow not faster than f)

Multiple Variables in Asymptotic Notation

Example 4

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6 Recurrences

Algorithm 2 mergesort(list L)

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$n \leftarrow \text{size}(L)$
if $n \le 1$ return $L$
$L_1 \leftarrow L[1 \cdots \lfloor \frac{n}{2} \rfloor]$
$L_2 \leftarrow L[\lfloor \frac{n}{2} \rfloor + 1 \cdots n]$
$\text{mergesort}(L_1)$
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This algorithm requires

$$T(n) = T\left(\left\lceil \frac{n}{2} \right\rceil\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \mathcal{O}(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + \mathcal{O}(n)$$

comparisons when $n > 1$ and 0 comparisons when $n \leq 1$.

Recurrences

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For this we need to solve the recurrence.

Methods for Solving Recurrences

1. **Guessing+Induction**
   Guess the right solution and prove that it is correct via induction. It needs experience to make the right guess.
2. **Master Theorem**
   For a lot of recurrences that appear in the analysis of algorithms this theorem can be used to obtain tight asymptotic bounds. It does not provide exact solutions.
3. **Characteristic Polynomial**
   Linear homogenous recurrences can be solved via this method.

Methods for Solving Recurrences

4. **Generating Functions**
   A more general technique that allows to solve certain types of linear inhomogenous relations and also sometimes non-linear recurrence relations.
5. **Transformation of the Recurrence**
   Sometimes one can transform the given recurrence relations so that it e.g. becomes linear and can therefore be solved with one of the other techniques.

6.1 Guessing+Induction

First we need to get rid of the $\mathcal{O}$-notation in our recurrence:

$$T(n) \leq \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

Informal way:

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One way of solving such a recurrence is to guess a solution, and check that it is correct by plugging it in.

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Formally, this is not correct if n is not a power of 2. Also even in this case one would need to do an induction proof.

6.1 Guessing+Induction

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

- Note that this proves the statement for $n = 2^k$, $k \in \mathbb{N}_{\geq 1}$, as the statement is wrong for $n = 1$.
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Hence, statement is true if we choose $d \geq c$.

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We consider the following recurrence instead of the original one:

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Note that we can do this as for constant-sized inputs the running time is always some constant (b in the above case).

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\boxed{\log\frac{9}{16}n = \log n + (\log 9 - 4)} & & = dn\log n + (\log 9 - 4)dn + 2d\log n + cn \\
\boxed{\log n \le \frac{n}{4}} & & \le dn\log n + (\log 9 - 3.5)dn + cn
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$$\begin{aligned}
& T(n) &\leq{}& 2T\left(\left\lceil \frac{n}{2}\right\rceil\right) + cn \\
& &\leq{}& 2\left(d\left\lceil \frac{n}{2}\right\rceil \log\left\lceil \frac{n}{2}\right\rceil\right) + cn \\
&\boxed{\left\lceil \tfrac{n}{2}\right\rceil \leq \tfrac{n}{2}+1} & \leq{}& 2(d(n/2+1)\log(n/2+1)) + cn \\
&\boxed{\tfrac{n}{2}+1 \leq \tfrac{9}{16}n} & \leq{}& dn\log\left(\frac{9}{16}n\right) + 2d\log n + cn \\
&\boxed{\log\tfrac{9}{16}n = \log n + (\log 9 - 4)} & ={}& dn\log n + (\log 9 - 4)dn + 2d\log n + cn \\
&\boxed{\log n \leq \tfrac{n}{4}} & \leq{}& dn\log n + (\log 9 - 3.5)dn + cn \\
& & \leq{}& dn\log n - 0.33dn + cn
\end{aligned}$$

6.1 Guessing+Induction

We also make a guess of $T(n) \le dn\log n$ and get

$$
\begin{aligned}
& T(n) &&\le 2T\left(\left\lceil\frac{n}{2}\right\rceil\right) + cn\\
& &&\le 2\left(d\left\lceil\frac{n}{2}\right\rceil\log\left\lceil\frac{n}{2}\right\rceil\right) + cn\\
\boxed{\left\lceil\tfrac{n}{2}\right\rceil \le \tfrac{n}{2}+1} & &&\le 2\big(d(n/2+1)\log(n/2+1)\big) + cn\\
\boxed{\tfrac{n}{2}+1 \le \tfrac{9}{16}n} & &&\le dn\log\left(\frac{9}{16}n\right) + 2d\log n + cn\\
\boxed{\log\tfrac{9}{16}n = \log n + (\log 9 - 4)} & &&= dn\log n + (\log 9 - 4)dn + 2d\log n + cn\\
\boxed{\log n \le \tfrac{n}{4}} & &&\le dn\log n + (\log 9 - 3.5)dn + cn\\
& &&\le dn\log n - 0.33dn + cn\\
& &&\le dn\log n
\end{aligned}
$$

for a suitable choice of d.

6.2 Master Theorem

Note that the cases do not cover all possibilities.

Lemma 5

Let $a \geq 1, b > 1$ *and* $\epsilon > 0$ *denote constants. Consider the recurrence*

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)\ .$$

Case 1.

If $f(n) = \mathcal{O}(n^{\log_b(a)-\epsilon})$ *then* $T(n) = \Theta(n^{\log_b a})$.

Case 2.

If $f(n) = \Theta(n^{\log_b(a)} \log^k n)$ *then* $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$, $k \geq 0$.

Case 3.

If $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ *and for sufficiently large* n $af(\frac{n}{b}) \leq cf(n)$ *for some constant* $c < 1$ *then* $T(n) = \Theta(f(n))$.

6.2 Master Theorem

We prove the Master Theorem for the case that n is of the form b^{ℓ}, and we assume that the non-recursive case occurs for problem size 1 and incurs cost 1.

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

The Recursion Tree

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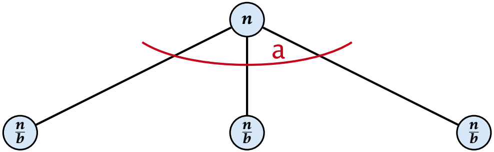
The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



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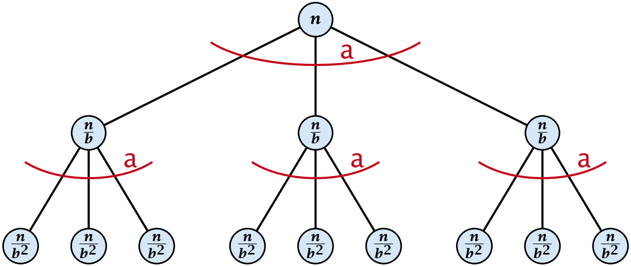
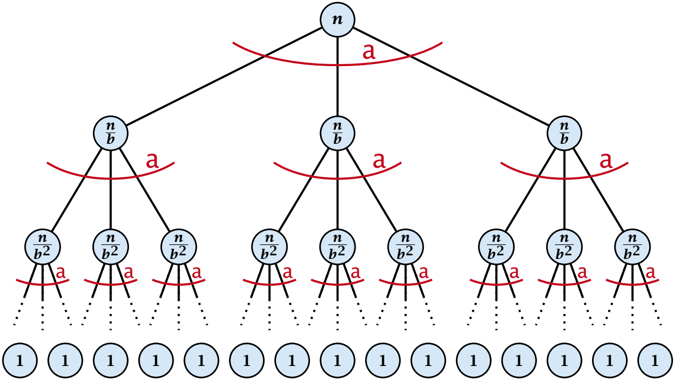
The running time of a recursive algorithm can be visualized by a recursion tree:



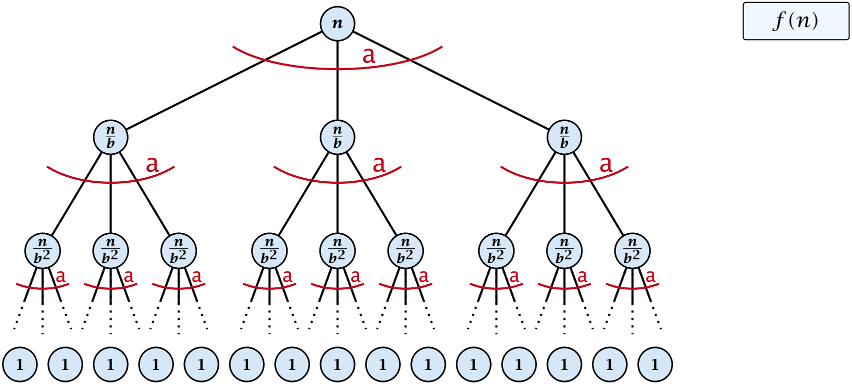
The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



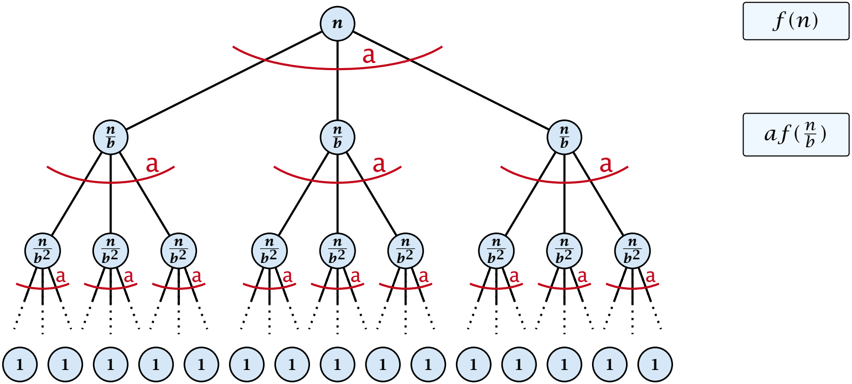
The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



The Recursion Tree

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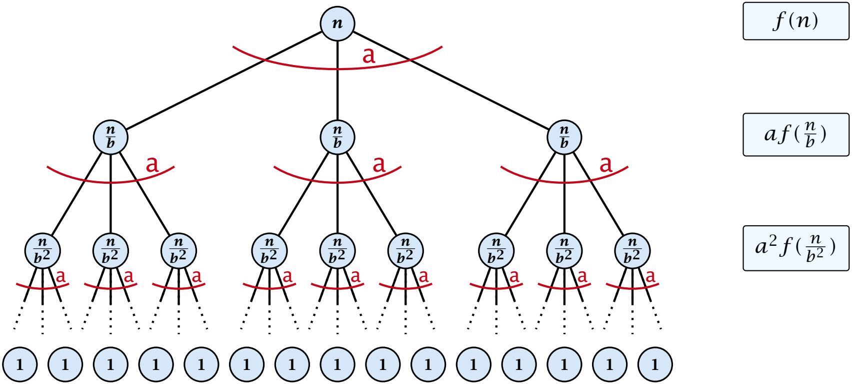

The Recursion Tree

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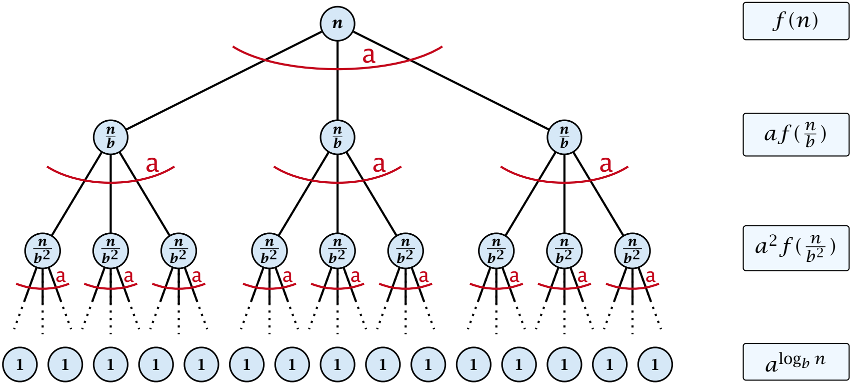
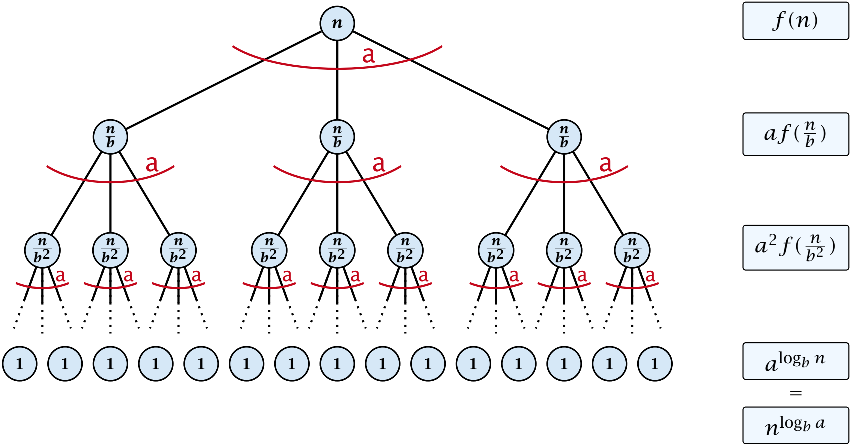


The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



$f(n)$

$af(\frac{n}{b})$

$a^2f(\frac{n}{b^2})$

$a^{\log_b n}$

$=$

$n^{\log_b a}$

6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) .$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a-\epsilon}$.

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a}$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a-\epsilon}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a-\epsilon}
\end{aligned}
$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a-\epsilon}$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a-\epsilon} \end{aligned}$$

$$b^{-i(\log_b a-\epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i}a^{-i}$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \\ &= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} \left(b^{\epsilon}\right)^i \end{aligned}$$

$$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}}$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \\
&= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} \left(b^{\epsilon}\right)^i
\end{aligned}
$$

$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}}$

$\boxed{\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}}$

Case 1. Now suppose that $f(n) \le cn^{\log_b a-\epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)$$

$$\le c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a-\epsilon}$$

$$\boxed{b^{-i(\log_b a-\epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i}a^{-i}} = cn^{\log_b a-\epsilon} \sum_{i=0}^{\log_b n-1} (b^{\epsilon})^i$$

$$\boxed{\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}} = cn^{\log_b a-\epsilon}(b^{\epsilon \log_b n} - 1)/(b^{\epsilon} - 1)$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a-\epsilon}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a-\epsilon} \\
\boxed{b^{-i(\log_b a-\epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i}a^{-i}} \quad &= cn^{\log_b a-\epsilon} \sum_{i=0}^{\log_b n-1} (b^{\epsilon})^i \\
\boxed{\textstyle\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}} \quad &= cn^{\log_b a-\epsilon}(b^{\epsilon\log_b n} - 1)/(b^{\epsilon} - 1) \\
&= cn^{\log_b a-\epsilon}(n^{\epsilon} - 1)/(b^{\epsilon} - 1)
\end{aligned}
$$

Case 1. Now suppose that $f(n) \le cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}$$

$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}}$

$$= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^\epsilon)^i$$

$\boxed{\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}}$

$$= cn^{\log_b a - \epsilon}(b^{\epsilon \log_b n} - 1)/(b^\epsilon - 1)$$

$$= cn^{\log_b a - \epsilon}(n^\epsilon - 1)/(b^\epsilon - 1)$$

$$= \frac{c}{b^\epsilon - 1} n^{\log_b a}(n^\epsilon - 1)/(n^\epsilon)$$

Case 1. Now suppose that $f(n) \le cn^{\log_b a-\epsilon}$.

$$\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
&\le c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a-\epsilon}
\end{aligned}$$

$$\boxed{b^{-i(\log_b a-\epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i}a^{-i}} \quad = cn^{\log_b a-\epsilon} \sum_{i=0}^{\log_b n-1} (b^\epsilon)^i$$

$$\boxed{\textstyle\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}} \quad = cn^{\log_b a-\epsilon}(b^{\epsilon\log_b n}-1)/(b^\epsilon-1)$$

$$\begin{aligned}
&= cn^{\log_b a-\epsilon}(n^\epsilon-1)/(b^\epsilon-1) \\
&= \frac{c}{b^\epsilon-1}n^{\log_b a}(n^\epsilon-1)/(n^\epsilon)
\end{aligned}$$

Hence,

$$T(n) \le \left(\frac{c}{b^\epsilon-1}+1\right)n^{\log_b(a)}$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a-\epsilon}$.

$$\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a-\epsilon} \\
\boxed{b^{-i(\log_b a-\epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i}a^{-i}} \quad &= cn^{\log_b a-\epsilon} \sum_{i=0}^{\log_b n-1} (b^\epsilon)^i \\
\boxed{\textstyle\sum_{i=0}^k q^i = \frac{q^{k+1}-1}{q-1}} \quad &= cn^{\log_b a-\epsilon}(b^{\epsilon \log_b n} - 1)/(b^\epsilon - 1) \\
&= cn^{\log_b a-\epsilon}(n^\epsilon - 1)/(b^\epsilon - 1) \\
&= \frac{c}{b^\epsilon - 1} n^{\log_b a}(n^\epsilon - 1)/(n^\epsilon)
\end{aligned}$$

Hence,

$$T(n) \leq \left(\frac{c}{b^\epsilon - 1} + 1\right) n^{\log_b (a)}$$

$$\boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a}).}$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}$.

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$$T(n) - n^{\log_b a}$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}
\end{aligned}
$$

Case 2. Now suppose that $f(n) \le cn^{\log_b a}$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\ &\le c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\ &= cn^{\log_b a} \sum_{i=0}^{\log_b n-1} 1 \end{aligned}$$

Case 2. Now suppose that $f(n) \le cn^{\log_b a}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
&\le c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\
&= cn^{\log_b a} \sum_{i=0}^{\log_b n-1} 1 \\
&= cn^{\log_b a} \log_b n
\end{aligned}
$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\ &= cn^{\log_b a} \sum_{i=0}^{\log_b n-1} 1 \\ &= cn^{\log_b a} \log_b n \end{aligned}$$

Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n)$$

Case 2. Now suppose that $f(n) \le cn^{\log_b a}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\
&= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\
&= cn^{\log_b a} \log_b n
\end{aligned}
$$

Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n)$$

$$\boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log n).}$$

Case 2. Now suppose that $f(n) \geq cn^{\log_b a}$.

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

Case 2. Now suppose that $f(n) \geq cn^{\log_b a}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
&\geq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}
\end{aligned}
$$

Case 2. Now suppose that $f(n) \geq cn^{\log_b a}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\geq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\
&= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1
\end{aligned}
$$

Case 2. Now suppose that $f(n) \geq cn^{\log_b a}$.

$$\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
&\geq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\
&= cn^{\log_b a} \sum_{i=0}^{\log_b n-1} 1 \\
&= cn^{\log_b a} \log_b n
\end{aligned}$$

Case 2. Now suppose that $f(n) \geq cn^{\log_b a}$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\ &\geq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\ &= cn^{\log_b a} \sum_{i=0}^{\log_b n-1} 1 \\ &= cn^{\log_b a} \log_b n \end{aligned}$$

Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n)$$

Case 2. Now suppose that $f(n) \geq cn^{\log_b a}$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\ &\geq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\ &= cn^{\log_b a} \sum_{i=0}^{\log_b n-1} 1 \\ &= cn^{\log_b a} \log_b n \end{aligned}$$

Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n) \qquad \boxed{\Rightarrow T(n) = \Omega(n^{\log_b a} \log n).}$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}(\log_b(n))^k$.

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$$T(n) - n^{\log_b a}$$

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

Case 2. Now suppose that $f(n) \le cn^{\log_b a}(\log_b(n))^k$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k
\end{aligned}
$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}(\log_b(n))^k$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k
\end{aligned}
$$

$$\boxed{n = b^\ell \Rightarrow \ell = \log_b n}$$

Case 2. Now suppose that $f(n) \le cn^{\log_b a}(\log_b(n))^k$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\ &\le c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \\ &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \end{aligned}$$

$$\boxed{n = b^\ell \Rightarrow \ell = \log_b n}$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}(\log_b(n))^k$.

$$\begin{aligned}T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)\\ &\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k\\ &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k\\ &= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k\end{aligned}$$

$\boxed{n = b^\ell \Rightarrow \ell = \log_b n}$

Case 2. Now suppose that $f(n) \le cn^{\log_b a}(\log_b(n))^k$.

$$\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)\\
&\le c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k\\
&= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k\\
&= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k\\
&= cn^{\log_b a} \sum_{i=1}^{\ell} i^k
\end{aligned}$$

$$\boxed{n = b^\ell \Rightarrow \ell = \log_b n}$$

Case 2. Now suppose that $f(n) \le cn^{\log_b a}(\log_b(n))^k$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \\
&= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\
&= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \\
&= cn^{\log_b a} \boxed{\sum_{i=1}^{\ell} i^k} \approx \tfrac{1}{k}\ell^{k+1}
\end{aligned}
$$

$n = b^\ell \Rightarrow \ell = \log_b n$

Case 2. Now suppose that $f(n) \le cn^{\log_b a}(\log_b(n))^k$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
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$$\boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n).}$$

Case 3. Now suppose that $f(n) \geq dn^{\log_b a+\epsilon}$, and that for sufficiently large n: $af(n/b) \leq cf(n)$, for $c < 1$.

Where did we use $f(n) \geq \Omega(n^{\log_b a+\epsilon})$?

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$$
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T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq \sum_{i=0}^{\log_b n-1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\
&\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a})
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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq \sum_{i=0}^{\log_b n-1} c^i f(n) + \mathcal{O}(n^{\log_b a})$$

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$$\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a})$$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

$$\boxed{\Rightarrow T(n) = \Theta(f(n)).}$$

Where did we use $f(n) \geq \Omega(n^{\log_b a+\epsilon})$?

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

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1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1_{1} & 1 & B \\
\hline
 & & & & & & & & 0 &
\end{array}$$

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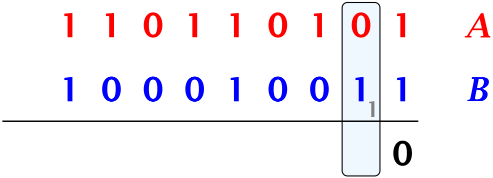

1	1	0	1	1	0	1	0	1	A
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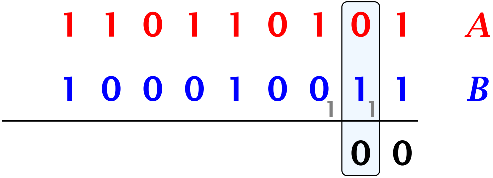

$$\begin{array}{ccccccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0_1 & 1_1 & 1 & & B \\ \hline & & & & & & & 0 & 0 & & \end{array}$$

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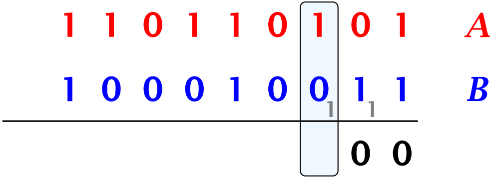

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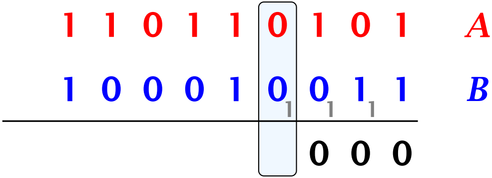

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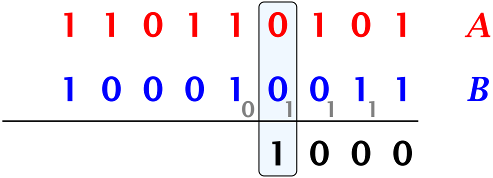

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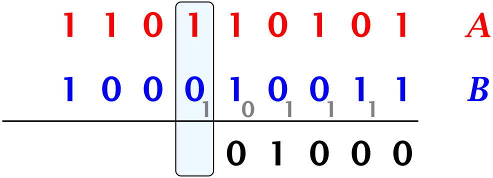

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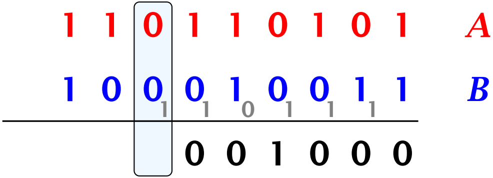

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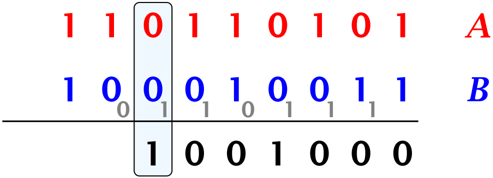

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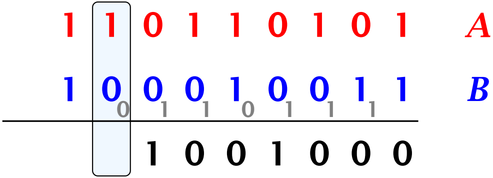

$$\begin{array}{ccccccccc l}
1 & \boxed{1} & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\
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 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & & \\
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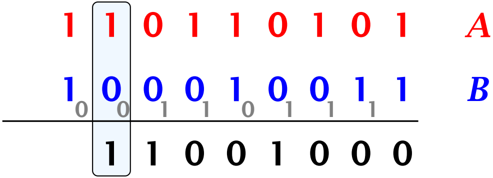

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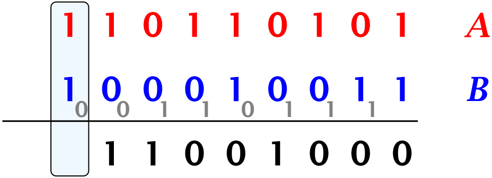

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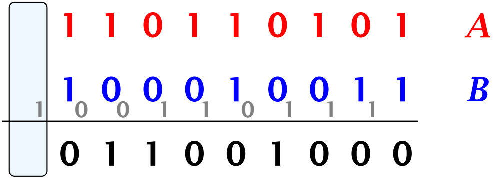

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\hline
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & &
\end{array}$$

This gives that two n-bit integers can be added in time $\mathcal{O}(n)$.

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

- This is also nown as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \le 2n$ bits.

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Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

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1 & 0 & 0 & 0 & 1 & \times & 1 & \boxed{0} & 1 & 1 \\ \hline
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- Computing intermediate results: $\mathcal{O}(nm)$.

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Time requirement:

- Computing intermediate results: $\mathcal{O}(nm)$.
- Adding m numbers of length $\leq 2n$: $\mathcal{O}((m + n)m) = \mathcal{O}(nm)$.

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A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k.

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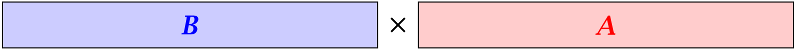

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$$b_{n-1} \quad \cdots \quad b_0 \times a_{n-1} \quad \cdots \quad a_0$$

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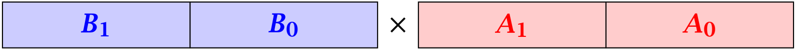

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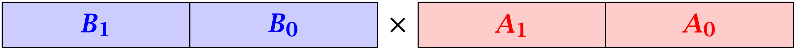

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Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

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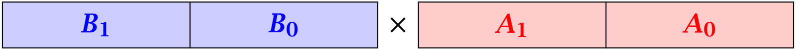

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Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 B_0$$

Example: Multiplying Two Integers

```
Algorithm 3 mult($(A, B)$)
if $|A| = |B| = 1$ then
    return $a_0 \cdot b_0$
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$Z_2 \leftarrow \mathrm{mult}(A_1, B_1)$
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Example: Multiplying Two Integers

Algorithm 3 mult(A, B)

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if $|A| = |B| = 1$ then                              $\mathcal{O}(1)$
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We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) \ .$$

Example: Multiplying Two Integers

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $T(n) = \Theta(n^{\log_b a})$
- Case 2: $f(n) = \Theta(n^{\log_b a}\log^k n)$ $T(n) = \Theta(n^{\log_b a}\log^{k+1} n)$
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In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$.

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$\Longrightarrow$ Not better then the “school method”.

Example: Multiplying Two Integers

We can use the following identity to compute Z_1:

A more precise (correct) analysis would say that computing Z_1 needs time $T(\frac{n}{2}+1)+\mathcal{O}(n)$.

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We can use the following identity to compute Z_1:

$$\begin{aligned} Z_1 &= A_1B_0 + A_0B_1 \\ &= (A_0 + A_1) \cdot (B_0 + B_1) - A_1B_1 - A_0B_0 \end{aligned}$$

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Hence,

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if $|A| = |B| = 1$ then
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$\mathcal{O}(1)$

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- Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $\quad T(n) = \Theta(n^{\log_b a})$
- Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $\quad T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
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Again we are in Case 1. We get a running time of $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$.

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A huge improvement over the "school method".

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Consider the recurrence relation:

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Note that we ignore boundary conditions for the moment.

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- First determine all solutions that satisfy recurrence relation.
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- First consider the homogenous case.

The Homogenous Case

The solution space

$$S = \left\{ \mathcal{T} = T[1], T[2], T[3], \dots \;\middle|\; \mathcal{T} \text{ fulfills recurrence relation} \right\}$$

is a vector space.

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How do we find a non-trivial solution?

We guess that the solution is of the form λ^n, $\lambda \neq 0$, and see what happens. In order for this guess to fulfill the recurrence we need

$$c_0\lambda^n + c_1\lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \cdots + c_k \cdot \lambda^{n-k} = 0$$

for all $n \geq k$.

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Dividing by λ^{n-k} gives that all these constraints are identical to

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Let $\lambda_1, \dots, \lambda_k$ be the k (complex) roots of $P[\lambda]$. Then, because of the vector space property

$$\alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \cdots + \alpha_k\lambda_k^n$$

is a solution for arbitrary values α_i.

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Lemma 6
Assume that the characteristic polynomial has k distinct roots $\lambda_1, \dots, \lambda_k$. Then all solutions to the recurrence relation are of the form

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We show that the above set of solutions contains one solution for every choice of boundary conditions.

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Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i's$ such that these conditions are met:

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We show that the column vectors are linearly independent. Then the above equation has a solution.

Computing the Determinant

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} =$$

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Computing the Determinant

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$$= \prod_{i=1}^{k} \lambda_i \cdot \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{k-2} & \lambda_1^{k-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{k-2} & \lambda_2^{k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-2} & \lambda_k^{k-1} \end{vmatrix}$$

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$$\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2} \end{vmatrix}$$

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$$\prod_{i=2}^{k}(\lambda_i-\lambda_1)\cdot\begin{vmatrix} 1 & \lambda_2 & \cdots & \lambda_2^{k-3} & \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-3} & \lambda_k^{k-2} \end{vmatrix}$$

Computing the Determinant

Repeating the above steps gives:

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^{k} \lambda_i \cdot \prod_{i>\ell} (\lambda_i - \lambda_\ell)$$

Hence, if all λ_i's are different, then the determinant is non-zero.

The Homogeneous Case

What happens if the roots are not all distinct?

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Suppose we have a root λ_i with multiplicity (Vielfachheit) at least 2. Then not only is λ_i^n a solution to the recurrence but also $n\lambda_i^n$.

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To see this consider the polynomial

$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_k\lambda^{n-k}$$

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Since λ_i is a root we can write this as $Q[\lambda] \cdot (\lambda - \lambda_i)^2$. Calculating the derivative gives a polynomial that still has root λ_i.

This means

$$c_0 n\lambda_i^{n-1} + c_1(n-1)\lambda_i^{n-2} + \cdots + c_k(n-k)\lambda_i^{n-k-1} = 0$$

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Hence,

$$c_0\underbrace{n\lambda_i^{n}}_{T[n]} + c_1\underbrace{(n-1)\lambda_i^{n-1}}_{T[n-1]} + \cdots + c_k\underbrace{(n-k)\lambda_i^{n-k}}_{T[n-k]} = 0$$

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We can continue $j-1$ times.

Hence, $n^\ell\lambda_i^n$ is a solution for $\ell \in 0,\dots,j-1$.

The Homogeneous Case

Lemma 7

Let $P[\lambda]$ denote the characteristic polynomial to the recurrence

$$c_0T[n] + c_1T[n-1] + \cdots + c_kT[n-k] = 0$$

Let λ_i, $i = 1, \ldots, m$ be the (complex) roots of $P[\lambda]$ with multiplicities ℓ_i. Then the general solution to the recurrence is given by

$$T[n] = \sum_{i=1}^{m} \sum_{j=0}^{\ell_i - 1} \alpha_{ij} \cdot (n^j \lambda_i^n) \ .$$

The full proof is omitted. We have only shown that any choice of α_{ij}'s is a solution to the recurrence.

Example: Fibonacci Sequence

$$
\begin{aligned}
T[0] &= 0 \\
T[1] &= 1 \\
T[n] &= T[n-1] + T[n-2] \text{ for } n \geq 2
\end{aligned}
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Finding the roots, gives

$$\lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$$

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$$\alpha\left(\frac{1+\sqrt{5}}{2}\right)+\beta\left(\frac{1-\sqrt{5}}{2}\right)=1\Longrightarrow\alpha-\beta=\frac{2}{\sqrt{5}}$$

Example: Fibonacci Sequence

Hence, the solution is

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

The Inhomogeneous Case

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \cdots + c_kT(n-k) = f(n)$$

with $f(n) \neq 0$.

While we have a fairly general technique for solving homogeneous, linear recurrence relations the inhomogeneous case is different.

The Inhomogeneous Case

The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n) \enspace ,$$

where T_h is any solution to the homogeneous equation, and T_p is one particular solution to the inhomogeneous equation.

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There is no general method to find a particular solution.

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$$T[n] = T[n-1] + 1 \qquad T[0] = 1$$

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I get a completely determined recurrence if I add $T[0] = 1$ and $T[1] = 2$.

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Example: Characteristic polynomial:

$$\lambda^2 - 2\lambda + 1 = 0$$

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$T[0] = 1$ gives $\alpha = 1$.

$T[1] = 2$ gives $1 + \beta = 2 \Longrightarrow \beta = 1$.

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Shift:

$$\begin{aligned} T[n-1] &= 2T[n-2] - T[n-3] + 2(n-1) - 1 \\ &= 2T[n-2] - T[n-3] + 2n - 3 \end{aligned}$$

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and so on...

6.4 Generating Functions

Definition 8 (Generating Function)

Let $(a_n)_{n\geq 0}$ be a sequence. The corresponding

- generating function (Erzeugendenfunktion) is
$$F(z) := \sum_{n\geq 0} a_n z^n \ ;$$

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- exponential generating function (exponentielle Erzeugendenfunktion) is
$$F(z) := \sum_{n\geq 0} \frac{a_n}{n!} z^n \;.$$

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There are no convergence issues here.

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We view a power series as a function $f : \mathbb{C} \to \mathbb{C}$.

Then, it is important to think about convergence/convergence radius etc.

6.4 Generating Functions

What does $\sum_{n\geq 0} z^n = \frac{1}{1-z}$ mean in the algebraic view?

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$$\big(1 - z\big) \cdot \Big(\sum_{n\ge 0}^{\infty} z^n\Big) = 1 \ .$$

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This is well-defined.

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Formally the derivative of a formal power series $\sum_{n\geq 0} a_n z^n$ is defined as $\sum_{n\geq 0} n a_n z^{n-1}$.

The known rules for differentiation work for this definition. In particular, e.g. the derivative of $\frac{1}{1-z}$ is $\frac{1}{(1-z)^2}$.

Note that this requires a proof if we consider power series as algebraic objects. However, we did not prove this in the lecture.

We can compute the derivative:

$$\underbrace{\sum_{n\geq 1} n z^{n-1}}_{\sum_{n\geq 0}(n+1)z^n} = \frac{1}{(1-z)^2}$$

Hence, the generating function of the sequence $a_n = n+1$ is $1/(1-z)^2$.

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Hence, the generating function of the sequence $a_n = (n+1)(n+2)$ is $\frac{2}{(1-z)^3}$.

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Computing the k-th derivative of $\sum z^n$.

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$$\begin{aligned}\sum_{n\geq k} n(n-1)\cdot\ldots\cdot(n-k+1)z^{n-k} &= \sum_{n\geq 0}(n+k)\cdot\ldots\cdot(n+1)z^n\\ &= \frac{k!}{(1-z)^{k+1}}\ .\end{aligned}$$

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$$= \frac{k!}{(1-z)^{k+1}} \enspace .$$

Hence:

$$\sum_{n\geq 0}\binom{n+k}{k}z^n = \frac{1}{(1-z)^{k+1}} \enspace .$$

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Hence:

$$\sum_{n\geq 0}\binom{n+k}{k}z^n = \frac{1}{(1-z)^{k+1}}\ .$$

The generating function of the sequence $a_n = \binom{n+k}{k}$ is $\frac{1}{(1-z)^{k+1}}$.

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The generating function of the sequence $a_n = n$ is $\frac{z}{(1-z)^2}$.

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Hence,

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The generating function of the sequence $f_n = a^n$ is $\frac{1}{1-az}$.

Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

$$A(z)$$

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Hence, $a_n = n + 1$.

Some Generating Functions

n-th sequence element	*generating function*

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$\frac{1}{n!}$	e^z

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$f_{n-k} \quad (n \geq k); \quad 0$ otw.	$z^k F$

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$c^n f_n$	$F(cz)$

Solving Recursions with Generating Functions

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6. The coefficients of the resulting power series are the a_n.

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

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$$A(z) = 1 + \sum_{n\geq 1} (2a_{n-1}) z^n$$

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$$\begin{aligned} A(z) &= 1 + \sum_{n\geq 1}(2a_{n-1})z^n \\ &= 1 + 2z\sum_{n\geq 1}a_{n-1}z^{n-1} \end{aligned}$$

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$$A(z) = \frac{1}{1-2z}$$

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5. Rewrite $f(z)$ as a power series:

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A(z) &= \sum_{n\geq 0} a_n z^n \\
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&= 1 + \sum_{n\geq 1} (3a_{n-1} + n) z^n
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Example: $a_n = 3a_{n-1} + n, a_0 = 1$

2./3. Transform right hand side:

$$\begin{aligned} A(z) &= \sum_{n\geq 0} a_n z^n \\ &= a_0 + \sum_{n\geq 1} a_n z^n \\ &= 1 + \sum_{n\geq 1} (3a_{n-1} + n) z^n \\ &= 1 + 3z \sum_{n\geq 1} a_{n-1} z^{n-1} + \sum_{n\geq 1} n z^n \end{aligned}$$

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&= 1 + 3z \sum_{n\geq 0} a_n z^n + \sum_{n\geq 0} n z^n \\
&= 1 + 3zA(z) + \frac{z}{(1-z)^2}
\end{aligned}$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

4. Solve for $A(z)$:

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4. Solve for $A(z)$:

$$A(z) = 1 + 3zA(z) + \frac{z}{(1-z)^2}$$

gives

$$A(z) = \frac{(1-z)^2 + z}{(1-3z)(1-z)^2} = \frac{z^2 - z + 1}{(1-3z)(1-z)^2}$$

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5. Write $f(z)$ as a formal power series:

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$$\frac{z^2 - z + 1}{(1-3z)(1-z)^2} \overset{!}{=} \frac{A}{1-3z} + \frac{B}{1-z} + \frac{C}{(1-z)^2}$$

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This gives

$$z^2 - z + 1 = A(1-z)^2 + B(1-3z)(1-z) + C(1-3z)$$

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This gives

$$\begin{aligned} z^2 - z + 1 &= A(1-z)^2 + B(1-3z)(1-z) + C(1-3z) \\ &= A(1-2z+z^2) + B(1-4z+3z^2) + C(1-3z) \end{aligned}$$

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$$\begin{aligned} z^2 - z + 1 &= A(1-z)^2 + B(1-3z)(1-z) + C(1-3z) \\ &= A(1-2z+z^2) + B(1-4z+3z^2) + C(1-3z) \\ &= (A+3B)z^2 + (-2A-4B-3C)z + (A+B+C) \end{aligned}$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

5. Write $f(z)$ as a formal power series:

This leads to the following conditions:

$$\begin{aligned} A + B + C &= 1 \\ 2A + 4B + 3C &= 1 \\ A + 3B &= 1 \end{aligned}$$

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This leads to the following conditions:

$$\begin{aligned} A + B + C &= 1 \\ 2A + 4B + 3C &= 1 \\ A + 3B &= 1 \end{aligned}$$

which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

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5. Write $f(z)$ as a formal power series:

$$A(z) = \frac{7}{4} \cdot \frac{1}{1-3z} - \frac{1}{4} \cdot \frac{1}{1-z} - \frac{1}{2} \cdot \frac{1}{(1-z)^2}$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

5. Write $f(z)$ as a formal power series:

$$\begin{aligned} A(z) &= \frac{7}{4} \cdot \frac{1}{1-3z} - \frac{1}{4} \cdot \frac{1}{1-z} - \frac{1}{2} \cdot \frac{1}{(1-z)^2} \\ &= \frac{7}{4} \cdot \sum_{n\geq 0} 3^n z^n - \frac{1}{4} \cdot \sum_{n\geq 0} z^n - \frac{1}{2} \cdot \sum_{n\geq 0} (n+1) z^n \end{aligned}$$

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&= \frac{7}{4} \cdot \sum_{n\geq 0} 3^n z^n - \frac{1}{4} \cdot \sum_{n\geq 0} z^n - \frac{1}{2} \cdot \sum_{n\geq 0} (n+1) z^n \\
&= \sum_{n\geq 0} \Big(\frac{7}{4} \cdot 3^n - \frac{1}{4} - \frac{1}{2}(n+1) \Big) z^n
\end{aligned}$$

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&= \sum_{n\ge 0}\Big(\frac{7}{4}\cdot 3^n - \frac{1}{4} - \frac{1}{2}(n+1)\Big)z^n\\
&= \sum_{n\ge 0}\Big(\frac{7}{4}\cdot 3^n - \frac{1}{2}n - \frac{3}{4}\Big)z^n
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5. Write $f(z)$ as a formal power series:

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6. This means $a_n = \frac{7}{4}3^n - \frac{1}{2}n - \frac{3}{4}$.

6.5 Transformation of the Recurrence

Example 10

$$\begin{aligned} f_0 &= 1 \\ f_1 &= 2 \\ f_n &= f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2\ . \end{aligned}$$

6.5 Transformation of the Recurrence

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$$g_n = g_{n-1} + g_{n-2} \text{ for } n \geq 2$$

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$$f_0 = 1$$
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$$f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 \ .$$

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$$g_n = g_{n-1} + g_{n-2} \text{ for } n \geq 2$$
$$g_1 = \log 2 = 1 (\text{for } \log = \log_2),\ g_0 = 0$$

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f_0 &= 1\\
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Define

$$g_n := \log f_n\ .$$

Then

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\begin{aligned}
g_n &= g_{n-1} + g_{n-2} \text{ for } n \geq 2\\
g_1 &= \log 2 = 1 (\text{for } \log = \log_2),\ g_0 = 0\\
g_n &= F_n \ (n\text{-th Fibonacci number})
\end{aligned}
$$

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Then

$$g_n = g_{n-1} + g_{n-2} \text{ for } n \geq 2$$
$$g_1 = \log 2 = 1 (\text{for } \log = \log_2), \; g_0 = 0$$
$$g_n = F_n \; (n\text{-th Fibonacci number})$$
$$f_n = 2^{F_n}$$

6.5 Transformation of the Recurrence

Example 11

$$f_1 = 1$$
$$f_n = 3f_{\frac{n}{2}} + n; \text{ for } n = 2^k,\, k \geq 1\ ;$$

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Define

$$g_k := f_{2^k}\ .$$

Then:

$$g_0 = 1$$
$$g_k = 3g_{k-1} + 2^k,\ k \geq 1$$

6 Recurrences

We get

$$g_k = 3\left[g_{k-1}\right] + 2^k$$

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$$\begin{aligned} g_k &= 3\left[g_{k-1}\right] + 2^k \\ &= 3\left[3g_{k-2} + 2^{k-1}\right] + 2^k \end{aligned}$$

6 Recurrences

We get

$$
\begin{aligned}
g_k &= 3\left[g_{k-1}\right] + 2^k \\
&= 3\left[3g_{k-2} + 2^{k-1}\right] + 2^k \\
&= 3^2\left[g_{k-2}\right] + 32^{k-1} + 2^k
\end{aligned}
$$

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g_k &= 3\left[g_{k-1}\right] + 2^k \\
&= 3\left[3g_{k-2} + 2^{k-1}\right] + 2^k \\
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&= 3^2\left[3g_{k-3} + 2^{k-2}\right] + 32^{k-1} + 2^k
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$$\begin{aligned} g_k &= 3\left[g_{k-1}\right] + 2^k \\ &= 3\left[3g_{k-2} + 2^{k-1}\right] + 2^k \\ &= 3^2\left[g_{k-2}\right] + 32^{k-1} + 2^k \\ &= 3^2\left[3g_{k-3} + 2^{k-2}\right] + 32^{k-1} + 2^k \\ &= 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k \end{aligned}$$

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&= 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k \\
&= 2^k \cdot \sum_{i=0}^{k} \left(\frac{3}{2}\right)^i
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&= 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k \\
&= 2^k \cdot \sum_{i=0}^{k} \left(\frac{3}{2}\right)^i \\
&= 2^k \cdot \frac{(\frac{3}{2})^{k+1} - 1}{1/2}
\end{aligned}
$$

6 Recurrences

We get

$$
\begin{aligned}
g_k &= 3\left[g_{k-1}\right] + 2^k \\
&= 3\left[3g_{k-2} + 2^{k-1}\right] + 2^k \\
&= 3^2\left[g_{k-2}\right] + 32^{k-1} + 2^k \\
&= 3^2\left[3g_{k-3} + 2^{k-2}\right] + 32^{k-1} + 2^k \\
&= 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k \\
&= 2^k \cdot \sum_{i=0}^{k} \left(\frac{3}{2}\right)^i \\
&= 2^k \cdot \frac{(\frac{3}{2})^{k+1} - 1}{1/2} = 3^{k+1} - 2^{k+1}
\end{aligned}
$$

6 Recurrences

Let $n = 2^k$:

$$\begin{aligned} g_k &= 3^{k+1} - 2^{k+1}, \text{ hence} \\ f_n &= 3 \cdot 3^k - 2 \cdot 2^k \end{aligned}$$

6 Recurrences

Let $n = 2^k$:

$$\begin{aligned} g_k &= 3^{k+1} - 2^{k+1}, \text{ hence} \\ f_n &= 3 \cdot 3^k - 2 \cdot 2^k \\ &= 3(2^{\log 3})^k - 2 \cdot 2^k \end{aligned}$$

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6 Recurrences

Let $n = 2^k$:

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Part III

Data Structures

Abstract Data Type

An abstract data type (ADT) is defined by an interface of operations or methods that can be performed and that have a defined behavior.

The data types in this lecture all operate on objects that are represented by a [key, value] pair.

- The key comes from a totally ordered set, and we assume that there is an efficient comparison function.
- The value can be anything; it usually carries satellite information important for the application that uses the ADT.

Dynamic Set Operations

- $\boldsymbol{S}.\textbf{search}(\boldsymbol{k})$: Returns pointer to object x from S with $\mathrm{key}[x] = k$ or null.

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- $\boldsymbol{S.\,\mathrm{successor}(x)}$: Return pointer to the next larger element in S or null if x is maximum.
- $\boldsymbol{S.\,\mathrm{predecessor}(x)}$: Return pointer to the next smaller element in S or null if x is minimum.

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  $S := \{x \in S \mid \mathrm{key}[x] \le k\}$, $S' := \{x \in S \mid \mathrm{key}[x] > k\}$.

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  Requires $\mathrm{key}[S.\,\mathrm{maximum}()] \le \mathrm{key}[S'.\,\mathrm{minimum}()]$.

Dynamic Set Operations

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- $S.\textbf{concatenate}(S')$: $S := S \cup S'$.
  Requires $\text{key}[S.\text{maximum}()] \le \text{key}[S'.\text{minimum}()]$.

- $S.\textbf{decrease-key}(x, k)$: Replace $\text{key}[x]$ by $k \le \text{key}[x]$.

Examples of ADTs

Stack:

- $\mathbf{S.\,push(x)}$: Insert an element.
- $\mathbf{S.\,pop()}$: Return the element from S that was inserted most recently; delete it from S.
- $\mathbf{S.\,empty()}$: Tell if S contains any object.

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Queue:

- $\mathbf{S.\,enqueue(x)}$: Insert an element.
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- $S.\mathbf{enqueue}(x)$: Insert an element.
- $S.\mathbf{dequeue}()$: Return the element that is longest in the structure; delete it from S.
- $S.\mathbf{empty}()$: Tell if S contains any object.

Priority-Queue:

- $S.\mathbf{insert}(x)$: Insert an element.
- $S.\mathbf{delete\text{-}min}()$: Return the element with lowest key-value; delete it from S.

7 Dictionary

Dictionary:

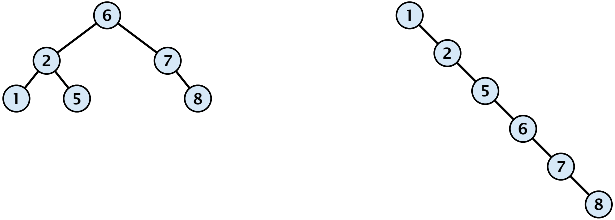
- $\boldsymbol{S.\,\textbf{insert}(x)}$: Insert an element x.
- $\boldsymbol{S.\,\textbf{delete}(x)}$: Delete the element pointed to by x.
- $\boldsymbol{S.\,\textbf{search}(k)}$: Return a pointer to an element e with $\mathrm{key}[e]=k$ in S if it exists; otherwise return null.

7.1 Binary Search Trees

An (internal) binary search tree stores the elements in a binary tree. Each tree-node corresponds to an element. All elements in the left sub-tree of a node v have a smaller key-value than $\mathrm{key}[v]$ and elements in the right sub-tree have a larger-key value. We assume that all key-values are different.

(External Search Trees store objects only at leaf-vertices)

Examples:

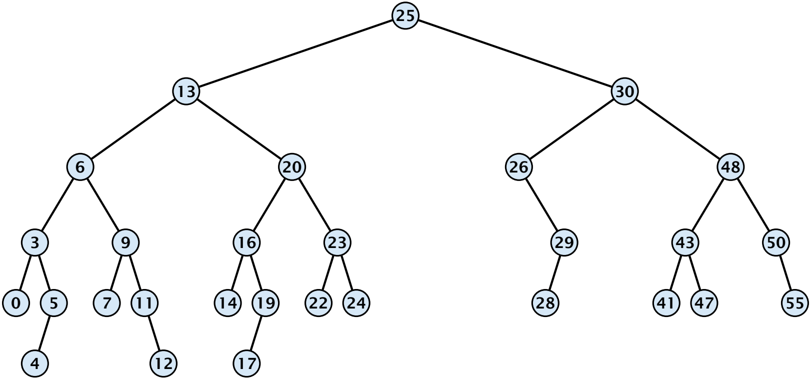


7.1 Binary Search Trees

We consider the following operations on binary search trees. Note that this is a super-set of the dictionary-operations.

- $T.\text{insert}(x)$
- $T.\text{delete}(x)$
- $T.\text{search}(k)$
- $T.\text{successor}(x)$
- $T.\text{predecessor}(x)$
- $T.\text{minimum}()$
- $T.\text{maximum}()$

Binary Search Trees: Searching

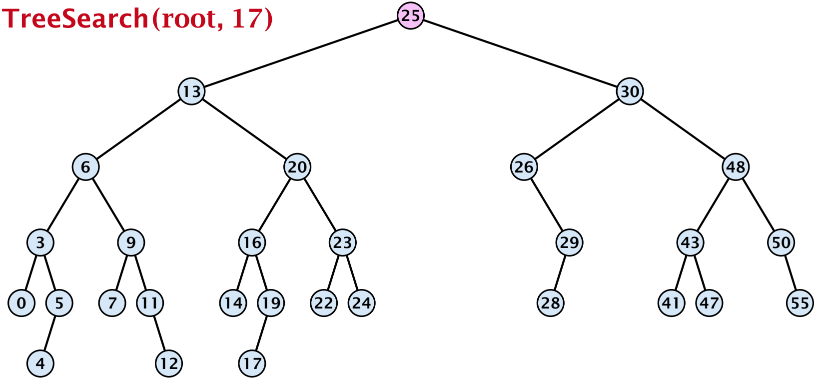


Algorithm 1 TreeSearch(x, k)

```
if $x$ = null or $k$ = key$[x]$ return $x$
if $k <$ key$[x]$ return TreeSearch(left$[x]$, $k$)
else return TreeSearch(right$[x]$, $k$)
```

Binary Search Trees: Searching

TreeSearch(root, 17)



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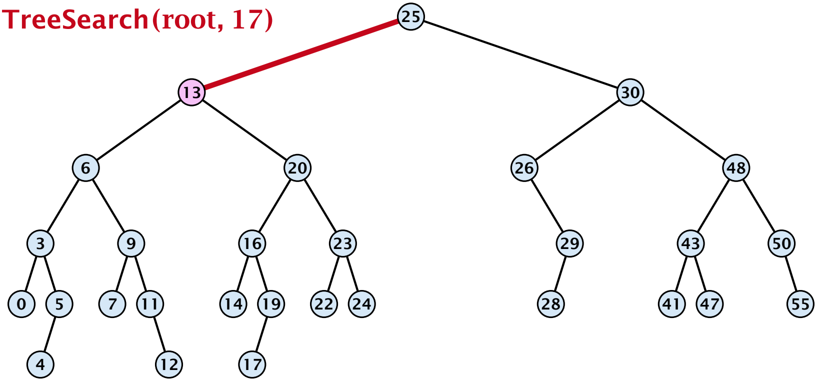
Binary Search Trees: Searching



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Binary Search Trees: Searching

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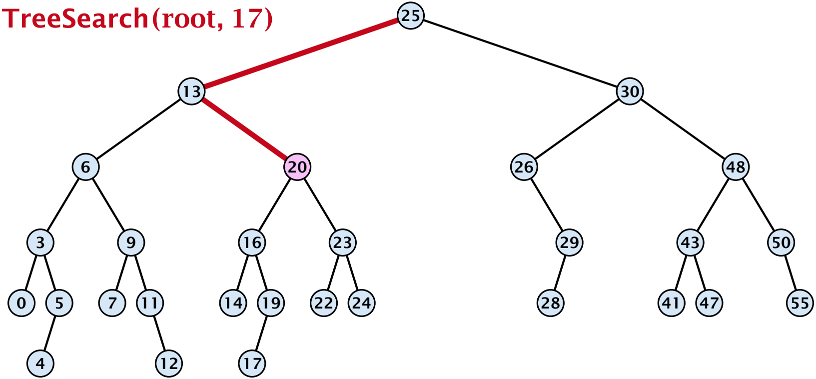

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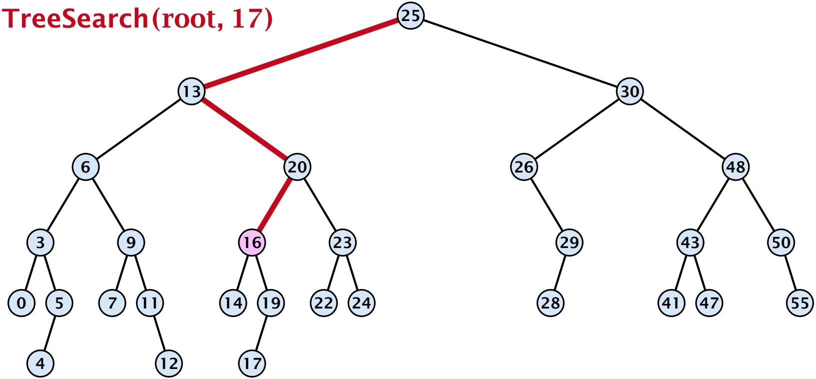
Binary Search Trees: Searching



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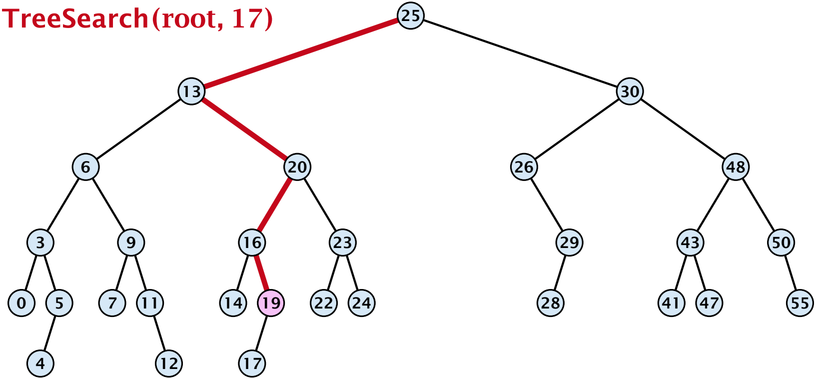

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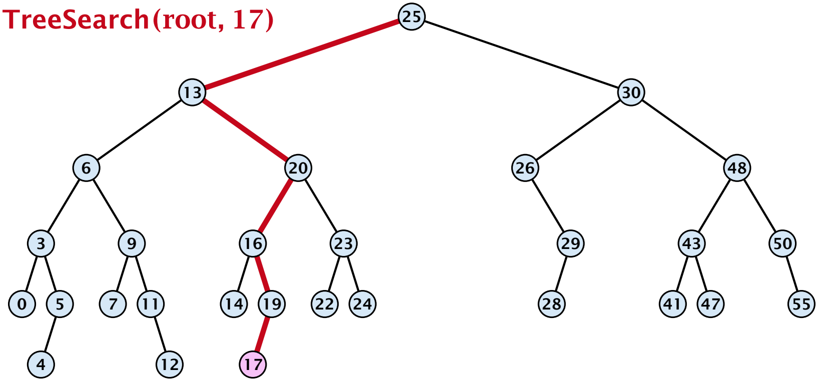

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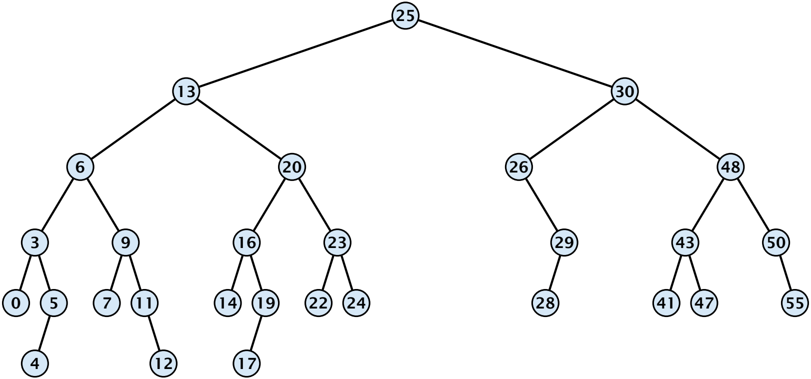

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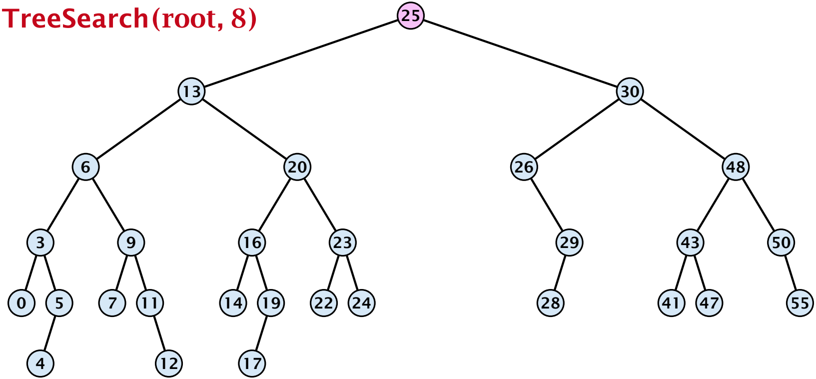
Binary Search Trees: Searching

TreeSearch(root, 8)



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Binary Search Trees: Searching

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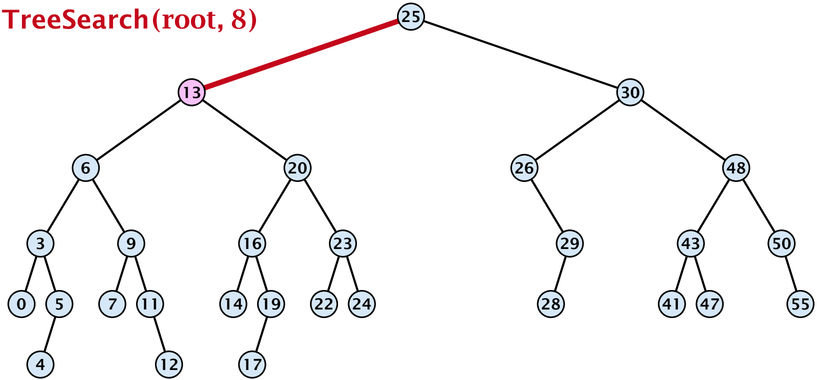

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Binary Search Trees: Searching

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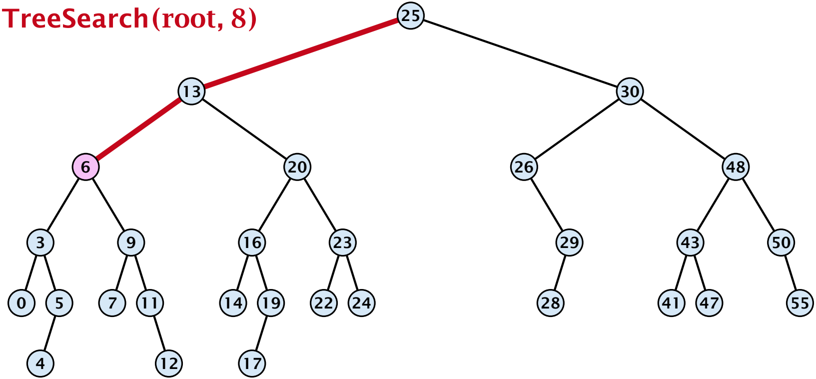

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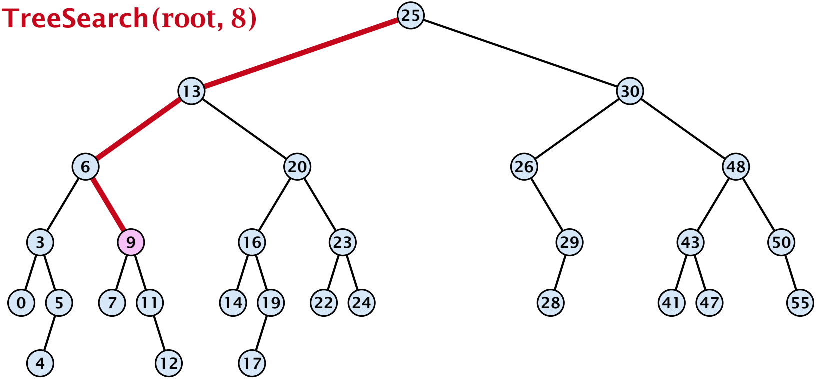

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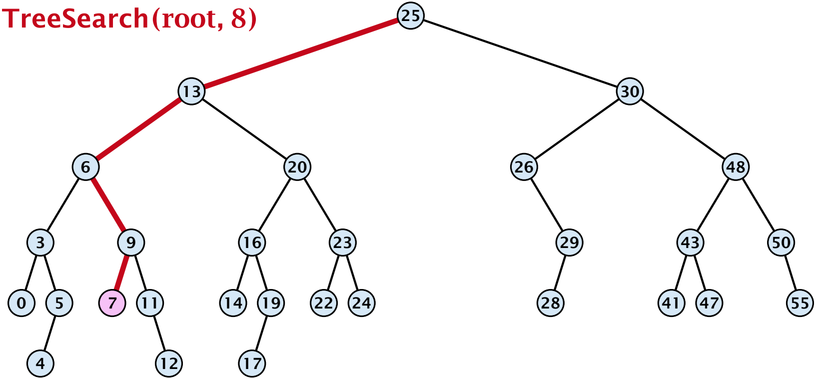

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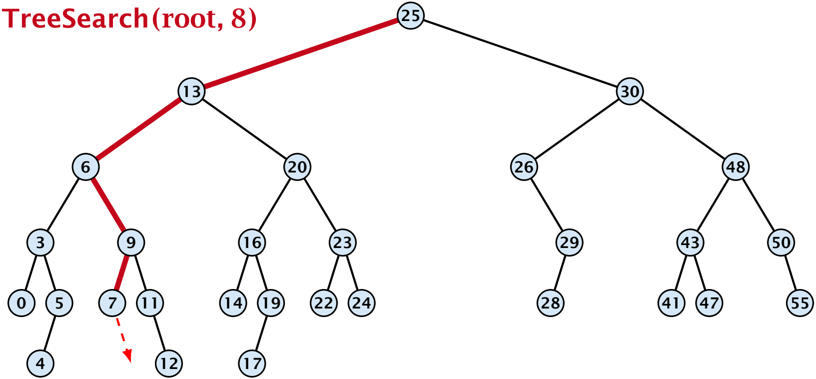
Binary Search Trees: Searching



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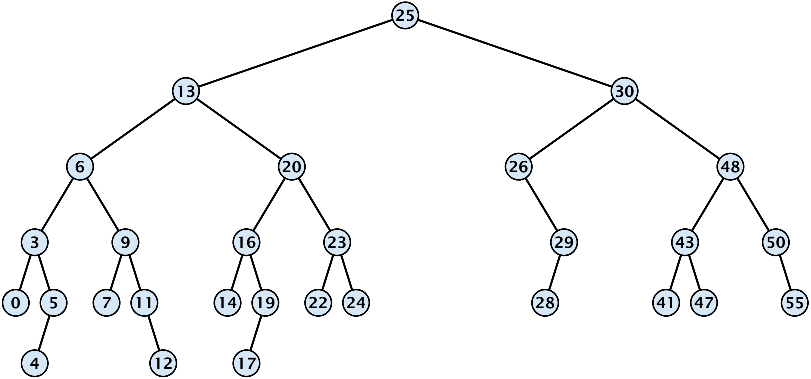
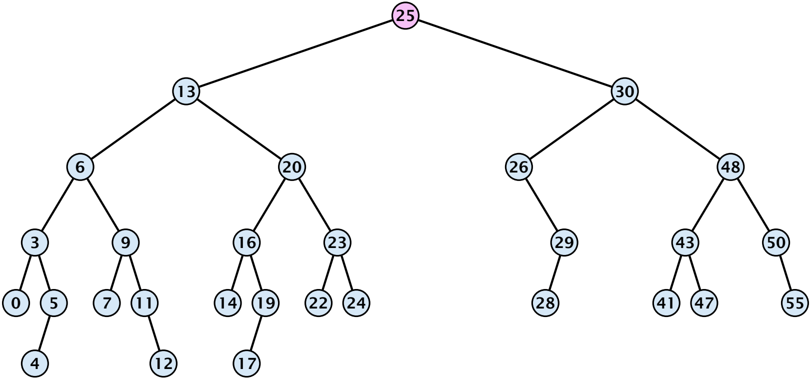
Binary Search Trees: Minimum



Algorithm 2 TreeMin(x)

```
if $x$ = null or left$[x]$ = null return $x$
return TreeMin(left$[x]$)
```

Binary Search Trees: Minimum



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```
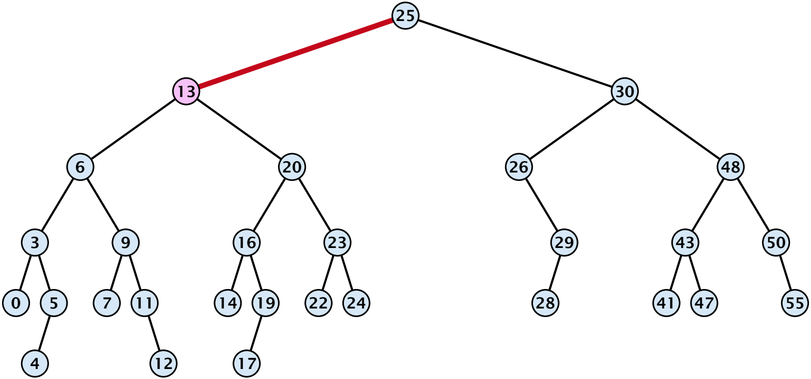

Binary Search Trees: Minimum



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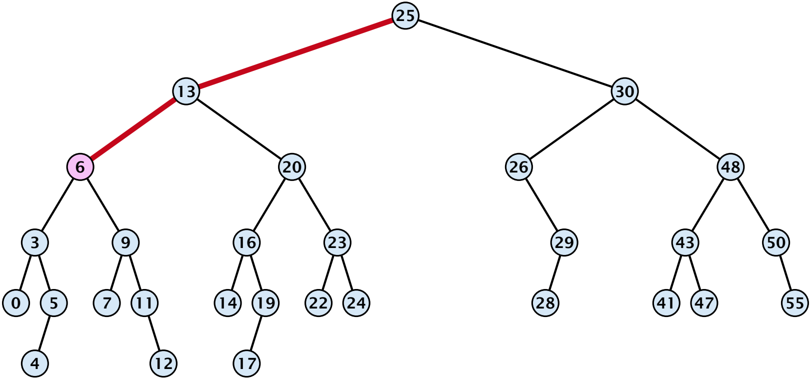

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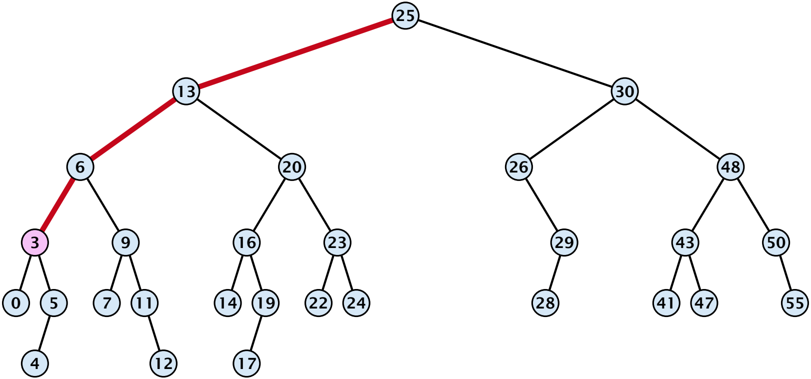

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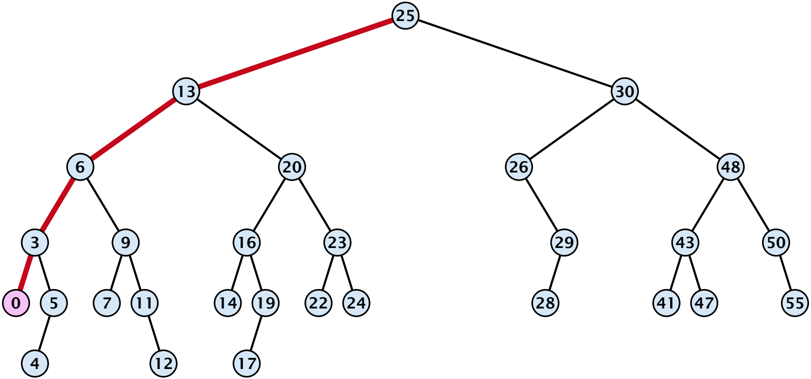

Algorithm 2 TreeMin(x)

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if $x$ = null or left[$x$] = null return $x$
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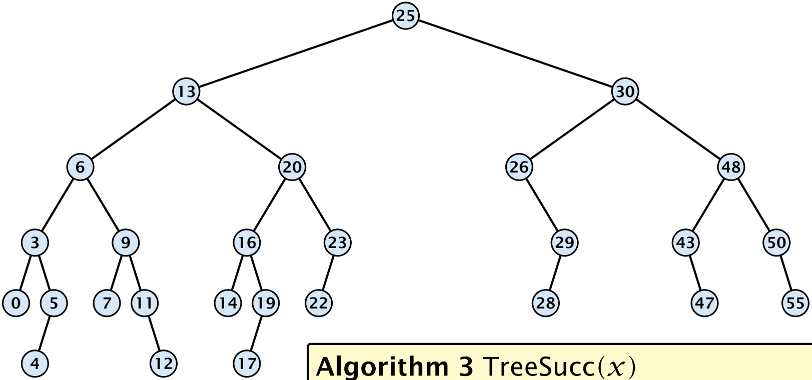
```

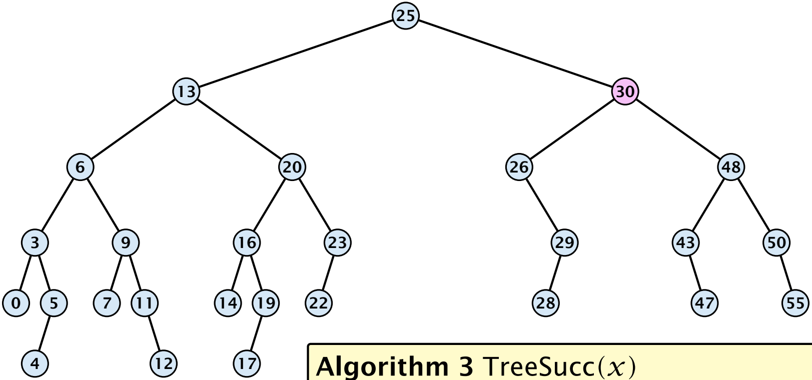
Binary Search Trees: Successor



Algorithm 3 TreeSucc(x)

```
if right[$x$] $\neq$ null return TreeMin(right[$x$])
$y \leftarrow$ parent[$x$]
while $y \neq$ null and $x =$ right[$y$] do
    $x \leftarrow y$; $y \leftarrow$ parent[$x$]
return $y$;
```

Binary Search Trees: Successor



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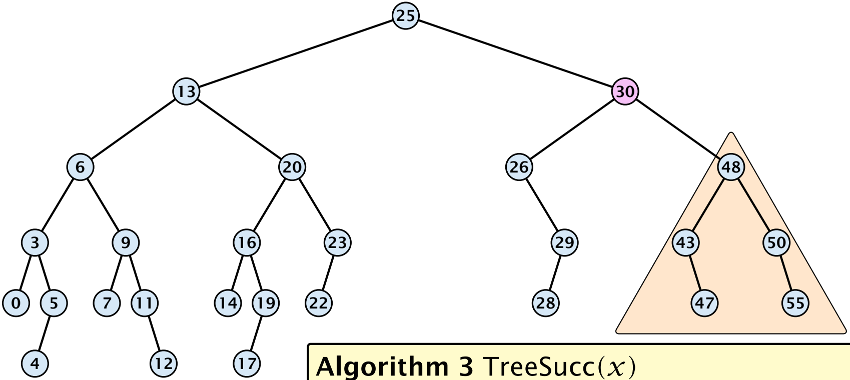

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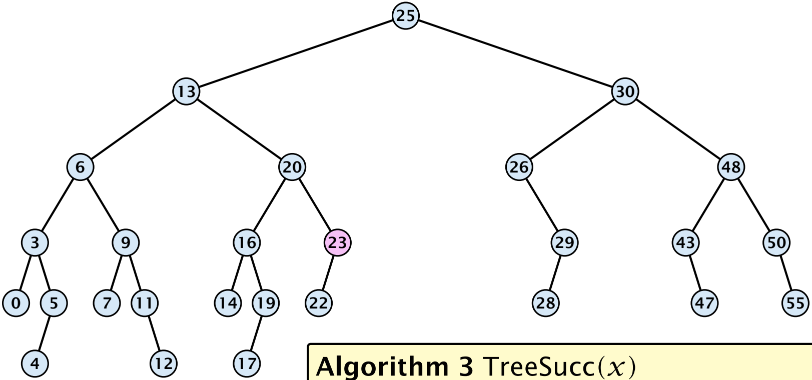

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```
if right[$x$] ≠ null return TreeMin(right[$x$])
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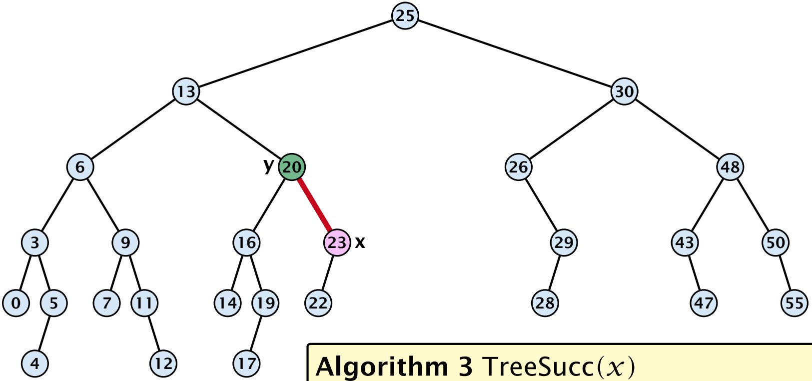

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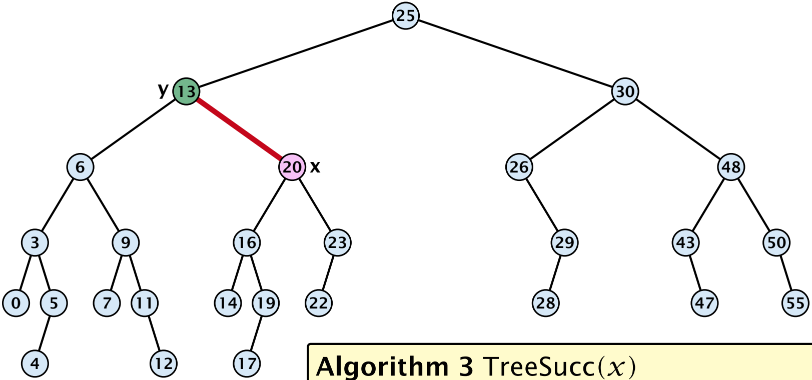

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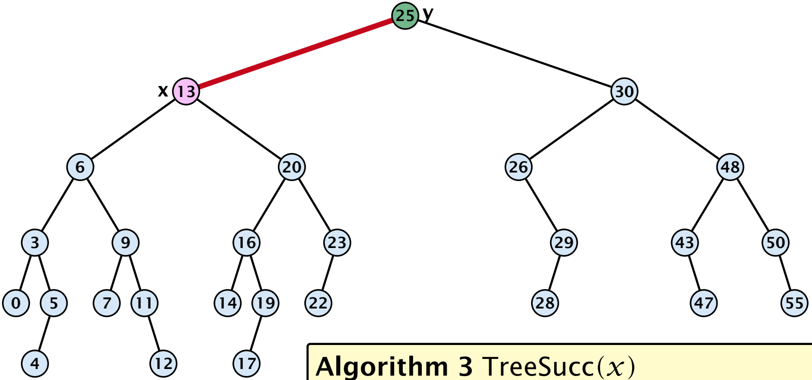

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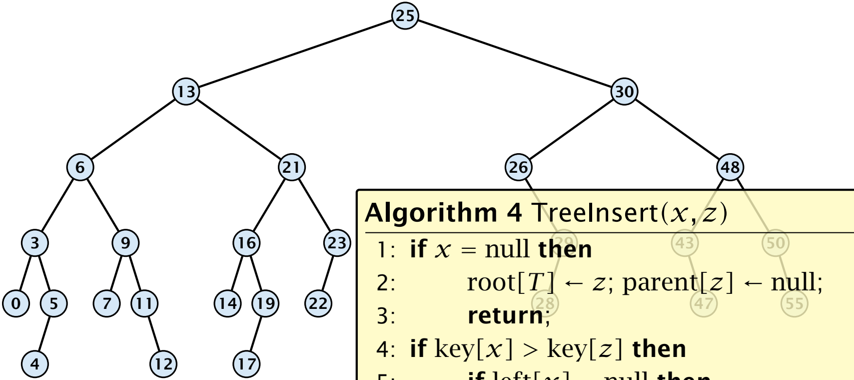
Binary Search Trees: Insert



Algorithm 4 TreeInsert(x, z)

```
if $x$ = null then
    root$[T] \leftarrow z$; parent$[z] \leftarrow$ null;
    return;
if key$[x] >$ key$[z]$ then
    if left$[x]$ = null then
        left$[x] \leftarrow z$; parent$[z] \leftarrow x$;
    else TreeInsert(left$[x], z$);
else
    if right$[x]$ = null then
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    else TreeInsert(right$[x], z$);
```
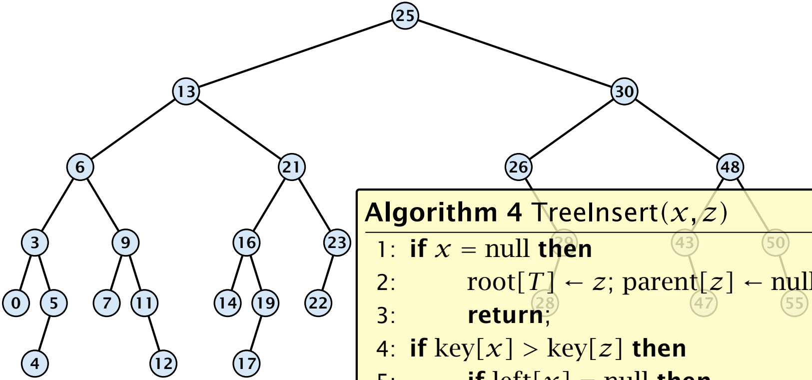

Binary Search Trees: Insert

Insert element **not** in the tree.



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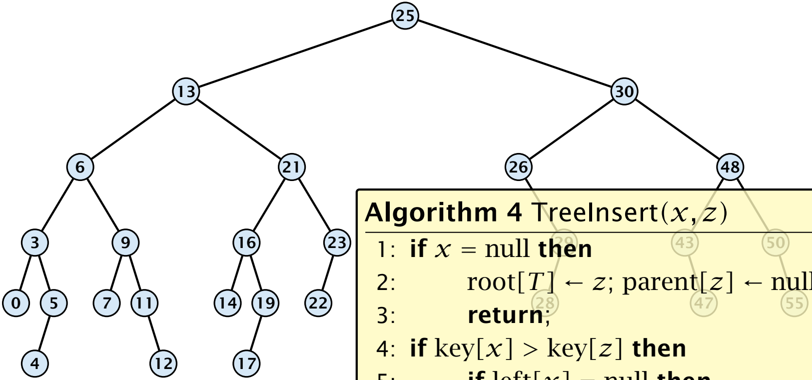

Search for z. At some point the search stops at a null-pointer. This is the place to insert z.

Algorithm 4 TreeInsert(x, z)

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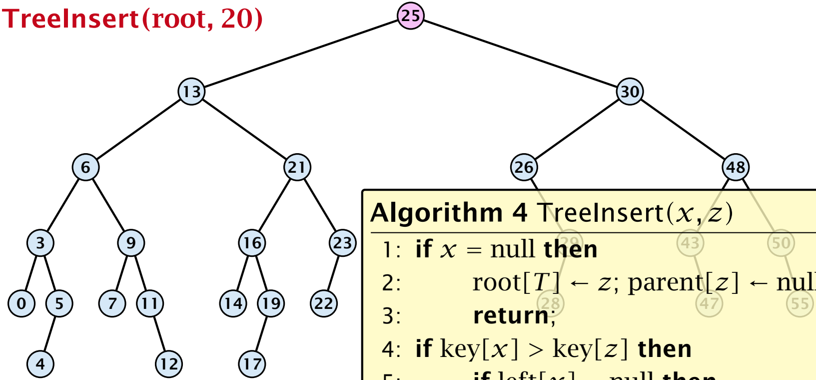
```

Binary Search Trees: Insert

Insert element **not** in the tree.

TreeInsert(root, 20)



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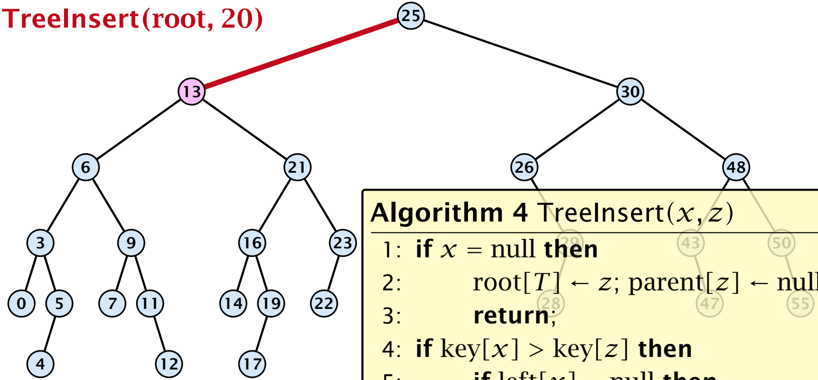
Binary Search Trees: Insert

Insert element **not** in the tree.

TreeInsert(root, 20)



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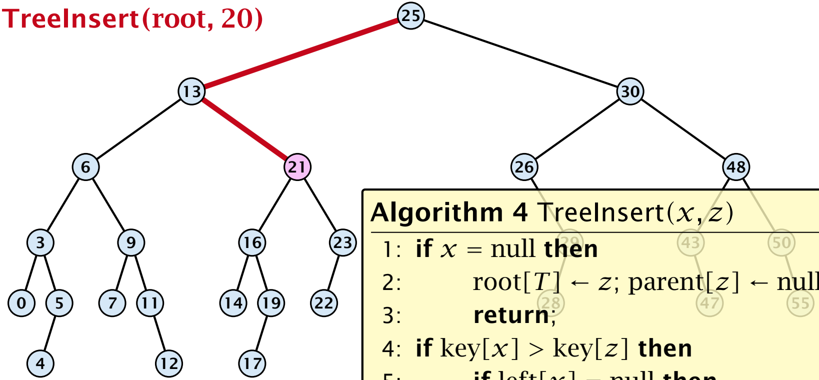

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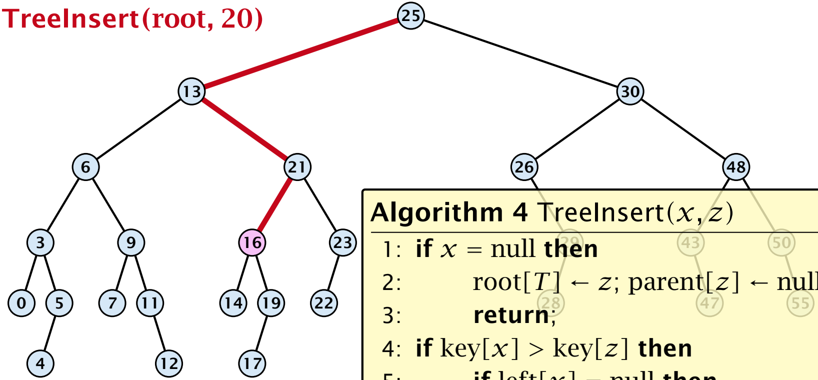

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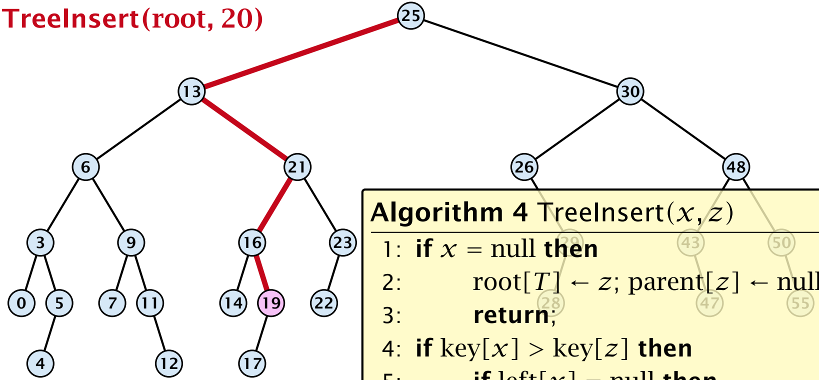

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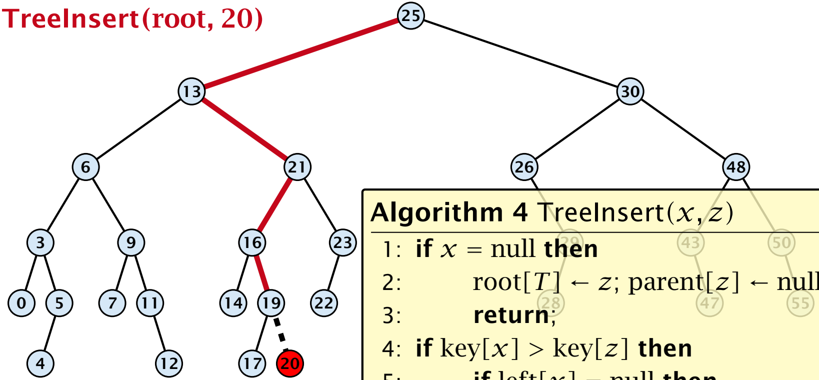
Insert element **not** in the tree.

TreeInsert(root, 20)

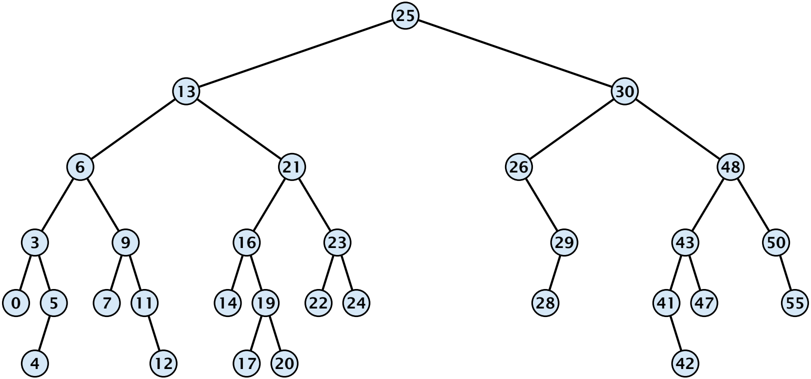


Search for z. At some point the search stops at a null-pointer. This is the place to insert z.

Algorithm 4 TreeInsert(x, z)

```
if $x$ = null then
    root[$T$] ← $z$; parent[$z$] ← null;
    return;
if key[$x$] > key[$z$] then
    if left[$x$] = null then
        left[$x$] ← $z$; parent[$z$] ← $x$;
    else TreeInsert(left[$x$], $z$);
else
    if right[$x$] = null then
        right[$x$] ← $z$; parent[$z$] ← $x$;
    else TreeInsert(right[$x$], $z$);
```

Binary Search Trees: Delete



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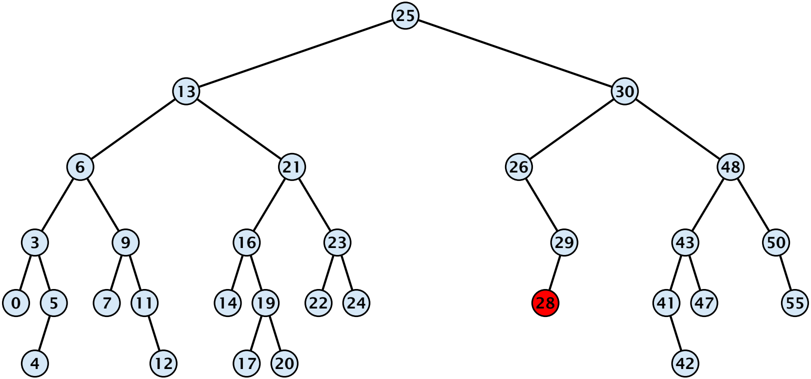

Case 1:

Element does not have any children

- Simply go to the parent and set the corresponding pointer to null.

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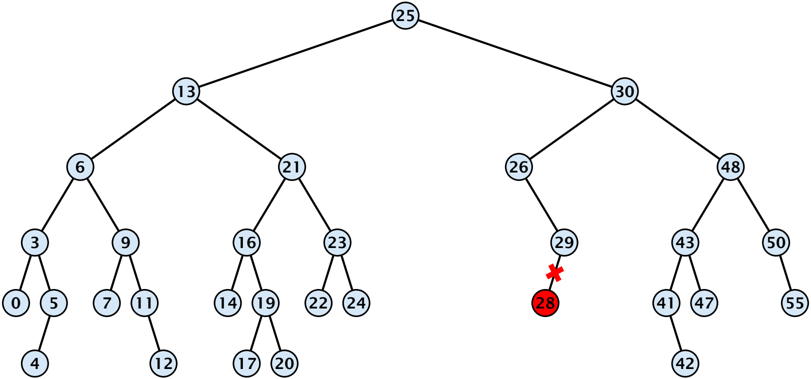

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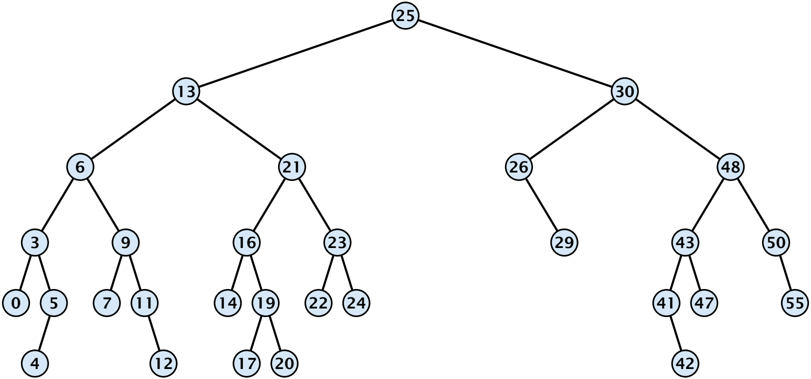

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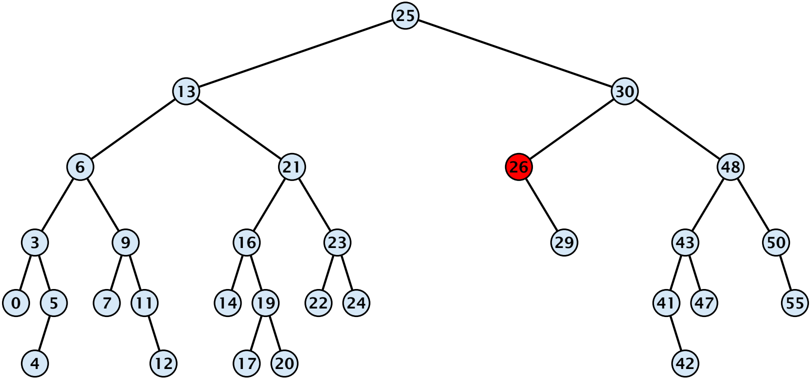
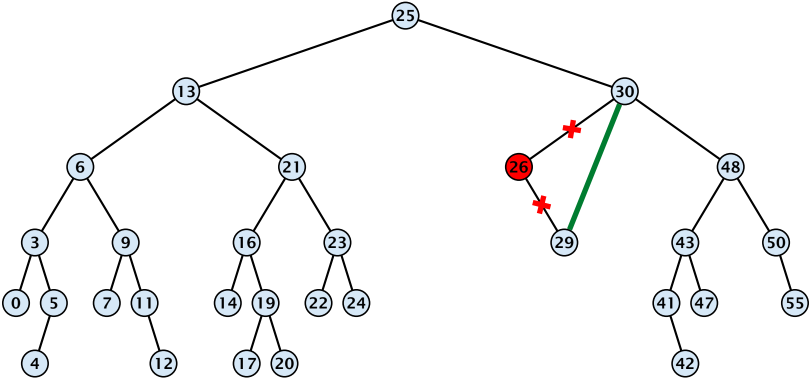


Case 2:

Element has exactly one child

- Splice the element out of the tree by connecting its parent to its successor.

Binary Search Trees: Delete



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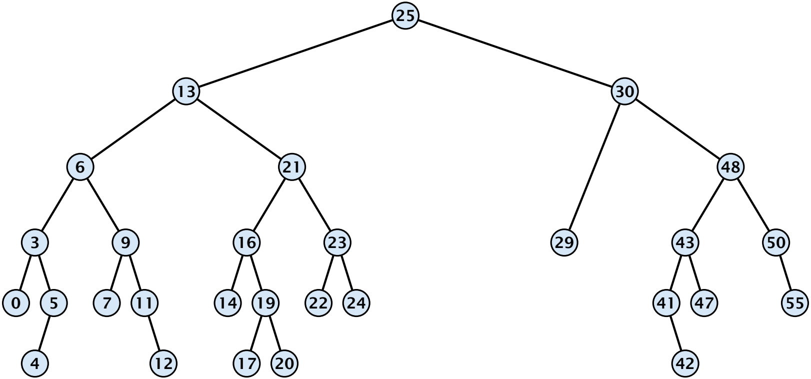

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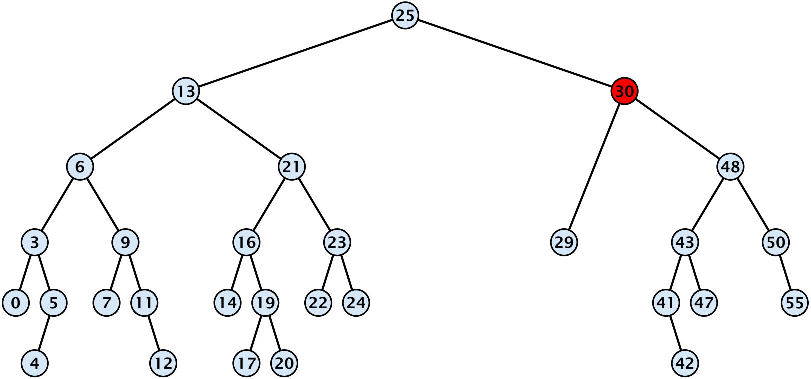

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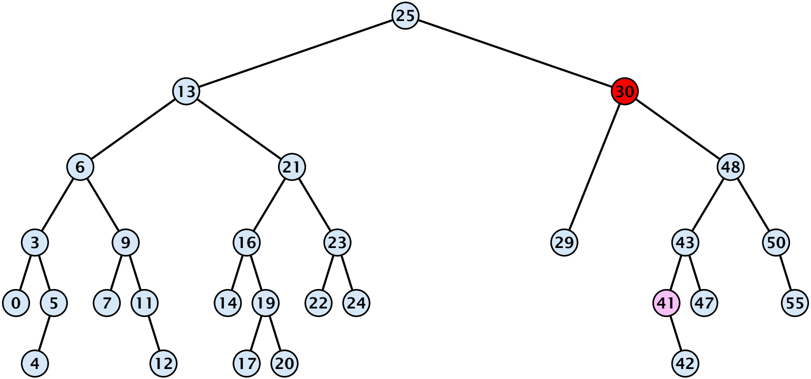
Element has two children

- Find the successor of the element
- Splice successor out of the tree
- Replace content of element by content of successor

Binary Search Trees: Delete



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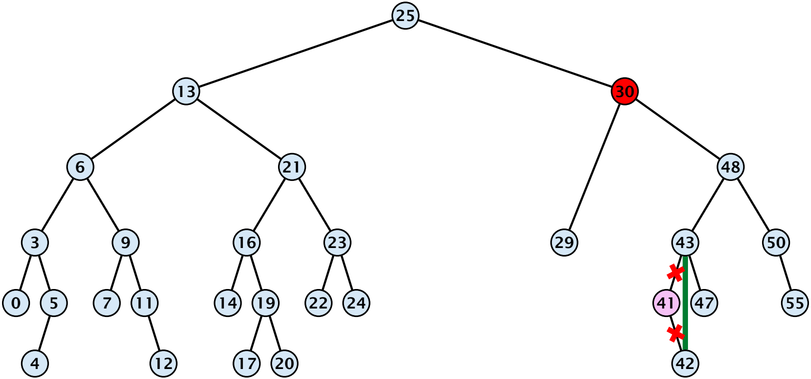

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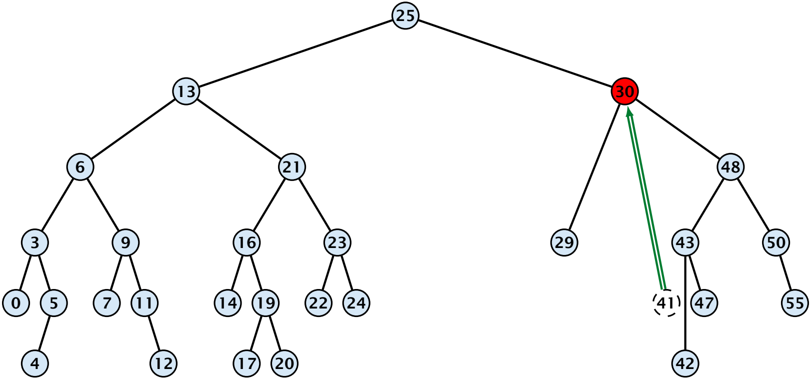

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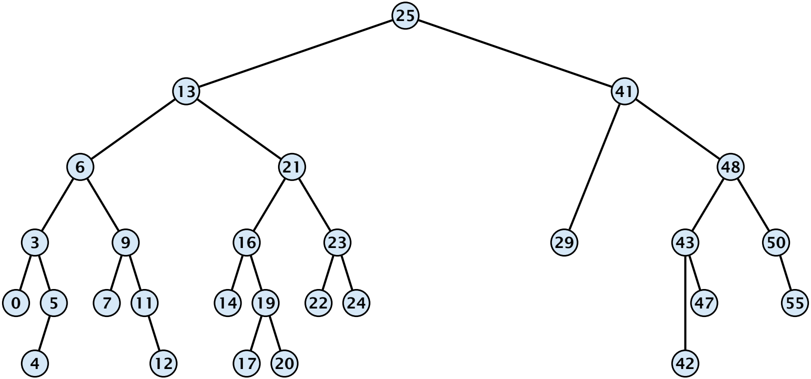

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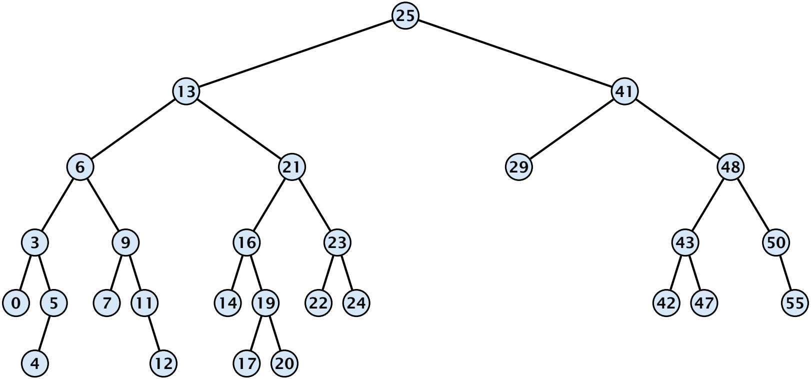

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Element has two children

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- Splice successor out of the tree
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Binary Search Trees: Delete

Algorithm 9 TreeDelete(z)

```
if $\text{left}[z] = \text{null}$ or $\text{right}[z] = \text{null}$
    then $y \leftarrow z$ else $y \leftarrow \text{TreeSucc}(z)$;          select $y$ to splice out
if $\text{left}[y] \neq \text{null}$
    then $x \leftarrow \text{left}[y]$ else $x \leftarrow \text{right}[y]$;  $x$ is child of $y$ (or null)
if $x \neq \text{null}$ then $\text{parent}[x] \leftarrow \text{parent}[y]$;      $\text{parent}[x]$ is correct
if $\text{parent}[y] = \text{null}$ then
    $\text{root}[T] \leftarrow x$
else
    if $y = \text{left}[\text{parent}[y]]$ then                            fix pointer to $x$
        $\text{left}[\text{parent}[y]] \leftarrow x$
    else
        $\text{right}[\text{parent}[y]] \leftarrow x$
if $y \neq z$ then copy $y$-data to $z$
```

Balanced Binary Search Trees

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AVL-trees, Red-black trees, Scapegoat trees, 2-3 trees, B-trees, AA trees, Treaps

similar: SPLAY trees.

7.2 Red Black Trees

Definition 12

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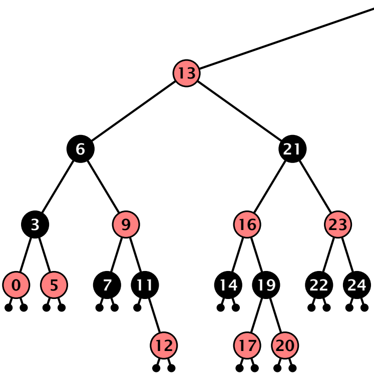
Red Black Trees: Example



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A red-black tree with n *internal nodes has height at most* $\mathcal{O}(\log n)$.

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The black height $\operatorname{bh}(v)$ of a node v in a red black tree is the number of black nodes on a path from v to a leaf vertex (not counting v).

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The black height $\text{bh}(v)$ of a node v in a red black tree is the number of black nodes on a path from v to a leaf vertex (not counting v).

We first show:

Lemma 15

A sub-tree of black height $\text{bh}(v)$ in a red black tree contains at least $2^{\text{bh}(v)} - 1$ internal vertices.

7.2 Red Black Trees

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- The sub-tree rooted at v contains $0 = 2^{\text{bh}(v)} - 1$ inner vertices.

7.2 Red Black Trees

Proof (cont.)

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induction step

- Supose v is a node with $\text{height}(v) > 0$.

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- Then T_v contains at least $2(2^{\mathrm{bh}(v)-1} - 1) + 1 \geq 2^{\mathrm{bh}(v)} - 1$ vertices.

☐

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The tree contains at least $2^{h/2} - 1$ internal vertices. Hence, $2^{h/2} - 1 \le n$.

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The tree contains at least $2^{h/2} - 1$ internal vertices. Hence, $2^{h/2} - 1 \leq n$.

Hence, $h \leq 2\log(n+1) = \mathcal{O}(\log n)$. □

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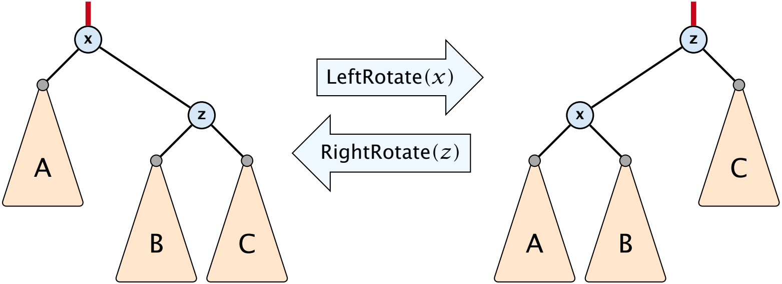
7.2 Red Black Trees

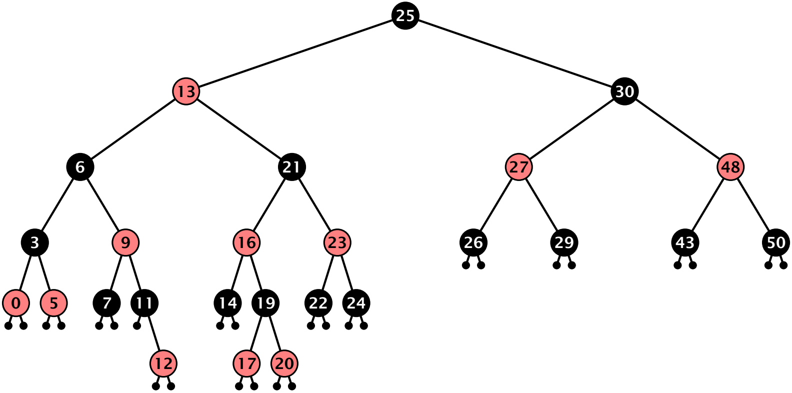
We need to adapt the insert and delete operations so that the red black properties are maintained.

Rotations

The properties will be maintained through rotations:



Red Black Trees: Insert

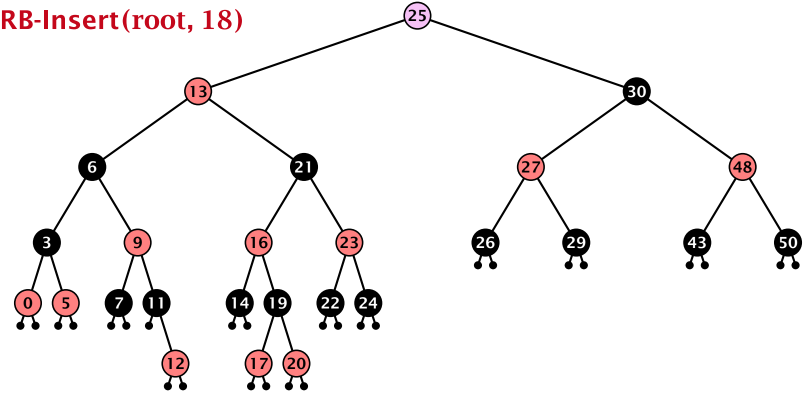


Insert:

- first make a normal insert into a binary search tree
- then fix red-black properties

Red Black Trees: Insert

RB-Insert(root, 18)



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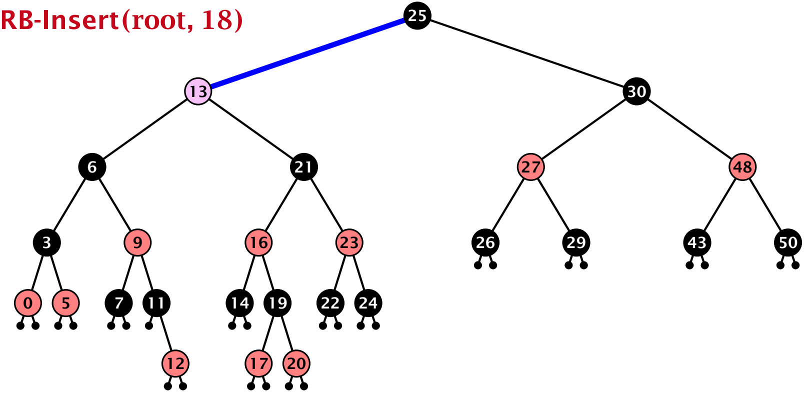

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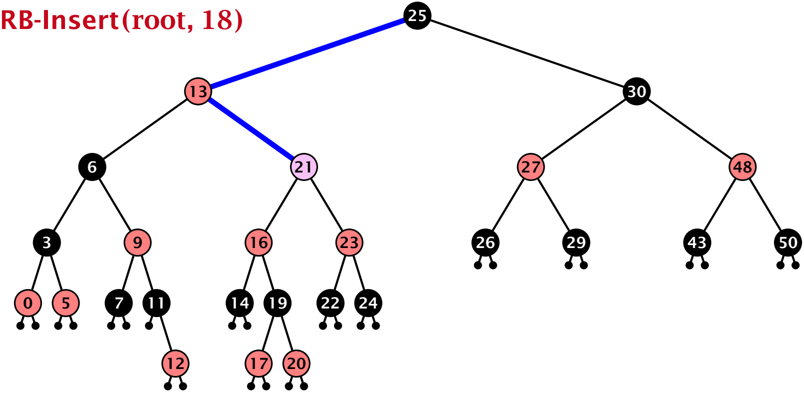

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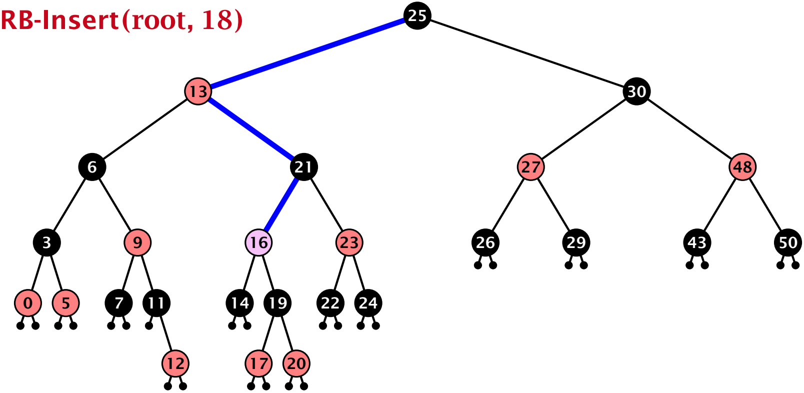

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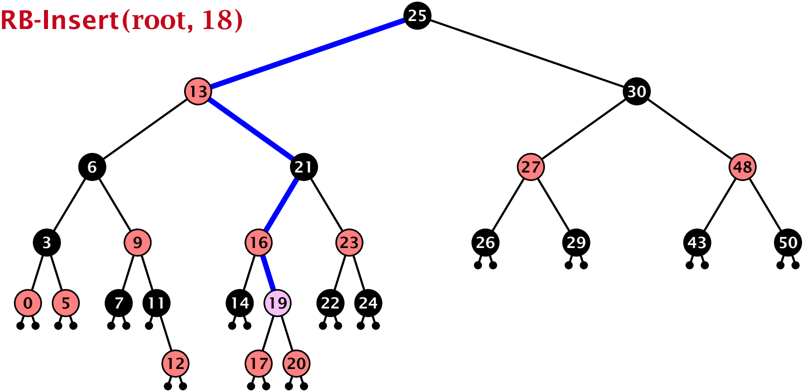

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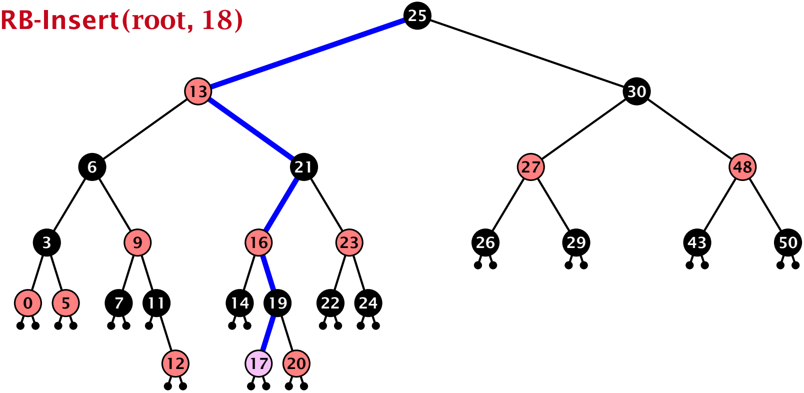

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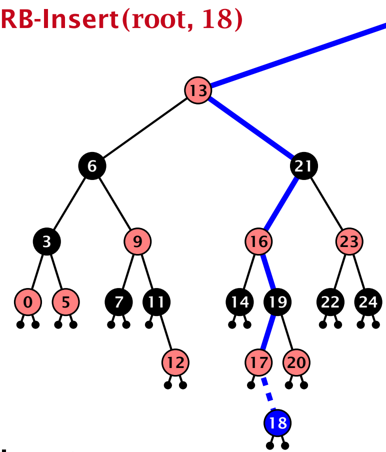

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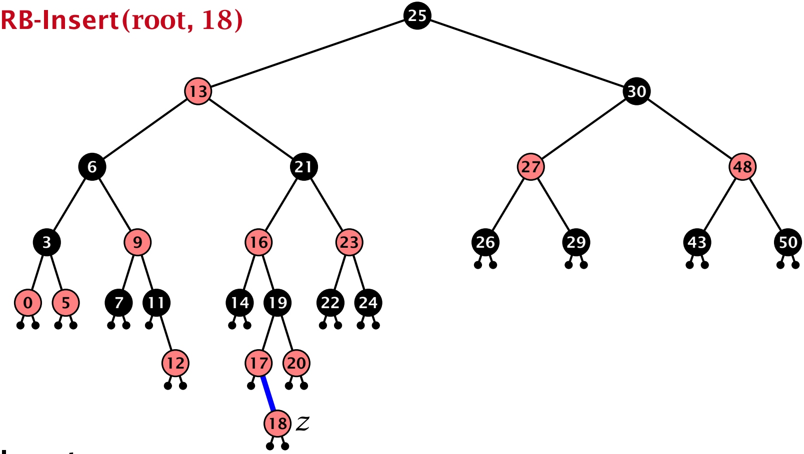

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Invariant of the fix-up algorithm:

- z is a red node

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If z has a parent but no grand-parent we could simply color the parent/root black; however this case never happens.

Red Black Trees: Insert

Algorithm 10 InsertFix(z)

```
while parent[$z$] $\neq$ null and col[parent[$z$]] = red do
    if parent[$z$] = left[gp[z]] then
        $uncle \leftarrow$ right[grandparent[$z$]]
        if col[$uncle$] = red then
            col[p[$z$]] $\leftarrow$ black; col[$u$] $\leftarrow$ black;
            col[gp[$z$]] $\leftarrow$ red; $z \leftarrow$ grandparent[$z$];
        else
            if $z$ = right[parent[$z$]] then
                $z \leftarrow$ p[$z$]; LeftRotate($z$);
            col[p[$z$]] $\leftarrow$ black; col[gp[$z$]] $\leftarrow$ red;
            RightRotate(gp[$z$]);
    else same as then-clause but right and left exchanged
col(root[$T$]) $\leftarrow$ black;
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    if parent[$z$] = left[gp[z]] then    $z$ in left subtree of grandparent
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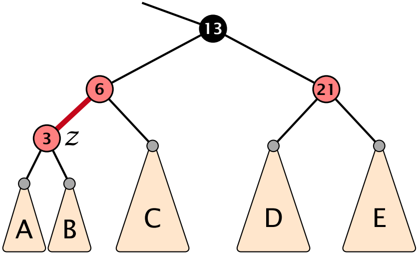
        $uncle \leftarrow$ right[grandparent[$z$]]
        if col[$uncle$] = red then
            col[p[$z$]] $\leftarrow$ black; col[$u$] $\leftarrow$ black;
            col[gp[$z$]] $\leftarrow$ red; $z \leftarrow$ grandparent[$z$];
        else
            if $z$ = right[parent[$z$]] then                2a: $z$ right child
                $z \leftarrow$ p[$z$]; LeftRotate($z$);
            col[p[$z$]] $\leftarrow$ black; col[gp[$z$]] $\leftarrow$ red;
            RightRotate(gp[$z$]);
    else same as then-clause but right and left exchanged
col(root[$T$]) $\leftarrow$ black;
```

Red Black Trees: Insert

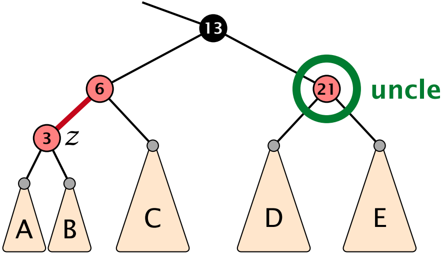
Algorithm 10 InsertFix(z)

```
while parent[$z$] $\neq$ null and col[parent[$z$]] = red do
    if parent[$z$] = left[gp[z]] then
        $uncle \leftarrow$ right[grandparent[$z$]]
        if col[$uncle$] = red then
            col[p[$z$]] $\leftarrow$ black; col[$u$] $\leftarrow$ black;
            col[gp[$z$]] $\leftarrow$ red; $z \leftarrow$ grandparent[$z$];
        else
            if $z$ = right[parent[$z$]] then
                $z \leftarrow$ p[$z$]; LeftRotate($z$);
            col[p[$z$]] $\leftarrow$ black; col[gp[$z$]] $\leftarrow$ red; 2b: $z$ left child
            RightRotate(gp[$z$]);
    else same as then-clause but right and left exchanged
col(root[$T$]) $\leftarrow$ black;
```

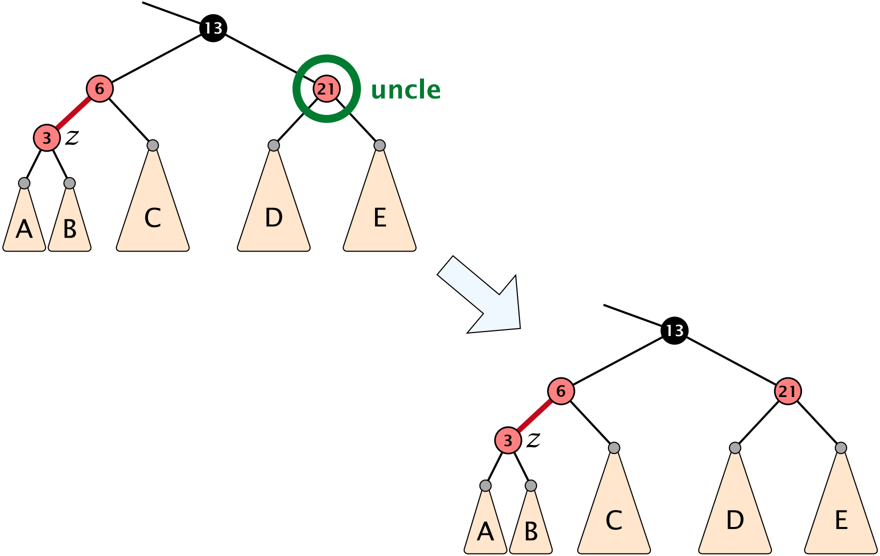
Case 1: Red Uncle



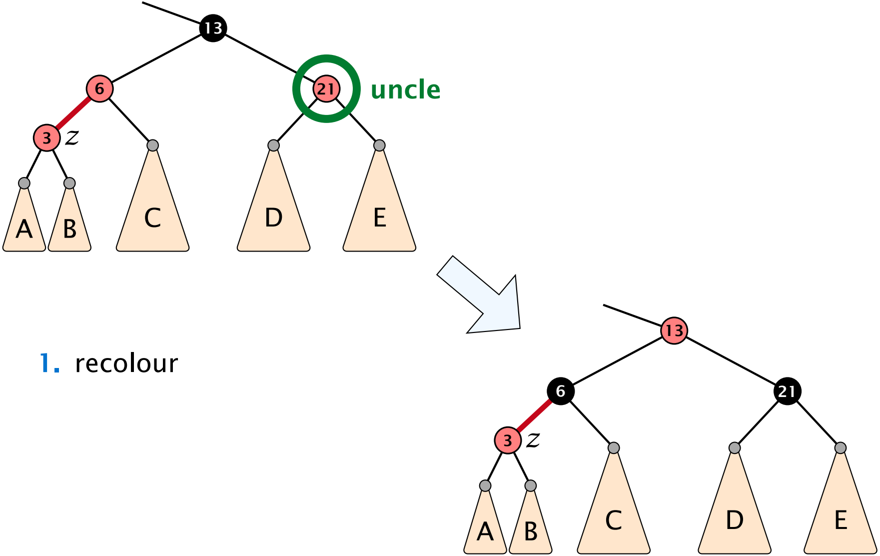
Case 1: Red Uncle



Case 1: Red Uncle

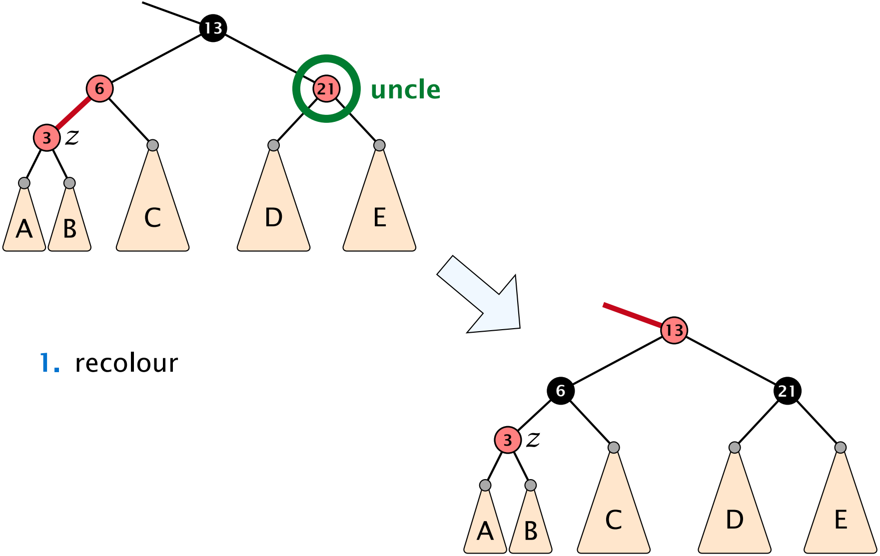


Case 1: Red Uncle



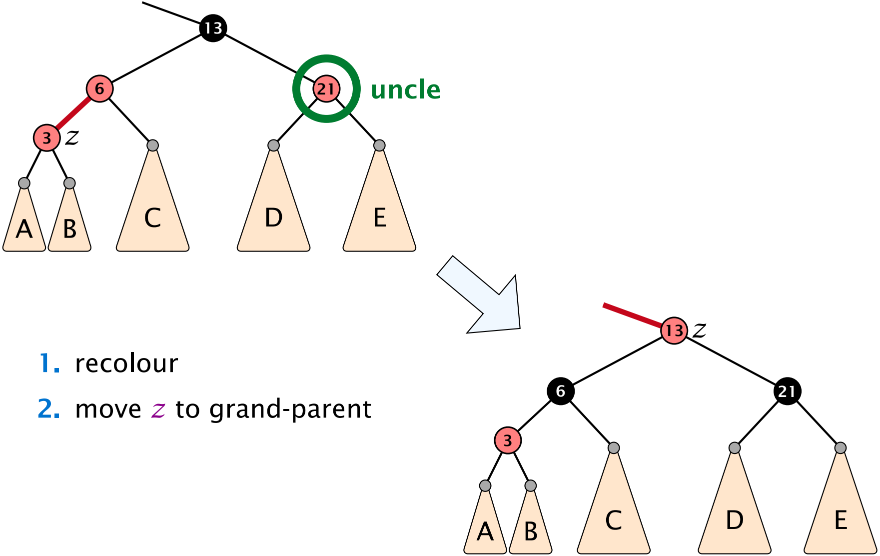
1. recolour

Case 1: Red Uncle



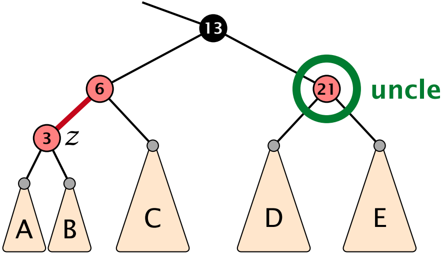
1. recolour

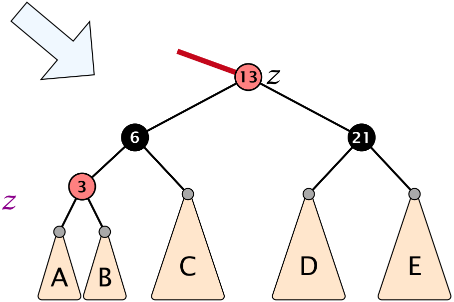
Case 1: Red Uncle



1. recolour
2. move z to grand-parent

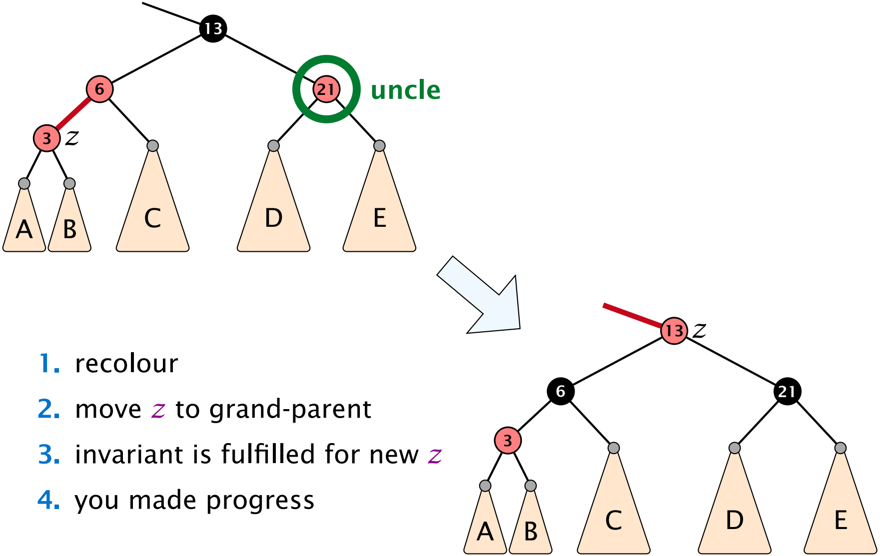
Case 1: Red Uncle





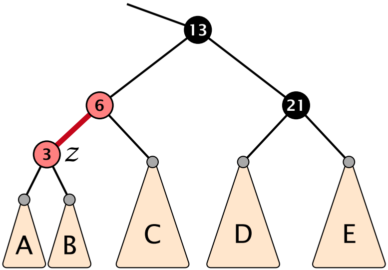
1. recolour
2. move z to grand-parent
3. invariant is fulfilled for new z

Case 1: Red Uncle

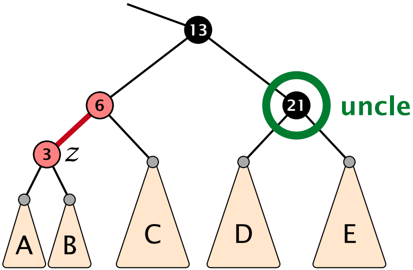


1. recolour
2. move z to grand-parent
3. invariant is fulfilled for new z
4. you made progress

Case 2b: Black uncle and z is left child

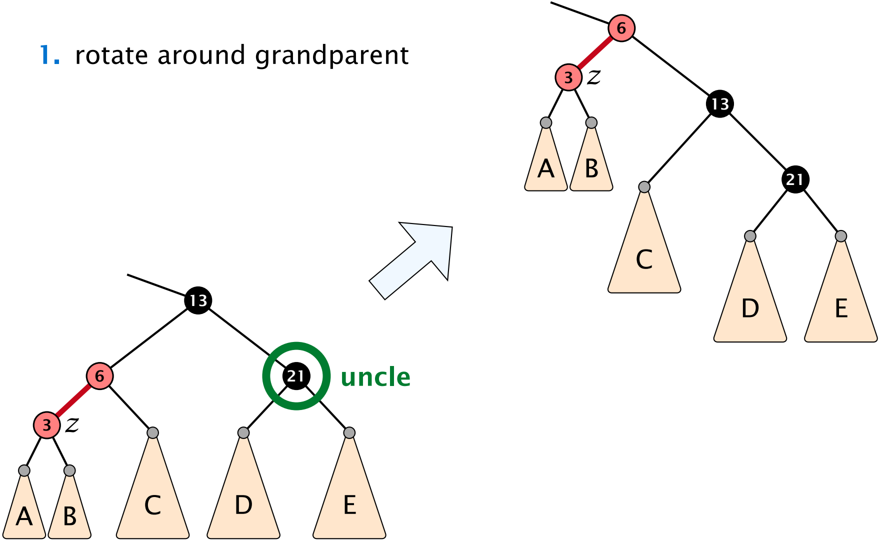


Case 2b: Black uncle and z is left child



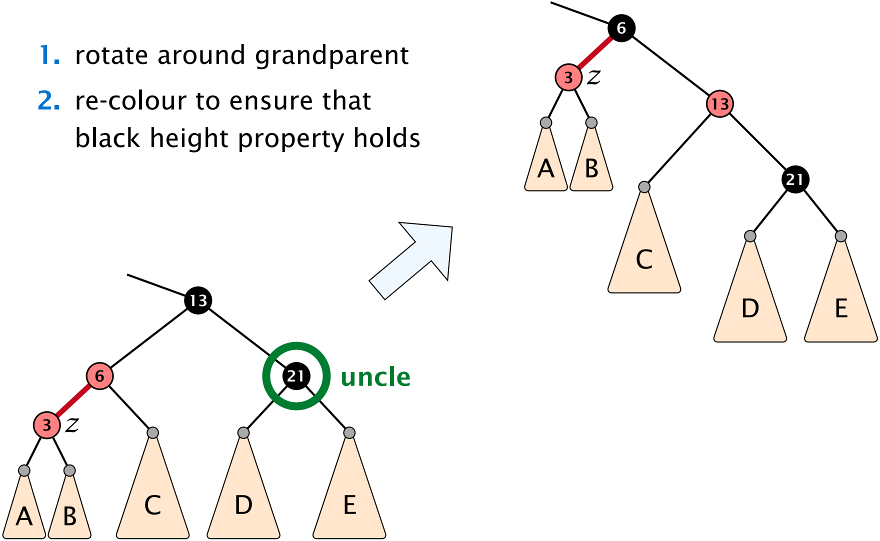
Case 2b: Black uncle and z is left child

1. rotate around grandparent



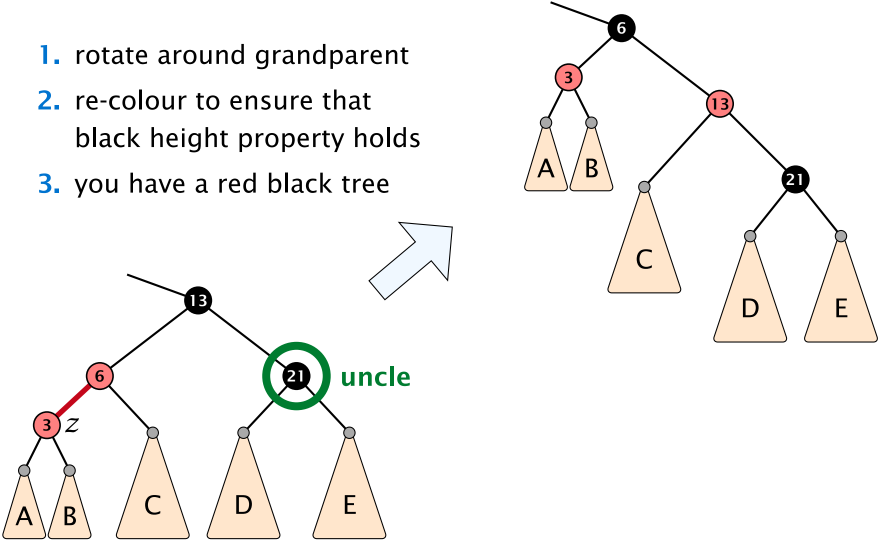
Case 2b: Black uncle and z is left child

1. rotate around grandparent
2. re-colour to ensure that black height property holds

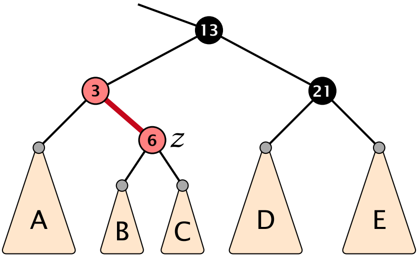


Case 2b: Black uncle and z is left child

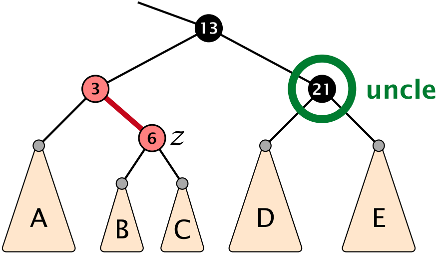
1. rotate around grandparent
2. re-colour to ensure that black height property holds
3. you have a red black tree



Case 2a: Black uncle and z is right child

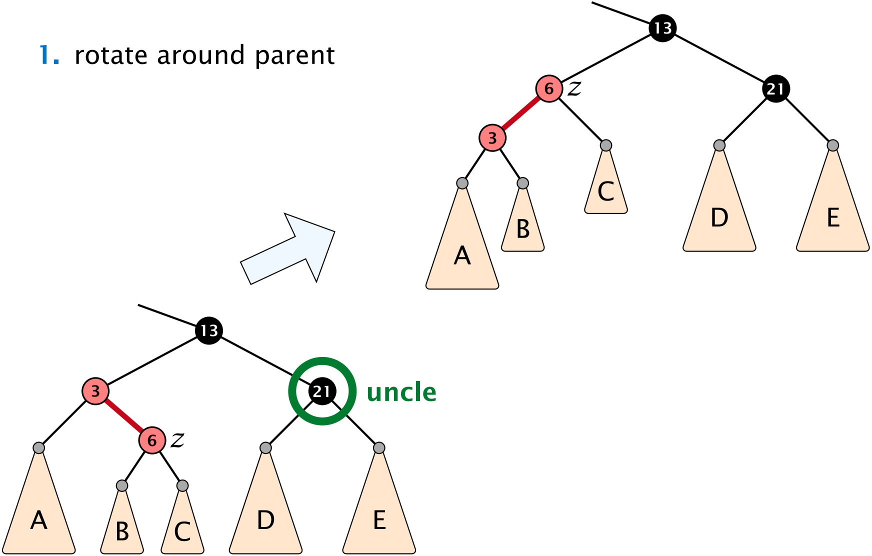


Case 2a: Black uncle and z is right child



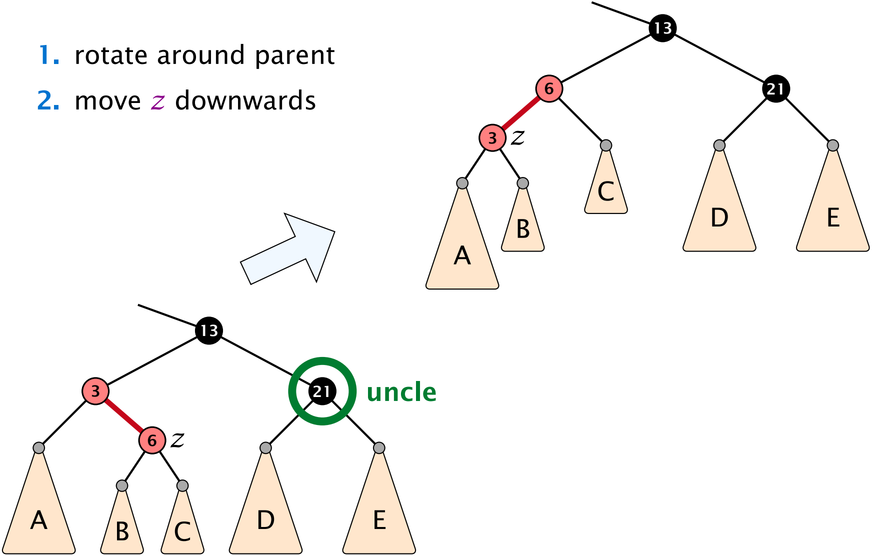
Case 2a: Black uncle and z is right child

1. rotate around parent



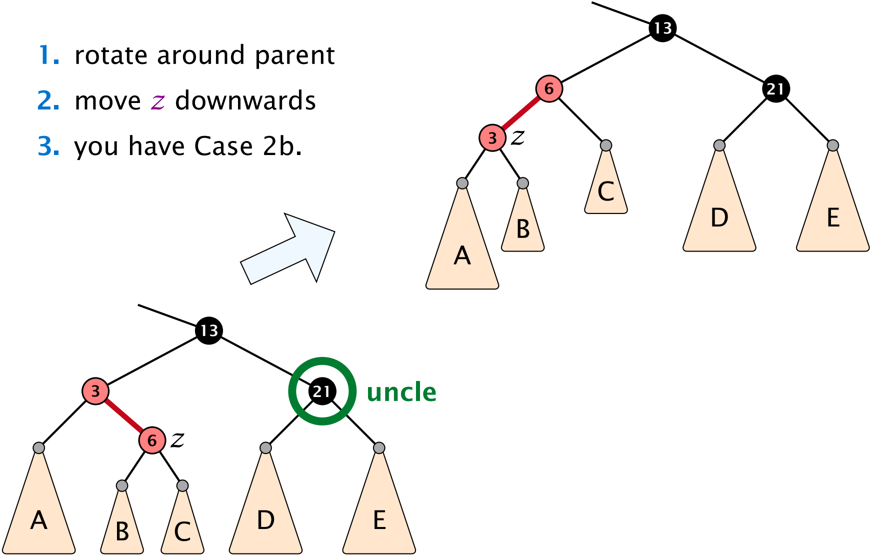
Case 2a: Black uncle and z is right child

1. rotate around parent
2. move z downwards



Case 2a: Black uncle and z is right child

1. rotate around parent
2. move z downwards
3. you have Case 2b.



Red Black Trees: Insert

Running time:

- Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.

Red Black Trees: Insert

Running time:

- Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- Case 2a $\rightarrow$ Case 2b $\rightarrow$ red-black tree

Red Black Trees: Insert

Running time:

- Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- Case 2a $\rightarrow$ Case 2b $\rightarrow$ red-black tree
- Case 2b $\rightarrow$ red-black tree

Red Black Trees: Insert

Running time:

- Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- Case 2a → Case 2b → red-black tree
- Case 2b → red-black tree

Performing Case 1 at most $\mathcal{O}(\log n)$ times and every other case at most once, we get a red-black tree. Hence $\mathcal{O}(\log n)$ re-colorings and at most 2 rotations.

Red Black Trees: Delete

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First do a standard delete.

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If the spliced out node x was red everything is fine.

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- Parent and child of x were red; two adjacent red vertices.

Red Black Trees: Delete

First do a standard delete.

If the spliced out node x was red everything is fine.

If it was black there may be the following problems.

- Parent and child of x were red; two adjacent red vertices.
- If you delete the root, the root may now be red.

Red Black Trees: Delete

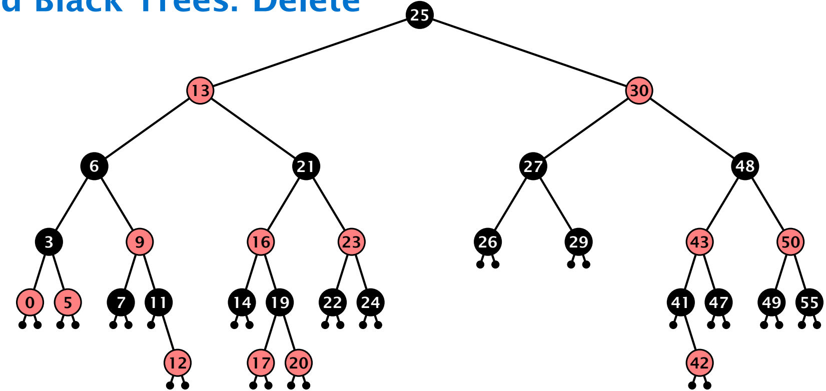
First do a standard delete.

If the spliced out node x was red everything is fine.

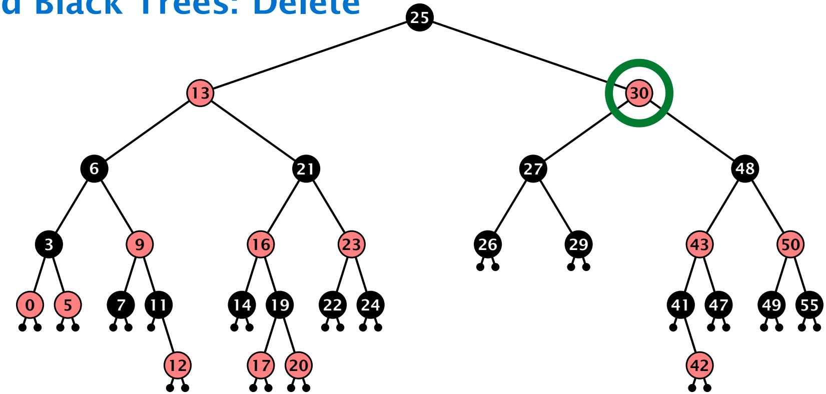
If it was black there may be the following problems.

- Parent and child of x were red; two adjacent red vertices.
- If you delete the root, the root may now be red.
- Every path from an ancestor of x to a descendant leaf of x changes the number of black nodes. Black height property might be violated.

Red Black Trees: Delete



Red Black Trees: Delete

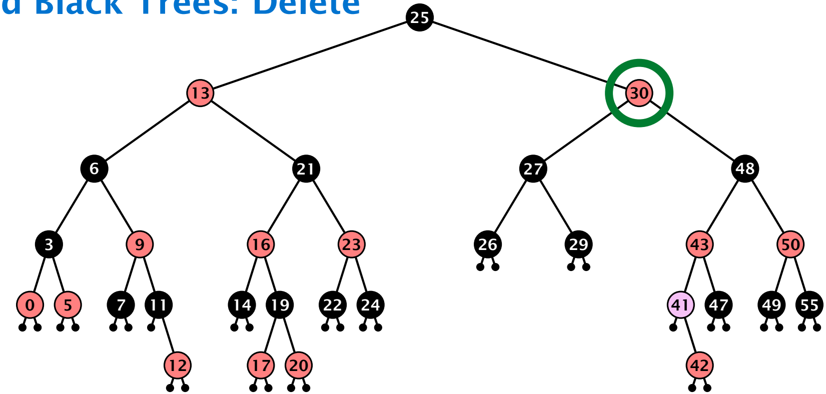


Case 3:

Element has two children

- do normal delete
- when replacing content by content of successor, don't change color of node

Red Black Trees: Delete



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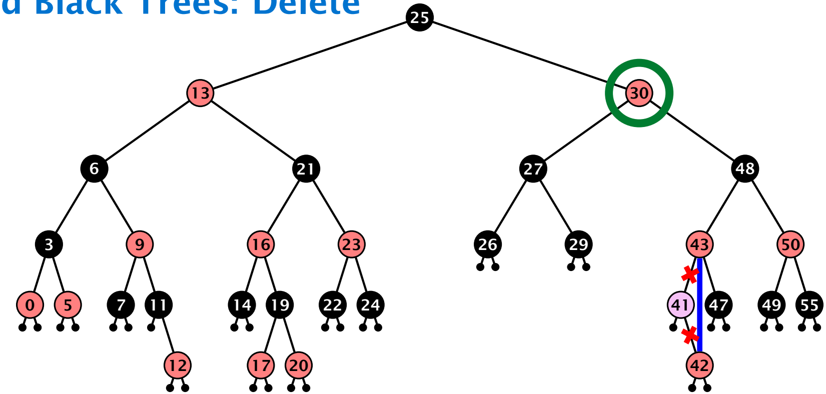

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Element has two children

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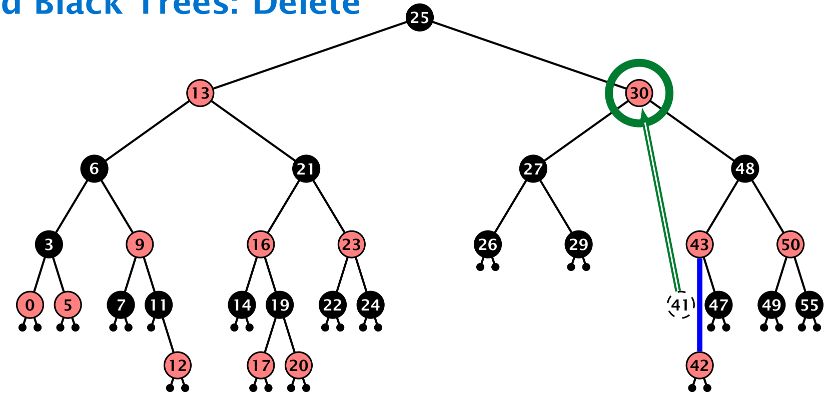

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Element has two children

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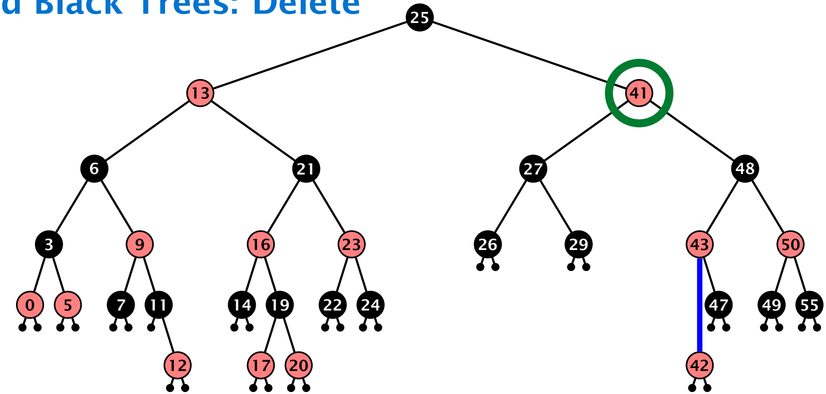
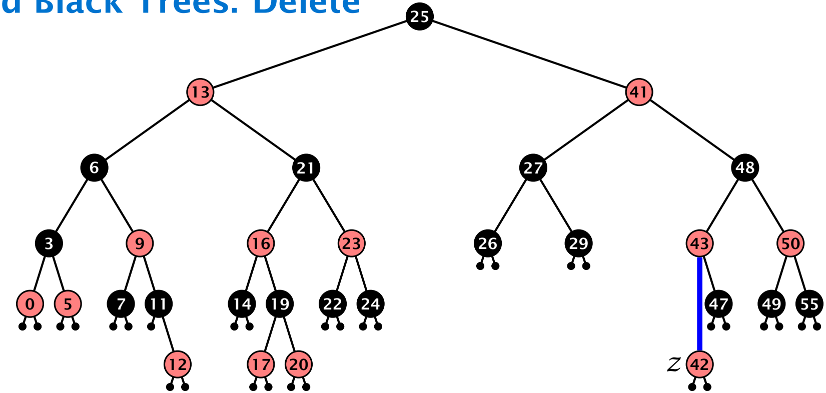


Case 3:

Element has two children

- do normal delete
- when replacing content by content of successor, don't change color of node

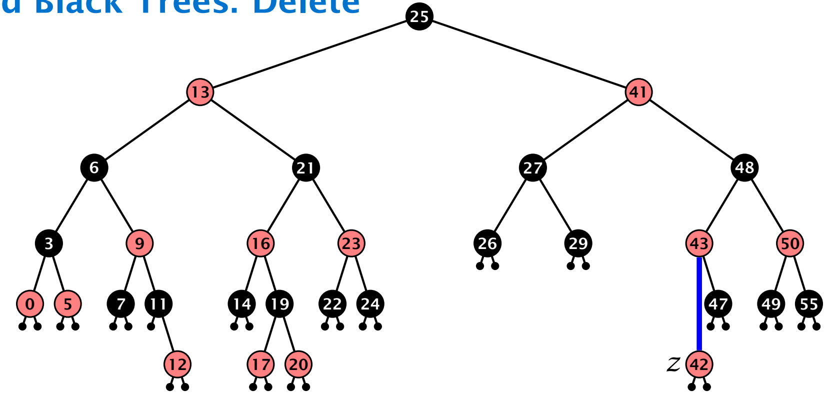
Red Black Trees: Delete



Delete:

- deleting black node messes up black-height property

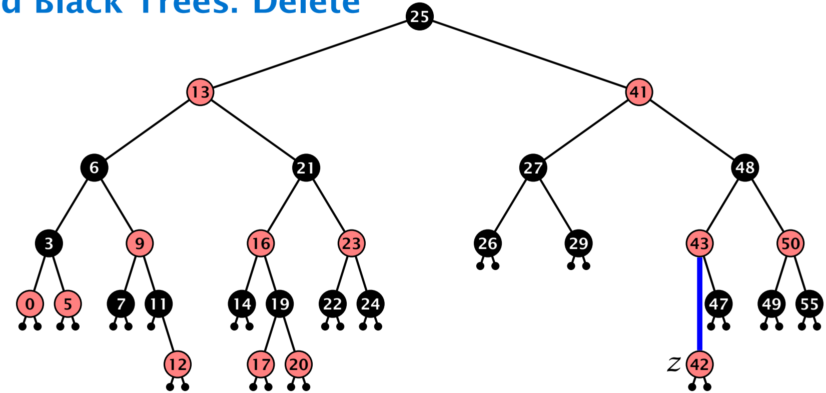
Red Black Trees: Delete



Delete:

- deleting black node messes up black-height property
- if z is red, we can simply color it black and everything is fine

Red Black Trees: Delete



Delete:

- deleting black node messes up black-height property
- if z is red, we can simply color it black and everything is fine
- the problem is if z is black (e.g. a dummy-leaf); we call a fix-up procedure to fix the problem.

Red Black Trees: Delete

Invariant of the fix-up algorithm

- the node z is black

Red Black Trees: Delete

Invariant of the fix-up algorithm

- the node z is black
- if we “assign” a fake black unit to the edge from z to its parent then the black-height property is fulfilled

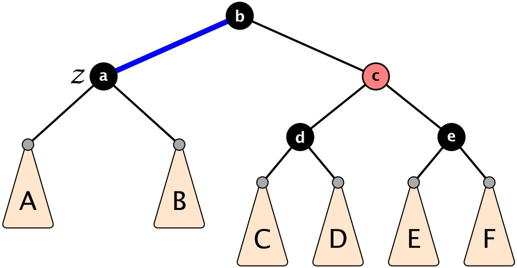
Red Black Trees: Delete

Invariant of the fix-up algorithm

- the node z is black
- if we “assign” a fake black unit to the edge from z to its parent then the black-height property is fulfilled

Goal: make rotations in such a way that you at some point can remove the fake black unit from the edge.

Case 1: Sibling of z is red



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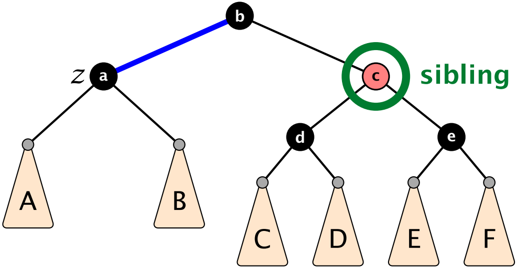

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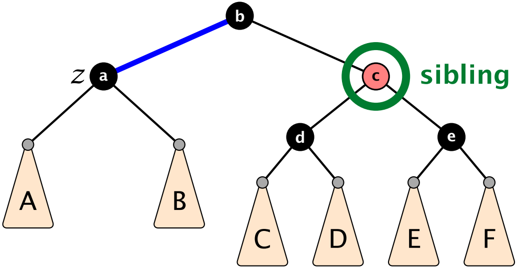
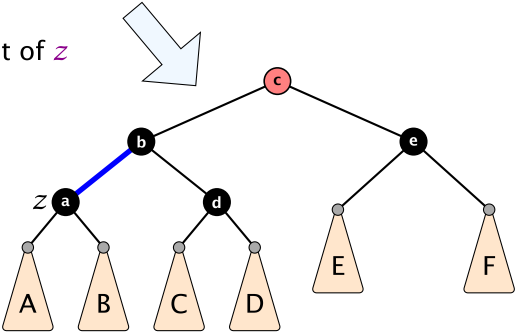


1. left-rotate around parent of z



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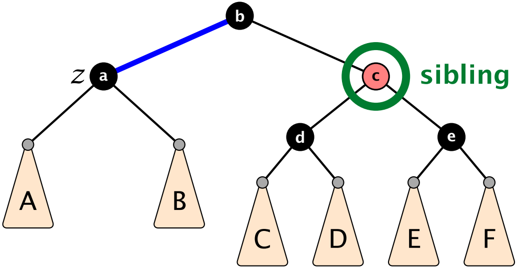
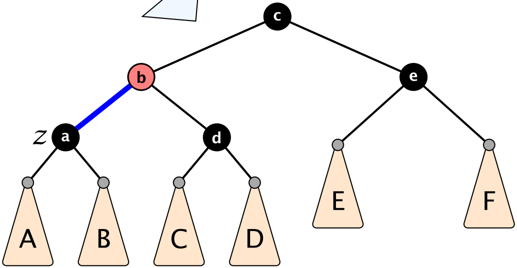
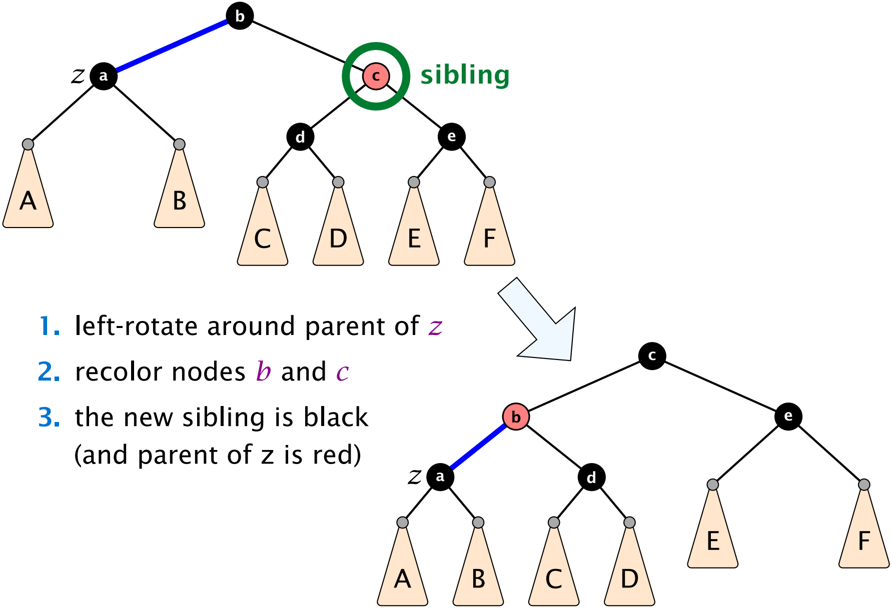


1. left-rotate around parent of z
2. recolor nodes b and c

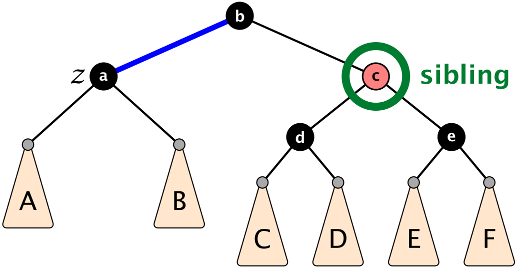


Case 1: Sibling of z is red

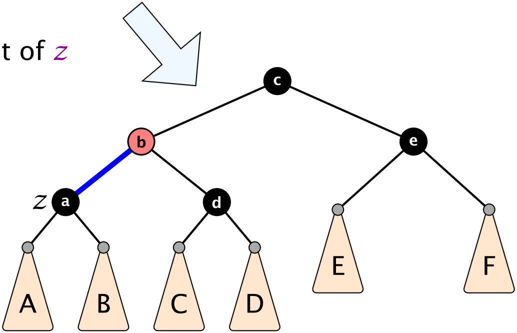


1. left-rotate around parent of z
2. recolor nodes b and c
3. the new sibling is black
   (and parent of z is red)

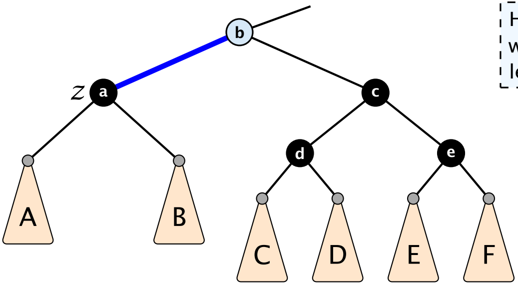
Case 1: Sibling of z is red



1. left-rotate around parent of z
2. recolor nodes b and c
3. the new sibling is black
   (and parent of z is red)
4. Case 2 (special),
   or Case 3, or Case 4

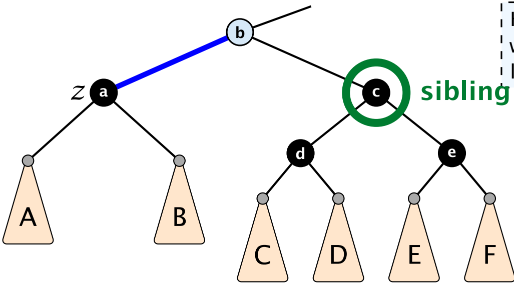


Case 2: Sibling is black with two black children



Here b is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

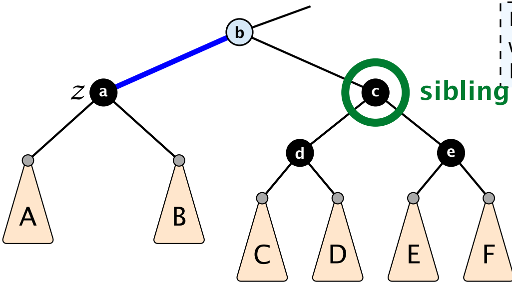
Case 2: Sibling is black with two black children

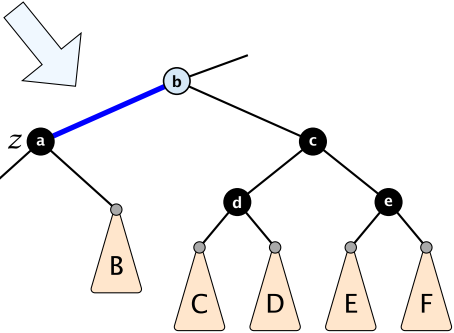


Here b is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

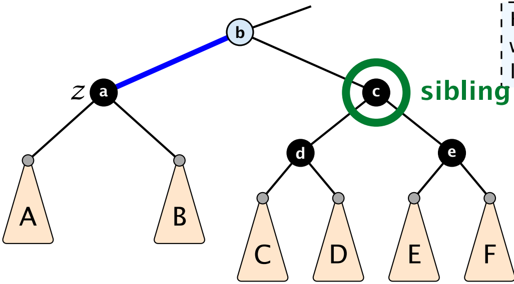
Case 2: Sibling is black with two black children

Here b is either black or red. If it is red we are in a special case that directly leads to a red-black tree.



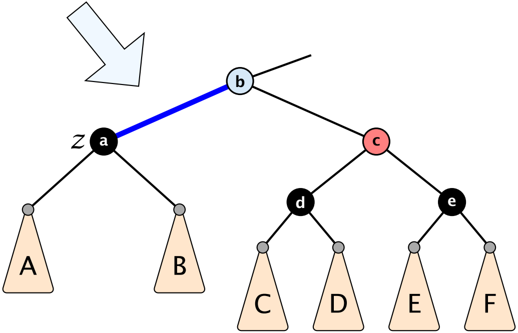


Case 2: Sibling is black with two black children

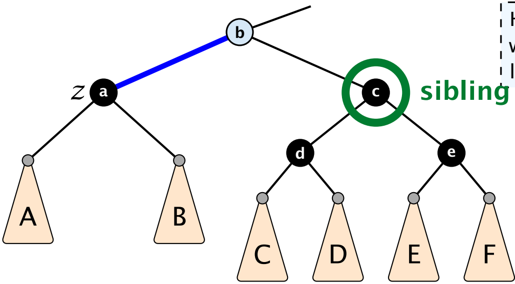


Here b is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

1. re-color node c

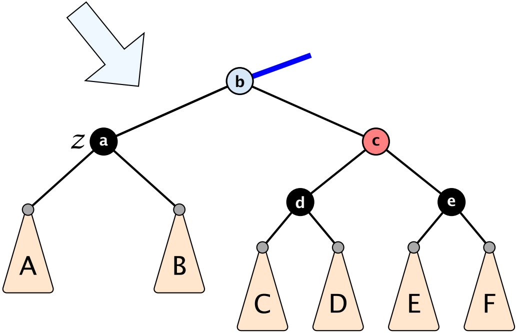


Case 2: Sibling is black with two black children

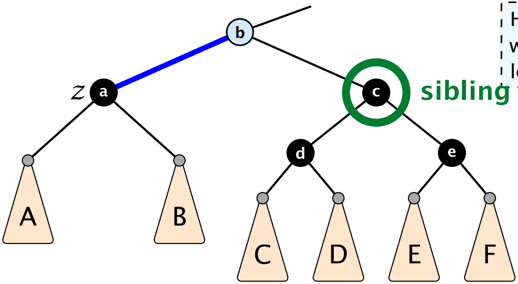


Here b is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

1. re-color node c
2. move fake black unit upwards

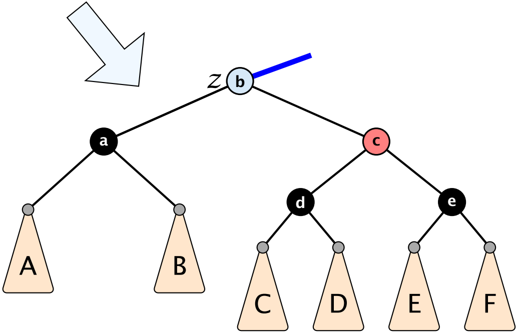


Case 2: Sibling is black with two black children

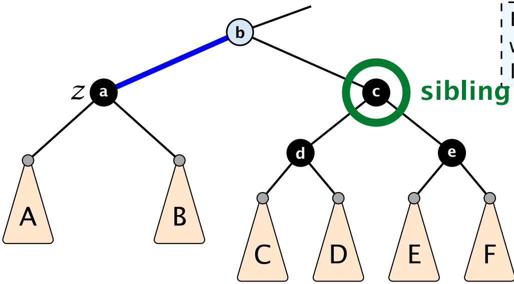


Here b is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

1. re-color node c
2. move fake black unit upwards
3. move z upwards

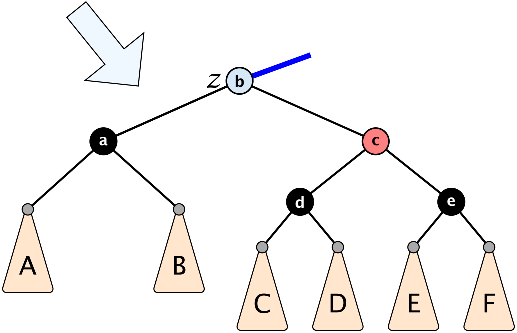


Case 2: Sibling is black with two black children

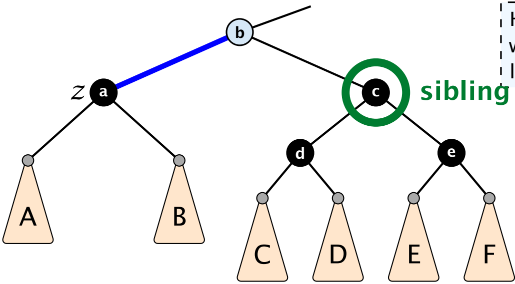


Here b is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

1. re-color node c
2. move fake black unit upwards
3. move z upwards
4. we made progress

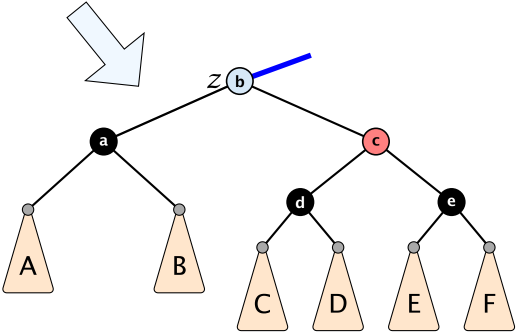


Case 2: Sibling is black with two black children

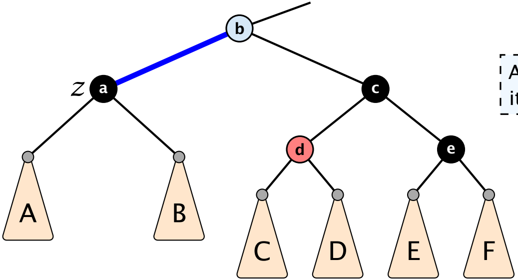


Here b is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

1. re-color node c
2. move fake black unit upwards
3. move z upwards
4. we made progress
5. if b is red we color it black and are done



Case 3: Sibling black with one black child to the right



Again the blue color of b indicates that it can either be black or red.

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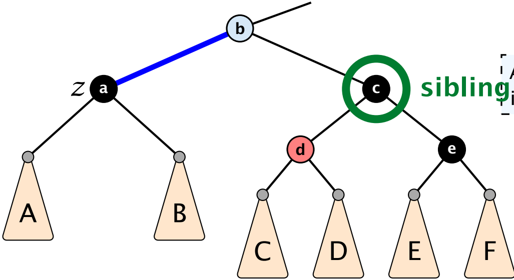

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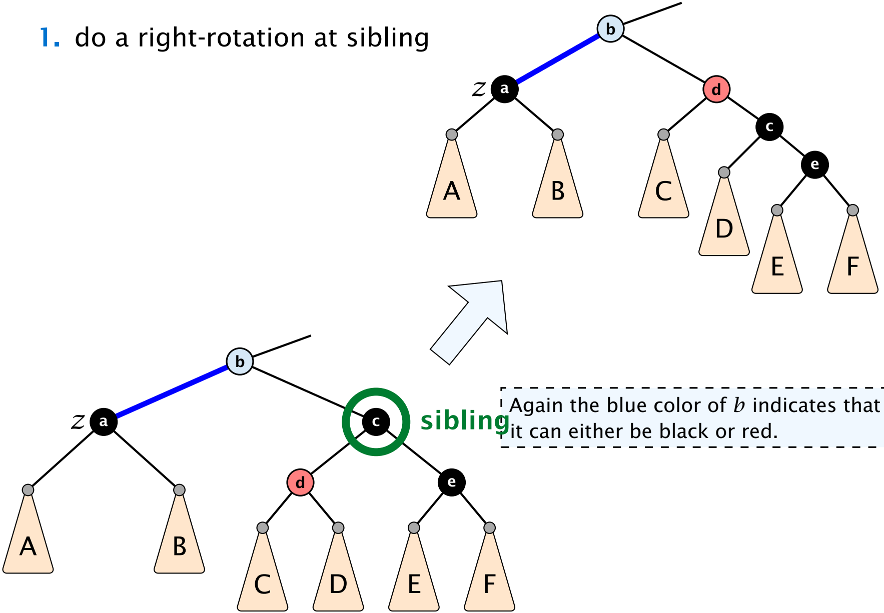

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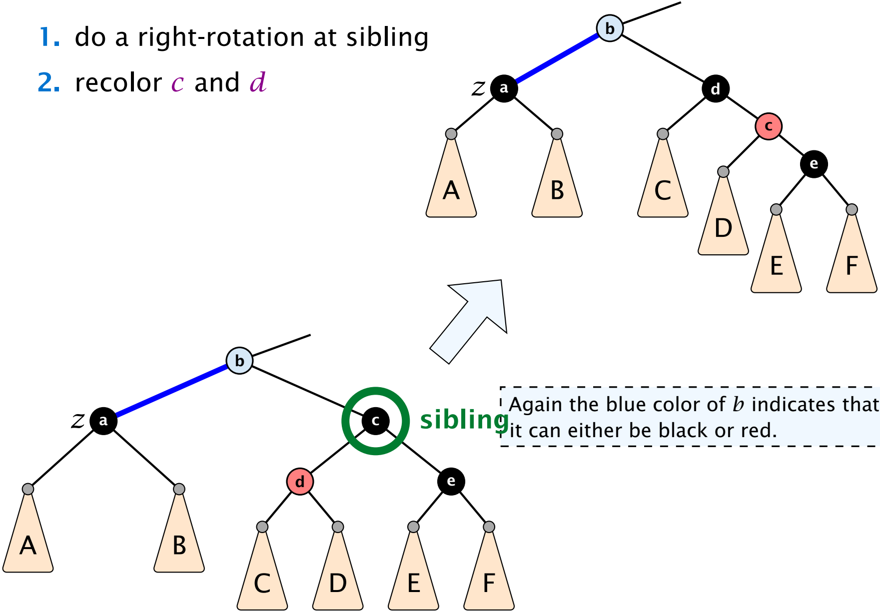
Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling
2. recolor c and d



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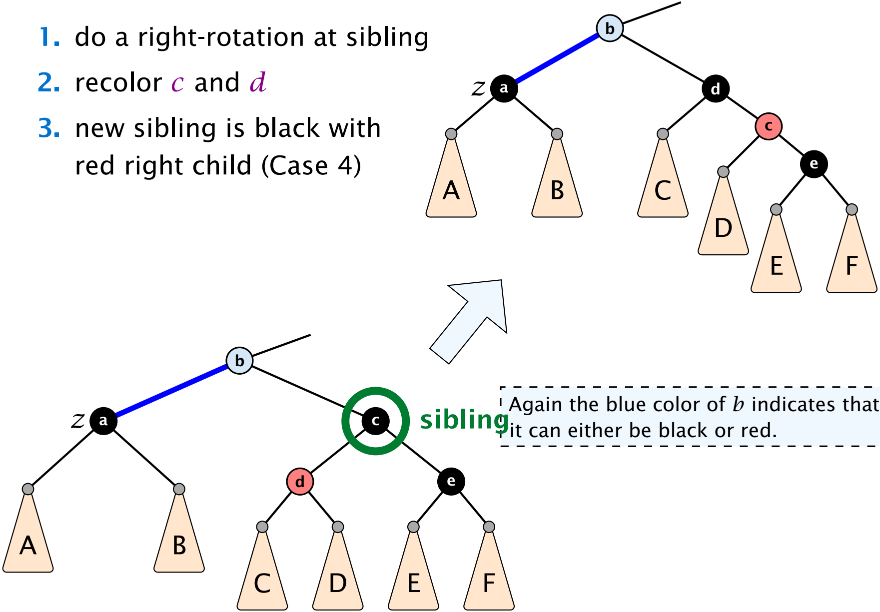
Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling
2. recolor c and d
3. new sibling is black with red right child (Case 4)



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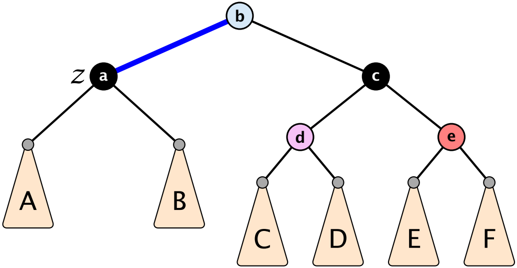

- Here b and d are either red or black but have possibly different colors.
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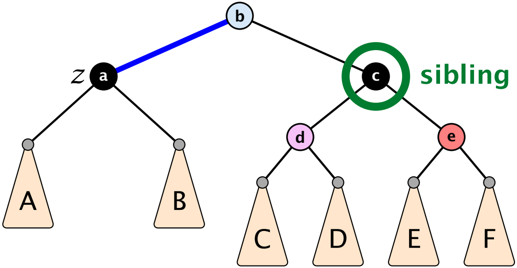

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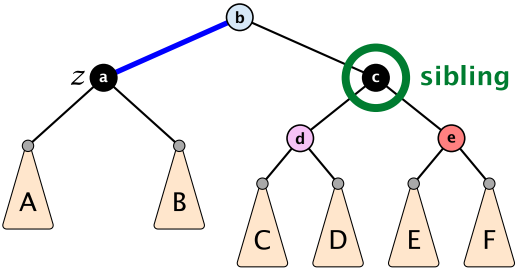

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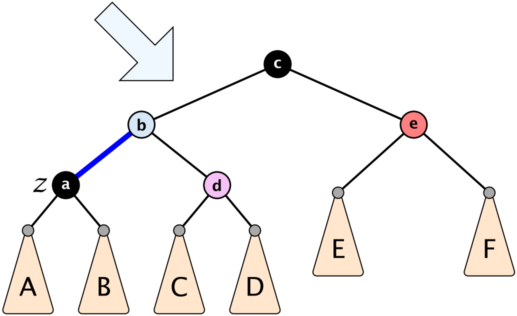
1. left-rotate around b



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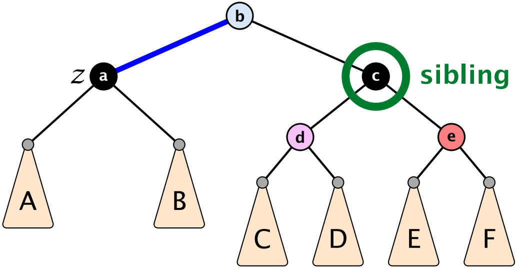

- Here b and d are either red or black but have possibly different colors.
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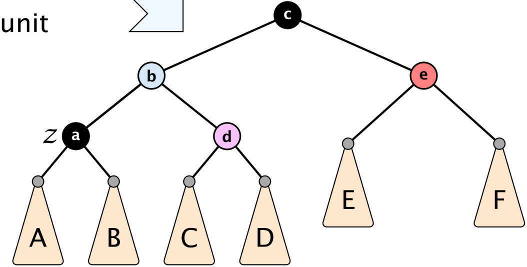
1. left-rotate around b
2. remove the fake black unit



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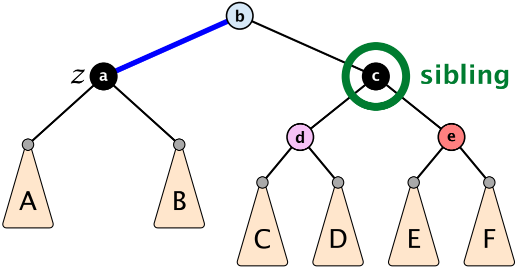

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1. left-rotate around b
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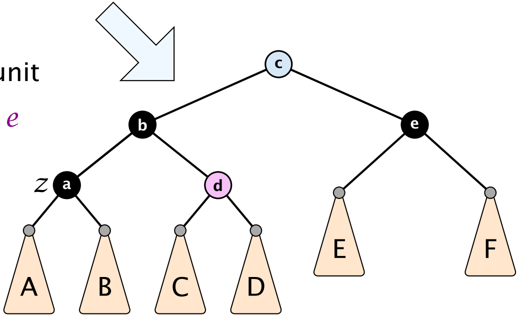
3. recolor nodes b, c, and e



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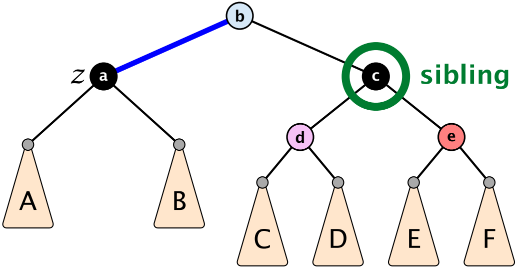

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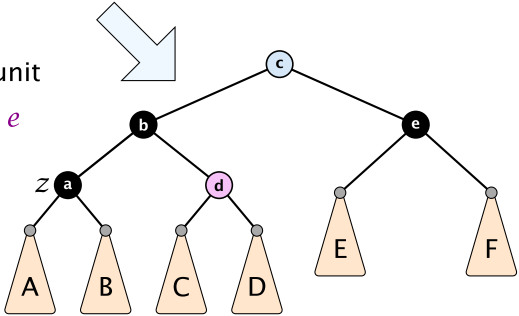
1. left-rotate around b
2. remove the fake black unit
3. recolor nodes b, c, and e
4. you have a valid red black tree



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Performing Case 2 at most $\mathcal{O}(\log n)$ times and every other step at most once, we get a red black tree. Hence, $\mathcal{O}(\log n)$ re-colorings and at most 3 rotations.

Splay Trees

Disadvantage of balanced search trees:

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Splay Trees:

- + after access, an element is moved to the root; splay(x)
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- – only amortized guarantee
- – read-operations change the tree

Splay Trees

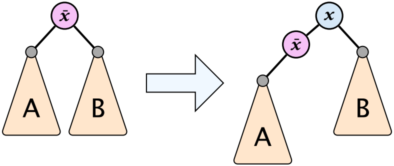
find(x)

- search for x according to a search tree
- let $\bar{x}$ be last element on search-path
- splay($\bar{x}$)

Splay Trees

insert(x)

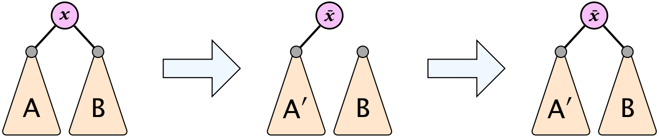
- search for x; $\bar{x}$ is last visited element during search (successer or predecessor of x)
- splay($\bar{x}$) moves $\bar{x}$ to the root
- insert x as new root



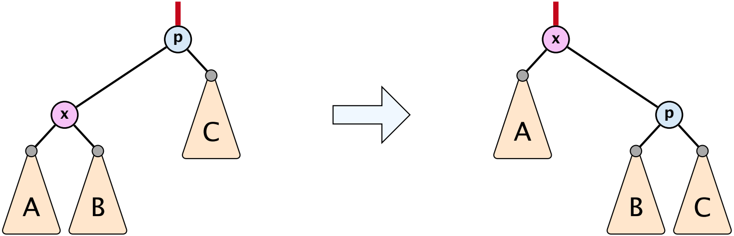
Splay Trees

delete(x)

- search for x; splay(x); remove x
- search largest element $\bar{x}$ in A
- splay($\bar{x}$) (on subtree A)
- connect root of B as right child of $\bar{x}$



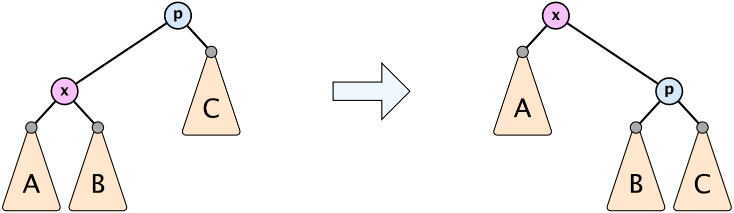
Move to Root



How to bring element to root?

- one (bad) option: moveToRoot(x)
- iteratively do rotation around parent of x until x is root
- if x is left child do right rotation otw. left rotation

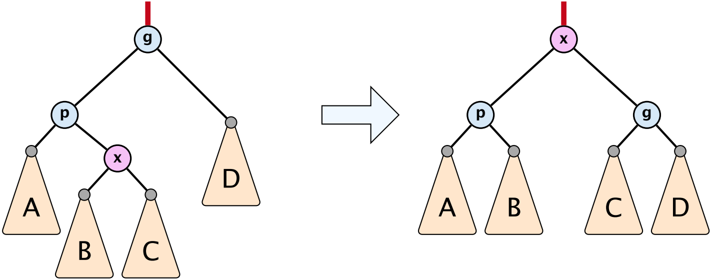
Splay: Zig Case



better option splay(x):

- zig case: if x is child of root do left rotation or right rotation around parent

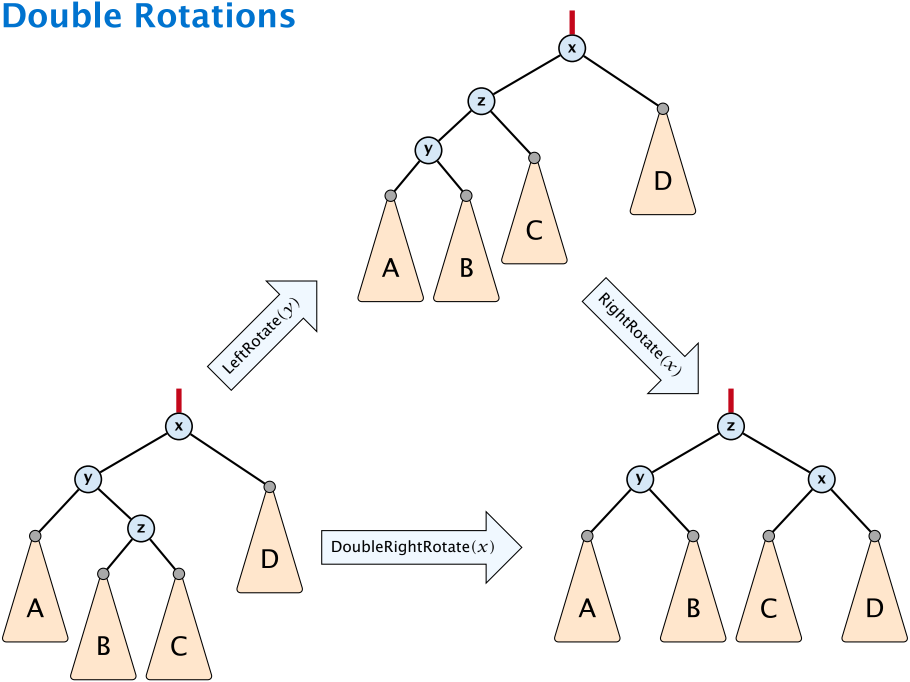
Splay: Zigzag Case



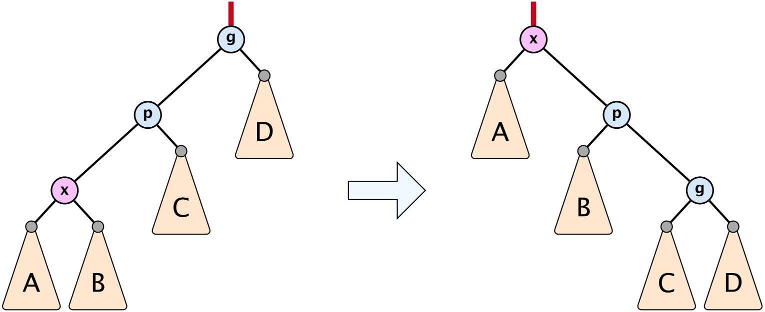
better option splay(x):

- zigzag case: if x is right child and parent of x is left child (or x left child parent of x right child)
- do double right rotation around grand-parent (resp. double left rotation)

Double Rotations



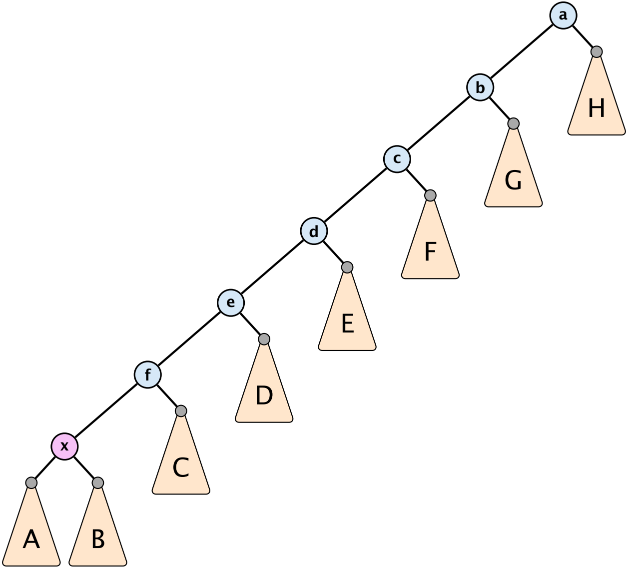
Splay: Zigzig Case



better option splay(x):

- zigzig case: if x is left child and parent of x is left child (or x right child, parent of x right child)
- do right roation around grand-parent followed by right rotation around parent (resp. left rotations)

Splay vs. Move to Root



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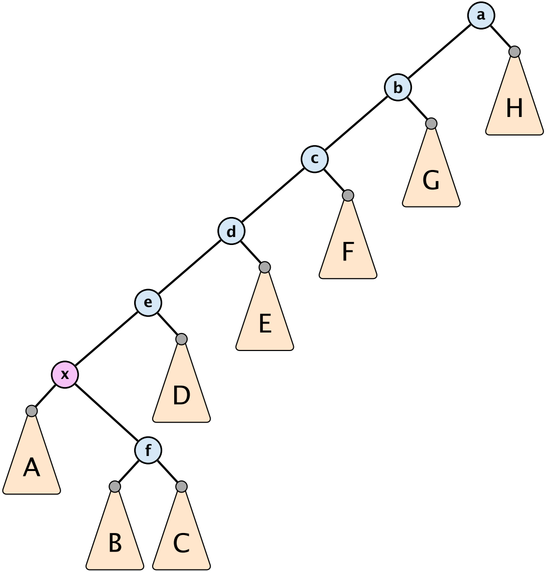

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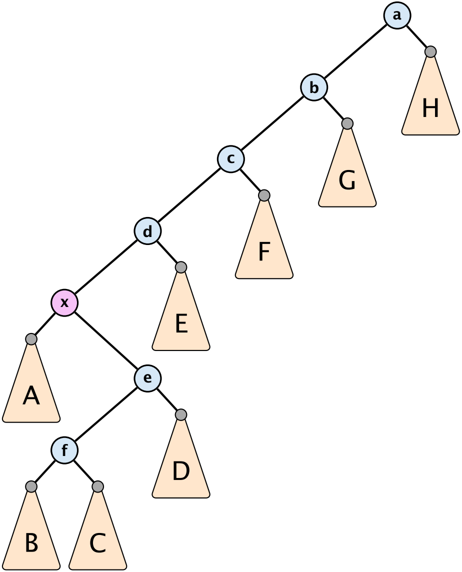

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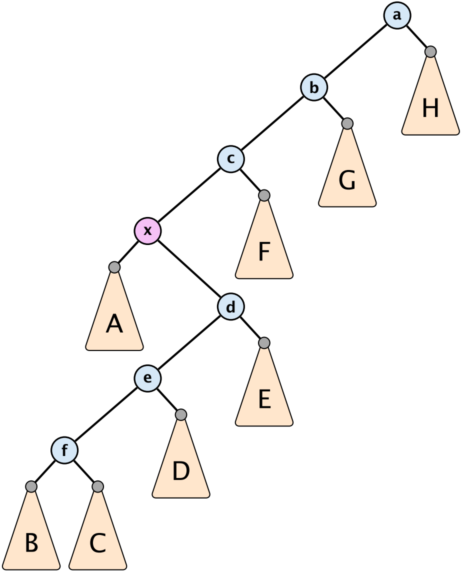

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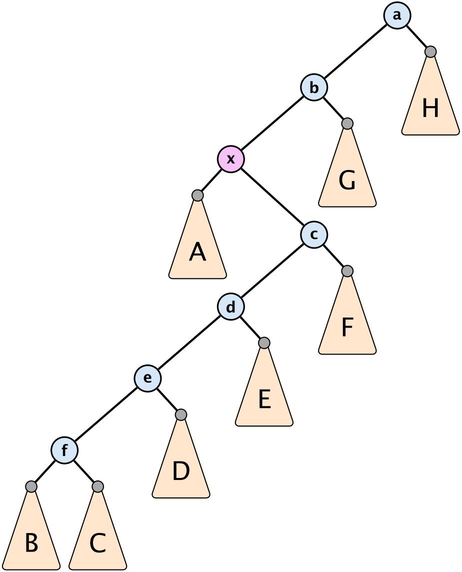

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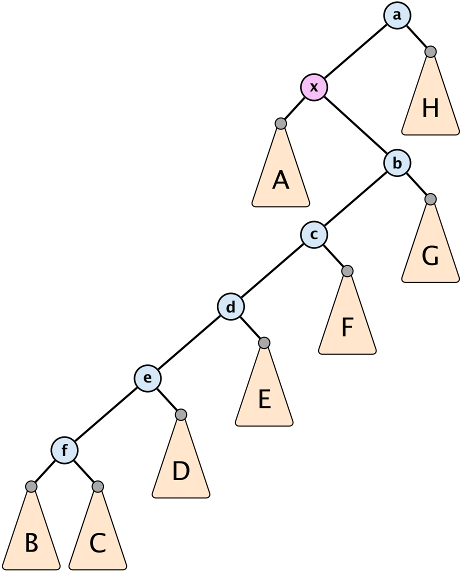

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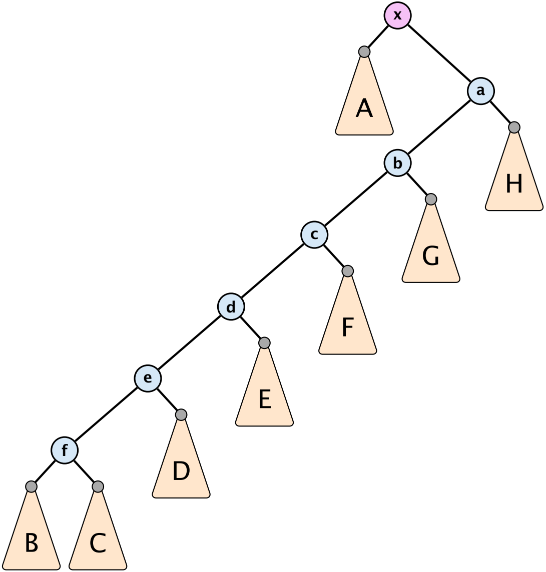

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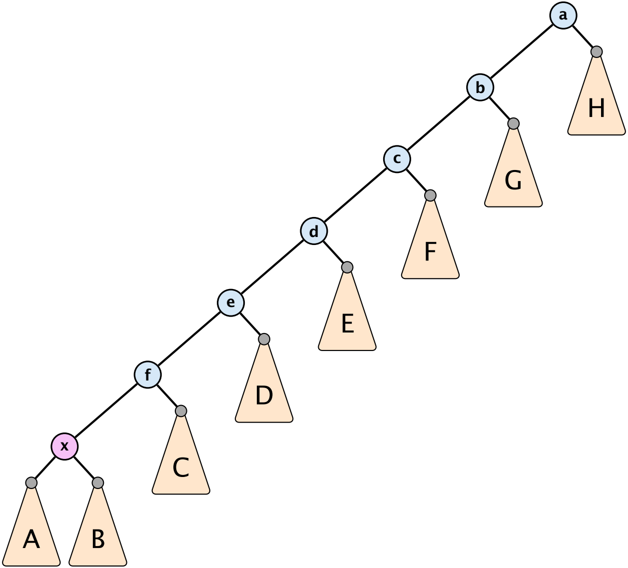

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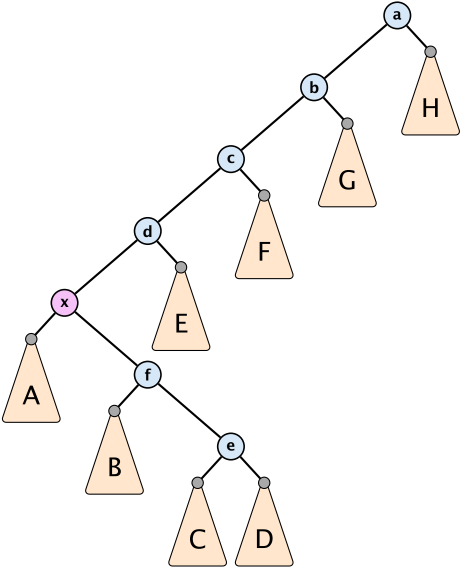

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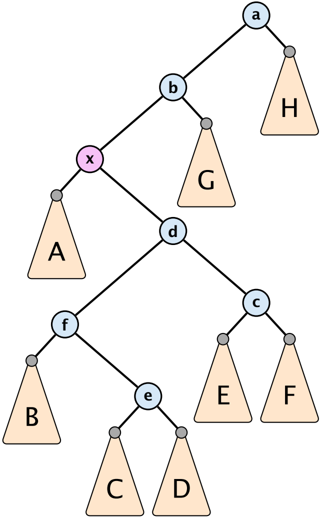

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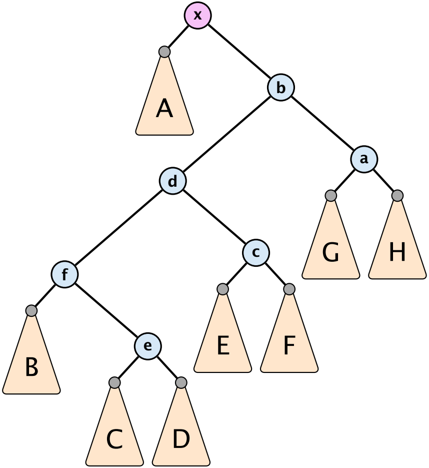

Static Optimality

Suppose we have a sequence of m find-operations. find(x) appears h_x times in this sequence.

The cost of a static search tree T is:

$$\operatorname{cost}(T) = m + \sum_x h_x \operatorname{depth}_T(x)$$

The total cost for processing the sequence on a splay-tree is $\mathcal{O}(\operatorname{cost}(T_{\text{min}}))$, where T_{min} is an optimal static search tree.

Dynamic Optimality

Let S be a sequence with m find-operations.

Let A be a data-structure based on a search tree:

- the cost for accessing element x is $1 + \mathrm{depth}(x)$;
- after accessing x the tree may be re-arranged through rotations;

Conjecture:
A splay tree that only contains elements from S has cost $\mathcal{O}(\mathrm{cost}(A, S))$, for processing S.

Lemma 16

Splay Trees have an amortized running time of $\mathcal{O}(\log n)$ *for all operations.*

Amortized Analysis

Definition 17

A data structure with operations $\mathrm{op}_1(), \dots, \mathrm{op}_k()$ has amortized running times $t_1, \dots, t_k$ for these operations if the following holds.

Suppose you are given a sequence of operations (starting with an empty data-structure) that operate on at most n elements, and let k_i denote the number of occurences of $\mathrm{op}_i()$ within this sequence. Then the actual running time must be at most $\sum_i k_i \cdot t_i(n)$.

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Introduce a potential for the data structure.

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$$\sum_{i=1}^{k} c_i$$

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- Show that $\Phi(D_i) \geq \Phi(D_0)$.

Then

$$\sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} c_i + \Phi(D_k) - \Phi(D_0) = \sum_{i=1}^{k} \hat{c}_i$$

This means the amortized costs can be used to derive a bound on the total cost.

Example: Stack

Stack

- $S.\mathrm{push}()$
- $S.\mathrm{pop}()$
- $S.\mathrm{multipop}(k)$: removes k items from the stack. If the stack currently contains less than k items it empties the stack.
- The user has to ensure that pop and multipop do not generate an underflow.

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- The user has to ensure that pop and multipop do not generate an underflow.

Actual cost:

- $\boldsymbol{S}.\,\mathbf{push()}$: cost 1.
- $\boldsymbol{S}.\,\mathbf{pop()}$: cost 1.
- $\boldsymbol{S}.\,\mathbf{multipop}(\boldsymbol{k})$: cost $\min\{\mathrm{size}, k\} = k$.

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- $\boldsymbol{S.}\,\mathbf{multipop(}\boldsymbol{k}\mathbf{)}$: cost

$$\hat{C}_{\text{mp}} = C_{\text{mp}} + \Delta\Phi = \min\{\text{size}, k\} - \min\{\text{size}, k\} \le 0 \enspace .$$

Example: Binary Counter

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Actual cost:

- Changing bit from 0 to 1: cost 1.
- Changing bit from 1 to 0: cost 1.
- Increment: cost is $k + 1$, where k is the number of consecutive ones in the least significant bit-positions (e.g, 001101 has $k = 1$).

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Choose potential function $\Phi(x) = k$, where k denotes the number of ones in the binary representation of x.

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- Changing bit from 1 to 0:

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Amortized cost:

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$$\hat{C}_{0\to1} = C_{0\to1} + \Delta\Phi = 1 + 1 \le 2 \ .$$

- Changing bit from 1 to 0:

$$\hat{C}_{1\to0} = C_{1\to0} + \Delta\Phi = 1 - 1 \le 0 \ .$$

- Increment: Let k denotes the number of consecutive ones in the least significant bit-positions. An increment involves k $(1 \to 0)$-operations, and one $(0 \to 1)$-operation.

  Hence, the amortized cost is $k\hat{C}_{1\to0} + \hat{C}_{0\to1} \le 2$.

Splay Trees

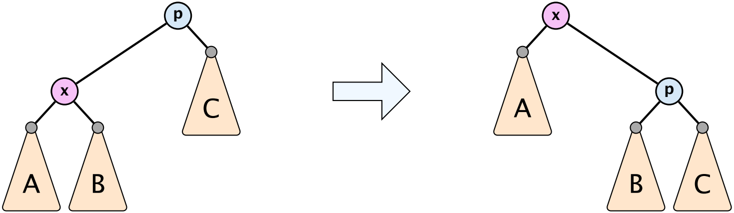
potential function for splay trees:

- size $\mathrm{s}(x) = |T_x|$
- rank $\mathrm{r}(x) = \log_2(s(x))$
- $\Phi(T) = \sum_{v \in T} r(v)$

amortized cost = real cost + potential change

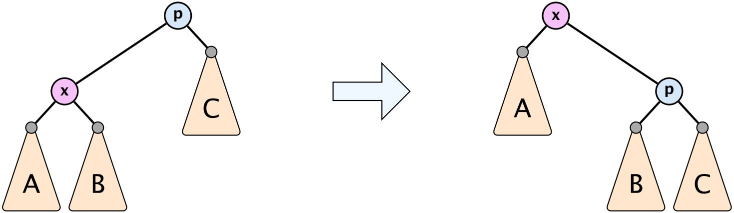
The cost is essentially the cost of the splay-operation, which is 1 plus the number of rotations.

Splay: Zig Case



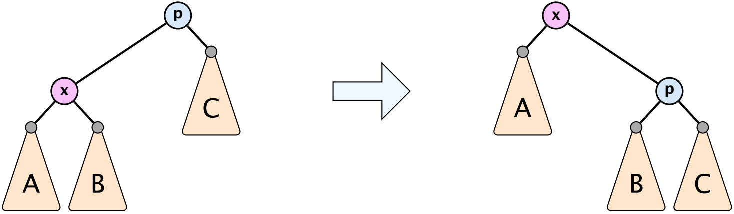
$\Delta\Phi =$

Splay: Zig Case



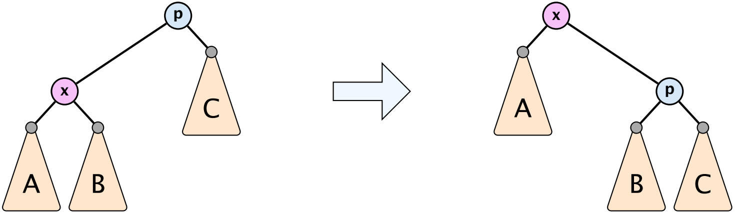
$$\Delta\Phi = r'(x) + r'(p) - r(x) - r(p)$$

Splay: Zig Case



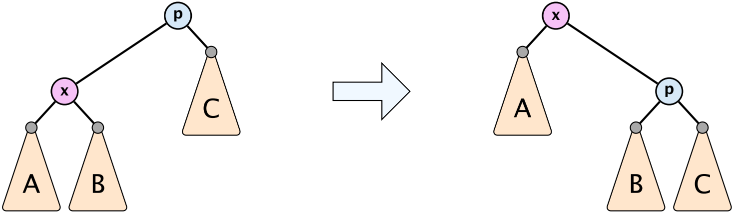
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p)\\ &= r'(p) - r(x)\end{aligned}$$

Splay: Zig Case



$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p) \\ &= r'(p) - r(x) \\ &\le r'(x) - r(x)\end{aligned}$$

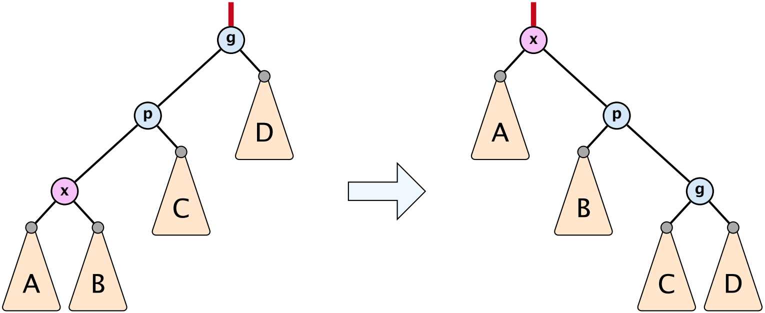
Splay: Zig Case



$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p)\\ &= r'(p) - r(x)\\ &\le r'(x) - r(x)\end{aligned}$$

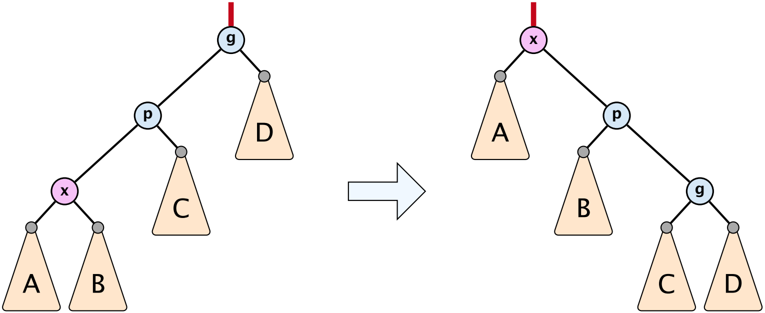
$$\mathrm{cost}_{\mathrm{zig}} \le 1 + 3(r'(x) - r(x))$$

Splay: Zigzig Case



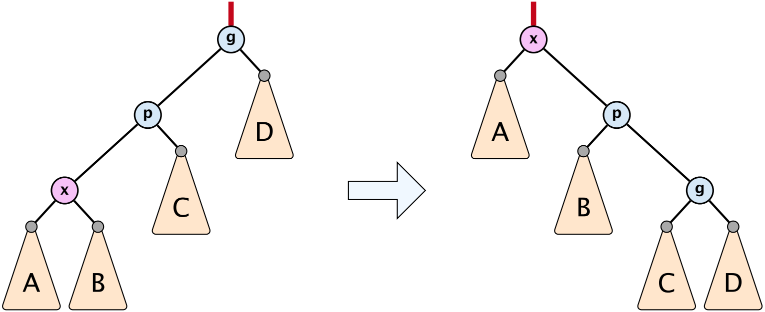
$$\Delta\Phi =$$

Splay: Zigzig Case



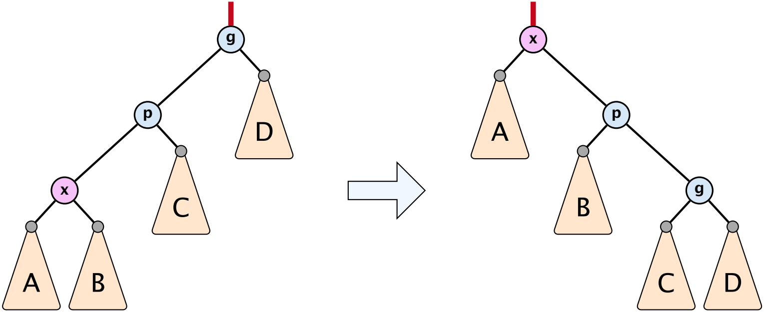
$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$

Splay: Zigzig Case



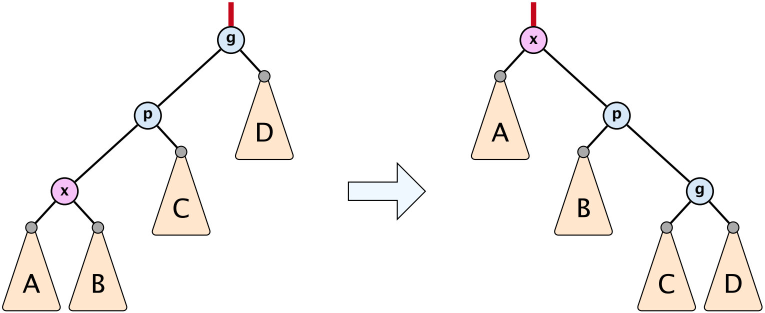
$$
\begin{aligned}
\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\
&= r'(p) + r'(g) - r(x) - r(p)
\end{aligned}
$$

Splay: Zigzig Case



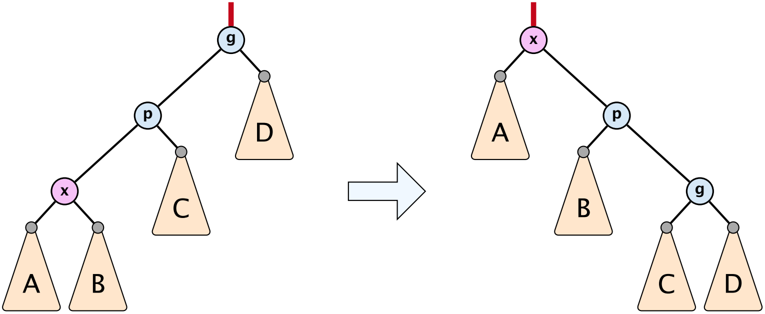
$$
\begin{aligned}
\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\
&= r'(p) + r'(g) - r(x) - r(p) \\
&\leq r'(x) + r'(g) - r(x) - r(x)
\end{aligned}
$$

Splay: Zigzig Case



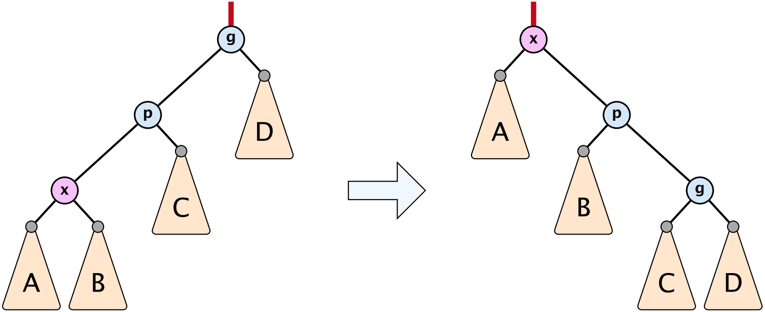
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)\\ &= r'(p) + r'(g) - r(x) - r(p)\\ &\leq r'(x) + r'(g) - r(x) - r(x)\\ &= r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x)\end{aligned}$$

Splay: Zigzig Case



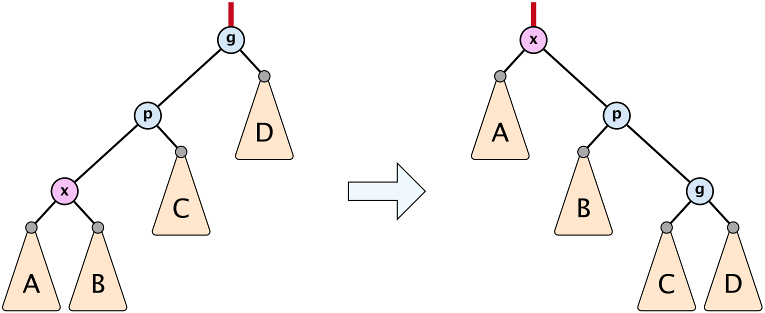
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)\\ &= r'(p) + r'(g) - r(x) - r(p)\\ &\le r'(x) + r'(g) - r(x) - r(x)\\ &= r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x)\\ &= -2r'(x) + r'(g) + r(x) + 3(r'(x) - r(x))\end{aligned}$$

Splay: Zigzig Case



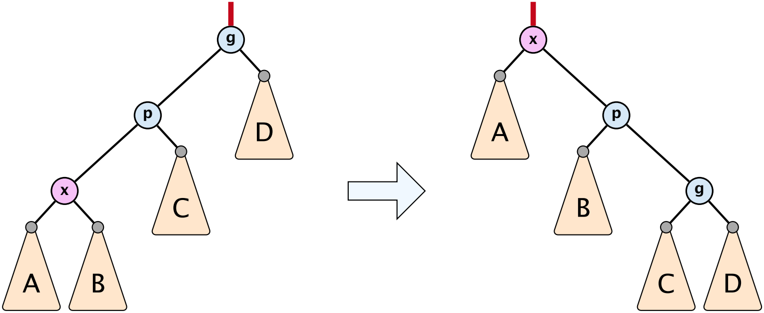
$$\begin{aligned}
\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\
&= r'(p) + r'(g) - r(x) - r(p) \\
&\le r'(x) + r'(g) - r(x) - r(x) \\
&= r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x) \\
&= -2r'(x) + r'(g) + r(x) + 3(r'(x) - r(x)) \\
&\le -2 + 3(r'(x) - r(x))
\end{aligned}$$

Splay: Zigzig Case



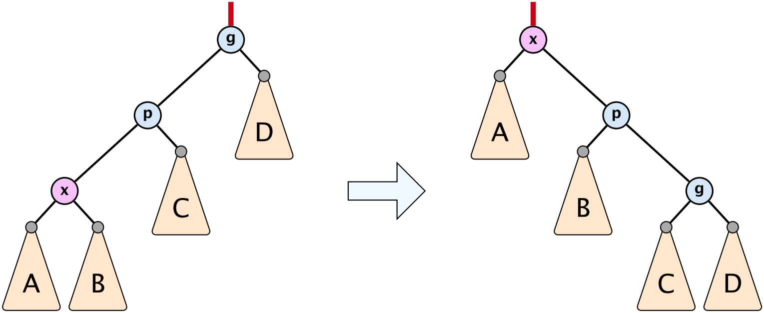
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)\\ &= r'(p) + r'(g) - r(x) - r(p)\\ &\le r'(x) + r'(g) - r(x) - r(x)\\ &= r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x)\\ &= -2r'(x) + r'(g) + r(x) + 3(r'(x) - r(x))\\ &\le -2 + 3(r'(x) - r(x)) \quad \Rightarrow \text{cost}_{\text{zigzig}} \le 3(r'(x) - r(x))\end{aligned}$$

Splay: Zigzig Case



$$\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)$$

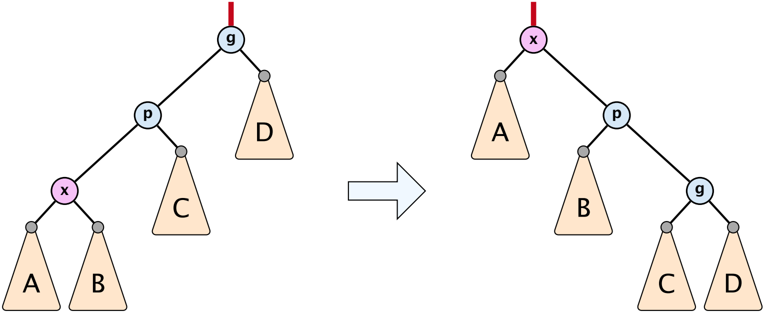
Splay: Zigzig Case



$$\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)$$

$$= \frac{1}{2}\Big(\log(s(x)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

Splay: Zigzig Case

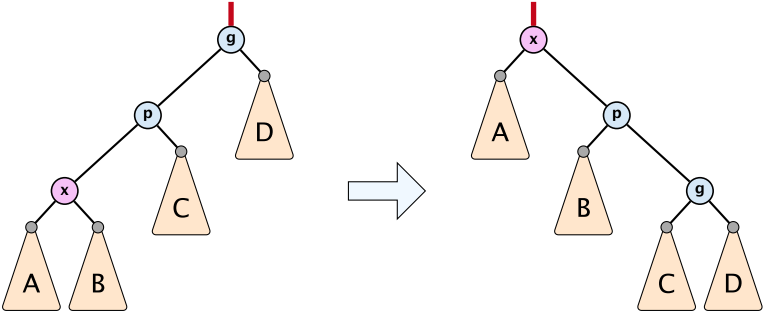


$$\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)$$

$$= \frac{1}{2}\Big(\log(s(x)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

$$= \frac{1}{2}\log\Big(\frac{s(x)}{s'(x)}\Big) + \frac{1}{2}\log\Big(\frac{s'(g)}{s'(x)}\Big)$$

Splay: Zigzig Case



$$\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)$$

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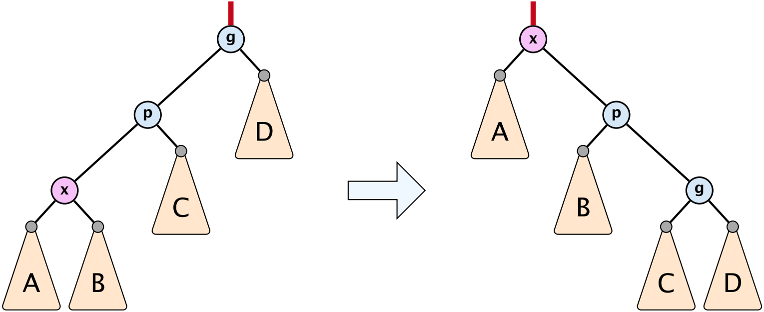
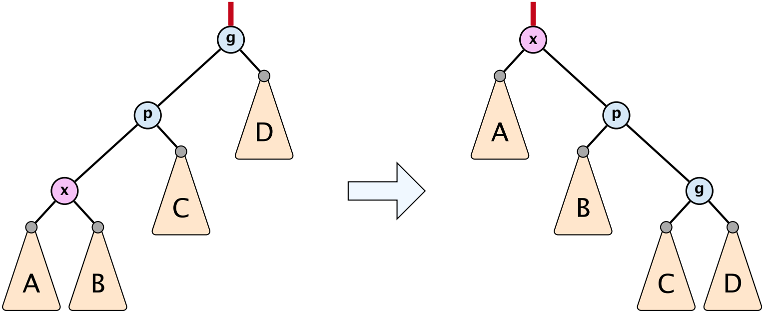
$$\le \log\Big(\frac{1}{2}\frac{s(x)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\Big)$$

Splay: Zigzig Case



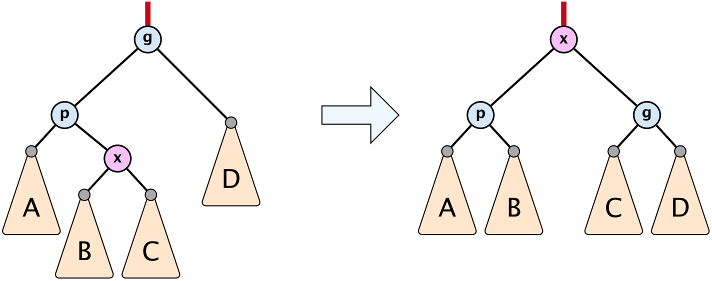
$$
\begin{aligned}
&\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big) \\
&\qquad = \frac{1}{2}\Big(\log(s(x)) + \log(s'(g)) - 2\log(s'(x))\Big) \\
&\qquad = \frac{1}{2}\log\Big(\frac{s(x)}{s'(x)}\Big) + \frac{1}{2}\log\Big(\frac{s'(g)}{s'(x)}\Big) \\
&\qquad \le \log\Big(\frac{1}{2}\frac{s(x)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\Big) \le \log\Big(\frac{1}{2}\Big)
\end{aligned}
$$

Splay: Zigzig Case



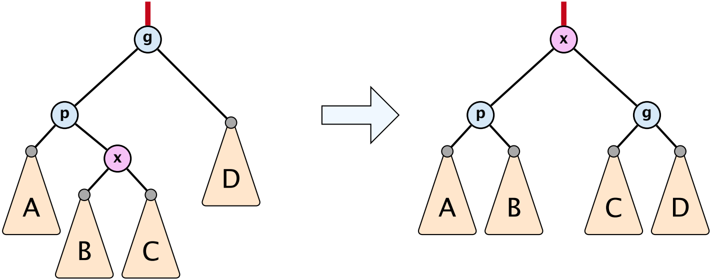
$$
\begin{aligned}
\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)
&= \frac{1}{2}\Big(\log(s(x)) + \log(s'(g)) - 2\log(s'(x))\Big)\\
&= \frac{1}{2}\log\Big(\frac{s(x)}{s'(x)}\Big) + \frac{1}{2}\log\Big(\frac{s'(g)}{s'(x)}\Big)\\
&\le \log\Big(\frac{1}{2}\frac{s(x)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\Big) \le \log\Big(\frac{1}{2}\Big) = -1
\end{aligned}
$$

Splay: Zigzag Case



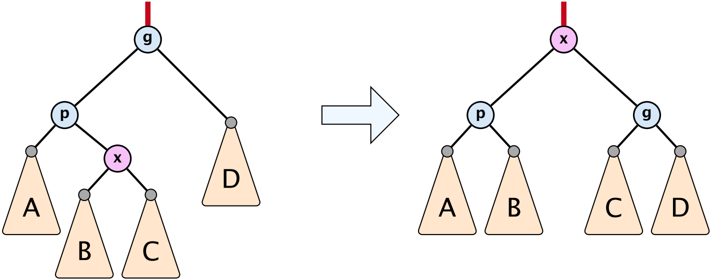
$\Delta\Phi =$

Splay: Zigzag Case



$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$

Splay: Zigzag Case



$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p)\end{aligned}$$

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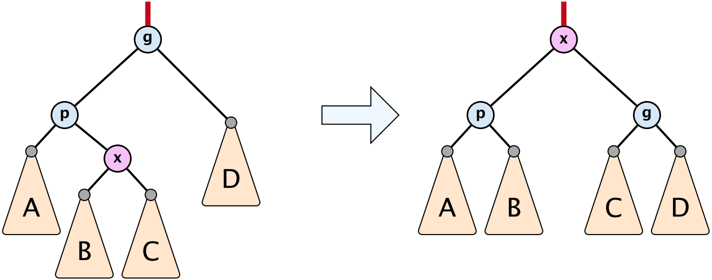

$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)\\ &= r'(p) + r'(g) - r(x) - r(p)\\ &\leq r'(p) + r'(g) - r(x) - r(x)\end{aligned}$$

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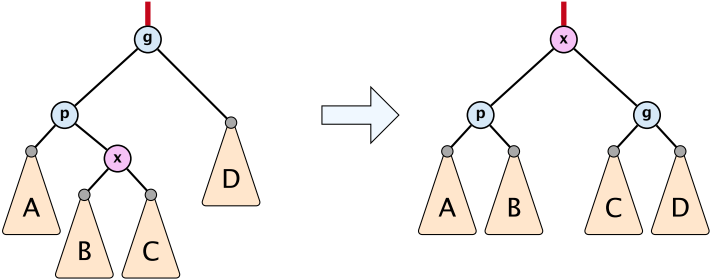

$$
\begin{aligned}
\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\
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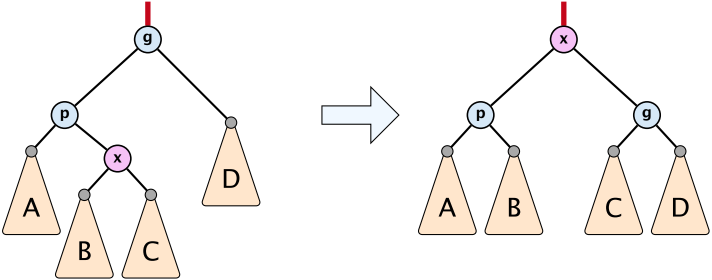
&= r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x)
\end{aligned}
$$

Splay: Zigzag Case



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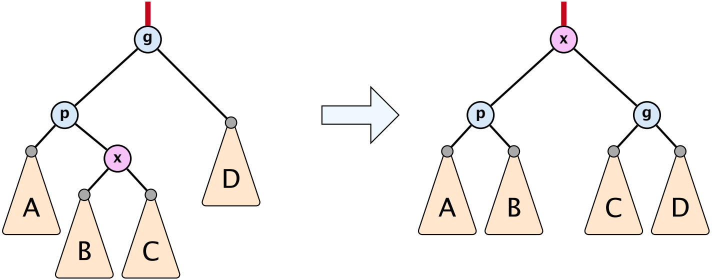
Splay: Zigzag Case



$$
\begin{aligned}
\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\
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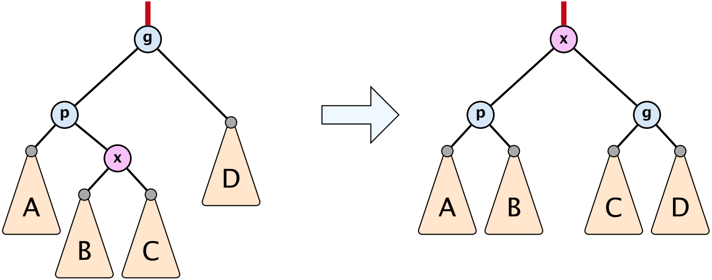
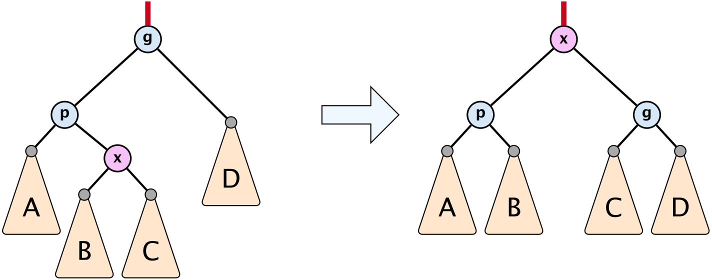
&= r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x) \\
&\le -2 + 2(r'(x) - r(x)) \quad \Rightarrow \text{cost}_{\text{zigzag}} \le 3(r'(x) - r(x))
\end{aligned}
$$

Splay: Zigzag Case



$$\frac{1}{2}\Big(r'(p) + r'(g) - 2r'(x)\Big)$$

Splay: Zigzag Case



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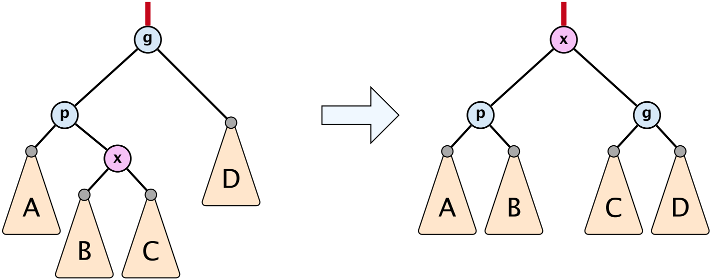
Splay: Zigzag Case



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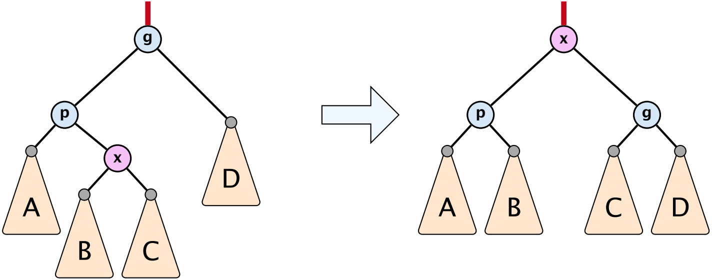

$$\frac{1}{2}\Big(r'(p)+r'(g)-2r'(x)\Big)$$

$$=\frac{1}{2}\Big(\log(s'(p))+\log(s'(g))-2\log(s'(x))\Big)$$

$$\leq \log\left(\frac{1}{2}\frac{s'(p)}{s'(x)}+\frac{1}{2}\frac{s'(g)}{s'(x)}\right)\leq \log\left(\frac{1}{2}\right)$$

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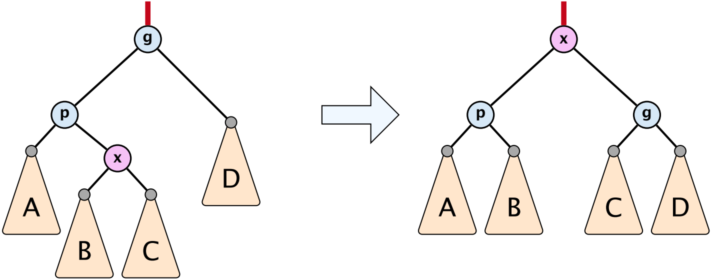

$$\frac{1}{2}\Big(r'(p) + r'(g) - 2r'(x)\Big)$$

$$= \frac{1}{2}\Big(\log(s'(p)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

$$\leq \log\left(\frac{1}{2}\frac{s'(p)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\right) \leq \log\left(\frac{1}{2}\right) = -1$$

Amortized cost of the whole splay operation:

$$
\begin{aligned}
&\leq 1 + 1 + \sum_{\text{steps } t} 3(r_t(x) - r_{t-1}(x)) \\
&= 2 + 3(r(\text{root}) - r_0(x)) \\
&\leq \mathcal{O}(\log n)
\end{aligned}
$$

7.4 Augmenting Data Structures

Suppose you want to develop a data structure with:

- **Insert(x)**: insert element x.
- **Search(k)**: search for element with key k.
- **Delete(x)**: delete element referenced by pointer x.
- **find-by-rank(ℓ)**: return the ℓ-th element; return "error" if the data-structure contains less than ℓ elements.

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Augment an existing data-structure instead of developing a new one.

7.4 Augmenting Data Structures

How to augment a data-structure

1. choose an underlying data-structure

- Of course, the above steps heavily depend on each other. For example it makes no sense to choose additional information to be stored (Step 2), and later realize that either the information cannot be maintained efficiently (Step 3) or is not sufficient to support the new operations (Step 4).
- However, the above outline is a good way to describe/document a new data-structure.

7.4 Augmenting Data Structures

How to augment a data-structure

1. choose an underlying data-structure
2. determine additional information to be stored in the underlying structure

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7.4 Augmenting Data Structures

How to augment a data-structure

1. choose an underlying data-structure
2. determine additional information to be stored in the underlying structure
3. verify/show how the additional information can be maintained for the basic modifying operations on the underlying structure.

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How to augment a data-structure

1. choose an underlying data-structure
2. determine additional information to be stored in the underlying structure
3. verify/show how the additional information can be maintained for the basic modifying operations on the underlying structure.
4. develop the new operations

- Of course, the above steps heavily depend on each other. For example it makes no sense to choose additional information to be stored (Step 2), and later realize that either the information cannot be maintained efficiently (Step 3) or is not sufficient to support the new operations (Step 4).
- However, the above outline is a good way to describe/document a new data-structure.

7.4 Augmenting Data Structures

Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

1. We choose a red-black tree as the underlying data-structure.

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1. We choose a red-black tree as the underlying data-structure.
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7.4 Augmenting Data Structures

Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

1. We choose a red-black tree as the underlying data-structure.
2. We store in each node v the size of the sub-tree rooted at v.
3. We need to be able to update the size-field in each node without asymptotically affecting the running time of insert, delete, and search. We come back to this step later...

7.4 Augmenting Data Structures

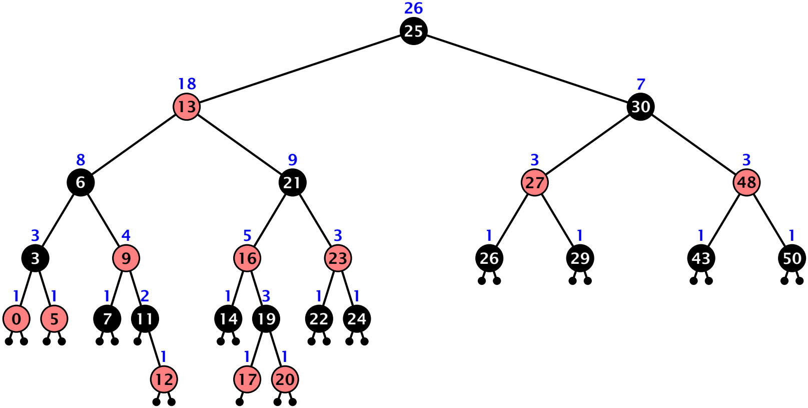
Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

4. How does find-by-rank work?
   Find-by-rank(k) := Select(root,k) with

Algorithm 1 Select(x, i)

```
if $x = \text{null}$ then return error
if $\text{left}[x] \neq \text{null}$ then $r \leftarrow \text{left}[x].\text{size} + 1$ else $r \leftarrow 1$
if $i = r$ then return $x$
if $i < r$ then
    return Select$(\text{left}[x], i)$
else
    return Select$(\text{right}[x], i - r)$
```

Select(x, i)

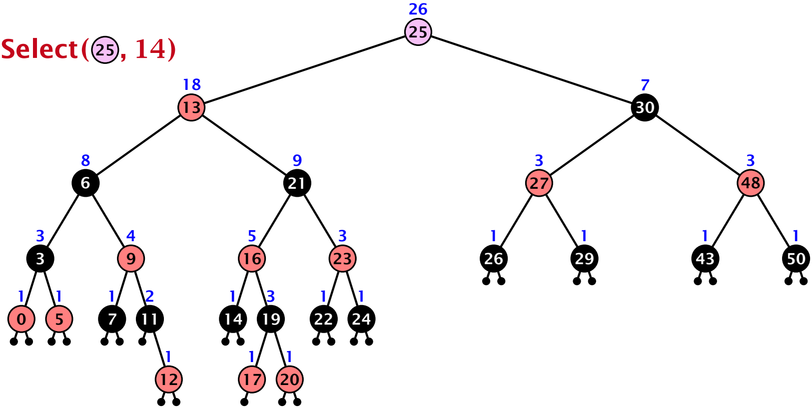


Find-by-rank:

- decide whether you have to proceed into the left or right sub-tree
- adjust the rank that you are searching for if you go right

Select(x, i)

Select(25, 14)



Find-by-rank:

- decide whether you have to proceed into the left or right sub-tree
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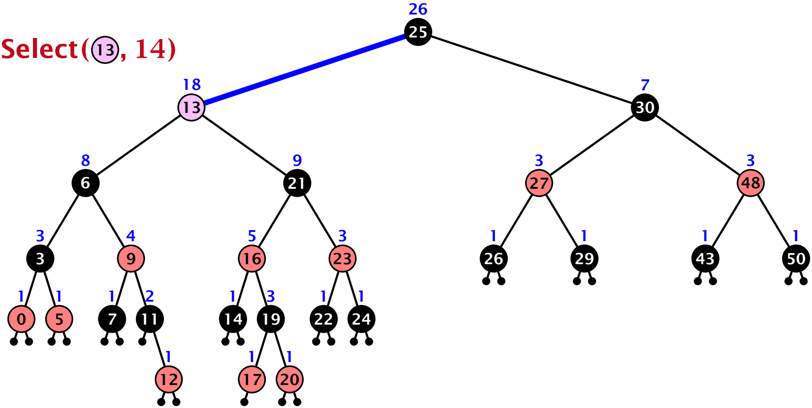

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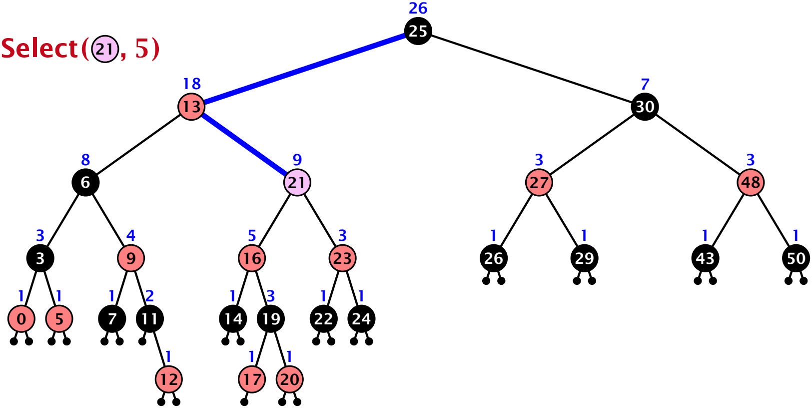

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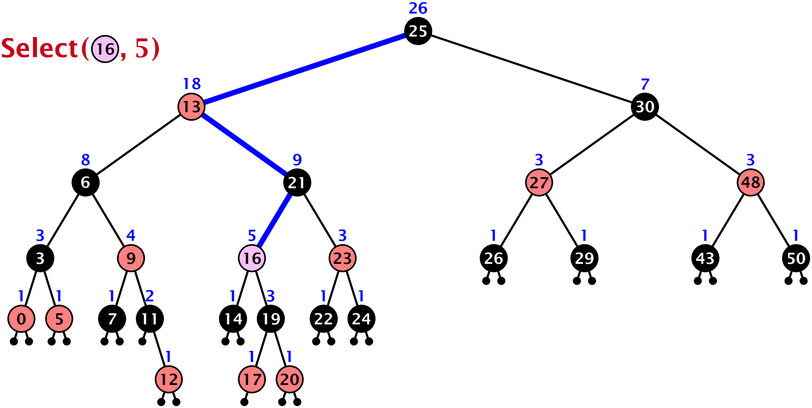

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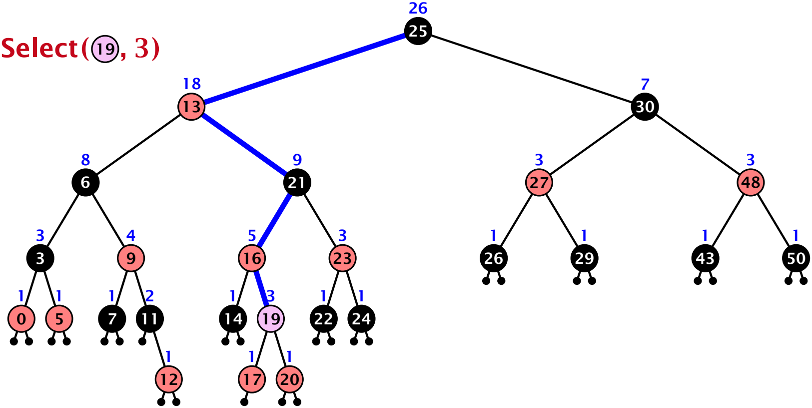

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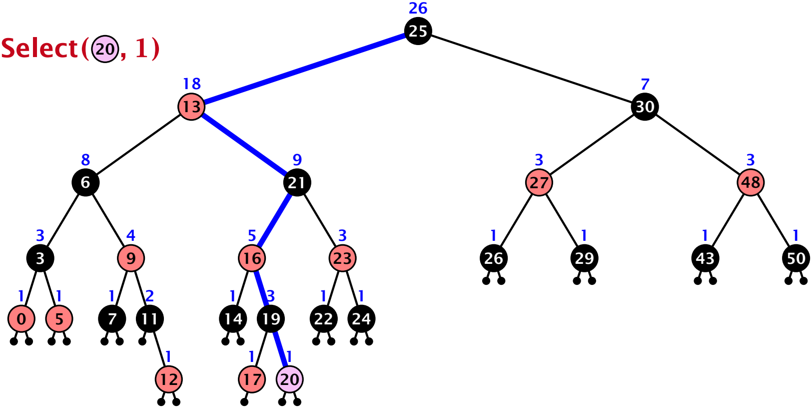

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7.4 Augmenting Data Structures

Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

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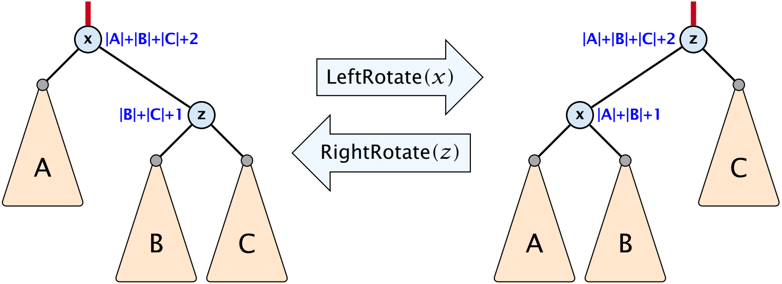
$\text{Search}(k)$: Nothing to do.

$\text{Insert}(x)$: When going down the search path increase the size field for each visited node. Maintain the size field during rotations.

$\text{Delete}(x)$: Directly after splicing out a node traverse the path from the spliced out node upwards, and decrease the size counter on every node on this path. Maintain the size field during rotations.

Rotations

The only operation during the fix-up procedure that alters the tree and requires an update of the size-field:



The nodes x and z are the only nodes changing their size-fields.

The new size-fields can be computed locally from the size-fields of the children.

7.5 Skip Lists

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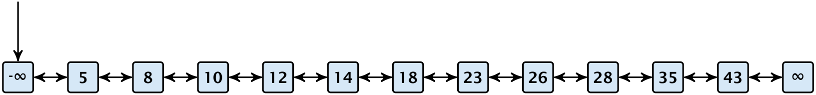

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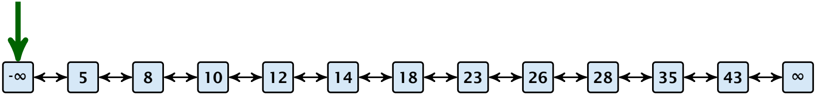

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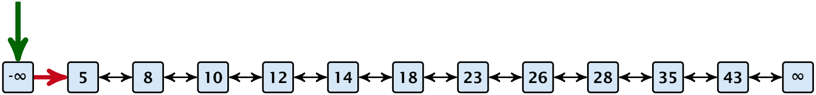

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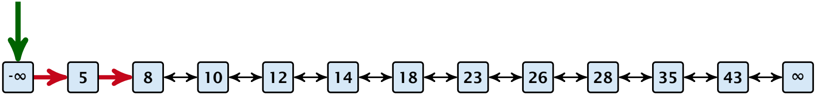

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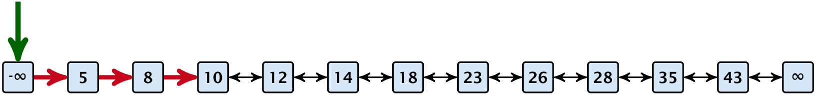

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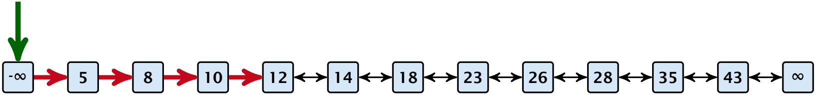

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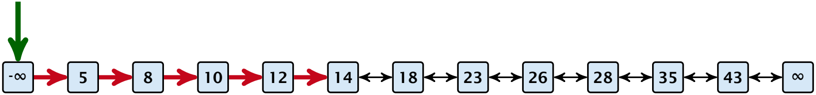

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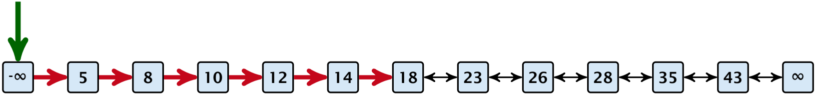

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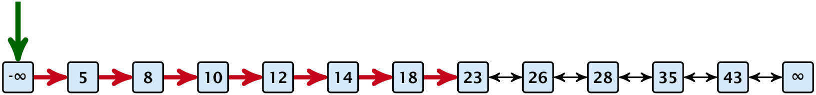

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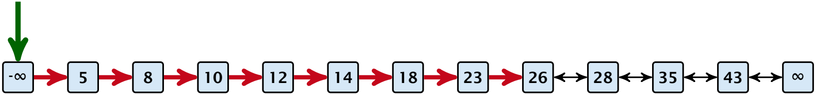

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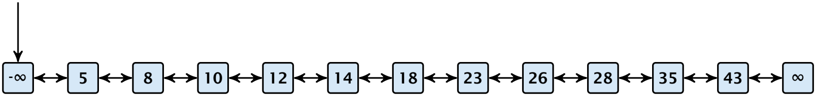

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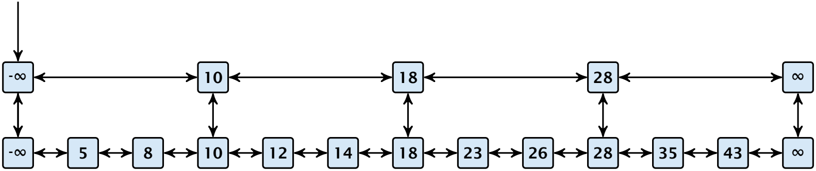

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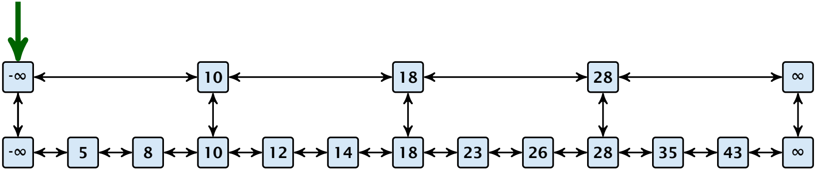

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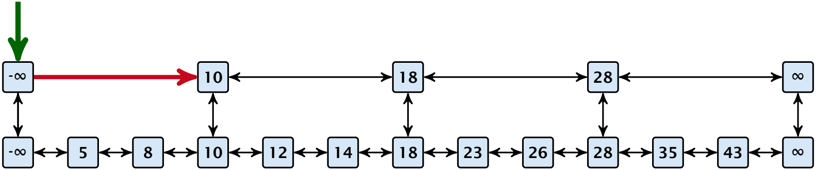

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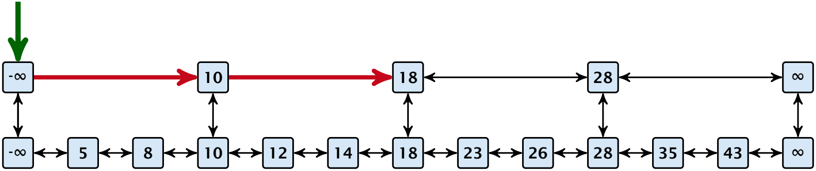

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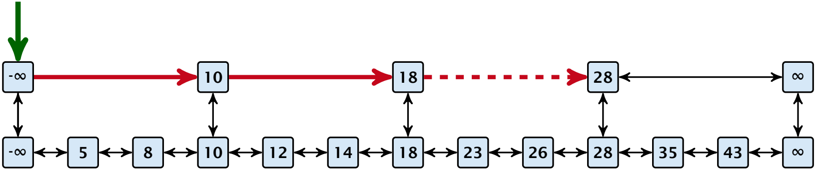

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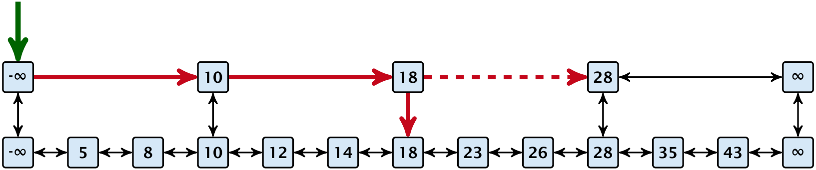

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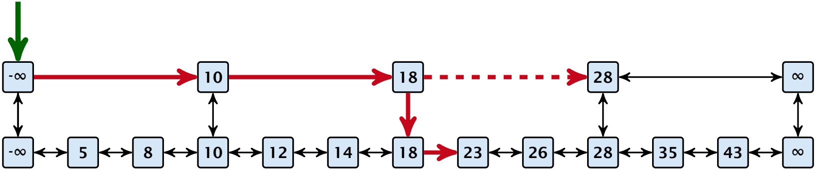

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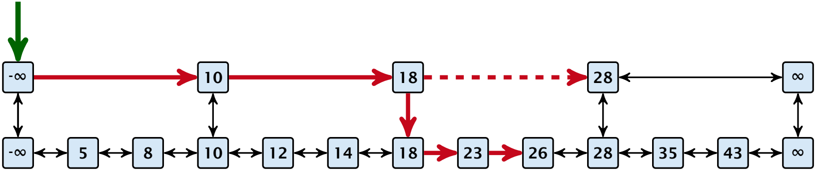

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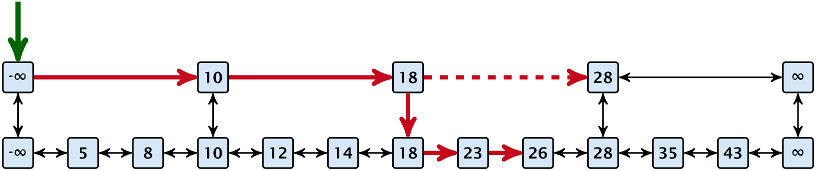

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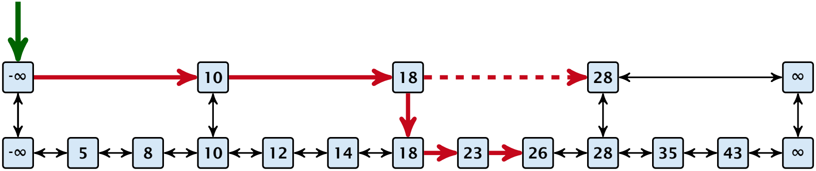

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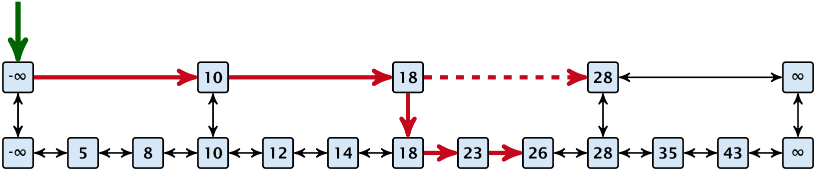

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Choose $|L_1| = \sqrt{n}$. Then search time $\Theta(\sqrt{n})$.

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Choosing $k = \Theta(\log n)$ gives a logarithmic running time.

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Use randomization instead!

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The time for both operations is dominated by the search time.

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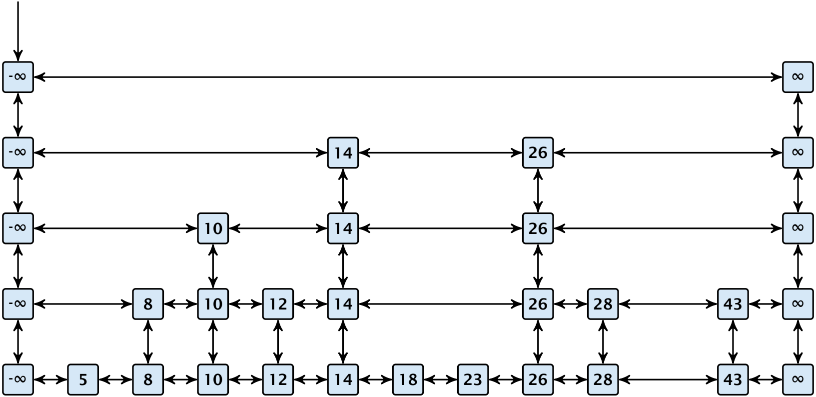
Insert (35):



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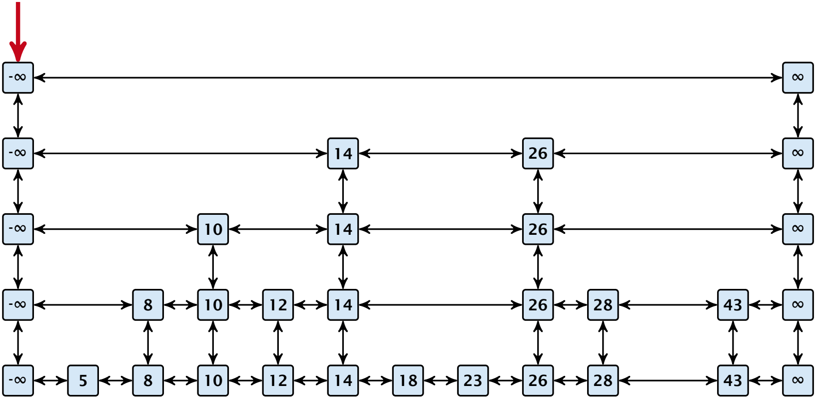

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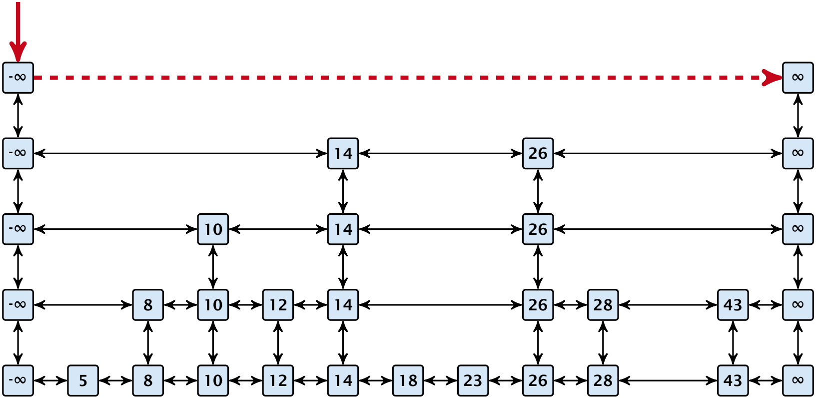

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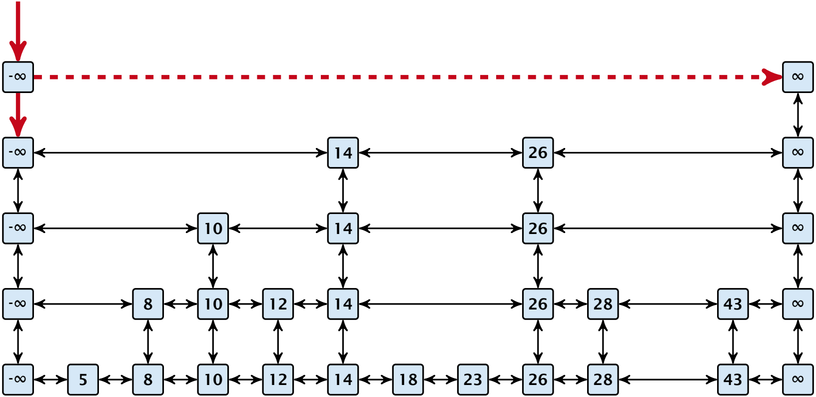

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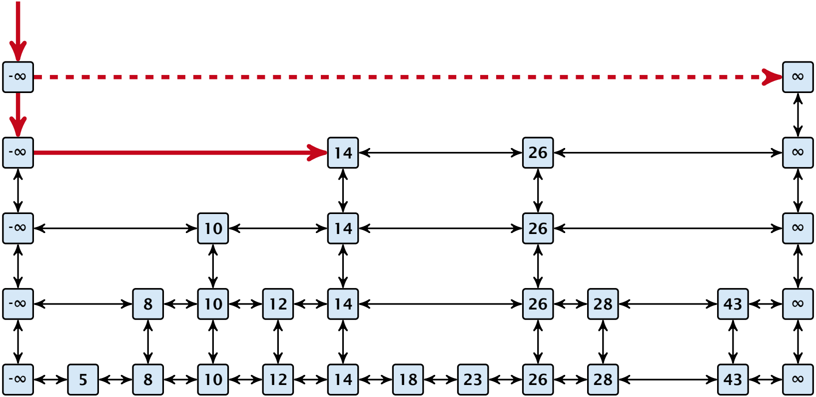

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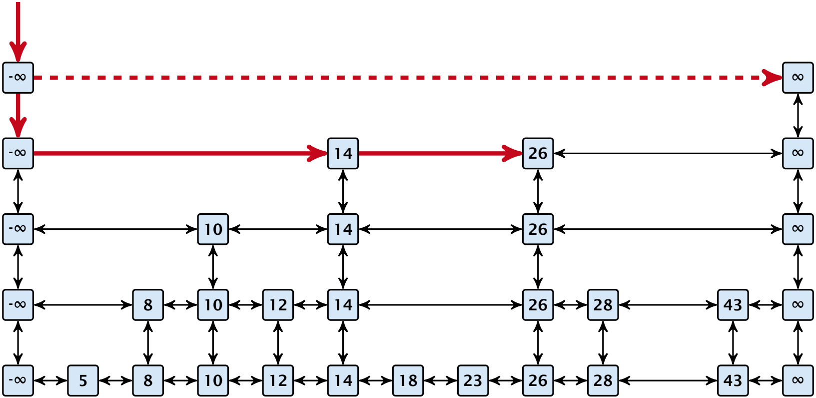

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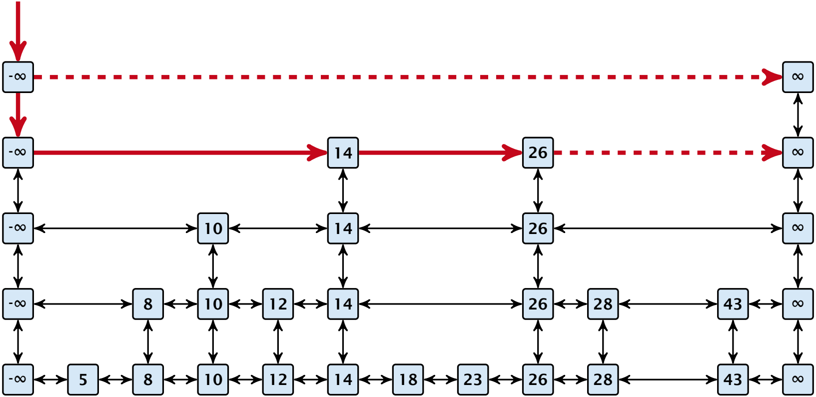

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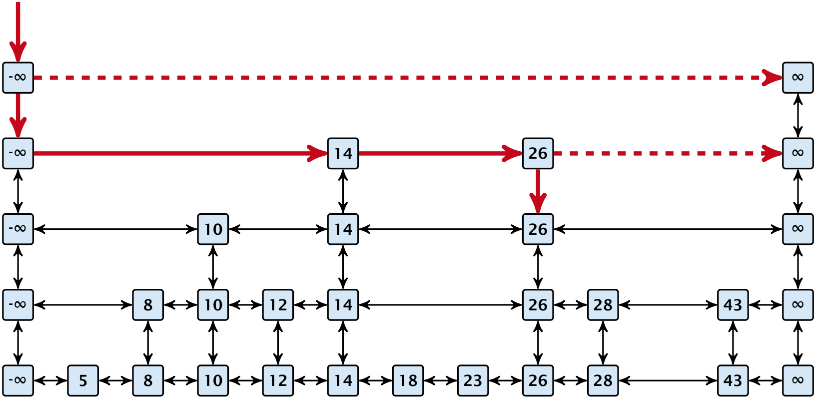

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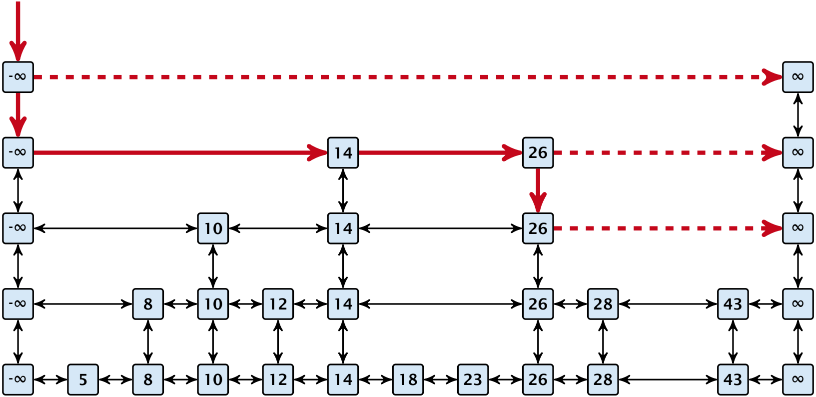

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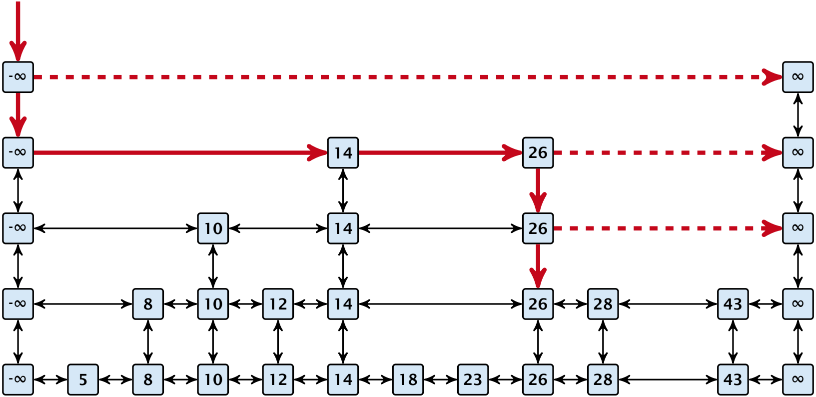

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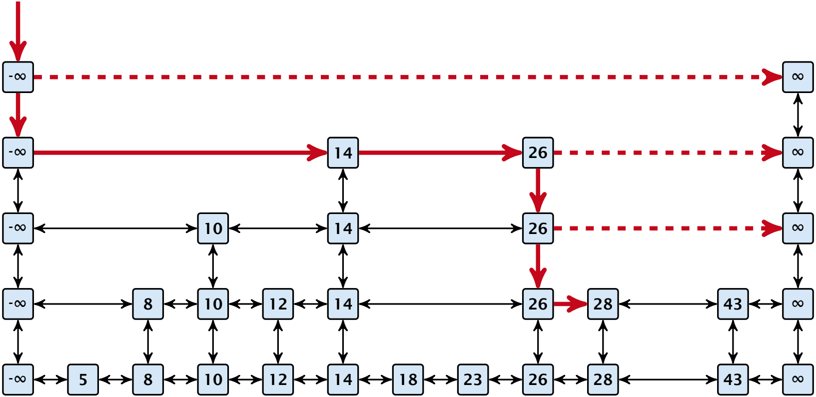

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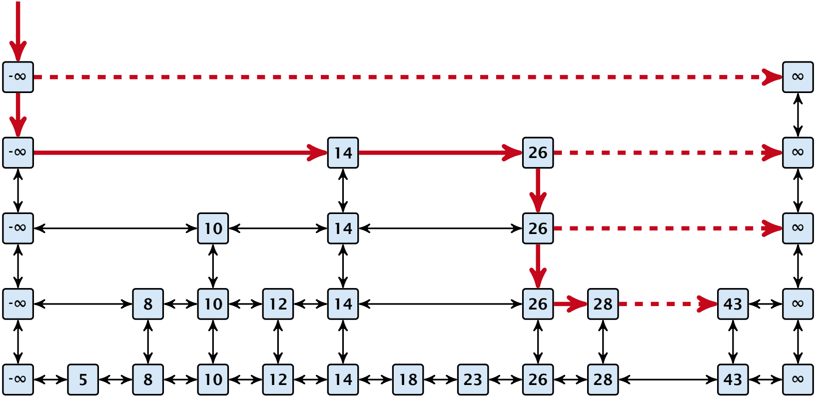

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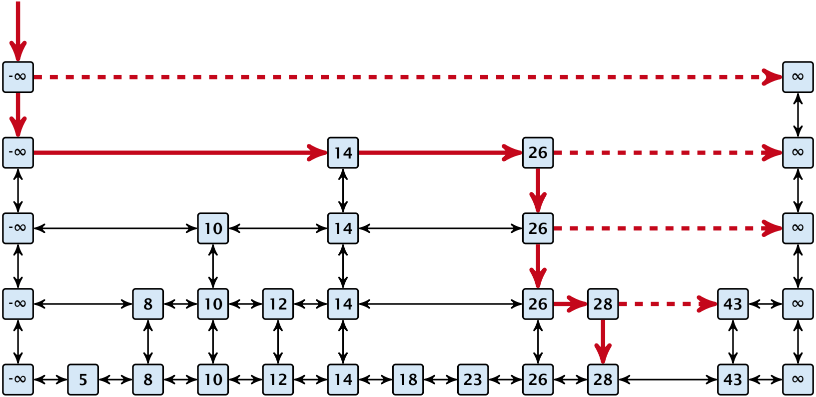

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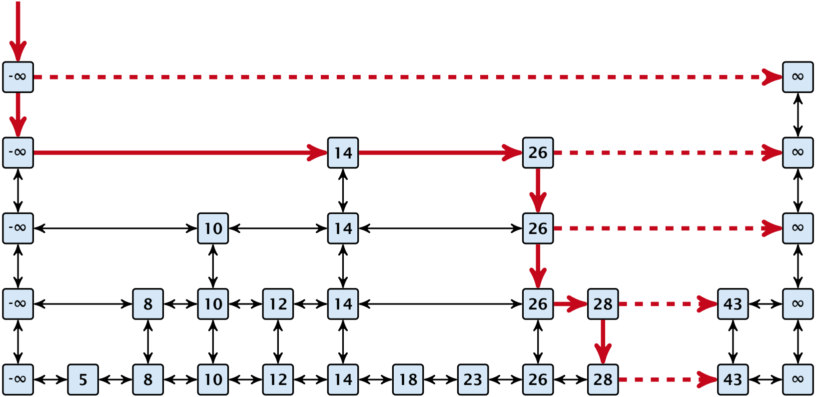

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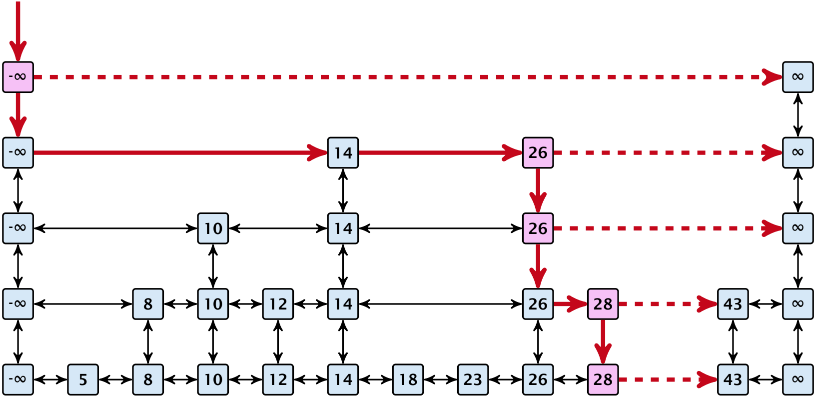

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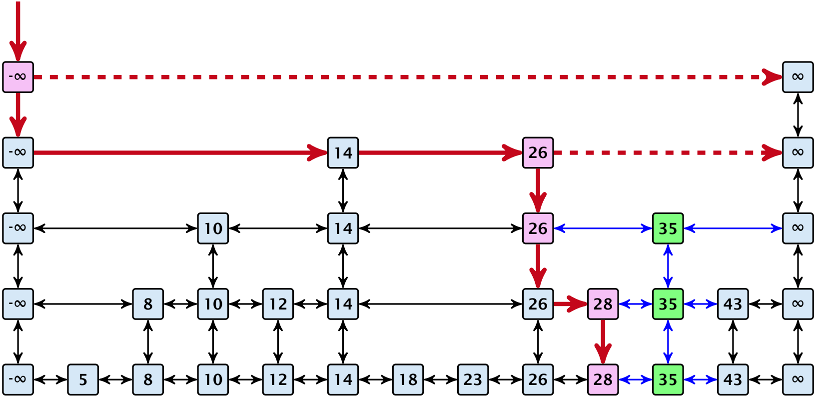

High Probability

Definition 18 (High Probability)

We say a **randomized** algorithm has running time $\mathcal{O}(\log n)$ with high probability if for any constant α the running time is at most $\mathcal{O}(\log n)$ with probability at least $1-\frac{1}{n^\alpha}$.

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Here the $\mathcal{O}$-notation hides a constant that may depend on α.

High Probability

Suppose there are polynomially many events $E_1, E_2, \ldots, E_\ell$, $\ell = n^c$ each holding with high probability (e.g. E_i may be the event that the i-th search in a skip list takes time at most $\mathcal{O}(\log n)$).

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Then the probability that all E_i hold is at least

$$\Pr[E_1 \wedge \cdots \wedge E_\ell]$$

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This means $E_1 \wedge \cdots \wedge E_\ell$ holds with high probability.

7.5 Skip Lists

Lemma 19

A search (and, hence, also insert and delete) in a skip list with n elements takes time $\mathcal{O}(\log n)$ with high probability (w. h. p.).

7.5 Skip Lists

Backward analysis:

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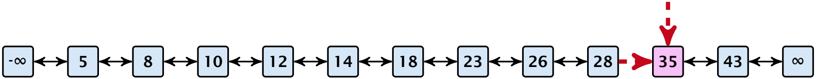

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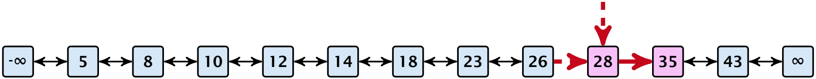

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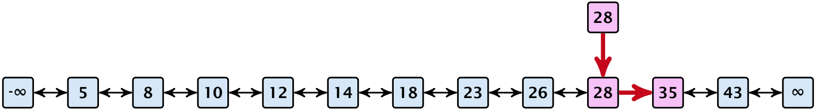

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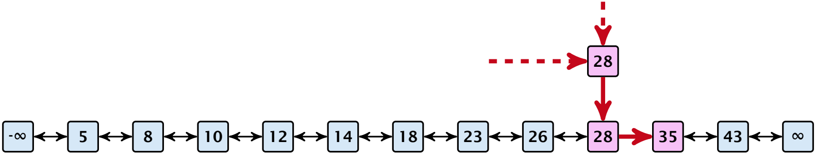

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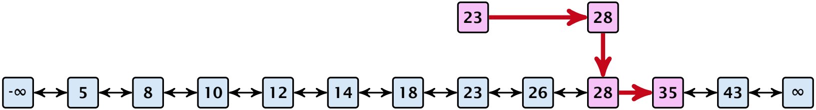

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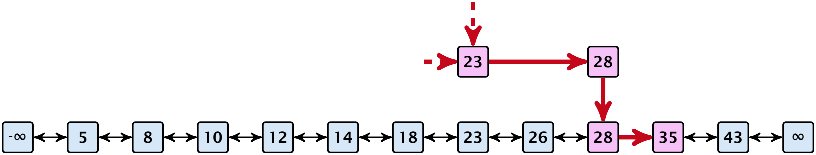

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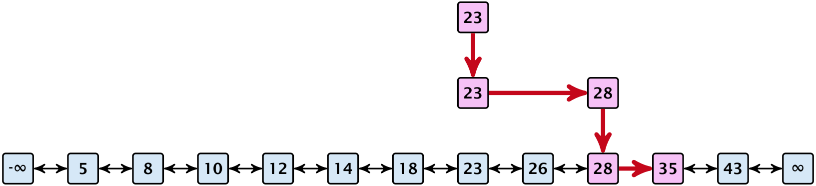

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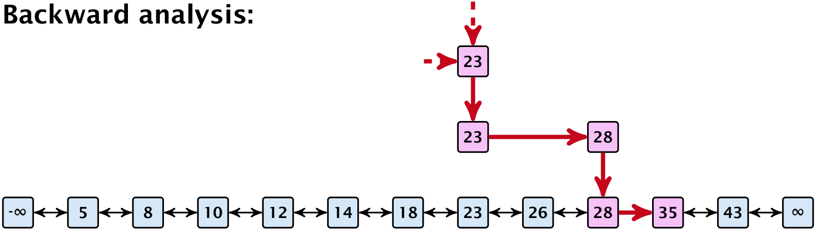

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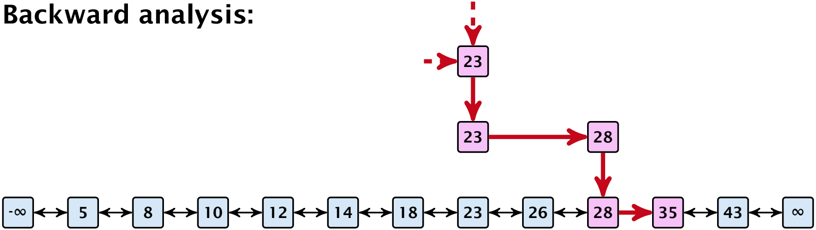

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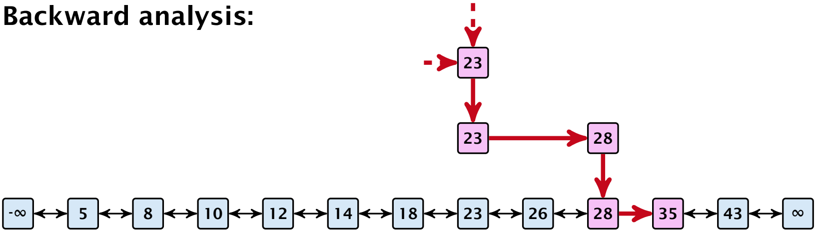

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- A “long” search path must also go very high.

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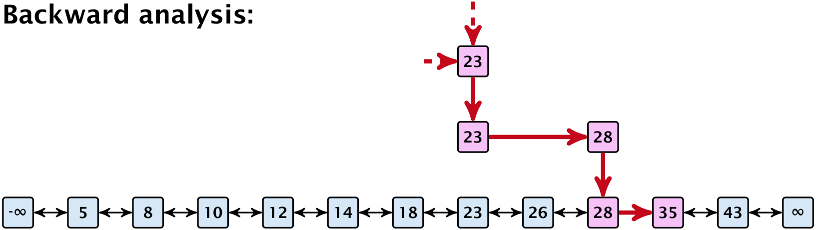

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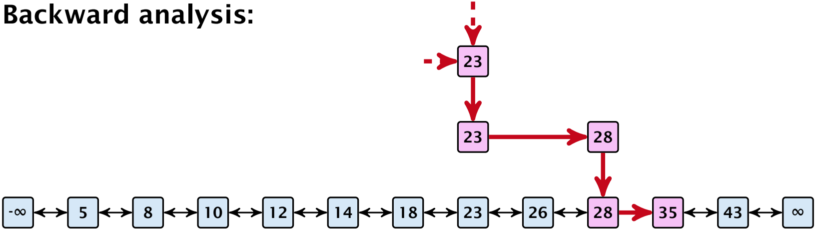

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- There are no elements in high lists.

From this it follows that w.h.p. there are no long paths.

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Estimation for Binomial Coefficients

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

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In particular, this means that during the construction in the backward analysis we see at most k heads (i.e., coin flips that tell you to go up) in z trials.

7.5 Skip Lists

$$\Pr[E_{z,k}]$$

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$$\leq \left(\frac{42\alpha}{64^\alpha}\right)^k n^{-\alpha} \leq n^{-\alpha}$$

7.5 Skip Lists

$$\Pr[E_{z,k}] \leq \Pr[\text{at most } k \text{ heads in } z \text{ trials}]$$

$$\leq \binom{z}{k} 2^{-(z-k)} \leq \left(\frac{ez}{k}\right)^k 2^{-(z-k)} \leq \left(\frac{2ez}{k}\right)^k 2^{-z}$$

choosing $k = \gamma \log n$ with $\gamma \geq 1$ and $z = (\beta + \alpha)\gamma \log n$

$$\leq \left(\frac{2ez}{k}\right)^k 2^{-\beta k} \cdot n^{-\gamma\alpha} \leq \left(\frac{2ez}{2^\beta k}\right)^k \cdot n^{-\alpha}$$

$$\leq \left(\frac{2e(\beta+\alpha)}{2^\beta}\right)^k n^{-\alpha}$$

now choosing $\beta = 6\alpha$ gives

$$\leq \left(\frac{42\alpha}{64^\alpha}\right)^k n^{-\alpha} \leq n^{-\alpha}$$

for $\alpha \geq 1$.

7.5 Skip Lists

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$$\Pr[A_{k+1}] \leq n2^{-(k+1)} \leq n^{-(\gamma-1)} \enspace .$$

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For the search to take at least $z = 7\alpha\gamma \log n$ steps either the event $E_{z,k}$ or the event A_{k+1} must hold.
Hence,

$$\Pr[\text{search requires } z \text{ steps}] \leq \Pr[E_{z,k}] + \Pr[A_{k+1}]$$

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Let A_{k+1} denote the event that the list L_{k+1} is non-empty. Then

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For the search to take at least $z = 7\alpha\gamma \log n$ steps either the event $E_{z,k}$ or the event A_{k+1} must hold.
Hence,

$$\begin{aligned} \Pr[\text{search requires } z \text{ steps}] &\leq \Pr[E_{z,k}] + \Pr[A_{k+1}] \\ &\leq n^{-\alpha} + n^{-(\gamma-1)} \end{aligned}$$

7.5 Skip Lists

So far we fixed $k = \gamma \log n$, $\gamma \geq 1$, and $z = 7\alpha\gamma \log n$, $\alpha \geq 1$.

This means that a search path of length $\Omega(\log n)$ visits a list on a level $\Omega(\log n)$, w.h.p.

Let A_{k+1} denote the event that the list L_{k+1} is non-empty. Then

$$\Pr[A_{k+1}] \leq n2^{-(k+1)} \leq n^{-(\gamma-1)} \ .$$

For the search to take at least $z = 7\alpha\gamma \log n$ steps either the event $E_{z,k}$ or the event A_{k+1} must hold.
Hence,

$$\begin{aligned}\Pr[\text{search requires } z \text{ steps}] &\leq \Pr[E_{z,k}] + \Pr[A_{k+1}] \\ &\leq n^{-\alpha} + n^{-(\gamma-1)}\end{aligned}$$

This means, the search requires at most z steps, w. h. p.

7.6 van Emde Boas Trees

Dynamic Set Data Structure S:

- $S.\text{insert}(x)$
- $S.\text{delete}(x)$
- $S.\text{search}(x)$
- $S.\text{min}()$
- $S.\text{max}()$
- $S.\text{succ}(x)$
- $S.\text{pred}(x)$

7.6 van Emde Boas Trees

For this chapter we ignore the problem of storing satellite data:

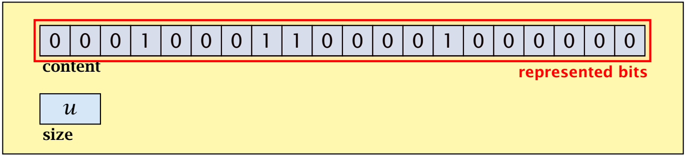
- **S. insert(x):** Inserts x into S.
- **S. delete(x):** Deletes x from S. Usually assumes that $x \in S$.
- **S. member(x):** Returns 1 if $x \in S$ and 0 otw.
- **S. min():** Returns the value of the minimum element in S.
- **S. max():** Returns the value of the maximum element in S.
- **S. succ(x):** Returns successor of x in S. Returns null if x is maximum or larger than any element in S. Note that x needs not to be in S.
- **S. pred(x):** Returns the predecessor of x in S. Returns null if x is minimum or smaller than any element in S. Note that x needs not to be in S.

7.6 van Emde Boas Trees

Can we improve the existing algorithms when the keys are from a restricted set?

In the following we assume that the keys are from $\{0, 1, \ldots, u - 1\}$, where u denotes the size of the universe.

Implementation 1: Array



one array of u bits

Use an array that encodes the indicator function of the dynamic set.

Implementation 1: Array

```
Algorithm 1 array.insert($x$)
    content[$x$] $\leftarrow 1$;
```

```
Algorithm 2 array.delete($x$)
    content[$x$] $\leftarrow 0$;
```

```
Algorithm 3 array.member($x$)
    return content[$x$];
```

- Note that we assume that x is valid, i.e., it falls within the array boundaries.
- Obviously(?) the running time is constant.

Implementation 1: Array

Algorithm 4 array.max()

```
for ($i$ = size $-1$; $i \geq 0$; $i$--) do
    if content[$i$] = 1 then return $i$;
return null;
```

Implementation 1: Array

Algorithm 4 array.max()

```
for ($i$ = size − 1; $i \geq 0$; $i$--) do
    if content[$i$] = 1 then return $i$;
return null;
```

Algorithm 5 array.min()

```
for ($i = 0$; $i <$ size; $i$++) do
    if content[$i$] = 1 then return $i$;
return null;
```

Implementation 1: Array

Algorithm 4 array.max()

```
for ($i$ = size $-1$; $i \geq 0$; $i$--) do
    if content[$i$] = 1 then return $i$;
return null;
```

Algorithm 5 array.min()

```
for ($i = 0$; $i <$ size; $i$++) do
    if content[$i$] = 1 then return $i$;
return null;
```

- Running time is $\mathcal{O}(u)$ in the worst case.

Implementation 1: Array

Algorithm 6 array.succ(x)

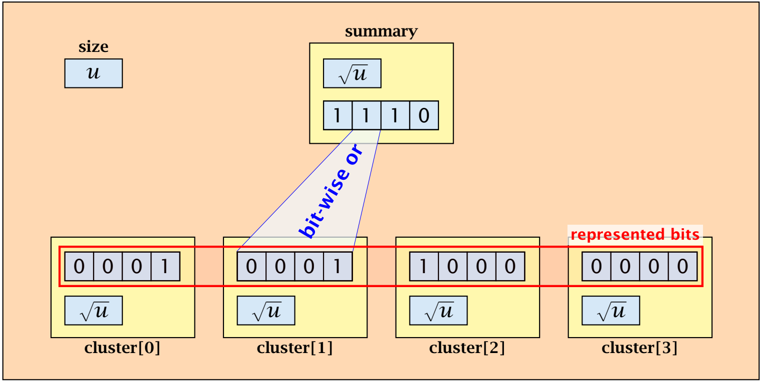
```
for ($i = x + 1$; $i <$ size; $i$++) do
    if content[$i$] $= 1$ then return $i$;
return null;
```

Algorithm 7 array.pred(x)

```
for ($i = x - 1$; $i \geq 0$; $i$--) do
    if content[$i$] $= 1$ then return $i$;
return null;
```

- Running time is $\mathcal{O}(u)$ in the worst case.

Implementation 2: Summary Array



- $\sqrt{u}$ cluster-arrays of $\sqrt{u}$ bits.
- One summary-array of $\sqrt{u}$ bits. The i-th bit in the summary array stores the bit-wise or of the bits in the i-th cluster.

Implementation 2: Summary Array

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The bit for a key x is contained in cluster number $\left\lfloor \frac{x}{\sqrt{u}} \right\rfloor$.

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Within the cluster-array the bit is at position $x \bmod \sqrt{u}$.

For simplicity we assume that $u = 2^{2k}$ for some $k \geq 1$. Then we can compute the cluster-number for an entry x as $\mathrm{high}(x)$ (the upper half of the dual representation of x) and the position of x within its cluster as $\mathrm{low}(x)$ (the lower half of the dual representation).

Implementation 2: Summary Array

Algorithm 8 member(x)

```
return cluster[high($x$)].member(low($x$));
```

Implementation 2: Summary Array

Algorithm 8 member(x)

```
return cluster[high($x$)].member(low($x$));
```

Algorithm 9 insert(x)

```
cluster[high($x$)].insert(low($x$));
summary.insert(high($x$));
```

Implementation 2: Summary Array

```
Algorithm 8 member($x$)
return cluster[high($x$)].member(low($x$));
```

```
Algorithm 9 insert($x$)
cluster[high($x$)].insert(low($x$));
summary.insert(high($x$));
```

- The running times are constant, because the corresponding array-functions have constant running times.

Implementation 2: Summary Array

Algorithm 10 delete(x)

```
cluster[high($x$)].delete(low($x$));
if cluster[high($x$)].min() = null then
    summary.delete(high($x$));
```

Implementation 2: Summary Array

Algorithm 10 delete(x)

```
cluster[high($x$)].delete(low($x$));
if cluster[high($x$)].min() = null then
    summary.delete(high($x$));
```

- The running time is dominated by the cost of a minimum computation on an array of size $\sqrt{u}$. Hence, $\mathcal{O}(\sqrt{u})$.

Implementation 2: Summary Array

Algorithm 11 max()

```
$maxcluster \leftarrow \text{summary.max}()$;
if $maxcluster = \text{null}$ return null;
$offs \leftarrow \text{cluster}[maxcluster].\text{max}()$
return $maxcluster \circ offs$;
```

Implementation 2: Summary Array

Algorithm 11 max()

```
$maxcluster \leftarrow$ summary.max();
if $maxcluster$ = null return null;
$offs \leftarrow$ cluster[$maxcluster$].max()
return $maxcluster \circ offs$;
```

Algorithm 12 min()

```
$mincluster \leftarrow$ summary.min();
if $mincluster$ = null return null;
$offs \leftarrow$ cluster[$mincluster$].min();
return $mincluster \circ offs$;
```

Implementation 2: Summary Array

Algorithm 11 max()

```
$maxcluster \leftarrow$ summary.max();
if $maxcluster$ = null return null;
$offs \leftarrow$ cluster[$maxcluster$].max()
return $maxcluster \circ offs$;
```

Algorithm 12 min()

```
$mincluster \leftarrow$ summary.min();
if $mincluster$ = null return null;
$offs \leftarrow$ cluster[$mincluster$].min();
return $mincluster \circ offs$;
```

The operator $\circ$ stands for the concatenation of two bitstrings.
This means if
$x = 0111_2$ and
$y = 0001_2$ then
$x \circ y = 01110001_2$.

- Running time is roughly $2\sqrt{u} = \mathcal{O}(\sqrt{u})$ in the worst case.

Implementation 2: Summary Array

Algorithm 13 succ(x)

```
m ← cluster[high(x)].succ(low(x))
if $m \neq$ null then return high($x$) $\circ$ $m$;
$\mathit{succcluster}$ ← summary.succ(high($x$));
if $\mathit{succcluster} \neq$ null then
    $\mathit{offs}$ ← cluster[$\mathit{succcluster}$].min();
    return $\mathit{succcluster} \circ \mathit{offs}$;
return null;
```

Implementation 2: Summary Array

```
Algorithm 13 succ(x)
$m \leftarrow$ cluster[high(x)].succ(low($x$))
if $m \neq$ null then return high($x$) $\circ$ $m$;
$succcluster \leftarrow$ summary.succ(high($x$));
if $succcluster \neq$ null then
    $offs \leftarrow$ cluster[$succcluster$].min();
    return $succcluster \circ offs$;
return null;
```

- Running time is roughly $3\sqrt{u} = \mathcal{O}(\sqrt{u})$ in the worst case.

Implementation 2: Summary Array

Algorithm 14 pred(x)

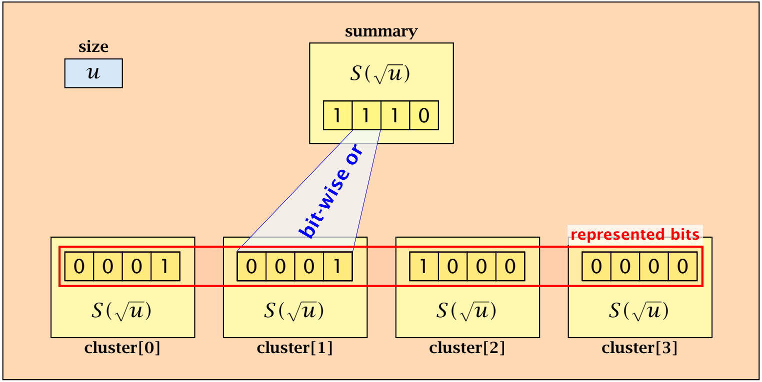
```
$m \leftarrow \text{cluster}[\text{high}(x)].\text{pred}(\text{low}(x))$
if $m \neq$ null then return $\text{high}(x) \circ m$;
$\mathit{predcluster} \leftarrow \text{summary}.\text{pred}(\text{high}(x))$;
if $\mathit{predcluster} \neq$ null then
    $\mathit{offs} \leftarrow \text{cluster}[\mathit{predcluster}].\max()$;
    return $\mathit{predcluster} \circ \mathit{offs}$;
return null;
```

- Running time is roughly $3\sqrt{u} = \mathcal{O}(\sqrt{u})$ in the worst case.

Implementation 3: Recursion

Instead of using sub-arrays, we build a recursive data-structure.

$S(u)$ is a dynamic set data-structure representing u bits:



Implementation 3: Recursion

We assume that $u = 2^{2^k}$ for some k.

The data-structure $S(2)$ is defined as an array of 2-bits (end of the recursion).

Implementation 3: Recursion

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The code from Implementation 2 can be used unchanged. We only need to redo the analysis of the running time.

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Note that in the code we do not need to specifically address the non-recursive case. This is achieved by the fact that an $S(4)$ will contain $S(2)$'s as sub-datastructures, which are arrays. Hence, a call like $\text{cluster}[1].\text{min}()$ from within the data-structure $S(4)$ is not a recursive call as it will call the function $\text{array}.\text{min}()$.

Implementation 3: Recursion

The code from Implementation 2 can be used unchanged. We only need to redo the analysis of the running time.

Note that in the code we do not need to specifically address the non-recursive case. This is achieved by the fact that an $S(4)$ will contain $S(2)$'s as sub-datastructures, which are arrays. Hence, a call like $\mathrm{cluster}[1].\mathrm{min}()$ from within the data-structure $S(4)$ is not a recursive call as it will call the function $\mathrm{array}.\mathrm{min}()$.

This means that the non-recursive case is been dealt with while initializing the data-structure.

Implementation 3: Recursion

Algorithm 15 member(x)

```
return cluster[high($x$)].member(low($x$));
```

- $T_{\mathrm{mem}}(u) = T_{\mathrm{mem}}(\sqrt{u}) + 1$.

Implementation 3: Recursion

Algorithm 16 insert(x)

```
cluster[high($x$)].insert(low($x$));
summary.insert(high($x$));
```

- $T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1$.

Implementation 3: Recursion

Algorithm 17 delete(x)

```
cluster[high(x)].delete(low(x));
if cluster[high(x)].min() = null then
    summary.delete(high(x));
```

- $T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1$.

Implementation 3: Recursion

Algorithm 18 min()

```
$mincluster \leftarrow \text{summary}.\text{min}()$;
if $mincluster = \text{null}$ return null;
$offs \leftarrow \text{cluster}[mincluster].\text{min}()$;
return $mincluster \circ offs$;
```

- $T_{\min}(u) = 2T_{\min}(\sqrt{u}) + 1$.

Implementation 3: Recursion

Algorithm 19 succ(x)

```
$m \leftarrow \text{cluster}[\text{high}(x)].\text{succ}(\text{low}(x))$
if $m \neq \text{null}$ then return $\text{high}(x) \circ m$;
$\mathit{succcluster} \leftarrow \text{summary}.\text{succ}(\text{high}(x))$;
if $\mathit{succcluster} \neq \text{null}$ then
    $\mathit{offs} \leftarrow \text{cluster}[\mathit{succcluster}].\text{min}()$;
    return $\mathit{succcluster} \circ \mathit{offs}$;
return null;
```

- $T_{\text{succ}}(u) = 2T_{\text{succ}}(\sqrt{u}) + T_{\text{min}}(\sqrt{u}) + 1$.

Implementation 3: Recursion

$$T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1:$$

Implementation 3: Recursion

$\boldsymbol{T_{\mathrm{mem}}(u) = T_{\mathrm{mem}}(\sqrt{u}) + 1}$**:**

Set $\ell := \log u$ and $X(\ell) := T_{\mathrm{mem}}(2^{\ell})$.

Implementation 3: Recursion

$\boldsymbol{T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1}$**:**

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^{\ell})$.Then

Implementation 3: Recursion

$\boldsymbol{T_{\mathrm{mem}}(u) = T_{\mathrm{mem}}(\sqrt{u}) + 1}$**:**

Set $\ell := \log u$ and $X(\ell) := T_{\mathrm{mem}}(2^\ell)$.Then

$$X(\ell)$$

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$$X(\ell) = T_{\text{mem}}(2^\ell)$$

Implementation 3: Recursion

$\boldsymbol{T_{\mathrm{mem}}(u) = T_{\mathrm{mem}}(\sqrt{u}) + 1}$**:**

Set $\ell := \log u$ and $X(\ell) := T_{\mathrm{mem}}(2^{\ell})$.Then

$$X(\ell) = T_{\mathrm{mem}}(2^{\ell}) = T_{\mathrm{mem}}(u)$$

Implementation 3: Recursion

$\boldsymbol{T_{\mathrm{mem}}(u) = T_{\mathrm{mem}}(\sqrt{u}) + 1}$**:**

Set $\ell := \log u$ and $X(\ell) := T_{\mathrm{mem}}(2^\ell)$.Then

$$X(\ell) = T_{\mathrm{mem}}(2^\ell) = T_{\mathrm{mem}}(u) = T_{\mathrm{mem}}(\sqrt{u}) + 1$$

Implementation 3: Recursion

$T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1$:

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^\ell)$.Then

$$\begin{aligned} X(\ell) = T_{\text{mem}}(2^\ell) &= T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1 \\ &= T_{\text{mem}}(2^{\frac{\ell}{2}}) + 1 \end{aligned}$$

Implementation 3: Recursion

$\boldsymbol{T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1}$**:**

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^{\ell})$.Then

$$\begin{aligned} X(\ell) = T_{\text{mem}}(2^{\ell}) &= T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1 \\ &= T_{\text{mem}}(2^{\frac{\ell}{2}}) + 1 = X(\tfrac{\ell}{2}) + 1 \ . \end{aligned}$$

Implementation 3: Recursion

$T_{\mathrm{mem}}(u) = T_{\mathrm{mem}}(\sqrt{u}) + 1$:

Set $\ell := \log u$ and $X(\ell) := T_{\mathrm{mem}}(2^{\ell})$.Then

$$\begin{aligned} X(\ell) = T_{\mathrm{mem}}(2^{\ell}) = T_{\mathrm{mem}}(u) &= T_{\mathrm{mem}}(\sqrt{u}) + 1 \\ &= T_{\mathrm{mem}}(2^{\frac{\ell}{2}}) + 1 = X(\tfrac{\ell}{2}) + 1 \enspace . \end{aligned}$$

Using Master theorem gives $X(\ell) = \mathcal{O}(\log \ell)$, and hence $T_{\mathrm{mem}}(u) = \mathcal{O}(\log\log u)$.

Implementation 3: Recursion

$$T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$$

Implementation 3: Recursion

$$\boldsymbol{T_{\mathrm{ins}}(u) = 2T_{\mathrm{ins}}(\sqrt{u}) + 1.}$$

Set $\ell := \log u$ and $X(\ell) := T_{\mathrm{ins}}(2^\ell)$.

Implementation 3: Recursion

$$\boldsymbol{T_{\mathrm{ins}}(u) = 2T_{\mathrm{ins}}(\sqrt{u}) + 1.}$$

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Implementation 3: Recursion

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Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$. Then

$$X(\ell)$$

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$$\boldsymbol{T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.}$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$. Then

$$X(\ell) = T_{\text{ins}}(2^\ell) = T_{\text{ins}}(u)$$

Implementation 3: Recursion

$$\boldsymbol{T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.}$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^{\ell})$. Then

$$X(\ell) = T_{\text{ins}}(2^{\ell}) = T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1$$

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Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^{\ell})$. Then

$$\begin{aligned} X(\ell) = T_{\text{ins}}(2^{\ell}) = T_{\text{ins}}(u) &= 2T_{\text{ins}}(\sqrt{u}) + 1 \\ &= 2T_{\text{ins}}(2^{\frac{\ell}{2}}) + 1 \end{aligned}$$

Implementation 3: Recursion

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Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^{\ell})$. Then

$$\begin{aligned} X(\ell) = T_{\text{ins}}(2^{\ell}) &= T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1 \\ &= 2T_{\text{ins}}(2^{\frac{\ell}{2}}) + 1 = 2X(\tfrac{\ell}{2}) + 1 \ . \end{aligned}$$

Implementation 3: Recursion

$$\boldsymbol{T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.}$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$. Then

$$\begin{aligned} X(\ell) = T_{\text{ins}}(2^\ell) = T_{\text{ins}}(u) &= 2T_{\text{ins}}(\sqrt{u}) + 1 \\ &= 2T_{\text{ins}}(2^{\frac{\ell}{2}}) + 1 = 2X(\tfrac{\ell}{2}) + 1 \enspace . \end{aligned}$$

Using Master theorem gives $X(\ell) = \mathcal{O}(\ell)$, and hence $T_{\text{ins}}(u) = \mathcal{O}(\log u)$.

Implementation 3: Recursion

$\boldsymbol{T_{\mathrm{ins}}(u) = 2T_{\mathrm{ins}}(\sqrt{u}) + 1}$.

Set $\ell := \log u$ and $X(\ell) := T_{\mathrm{ins}}(2^{\ell})$. Then

$$\begin{aligned} X(\ell) = T_{\mathrm{ins}}(2^{\ell}) = T_{\mathrm{ins}}(u) &= 2T_{\mathrm{ins}}(\sqrt{u}) + 1 \\ &= 2T_{\mathrm{ins}}(2^{\frac{\ell}{2}}) + 1 = 2X(\tfrac{\ell}{2}) + 1 \enspace . \end{aligned}$$

Using Master theorem gives $X(\ell) = \mathcal{O}(\ell)$, and hence $T_{\mathrm{ins}}(u) = \mathcal{O}(\log u)$.

The same holds for $T_{\max}(u)$ and $T_{\min}(u)$.

Implementation 3: Recursion

$$T_{\mathrm{del}}(u) = 2T_{\mathrm{del}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1 \leq 2T_{\mathrm{del}}(\sqrt{u}) + c\log(u).$$

Implementation 3: Recursion

$$T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1 \leq 2T_{\text{del}}(\sqrt{u}) + c\log(u).$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{del}}(2^\ell)$.

Implementation 3: Recursion

$$\boldsymbol{T_{\mathrm{del}}(u) = 2T_{\mathrm{del}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1 \leq 2T_{\mathrm{del}}(\sqrt{u}) + c\log(u)}.$$

Set $\ell := \log u$ and $X(\ell) := T_{\mathrm{del}}(2^{\ell})$. Then

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Using Master theorem gives $X(\ell) = \Theta(\ell\log\ell)$, and hence $T_{\text{del}}(u) = \mathcal{O}(\log u\log\log u)$.

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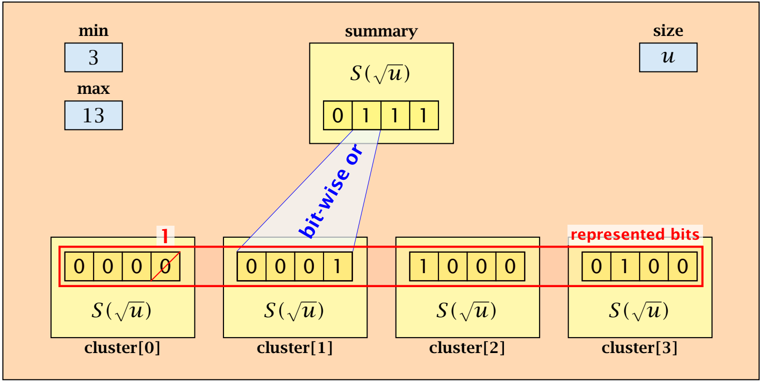
The same holds for $T_{\text{pred}}(u)$ and $T_{\text{succ}}(u)$.

Implementation 4: van Emde Boas Trees



- The bit referenced by min is not set within sub-datastructures.
- The bit referenced by max is set within sub-datastructures (if $\max \neq \min$).

Implementation 4: van Emde Boas Trees

Advantages of having max/min pointers:

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- We can insert into an empty datastructure in constant time by only setting $\text{min} = \text{max} = x$.
- We can delete from a data-structure that just contains one element in constant time by setting $\text{min} = \text{max} = \text{null}$.

Implementation 4: van Emde Boas Trees

```
Algorithm 20 max()
    return max;
```

```
Algorithm 21 min()
    return min;
```

- Constant time.

Implementation 4: van Emde Boas Trees

Algorithm 22 member(x)

```
if $x = \min$ then return 1; // TRUE
return $\text{cluster}[\text{high}(x)].\text{member}(\text{low}(x))$;
```

- $T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1 \Longrightarrow T(u) = \mathcal{O}(\log\log u)$.

Implementation 4: van Emde Boas Trees

Algorithm 23 succ(x)

```
if $\min \neq \text{null} \wedge x < \min$ then return min;
$maxincluster \leftarrow \text{cluster}[\text{high}(x)].\max()$;
if $maxincluster \neq \text{null} \wedge \text{low}(x) < maxincluster$ then
    $offs \leftarrow \text{cluster}[\text{high}(x)].\text{succ}(\text{low}(x))$;
    return $\text{high}(x) \circ offs$;
else
    $succcluster \leftarrow \text{summary}.\text{succ}(\text{high}(x))$;
    if $succcluster = \text{null}$ then return null;
    $offs \leftarrow \text{cluster}[succcluster].\min()$;
    return $succcluster \circ offs$;
```

- $T_{\text{succ}}(u) = T_{\text{succ}}(\sqrt{u}) + 1 \Longrightarrow T_{\text{succ}}(u) = \mathcal{O}(\log\log u)$.

Implementation 4: van Emde Boas Trees

Algorithm 35 insert(x)

```
if $\text{min} = \text{null}$ then
    $\text{min} = x$; $\text{max} = x$;
else
    if $x < \text{min}$ then exchange $x$ and min;
    if $x > \text{max}$ then $\text{max} = x$;
    if $\text{cluster}[\text{high}(x)].\text{min} = \text{null}$; then
        $\text{summary}.\text{insert}(\text{high}(x))$;
        $\text{cluster}[\text{high}(x)].\text{insert}(\text{low}(x))$;
    else
        $\text{cluster}[\text{high}(x)].\text{insert}(\text{low}(x))$;
```

- $T_{\text{ins}}(u) = T_{\text{ins}}(\sqrt{u}) + 1 \Longrightarrow T_{\text{ins}}(u) = \mathcal{O}(\log\log u)$.

Implementation 4: van Emde Boas Trees

Note that the recusive call in Line 8 takes constant time as the if-condition in Line 6 ensures that we are inserting in an empty sub-tree.

The only non-constant recursive calls are the call in Line 7 and in Line 10. These are mutually exclusive, i.e., only one of these calls will actually occur.

From this we get that $T_{\text{ins}}(u) = T_{\text{ins}}(\sqrt{u}) + 1$.

Implementation 4: van Emde Boas Trees

- **Assumes that x is contained in the structure.**

Algorithm 36 delete(x)

```
if $\min = \max$ then
    $\min = \max = \text{null}$;
else
    if $x = \min$ then
        $\mathit{firstcluster} \leftarrow \text{summary.min}()$;
        $\mathit{offs} \leftarrow \text{cluster}[\mathit{firstcluster}].\min()$;
        $x \leftarrow \mathit{firstcluster} \circ \mathit{offs}$;
        $\min \leftarrow x$;
    $\text{cluster}[\text{high}(x)].\text{delete}(\text{low}(x))$;
```

continued...

Implementation 4: van Emde Boas Trees

- **Assumes that x is contained in the structure.**

Algorithm 36 delete(x)

```
if $\text{min} = \text{max}$ then
    $\text{min} = \text{max} = \text{null}$;
else
    if $x = \text{min}$ then                          find new minimum
        $\mathit{firstcluster} \leftarrow \text{summary.min}()$;
        $\mathit{offs} \leftarrow \text{cluster}[\mathit{firstcluster}].\text{min}()$;
        $x \leftarrow \mathit{firstcluster} \circ \mathit{offs}$;
        $\text{min} \leftarrow x$;
    $\text{cluster}[\text{high}(x)].\text{delete}(\text{low}(x))$;
```

continued...

Implementation 4: van Emde Boas Trees

- **Assumes that x is contained in the structure.**

Algorithm 36 delete(x)

```
if min = max then
    min = max = null;
else
    if $x$ = min then
        $firstcluster$ ← summary.min();
        $offs$ ← cluster[$firstcluster$].min();
        $x$ ← $firstcluster \circ offs$;
        min ← $x$;
    cluster[high($x$)].delete(low($x$));        delete
```

continued...

Implementation 4: van Emde Boas Trees

Algorithm 36 delete(x)

...continued

```
if cluster[high($x$)].min() = null then
    summary.delete(high($x$));
    if $x$ = max then
        $summax$ ← summary.max();
        if $summax$ = null then max ← min;
        else
            $offs$ ← cluster[$summax$].max();
            max ← $summax \circ offs$
else
    if $x$ = max then
        $offs$ ← cluster[high($x$)].max();
        max ← high($x$) $\circ$ $offs$;
```

Implementation 4: van Emde Boas Trees

Algorithm 36 delete(x)

...continued

fix maximum

```
    if cluster[high($x$)].min() $=$ null then
        summary.delete(high($x$));
        if $x = \max$ then
            $summax \leftarrow$ summary.max();
            if $summax =$ null then max $\leftarrow$ min;
            else
                $offs \leftarrow$ cluster[$summax$].max();
                max $\leftarrow summax \circ offs$
    else
        if $x = \max$ then
            $offs \leftarrow$ cluster[high($x$)].max();
            max $\leftarrow$ high($x$) $\circ\, offs$;
```

Implementation 4: van Emde Boas Trees

Note that only one of the possible recusive calls in Line 9 and Line 11 in the deletion-algorithm may take non-constant time.

To see this observe that the call in Line 11 only occurs if the cluster where x was deleted is now empty. But this means that the call in Line 9 deleted the last element in $\mathrm{cluster}[\mathrm{high}(x)]$. Such a call only takes constant time.

Hence, we get a recurrence of the form

$$T_{\mathrm{del}}(u) = T_{\mathrm{del}}(\sqrt{u}) + c \ .$$

This gives $T_{\mathrm{del}}(u) = \mathcal{O}(\log\log u)$.

7.6 van Emde Boas Trees

Space requirements:

- The space requirement fulfills the recurrence

$$S(u) = (\sqrt{u} + 1)S(\sqrt{u}) + \mathcal{O}(\sqrt{u}) \ .$$

- Note that we cannot solve this recurrence by the Master theorem as the branching factor is not constant.
- One can show by induction that the space requirement is $S(u) = \mathcal{O}(u)$. Exercise.

- Let the “real” recurrence relation be

$$S(k^2) = (k+1)S(k) + c_1 \cdot k;\ S(4) = c_2$$

- Replacing $S(k)$ by $R(k) := S(k)/c_2$ gives the recurrence

$$R(k^2) = (k+1)R(k) + ck;\ R(4) = 1$$

  where $c = c_1/c_2 < 1$.
- Now, we show $R(k^2) \leq k^2 - 2$ for $k^2 \geq 4$.
  - Obviously, this holds for $k^2 = 4$.
  - For $k^2 > 4$ we have

$$\begin{aligned} R(k^2) &= (1+k)R(k) + ck \\ &\leq (1+k)(k-2) + k \leq k^2 - 2 \end{aligned}$$

  - This shows that $R(k)$ and, hence, $S(k)$ grows linearly.

7.7 Hashing

Dictionary:

- $\boldsymbol{S.\,\mathrm{insert}(x)}$: Insert an element x.
- $\boldsymbol{S.\,\mathrm{delete}(x)}$: Delete the element pointed to by x.
- $\boldsymbol{S.\,\mathrm{search}(k)}$: Return a pointer to an element e with $\mathrm{key}[e] = k$ in S if it exists; otherwise return null.

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So far we have implemented the search for a key by carefully choosing split-elements.

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So far we have implemented the search for a key by carefully choosing split-elements.

Then the memory location of an object x with key k is determined by successively comparing k to split-elements.

Hashing tries to directly compute the memory location from the given key. The goal is to have constant search time.

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Definitions:

- Universe U of keys, e.g., $U \subseteq \mathbb{N}_0$. U very large.

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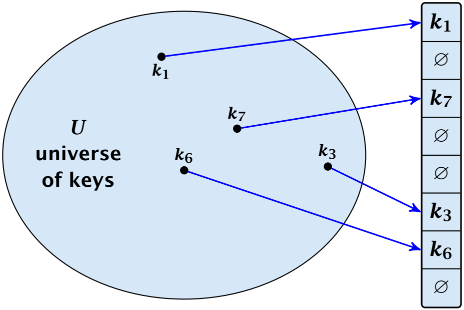
- Hash function $h : U \to [0,\dots,n-1]$.

The hash-function h should fulfill:

- Fast to evaluate.
- Small storage requirement.
- Good distribution of elements over the whole table.

Direct Addressing

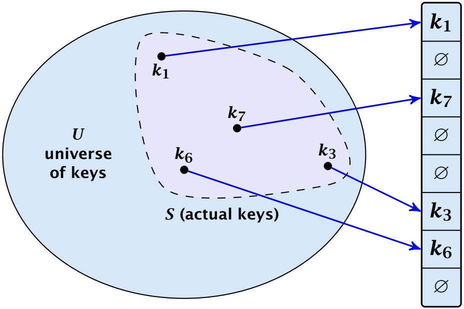
Ideally the hash function maps all keys to different memory locations.



This special case is known as Direct Addressing. It is usually very unrealistic as the universe of keys typically is quite large, and in particular larger than the available memory.

Perfect Hashing

Suppose that we know the set S of actual keys (no insert/no delete). Then we may want to design a simple hash-function that maps all these keys to different memory locations.



Such a hash function h is called a perfect hash function for set S.

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If we do not know the keys in advance, the best we can hope for is that the hash function distributes keys evenly across the table.

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Hence, there may be two elements k_1, k_2 from the set S that map to the same memory location (i.e., $h(k_1) = h(k_2)$). This is called a collision.

Collisions

Typically, collisions do not appear once the size of the set S of actual keys gets close to n, but already when $|S| \geq \omega(\sqrt{n})$.

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Lemma 20

The probability of having a collision when hashing m elements into a table of size n under uniform hashing is at least

$$1 - e^{-\frac{m(m-1)}{2n}} \approx 1 - e^{-\frac{m^2}{2n}} \quad .$$

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Uniform hashing:

Choose a hash function uniformly at random from all functions $f : U \to [0, \dots, n-1]$.

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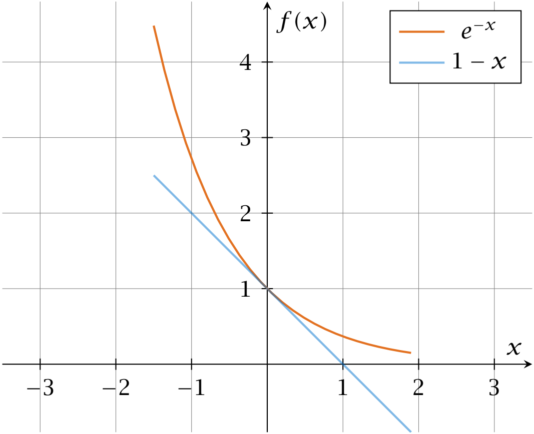
Here the first equality follows since the ℓ-th element that is hashed has a probability of $\frac{n-\ell+1}{n}$ to not generate a collision under the condition that the previous elements did not induce collisions. □

Collisions



The inequality $1 - x \leq e^{-x}$ is derived by stopping the Taylor-expansion of e^{-x} after the second term.

Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

- **open addressing**, aka. closed hashing
- **hashing with chaining**, aka. closed addressing, open hashing.

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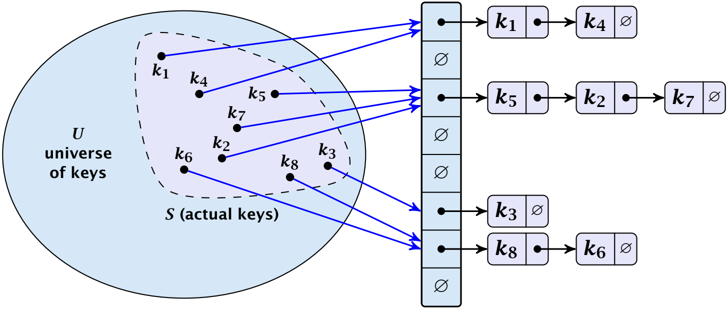
There are applications e.g. computer chess where you do not resolve collisions at all.

Hashing with Chaining

Arrange elements that map to the same position in a linear list.

- Access: compute $h(x)$ and search list for $\mathrm{key}[x]$.
- Insert: insert at the front of the list.



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We assume uniform hashing for the following analysis.

Hashing with Chaining

The time required for an unsuccessful search is 1 plus the length of the list that is examined.

Hashing with Chaining

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Hashing with Chaining

The time required for an unsuccessful search is 1 plus the length of the list that is examined. The average length of a list is $\alpha = \frac{m}{n}$. Hence, if A is the collision resolving strategy “Hashing with Chaining” we have

$$A^- = 1 + \alpha \ .$$

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Hashing with Chaining

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$$\mathrm{E}\left[\frac{1}{m}\sum_{i=1}^{m}\left(1+\sum_{j=i+1}^{m}X_{ij}\right)\right]=\frac{1}{m}\sum_{i=1}^{m}\left(1+\sum_{j=i+1}^{m}\mathrm{E}\left[X_{ij}\right]\right)$$

Hashing with Chaining

$$\begin{aligned} \mathrm{E}\left[\frac{1}{m}\sum_{i=1}^{m}\left(1+\sum_{j=i+1}^{m}X_{ij}\right)\right] &= \frac{1}{m}\sum_{i=1}^{m}\left(1+\sum_{j=i+1}^{m}\mathrm{E}\left[X_{ij}\right]\right) \\ &= \frac{1}{m}\sum_{i=1}^{m}\left(1+\sum_{j=i+1}^{m}\frac{1}{n}\right) \end{aligned}$$

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&= \frac{1}{m}\sum_{i=1}^{m}\Big(1+\sum_{j=i+1}^{m}\frac{1}{n}\Big)\\
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\end{aligned}
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Hence, the expected cost for a successful search is $A^+ \leq 1 + \frac{\alpha}{2}$.

Hashing with Chaining

Disadvantages:

- pointers increase memory requirements
- pointers may lead to bad cache efficiency

Advantages:

- no à priori limit on the number of elements
- deletion can be implemented efficiently
- by using balanced trees instead of linked list one can also obtain worst-case guarantees.

Open Addressing

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All objects are stored in the table itself.

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Define a function $h(k,j)$ that determines the table-position to be examined in the j-th step. The values $h(k,0)$,. . . ,$h(k,n-1)$ must form a permutation of 0, . . . , $n-1$.

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Search(k): Try position $h(k,0)$; if it is empty your search fails; otw. continue with $h(k,1)$, $h(k,2)$, ...

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Insert(x): Search until you find an empty slot; insert your element there. If your search reaches $h(k, n-1)$, and this slot is non-empty then your table is full.

Open Addressing

Choices for $h(k, j)$:

- Linear probing:
  $h(k, i) = h(k) + i \mod n$
  (sometimes: $h(k, i) = h(k) + ci \mod n$).

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For quadratic probing and double hashing one has to ensure that the search covers all positions in the table (i.e., for double hashing $h_2(k)$ must be relatively prime to n (teilerfremd); for quadratic probing c_1 and c_2 have to be chosen carefully).

Linear Probing

- Advantage: Cache-efficiency. The new probe position is very likely to be in the cache.

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Lemma 21

Let L *be the method of linear probing for resolving collisions:*

$$L^{+} \approx \frac{1}{2}\left(1 + \frac{1}{1-\alpha}\right)$$

$$L^{-} \approx \frac{1}{2}\left(1 + \frac{1}{(1-\alpha)^2}\right)$$

Quadratic Probing

- Not as cache-efficient as Linear Probing.
- Secondary clustering: caused by the fact that all keys mapped to the same position have the same probe sequence.

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Lemma 22

Let Q be the method of quadratic probing for resolving collisions:

$$Q^+ \approx 1 + \ln\left(\frac{1}{1-\alpha}\right) - \frac{\alpha}{2}$$

$$Q^- \approx \frac{1}{1-\alpha} + \ln\left(\frac{1}{1-\alpha}\right) - \alpha$$

Double Hashing

- Any probe into the hash-table usually creates a cache-miss.

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Lemma 23

Let D be the method of double hashing for resolving collisions:

$$D^+ \approx \frac{1}{\alpha}\ln\left(\frac{1}{1-\alpha}\right)$$

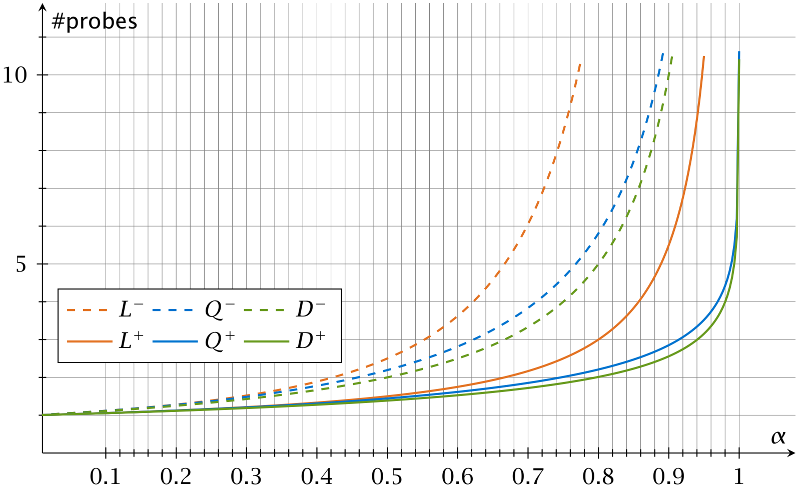
$$D^- \approx \frac{1}{1-\alpha}$$

Open Addressing

Some values:

α	*Linear Probing*		*Quadratic Probing*		*Double Hashing*	
	L^+	L^-	Q^+	Q^-	D^+	D^-
0.5	1.5	2.5	1.44	2.19	1.39	2
0.9	5.5	50.5	2.85	11.40	2.55	10
0.95	10.5	200.5	3.52	22.05	3.15	20

Open Addressing



Analysis of Idealized Open Address Hashing

We analyze the time for a search in a very idealized Open Addressing scheme.

- The probe sequence $h(k,0), h(k,1), h(k,2), \ldots$ is equally likely to be any permutation of $\langle 0, 1, \ldots, n-1 \rangle$.

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$$\Pr[A_1 \cap A_2 \cap \cdots \cap A_{i-1}]$$

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$$\Pr[X \geq i]$$

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$$\Pr[X \geq i] = \frac{m}{n} \cdot \frac{m-1}{n-1} \cdot \frac{m-2}{n-2} \cdot \ldots \cdot \frac{m-i+2}{n-i+2}$$

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$\mathrm{E}[X]$

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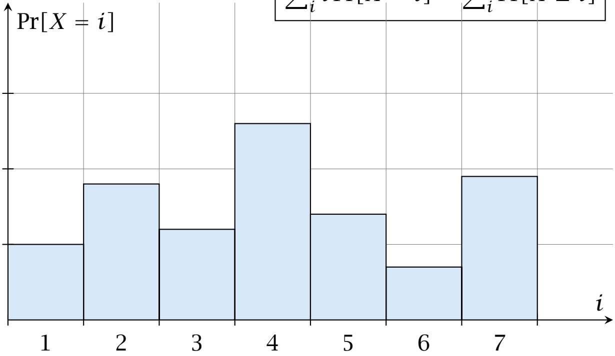
$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \dots$$

Analysis of Idealized Open Address Hashing

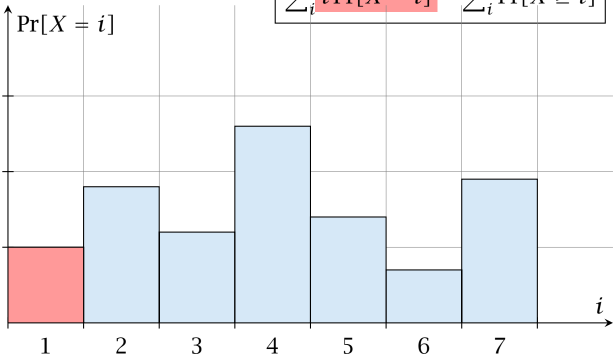


$$\sum_i i \Pr[X = i] = \sum_i \Pr[X \geq i]$$

Analysis of Idealized Open Address Hashing

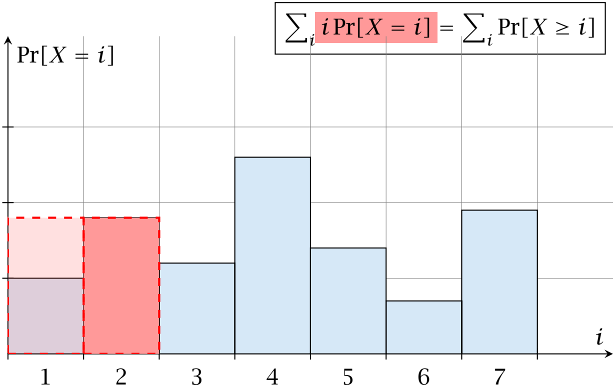
$i = 1$

$$\sum_i i \Pr[X = i] = \sum_i \Pr[X \geq i]$$



Analysis of Idealized Open Address Hashing

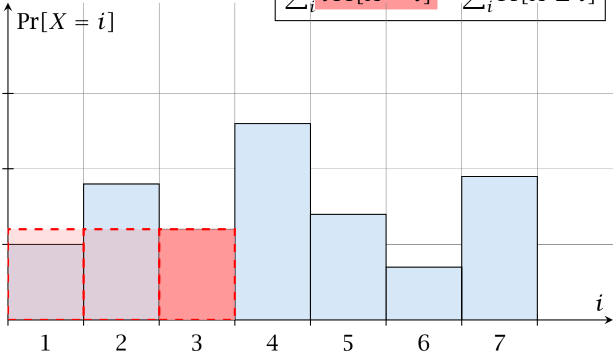
$i = 2$



Analysis of Idealized Open Address Hashing

$i = 3$

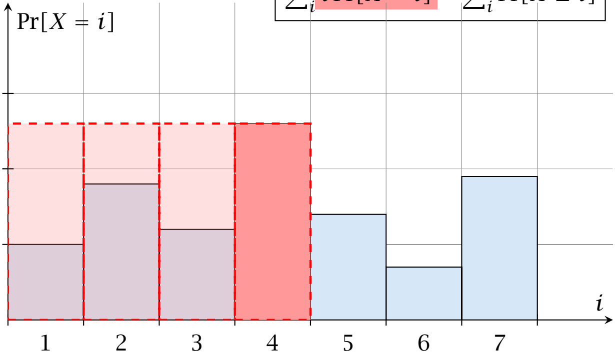
$\sum_i i \Pr[X = i] = \sum_i \Pr[X \geq i]$



Analysis of Idealized Open Address Hashing

$\boldsymbol{i = 4}$

$$\sum_i i \Pr[X = i] = \sum_i \Pr[X \ge i]$$



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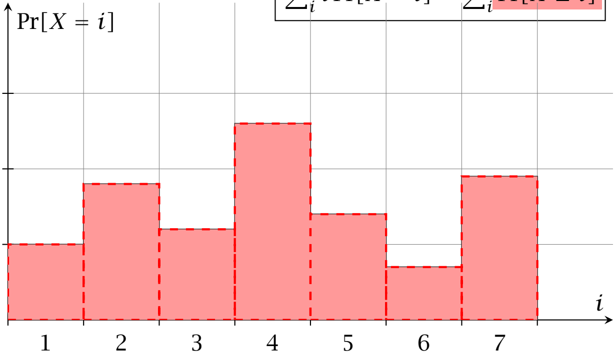

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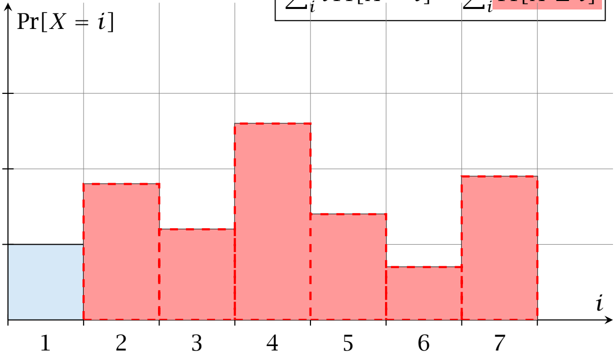

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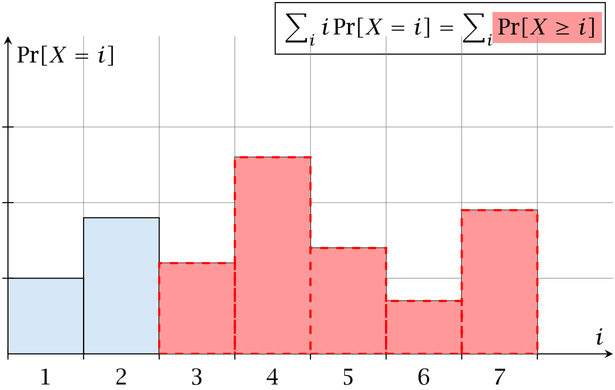

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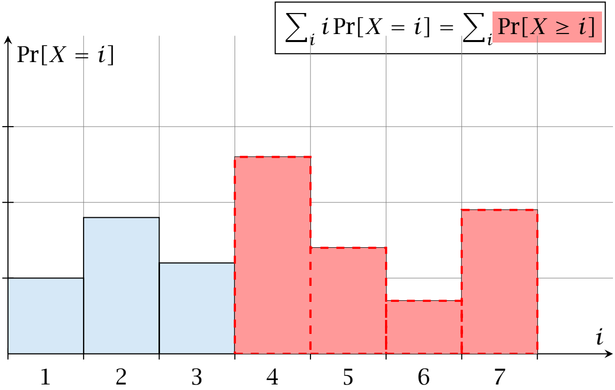
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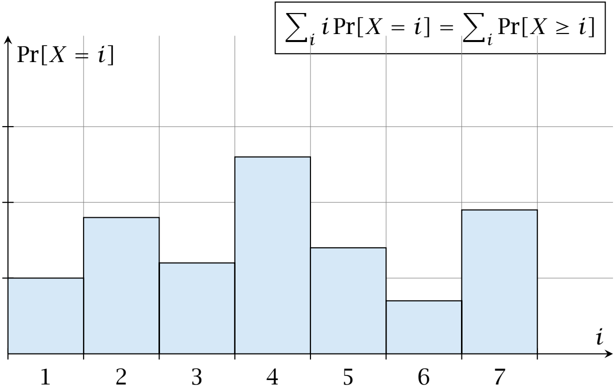

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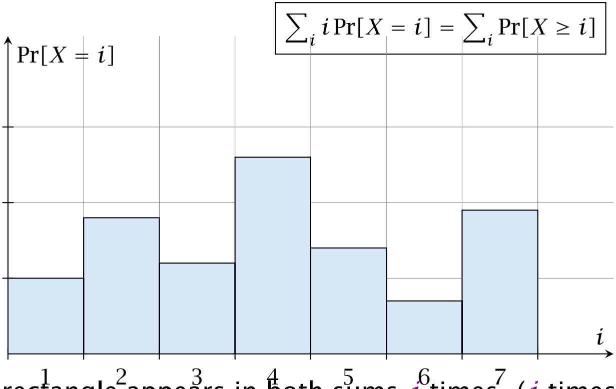

The j-th rectangle appears in both sums j times. (j times in the first due to multiplication with j; and j times in the second for summands $i = 1, 2, \ldots, j$)

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$$\frac{1}{m}\sum_{i=0}^{m-1}\frac{n}{n-i} = \frac{n}{m}\sum_{i=0}^{m-1}\frac{1}{n-i} = \frac{1}{\alpha}\sum_{k=n-m+1}^{n}\frac{1}{k}$$

Analysis of Idealized Open Address Hashing

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Let k be the $i+1$-st element. The expected time for a search for k is at most $\frac{1}{1-i/n} = \frac{n}{n-i}$.

$$\frac{1}{m}\sum_{i=0}^{m-1}\frac{n}{n-i} = \frac{n}{m}\sum_{i=0}^{m-1}\frac{1}{n-i} = \frac{1}{\alpha}\sum_{k=n-m+1}^{n}\frac{1}{k}$$

$$\leq \frac{1}{\alpha}\int_{n-m}^{n}\frac{1}{x}\mathrm{d}x$$

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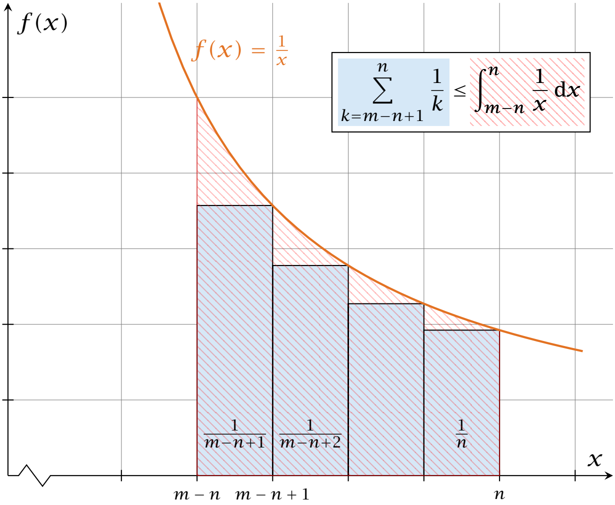
$$\leq \frac{1}{\alpha}\int_{n-m}^{n}\frac{1}{x}\mathrm{d}x = \frac{1}{\alpha}\ln\frac{n}{n-m} = \frac{1}{\alpha}\ln\frac{1}{1-\alpha}\ .$$

Analysis of Idealized Open Address Hashing



$$\sum_{k=m-n+1}^{n} \frac{1}{k} \le \int_{m-n}^{n} \frac{1}{x}\,\mathrm{d}x$$

Deletions in Hashtables

How do we delete in a hash-table?

- For hashing with chaining this is not a problem. Simply search for the key, and delete the item in the corresponding list.

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- For hashing with chaining this is not a problem. Simply search for the key, and delete the item in the corresponding list.
- For open addressing this is difficult.

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    - During a search a deleted-marker must not be used to terminate the probe sequence.

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- The table could fill up with deleted-markers leading to bad performance.

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- One can delete an element by replacing it with a deleted-marker.
    - During an insertion if a deleted-marker is encountered an element can be inserted there.
    - During a search a deleted-marker must not be used to terminate the probe sequence.
- The table could fill up with deleted-markers leading to bad performance.
- If a table contains many deleted-markers (linear fraction of the keys) one can rehash the whole table and amortize the cost for this rehash against the cost for the deletions.

Deletions for Linear Probing

- For Linear Probing one can delete elements without using deletion-markers.

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- Upon a deletion elements that are further down in the probe-sequence may be moved to guarantee that they are still found during a search.

Deletions for Linear Probing

Algorithm 37 delete(p)

```
$T[p] \leftarrow \text{null}$
$p \leftarrow \text{succ}(p)$
while $T[p] \neq \text{null}$ do
    $y \leftarrow T[p]$
    $T[p] \leftarrow \text{null}$
    $p \leftarrow \text{succ}(p)$
    insert($y$)
```

p is the index into the table-cell that contains the object to be deleted.

Deletions for Linear Probing

Algorithm 37 delete(p)

```
T[p] ← null
p ← succ(p)
while T[p] ≠ null do
    y ← T[p]
    T[p] ← null
    p ← succ(p)
    insert(y)
```

p is the index into the table-cell that contains the object to be deleted.

Pointers into the hash-table become invalid.

Universal Hashing

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However, the assumption of uniform hashing that h is chosen randomly from all functions $f : U \to [0, \dots, n-1]$ is clearly unrealistic as there are $n^{|U|}$ such functions. Even writing down such a function would take $|U| \log n$ bits.

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Universal hashing tries to define a set $\mathcal{H}$ of functions that is much smaller but still leads to good average case behaviour when selecting a hash-function uniformly at random from $\mathcal{H}$.

Universal Hashing

Definition 24

A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0,\dots,n-1\}$ is called universal if for all $u_1, u_2 \in U$ with $u_1 \neq u_2$

$$\Pr[h(u_1) = h(u_2)] \leq \frac{1}{n} \ ,$$

where the probability is w. r. t. the choice of a random hash-function from set $\mathcal{H}$.

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where the probability is w. r. t. the choice of a random hash-function from set $\mathcal{H}$.

Note that this means that the probability of a collision between two arbitrary elements is at most $\frac{1}{n}$.

Universal Hashing

Definition 25

A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0,\ldots,n-1\}$ is called 2-independent (pairwise independent) if the following two conditions hold

- For any key $u \in U$, and $t \in \{0,\ldots,n-1\}$ $\Pr[h(u) = t] = \frac{1}{n}$, i.e., a key is distributed uniformly within the hash-table.
- For all $u_1, u_2 \in U$ with $u_1 \neq u_2$, and for any two hash-positions t_1, t_2:

$$\Pr[h(u_1) = t_1 \wedge h(u_2) = t_2] \leq \frac{1}{n^2} \ .$$

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This requirement clearly implies a universal hash-function.

Universal Hashing

Definition 26
A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0,\dots,n-1\}$ is called k-independent if for any choice of $\ell \leq k$ distinct keys $u_1,\dots,u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions $t_1,\dots,t_\ell$:

$$\Pr[h(u_1) = t_1 \wedge \cdots \wedge h(u_\ell) = t_\ell] \leq \frac{1}{n^\ell} \ ,$$

where the probability is w. r. t. the choice of a random hash-function from set $\mathcal{H}$.

Universal Hashing

Definition 27

A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0,\dots,n-1\}$ is called (μ,k)-independent if for any choice of $\ell \le k$ distinct keys $u_1,\dots,u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions $t_1,\dots,t_\ell$:

$$\Pr[h(u_1)=t_1 \wedge \cdots \wedge h(u_\ell)=t_\ell] \le \frac{\mu}{n^\ell} \ ,$$

where the probability is w. r. t. the choice of a random hash-function from set $\mathcal{H}$.

Universal Hashing

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Let $U := \{0, \dots, p-1\}$ for a prime p. Let $\mathbb{Z}_p := \{0, \dots, p-1\}$, and let $\mathbb{Z}_p^* := \{1, \dots, p-1\}$ denote the set of invertible elements in $\mathbb{Z}_p$.

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Define

$$h_{a,b}(x) := (ax + b \bmod p) \bmod n$$

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Define

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Lemma 28

The class

$$\mathcal{H} = \{h_{a,b} \mid a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$$

is a universal class of hash-functions from U *to* $\{0, \dots, n-1\}$.

Universal Hashing

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Proof.

Let $x, y \in U$ be two distinct keys. We have to show that the probability of a collision is only $1/n$.

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  If $x \neq y$ then $(x - y) \not\equiv 0 \pmod{p}$.

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  Multiplying with $a \not\equiv 0 \pmod{p}$ gives

$$a(x - y) \not\equiv 0 \pmod{p}$$

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  Multiplying with $a \not\equiv 0 \pmod{p}$ gives

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  where we use that $\mathbb{Z}_p$ is a field (Körper) and, hence, has no zero divisors (nullteilerfrei).

Universal Hashing

- The hash-function does not generate collisions before the $\pmod{n}$-operation. Furthermore, every choice (a, b) is mapped to a different pair (t_x, t_y) with $t_x := ax + b$ and $t_y := ay + b$.

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- The hash-function does not generate collisions before the $(\mathrm{mod}\ n)$-operation. Furthermore, every choice (a, b) is mapped to a different pair (t_x, t_y) with $t_x := ax + b$ and $t_y := ay + b$.

  This holds because we can compute a and b when given t_x and t_y:

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$$\begin{aligned} t_x &\equiv ax + b && \pmod{p} \\ t_y &\equiv ay + b && \pmod{p} \end{aligned}$$

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$$\begin{aligned} t_x &\equiv ax + b && \pmod{p} \\ t_y &\equiv ay + b && \pmod{p} \end{aligned}$$

$$\begin{aligned} t_x - t_y &\equiv a(x - y) && \pmod{p} \\ t_y &\equiv ay + b && \pmod{p} \end{aligned}$$

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$$\begin{aligned} a &\equiv (t_x-t_y)(x-y)^{-1} && \pmod{p} \\ b &\equiv t_y-ay && \pmod{p} \end{aligned}$$

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There is a one-to-one correspondence between hash-functions (pairs (a,b), $a \neq 0$) and pairs (t_x,t_y), $t_x \neq t_y$.

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Fix a value t_x. There are $p-1$ possible values for choosing t_y.

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What happens when we do the $\bmod n$ operation?

Fix a value t_x. There are $p - 1$ possible values for choosing t_y.

From the range $0, \dots, p-1$ the values $t_x, t_x + n, t_x + 2n, \dots$ map to t_x after the modulo-operation. These are at most $\lceil p/n \rceil$ values.

Universal Hashing

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As $t_y \neq t_x$ there are

$$\left\lceil \frac{p}{n} \right\rceil - 1$$

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possibilities for choosing t_y such that the final hash-value creates a collision.

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possibilities for choosing t_y such that the final hash-value creates a collision.

This happens with probability at most $\frac{1}{n}$.

Universal Hashing

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It is also possible to show that $\mathcal{H}$ is an (almost) pairwise independent class of hash-functions.

$$\Pr_{t_x \neq t_y \in \mathbb{Z}_p^2}\left[\begin{array}{c} t_x \bmod n = h_1 \\ \wedge \\ t_y \bmod n = h_2 \end{array}\right]$$

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$$\frac{\left\lfloor\frac{p}{n}\right\rfloor^2}{p(p-1)} \le \Pr_{t_x \neq t_y \in \mathbb{Z}_p^2}\left[\begin{array}{c} t_x \bmod n = h_1 \\ \wedge \\ t_y \bmod n = h_2 \end{array}\right] \le \frac{\left\lceil\frac{p}{n}\right\rceil^2}{p(p-1)}$$

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Note that the middle is the probability that $h(x) = h_1$ and $h(y) = h_2$. The total number of choices for (t_x, t_y) is $p(p-1)$. The number of choices for t_x (t_y) such that $t_x \bmod n = h_1$ ($t_y \bmod n = h_2$) lies between $\lfloor \frac{p}{n} \rfloor$ and $\lceil \frac{p}{n} \rceil$.

Universal Hashing

Definition 29

Let $d \in \mathbb{N}$; $q \geq (d+1)n$ be a prime; and let $\bar{a} \in \{0, \dots, q-1\}^{d+1}$. Define for $x \in \{0, \dots, q-1\}$

$$h_{\bar{a}}(x) := \Big(\sum_{i=0}^{d} a_i x^i \bmod q \Big) \bmod n \enspace .$$

Let $\mathcal{H}_n^d := \{h_{\bar{a}} \mid \bar{a} \in \{0, \dots, q-1\}^{d+1}\}$. The class $\mathcal{H}_n^d$ is $(e, d+1)$-independent.

Note that in the previous case we had $d = 1$ and chose $a_d \neq 0$.

Universal Hashing

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For the coefficients $\bar{a} \in \{0, \dots, q-1\}^{d+1}$ let $f_{\bar{a}}$ denote the polynomial

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$$f_{\bar{a}}(x) = \Big(\sum_{i=0}^{d} a_i x^i \Big) \bmod q$$

The polynomial is defined by $d+1$ distinct points.

Universal Hashing

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Fix $\ell \leq d+1$; let $x_1, \ldots, x_\ell \in \{0, \ldots, q-1\}$ be keys, and let $t_1, \ldots, t_\ell$ denote the corresponding hash-function values.

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Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

$$f_{\bar{a}}(x_i) \in \underbrace{\{t_i + \alpha \cdot n \mid \alpha \in \{0,\dots,\lceil \tfrac{q}{n} \rceil - 1\}\}}_{=:B_i}$$

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In order to obtain the cardinality of A^ℓ we choose our polynomial by fixing $d+1$ points.

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We first fix the values for inputs $x_1, \dots, x_\ell$.
We have

$$|B_1| \cdot \ldots \cdot |B_\ell|$$

possibilities to do this (so that $h_{\bar{a}}(x_i) = t_i$).

Universal Hashing

Now, we choose $d - \ell + 1$ other inputs and choose their value arbitrarily. We have $q^{d-\ell+1}$ possibilities to do this.

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Therefore we have

$$|B_1|\cdot\ldots\cdot|B_\ell|\cdot q^{d-\ell+1}\le\lceil\frac{q}{n}\rceil^\ell\cdot q^{d-\ell+1}$$

possibilities to choose $\bar{a}$ such that $h_{\bar{a}}\in A_\ell$.

Universal Hashing

Therefore the probability of choosing $h_{\bar{a}}$ from A_ℓ is only

$$\frac{\lceil \frac{q}{n} \rceil^{\ell} \cdot q^{d-\ell+1}}{q^{d+1}}$$

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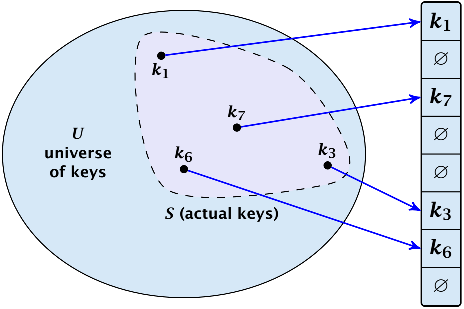
$$\leq \Big(1 + \frac{1}{\ell}\Big)^\ell \cdot \frac{1}{n^\ell} \leq \frac{e}{n^\ell} \ .$$

This shows that the $\mathcal{H}$ is $(e, d+1)$-universal.

The last step followed from $q \geq (d+1)n$, and $\ell \leq d+1$.

Perfect Hashing

Suppose that we **know** the set S of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



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The probability of having 1 or more collisions can be at most $\frac{1}{2}$ as otherwise the expectation would be larger than $\frac{1}{2}$.

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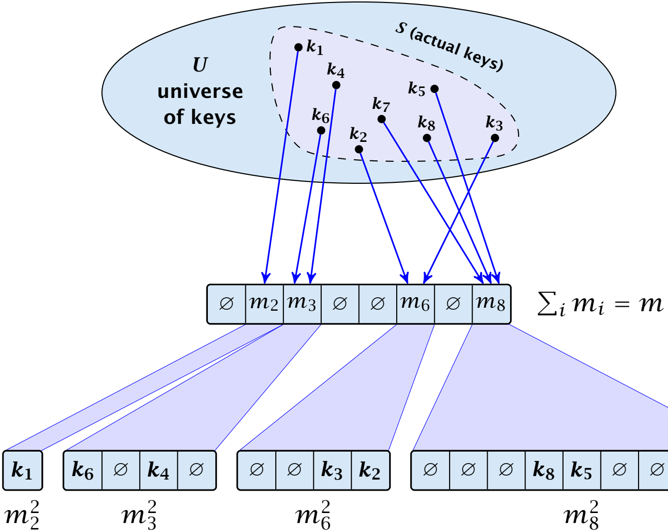
Let m_j denote the number of items that are hashed to the j-th bucket. For each bucket we choose a second hash-function that maps the elements of the bucket into a table of size m_j^2. The second function can be chosen such that all elements are mapped to different locations.

Perfect Hashing



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$$= 2\binom{m}{2}\frac{1}{m} + m = 2m - 1 \ .$$

Perfect Hashing

We need only $\mathcal{O}(m)$ time to construct a hash-function h with $\sum_j m_j^2 = \mathcal{O}(4m)$, because with probability at least $1/2$ a random function from a universal family will have this property.

Then we construct a hash-table h_j for every bucket. This takes expected time $\mathcal{O}(m_j)$ for every bucket. A random function h_j is collision-free with probability at least $1/2$. We need $\mathcal{O}(m_j)$ to test this.

We only need that the hash-functions are chosen from a universal family!!!

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Goal:
Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

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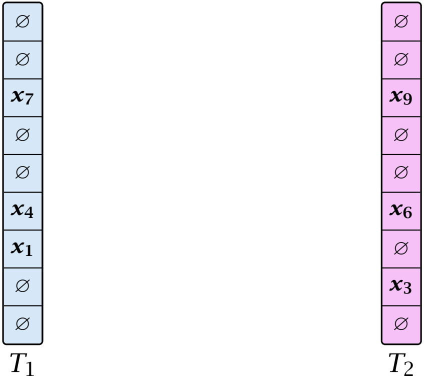
- A search clearly takes constant time if the above constraint is met.

Cuckoo Hashing

Insert:



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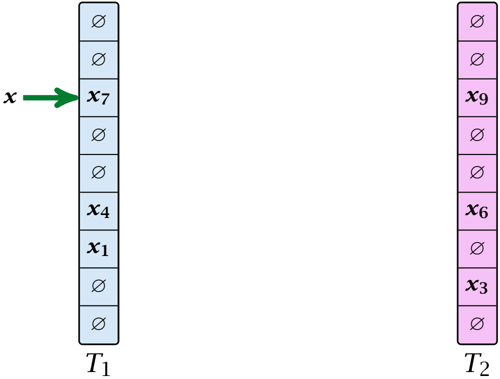

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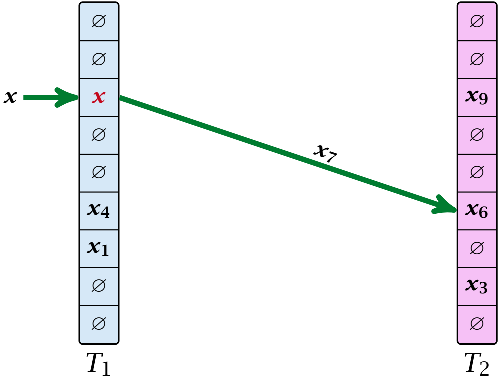

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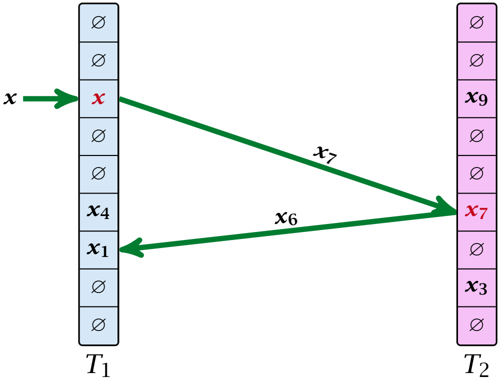

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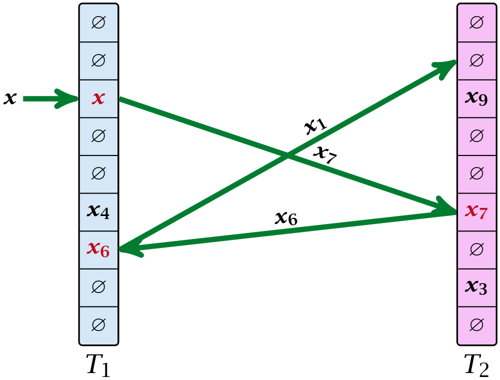

Cuckoo Hashing

Algorithm 38 Cuckoo-Insert(x)

```
if $T_1[h_1(x)] = x \vee T_2[h_2(x)] = x$ then return
$\text{steps} \leftarrow 1$
while $\text{steps} \le \text{maxsteps}$ do
    exchange $x$ and $T_1[h_1(x)]$
    if $x = \text{null}$ then return
    exchange $x$ and $T_2[h_2(x)]$
    if $x = \text{null}$ then return
    $\text{steps} \leftarrow \text{steps} + 1$
rehash() // change hash-functions; rehash everything
Cuckoo-Insert$(x)$
```

Cuckoo Hashing

- We call one iteration through the while-loop a step of the algorithm.

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- We call a sequence of iterations through the while-loop without the termination condition becoming true a phase of the algorithm.
- We say a phase is successful if it is not terminated by the maxstep-condition, but the while loop is left because $x = \text{null}$.

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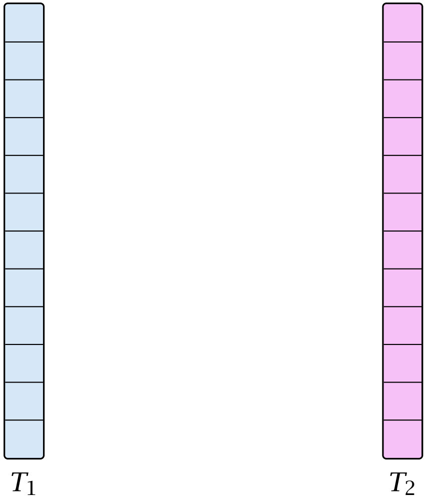
Formally what is the probability to enter an infinite loop that touches s different keys?

Cuckoo Hashing: Insert



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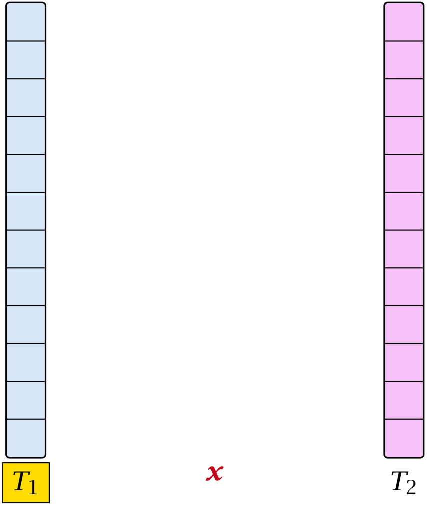

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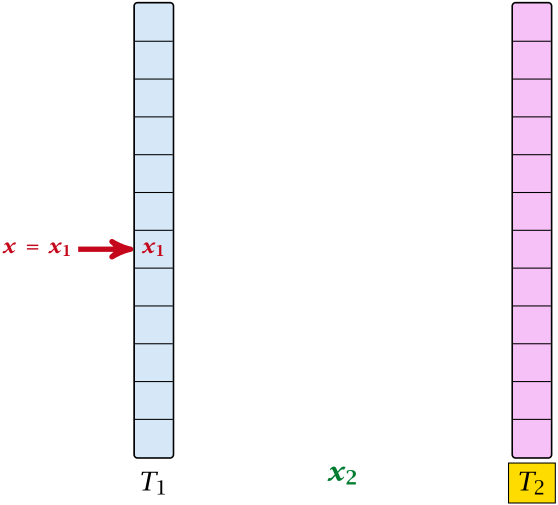

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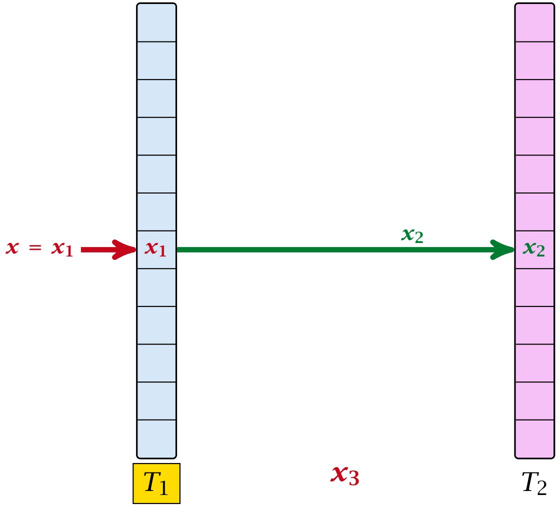

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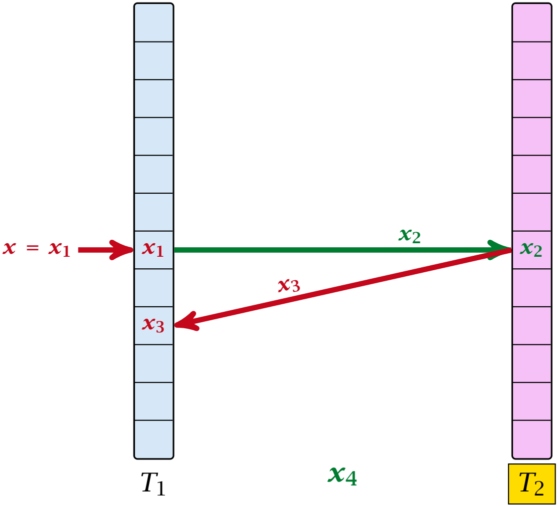

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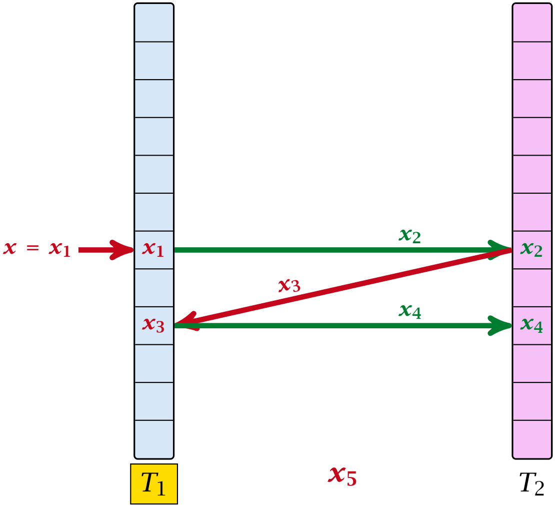

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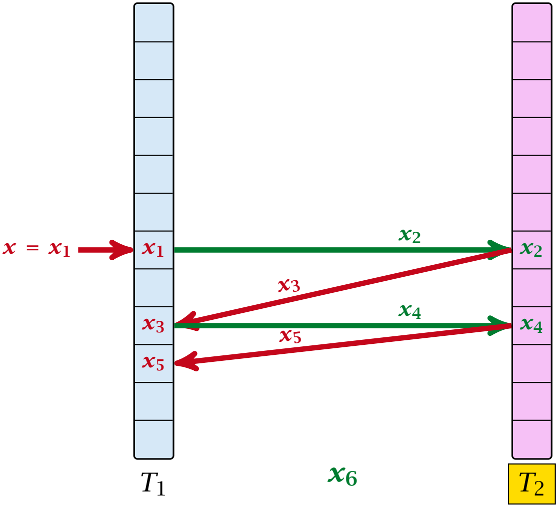

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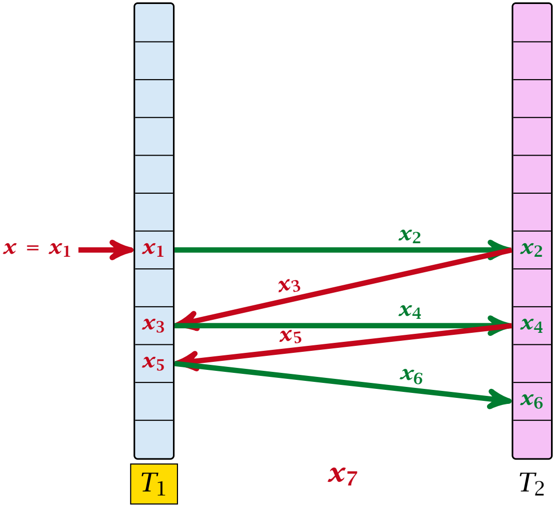

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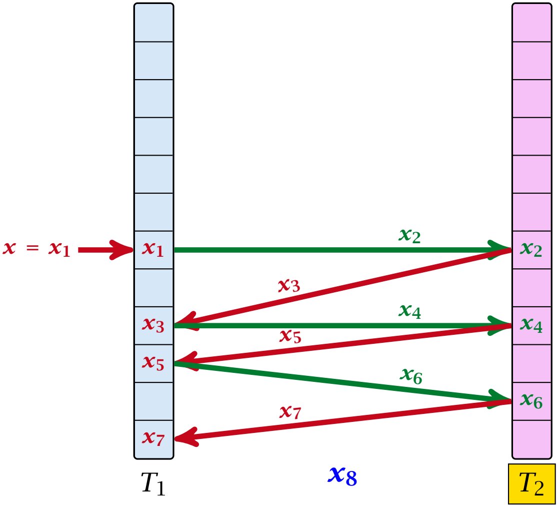

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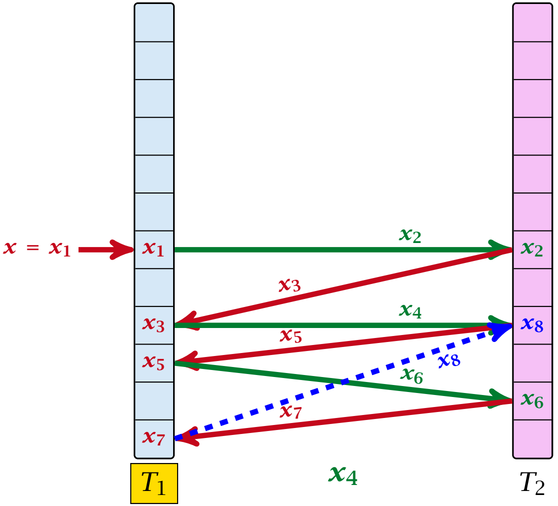

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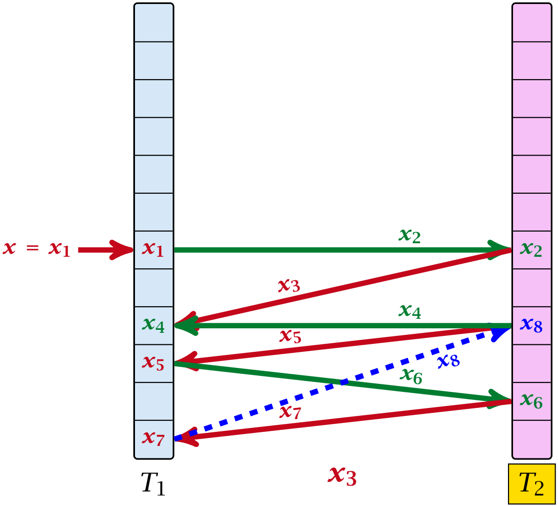

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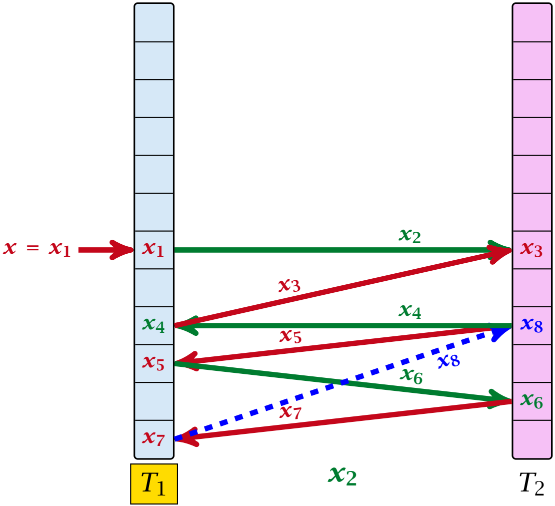

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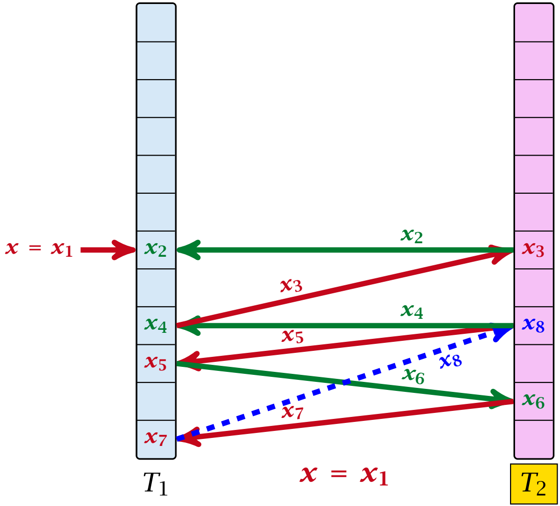

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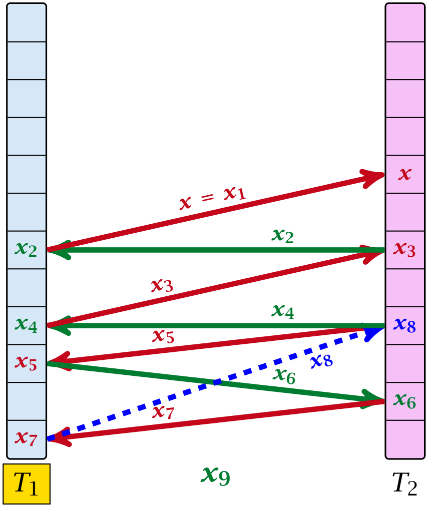

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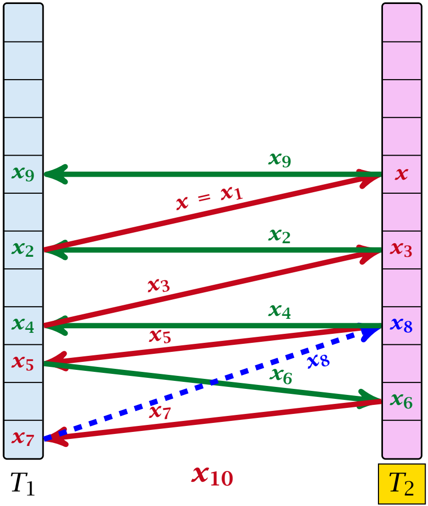

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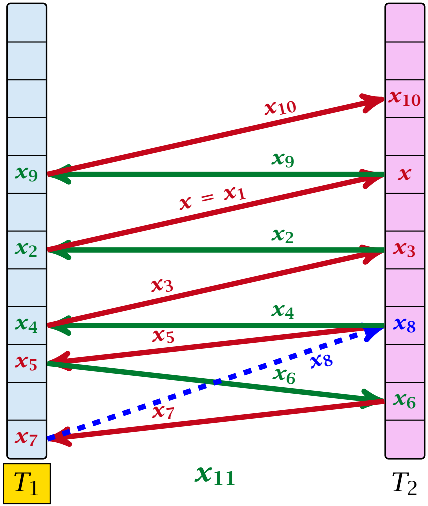

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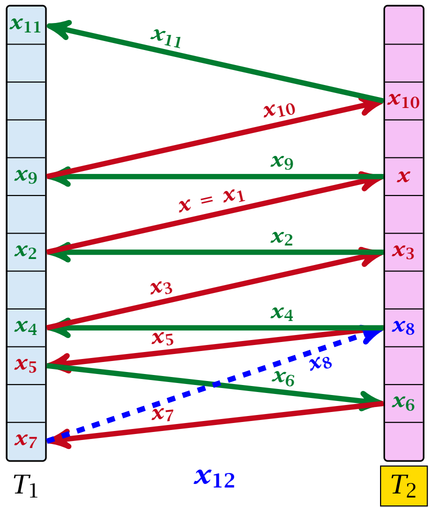

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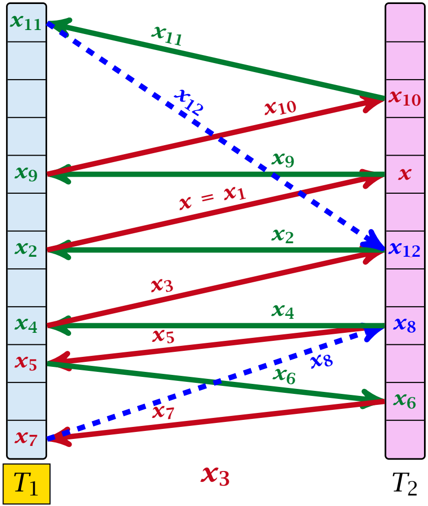

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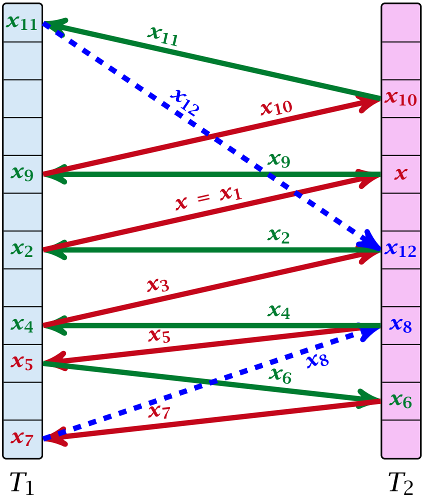

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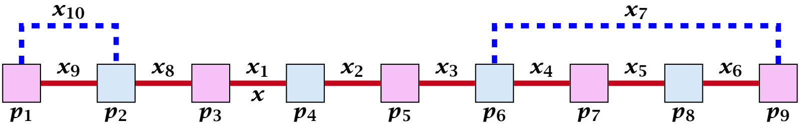

A cycle-structure of size s is defined by

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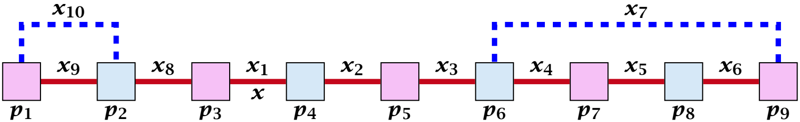

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- $s - 1$ different cells (alternating btw. cells from T_1 and T_2).

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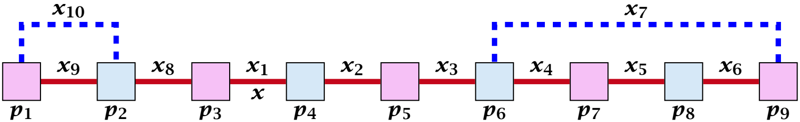

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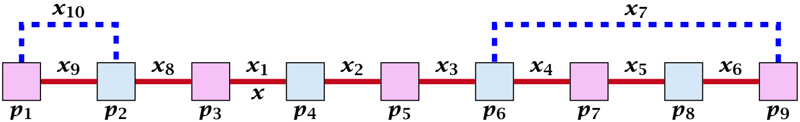

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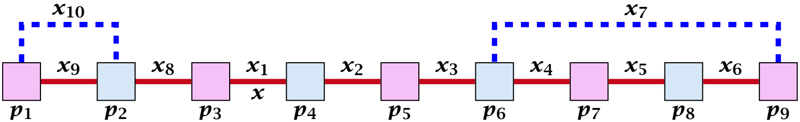

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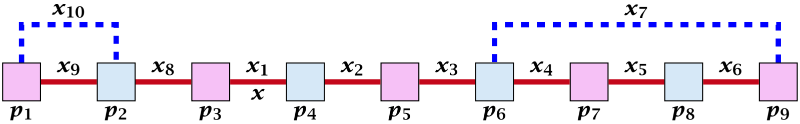

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- The leftmost cell is "linked forward" to some cell on the right.
- The rightmost cell is "linked backward" to a cell on the left.
- One link represents key x; this is where the counting starts.

Cuckoo Hashing

A cycle-structure is active if for every key x_ℓ (linking a cell p_i from T_1 and a cell p_j from T_2) we have

$$h_1(x_\ell) = p_i \qquad \text{and} \qquad h_2(x_\ell) = p_j$$

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Observation:
If during a phase the insert-procedure runs into a cycle there must exist an active cycle structure of size $s \geq 3$.

Cuckoo Hashing

What is the probability that all keys in a cycle-structure of size s correctly map into their T_1-cell?

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This probability is at most $\frac{\mu}{n^s}$ since h_1 is a (μ, s)-independent hash-function.

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This probability is at most $\frac{\mu}{n^s}$ since h_2 is a (μ, s)-independent hash-function.

These events are independent.

Cuckoo Hashing

The probability that a given cycle-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

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The probability that a given cycle-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

What is the probability that there exists an active cycle structure of size s?

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The number of cycle-structures of size s is at most

$$s^3 \cdot n^{s-1} \cdot m^{s-1} \ .$$

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- There are at most s^2 possibilities where to attach the forward and backward links.

Cuckoo Hashing

The number of cycle-structures of size s is at most

$$s^3 \cdot n^{s-1} \cdot m^{s-1} \ .$$

- There are at most s^2 possibilities where to attach the forward and backward links.
- There are at most s possibilities to choose where to place key x.

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- There are m^{s-1} possibilities to choose the keys apart from x.
- There are n^{s-1} possibilities to choose the cells.

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$

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Here we used the fact that $(1+\epsilon)m \le n$.

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Here we used the fact that $(1+\epsilon)m \leq n$.

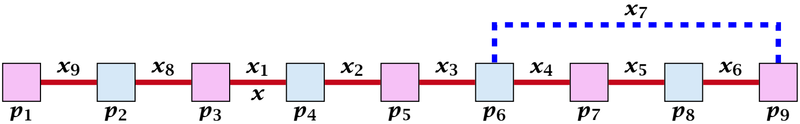
Hence,

$$\Pr[\text{cycle}] = \mathcal{O}\left(\frac{1}{m^2}\right) \enspace .$$

Cuckoo Hashing

Now, we analyze the probability that a phase is not successful without running into a closed cycle.

Cuckoo Hashing



Sequence of visited keys:

$x = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_3, x_2, x_1 = x, x_8, x_9, \ldots$

Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with x in the order that they are visited during the phase.

Cuckoo Hashing

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Lemma 30

If the sequence is of length p then there exists a sub-sequence of at least $\frac{p+2}{3}$ keys starting with x of distinct keys.

Cuckoo Hashing

Proof.

Let i be the number of keys (including x) that we see before the first repeated key. Let j denote the total number of distinct keys.

The sequence is of the form:

$$x = x_1 \to x_2 \to \cdots \to x_i \to x_r \to x_{r-1} \to \cdots \to x_1 \to x_{i+1} \to \cdots \to x_j$$

As $r \leq i-1$ the length p of the sequence is

$$p = i + r + (j - i) \leq i + j - 1 \ .$$

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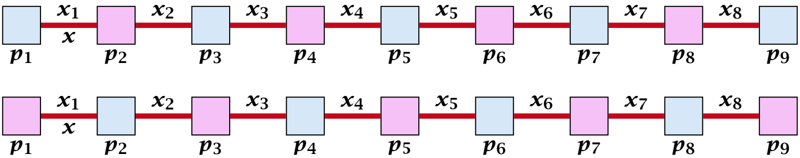
As $r \le i - 1$ the length p of the sequence is

$$p = i + r + (j - i) \le i + j - 1 \ .$$

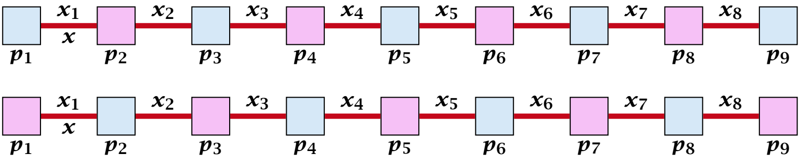
Either sub-sequence $x_1 \to x_2 \to \cdots \to x_i$ or sub-sequence $x_1 \to x_{i+1} \to \cdots \to x_j$ has at least $\frac{p+2}{3}$ elements. □

Cuckoo Hashing



A path-structure of size s is defined by

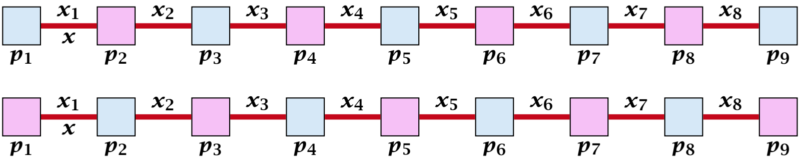
Cuckoo Hashing



A path-structure of size s is defined by

- $s + 1$ different cells (alternating btw. cells from T_1 and T_2).

Cuckoo Hashing



A path-structure of size s is defined by

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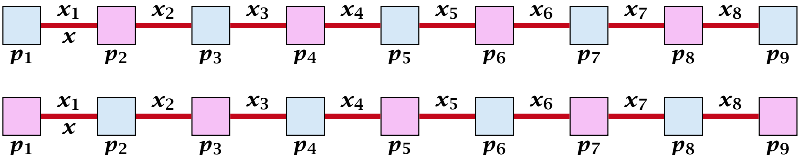

A path-structure of size s is defined by

- $s + 1$ different cells (alternating btw. cells from T_1 and T_2).
- s distinct keys $x = x_1, x_2, \ldots, x_s$, linking the cells.
- The leftmost cell is either from T_1 or T_2.

Cuckoo Hashing

A path-structure is active if for every key x_ℓ (linking a cell p_i from T_1 and a cell p_j from T_2) we have

$$h_1(x_\ell) = p_i \qquad \text{and} \qquad h_2(x_\ell) = p_j$$

Observation:
If a phase takes at least t steps without running into a cycle there must exist an active path-structure of size $(2t + 2)/3$.

Cuckoo Hashing

The probability that a given path-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

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by choosing $\ell \geq \log\left(\frac{1}{2\mu^2 m^2}\right) / \log\left(\frac{1}{1+\epsilon}\right) = \log\left(2\mu^2 m^2\right) / \log\left(1+\epsilon\right)$

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This gives $\mathrm{maxsteps} = \Theta(\log m)$.

Cuckoo Hashing

So far we estimated

$$\Pr[\text{cycle}] \leq \mathcal{O}\Big(\frac{1}{m^2}\Big)$$

and

$$\Pr[\text{unsuccessful} \mid \text{no cycle}] \leq \mathcal{O}\Big(\frac{1}{m^2}\Big)$$

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for a suitable constant $c > 0$.

Cuckoo Hashing

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&\qquad \le \frac{1}{c} \Pr[\text{search at least } t \text{ steps} \wedge \text{successful}] / \Pr[\text{no cycle}]
\end{aligned}$$

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&\qquad \le \frac{1}{c}\sum_{t\ge 1} 2\mu^2\Big(\frac{1}{1+\epsilon}\Big)^{(2t-1)/3}
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Cuckoo Hashing

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\end{aligned}$$

This means the expected cost for a successful phase is constant (even after accounting for the cost of the incomplete step that finishes the phase).

Cuckoo Hashing

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A phase that is not successful induces cost for doing a complete rehash (this dominates the cost for the steps in the phase).

Cuckoo Hashing

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The probability that a phase is not successful is $q = \mathcal{O}(1/m^2)$ (probability $\mathcal{O}(1/m^2)$ of running into a cycle and probability $\mathcal{O}(1/m^2)$ of reaching $\mathrm{maxsteps}$ without running into a cycle).

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Therefore the expected cost for re-hashes is $\mathcal{O}(m) \cdot \mathcal{O}(p) = \mathcal{O}(1)$.

Formal Proof

Let Y_i denote the event that the i-th rehash occurs and does not lead to a valid configuration (i.e., one of the $m+1$ insertions fails):

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Therefore, it is sufficient to have $(\mu, \Theta(\log m))$-independent hash-functions.

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- Note that right after a change in table-size we have $m = \alpha n/2$. In order for a table-expand to occur at least $\alpha n/2$ insertions are required. Similar, for a table-shrink at least $\alpha n/4$ deletions must occur.

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- Note that right after a change in table-size we have $m = \alpha n/2$. In order for a table-expand to occur at least $\alpha n/2$ insertions are required. Similar, for a table-shrink at least $\alpha n/4$ deletions must occur.
- Therefore we can amortize the rehash cost after a change in table-size against the cost for insertions and deletions.

Cuckoo Hashing

Lemma 31

Cuckoo Hashing has an expected constant insert-time and a worst-case constant search-time.

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Cuckoo Hashing has an expected constant insert-time and a worst-case constant search-time.

Note that the above lemma only holds if the fill-factor (number of keys/total number of hash-table slots) is at most $\frac{1}{2(1+\epsilon)}$.

The $1/(2(1+\epsilon))$ fill-factor comes from the fact that the total hash-table is of size $2n$ (because we have two tables of size n); moreover $m \leq (1+\epsilon)n$.

8 Priority Queues

A Priority Queue S is a dynamic set data structure that supports the following operations:

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- $\boldsymbol{\mathrm{boolean}\ S.\,\mathrm{is\text{-}empty}()}$: Returns true if the data-structure is empty and false otherwise.

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- **boolean** $\boldsymbol{S.}$ **is-empty()**: Returns true if the data-structure is empty and false otherwise.

Sometimes we also have

- $\boldsymbol{S.}$ **merge**$(\boldsymbol{S'})$: $S := S \cup S'$; $S' := \emptyset$.

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An addressable Priority Queue also supports:

- **handle $S.\mathrm{insert}(x)$:** Adds element x to the data-structure, and returns a handle to the object for future reference.
- **$S.\mathrm{delete}(h)$:** Deletes element specified through handle h.
- **$S.\mathrm{decrease\text{-}key}(h,k)$:** Decreases the key of the element specified by handle h to k. Assumes that the key is at least k before the operation.

Dijkstra's Shortest Path Algorithm

Algorithm 39 Shortest-Path($G = (V, E, d), s \in V$)

```
Input: weighted graph $G = (V, E, d)$; start vertex $s$;
Output: key-field of every node contains distance from $s$;
$S$.build(); // build empty priority queue
for all $v \in V \setminus \{s\}$ do
    $v.\mathrm{key} \leftarrow \infty$;
    $h_v \leftarrow S$.insert($v$);
$s.\mathrm{key} \leftarrow 0$; $S$.insert($s$);
while $S$.is-empty() $=$ false do
    $v \leftarrow S$.delete-min();
    for all $x \in V$ s.t. $(v, x) \in E$ do
        if $x.\mathrm{key} > v.\mathrm{key} + d(v, x)$ then
            $S$.decrease-key($h_x$, $v.\mathrm{key} + d(v, x)$);
            $x.\mathrm{key} \leftarrow v.\mathrm{key} + d(v, x)$;
```

Prim's Minimum Spanning Tree Algorithm

```
Algorithm 40 Prim-MST($G = (V, E, d), s \in V$)
Input: weighted graph $G = (V, E, d)$; start vertex $s$;
Output: pred-fields encode MST;
$S$.build(); // build empty priority queue
for all $v \in V \setminus \{s\}$ do
    $v$.key $\leftarrow \infty$;
    $h_v \leftarrow S$.insert($v$);
$s$.key $\leftarrow 0$; $S$.insert($s$);
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    $v \leftarrow S$.delete-min();
    for all $x \in V$ s.t. $\{v, x\} \in E$ do
        if $x$.key $> d(v, x)$ then
            $S$.decrease-key($h_x$,$d(v, x)$);
            $x$.key $\leftarrow d(v, x)$;
            $x$.pred $\leftarrow v$;
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Analysis of Dijkstra and Prim

Both algorithms require:

- 1 build() operation
- $|V|$ insert() operations
- $|V|$ delete-min() operations
- $|V|$ is-empty() operations
- $|E|$ decrease-key() operations

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How good a running time can we obtain?

8 Priority Queues

Operation	*Binary Heap*	*BST*	*Binomial Heap*	*Fibonacci Heap**
build	n	$n \log n$	$n \log n$	n
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n$**	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	n	$n \log n$	$\log n$	1

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delete-min	$\log n$	$\log n$	$\log n$	$\log n$
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Note that most applications use **build()** only to create an empty heap which then costs time 1.

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The standard version of binary heaps is not addressable, and hence does not support a delete operation.

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Note that most applications use **build()** only to create an empty heap which then costs time 1.

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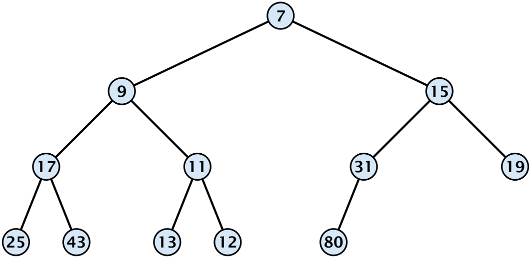
Fibonacci heaps only give an amortized guarantee.

8 Priority Queues

Using Binary Heaps, Prim and Dijkstra run in time $\mathcal{O}((|V| + |E|) \log |V|)$.

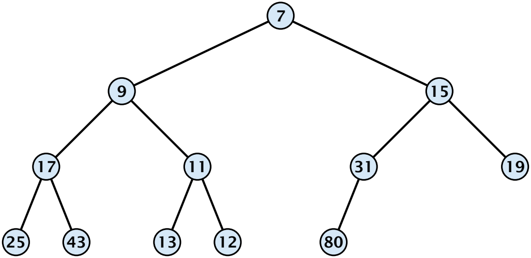
Using Fibonacci Heaps, Prim and Dijkstra run in time $\mathcal{O}(|V| \log |V| + |E|)$.

8.1 Binary Heaps



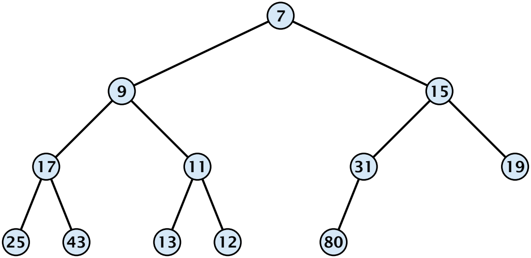
8.1 Binary Heaps

- Nearly complete binary tree; only the last level is not full, and this one is filled from left to right.



8.1 Binary Heaps

- Nearly complete binary tree; only the last level is not full, and this one is filled from left to right.
- Heap property: A node's key is not larger than the key of one of its children.



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Binary Heaps

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- **minimum()**: return the root-element. Time $\mathcal{O}(1)$.
- **is-empty()**: check whether root-pointer is null. Time $\mathcal{O}(1)$.

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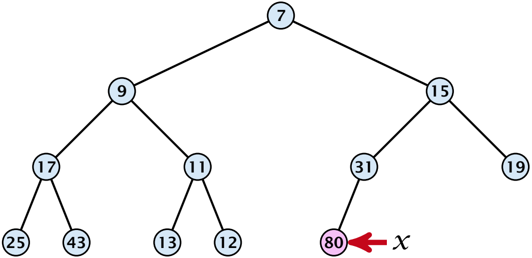
Maintain a pointer to the last element x.



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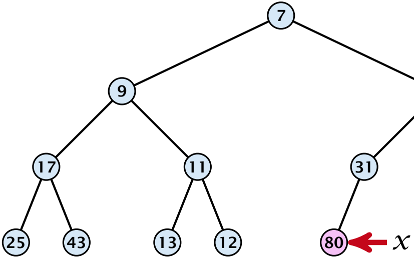
Maintain a pointer to the last element x.

- We can compute the predecessor of x (last element when x is deleted) in time $\mathcal{O}(\log n)$.



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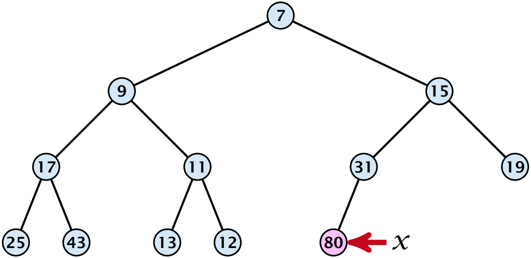
  go left; go right until you reach a leaf



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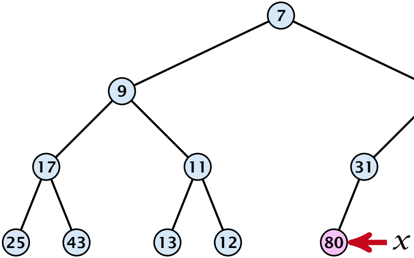

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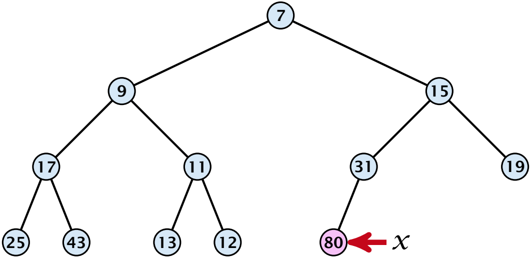

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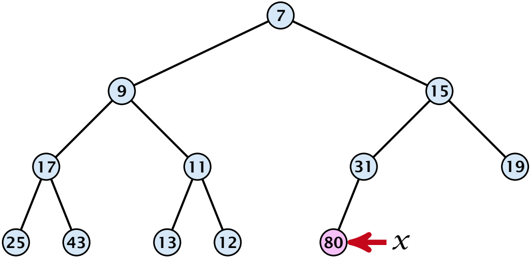

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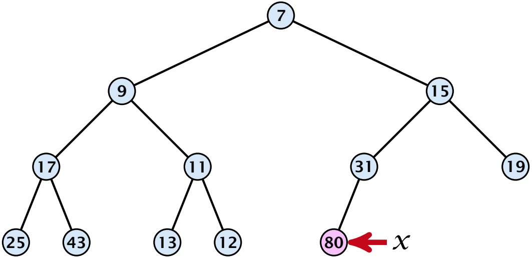

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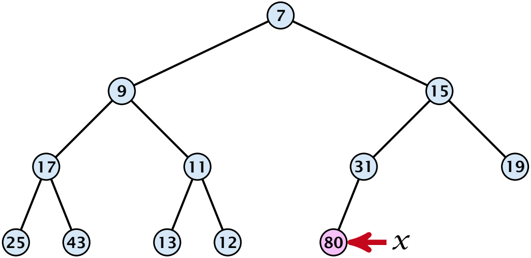
  go right; go left until you reach a null-pointer.

  if you hit the root on the way up, go to the leftmost element;
  insert a new element as a left child;



Insert

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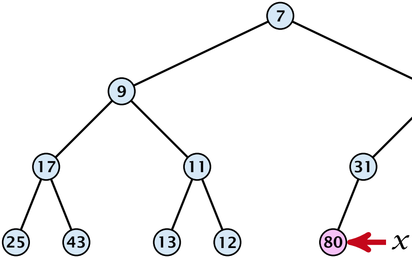

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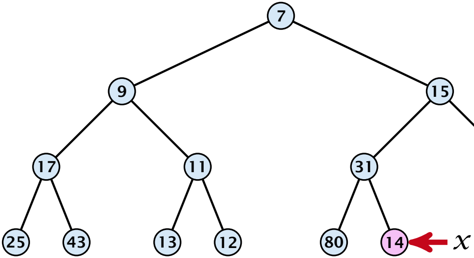
2. Exchange with parent until heap property is fulfilled.



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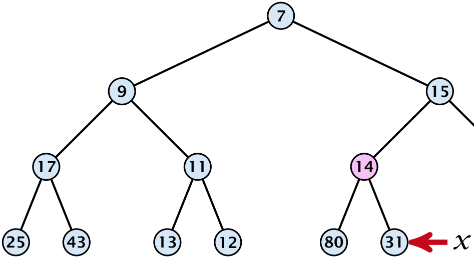

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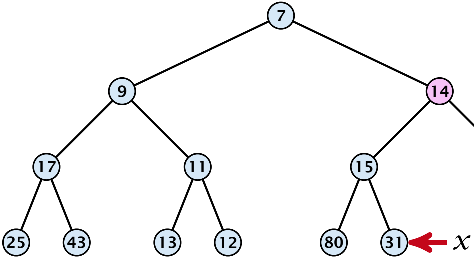

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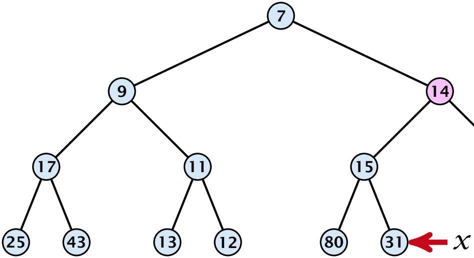
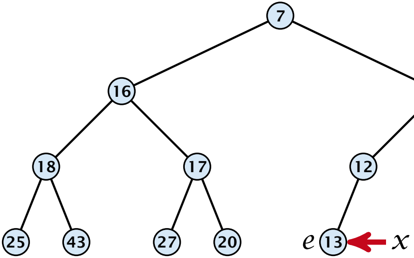


Note that an exchange can either be done by moving the data or by changing pointers. The latter method leads to an addressable priority queue.

Delete

1. Exchange the element to be deleted with the element e pointed to by x.



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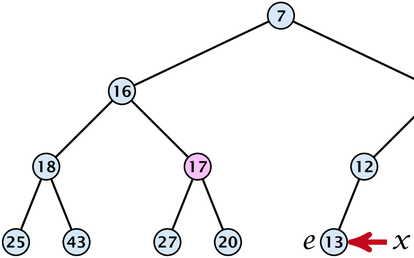
2. Restore the heap-property for the element e.



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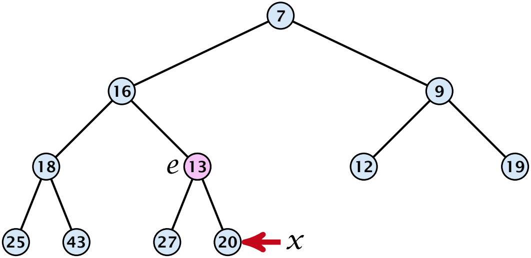

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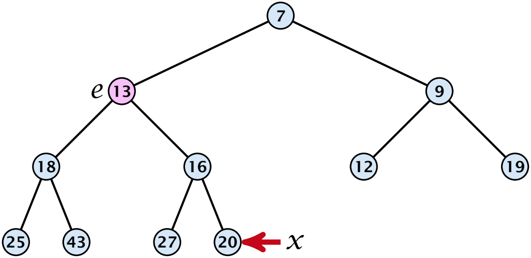

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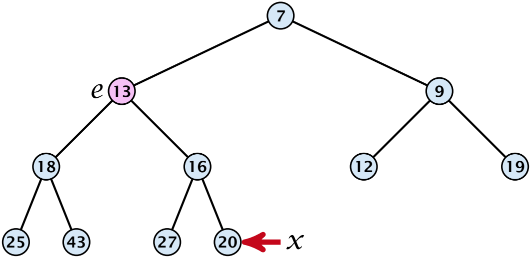

At its new position e may either travel up or down in the tree (but not both directions).

Binary Heaps

Operations:

- **minimum()**: return the root-element. Time $\mathcal{O}(1)$.
- **is-empty()**: check whether root-pointer is null. Time $\mathcal{O}(1)$.
- **insert(k)**: insert at successor of x and bubble up. Time $\mathcal{O}(\log n)$.
- **delete(h)**: swap with x and bubble up or sift-down. Time $\mathcal{O}(\log n)$.

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- **delete(h)**: Swap with x and bubble up or sift-down. Time $\mathcal{O}(\log n)$.
- **build($x_1, \dots, x_n$)**: Insert elements arbitrarily; then do sift-down operations starting with the lowest layer in the tree. Time $\mathcal{O}(n)$.

Binary Heaps

Binary Heaps

The standard implementation of binary heaps is via arrays. Let $A[0,\ldots,n-1]$ be an array

- The parent of i-th element is at position $\lfloor\frac{i-1}{2}\rfloor$.
- The left child of i-th element is at position $2i+1$.
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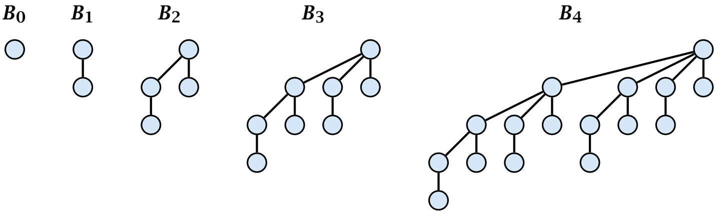
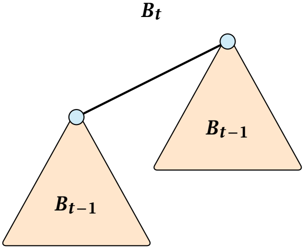
The resulting binary heap is not addressable. The elements don't maintain their positions and therefore there are no stable handles.

8.2 Binomial Heaps

Operation	*Binary Heap*	*BST*	*Binomial Heap*	*Fibonacci Heap**
build	n	$n \log n$	$n \log n$	n
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n$**	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	n	$n \log n$	**$\log n$**	1

Binomial Trees





Binomial Trees

Properties of Binomial Trees

- B_k has 2^k nodes.

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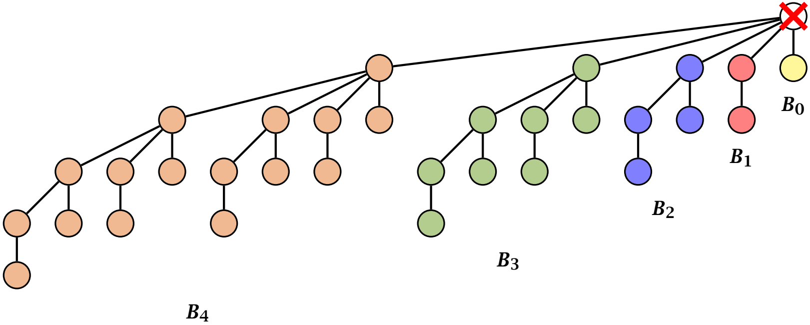
Binomial Trees

Properties of Binomial Trees

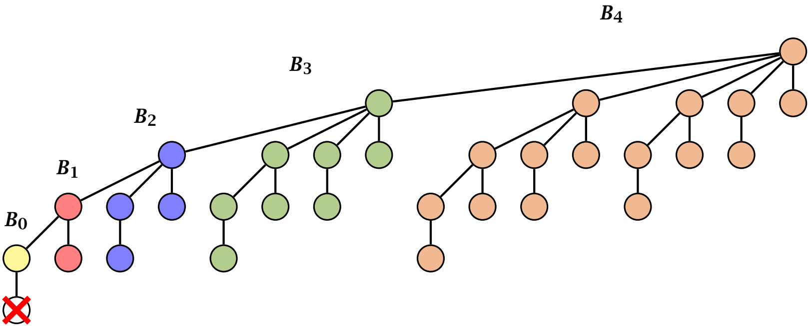
- B_k has 2^k nodes.
- B_k has height k.
- The root of B_k has degree k.
- B_k has $\binom{k}{\ell}$ nodes on level ℓ.
- Deleting the root of B_k gives trees $B_0, B_1, \dots, B_{k-1}$.

Binomial Trees



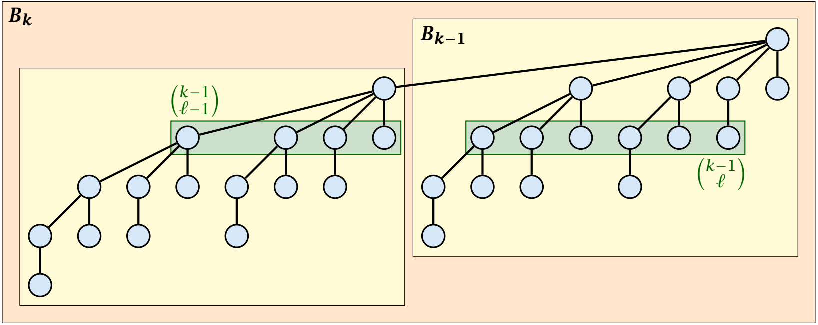
Deleting the root of B_5 leaves sub-trees B_4, B_3, B_2, B_1, and B_0.

Binomial Trees



Deleting the leaf furthest from the root (in B_5) leaves a path that connects the roots of sub-trees B_4, B_3, B_2, B_1, and B_0.

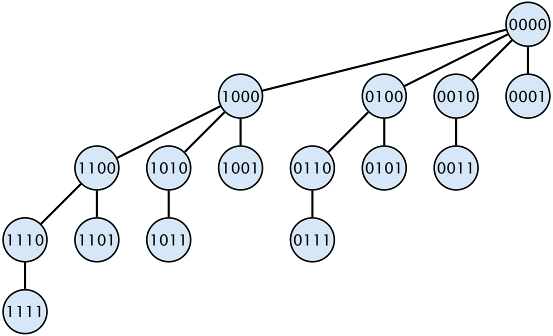
Binomial Trees



The number of nodes on level ℓ in tree B_k is therefore

$$\binom{k-1}{\ell-1} + \binom{k-1}{\ell} = \binom{k}{\ell}$$

Binomial Trees



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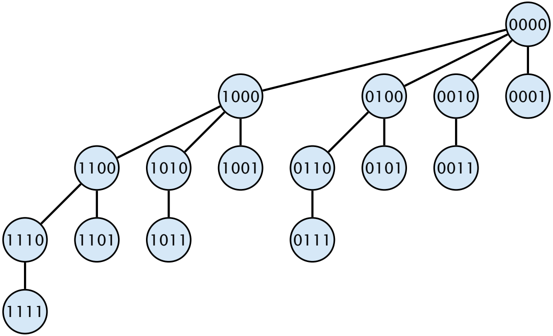

The binomial tree B_k is a sub-graph of the hypercube H_k.

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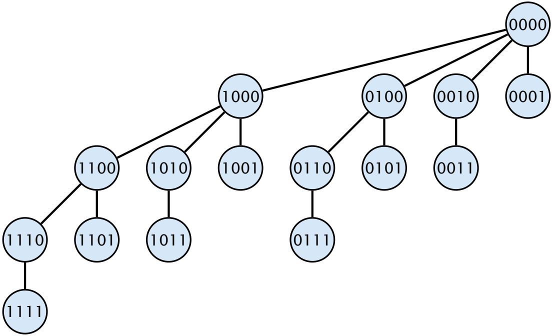

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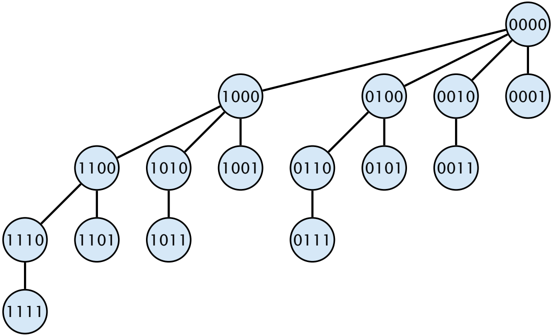
The parent of a node with label $b_k, \dots, b_1$ is obtained by setting the least significant 1-bit to 0.

Binomial Trees



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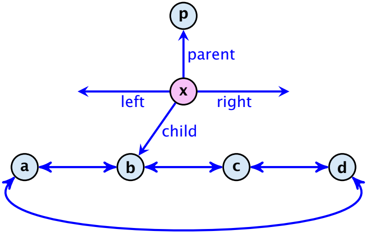
The ℓ-th level contains nodes that have ℓ 1's in their label.

8.2 Binomial Heaps

How do we implement trees with non-constant degree?

- The children of a node are arranged in a circular linked list.



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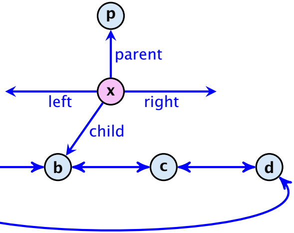

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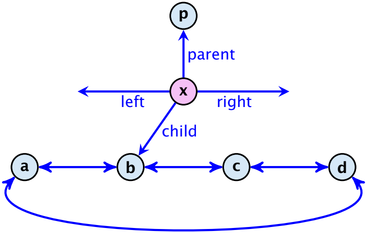

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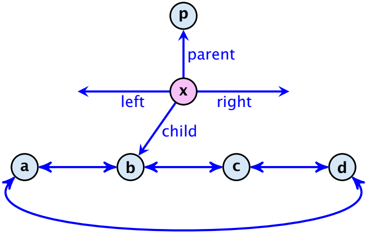
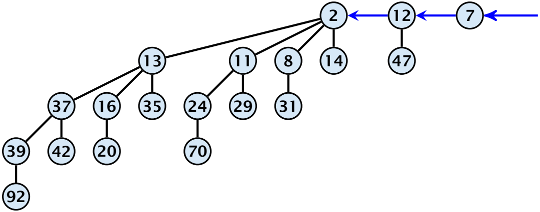
- A parent-pointer points to the parent node.
- Pointers $x.\text{left}$ and $x.\text{right}$ point to the left and right sibling of x (if x does not have siblings then $x.\text{left} = x.\text{right} = x$).



8.2 Binomial Heaps

- Given a pointer to a node x we can splice out the sub-tree rooted at x in constant time.
- We can add a child-tree T to a node x in constant time if we are given a pointer to x and a pointer to the root of T.

Binomial Heap



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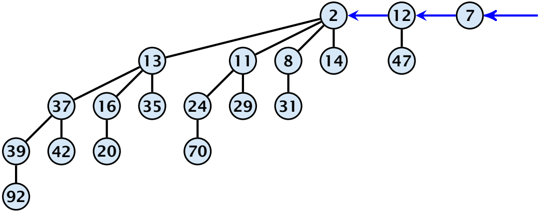
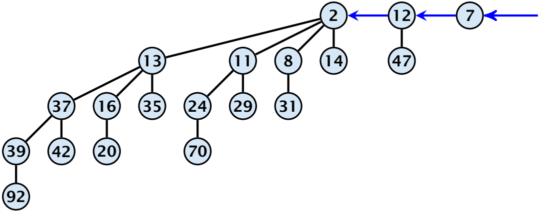


In a binomial heap the keys are arranged in a collection of binomial trees.

Binomial Heap



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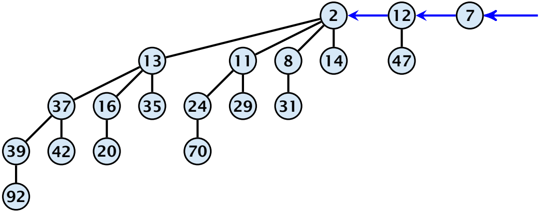

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There is at most one tree for every dimension/order. For example the above heap contains trees B_0, B_1, and B_4.

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Let B_{k_1}, B_{k_2}, B_{k_3}, $k_i < k_{i+1}$ denote the binomial trees in the collection and recall that every tree may be contained at most once.

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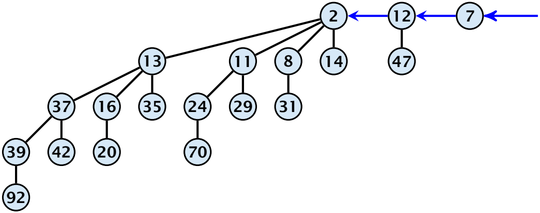
Then $n = \sum_i 2^{k_i}$ must hold. But since the k_i are all distinct this means that the k_i define the non-zero bit-positions in the binary representation of n.

Binomial Heap

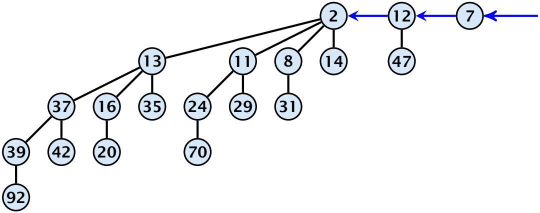
Properties of a heap with n keys:



Binomial Heap

Properties of a heap with n keys:

- Let $n = b_d b_{d-1}, \dots, b_0$ denote binary representation of n.



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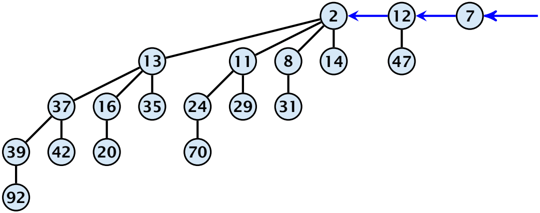

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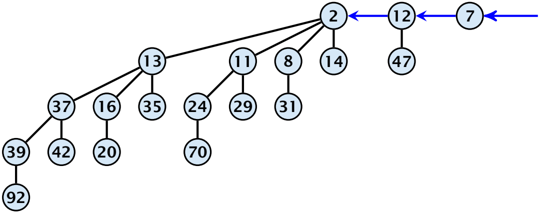

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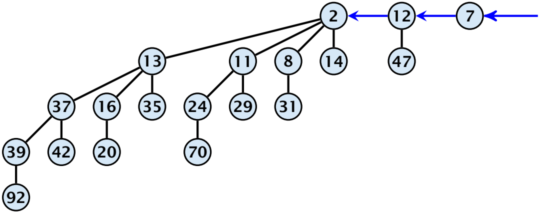

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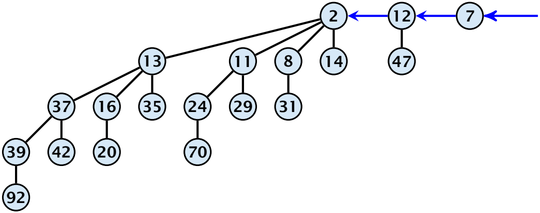

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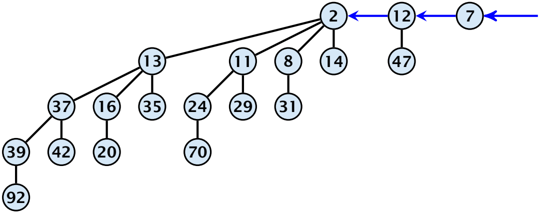
- Hence, at most $\lfloor \log n \rfloor + 1$ trees.
- The minimum must be contained in one of the roots.
- The height of the largest tree is at most $\lfloor \log n \rfloor$.
- The trees are stored in a single-linked list; ordered by dimension/size.



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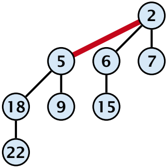

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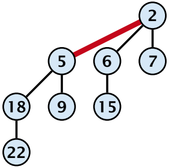
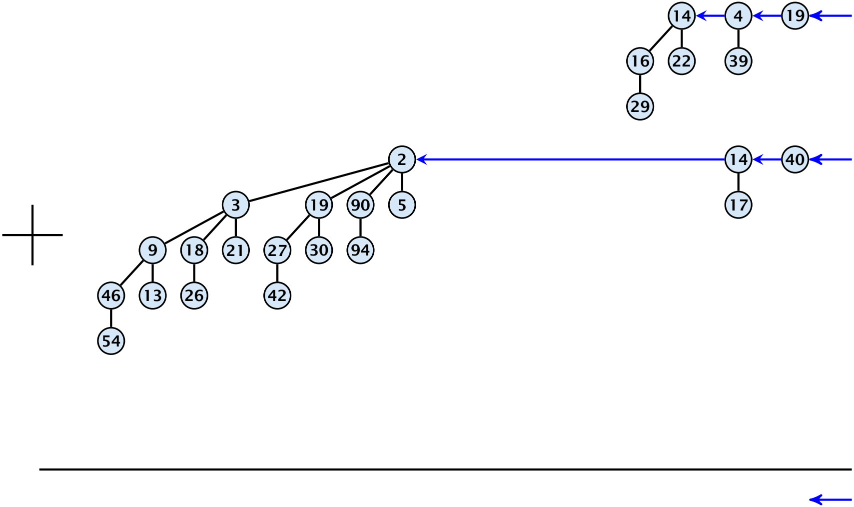


For more trees the technique is analogous to binary addition.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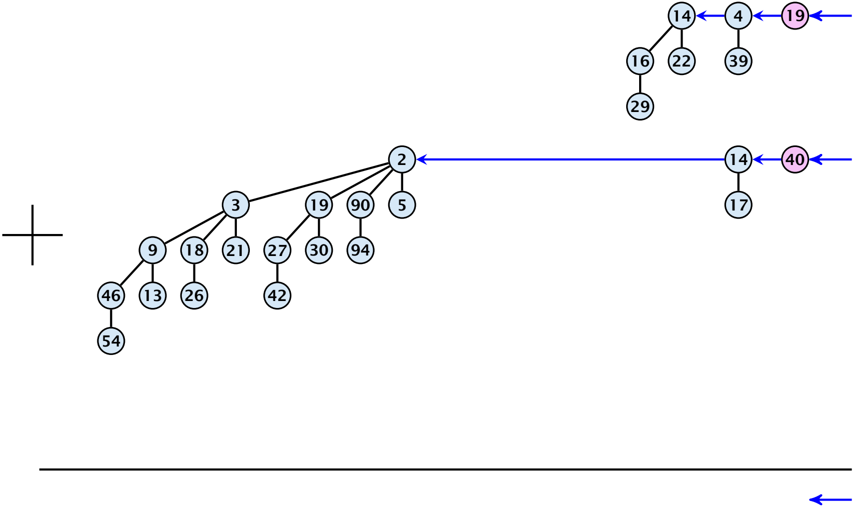
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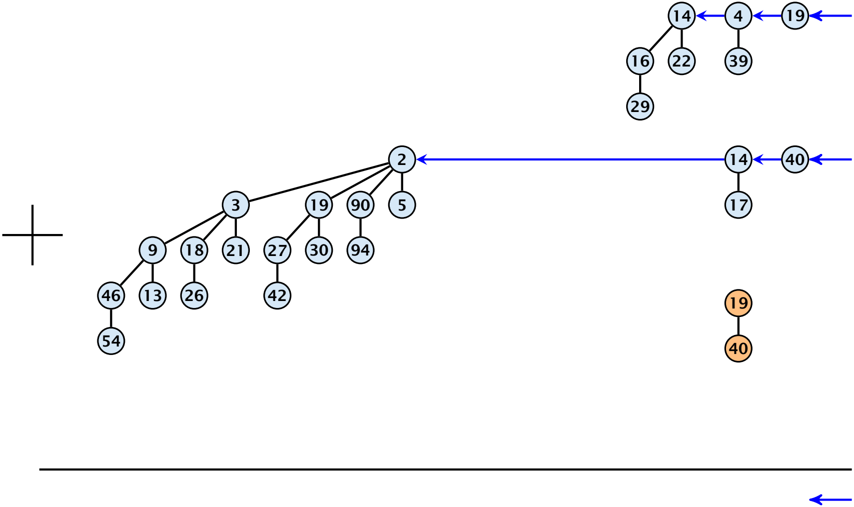

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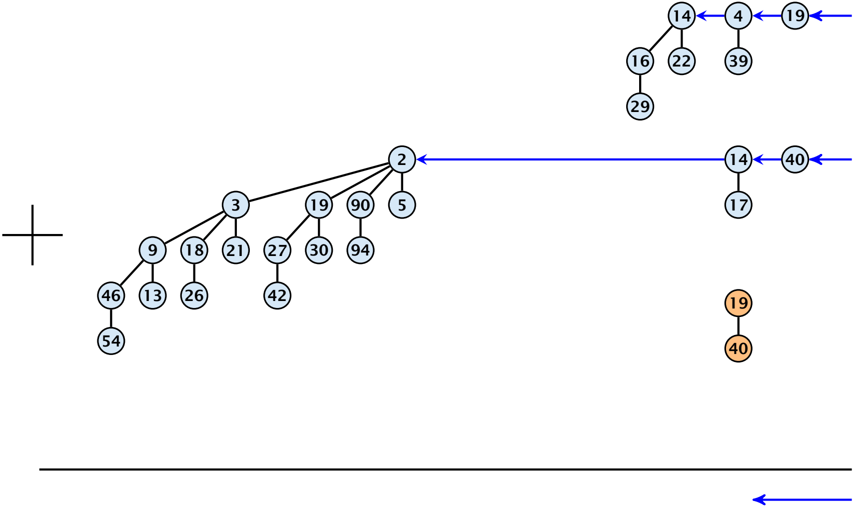
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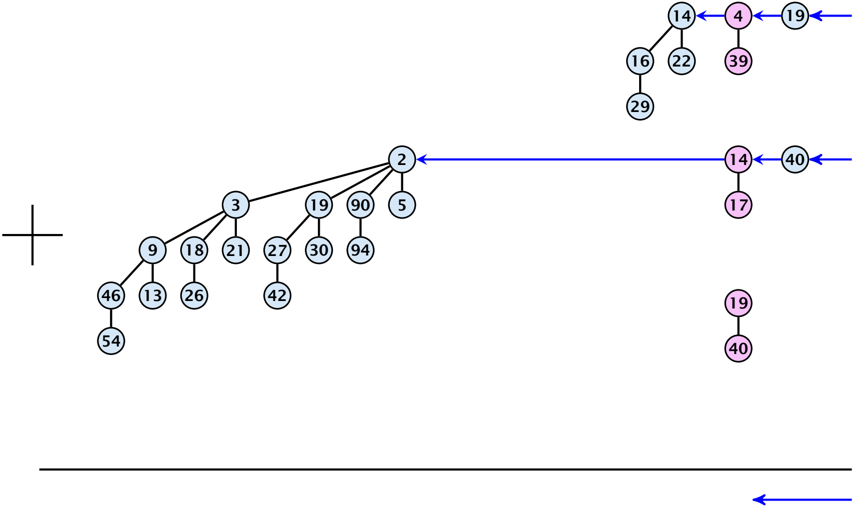
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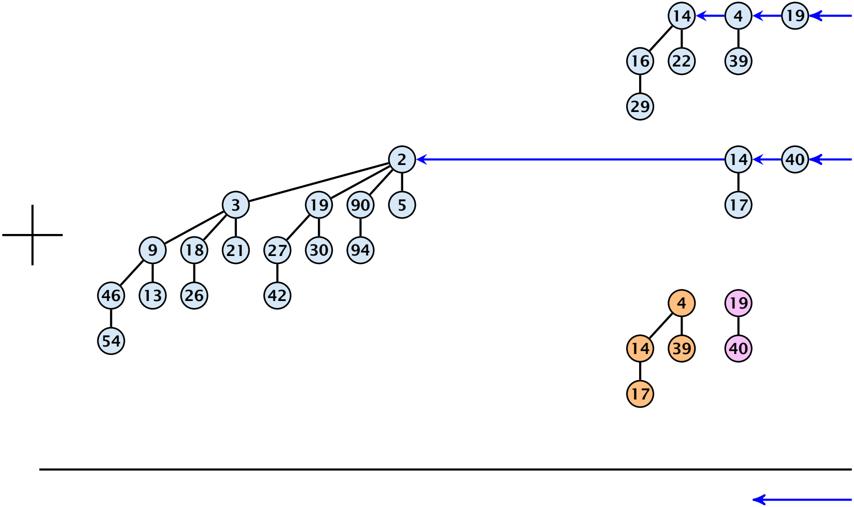
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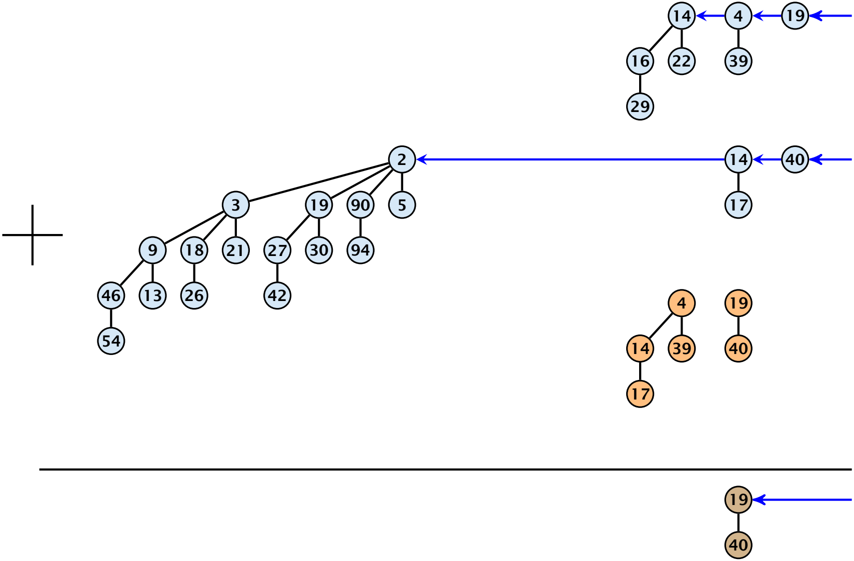
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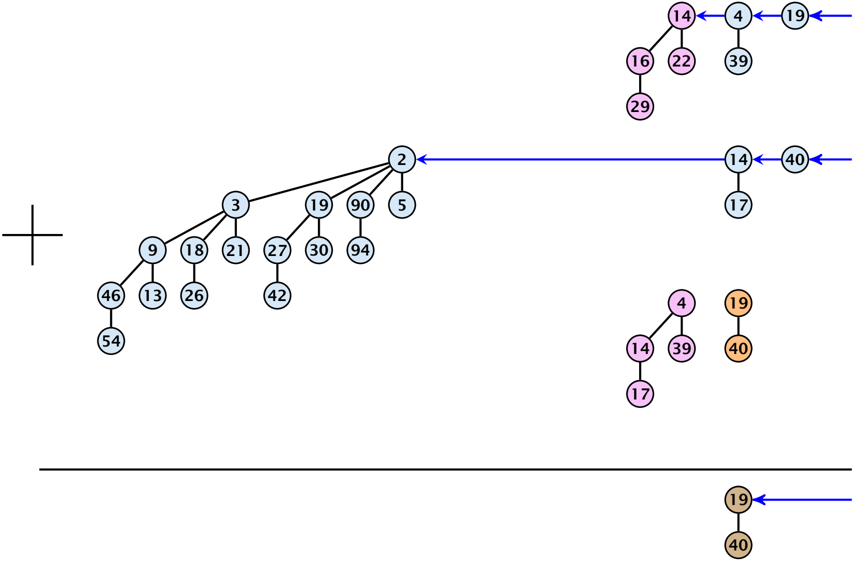
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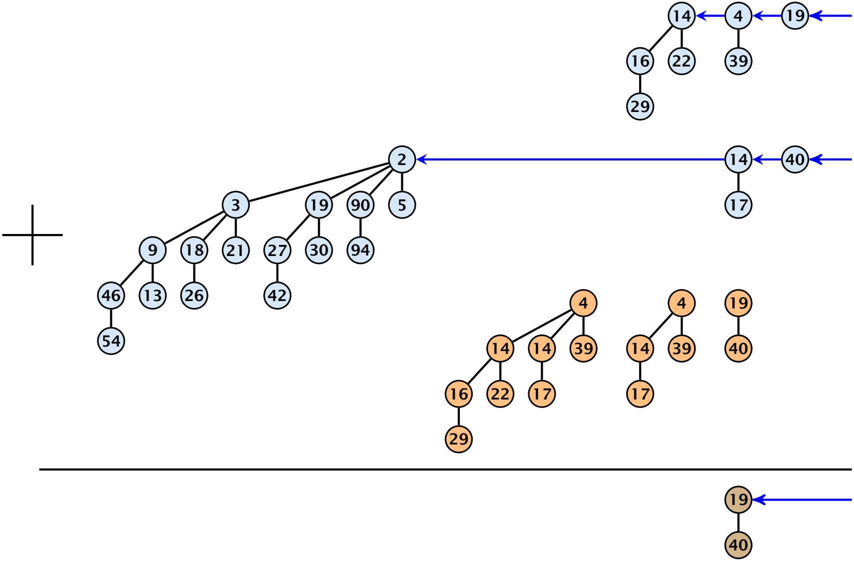
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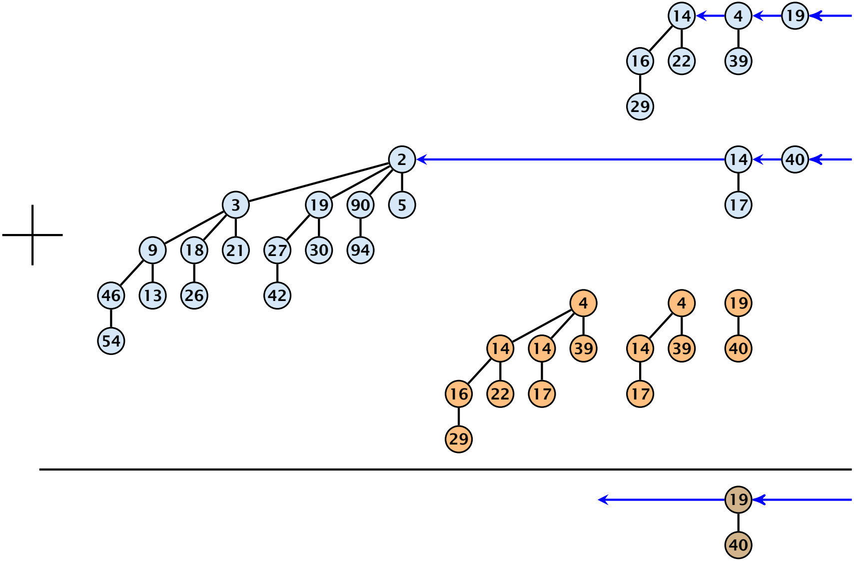
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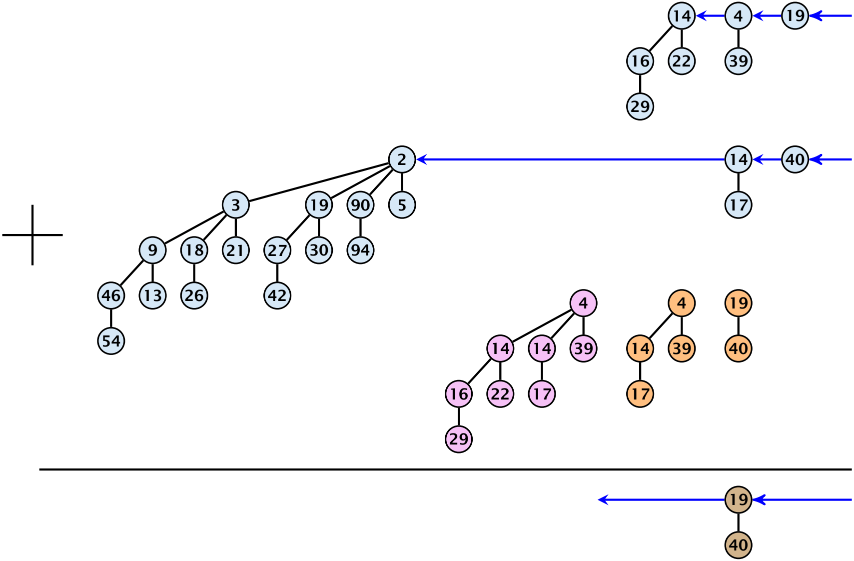

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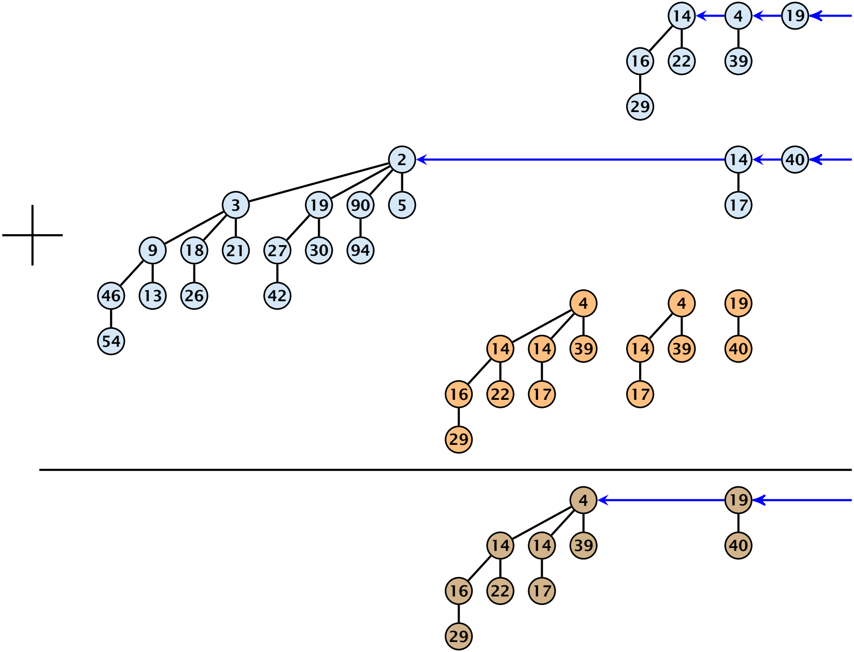
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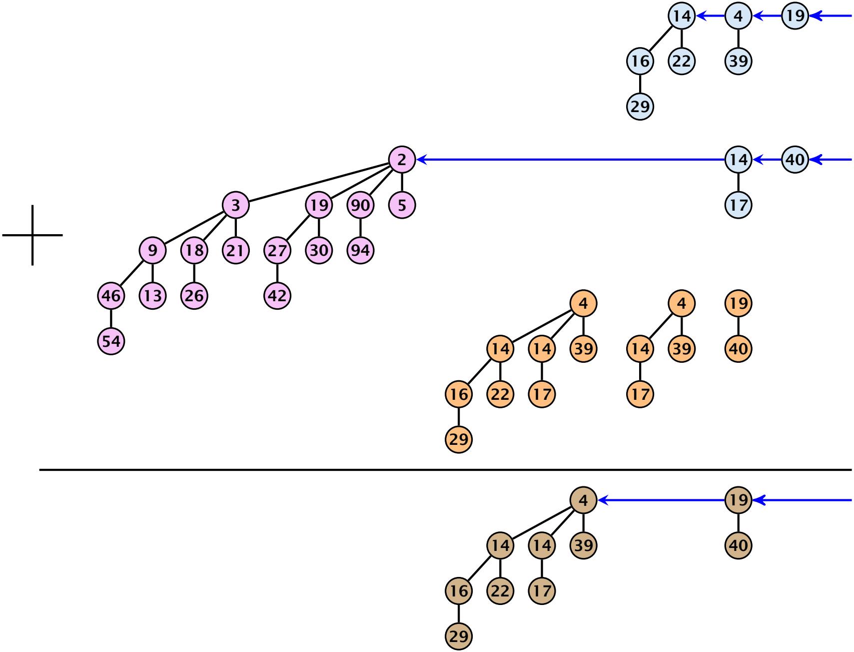
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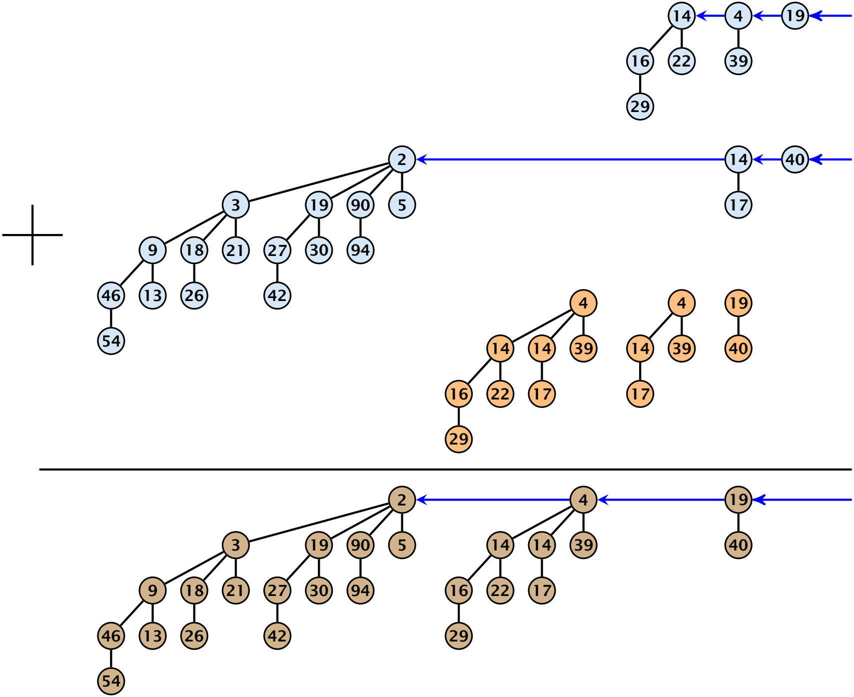
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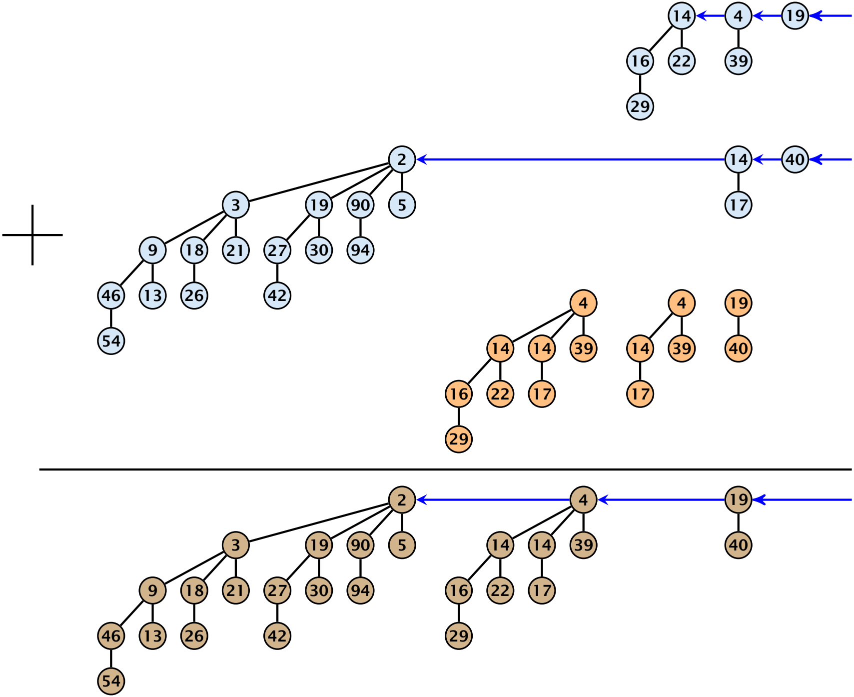
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8.2 Binomial Heaps

S_1.merge(S_2):

- Analogous to binary addition.

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8.2 Binomial Heaps

All other operations can be reduced to $\mathrm{merge}()$.

$S.\mathrm{insert}(x)$:

- Create a new heap S' that contains just the element x.

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$S.\mathrm{insert}(x)$:

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- Execute $S.\mathrm{merge}(S')$.

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All other operations can be reduced to $\text{merge}()$.

$S.\text{insert}(x)$:

- Create a new heap S' that contains just the element x.
- Execute $S.\text{merge}(S')$.
- Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

S. minimum():

- Find the minimum key-value among all roots.
- Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

S. delete-min():

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- Find the minimum key-value among all roots.
- Remove the corresponding tree $T_{\min}$ from the heap.
- Create a new heap S' that contains the trees obtained from $T_{\min}$ after deleting the root (note that these are just $\mathcal{O}(\log n)$ trees).

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- Create a new heap S' that contains the trees obtained from $T_{\min}$ after deleting the root (note that these are just $\mathcal{O}(\log n)$ trees).
- Compute $S.\mathrm{merge}(S')$.
- Time: $\mathcal{O}(\log n)$.

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8.2 Binomial Heaps

S. decrease-key(handle h):

- Decrease the key of the element pointed to by h.
- Bubble the element up in the tree until the heap property is fulfilled.
- Time: $\mathcal{O}(\log n)$ since the trees have height $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

S. delete(handle h):

8.2 Binomial Heaps

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- Execute $S.\text{decrease-key}(h, -\infty)$.

8.2 Binomial Heaps

S. delete(handle h):

- Execute S. decrease-key$(h, -\infty)$.
- Execute S. delete-min().

8.2 Binomial Heaps

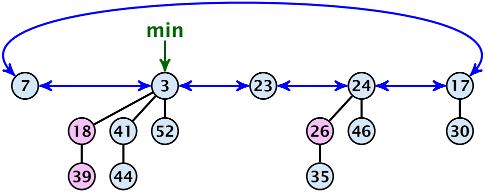
S. delete(handle h):

- Execute $S.\text{decrease-key}(h, -\infty)$.
- Execute $S.\text{delete-min}()$.
- Time: $\mathcal{O}(\log n)$.

8.3 Fibonacci Heaps

Collection of trees that fulfill the heap property.

Structure is much more relaxed than binomial heaps.



8.3 Fibonacci Heaps

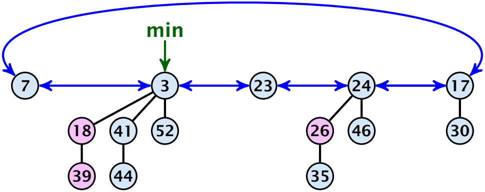
Additional implementation details:

- Every node x stores its degree in a field $x.\mathrm{degree}$. Note that this can be updated in constant time when adding a child to x.
- Every node stores a boolean value $x.\mathrm{marked}$ that specifies whether x is marked or not.

8.3 Fibonacci Heaps

The potential function:

- $t(S)$ denotes the number of trees in the heap.
- $m(S)$ denotes the number of marked nodes.
- We use the potential function $\Phi(S) = t(S) + 2m(S)$.



The potential is $\Phi(S) = 5 + 2 \cdot 3 = 11$.

8.3 Fibonacci Heaps

We assume that one unit of potential can pay for a constant amount of work, where the constant is chosen “big enough” (to take care of the constants that occur).

To make this more explicit we use c to denote the amount of work that a unit of potential can pay for.

8.3 Fibonacci Heaps

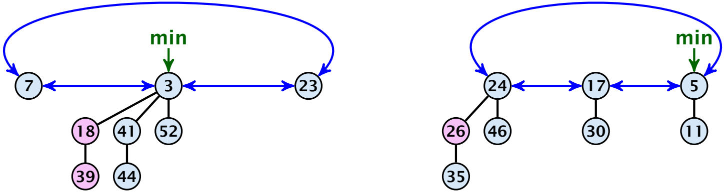
S. minimum()

- Access through the min-pointer.
- Actual cost $\mathcal{O}(1)$.
- No change in potential.
- Amortized cost $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

$S.\mathbf{merge}(S')$

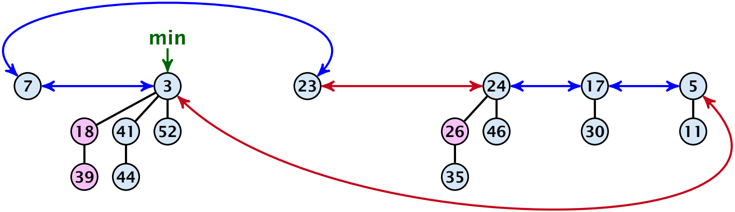
- Merge the root lists.
- Adjust the min-pointer



8.3 Fibonacci Heaps

$S.\mathbf{merge}(S')$

- Merge the root lists.
- Adjust the min-pointer



Running time:

- Actual cost $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

$S.\mathrm{merge}(S')$

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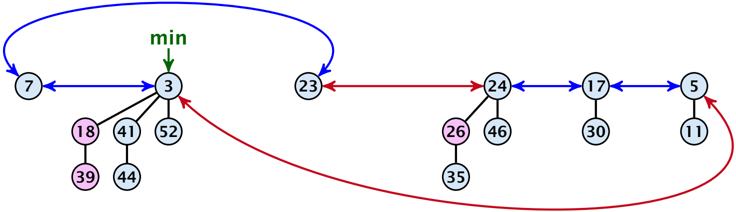

Running time:

- Actual cost $\mathcal{O}(1)$.
- No change in potential.

8.3 Fibonacci Heaps

$S.\,\mathbf{merge}(S')$

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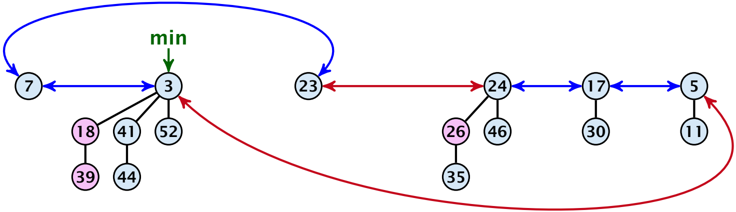
- Adjust the min-pointer



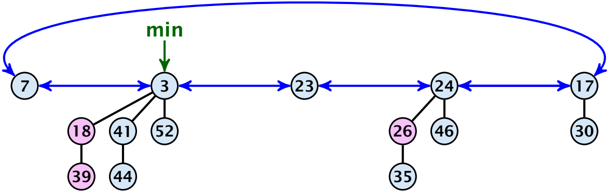
Running time:

- Actual cost $\mathcal{O}(1)$.
- No change in potential.
- Hence, amortized cost is $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

S. insert(x)

- Create a new tree containing x.
- Insert x into the root-list.
- Update min-pointer, if necessary.



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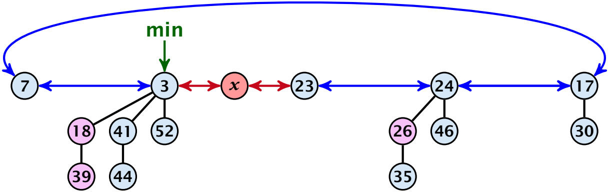

8.3 Fibonacci Heaps

$S.\text{insert}(x)$

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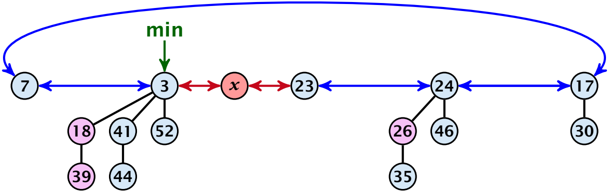
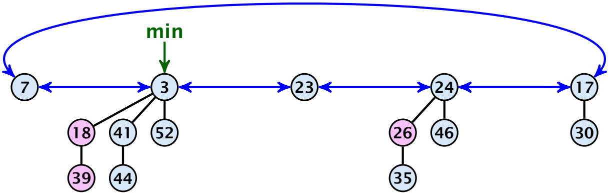


Running time:

- Actual cost $\mathcal{O}(1)$.
- Change in potential is $+1$.
- Amortized cost is $c + \mathcal{O}(1) = \mathcal{O}(1)$.

8.3 Fibonacci Heaps

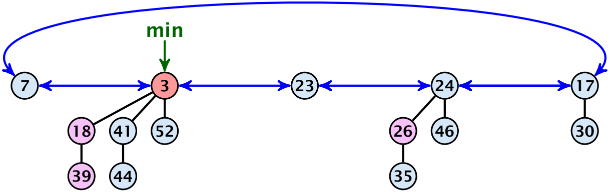
S. delete-min(x)



8.3 Fibonacci Heaps

$S.\,\mathbf{delete\text{-}min}(x)$

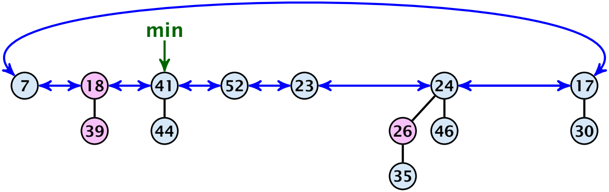
- Delete minimum; add child-trees to heap; time: $D(\min) \cdot \mathcal{O}(1)$.



8.3 Fibonacci Heaps

S. delete-min(x)

- Delete minimum; add child-trees to heap; time: $D(\min) \cdot \mathcal{O}(1)$.
- Update min-pointer; time: $(t + D(\min)) \cdot \mathcal{O}(1)$.



8.3 Fibonacci Heaps

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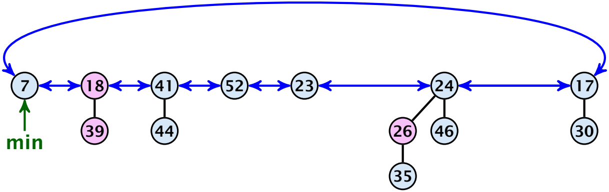

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- Delete minimum; add child-trees to heap; time: $D(\min) \cdot \mathcal{O}(1)$.
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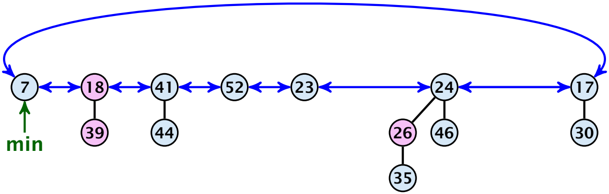
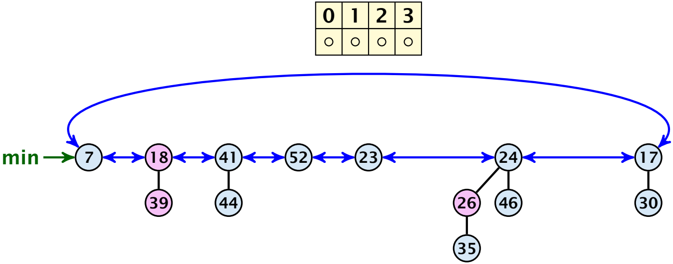


- Consolidate root-list so that no roots have the same degree. Time $t \cdot \mathcal{O}(1)$ (see next slide).

8.3 Fibonacci Heaps

Consolidate:



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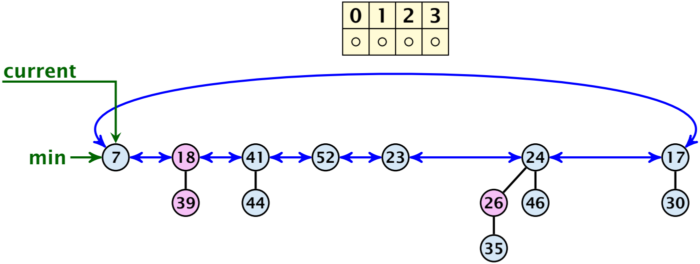

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8.3 Fibonacci Heaps

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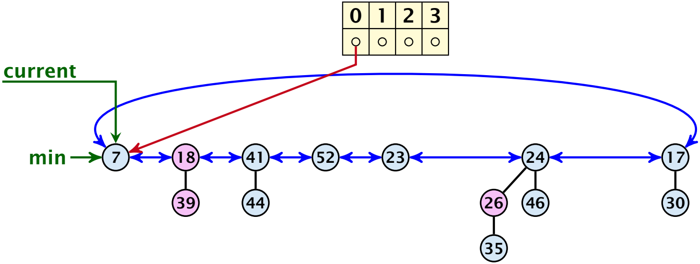

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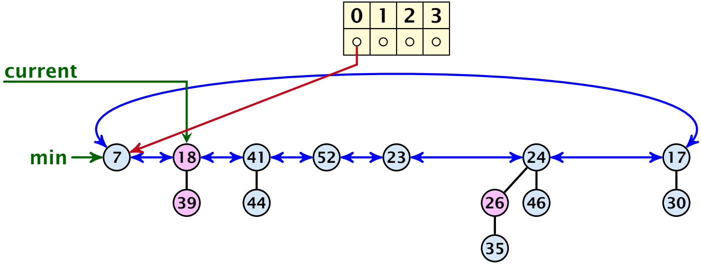

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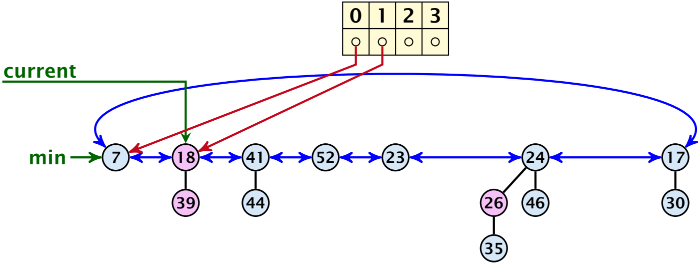

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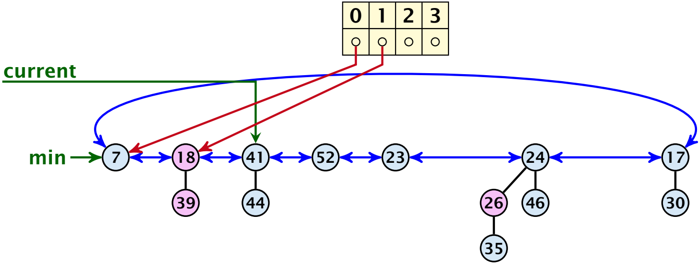

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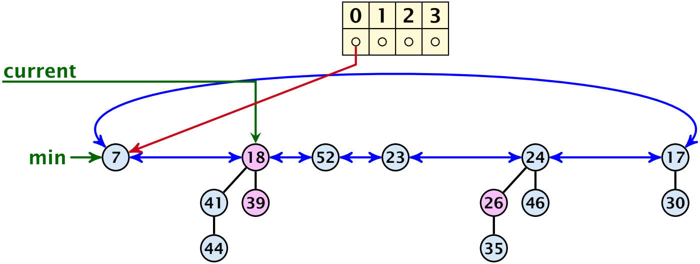

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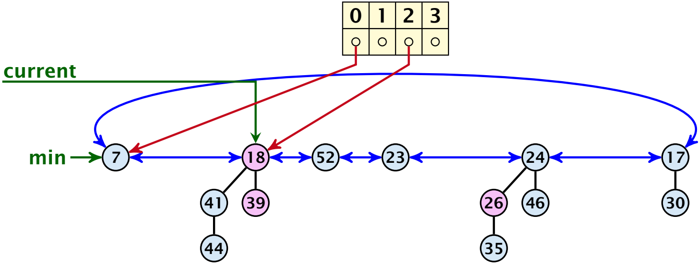

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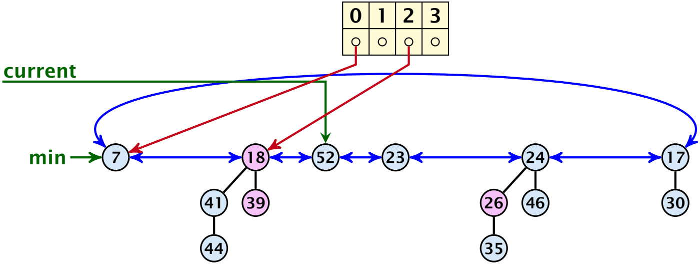

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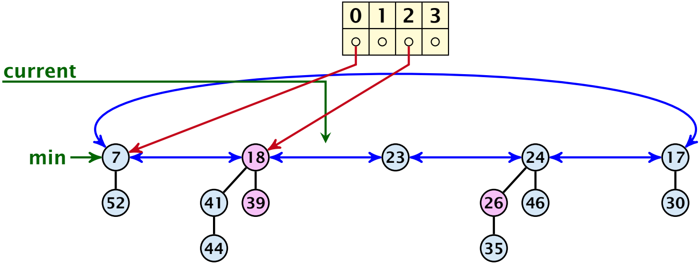

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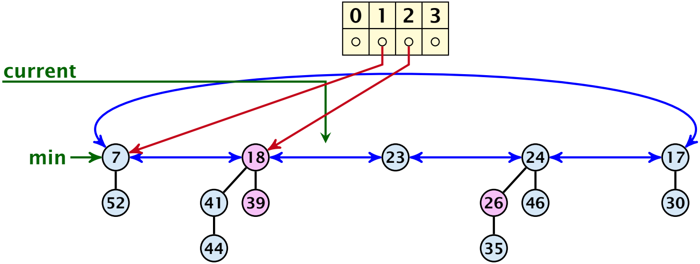

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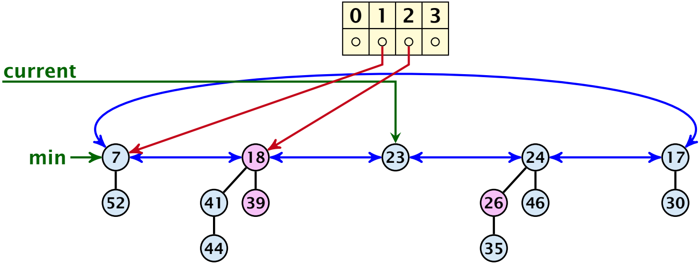

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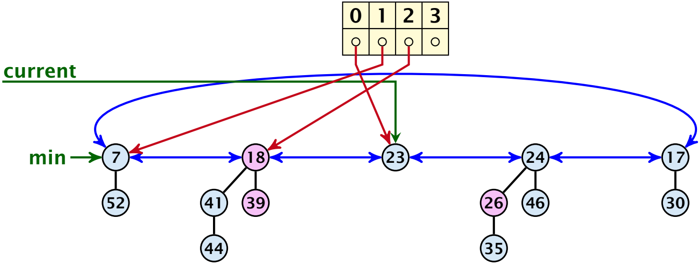

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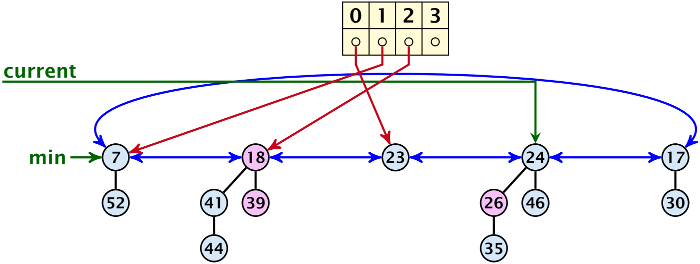

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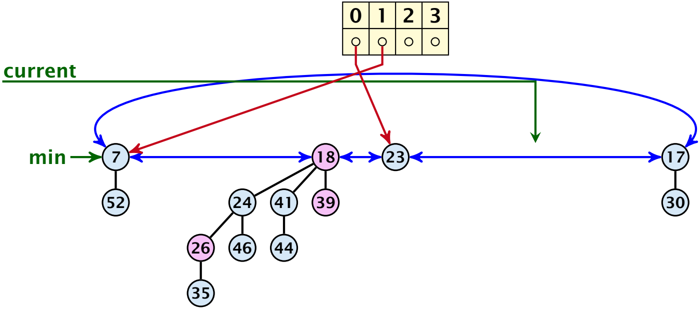
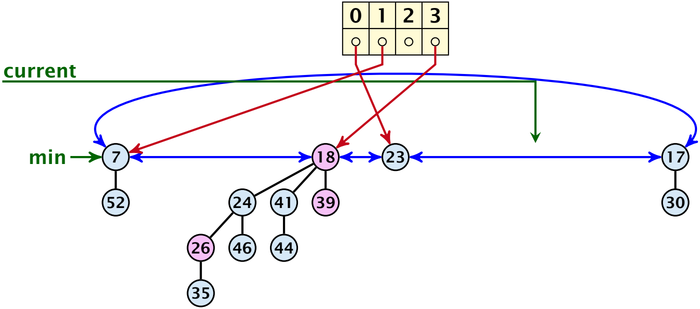


8.3 Fibonacci Heaps

Consolidate:



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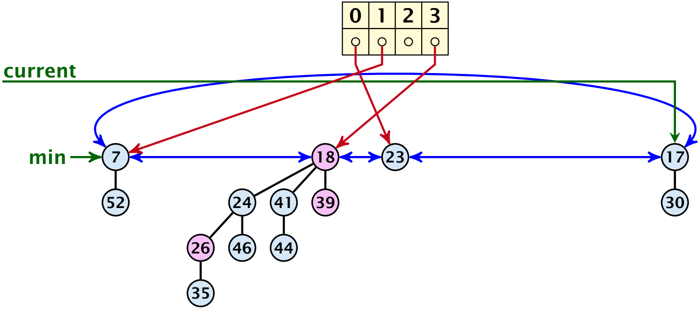

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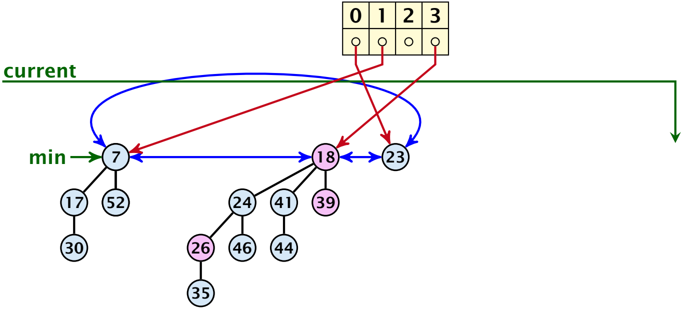

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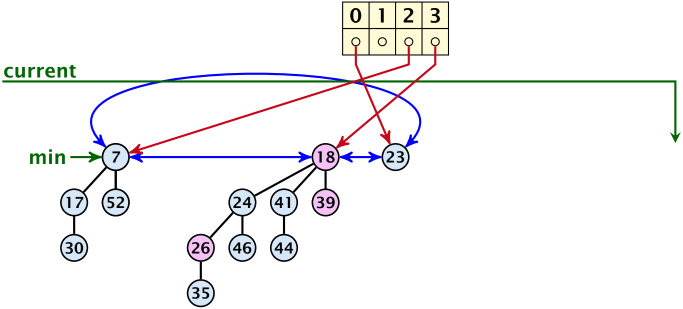

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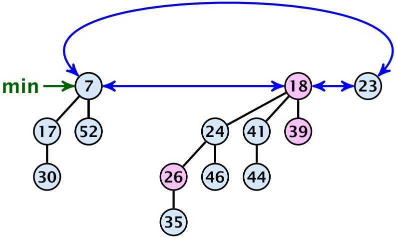

8.3 Fibonacci Heaps

Actual cost for delete-min()

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for $c \geq c_1$.

8.3 Fibonacci Heaps

If the input trees of the consolidation procedure are binomial trees (for example only singleton vertices) then the output will be a set of distinct binomial trees, and, hence, the Fibonacci heap will be (more or less) a Binomial heap right after the consolidation.

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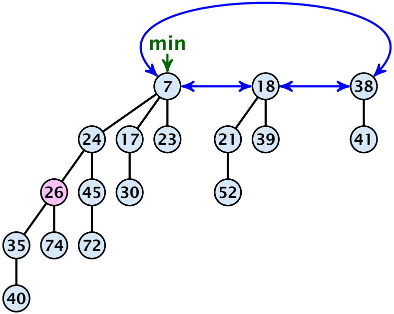
If we do not have delete or decrease-key operations then $D_n \leq \log n$.

Fibonacci Heaps: decrease-key(handle h, v)



Case 1: decrease-key does not violate heap-property

- Just decrease the key-value of element referenced by h. Nothing else to do.

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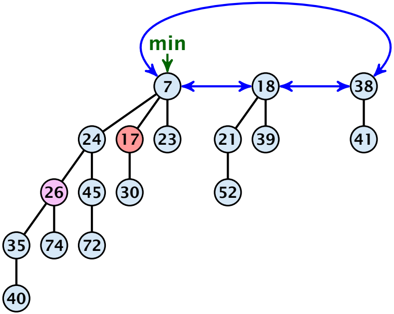

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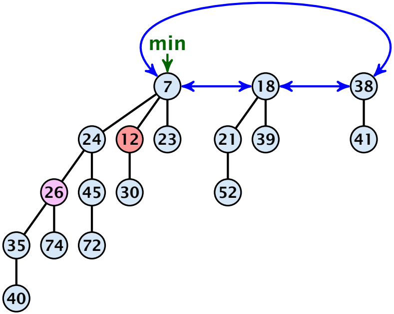

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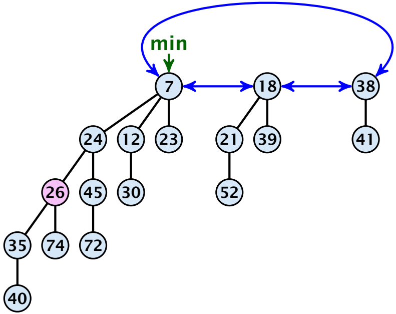

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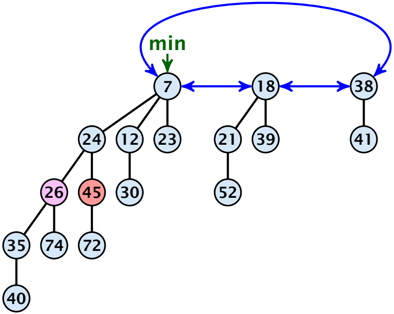

Case 2: heap-property is violated, but parent is not marked

- Decrease key-value of element x reference by h.
- If the heap-property is violated, cut the parent edge of x, and make x into a root.
- Adjust min-pointers, if necessary.
- Mark the (previous) parent of x (unless it's a root).

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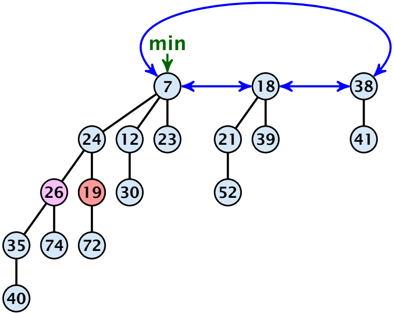

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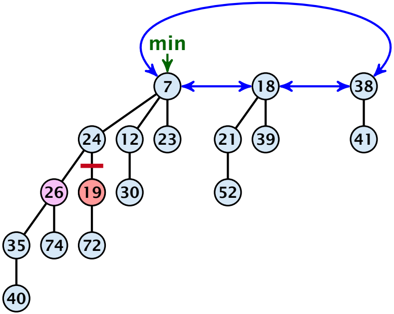

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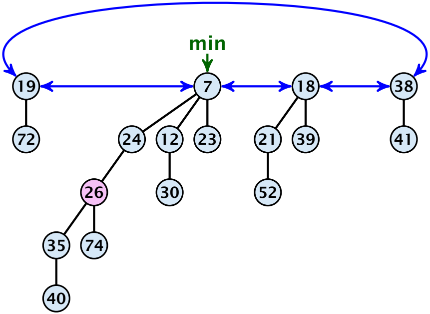

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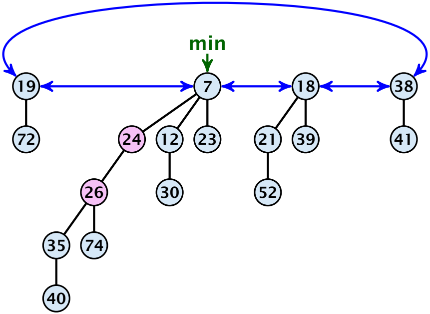

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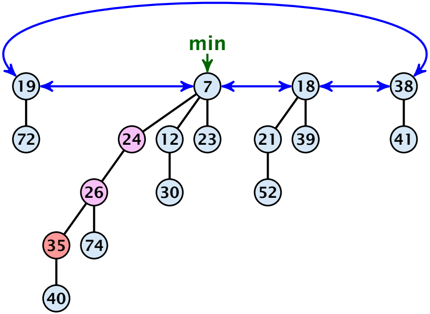

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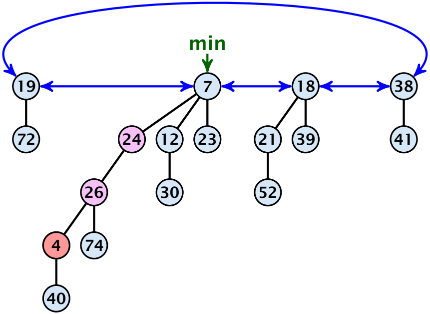

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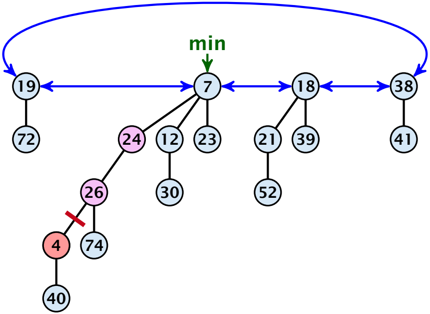

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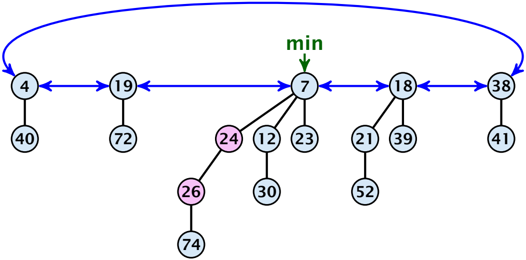

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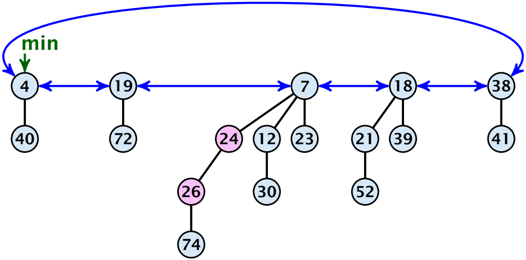

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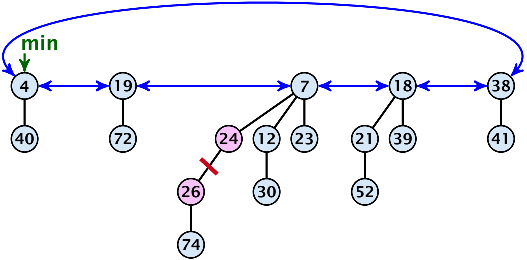

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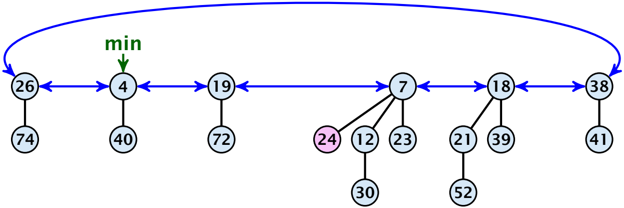

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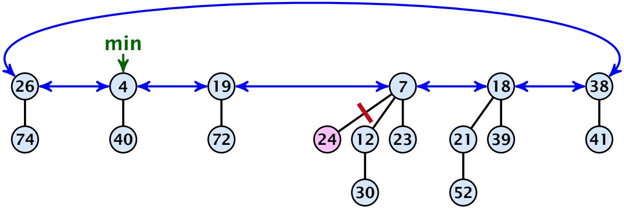

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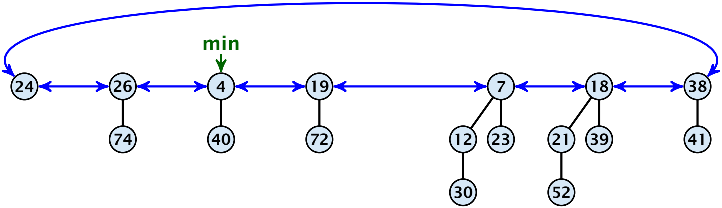

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- Cut the parent edge of x, and make x into a root.
- Adjust min-pointers, if necessary.
- Execute the following:

```
$p \leftarrow \text{parent}[x]$;
while ($p$ is marked)
    $pp \leftarrow \text{parent}[p]$;
    cut of $p$; make it into a root; unmark it;
    $p \leftarrow pp$;
if $p$ is unmarked and not a root mark it;
```

Fibonacci Heaps: decrease-key(handle h, v)

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- $m' \le m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.

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  $$c_2(\ell+1)+c(4-\ell) \leq (c_2-c)\ell+4c+c_2 = \mathcal{O}(1)\,,$$

  if $c \geq c_2$.

Delete node

$H.\mathbf{delete}(x)$:

- decrease value of x to $-\infty$.
- delete-min.

Amortized cost: $\mathcal{O}(D_n)$

- $\mathcal{O}(1)$ for decrease-key.
- $\mathcal{O}(D_n)$ for delete-min.

8.3 Fibonacci Heaps

Lemma 32

Let x be a node with degree k and let $y_1, \ldots, y_k$ denote the children of x in the order that they were linked to x. Then

$$\mathrm{degree}(y_i) \geq \begin{cases} 0 & \textit{if } i = 1 \\ i - 2 & \textit{if } i > 1 \end{cases}$$

8.3 Fibonacci Heaps

Proof

- When y_i was linked to x, at least $y_1, \dots, y_{i-1}$ were already linked to x.

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- Therefore, $\mathrm{degree}(y_i) \geq i - 2$.

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$\phi = \frac{1}{2}(1+\sqrt{5})$ denotes the *golden ratio*. Note that $\phi^2 = 1+\phi$.

Definition 33

Consider the following non-standard Fibonacci type sequence:

$$F_k = \begin{cases} 1 & \text{if } k = 0 \\ 2 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

Facts:

1. $F_k \geq \phi^k$.
2. For $k \geq 2$: $F_k = 2 + \sum_{i=0}^{k-2} F_i$.

The above facts can be easily proved by induction. From this it follows that $s_k \geq F_k \geq \phi^k$, which gives that the maximum degree in a Fibonacci heap is logarithmic.

k=0: $1 = F_0 \geq \Phi^0 = 1$

k=1: $2 = F_1 \geq \Phi^1 \approx 1.61$

k-2,k-1→ k: $F_k = F_{k-1} + F_{k-2} \geq \Phi^{k-1} + \Phi^{k-2} = \Phi^{k-2}\overbrace{(\Phi+1)}^{\Phi^2} = \Phi^k$

k=2: $3 = F_2 = 2 + 1 = 2 + F_0$

k-1→ k: $F_k = F_{k-1} + F_{k-2} = 2 + \sum_{i=0}^{k-3} F_i + F_{k-2} = 2 + \sum_{i=0}^{k-2} F_i$

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- **$\mathcal{P}.\mathbf{union}(x, y)$:** Given two elements x, and y that are currently in sets S_x and S_y, respectively, the function replaces S_x and S_y by $S_x \cup S_y$ and returns an identifier for the new set.

9 Union Find

Applications:

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- Kruskals Minimum Spanning Tree Algorithm

9 Union Find

Algorithm 1 Kruskal-MST($G = (V, E), w$)

```
A ← ∅;
for all v ∈ V do
    v.set ← 𝒫.makeset(v.label)
sort edges in non-decreasing order of weight w
for all (u, v) ∈ E in non-decreasing order do
    if 𝒫.find(u.set) ≠ 𝒫.find(v.set) then
        A ← A ∪ {(u, v)}
        𝒫.union(u.set, v.set)
```

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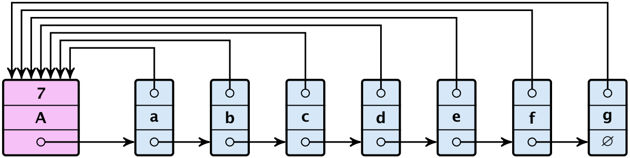

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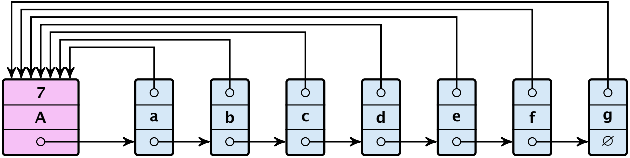

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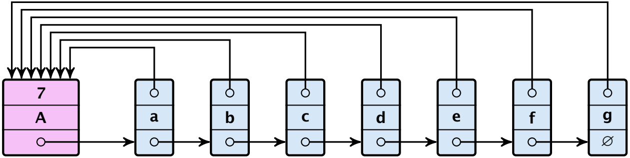

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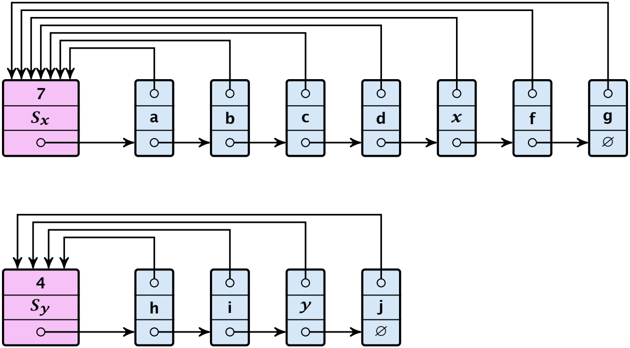

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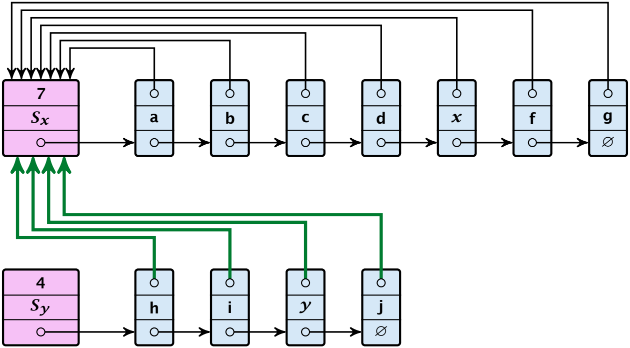

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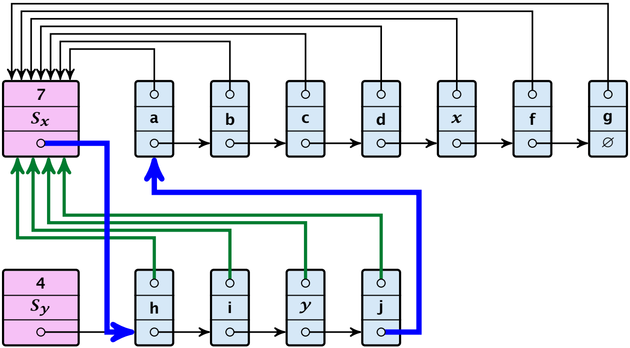

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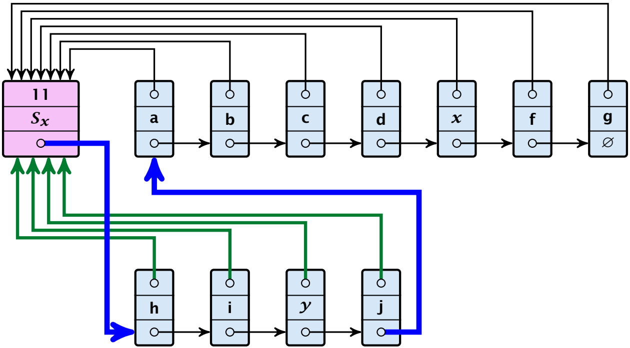

List Implementation

Running times:

- $\text{find}(x)$: constant
- $\text{makeset}(x)$: constant
- $\text{union}(x, y)$: $\mathcal{O}(n)$, where n denotes the number of elements contained in the set system.

List Implementation

Lemma 34

The list implementation for the ADT union find fulfills the following amortized time bounds:

- $\text{find}(x)$: $\mathcal{O}(1)$.
- $\text{makeset}(x)$: $\mathcal{O}(\log n)$.
- $\text{union}(x, y)$: $\mathcal{O}(1)$.

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- If we can find a charging scheme that guarantees that balances always stay positive the amortized time bounds are proven.

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- Later operations charge the account but the balance never drops below zero.

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- Charge c to every element in set S_x.

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Proof.

Whenever an element x is charged the number of elements in x's set doubles. This can happen at most $\lfloor \log n \rfloor$ times. □

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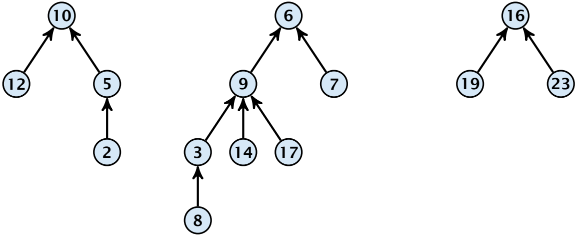

Set system $\{2, 5, 10, 12\}$, $\{3, 6, 7, 8, 9, 14, 17\}$, $\{16, 19, 23\}$.

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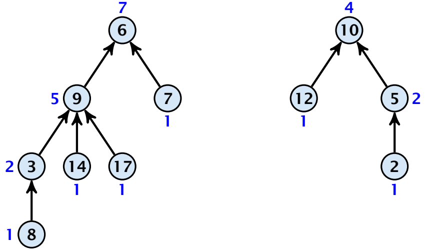

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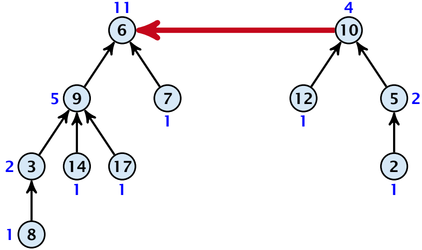

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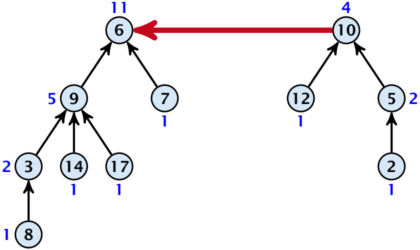

- Time: constant for $\text{link}(a, b)$ plus two find-operations.

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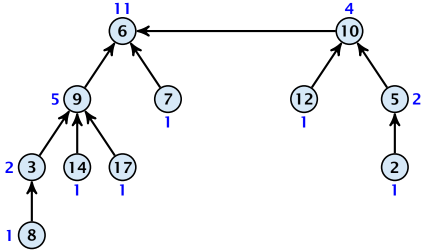

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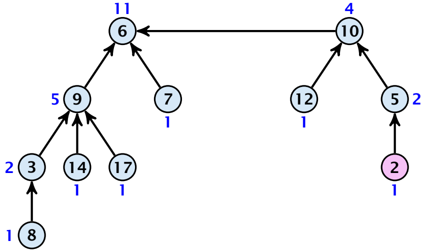

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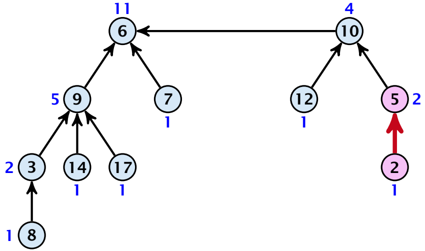

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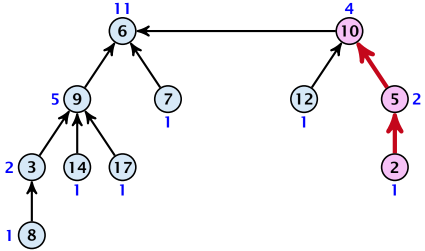

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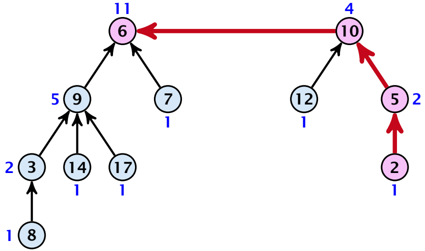

Path Compression

find(x):

- Go upward until you find the root.
- Re-attach all visited nodes as children of the root.
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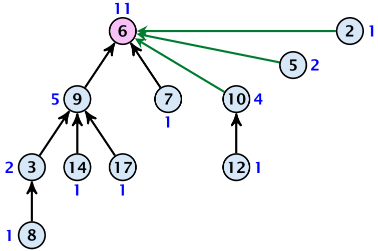

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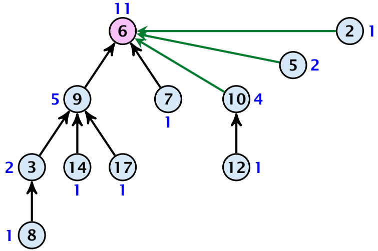

- Note that the size-fields now only give an upper bound on the size of a sub-tree.

Path Compression

Asymptotically the cost for a find-operation does not increase due to the path compression heuristic.

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Asymptotically the cost for a find-operation does not increase due to the path compression heuristic.

However, for a worst-case analysis there is no improvement on the running time. It can still happen that a find-operation takes time $\mathcal{O}(\log n)$.

Amortized Analysis

Definitions:

Amortized Analysis

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- $\text{size}(v) :=$ the number of nodes that were in the sub-tree rooted at v when v became the child of another node (or the number of nodes if v is the root).

  Note that this is the same as the size of v's subtree in the case that there are no find-operations.

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- $\text{rank}(v) := \lfloor \log(\text{size}(v)) \rfloor$.
- $\Longrightarrow \text{size}(v) \geq 2^{\text{rank}(v)}$.

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Lemma 37

The rank of a parent must be strictly larger than the rank of a child.

Amortized Analysis

Lemma 38

There are at most $n/2^s$ *nodes of rank* s*.*

Amortized Analysis

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Proof.

- Let's say a node v sees node x if v is in x's sub-tree at the time that x becomes a child.

☐

Amortized Analysis

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Proof.

- Let's say a node v sees node x if v is in x's sub-tree at the time that x becomes a child.
- A node v sees at most one node of rank s during the running time of the algorithm.

☐

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- A node v sees at most one node of rank s during the running time of the algorithm.
- This holds because the rank-sequence of the roots of the different trees that contain v during the running time of the algorithm is a strictly increasing sequence.

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Amortized Analysis

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Proof.

- Let's say a node v sees node x if v is in x's sub-tree at the time that x becomes a child.
- A node v sees at most one node of rank s during the running time of the algorithm.
- This holds because the rank-sequence of the roots of the different trees that contain v during the running time of the algorithm is a strictly increasing sequence.
- Hence, every node *sees* at most one rank s node, but every rank s node is seen by at least 2^s different nodes. ☐

Amortized Analysis

We define

$$\operatorname{tow}(i) := \begin{cases} 1 & \text{if } i = 0 \\ 2^{\operatorname{tow}(i-1)} & \text{otw.} \end{cases}$$

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$$\log^*(n) := \min\{i \mid \operatorname{tow}(i) \geq n\} \;.$$

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and

$$\log^*(n) := \min\{i \mid \mathrm{tow}(i) \geq n\} \; .$$

Theorem 39

Union find with path compression fulfills the following amortized running times:

- $\mathrm{makeset}(x) : \mathcal{O}(\log^*(n))$
- $\mathrm{find}(x) : \mathcal{O}(\log^*(n))$
- $\mathrm{union}(x, y) : \mathcal{O}(\log^*(n))$

Amortized Analysis

In the following we assume $n \geq 2$.

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- The rank-group $g = 0$ contains only nodes with rank 0 or rank 1.
- A rank group $g \geq 1$ contains ranks $\text{tow}(g-1)+1, \dots, \text{tow}(g)$.
- The maximum non-empty rank group is $\log^*(\lfloor \log n \rfloor) \leq \log^*(n) - 1$ (which holds for $n \geq 2$).

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- The maximum non-empty rank group is $\log^*(\lfloor \log n \rfloor) \leq \log^*(n) - 1$ (which holds for $n \geq 2$).
- Hence, the total number of rank-groups is at most $\log^* n$.

Amortized Analysis

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Accounting Scheme:

- create an account for every find-operation

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- If $\text{parent}[v]$ is the root we charge the cost to the find-account.

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- If $\text{parent}[v]$ is the root we charge the cost to the find-account.
- If the group-number of $\text{rank}(v)$ is the same as that of $\text{rank}(\text{parent}[v])$ (before starting path compression) we charge the cost to the node-account of v.

Amortized Analysis

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- create an account for every find-operation
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- If the group-number of $\text{rank}(v)$ is the same as that of $\text{rank}(\text{parent}[v])$ (before starting path compression) we charge the cost to the node-account of v.
- Otherwise we charge the cost to the find-account.

Amortized Analysis

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- A find-account is charged at most $\log^*(n)$ times (once for the root and at most $\log^*(n) - 1$ times when increasing the rank-group).

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- After a node v is charged its parent-edge is re-assigned. The rank of the parent strictly increases.
- After some charges to v the parent will be in a larger rank-group. $\Longrightarrow$ v will never be charged again.
- The total charge made to a node in rank-group g is at most $\mathrm{tow}(g) - \mathrm{tow}(g-1) - 1 \leq \mathrm{tow}(g)$.

Amortized Analysis

What is the total charge made to nodes?

Amortized Analysis

What is the total charge made to nodes?

- The total charge is at most

  $$\sum_g n(g) \cdot \mathrm{tow}(g) \ ,$$

  where $n(g)$ is the number of nodes in group g.

Amortized Analysis

For $g \geq 1$ we have

$$n(g)$$

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Hence,

$$\sum_g n(g)\,\mathrm{tow}(g)$$

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$$\sum_g n(g)\,\mathrm{tow}(g) \leq n(0)\,\mathrm{tow}(0) + \sum_{g \geq 1} n(g)\,\mathrm{tow}(g)$$

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Hence,

$$\sum_g n(g)\,\mathrm{tow}(g) \leq n(0)\,\mathrm{tow}(0) + \sum_{g\geq 1} n(g)\,\mathrm{tow}(g) \leq n\log^*(n)$$

Amortized Analysis

Without loss of generality we can assume that all makeset-operations occur at the start.

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Without loss of generality we can assume that all makeset-operations occur at the start.

This means if we inflate the cost of makeset to $\log^* n$ and add this to the node account of v then the balances of all node accounts will sum up to a positive value (this is sufficient to obtain an amortized bound).

Amortized Analysis

Amortized Analysis

The analysis is not tight. In fact it has been shown that the amortized time for the union-find data structure with path compression is $\mathcal{O}(\alpha(m,n))$, where $\alpha(m,n)$ is the inverse Ackermann function which grows a lot lot slower than $\log^* n$. (Here, we consider the average running time of m operations on at most n elements).

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There is also a lower bound of $\Omega(\alpha(m,n))$.

Amortized Analysis

$$A(x,y) = \begin{cases} y+1 & \text{if } x = 0 \\ A(x-1,1) & \text{if } y = 0 \\ A(x-1,A(x,y-1)) & \text{otw.} \end{cases}$$

$$\alpha(m,n) = \min\{i \geq 1 : A(i,\lfloor m/n \rfloor) \geq \log n\}$$

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$$\alpha(m,n) = \min\{i \geq 1 : A(i, \lfloor m/n \rfloor) \geq \log n\}$$

- $A(0,y) = y+1$
- $A(1,y) = y+2$
- $A(2,y) = 2y+3$
- $A(3,y) = 2^{y+3} - 3$
- $A(4,y) = \underbrace{2^{2^{2^{2}}}}_{y+3 \text{ times}} - 3$

Part IV

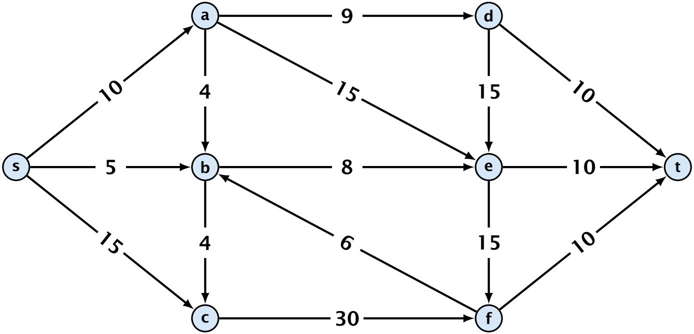
Flows and Cuts

The following slides are partially based on slides by Kevin Wayne.

10 Introduction

Flow Network

- directed graph $G = (V, E)$; edge capacities $c(e)$



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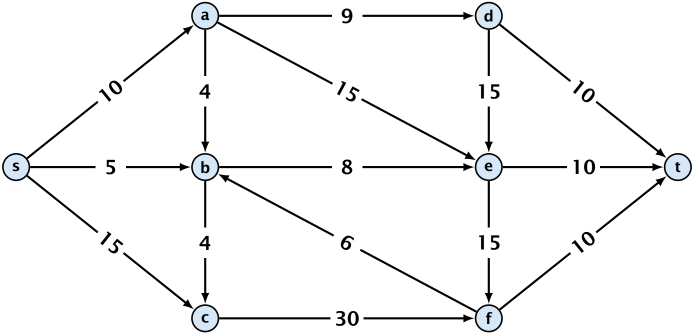

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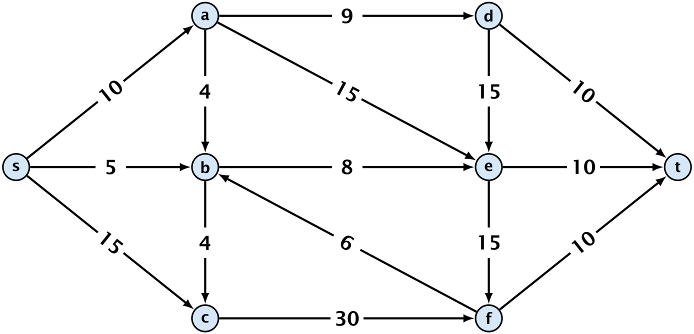
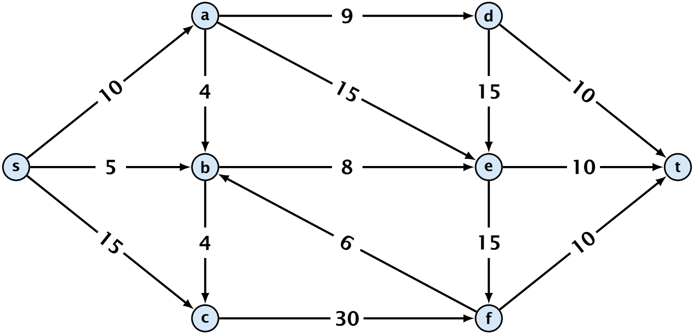
- two special nodes: source s; target t;
- no edges entering s or leaving t;



10 Introduction

Flow Network

- directed graph $G = (V, E)$; edge capacities $c(e)$
- two special nodes: source s; target t;
- no edges entering s or leaving t;
- at least for now: no parallel edges;



Cuts

Definition 40

An (s,t)-cut in the graph G is given by a set $A \subset V$ with $s \in A$ and $t \in V \setminus A$.

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Definition 41

The capacity of a cut A is defined as

$$\mathrm{cap}(A, V \setminus A) := \sum_{e \in \mathrm{out}(A)} c(e) \enspace ,$$

where $\mathrm{out}(A)$ denotes the set of edges of the form $A \times V \setminus A$ (i.e. edges leaving A).

Cuts

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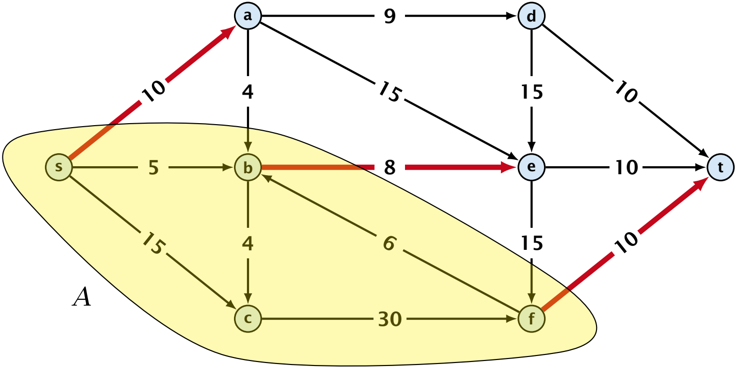
Minimum Cut Problem: Find an (s,t)-cut with minimum capacity.

Cuts

Example 42



The capacity of the cut is $\mathrm{cap}(A, V \setminus A) = 28$.

Flows

Definition 43

An (s,t)-flow is a function $f : E \mapsto \mathbb{R}^+$ that satisfies

1. For each edge e

$$0 \leq f(e) \leq c(e) \enspace .$$

   (capacity constraints)

Flows

Definition 43

An (s,t)-flow is a function $f : E \mapsto \mathbb{R}^+$ that satisfies

1. For each edge e
   $$0 \le f(e) \le c(e) \enspace .$$
   (capacity constraints)
2. For each $v \in V \setminus \{s,t\}$
   $$\sum_{e \in \text{out}(v)} f(e) = \sum_{e \in \text{into}(v)} f(e) \enspace .$$
   (flow conservation constraints)

Flows

Definition 44

The value of an (s,t)-flow f is defined as

$$\mathrm{val}(f) = \sum_{e \in \mathrm{out}(s)} f(e) \enspace .$$

Flows

Definition 44

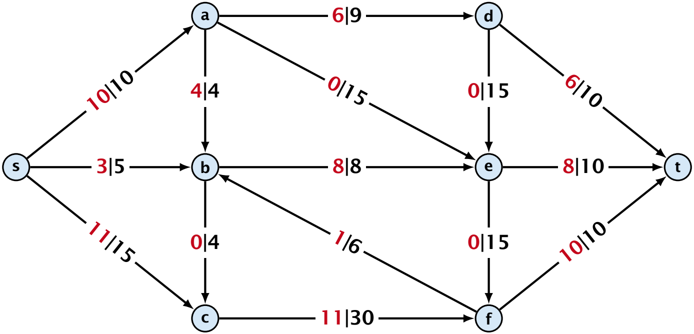
The value of an (s,t)-flow f is defined as

$$\mathrm{val}(f) = \sum_{e \in \mathrm{out}(s)} f(e) \ .$$

Maximum Flow Problem: Find an (s,t)-flow with maximum value.

Flows

Example 45



The value of the flow is $\text{val}(f) = 24$.

Flows

Lemma 46 (Flow value lemma)
Let f be a flow, and let $A \subseteq V$ be an (s,t)-cut. Then the net-flow across the cut is equal to the amount of flow leaving s, i.e.,

$$\mathrm{val}(f) = \sum_{e \in \mathrm{out}(A)} f(e) - \sum_{e \in \mathrm{into}(A)} f(e) \enspace .$$

Proof.

$$\mathrm{val}(f)$$

Proof.

$$\mathrm{val}(f) = \sum_{e \in \mathrm{out}(s)} f(e)$$

Proof.

$$\begin{aligned}\mathrm{val}(f) &= \sum_{e\in\mathrm{out}(s)} f(e)\\ &= \sum_{e\in\mathrm{out}(s)} f(e) + \sum_{v\in A\setminus\{s\}}\left(\sum_{e\in\mathrm{out}(v)} f(e) - \sum_{e\in\mathrm{in}(v)} f(e)\right)\end{aligned}$$

Proof.

$$
\begin{aligned}
\mathrm{val}(f) &= \sum_{e\in\mathrm{out}(s)} f(e) \\
&= \sum_{e\in\mathrm{out}(s)} f(e) + \sum_{v\in A\setminus\{s\}} \overset{\mathbf{=\ 0}}{\boxed{\left(\sum_{e\in\mathrm{out}(v)} f(e) - \sum_{e\in\mathrm{in}(v)} f(e)\right)}}
\end{aligned}
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Proof.

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\mathrm{val}(f) &= \sum_{e\in\mathrm{out}(s)} f(e) \\
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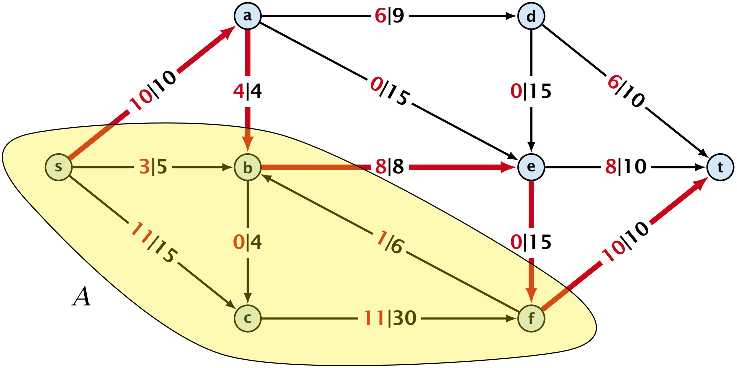
&= \sum_{e\in\mathrm{out}(A)} f(e) - \sum_{e\in\mathrm{into}(A)} f(e)
\end{aligned}$$

Proof.

$$\begin{aligned}\mathrm{val}(f) &= \sum_{e\in \mathrm{out}(s)} f(e)\\ &= \sum_{e\in \mathrm{out}(s)} f(e) + \sum_{v\in A\setminus\{s\}} \Bigg(\sum_{e\in \mathrm{out}(v)} f(e) - \sum_{e\in \mathrm{in}(v)} f(e)\Bigg)\\ &= \sum_{e\in \mathrm{out}(A)} f(e) - \sum_{e\in \mathrm{into}(A)} f(e)\end{aligned}$$

The last equality holds since every edge with both end-points in A contributes negatively as well as positively to the sum in Line 2. The only edges whose contribution doesn't cancel out are edges leaving or entering A. □

Example 47



The net-flow across the cut is $\mathrm{val}(f) = 24$.

Corollary 48

Let f be an (s,t)-flow and let A be an (s,t)-cut, such that

$$\mathrm{val}(f) = \mathrm{cap}(A, V \setminus A).$$

Then f is a maximum flow.

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Proof.

Suppose that there is a flow f' with larger value. Then

$$\begin{aligned} \mathrm{cap}(A, V \setminus A) &< \mathrm{val}(f') \\ &= \sum_{e \in \mathrm{out}(A)} f'(e) - \sum_{e \in \mathrm{into}(A)} f'(e) \end{aligned}$$

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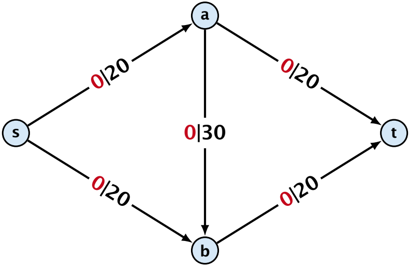
☐

11 Augmenting Path Algorithms

Greedy-algorithm:

- start with $f(e) = 0$ everywhere
- find an s-t path with $f(e) < c(e)$ on every edge
- augment flow along the path
- repeat as long as possible



flow value: 0

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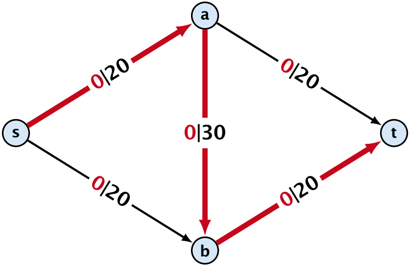

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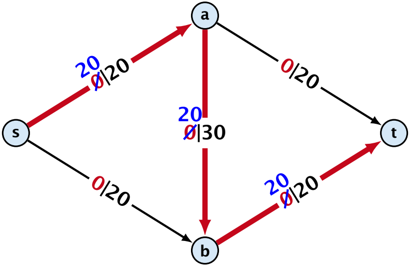

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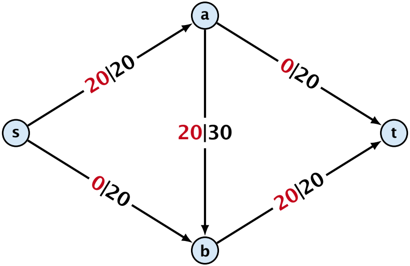

flow value: 20

The Residual Graph

From the graph $G = (V, E, c)$ and the current flow f we construct an auxiliary graph $G_f = (V, E_f, c_f)$ (the residual graph):

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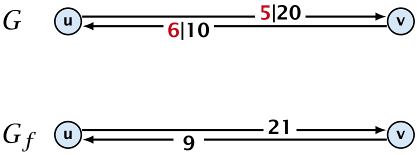

Augmenting Path Algorithm

Definition 49

An augmenting path with respect to flow f, is a path from s to t in the auxiliary graph G_f that contains only edges with non-zero capacity.

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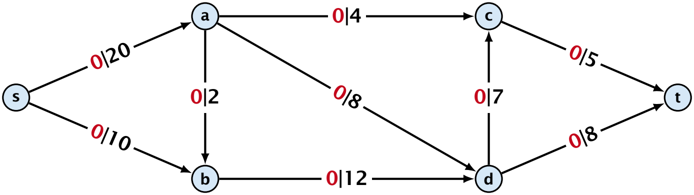
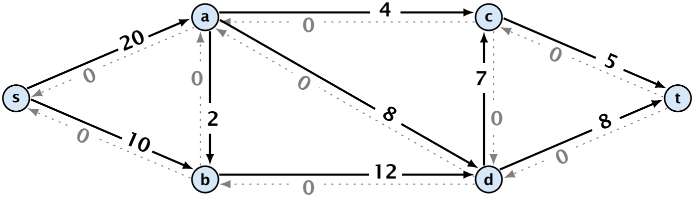
Algorithm 1 FordFulkerson$(G = (V, E, c))$

```
Initialize $f(e) \leftarrow 0$ for all edges.
while $\exists$ augmenting path $p$ in $G_f$ do
    augment as much flow along $p$ as possible.
```

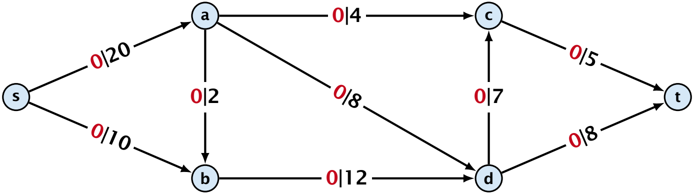
Augmenting Paths



flow value: 0



Augmenting Paths



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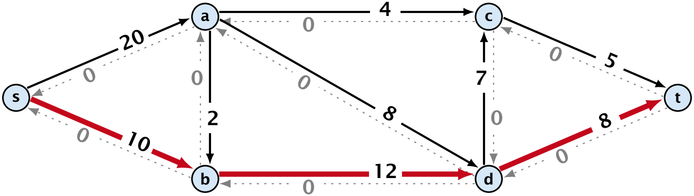

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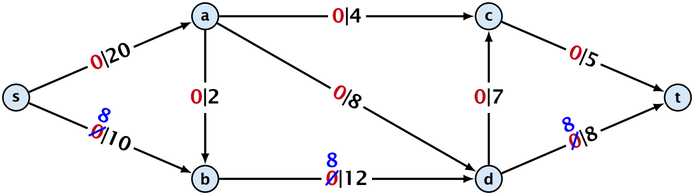
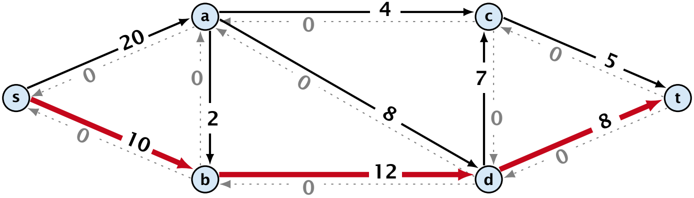


flow value: 0



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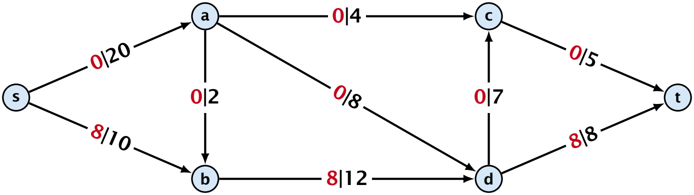
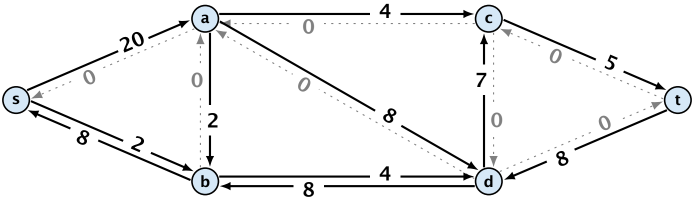


flow value: 8



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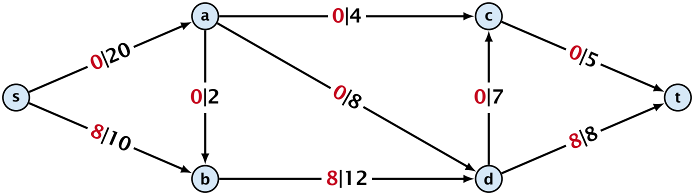

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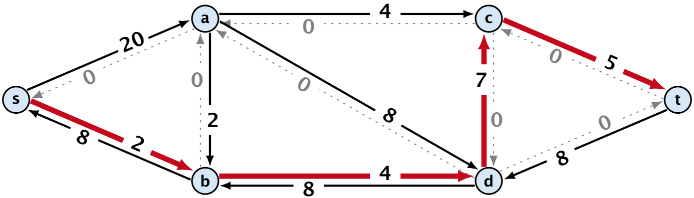

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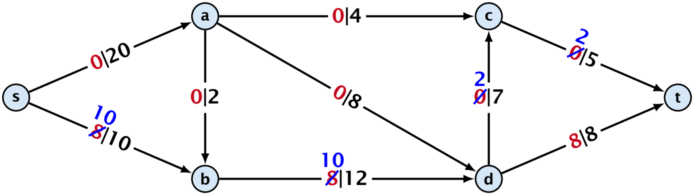

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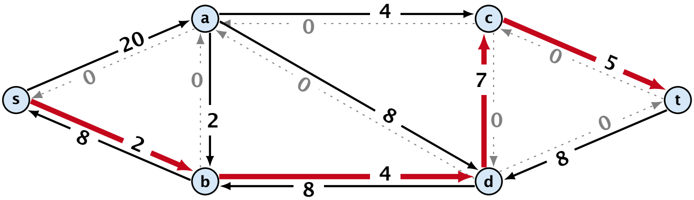

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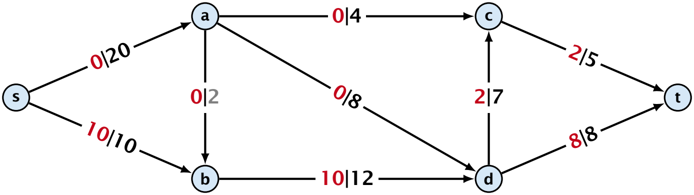
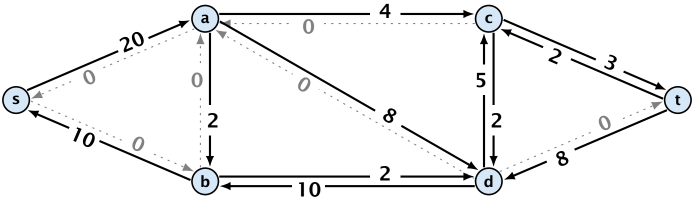


flow value: 10



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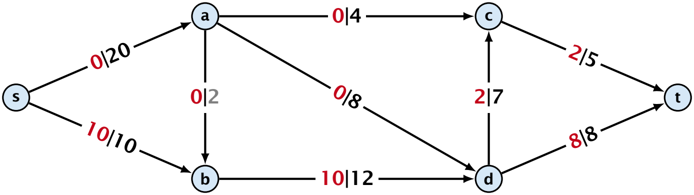

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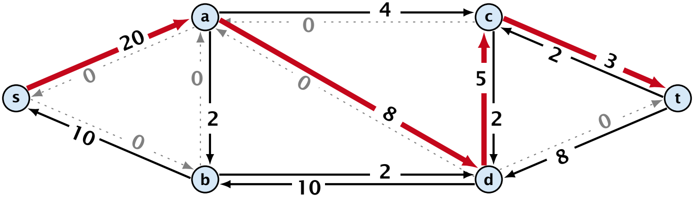

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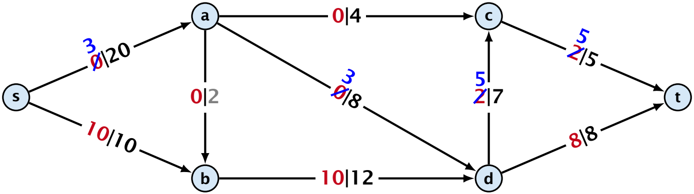

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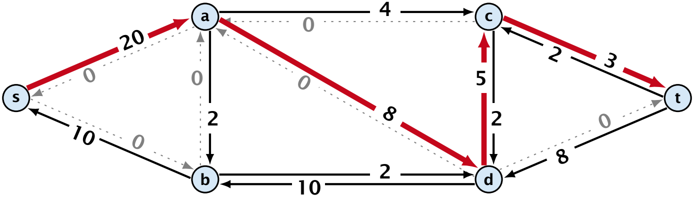

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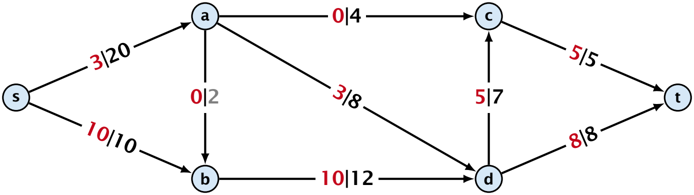
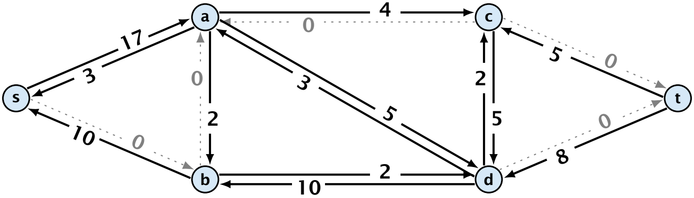


flow value: 13



Augmenting Path Algorithm

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Theorem 50

A flow f is a maximum flow ***iff*** *there are no augmenting paths.*

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Proof.

Let f be a flow. The following are equivalent:

1. There exists a cut A such that $\mathrm{val}(f) = \mathrm{cap}(A, V \setminus A)$.

□

Augmenting Path Algorithm

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2. Flow f is a maximum flow.

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Proof.

Let f be a flow. The following are equivalent:

1. There exists a cut A such that $\mathrm{val}(f) = \mathrm{cap}(A, V \setminus A)$.
2. Flow f is a maximum flow.
3. There is no augmenting path w.r.t. f.

□

Augmenting Path Algorithm

Augmenting Path Algorithm

$1. \implies 2.$

This we already showed.

Augmenting Path Algorithm

1\. $\Longrightarrow$ 2.

This we already showed.

2\. $\Longrightarrow$ 3.

If there were an augmenting path, we could improve the flow. Contradiction.

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- Let f be a flow with no augmenting paths.

Augmenting Path Algorithm

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Augmenting Path Algorithm

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This we already showed.

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If there were an augmenting path, we could improve the flow. Contradiction.

3. $\Longrightarrow$ 1.

- Let f be a flow with no augmenting paths.
- Let A be the set of vertices reachable from s in the residual graph along non-zero capacity edges.
- Since there is no augmenting path we have $s \in A$ and $t \notin A$.

Augmenting Path Algorithm

$\mathrm{val}(f)$

Augmenting Path Algorithm

$$\mathrm{val}(f) = \sum_{e\in\mathrm{out}(A)} f(e) - \sum_{e\in\mathrm{into}(A)} f(e)$$

Augmenting Path Algorithm

$$\begin{aligned}\mathrm{val}(f) &= \sum_{e\in\mathrm{out}(A)} f(e) - \sum_{e\in\mathrm{into}(A)} f(e)\\ &= \sum_{e\in\mathrm{out}(A)} c(e)\end{aligned}$$

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This finishes the proof.

Here the first equality uses the flow value lemma, and the second exploits the fact that the flow along incoming edges must be 0 as the residual graph does not have edges leaving A.

Analysis

Assumption:
All capacities are integers between 1 and C.

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Invariant:
Every flow value $f(e)$ and every residual capacity $c_f(e)$ remains integral troughout the algorithm.

Lemma 52

The algorithm terminates in at most $\mathrm{val}(f^*) \le nC$ *iterations, where* f^* *denotes the maximum flow. Each iteration can be implemented in time* $\mathcal{O}(m)$*. This gives a total running time of* $\mathcal{O}(nmC)$*.*

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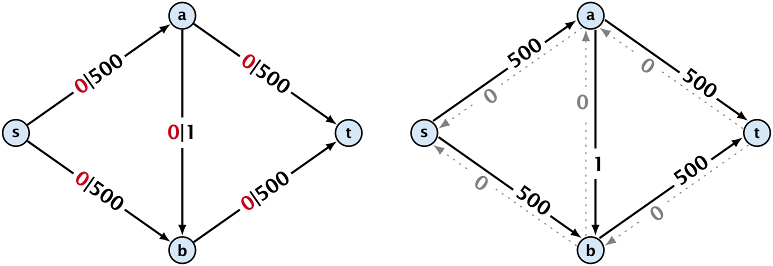
Theorem 53
If all capacities are integers, then there exists a maximum flow for which every flow value $f(e)$ *is integral.*

A Bad Input

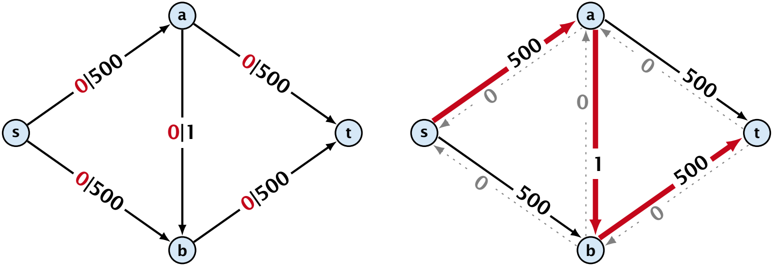
Problem: The running time may not be polynomial



flow value: 0

A Bad Input

Problem: The running time may not be polynomial



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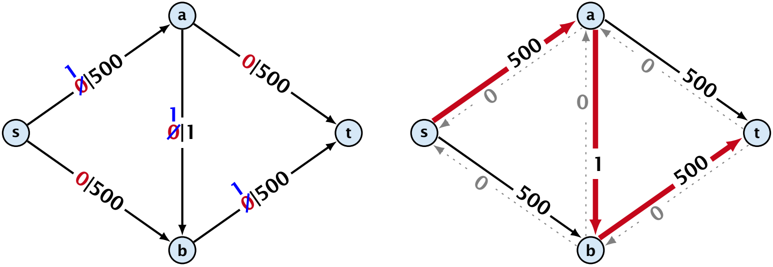

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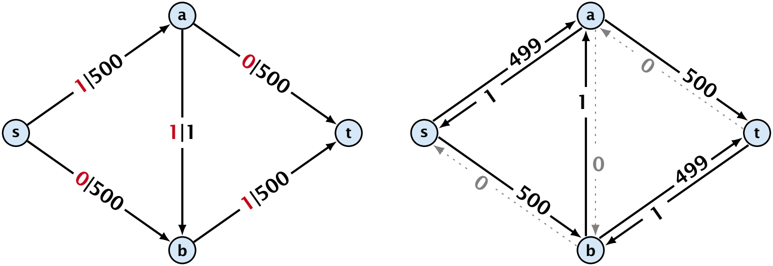

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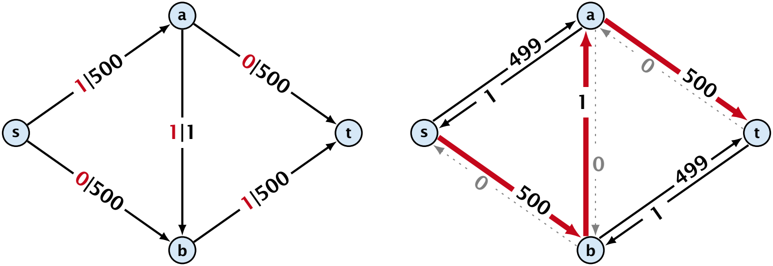

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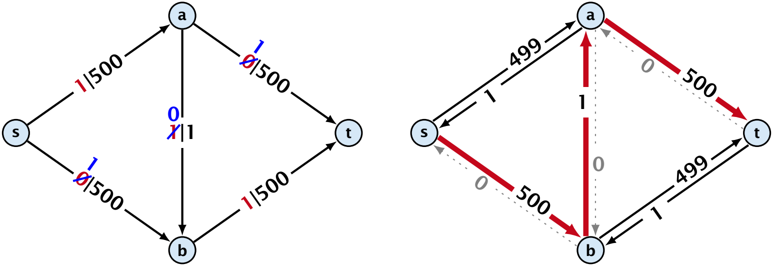

flow value: 1

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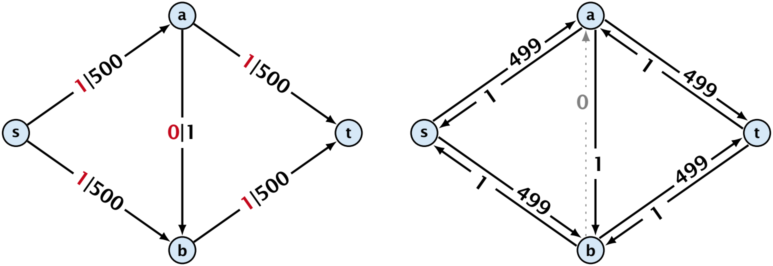
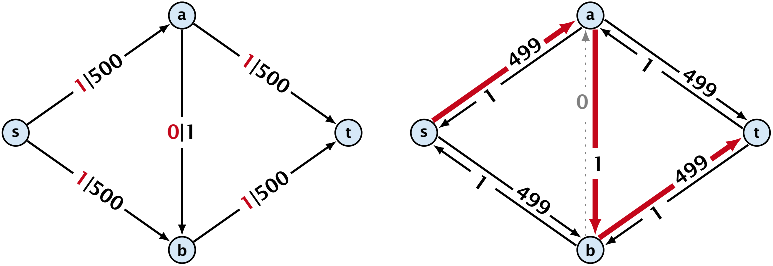
Problem: The running time may not be polynomial



flow value: 2

A Bad Input

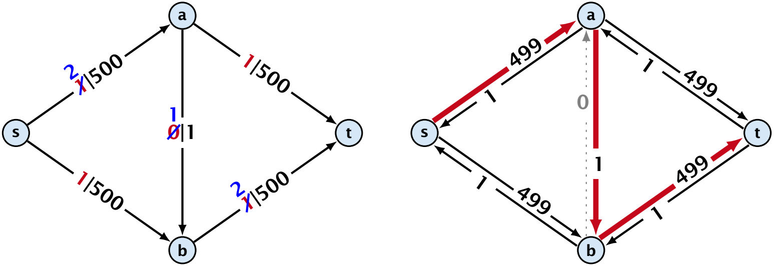
Problem: The running time may not be polynomial



flow value: 2

A Bad Input

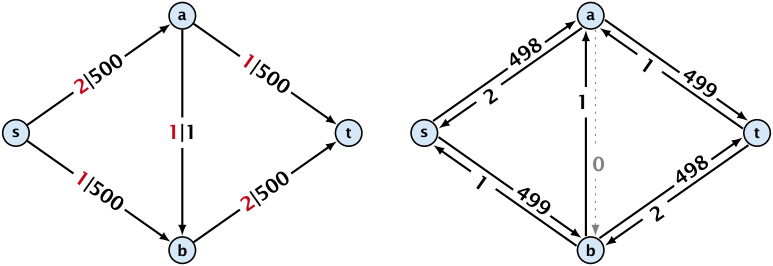
Problem: The running time may not be polynomial



flow value: 2

A Bad Input

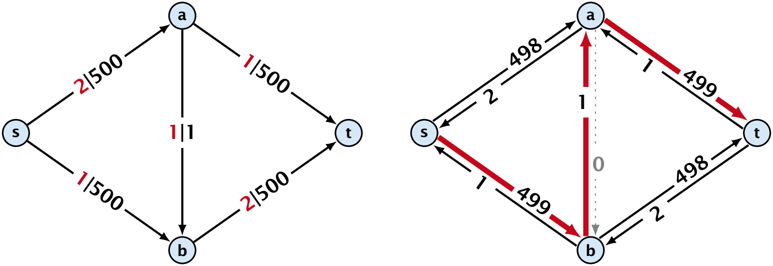
Problem: The running time may not be polynomial



flow value: 3

A Bad Input

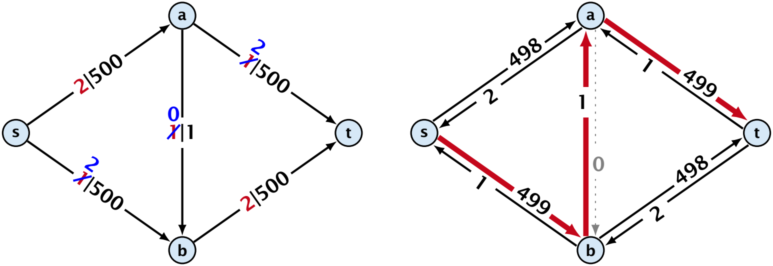
Problem: The running time may not be polynomial



flow value: 3

A Bad Input

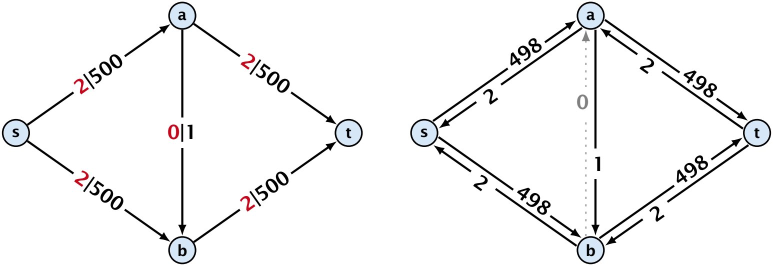
Problem: The running time may not be polynomial



flow value: 3

A Bad Input

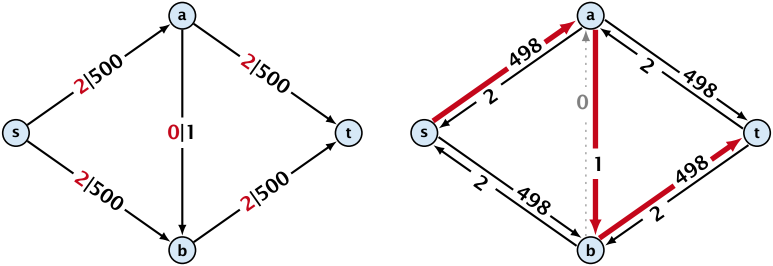
Problem: The running time may not be polynomial



flow value: 4

A Bad Input

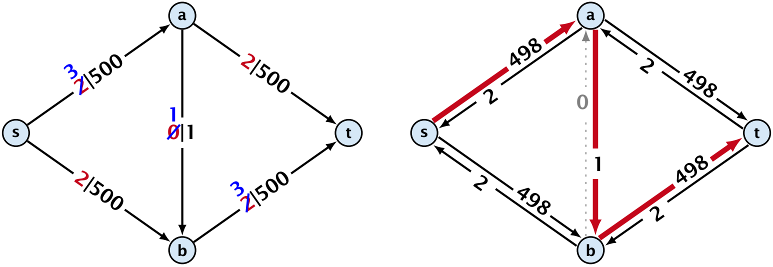
Problem: The running time may not be polynomial



flow value: 4

A Bad Input

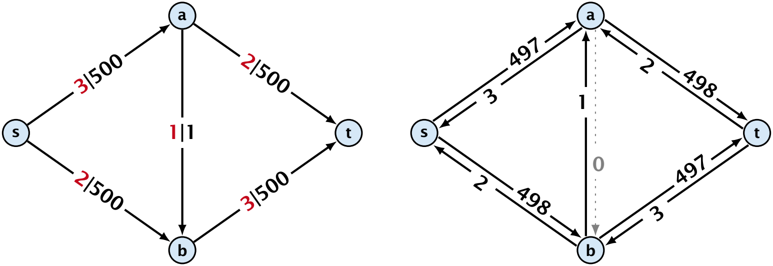
Problem: The running time may not be polynomial



flow value: 4

A Bad Input

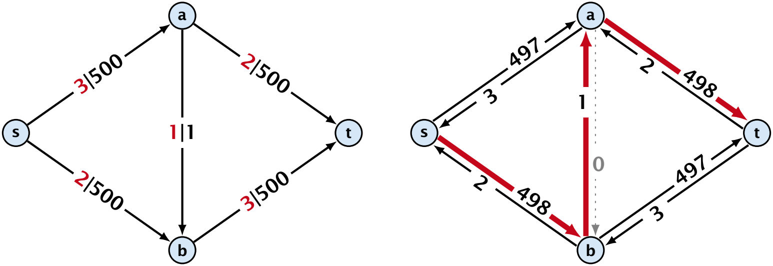
Problem: The running time may not be polynomial



flow value: 5

A Bad Input

Problem: The running time may not be polynomial



flow value: 5

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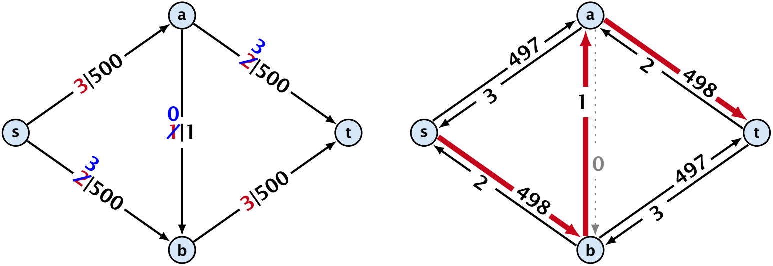

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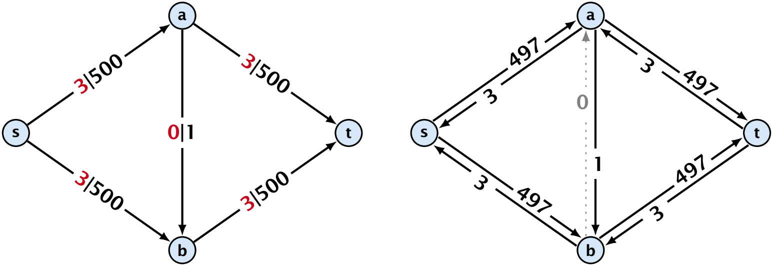

flow value: 6

A Bad Input

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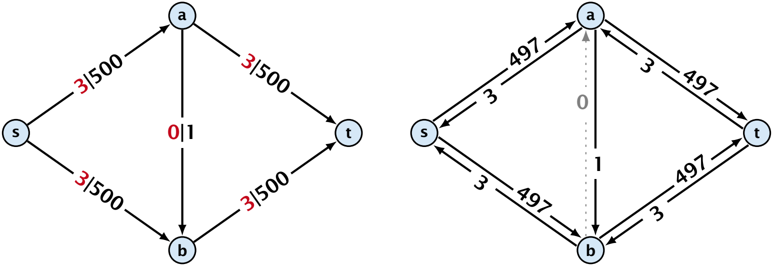


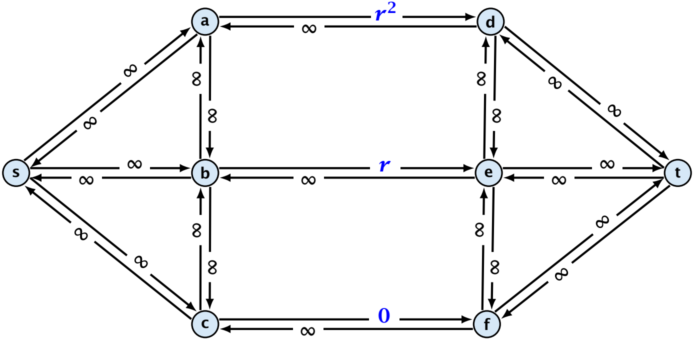
flow value: 6

Question:

Can we tweak the algorithm so that the running time is polynomial in the input length?

A Pathological Input

Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.



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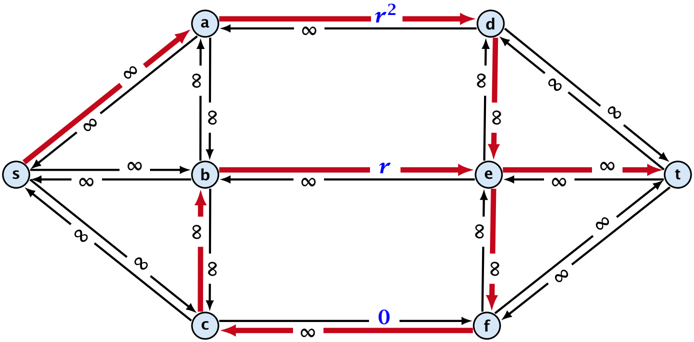

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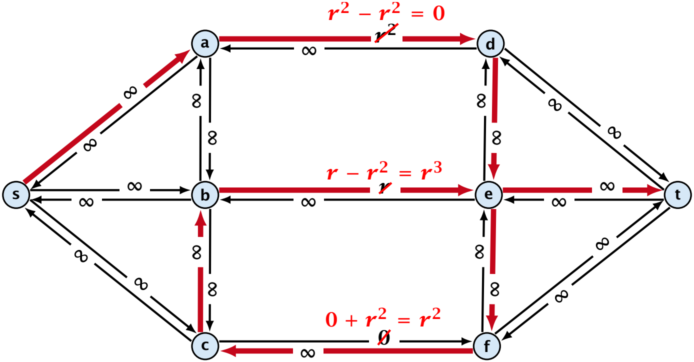

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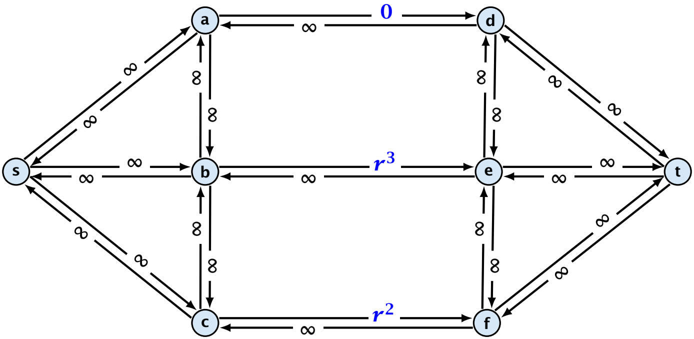

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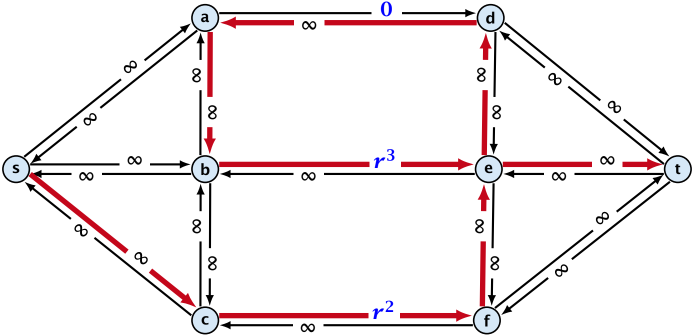

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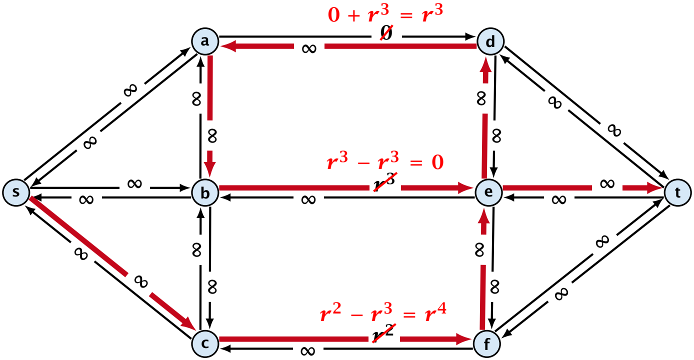

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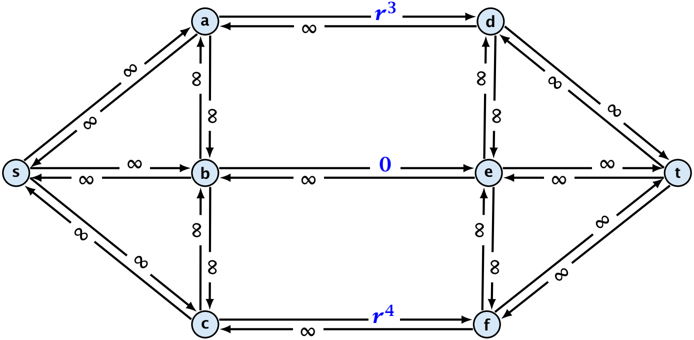

flow value: $r^2 + r^3$

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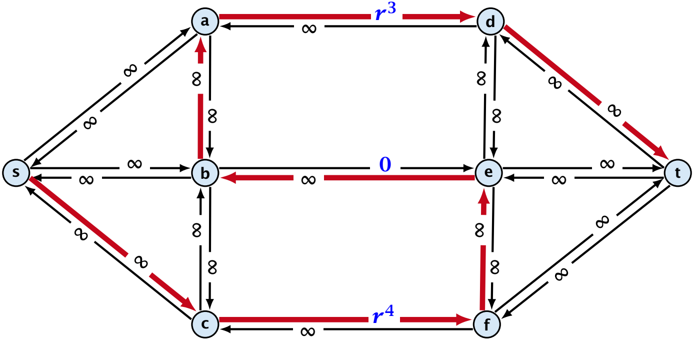

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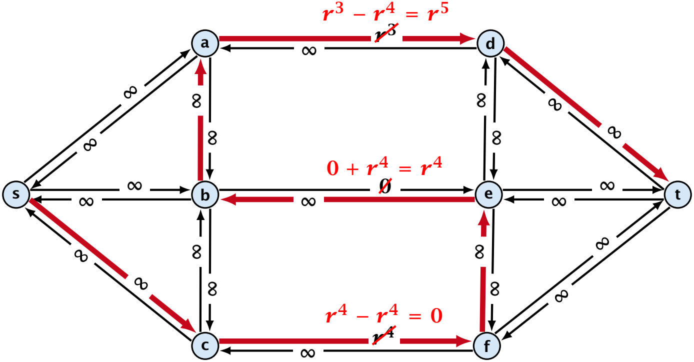

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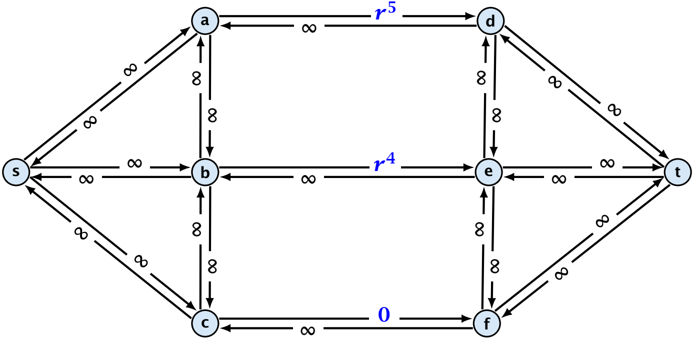

flow value: $r^2 + r^3 + r^4$

Running time may be infinite!!!

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Overview: Shortest Augmenting Paths

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Lemma 54

The length of the shortest augmenting path never decreases.

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After at most $\mathcal{O}(m)$ *augmentations, the length of the shortest augmenting path strictly increases.*

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Proof.

- We can find the shortest augmenting paths in time $\mathcal{O}(m)$ via BFS.

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Proof.

- We can find the shortest augmenting paths in time $\mathcal{O}(m)$ via BFS.
- $\mathcal{O}(m)$ augmentations for paths of exactly $k < n$ edges.

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Define the level $\ell(v)$ of a node as the length of the shortest s-v path in G_f (along non-zero edges).

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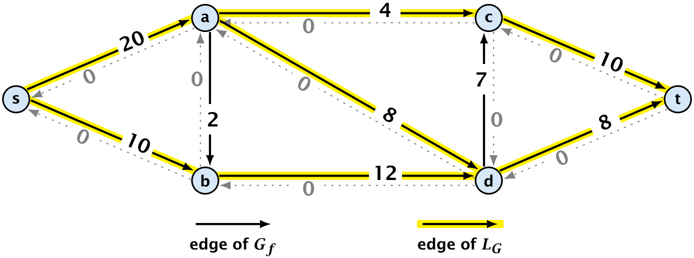

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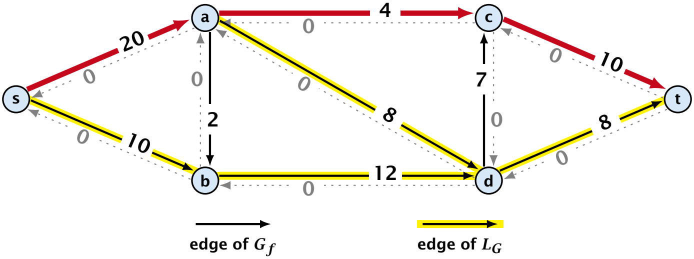

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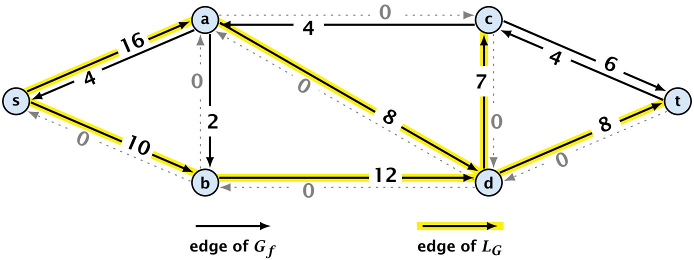

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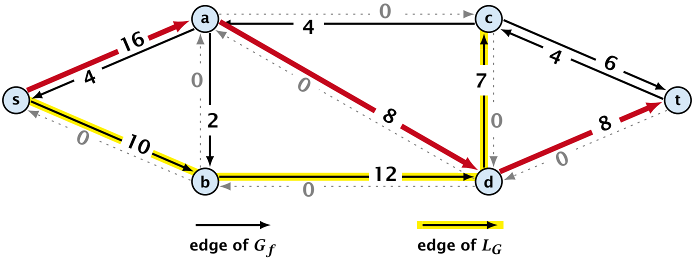

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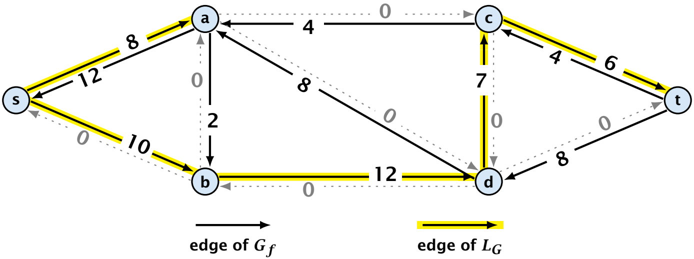

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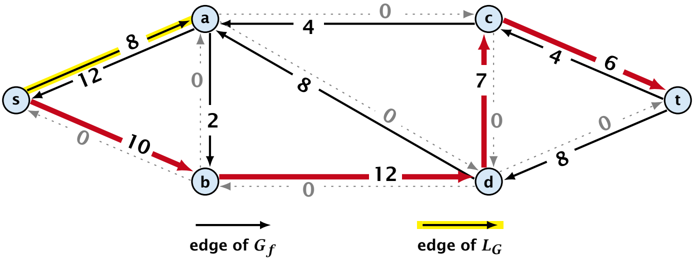

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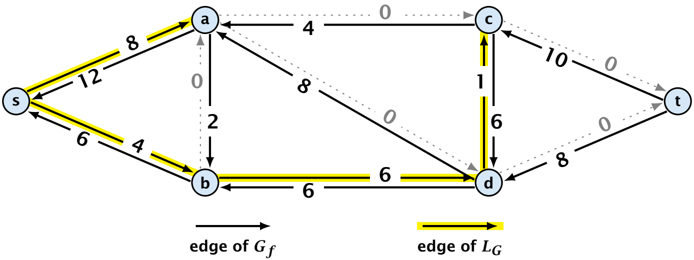

In the following we assume that the residual graph G_f does not contain zero capacity edges.

This means, we construct it in the usual sense and then delete edges of zero capacity.

Shortest Augmenting Path

First Lemma:

The length of the shortest augmenting path never decreases.

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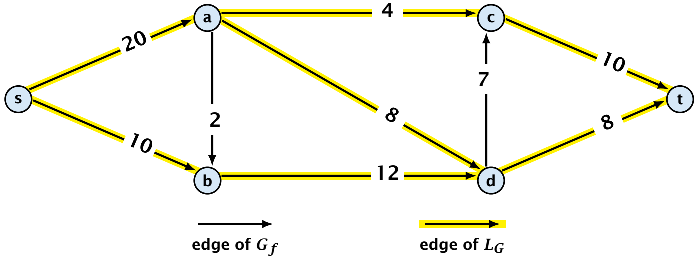

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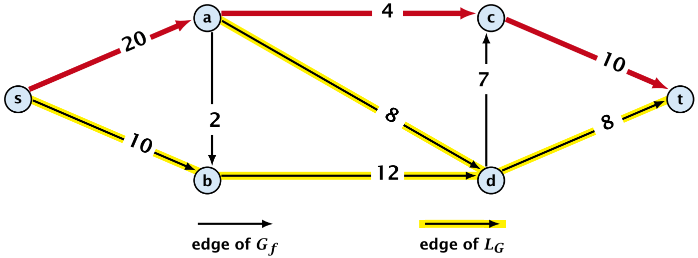

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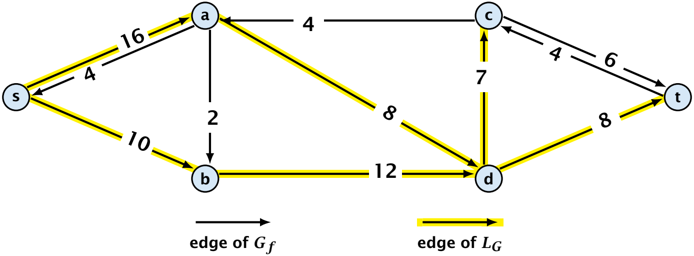

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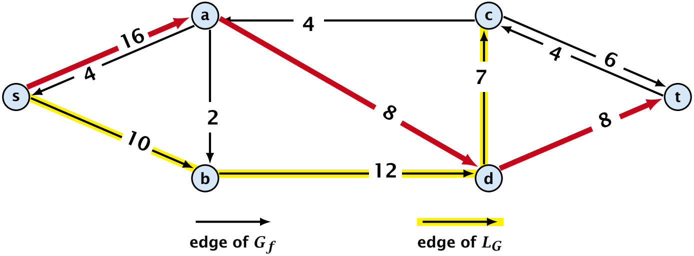

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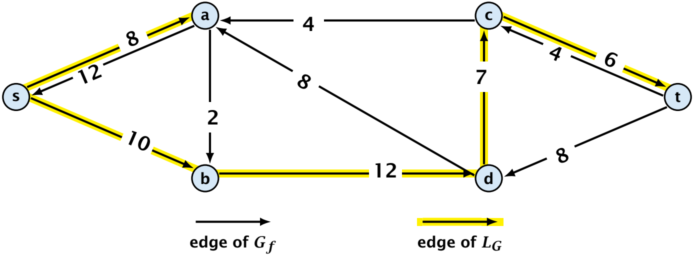

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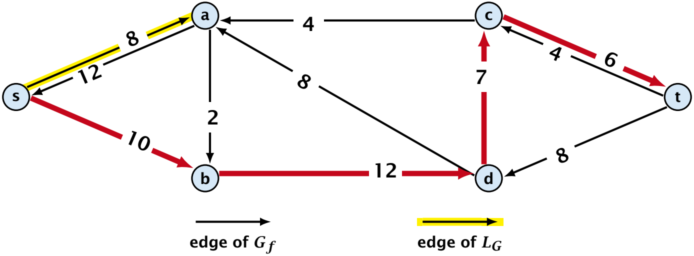

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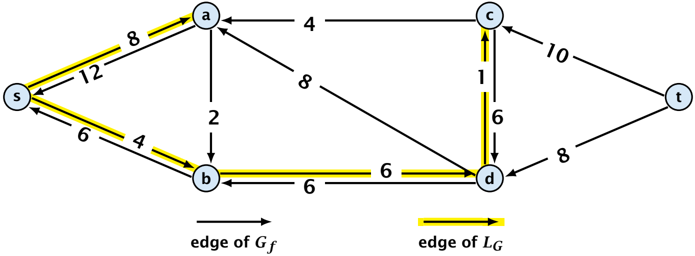

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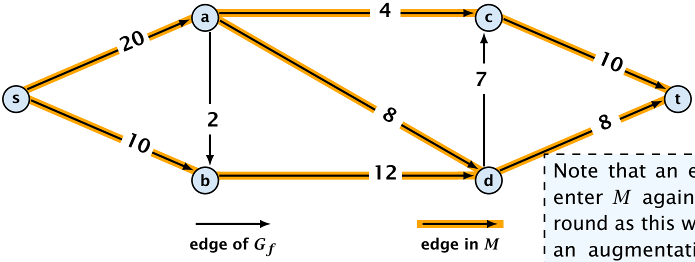
In each augmentation an edge is deleted from M.



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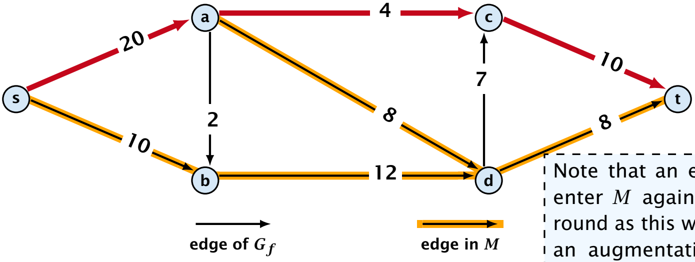

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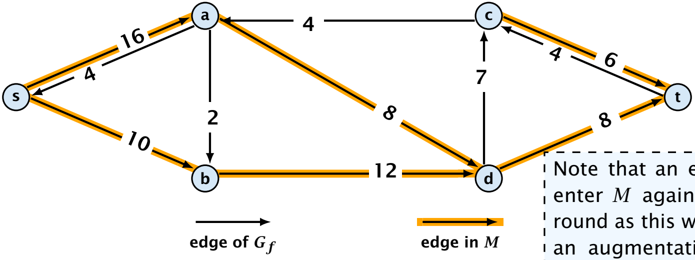

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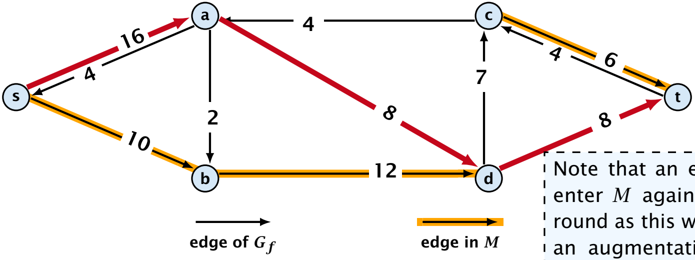

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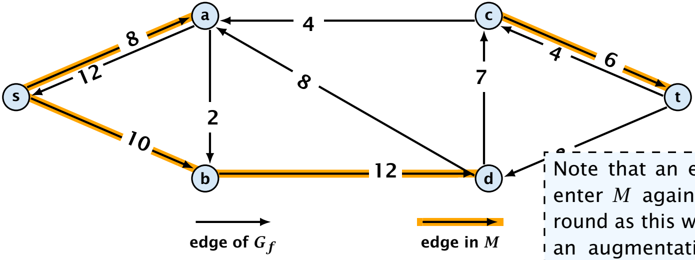

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Note:

There always exists a set of m augmentations that gives a maximum flow (why?).

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When sticking to shortest augmenting paths we cannot improve (asymptotically) on the number of augmentations.

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However, we can improve the running time to $\mathcal{O}(mn^2)$ by improving the running time for finding an augmenting path (currently we assume $\mathcal{O}(m)$ per augmentation for this).

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We maintain a subset M of the edges of G_f with the guarantee that a shortest s-t path using only edges from M is a shortest augmenting path.

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Note that M is not the set of edges of the level graph but a subset of level-graph edges.

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There are at most n phases. Hence, total cost is $\mathcal{O}(mn^2)$.

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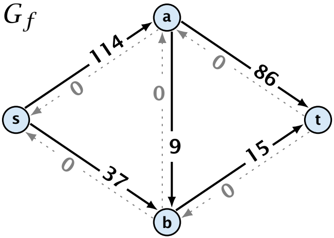

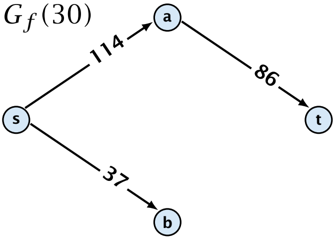

Capacity Scaling

```
Algorithm 1 maxflow($G, s, t, c$)
foreach $e \in E$ do $f_e \leftarrow 0$;
$\Delta \leftarrow 2^{\lceil \log_2 C \rceil}$
while $\Delta \geq 1$ do
    $G_f(\Delta) \leftarrow \Delta$-residual graph
    while there is augmenting path $P$ in $G_f(\Delta)$ do
        $f \leftarrow$ augment($f, c, P$)
        update($G_f(\Delta)$)
    $\Delta \leftarrow \Delta/2$
return $f$
```

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- therefore after the last phase there are no augmenting paths anymore
- this means we have a maximum flow.

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There are $\lceil \log C \rceil + 1$ *iterations over* Δ.

Proof: obvious.

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- In G_f this cut can have capacity at most $m\Delta$.
- This gives me an upper bound on the flow that I can still add.

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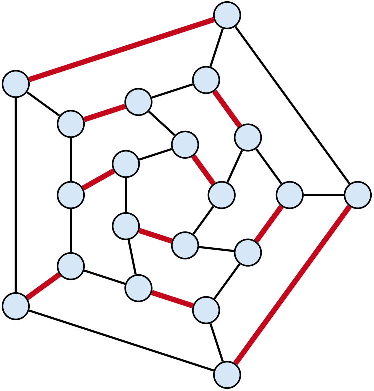
Theorem 62

We need $\mathcal{O}(m \log C)$ augmentations. The algorithm can be implemented in time $\mathcal{O}(m^2 \log C)$.

Matching

- Input: undirected graph $G = (V, E)$.
- $M \subseteq E$ is a matching if each node appears in at most one edge in M.
- Maximum Matching: find a matching of maximum cardinality



Bipartite Matching

- Input: undirected, bipartite graph $G = (L \uplus R, E)$.
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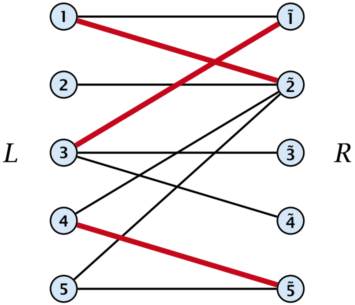

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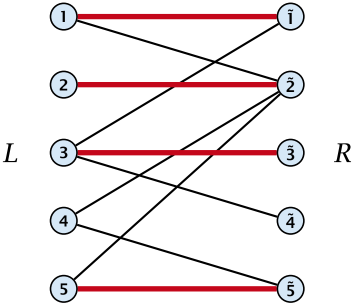
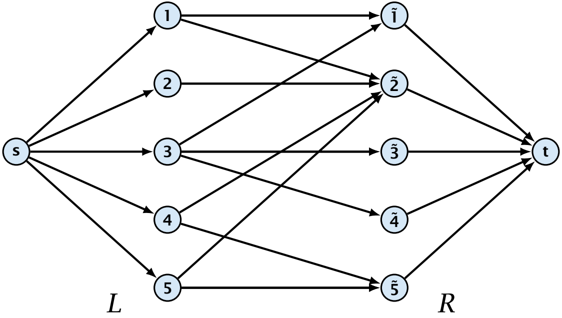


Maxflow Formulation

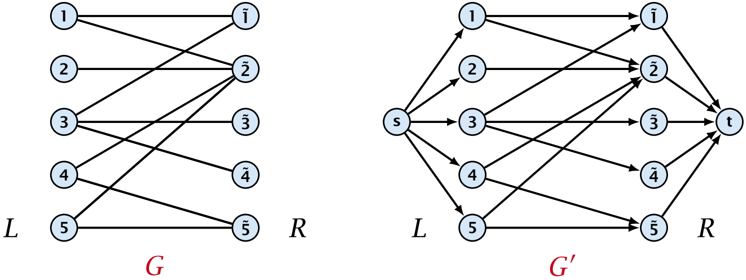
- Input: undirected, bipartite graph $G = (L \uplus R \uplus \{s,t\}, E')$.
- Direct all edges from L to R.
- Add source s and connect it to all nodes on the left.
- Add t and connect all nodes on the right to t.
- All edges have unit capacity.



Proof

Max cardinality matching in G ≤ value of maxflow in G'

- Given a maximum matching M of cardinality k.
- Consider flow f that sends one unit along each of k paths.
- f is a flow and has cardinality k.



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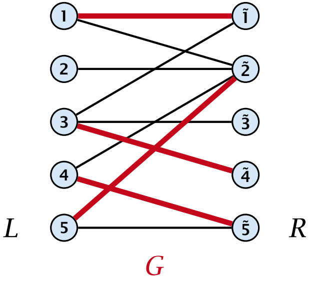

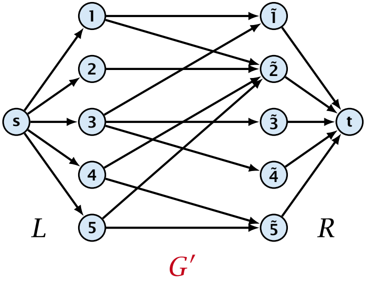

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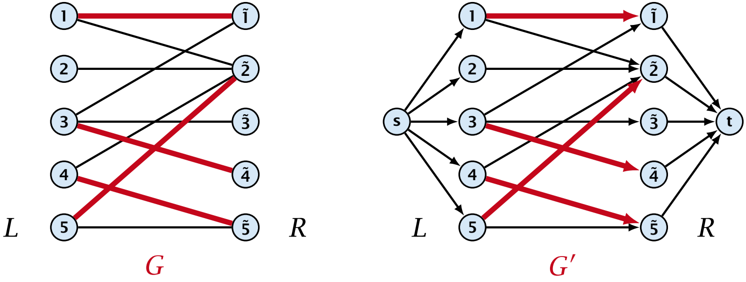

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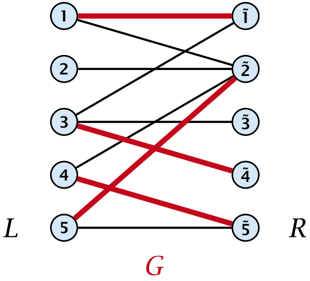

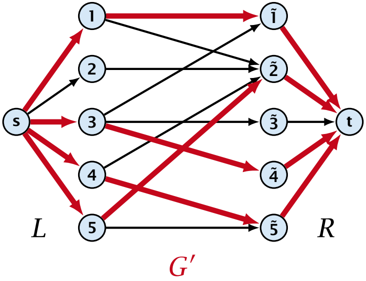

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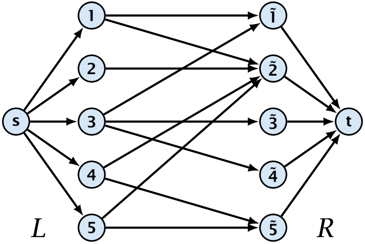
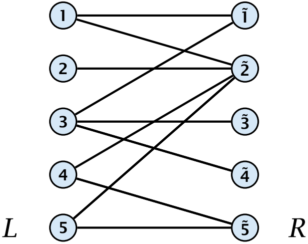
- Each node in L and R participates in at most one edge in M.
- $|M| = k$, as the flow must use at least k middle edges.



G'



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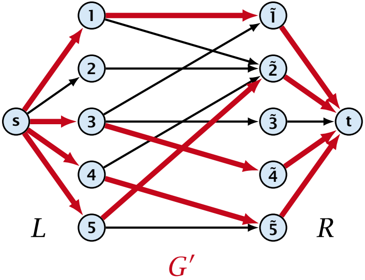

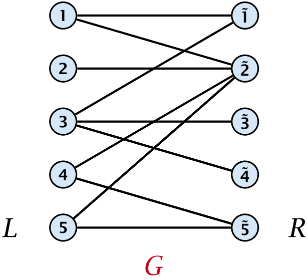

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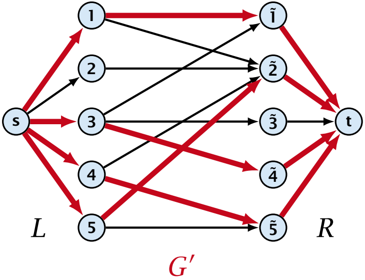
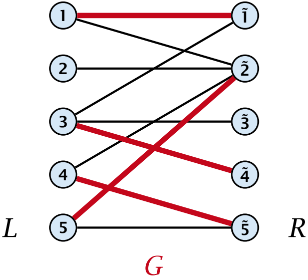




12.1 Matching

Which flow algorithm to use?

- Generic augmenting path: $\mathcal{O}(m\,\mathrm{val}(f^*)) = \mathcal{O}(mn)$.
- Capacity scaling: $\mathcal{O}(m^2 \log C) = \mathcal{O}(m^2)$.
- Shortest augmenting path: $\mathcal{O}(mn^2)$.

For unit capacity simple graphs shortest augmenting path can be implemented in time $\mathcal{O}(m\sqrt{n})$.

A graph is a unit capacity simple graph if

- every edge has capacity 1
- a node has either at most one leaving edge **or** at most one entering edge

Baseball Elimination

team	***wins***	***losses***	***remaining games***			
i	w_i	ℓ_i	***Atl***	***Phi***	***NY***	***Mon***
Atlanta	83	71	–	1	6	1
Philadelphia	80	79	1	–	0	2
New York	78	78	6	0	–	0
Montreal	77	82	1	2	0	–

Which team can end the season with most wins?

- Montreal is eliminated, since even after winning all remaining games there are only 80 wins.
- But also Philadelphia is eliminated. Why?

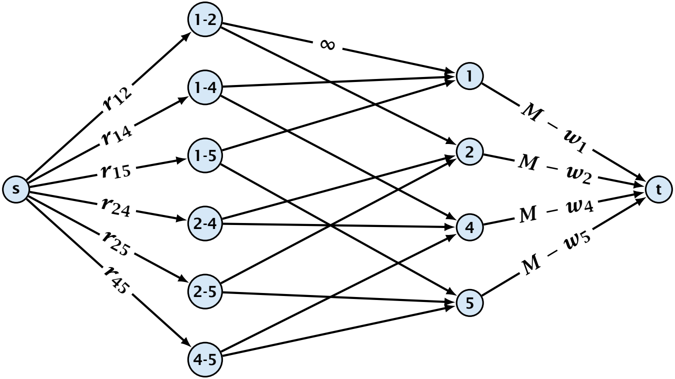
Baseball Elimination

Formal definition of the problem:

- Given a set S of teams, and one specific team $z \in S$.
- Team x has already won w_x games.
- Team x still has to play team y, r_{xy} times.
- Does team z still have a chance to finish with the most number of wins.

Baseball Elimination

Flow network for $z = 3$. M is number of wins Team 3 can still obtain.



Idea. Distribute the results of remaining games in such a way that no team gets too many wins.

Certificate of Elimination

Let $T \subseteq S$ be a subset of teams. Define

$$w(T) := \sum_{i\in T} w_i, \qquad r(T) := \sum_{i,j\in T, i<j} r_{ij}$$

wins of teams in T

remaining games among teams in T

If $\frac{w(T)+r(T)}{|T|} > M$ then one of the teams in T will have more than M wins in the end. A team that can win at most M games is therefore eliminated.

Theorem 63

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{ij\in S\setminus\{z\},i<j} r_{ij}$.

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$$r(S\setminus\{z\})$$

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$$r(S \setminus \{z\}) > \mathrm{cap}(A, V \setminus A)$$

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- This gives $M < (w(T) + r(T))/|T|$, i.e., z is eliminated.

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- Hence, team z is not eliminated.

Project Selection

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- Set P of possible projects. Project v has an associated profit p_v (can be positive or negative).

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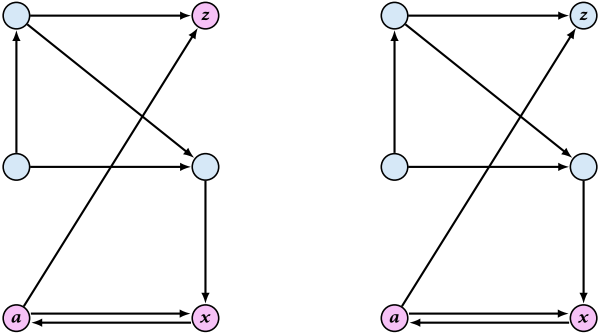
- A subset A of projects is feasible if the prerequisites of every project in A also belong to A.

Goal: Find a feasible set of projects that maximizes the profit.

Project Selection

The prerequisite graph:

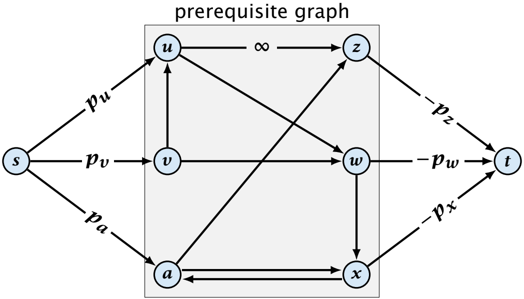
- $\{x, a, z\}$ is a feasible subset.
- $\{x, a\}$ is infeasible.



Project Selection

Mincut formulation:

- Edges in the prerequisite graph get infinite capacity.
- Add edge (s, v) with capacity p_v for nodes v with positive profit.
- Create edge (v, t) with capacity $-p_v$ for nodes v with negative profit.



Theorem 64

A is a mincut if $A \setminus \{s\}$ is the optimal set of projects.

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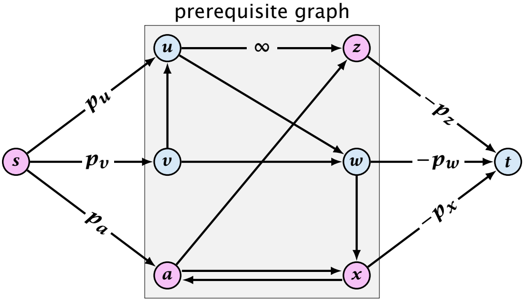
Proof.

- A is feasible because of capacity infinity edges.

For the formula we define $p_s := 0$.

The step follows by adding $\sum_{v\in A: p_v>0} p_v - \sum_{v\in A: p_v>0} p_v = 0$.

Note that minimizing the capacity of the cut $(A, V \setminus A)$ corresponds to maximizing profits of projects in A.



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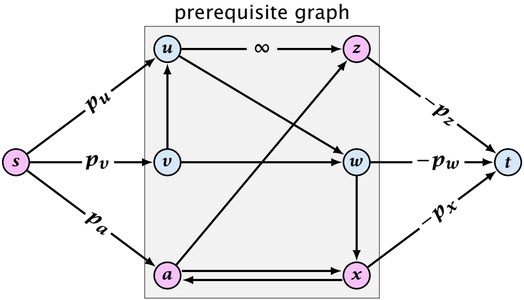

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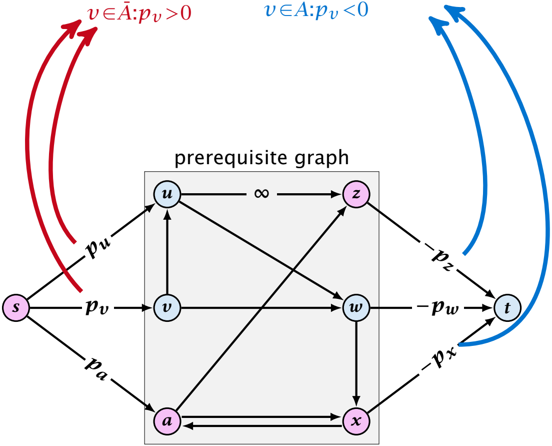

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  $$= \sum_{v: p_v > 0} p_v - \sum_{v \in A} p_v$$

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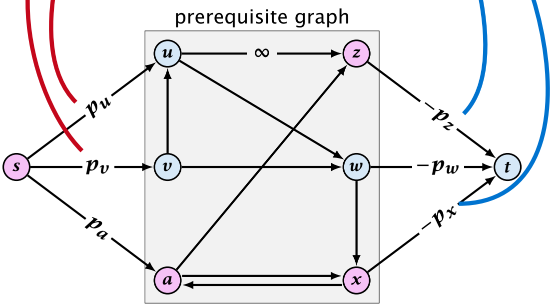

Preflows

Preflows

Definition 65

An (s,t)-preflow is a function $f : E \mapsto \mathbb{R}^+$ that satisfies

1. For each edge e

$$0 \leq f(e) \leq c(e) \enspace .$$

(capacity constraints)

Preflows

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An (s,t)-preflow is a function $f : E \mapsto \mathbb{R}^+$ that satisfies

1. For each edge e

$$0 \leq f(e) \leq c(e) \ .$$

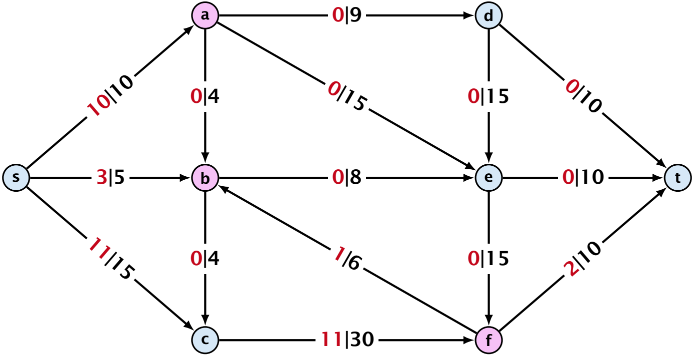
   (capacity constraints)

2. For each $v \in V \setminus \{s,t\}$

$$\sum_{e \in \mathrm{out}(v)} f(e) \leq \sum_{e \in \mathrm{into}(v)} f(e) \ .$$

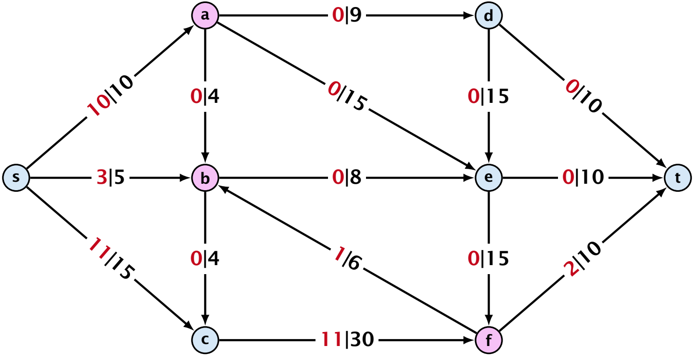
Preflows

Example 66



Preflows

Example 66



A node that has $\sum_{e \in \text{out}(v)} f(e) < \sum_{e \in \text{into}(v)} f(e)$ is called an active node.

Preflows

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Definition:

A labelling is a function $\ell : V \to \mathbb{N}$. It is valid for preflow f if

- $\ell(u) \le \ell(v) + 1$ for all edges (u, v) in the residual graph G_f (only non-zero capacity edges!!!)

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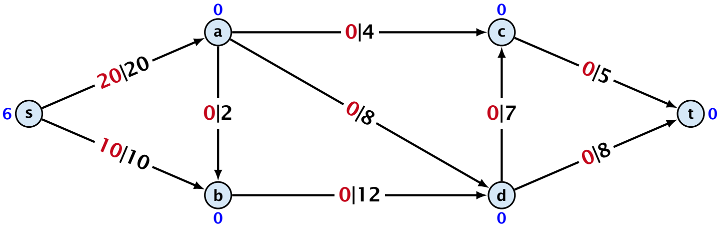
Intuition:

The labelling can be viewed as a height function. Whenever the height from node u to node v decreases by more than 1 (i.e., it goes very steep downhill from u to v), the corresponding edge must be saturated.

Preflows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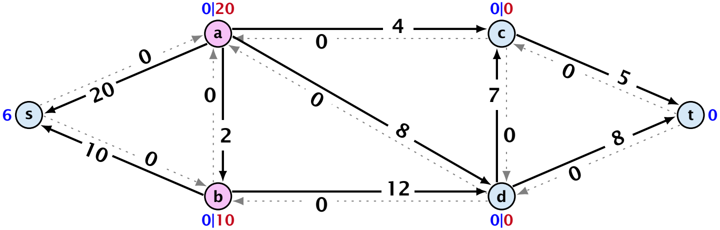
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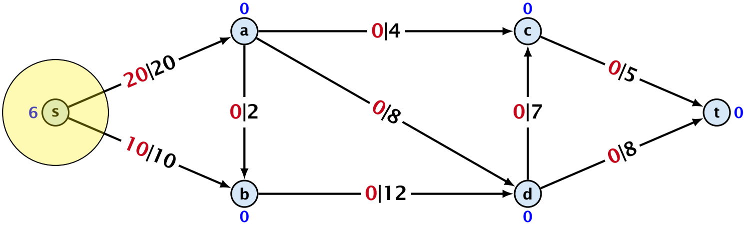
Preflows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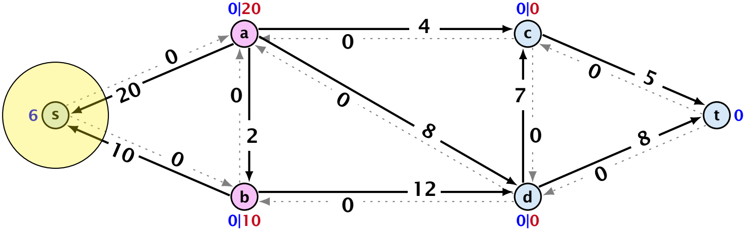
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Preflows

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Lemma 67

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Lemma 68

A flow that has a valid labelling is a maximum flow.

Push Relabel Algorithms

Push Relabel Algorithms

Idea:

- start with some preflow and some valid labelling

Note that this is somewhat dual to an augmenting path algorithm. The former maintains the property that it has a feasible flow. It successively changes this flow until it saturates some cut in which case we conclude that the flow is maximum. A preflow push algorithm maintains the property that it has a saturated cut. The preflow is changed iteratively until it fulfills conservation constraints in which case we can conclude that we have a maximum flow.

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- successively change the preflow while maintaining a valid labelling

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- successively change the preflow while maintaining a valid labelling
- stop when you have a flow (i.e., no more active nodes)

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An arc (u,v) with $c_f(u,v) > 0$ in the residual graph is admissible if $\ell(u) = \ell(v) + 1$ (i.e., it goes downwards w.r.t. labelling ℓ).

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The push operation

Consider an active node u with excess flow $f(u) = \sum_{e\in\text{into}(u)} f(e) - \sum_{e\in\text{out}(u)} f(e)$ and suppose $e = (u,v)$ is an admissible arc with residual capacity $c_f(e)$.

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We can send flow $\min\{c_f(e), f(u)\}$ along e and obtain a new preflow. The old labelling is still valid (!!!).

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  the arc e is deleted from the residual graph

Note that a push-operation may be saturating **and** deactivating at the same time.

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- deactivating push: $\min\{f(u), c_f(e)\} = f(u)$
  the node u becomes inactive

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Push Relabel Algorithms

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- Edges (w,u) incoming to u still fulfill their constraint $\ell(w) \leq \ell(u) + 1$.

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The relabel operation
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Increasing the label of u by 1 results in a valid labelling.

- Edges (w,u) incoming to u still fulfill their constraint $\ell(w) \le \ell(u) + 1$.
- An outgoing edge (u,w) had $\ell(u) < \ell(w) + 1$ before since it was not admissible. Now: $\ell(u) \le \ell(w) + 1$.

Push Relabel Algorithms

Intuition:
We want to send flow downwards, since the source has a height/label of n and the target a height/label of 0. If we see an active node u with an admissible arc we push the flow at u towards the other end-point that has a lower height/label. If we do not have an admissible arc but excess flow into u it should roughly mean that the level/height/label of u should rise. (If we consider the flow to be water then this would be natural.)

Note that the above intuition is very incorrect as the labels are integral, i.e., they cannot really be seen as the height of a node.

Reminder

- In a preflow nodes may not fulfill conservation constraints; a node may have more incoming flow than outgoing flow.
- Such a node is called active.
- A labelling is valid if for every edge (u,v) in the residual graph $\ell(u) \leq \ell(v) + 1$.
- An arc (u,v) in residual graph is admissible if $\ell(u) = \ell(v) + 1$.
- A saturating push along e pushes an amount of $c(e)$ flow along the edge, thereby saturating the edge (and making it dissappear from the residual graph).
- A deactivating push along $e = (u,v)$ pushes a flow of $f(u)$, where $f(u)$ is the excess flow of u. This makes u inactive.

Push Relabel Algorithms

Algorithm 1 maxflow(G, s, t, c)

```
find initial preflow $f$
while there is active node $u$ do
    if there is admiss. arc $e$ out of $u$ then
        push$(G, e, f, c)$
    else
        relabel$(u)$
return $f$
```

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return $f$
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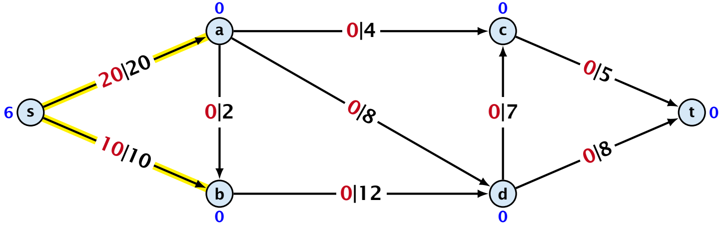
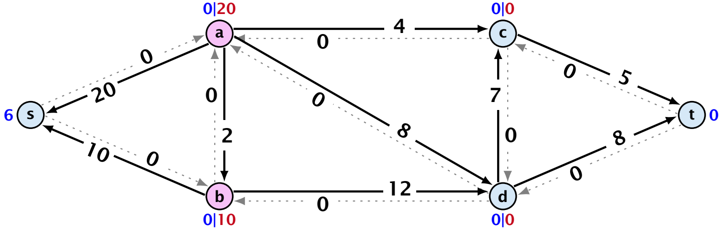
In the following example we always stick to the same active node u until it becomes inactive but this is not required.

Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.





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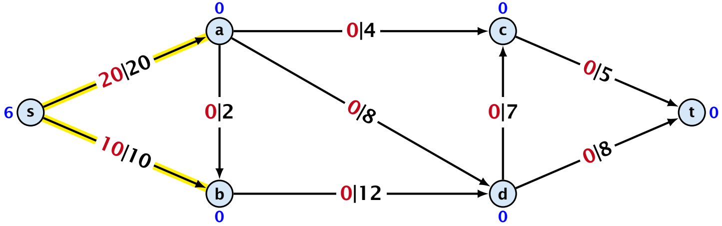

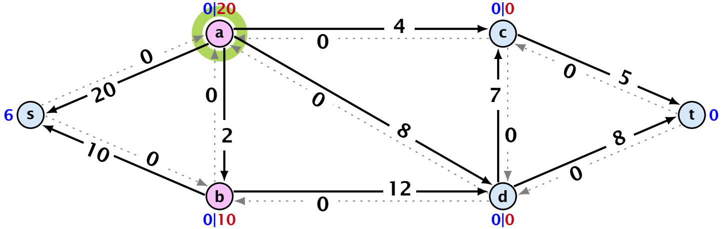

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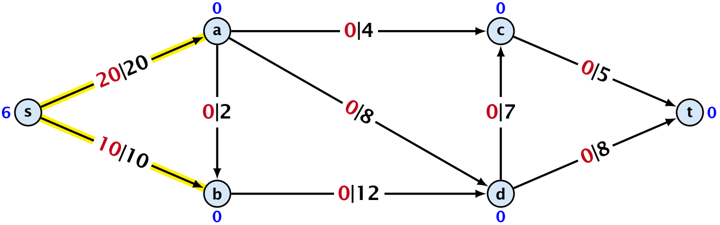
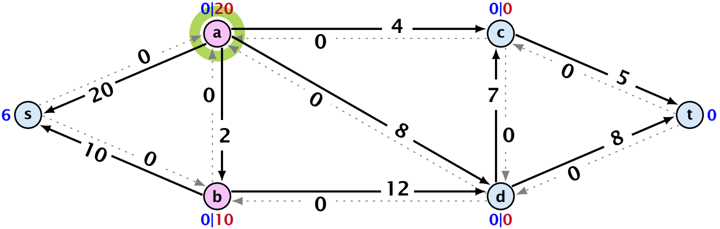


relabel to 1



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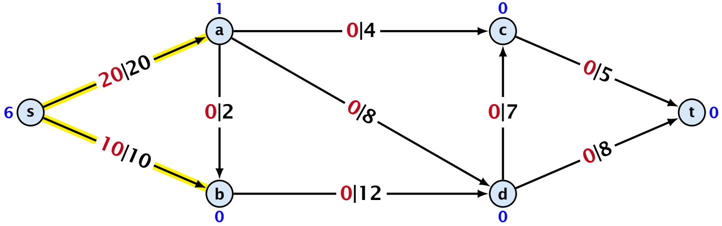

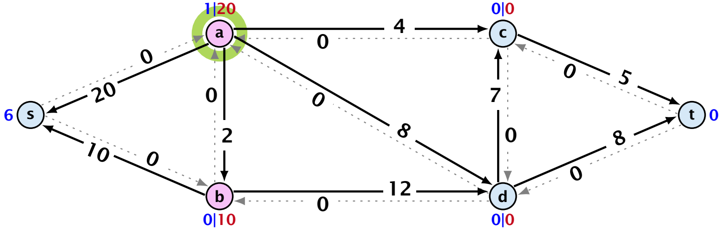

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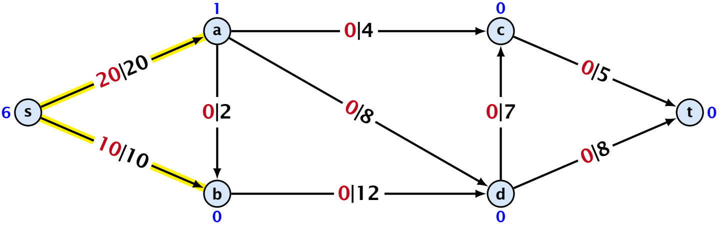
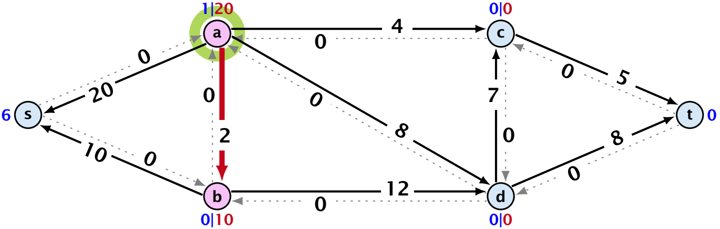


saturating push



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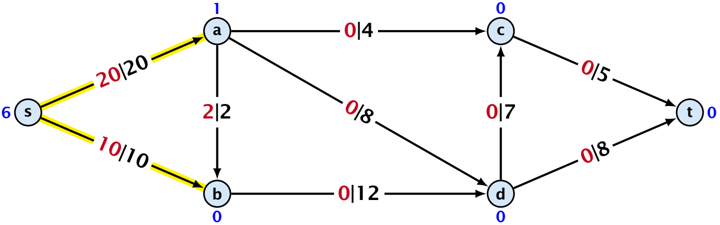

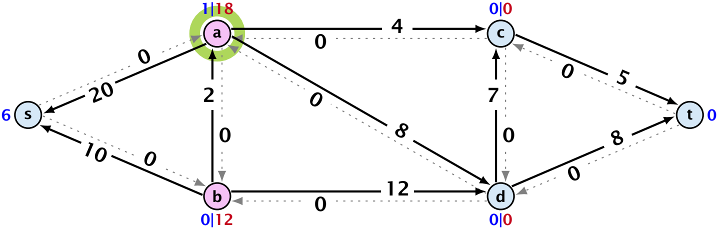

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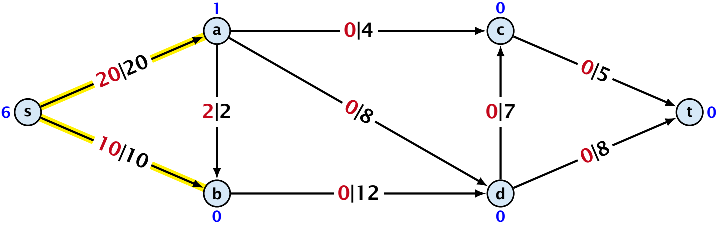

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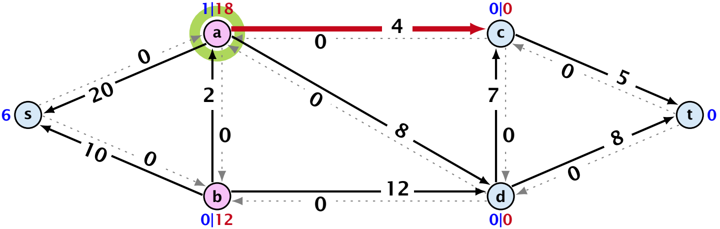

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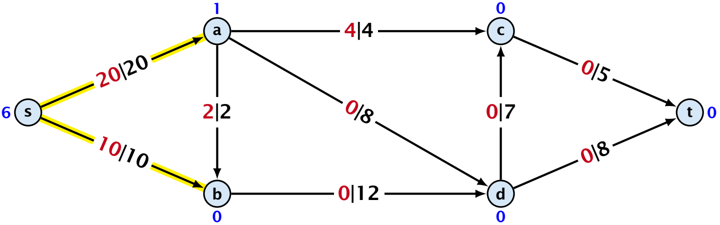
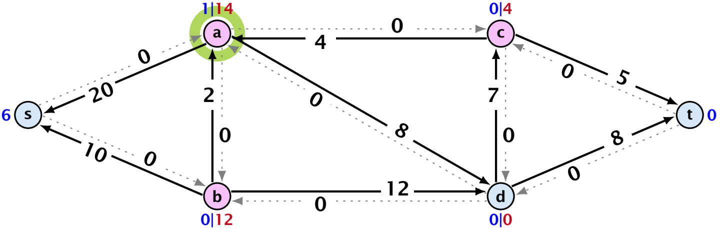
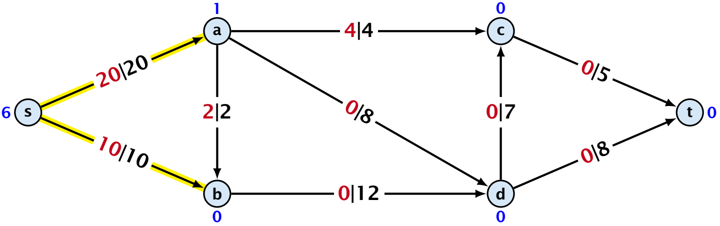


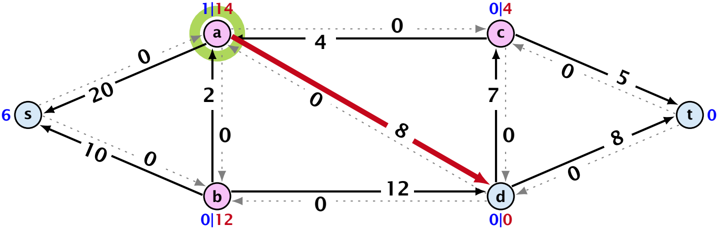


Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.



saturating push



Preflow Push

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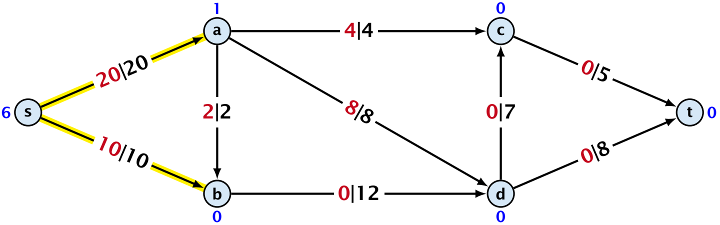

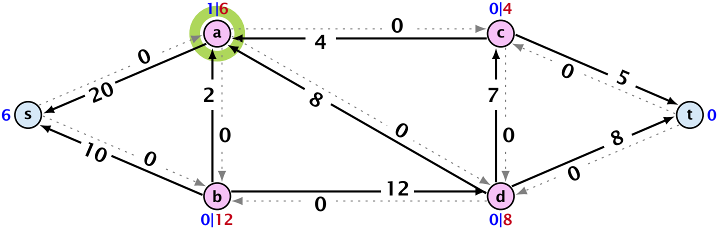

Preflow Push

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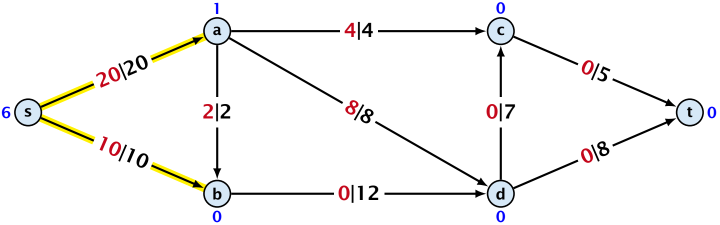
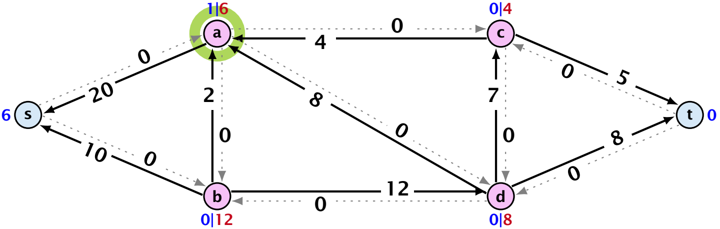


relabel to 7



Preflow Push

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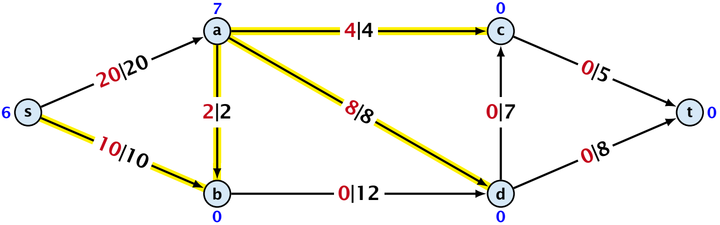

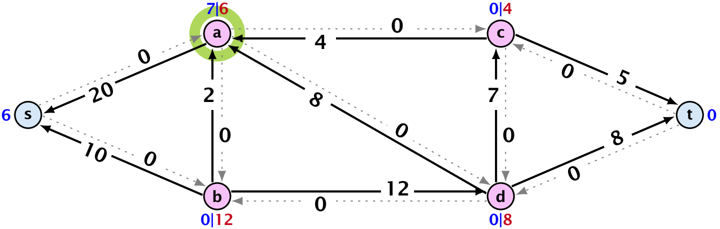

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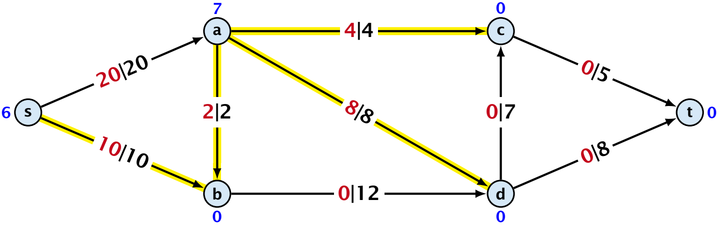
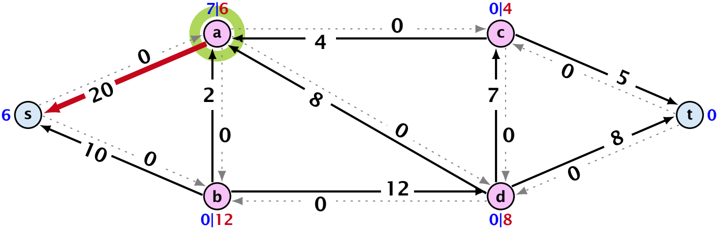


deactivating push



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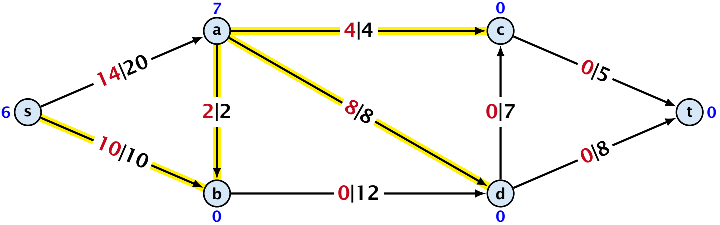

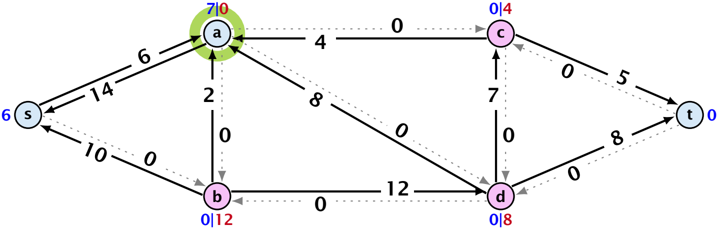

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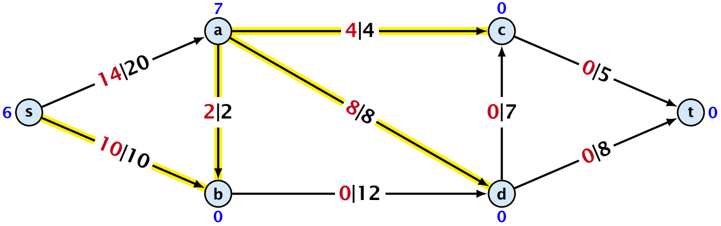

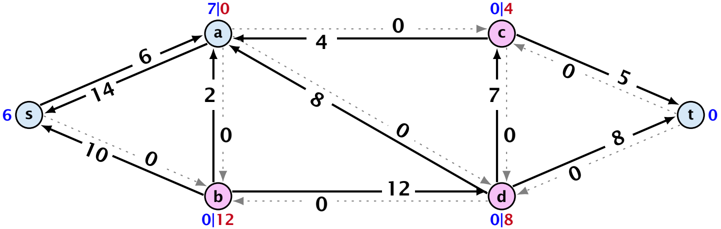

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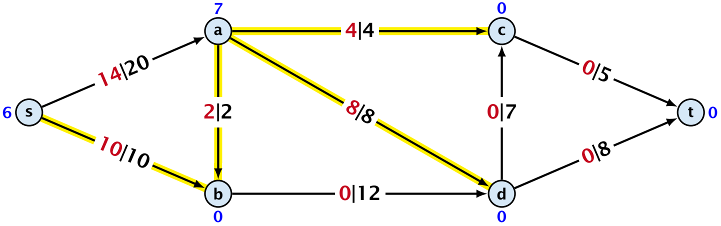

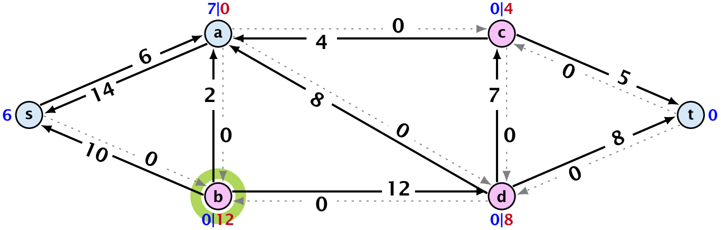

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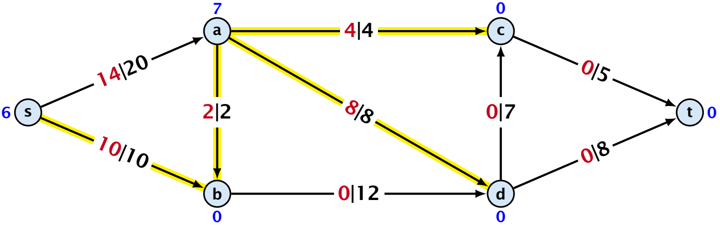
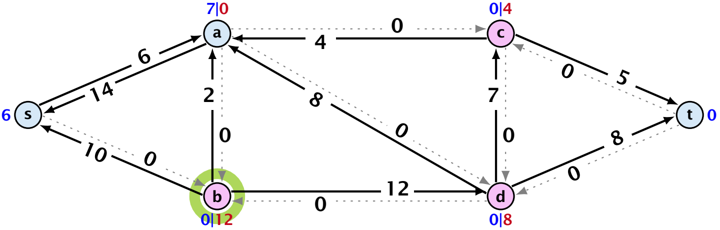


relabel to 1



Preflow Push

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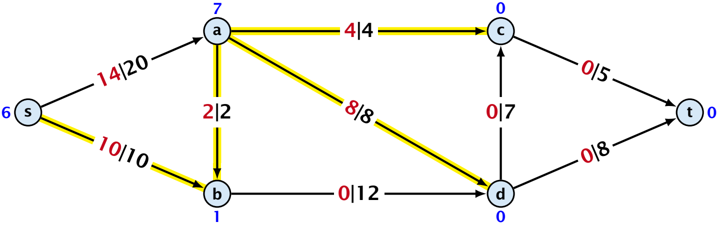

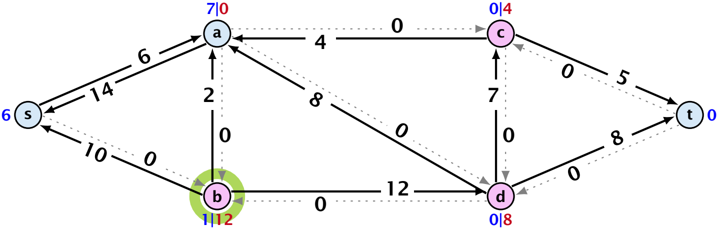

Preflow Push

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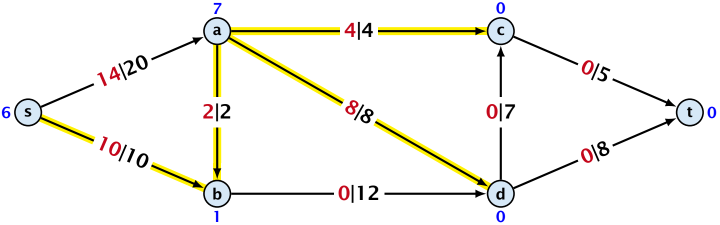
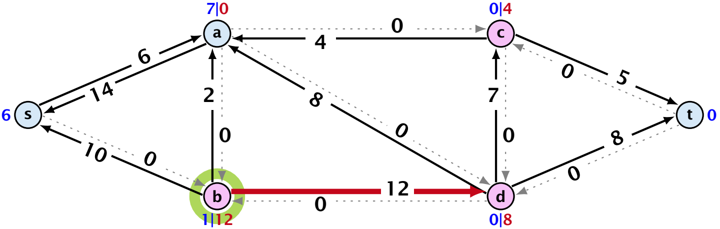


saturating and deactivating push



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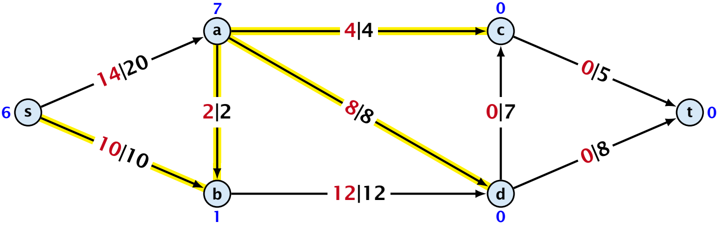

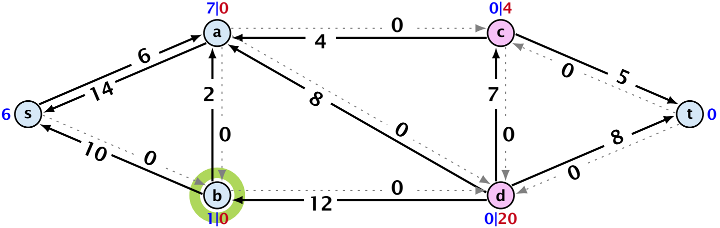

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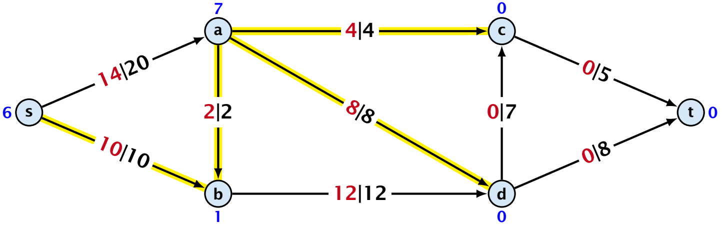

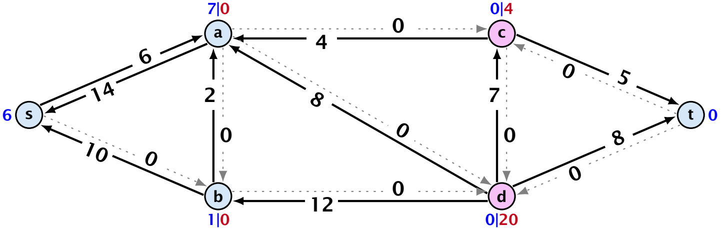

Preflow Push

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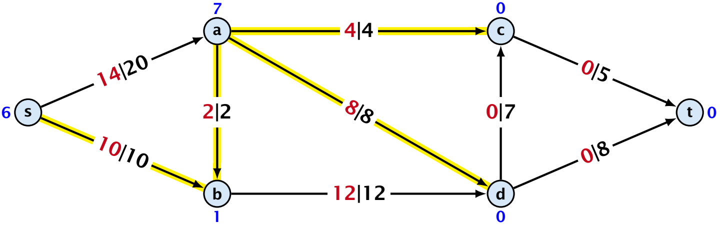

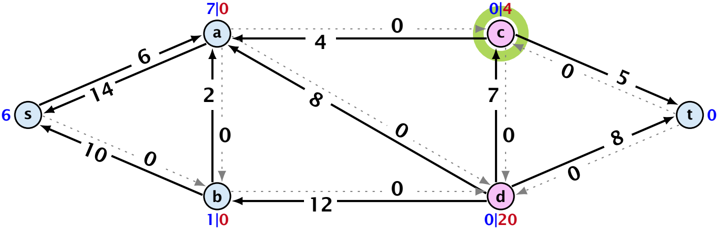

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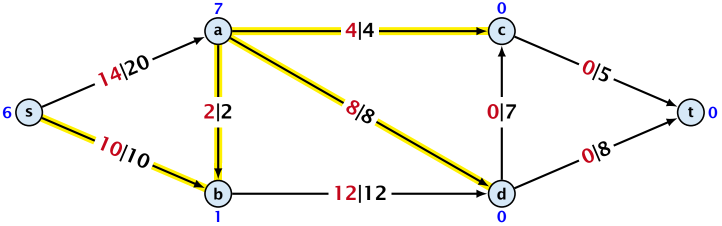
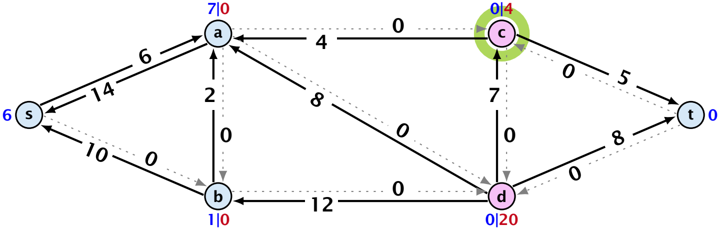


relabel to 1



Preflow Push

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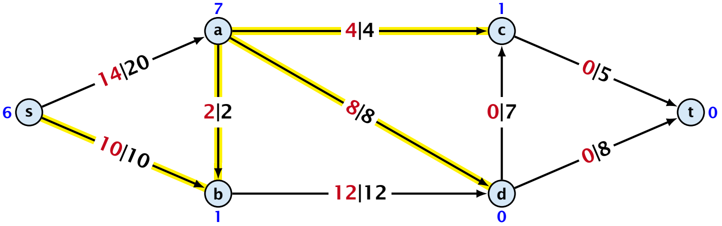

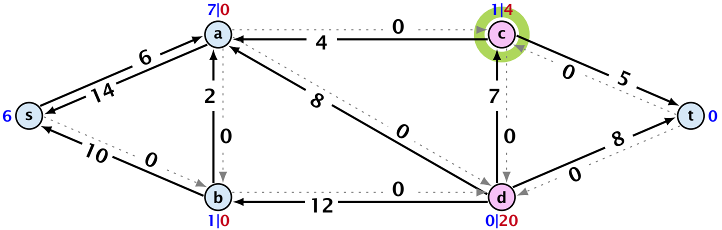

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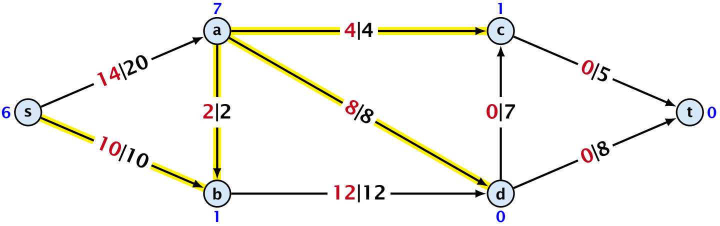
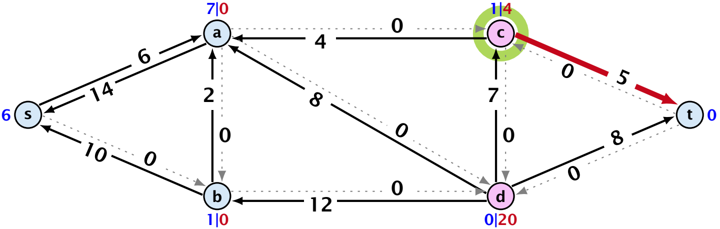


deactivating push



Preflow Push

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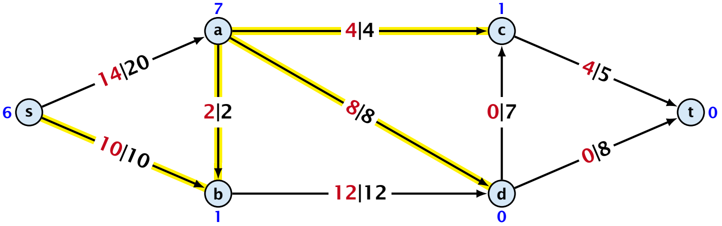

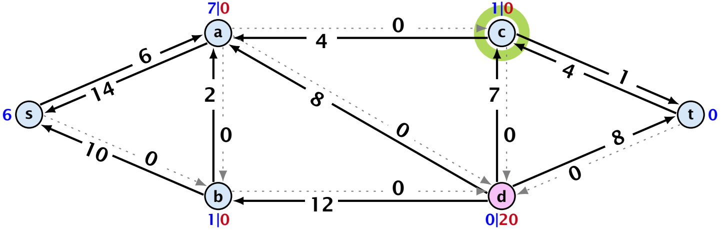

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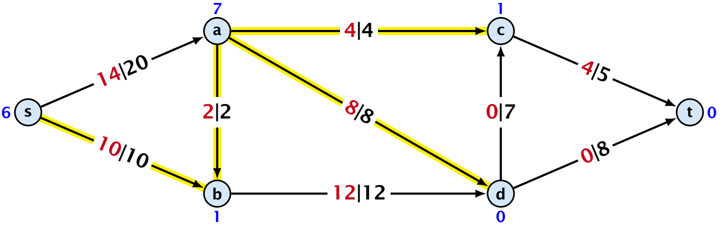

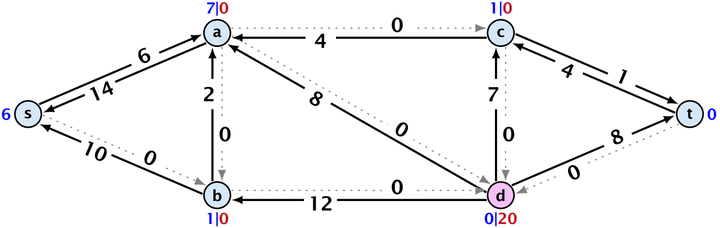

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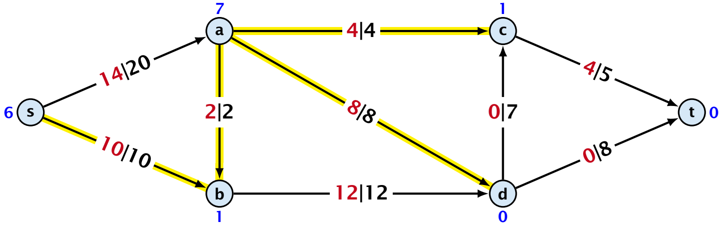

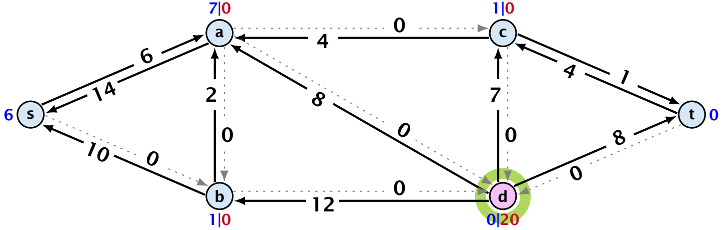

Preflow Push

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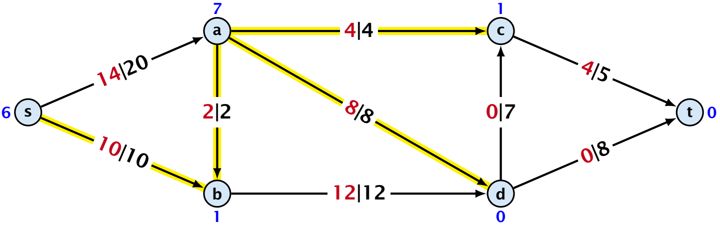
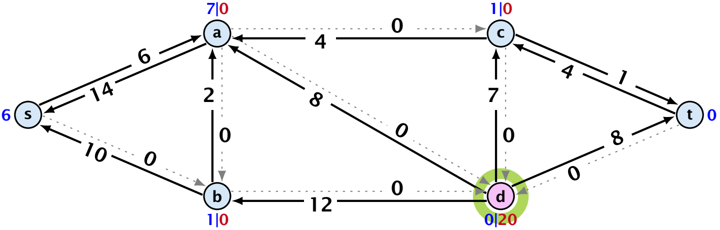


relabel to 1



Preflow Push

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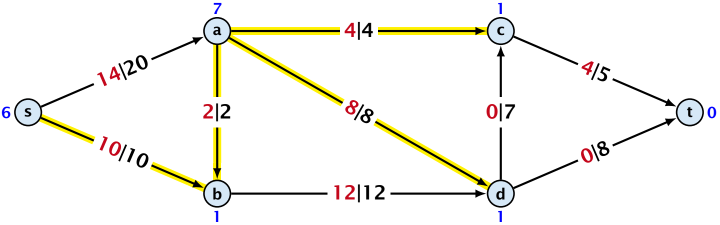

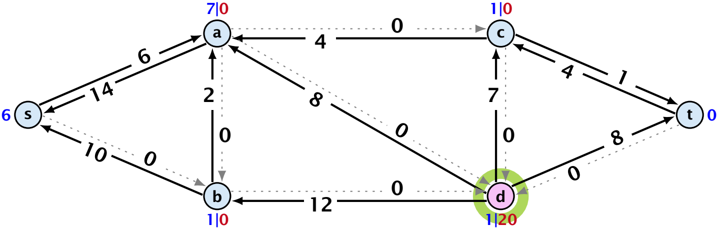

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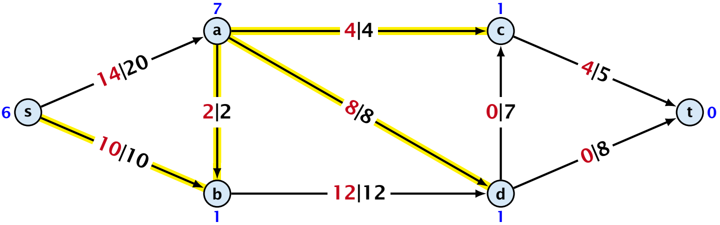
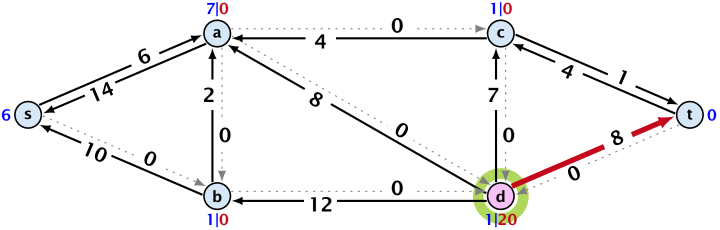


saturating push



Preflow Push

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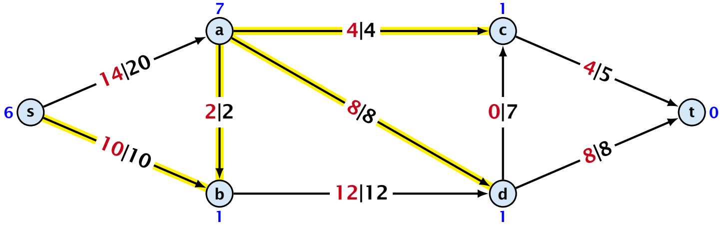

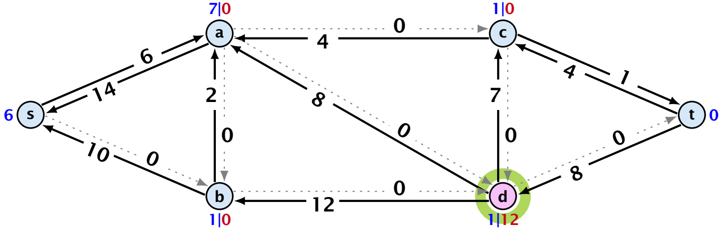

Preflow Push

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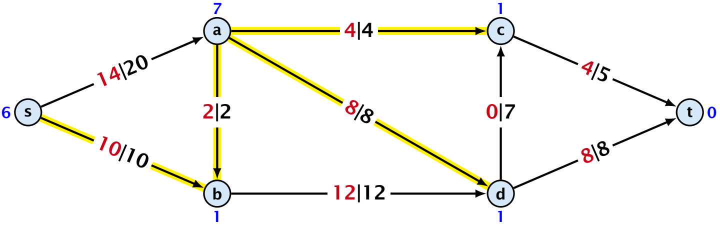
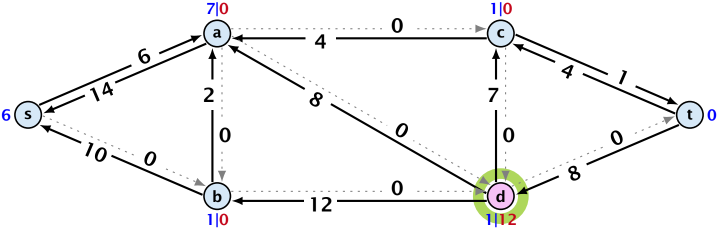


relabel to 2



Preflow Push

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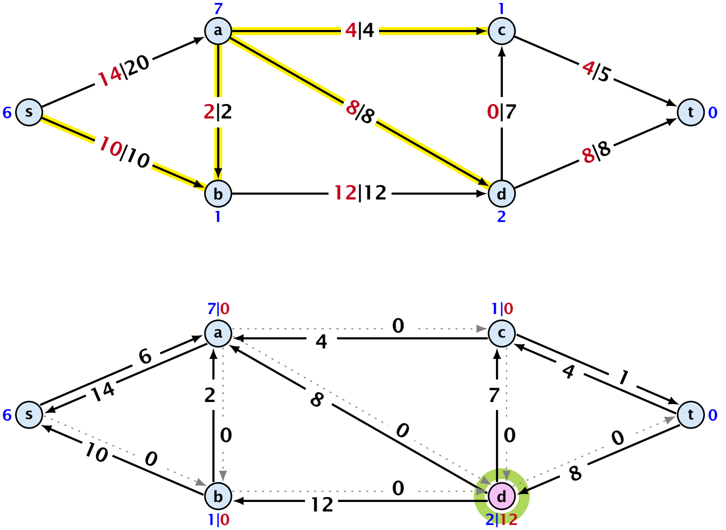

Preflow Push

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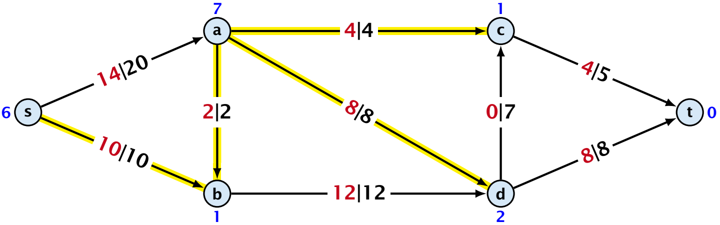
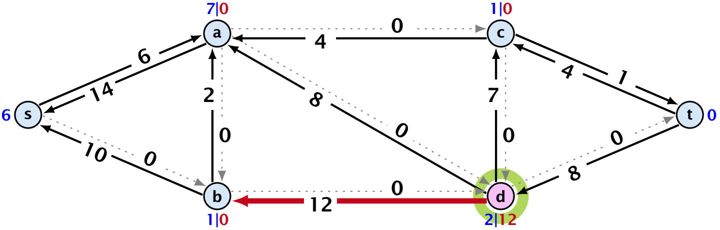


saturating and deactivating push



Preflow Push

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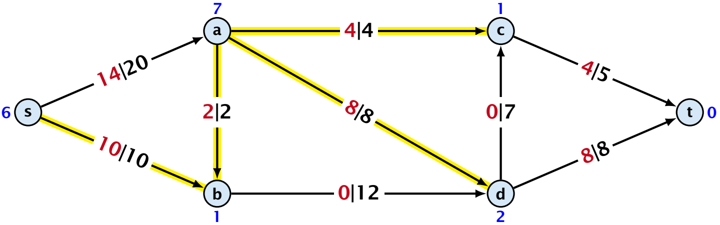

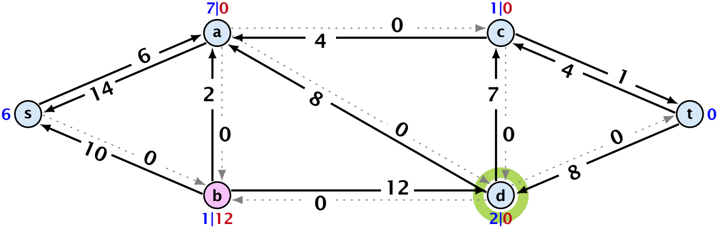

Preflow Push

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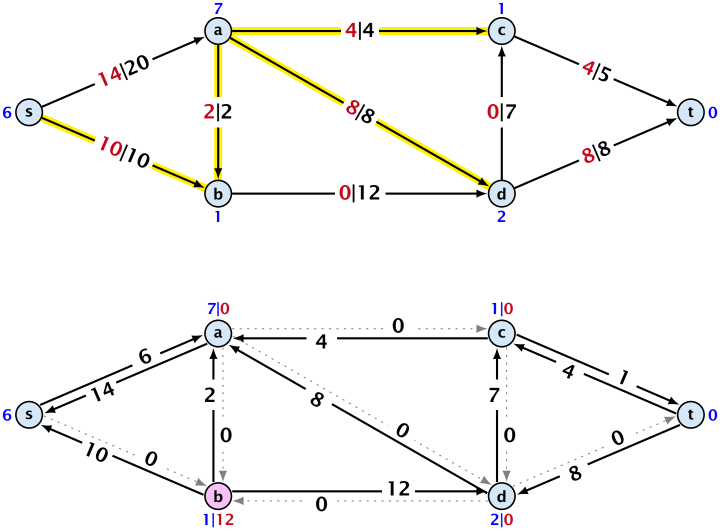

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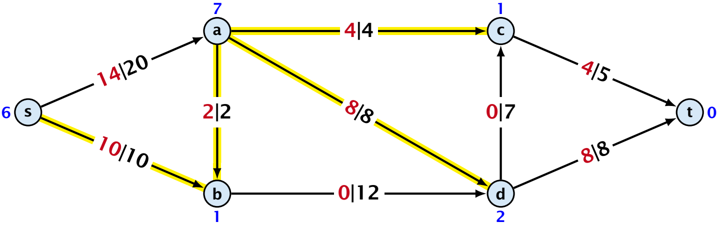

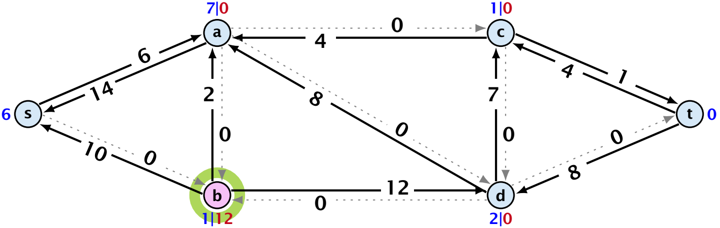

Preflow Push

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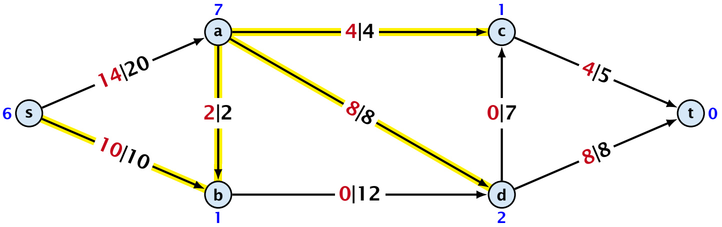
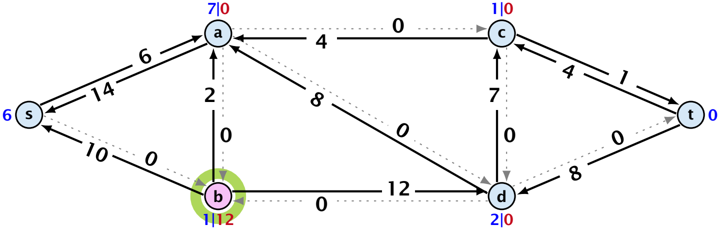


relabel to 3



Preflow Push

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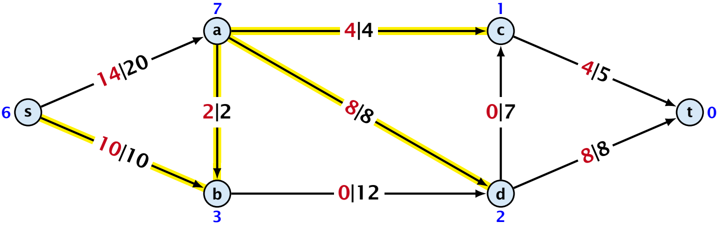

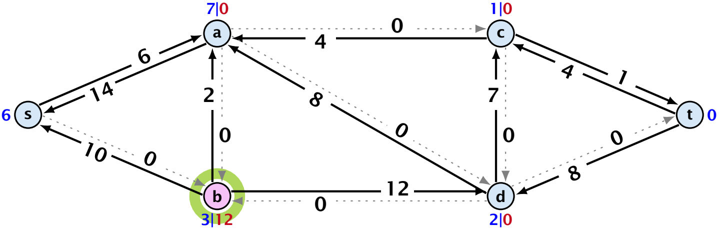

Preflow Push

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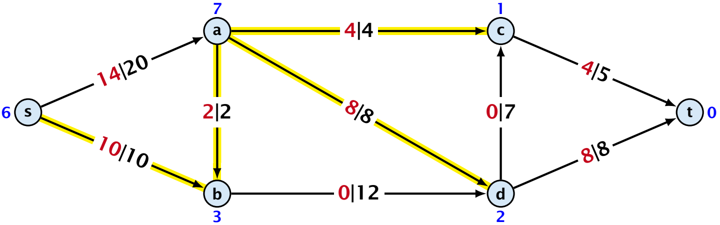
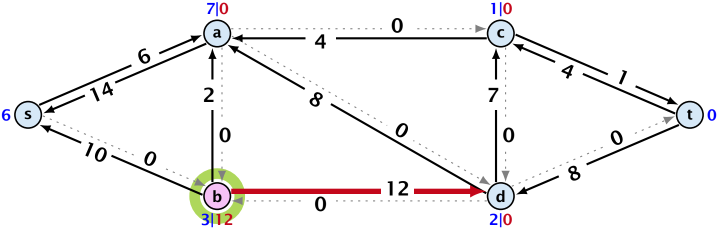


saturating and deactivating push



Preflow Push

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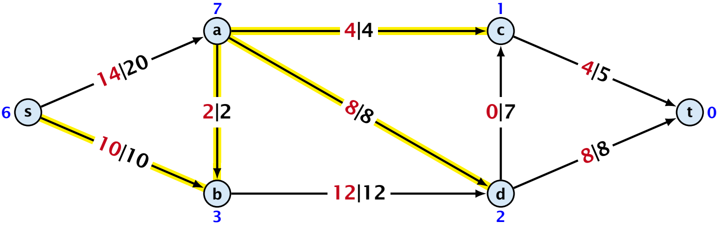

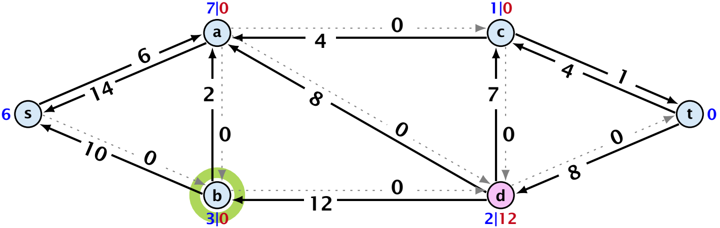

Preflow Push

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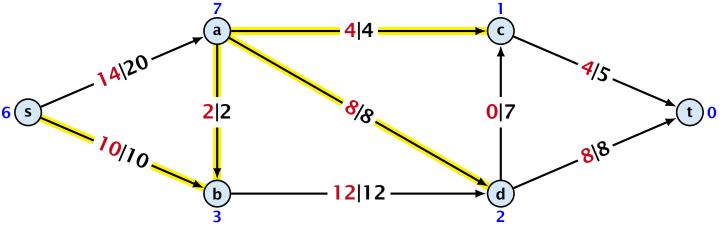
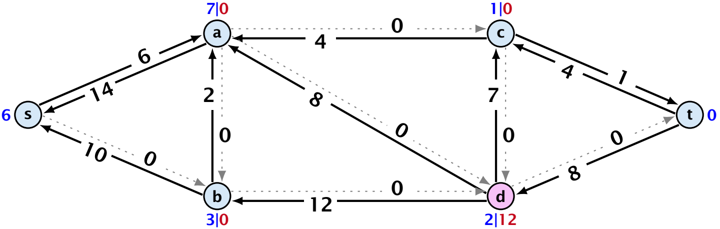
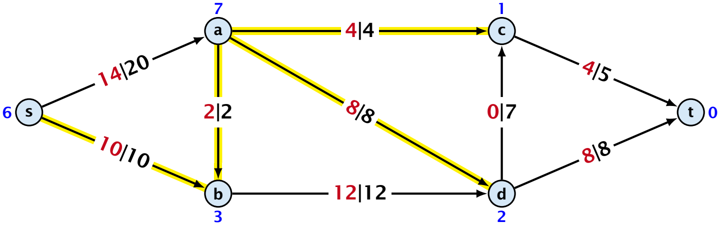
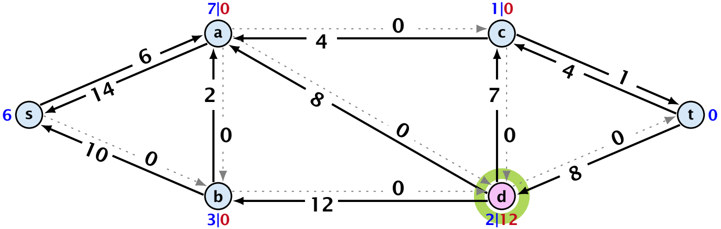




Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.





Preflow Push

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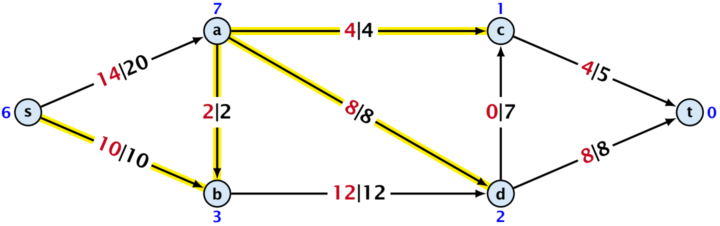
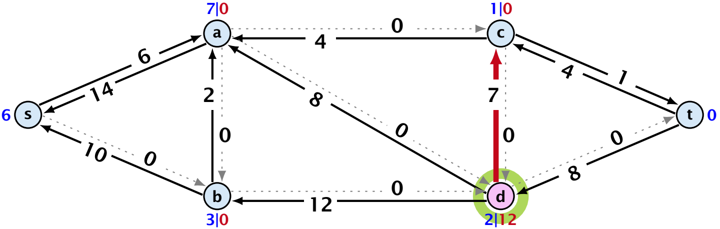


saturating push



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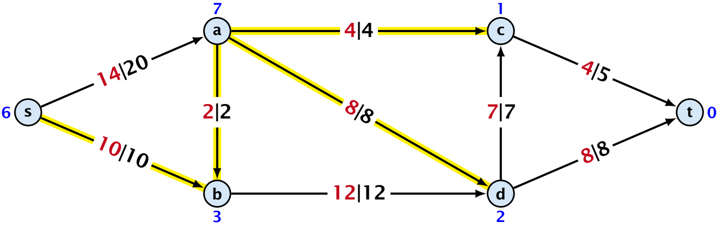

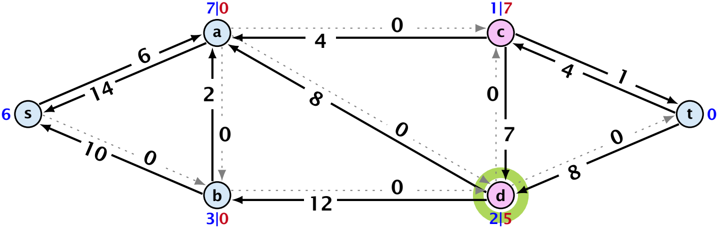

Preflow Push

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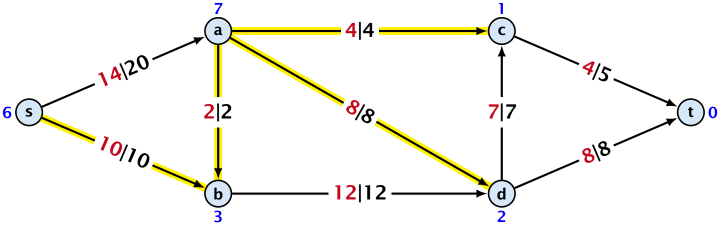
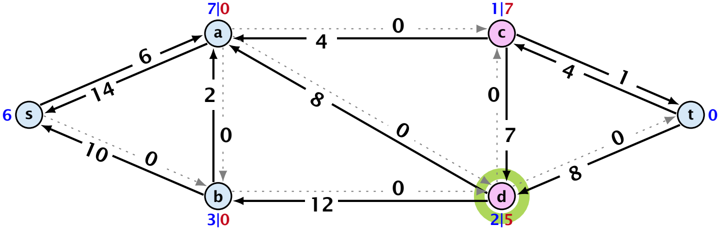


relabel to 4



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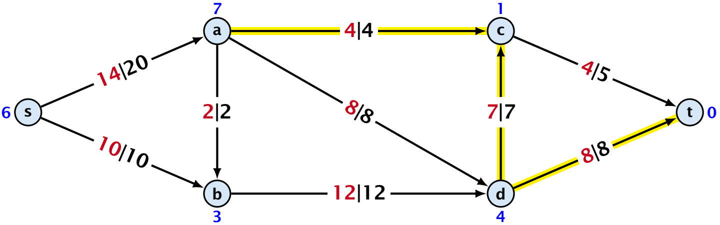

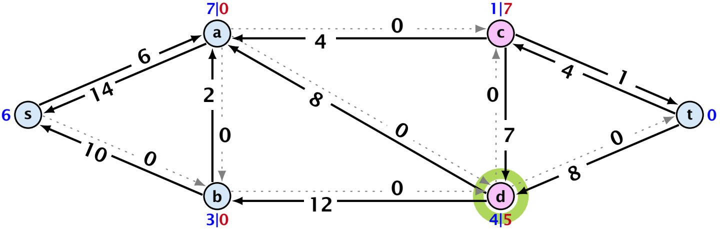

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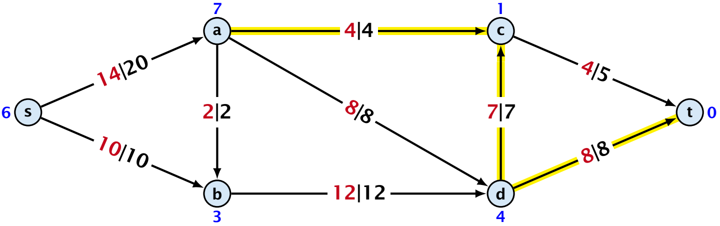
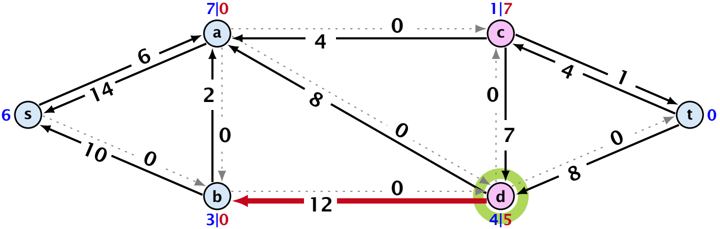


deactivating push



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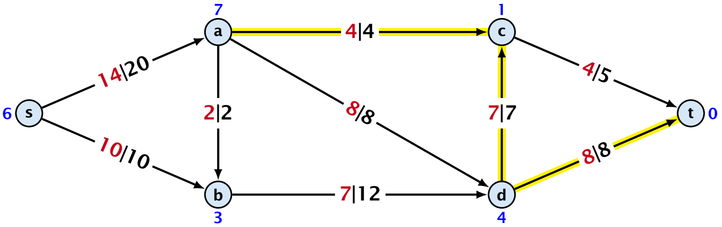

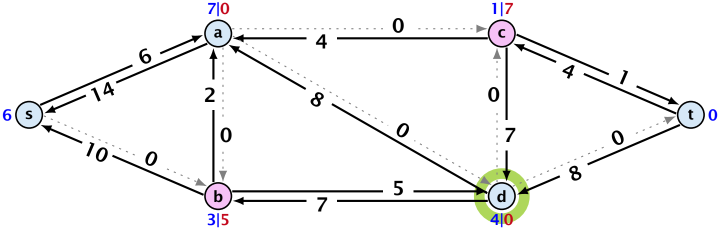

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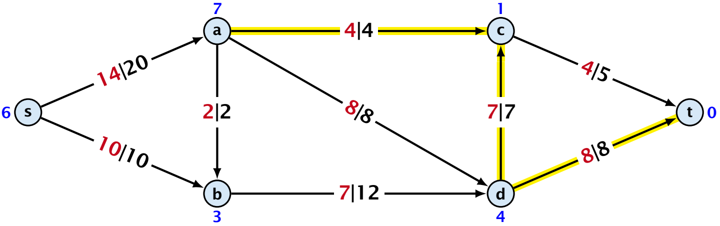

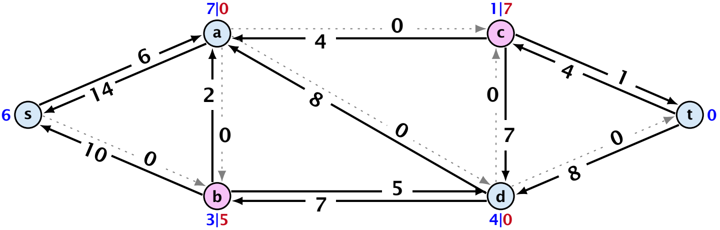

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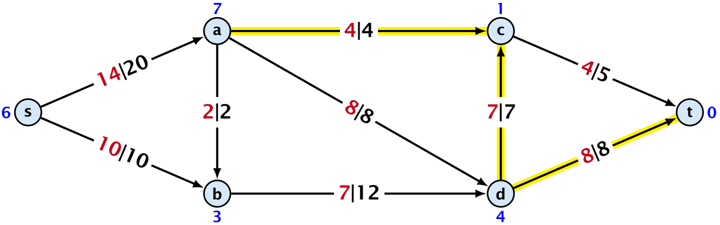

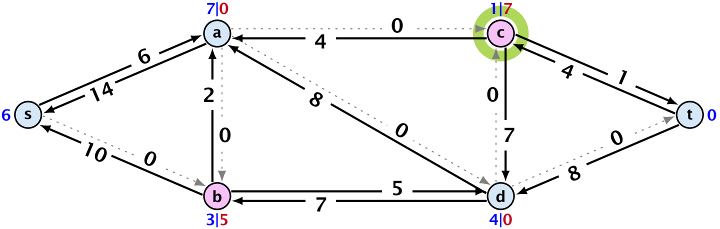

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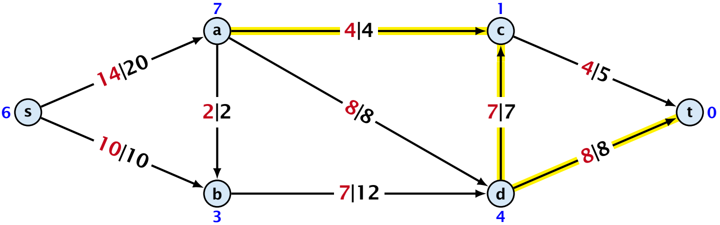
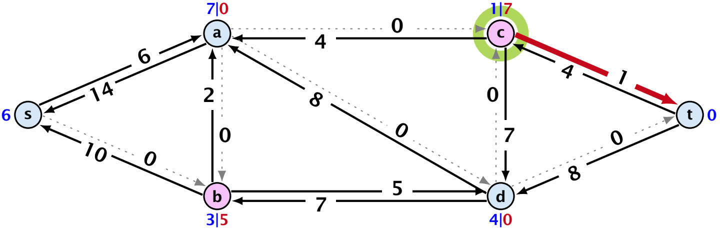


saturating push



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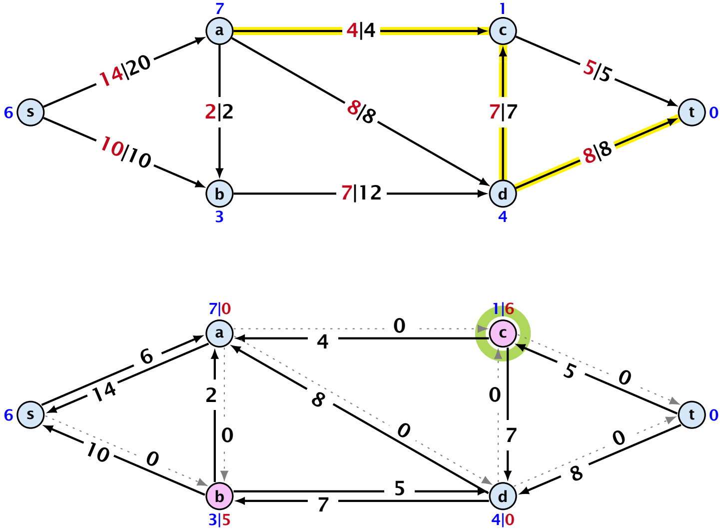

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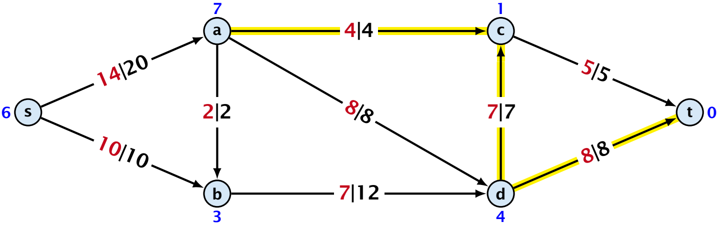
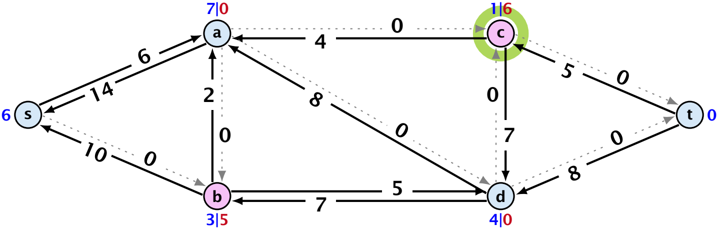


relabel to 5



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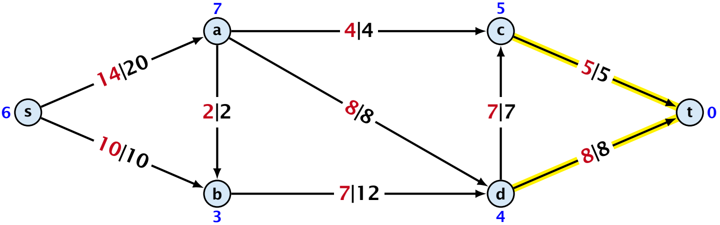

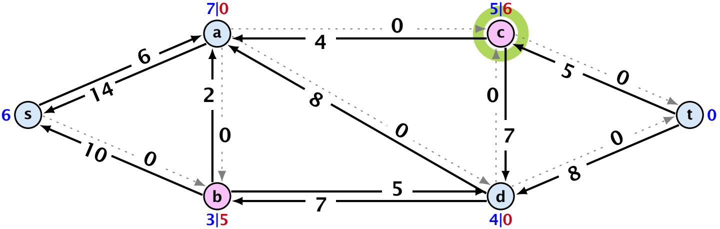

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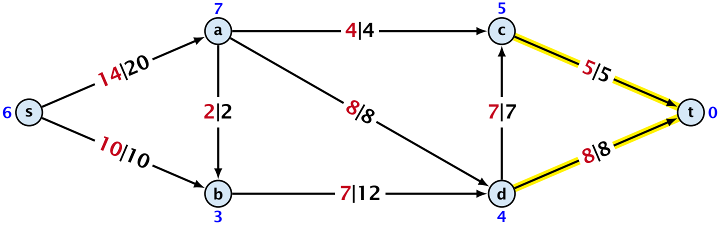
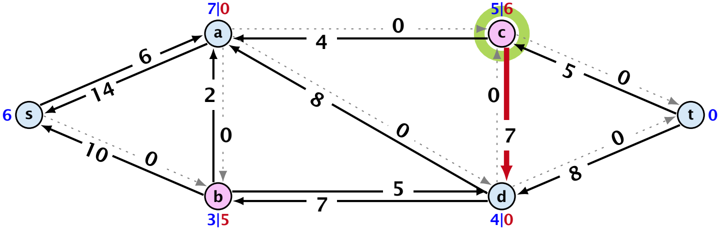


deactivating push



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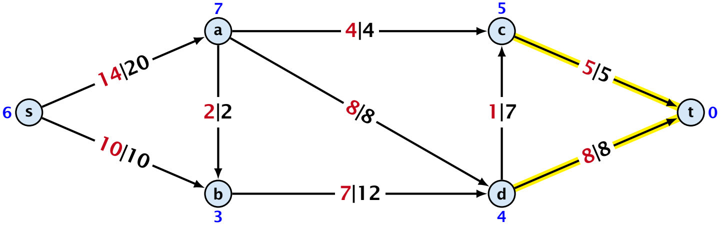

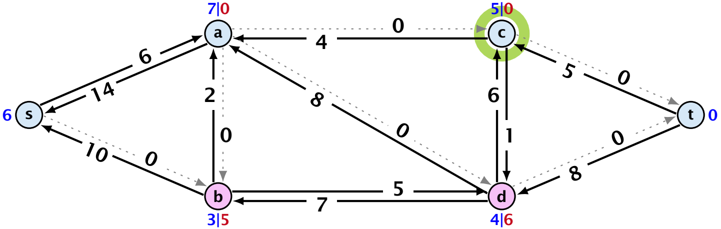

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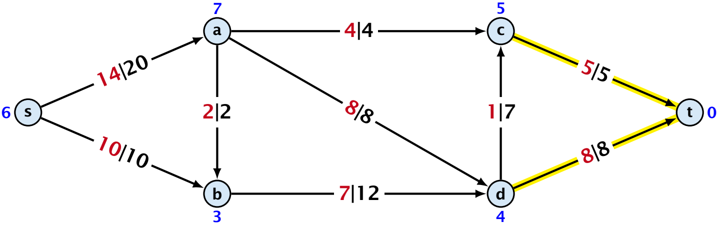

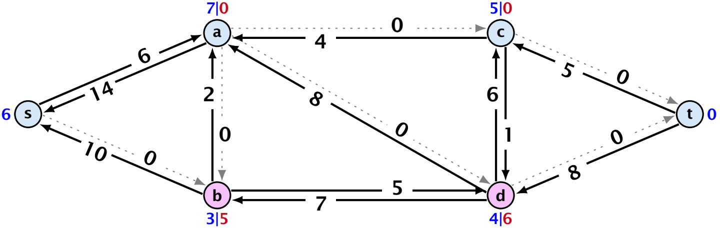

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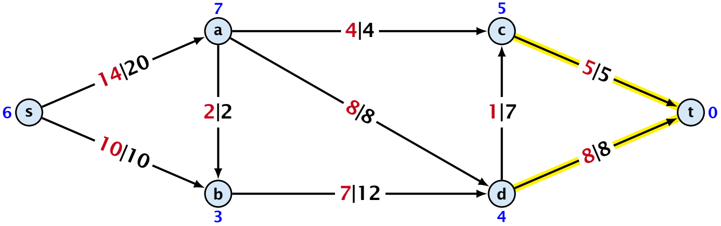

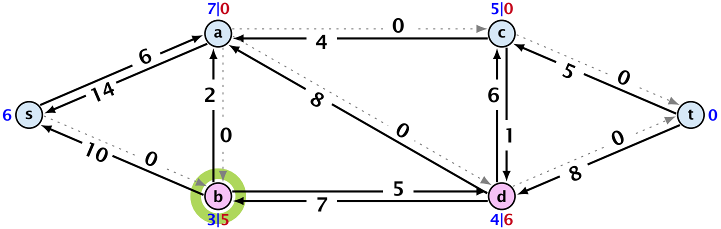

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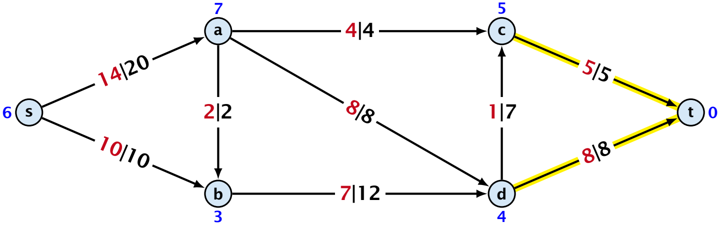
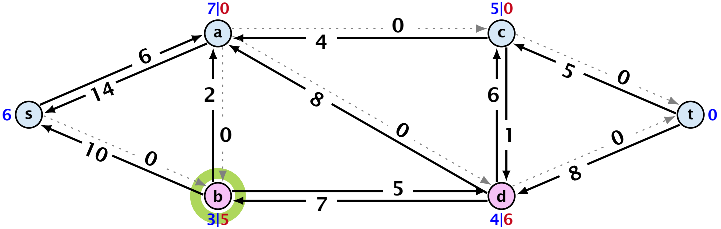


relabel to 5



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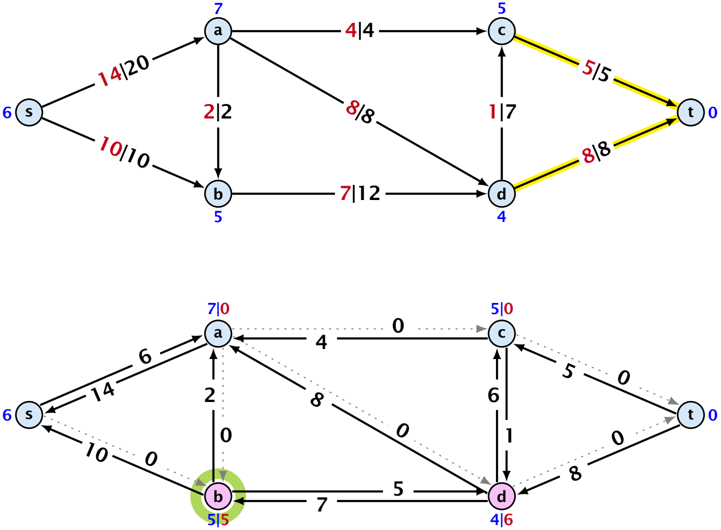

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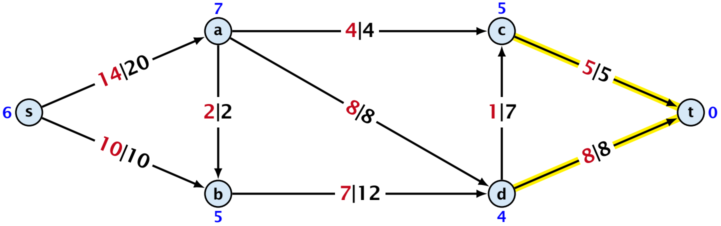
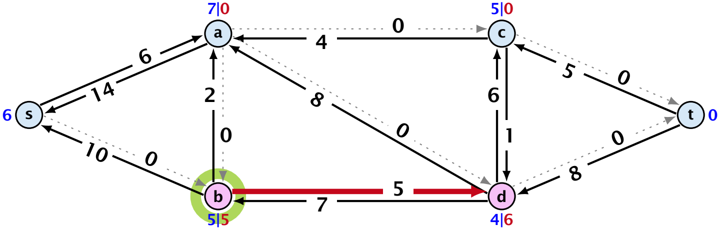


saturating and deactivating push



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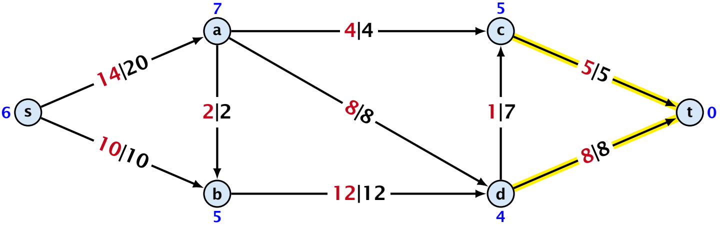

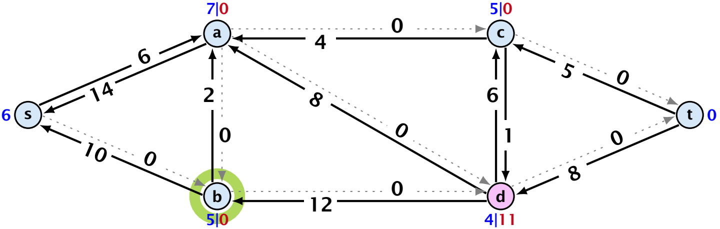

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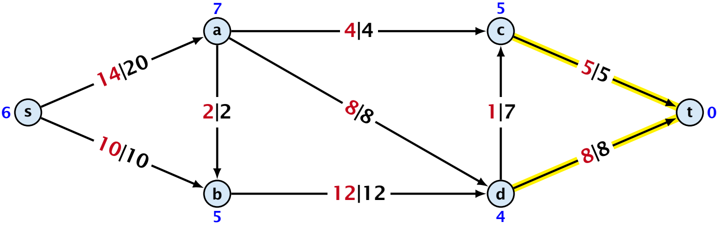

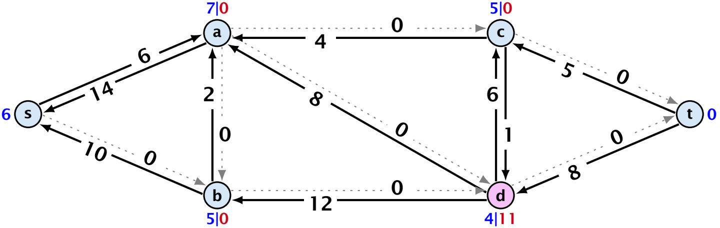

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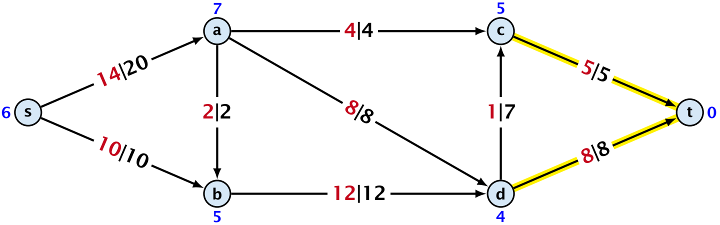

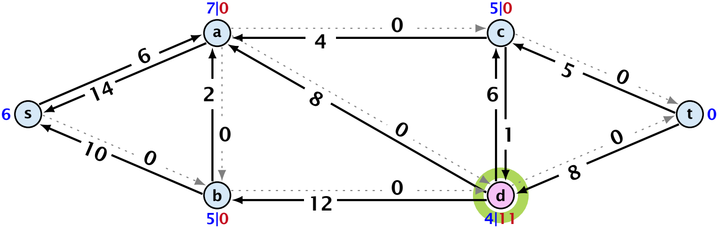

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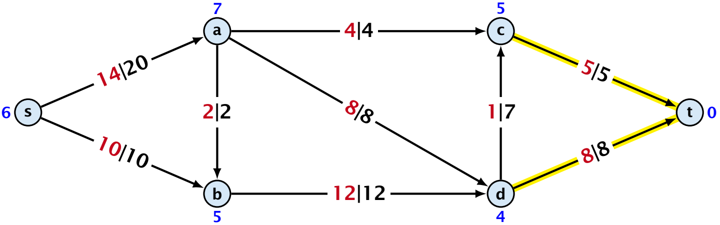
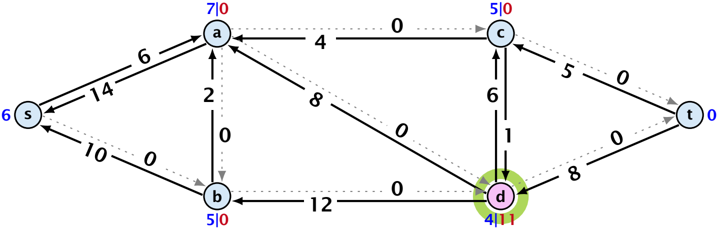


relabel to 6



Preflow Push

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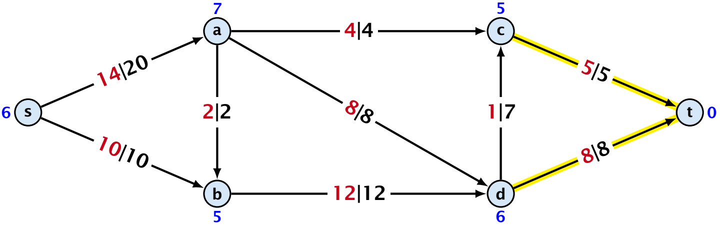

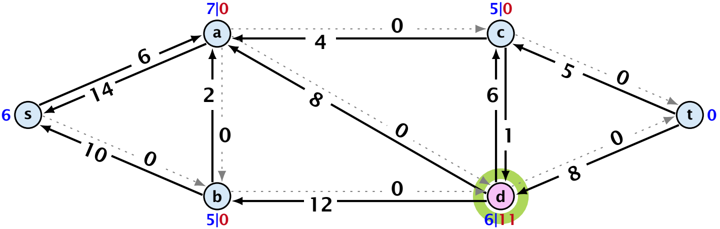

Preflow Push

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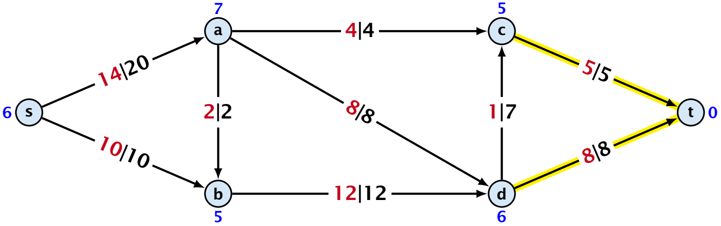
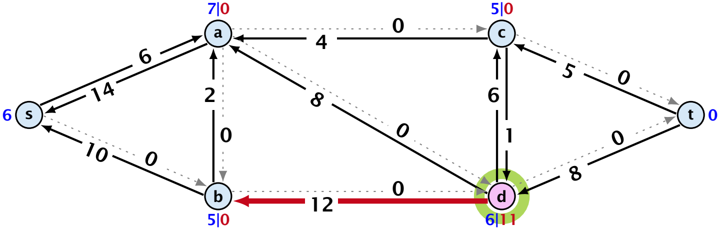


deactivating push



Preflow Push

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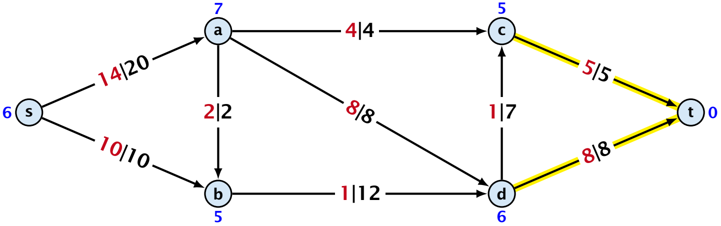

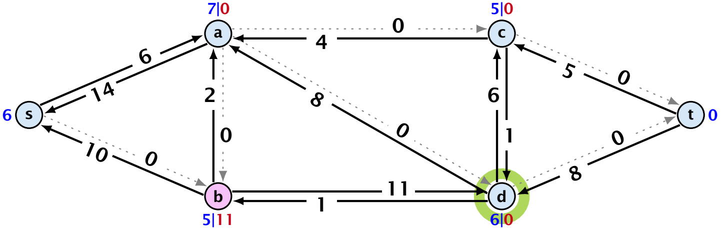

Preflow Push

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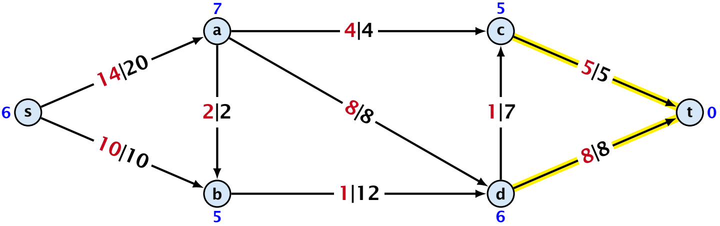

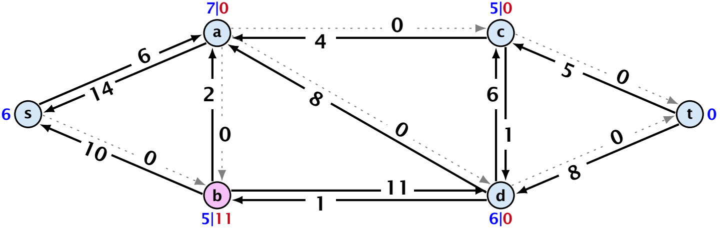

Preflow Push

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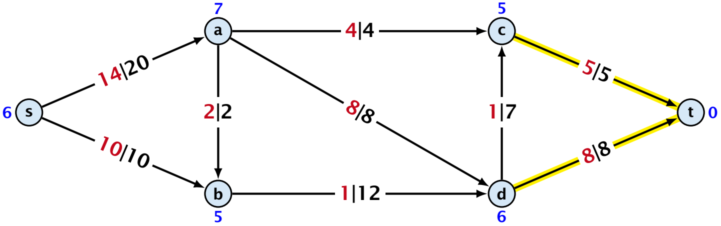

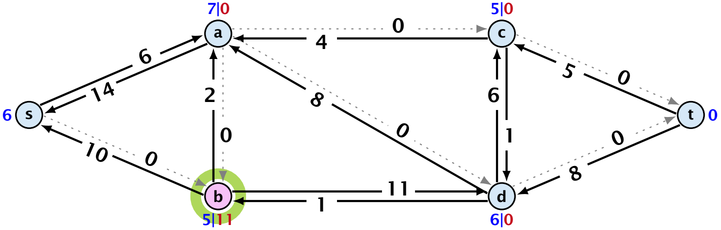

Preflow Push

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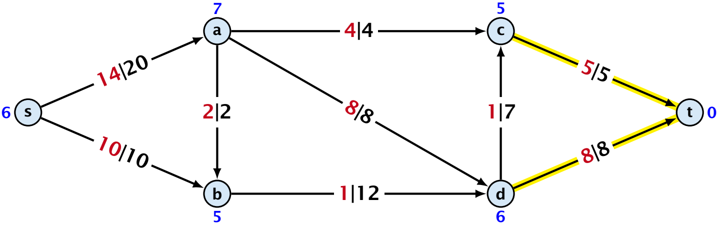
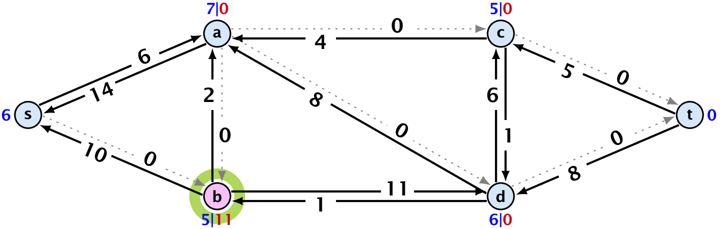


relabel to 7



Preflow Push

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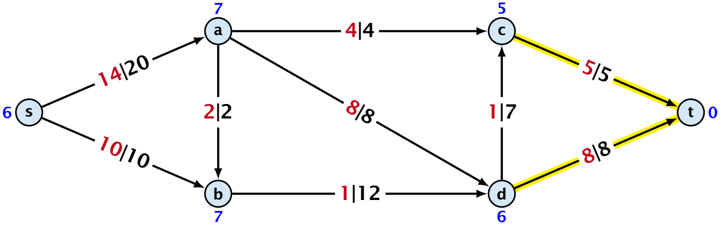

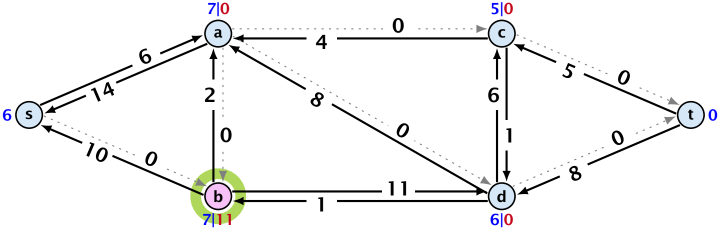

Preflow Push

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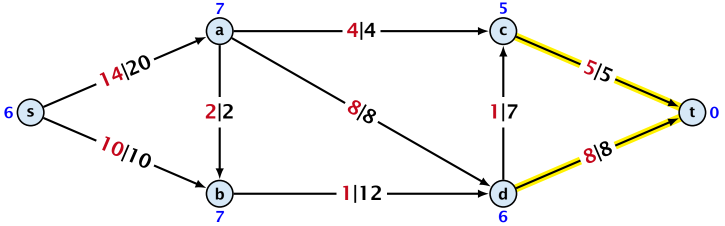
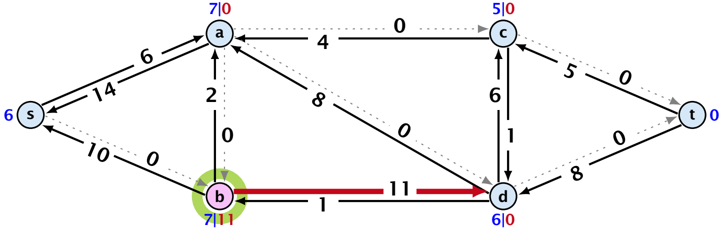


saturating and deactivating push



Preflow Push

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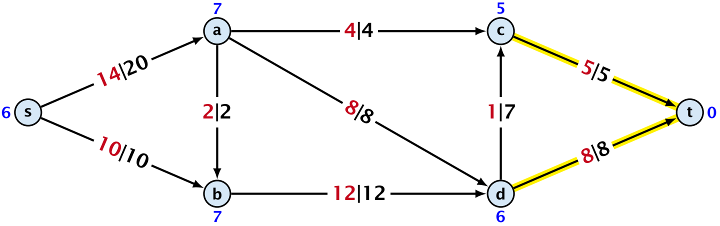

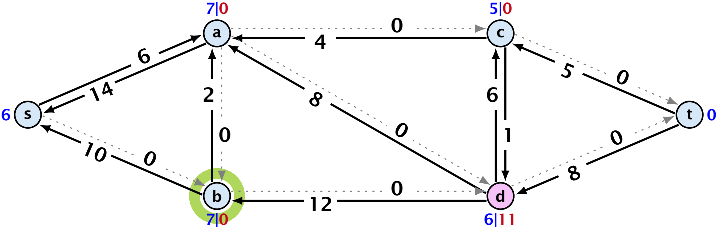

Preflow Push

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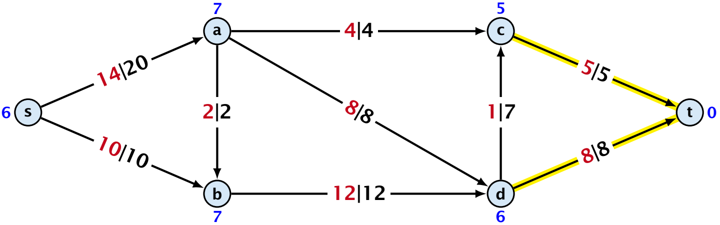

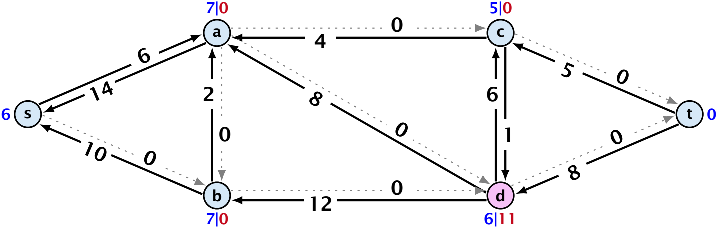

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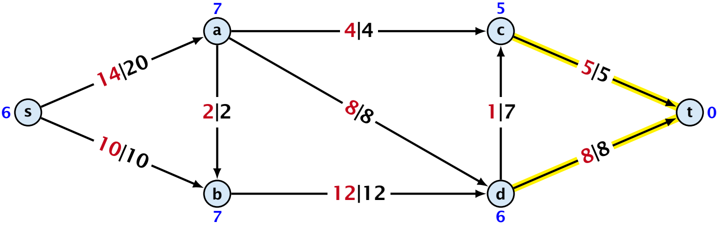

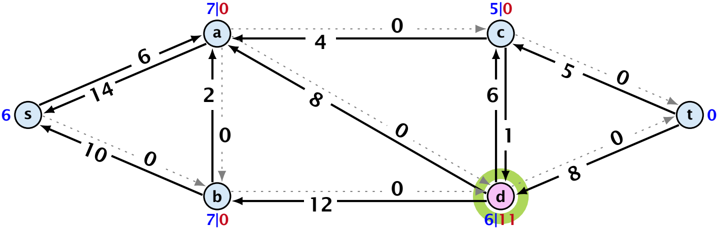

Preflow Push

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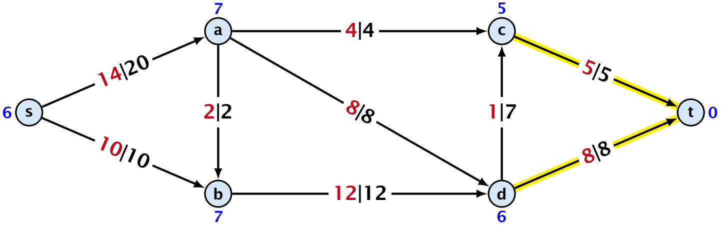
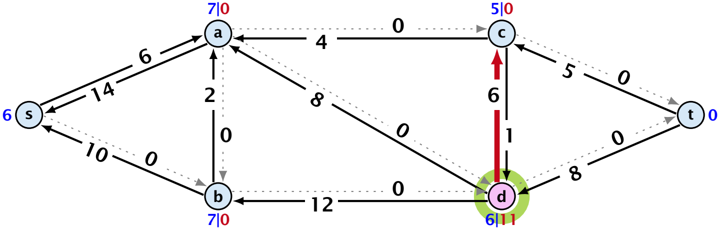


saturating push



Preflow Push

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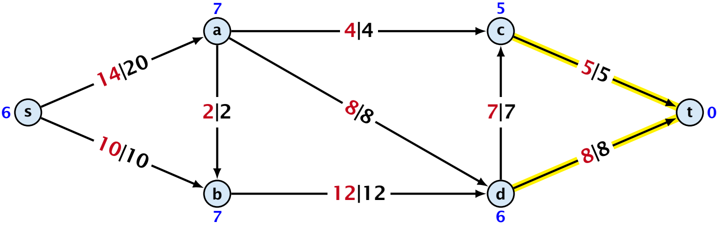

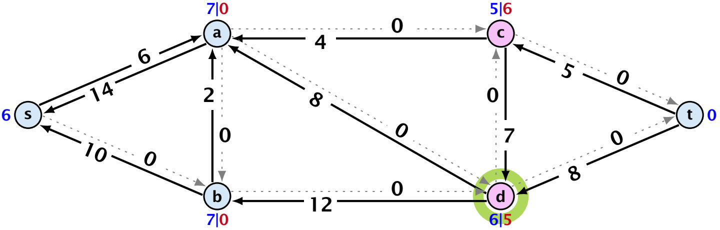

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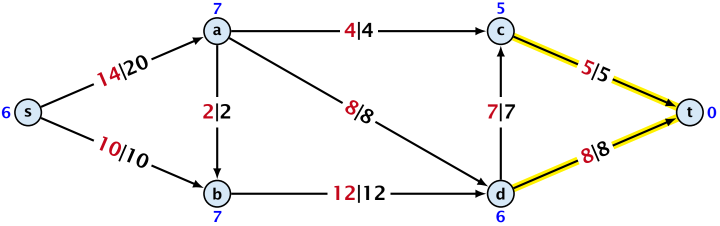
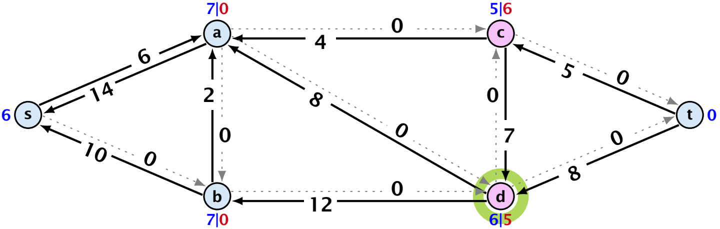


relabel to 8



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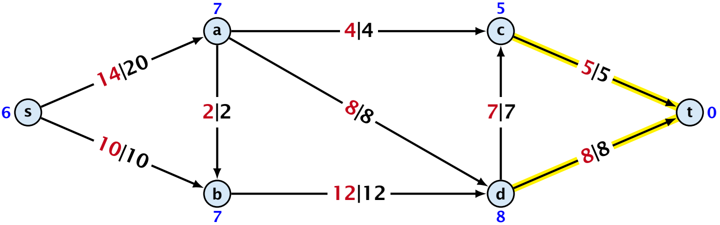

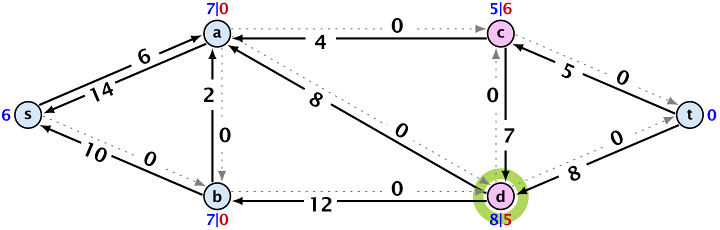

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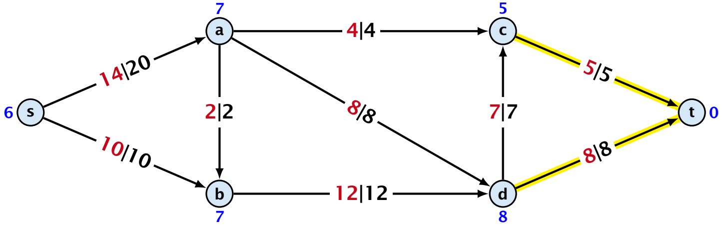
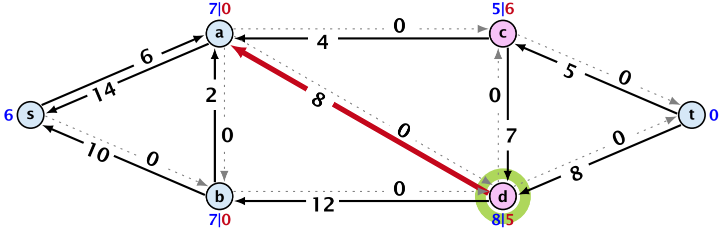


deactivating push



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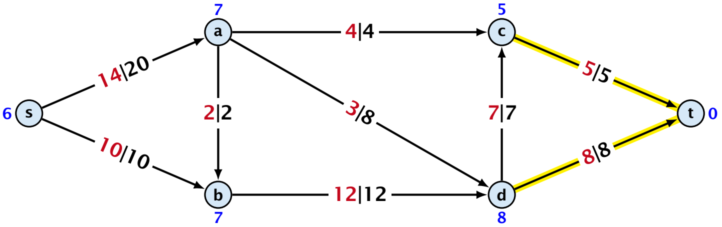

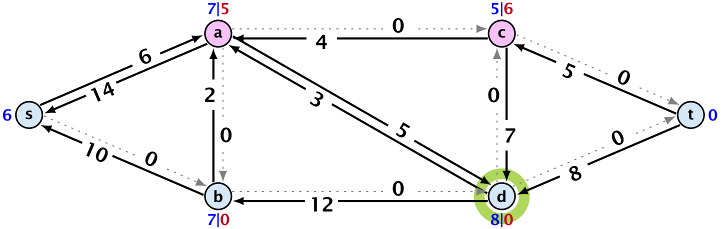

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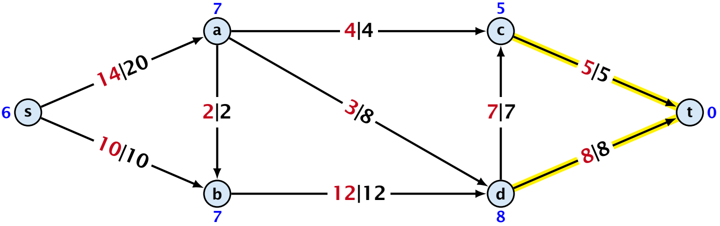

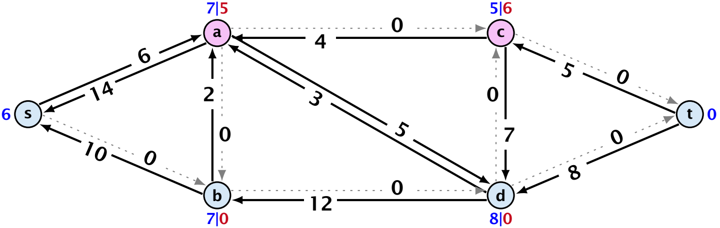

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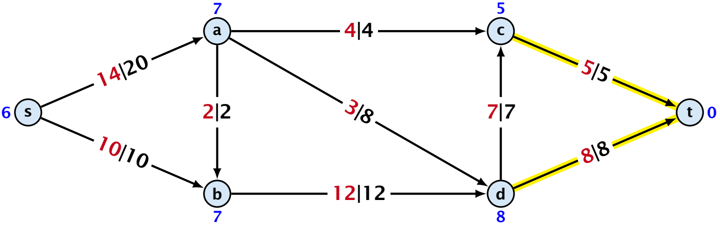

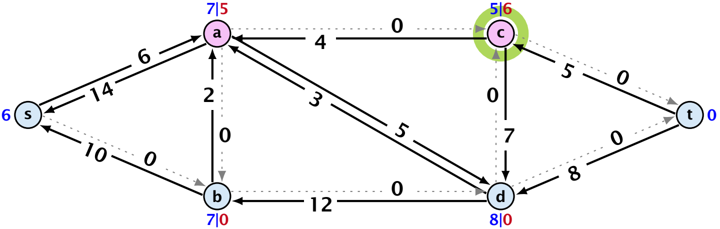

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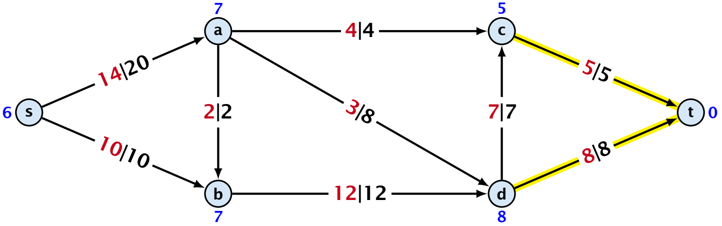
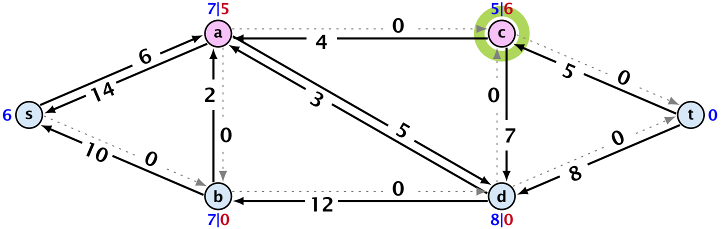


relabel to 8



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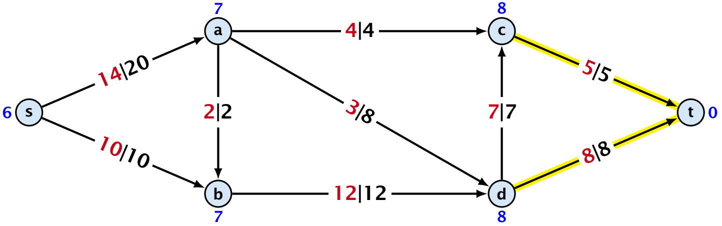

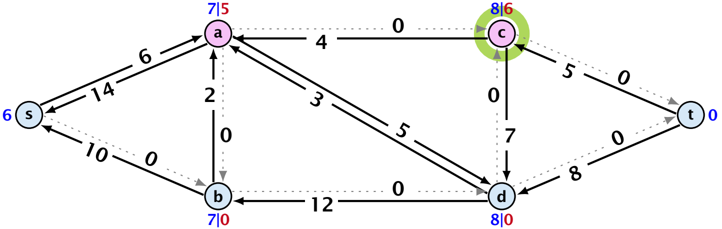

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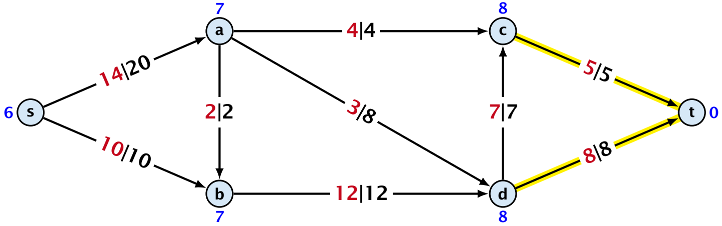
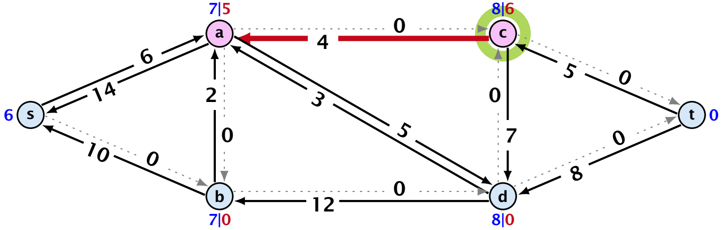


saturating push



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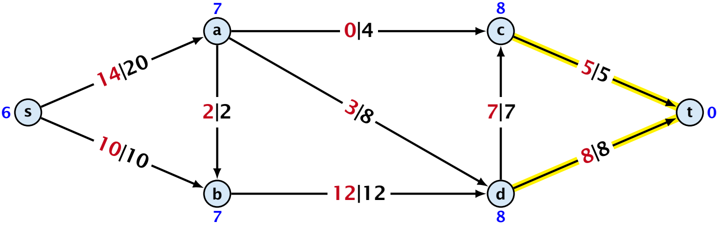

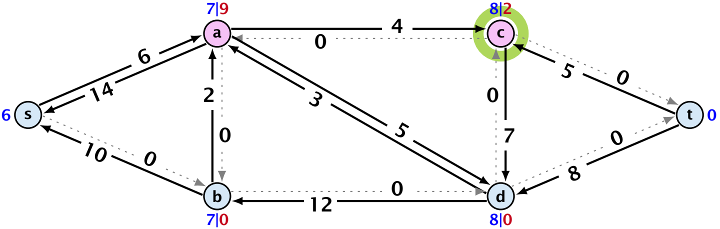

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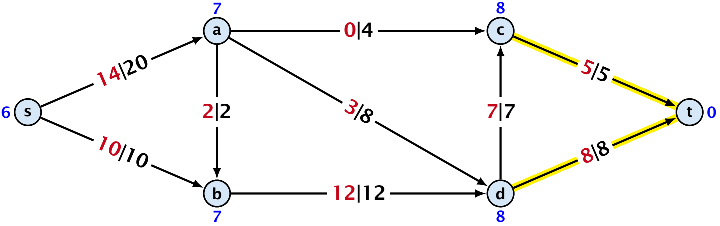
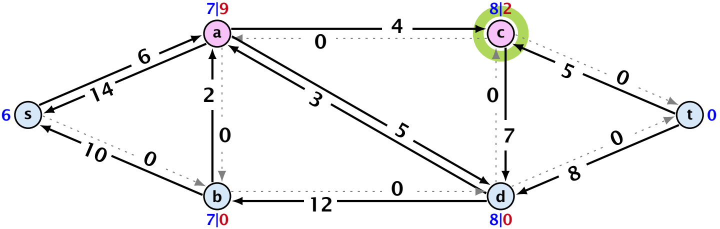


relabel to 9



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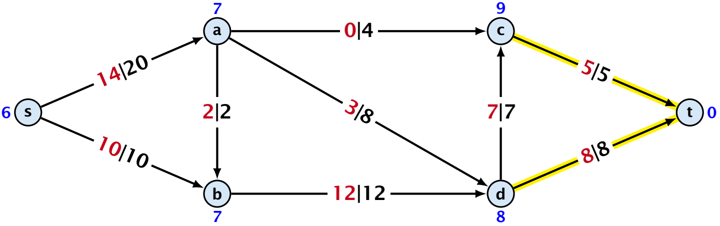

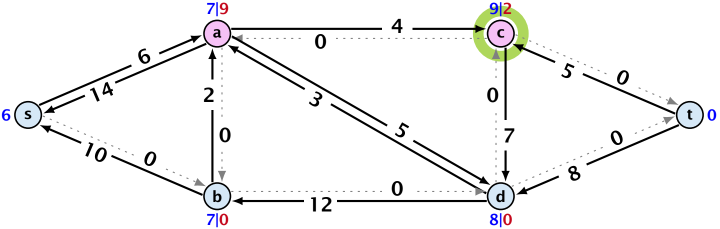

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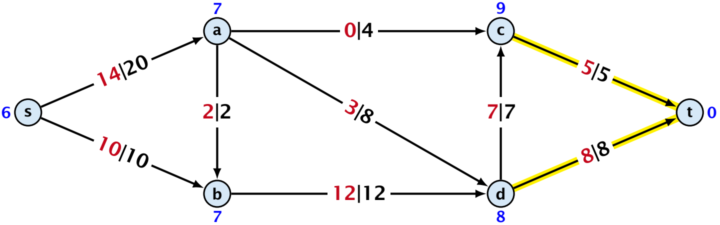
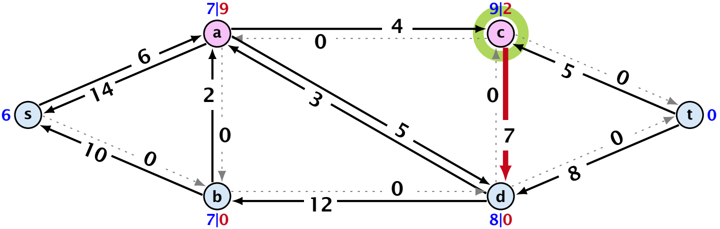


deactivating push



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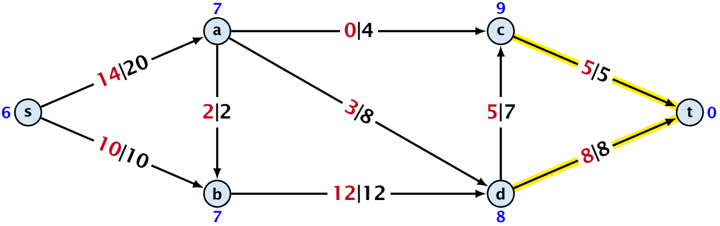

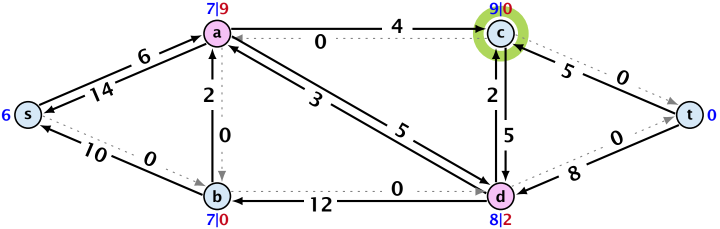

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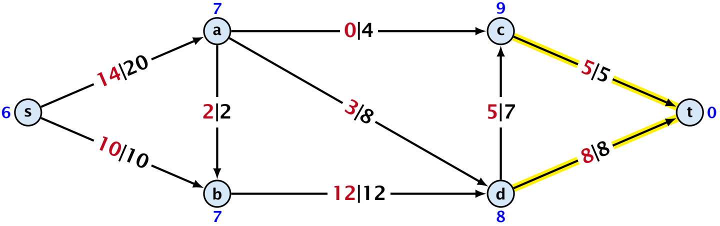

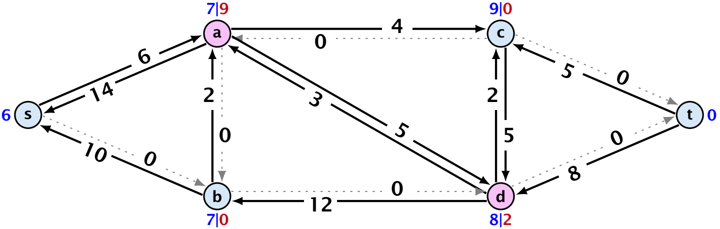

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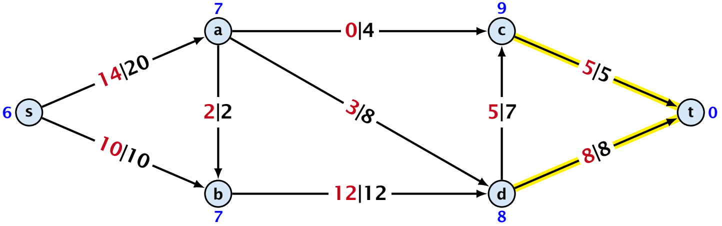

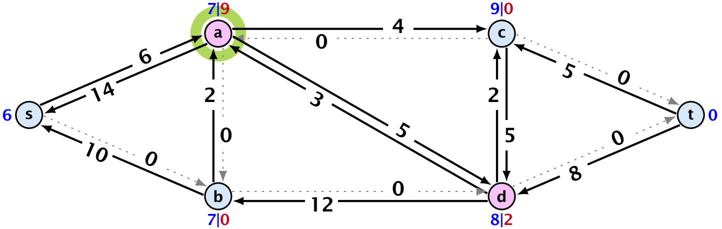

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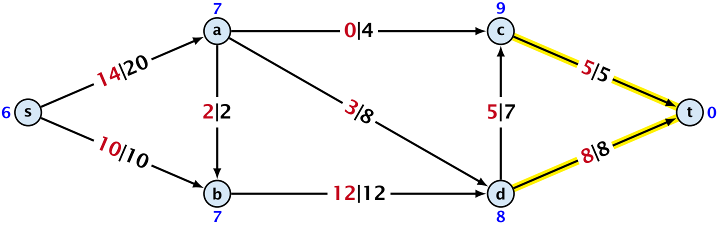
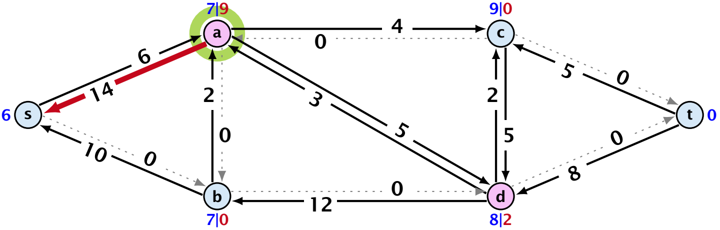


deactivating push



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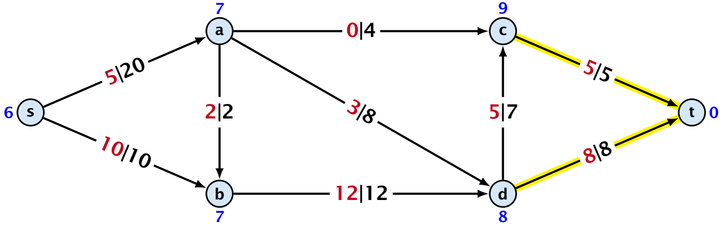

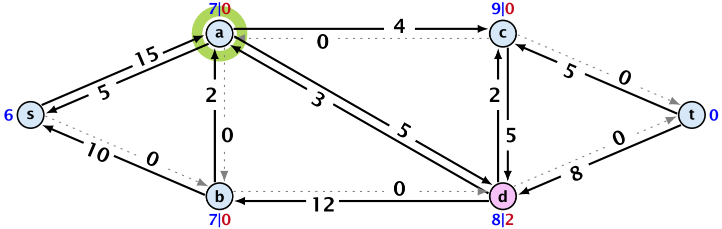

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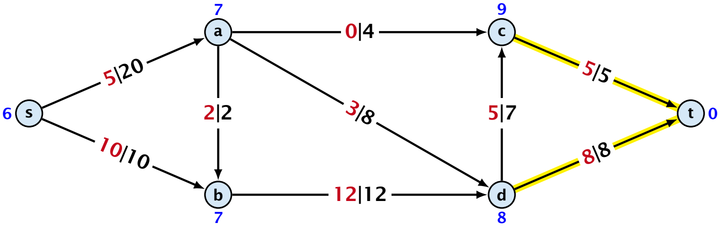

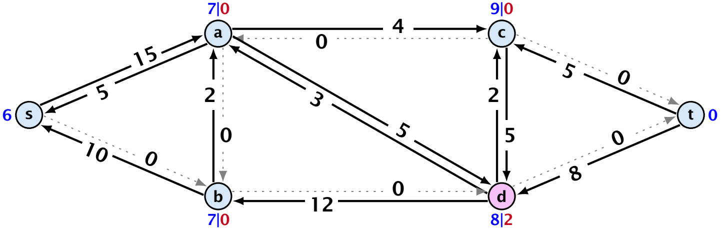

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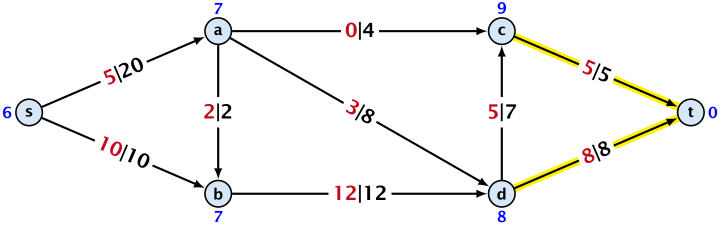

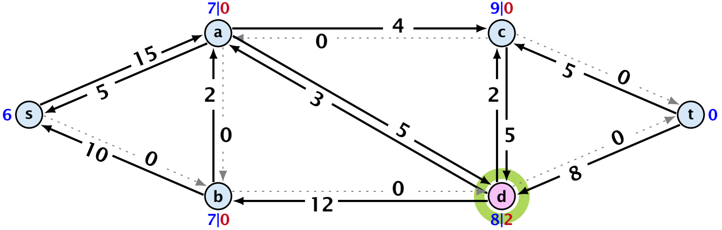

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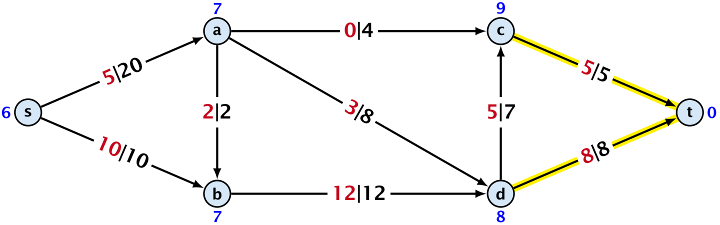
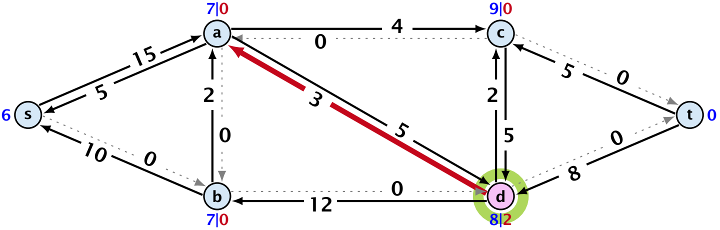


deactivating push



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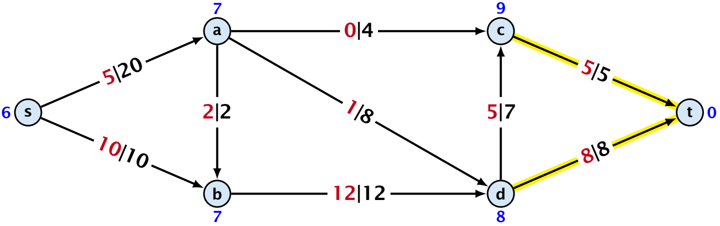

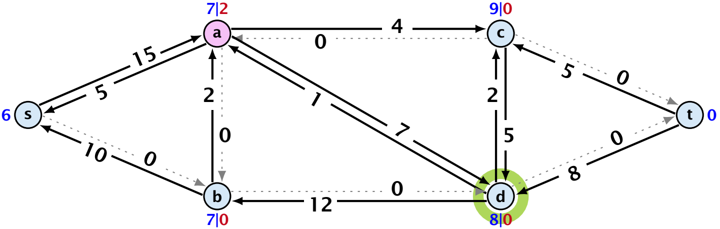

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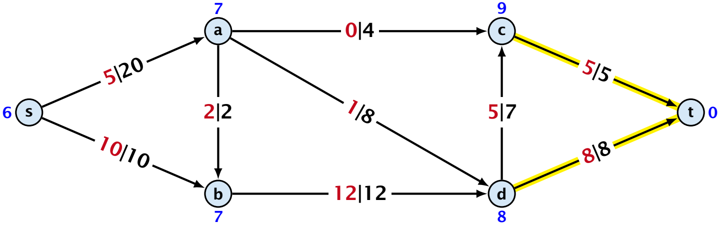

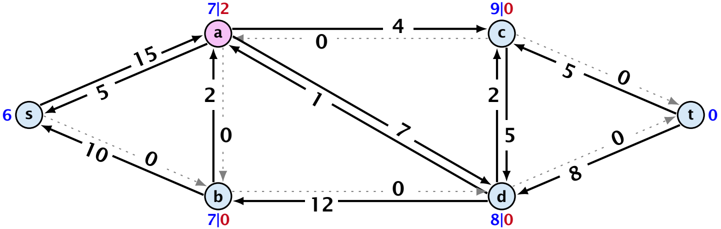

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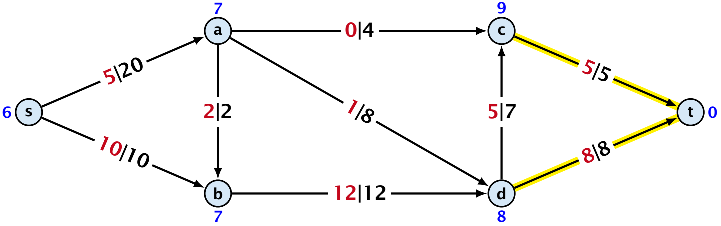

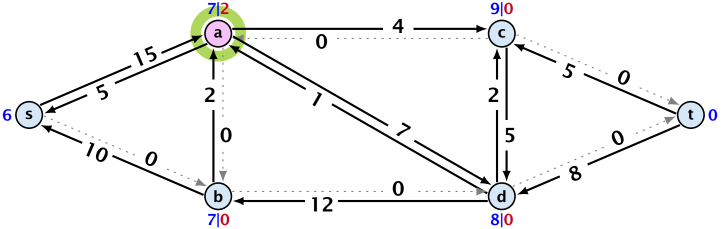

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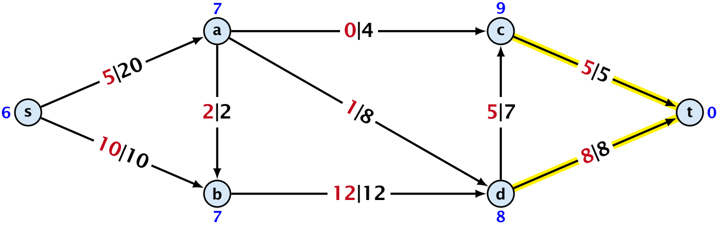

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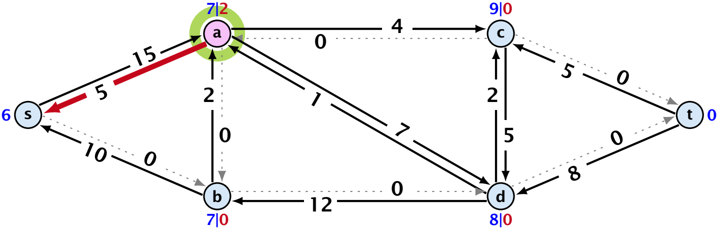

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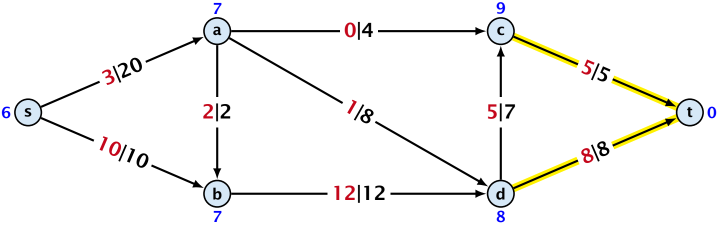

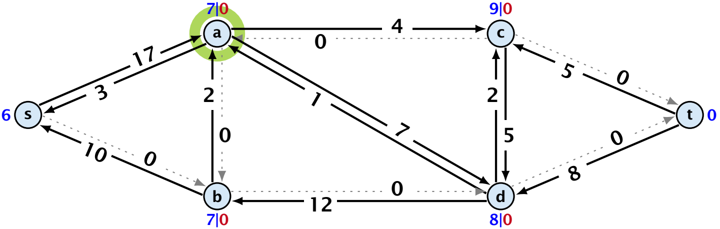

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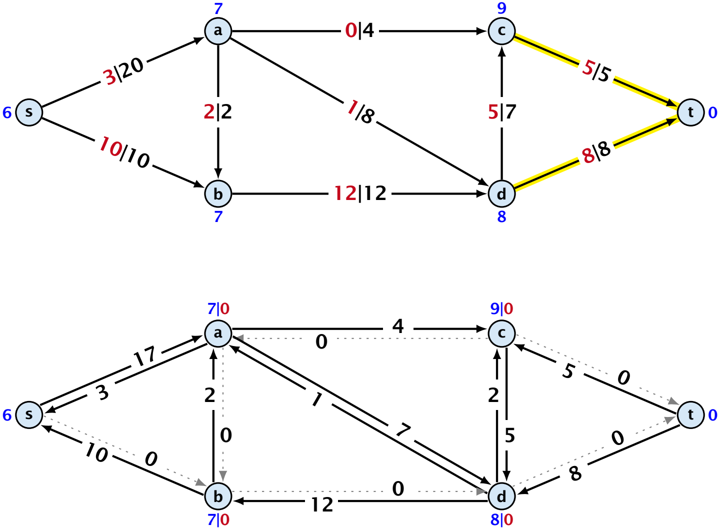

Analysis

Note that the lemma is almost trivial. A node v having excess flow means that the current preflow ships something to v. The residual graph allows to *undo* flow. Therefore, there must exist a path that can undo the shipment and move it back to s. However, a formal proof is required.

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An active node has a path to s in the residual graph.

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- Let $f(B) = \sum_{v \in B} f(v)$ be the excess flow of all nodes in B.

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

$$f(x,y) = \begin{cases} 0 & (x,y) \notin E \\ f((x,y)) & (x,y) \in E \end{cases}$$

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$$\begin{aligned} f(B) &= \sum_{b\in B} f(b) \\ &= \sum_{b\in B} \left(\sum_{v\in V} f(v,b) - \sum_{v\in V} f(b,v) \right) \\ &= \sum_{b\in B} \left(\sum_{v\in A} f(v,b) + \sum_{v\in B} f(v,b) - \sum_{v\in A} f(b,v) - \sum_{v\in B} f(b,v) \right) \\ &= \sum_{b\in B} \sum_{v\in A} f(v,b) - \sum_{b\in B} \sum_{v\in A} f(b,v) + \sum_{b\in B} \sum_{v\in B} f(v,b) - \sum_{b\in B} \sum_{v\in B} f(b,v) \end{aligned}$$

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

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Hence, the excess flow $f(b)$ must be 0 for every node $b \in B$.

Analysis

Lemma 70

The label of a node cannot become larger than $2n - 1$.

Analysis

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Proof.

- When increasing the label at a node u there exists a path from u to s of length at most $n - 1$. Along each edge of the path the height/label can at most drop by 1, and the label of the source is n.

Analysis

Lemma 70

The label of a node cannot become larger than $2n-1$*.*

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- When increasing the label at a node u there exists a path from u to s of length at most $n-1$. Along each edge of the path the height/label can at most drop by 1, and the label of the source is n.

Lemma 71

There are only $\mathcal{O}(n^2)$ *relabel operations.*

Analysis

Lemma 72

The number of saturating pushes performed is at most $\mathcal{O}(mn)$.

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- Suppose that we just made a saturating push along (u, v).
- Hence, the edge (u, v) is deleted from the residual graph.
- For the edge to appear again, a push from v to u is required.

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- Currently, $\ell(u) = \ell(v) + 1$, as we only make pushes along admissible edges.

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- Currently, $\ell(u) = \ell(v) + 1$, as we only make pushes along admissible edges.
- For a push from v to u the edge (v, u) must become admissible. The label of v must increase by at least 2.
- Since the label of v is at most $2n - 1$, there are at most n pushes along (u, v).

Lemma 73

The number of deactivating pushes performed is at most $\mathcal{O}(n^2m)$.

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- A relabel increases Φ by at most 1.

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- A relabel increases Φ by at most 1.
- A deactivating push decreases Φ by at least 1 as the node that is pushed from becomes inactive and has a label that is strictly larger than the target.
- Hence,

$$\begin{aligned}\text{\#deactivating\_pushes} &\leq \text{\#relabels} + 2n \cdot \text{\#saturating\_pushes} \\ &\leq \mathcal{O}(n^2 m) \enspace .\end{aligned}$$

Analysis

Theorem 74

There is an implementation of the generic push relabel algorithm with running time $\mathcal{O}(n^2m)$.

Analysis

Proof:

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For every node maintain a list of admissible edges starting at that node. Further maintain a list of active nodes.

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A push along an edge (u, v) can be performed in constant time

- check whether edge (v, u) needs to be added to G_f

Analysis

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A push along an edge (u,v) can be performed in constant time

- check whether edge (v,u) needs to be added to G_f
- check whether (u,v) needs to be deleted (saturating push)

Analysis

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- check whether edge (v,u) needs to be added to G_f
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- check whether u becomes inactive and has to be deleted from the set of active nodes

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A relabel at a node u can be performed in time $\mathcal{O}(n)$

- check for all outgoing edges if they become admissible

Analysis

Proof:

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- check whether u becomes inactive and has to be deleted from the set of active nodes

A relabel at a node u can be performed in time $\mathcal{O}(n)$

- check for all outgoing edges if they become admissible
- check for all incoming edges if they become non-admissible

Analysis

For special variants of push relabel algorithms we organize the neighbours of a node into a linked list (possible neighbours in the residual graph G_f). Then we use the discharge-operation:

Algorithm 2 discharge(u)

```
while $u$ is active do
    $v \leftarrow u.current\text{-}neighbour$
    if $v$ = null then
        relabel($u$)
        $u.current\text{-}neighbour \leftarrow u.neighbour\text{-}list\text{-}head$
    else
        if $(u, v)$ admissible then push($u, v$)
        else $u.current\text{-}neighbour \leftarrow v.next\text{-}in\text{-}list$
```

Note that $u.current\text{-}neighbour$ is a global variable. It is only changed within the discharge routine, but keeps its value between consecutive calls to discharge.

In order for e to become admissible the other end-point say v has to push flow to u (so that the edge (u, v) re-appears in the residual graph). For this the label of v needs to be larger than the label of u. Then in order to make (u, v) admissible the label of u has to increase.

Lemma 75

If $v = \mathrm{null}$ in Line 3, then there is no outgoing admissible edge from u.

Proof.

- While pushing from u the current-neighbour pointer is only advanced if the current edge is not admissible.
- The only thing that could make the edge admissible again would be a relabel at u.
- If we reach the end of the list ($v = \mathrm{null}$) all edges are not admissible. □

This shows that discharge(u) is correct, and that we can perform a relabel in Line 4.

13.2 Relabel to Front

Algorithm 1 relabel-to-front(G, s, t)

```
initialize preflow
initialize node list $L$ containing $V \setminus \{s,t\}$ in any order
foreach $u \in V \setminus \{s,t\}$ do
    $u.current\text{-}neighbour \leftarrow u.neighbour\text{-}list\text{-}head$
$u \leftarrow L.head$
while $u \neq \mathrm{null}$ do
    $old\text{-}height \leftarrow \ell(u)$
    discharge$(u)$
    if $\ell(u) > old\text{-}height$ then // relabel happened
        move $u$ to the front of $L$
    $u \leftarrow u.next$
```

13.2 Relabel to Front

Lemma 76 (Invariant)

In Line 6 of the relabel-to-front algorithm the following invariant holds.

1. *The sequence L is topologically sorted w.r.t. the set of admissible edges; this means for an admissible edge (x, y) the node x appears before y in sequence L.*
2. *No node before u in the list L is active.*

Proof:

- Initialization:
  1. In the beginning s has label $n \geq 2$, and all other nodes have label 0. Hence, no edge is admissible, which means that any ordering L is permitted.
  2. We start with u being the head of the list; hence no node before u can be active
- Maintenance:
  1. - Pushes do no create any new admissible edges. Therefore, if discharge$()$ does not relabel u, L is still topologically sorted.
     - After relabeling, u cannot have admissible incoming edges as such an edge (x,u) would have had a difference $\ell(x) - \ell(u) \geq 2$ before the re-labeling (such edges do not exist in the residual graph).
       Hence, moving u to the front does not violate the sorting property for any edge; however it fixes this property for all admissible edges leaving u that were generated by the relabeling.

13.2 Relabel to Front

Proof:

- Maintenance:
    2. If we do a relabel there is nothing to prove because the only node before u' (u in the next iteration) will be the current u; the discharge(u) operation only terminates when u is not active anymore.

       For the case that we do not relabel, observe that the only way a predecessor could be active is that we push flow to it via an admissible arc. However, all admissible arc point to successors of u.

Note that the invariant means that for $u = \text{null}$ we have a preflow with a valid labelling that does not have active nodes. This means we have a maximum flow.

13.2 Relabel to Front

Lemma 77
There are at most $\mathcal{O}(n^3)$ calls to discharge(u).

Every discharge operation without a relabel advances u (the current node within list L). Hence, if we have n discharge operations without a relabel we have $u = \mathrm{null}$ and the algorithm terminates.

Therefore, the number of calls to discharge is at most $n(\#relabels + 1) = \mathcal{O}(n^3)$.

13.2 Relabel to Front

Lemma 78
The cost for all relabel-operations is only $\mathcal{O}(n^2)$.

A relabel-operation at a node is constant time (increasing the label and resetting $u.\mathit{current\text{-}neighbour}$). In total we have $\mathcal{O}(n^2)$ relabel-operations.

13.2 Relabel to Front

Recall that a saturating push operation ($\min\{c_f(e), f(u)\} = c_f(e)$) can also be a deactivating push operation ($\min\{c_f(e), f(u)\} = f(u)$).

Lemma 79

The cost for all saturating push-operations that are ***not*** *deactivating is only* $\mathcal{O}(mn)$.

Note that such a push-operation leaves the node u active but makes the edge e disappear from the residual graph. Therefore the push-operation is immediately followed by an increase of the pointer $u.\mathit{current\text{-}neighbour}$.

This pointer can traverse the neighbour-list at most $\mathcal{O}(n)$ times (upper bound on number of relabels) and the neighbour-list has only $\mathit{degree}(u) + 1$ many entries ($+1$ for null-entry).

13.2 Relabel to Front

Lemma 80
The cost for all deactivating push-operations is only $\mathcal{O}(n^3)$*.*

A deactivating push-operation takes constant time and ends the current call to discharge(). Hence, there are only $\mathcal{O}(n^3)$ such operations.

Theorem 81
The push-relabel algorithm with the rule relabel-to-front takes time $\mathcal{O}(n^3)$*.*

13.3 Highest Label

Algorithm 1 highest-label(G, s, t)

```
initialize preflow
foreach $u \in V \setminus \{s, t\}$ do
    $u.current\text{-}neighbour \leftarrow u.neighbour\text{-}list\text{-}head$
while $\exists$ active node $u$ do
    select active node $u$ with highest label
    discharge$(u)$
```

13.3 Highest Label

Lemma 82

When using highest label the number of deactivating pushes is only $\mathcal{O}(n^3)$.

A push from a node on level ℓ can only “activate” nodes on levels strictly less than ℓ.

This means, after a deactivating push from u a relabel is required to make u active again.

Hence, after n deactivating pushes without an intermediate relabel there are no active nodes left.

Therefore, the number of deactivating pushes is at most $n(\#relabels + 1) = \mathcal{O}(n^3)$.

13.3 Highest Label

Since a discharge-operation is terminated by a deactivating push this gives an upper bound of $\mathcal{O}(n^3)$ on the number of discharge-operations.

The cost for relabels and saturating pushes can be estimated in exactly the same way as in the case of the generic push-relabel algorithm.

Question:

How do we find the next node for a discharge operation?

13.3 Highest Label

Maintain lists L_i, $i \in \{0, \dots, 2n\}$, where list L_i contains active nodes with label i (maintaining these lists induces only constant additional cost for every push-operation and for every relabel-operation).

After a discharge operation terminated for a node u with label k, traverse the lists $L_k, L_{k-1}, \dots, L_0$, (in that order) until you find a non-empty list.

Unless the last (deactivating) push was to s or t the list $k - 1$ must be non-empty (i.e., the search takes constant time).

13.3 Highest Label

Hence, the total time required for searching for active nodes is at most

$$\mathcal{O}(n^3) + n(\#\mathit{deactivating\text{-}pushes\text{-}to\text{-}s\text{-}or\text{-}t})$$

Lemma 83

The number of deactivating pushes to s or t is at most $\mathcal{O}(n^2)$.

With this lemma we get

Theorem 84

The push-relabel algorithm with the rule highest-label takes time $\mathcal{O}(n^3)$.

13.3 Highest Label

Proof of the Lemma.

- We only show that the number of pushes to the source is at most $\mathcal{O}(n^2)$. A similar argument holds for the target.
- After a node v (which must have $\ell(v) = n + 1$) made a deactivating push to the source there needs to be another node whose label is increased from $\leq n + 1$ to $n + 2$ before v can become active again.
- This happens for every push that v makes to the source. Since, every node can pass the threshold $n + 2$ at most once, v can make at most n pushes to the source.
- As this holds for every node the total number of pushes to the source is at most $\mathcal{O}(n^2)$.

Mincost Flow

Problem Definition:

$$\begin{array}{ll} \text{min} & \sum_e c(e)f(e) \\ \text{s.t.} & \forall e\in E:\;\; 0\le f(e)\le u(e) \\ & \forall v\in V:\;\; f(v)=b(v) \end{array}$$

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- $G = (V, E)$ is a directed graph.

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- $G = (V, E)$ is a directed graph.
- $u : E \to \mathbb{R}_0^+ \cup \{\infty\}$ is the capacity function.

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- $c : E \to \mathbb{R}$ is the cost function (note that $c(e)$ may be negative).

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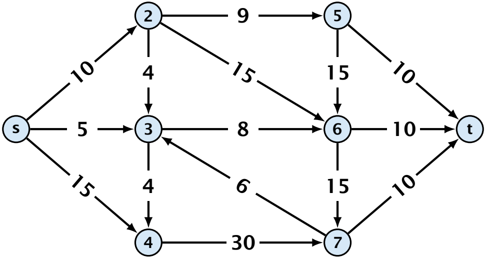
- $c : E \to \mathbb{R}$ is the cost function
  (note that $c(e)$ may be negative).
- $b : V \to \mathbb{R}$, $\sum_{v \in V} b(v) = 0$ is a demand function.

Solve Maxflow Using Mincost Flow



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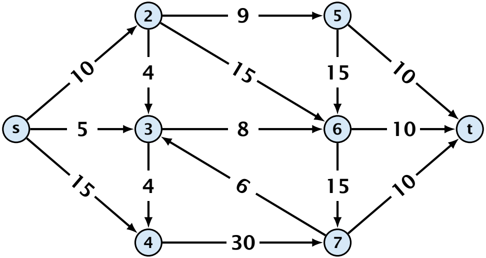

- Given a flow network for a standard maxflow problem.

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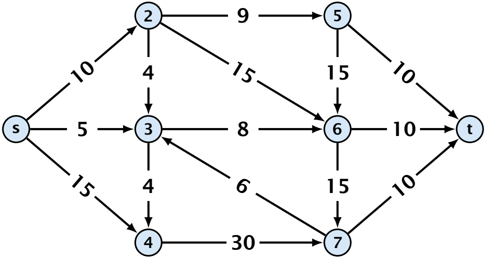
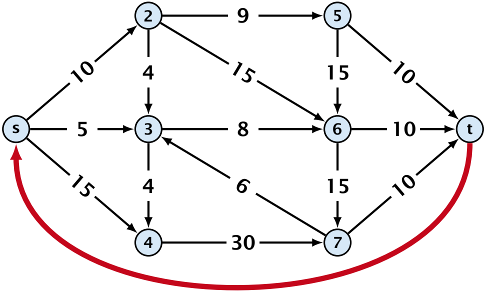


- Given a flow network for a standard maxflow problem.
- Set $b(v) = 0$ for every node. Keep the capacity function u for all edges. Set the cost $c(e)$ for every edge to 0.

Solve Maxflow Using Mincost Flow



- Given a flow network for a standard maxflow problem.
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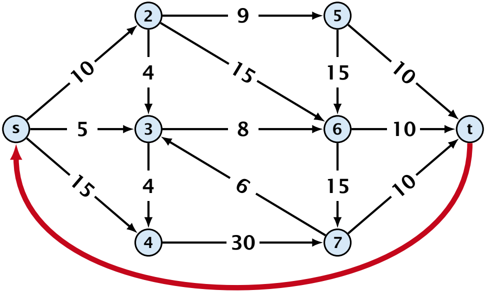

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- Add an edge from t to s with infinite capacity and cost -1.
- Then, $\mathrm{val}(f^*) = -\mathrm{cost}(f_{\min})$, where f^* is a maxflow, and $f_{\min}$ is a mincost-flow.

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Solve decision version of maxflow:

- Given a flow network for a standard maxflow problem, and a value k.

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- Set edge-costs to zero, and keep the capacities.
- There exists a maxflow of value at least k if and only if the mincost-flow problem is feasible.

Generalization

Our model:

$$\begin{array}{ll} \text{min} & \sum_e c(e)f(e) \\ \text{s.t.} & \forall e \in E: \;\; 0 \le f(e) \le u(e) \\ & \forall v \in V: \;\; f(v) = b(v) \end{array}$$

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A more general model?

$$\begin{array}{ll} \text{min} & \sum_e c(e)f(e) \\ \text{s.t.} & \forall e \in E:\ \ \ell(e) \le f(e) \le u(e) \\ & \forall v \in V:\ \ a(v) \le f(v) \le b(v) \end{array}$$

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Generalization

Differences

- Flow along an edge e may have non-zero lower bound $\ell(e)$.
- Flow along e may have negative upper bound $u(e)$.
- The demand at a node v may have lower bound $a(v)$ and upper bound $b(v)$ instead of just lower bound = upper bound = $b(v)$.

Reduction I

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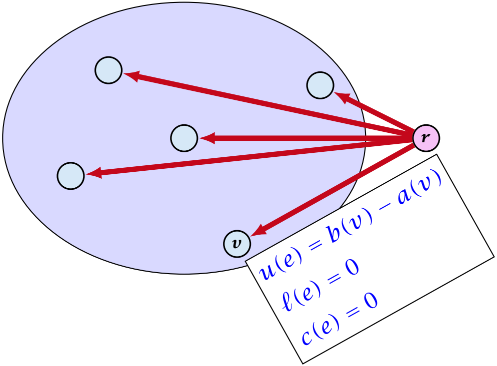

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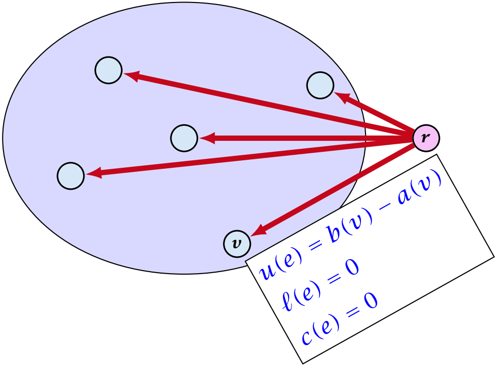

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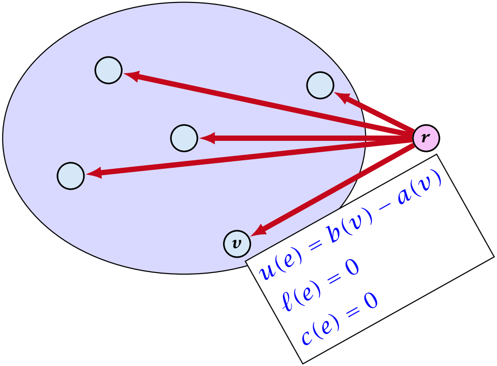

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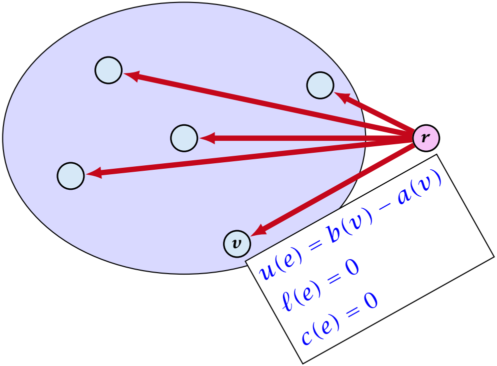

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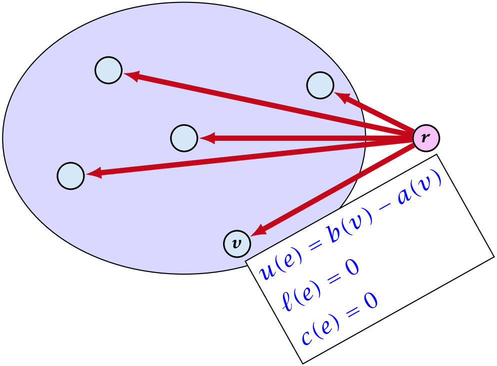

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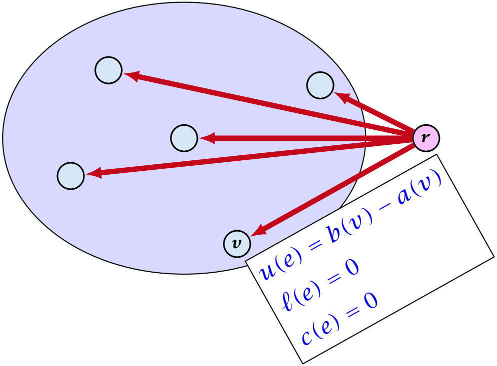

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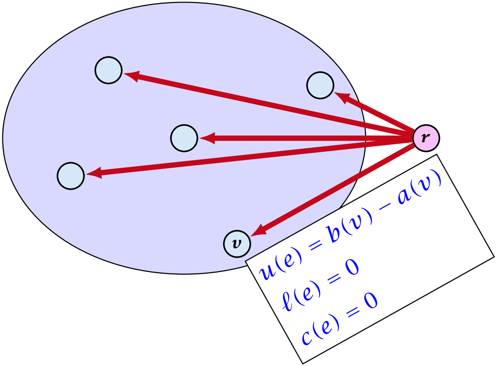

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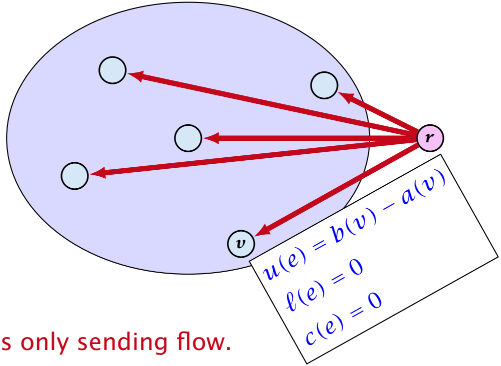
$-\sum_v b(v)$ is negative; hence r is only sending flow.



Reduction II

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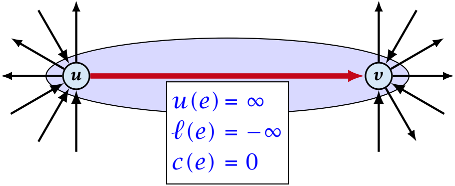

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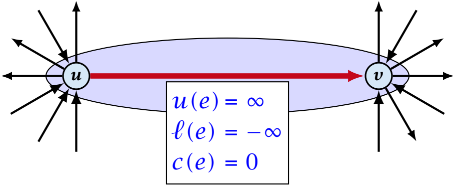

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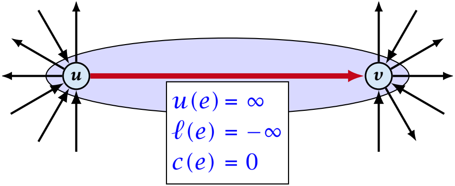
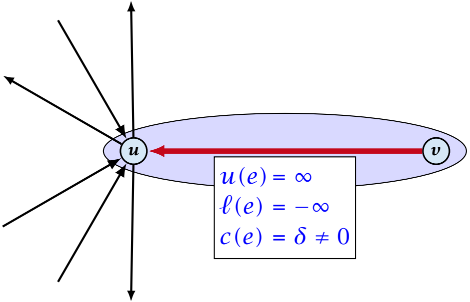


If $c(e) = 0$ we can contract the edge/identify nodes u and v.

If $c(e) \neq 0$ we can transform the graph so that $c(e) = 0$.

Reduction II

We can transform any network so that a particular edge has cost $c(e) = 0$:



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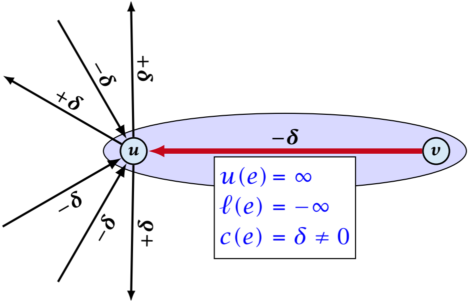

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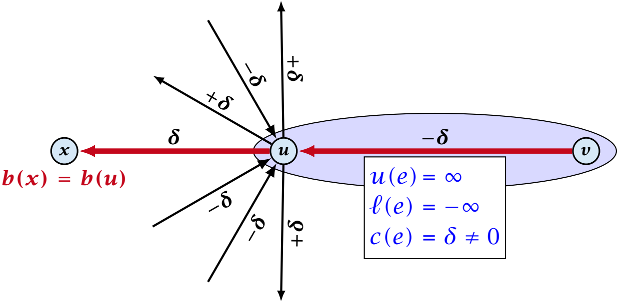

Additionally we set $b(u) = 0$.

Reduction III

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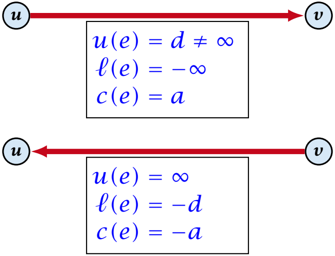
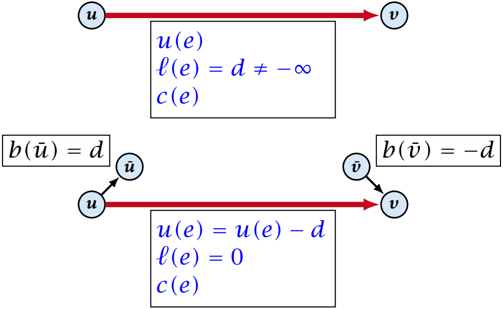


Replace the edge by an edge in opposite direction.

Reduction IV

$$\begin{array}{ll} \text{min} & \sum_e c(e) f(e) \\ \text{s.t.} & \forall e \in E: \;\; \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: \;\; f(v) = b(v) \end{array}$$

We can assume that $\ell(e) = 0$:



The added edges have infinite capacity and cost $c(e)/2$.

Applications

Caterer Problem

- She needs to supply r_i napkins on N successive days.

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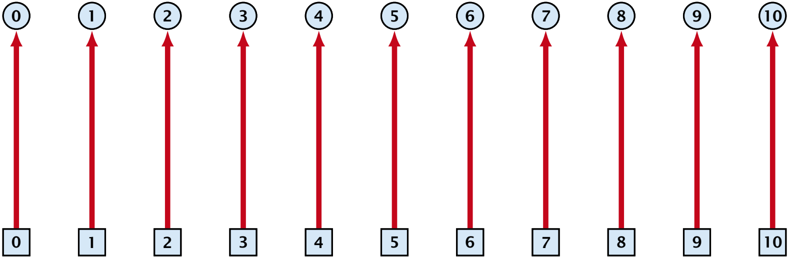
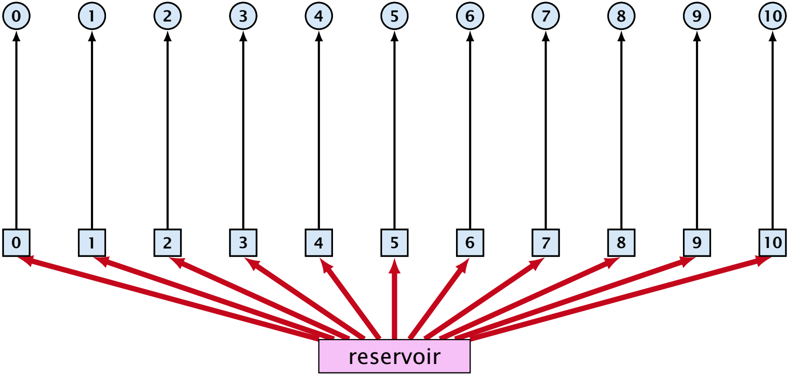
- At the end of each day she should determine how many to send to each laundry and how many to buy in order to fulfill demand.
- Minimize cost.



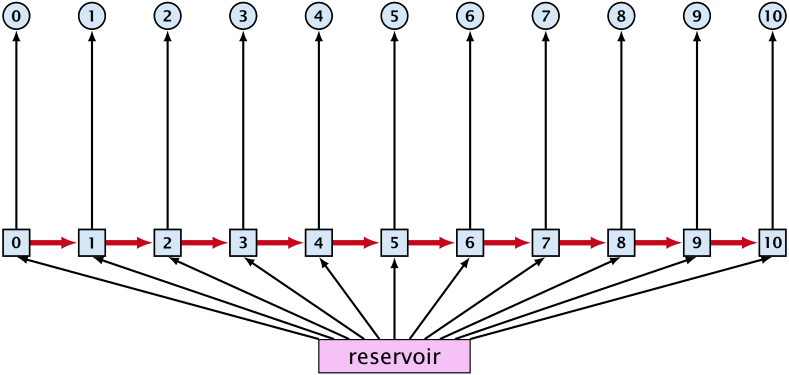
day edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = r_i$;
cost: $c(e) = 0$



buy edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = p$



forward edges:

upper bound: $u(e_i) = \infty$;
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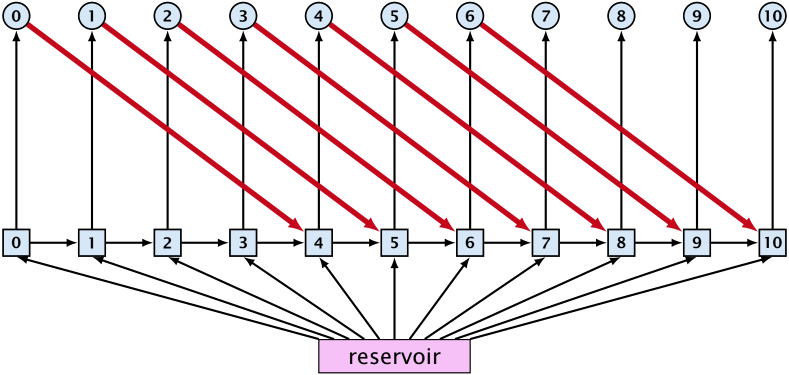

slow edges:

upper bound: $u(e_i) = \infty$;
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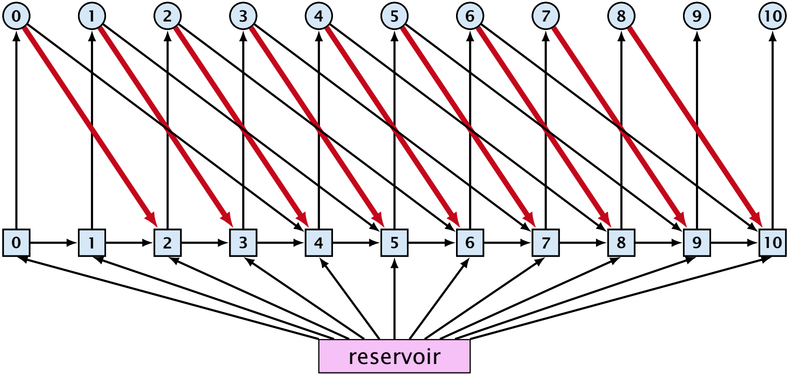
cost: $c(e) = s$



fast edges:

upper bound: $u(e_i) = \infty$;
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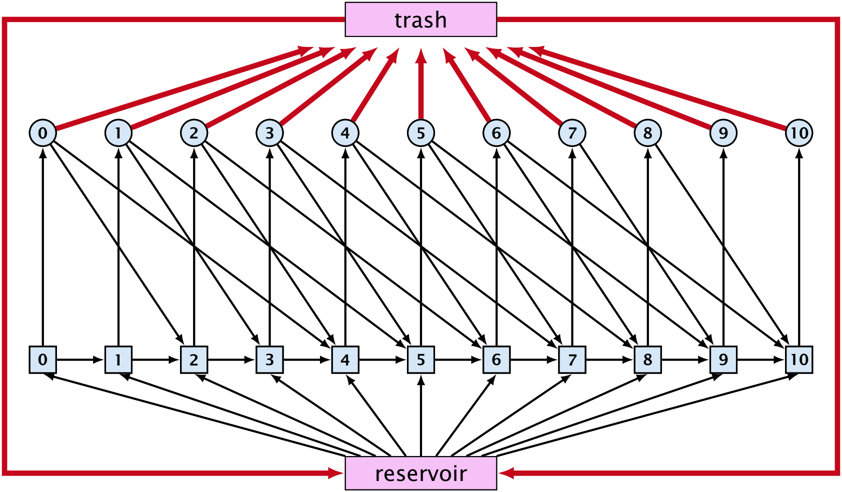
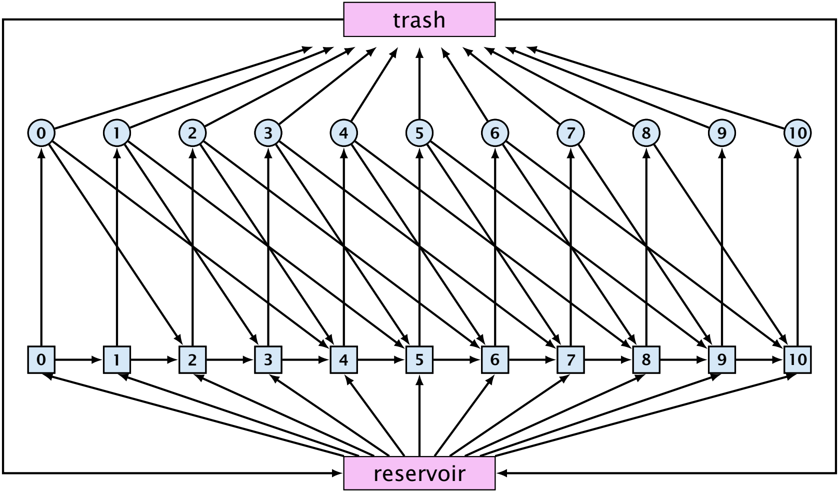
cost: $c(e) = f$



trash edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = 0$


trash
0
1
2
3
4
5
6
7
8
9
10
0
1
2
3
4
5
6
7
8
9
10
reservoir

Residual Graph

Version A:
The residual graph G' for a mincost flow is just a copy of the graph G.

If we send $f(e)$ along an edge, the corresponding edge e' in the residual graph has its lower and upper bound changed to $\ell(e') = \ell(e) - f(e)$ and $u(e') = u(e) - f(e)$.

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Version B:
The residual graph for a mincost flow is exactly defined as the residual graph for standard flows, with the only exception that one needs to define a cost for the residual edge.

For a flow of z from u to v the residual edge (v, u) has capacity z and a cost of $-c((u, v))$.

14 Mincost Flow

A circulation in a graph $G = (V, E)$ is a function $f : E \to \mathbb{R}^+$ that has an excess flow $f(v) = 0$ for every node $v \in V$.

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A circulation is feasible if it fulfills capacity constraints, i.e., $f(e) \le u(e)$ for every edge of G.

Lemma 85

A given flow is a mincost-flow if and only if the corresponding residual graph G_f *does not have a feasible circulation of negative cost.*

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  Then $f + g$ is a feasible flow with cost $\mathrm{cost}(f) + \mathrm{cost}(g) < \mathrm{cost}(f)$. Hence, f is not minimum cost.

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⇐ Let f be a non-mincost flow, and let f^* be a min-cost flow. We need to show that the residual graph has a feasible circulation with negative cost.

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- ⇐ Let f be a non-mincost flow, and let f^* be a min-cost flow. We need to show that the residual graph has a feasible circulation with negative cost.

  Clearly $f^* - f$ is a circulation of negative cost. One can also easily see that it is feasible for the residual graph. (after sending $-f$ in the residual graph (pushing all flow back) we arrive at the original graph; for this f^* is clearly feasible)

14 Mincost Flow

Lemma 86

A graph (without zero-capacity edges) has a feasible circulation of negative cost if and only if it has a negative cycle w.r.t. edge-weights $c : E \to \mathbb{R}$.

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Proof.

- Suppose that we have a negative cost circulation.

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Proof.

- Suppose that we have a negative cost circulation.
- Find directed cycle only using edges that have non-zero flow.

14 Mincost Flow

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A graph (without zero-capacity edges) has a feasible circulation of negative cost if and only if it has a negative cycle w.r.t. edge-weights $c : E \to \mathbb{R}$*.*

Proof.

- Suppose that we have a negative cost circulation.
- Find directed cycle only using edges that have non-zero flow.
- If this cycle has negative cost you are done.

14 Mincost Flow

Lemma 86

A graph (without zero-capacity edges) has a feasible circulation of negative cost if and only if it has a negative cycle w.r.t. edge-weights $c : E \to \mathbb{R}$.

Proof.

- Suppose that we have a negative cost circulation.
- Find directed cycle only using edges that have non-zero flow.
- If this cycle has negative cost you are done.
- Otherwise send flow in opposite direction along the cycle until the bottleneck edge(s) does not carry any flow.

14 Mincost Flow

Lemma 86

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- Suppose that we have a negative cost circulation.
- Find directed cycle only using edges that have non-zero flow.
- If this cycle has negative cost you are done.
- Otherwise send flow in opposite direction along the cycle until the bottleneck edge(s) does not carry any flow.
- You still have a circulation with negative cost.

14 Mincost Flow

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- Suppose that we have a negative cost circulation.
- Find directed cycle only using edges that have non-zero flow.
- If this cycle has negative cost you are done.
- Otherwise send flow in opposite direction along the cycle until the bottleneck edge(s) does not carry any flow.
- You still have a circulation with negative cost.
- Repeat.

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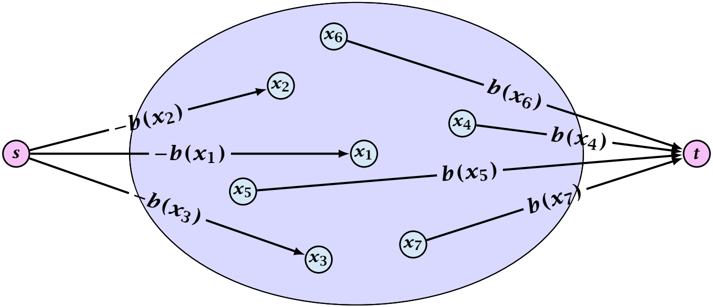
Algorithm 48 CycleCanceling$(G = (V, E), c, u, b)$

```
establish a feasible flow $f$ in $G$
while $G_f$ contains negative cycle do
    use Bellman-Ford to find a negative circuit $Z$
    $\delta \leftarrow \min\{u_f(e) \mid e \in Z\}$
    augment $\delta$ units along $Z$ and update $G_f$
```

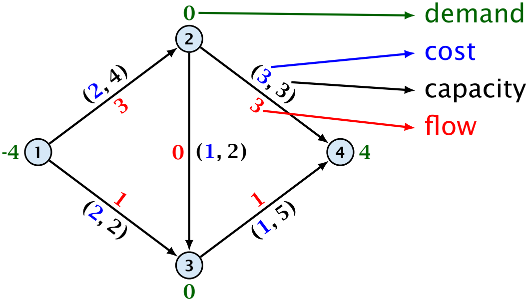
How do we find the initial feasible flow?



- Connect new node s to all nodes with negative $b(v)$-value.
- Connect nodes with positive $b(v)$-value to a new node t.
- There exist a feasible flow in the original graph iff in the resulting graph there exists an s-t flow of value

$$\sum_{v:b(v)<0} (-b(v)) = \sum_{v:b(v)>0} b(v) \ .$$

14 Mincost Flow



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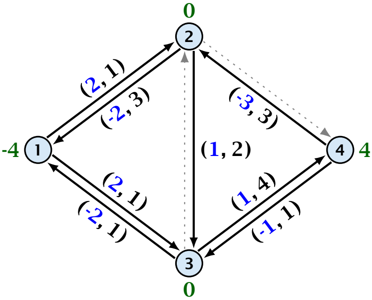

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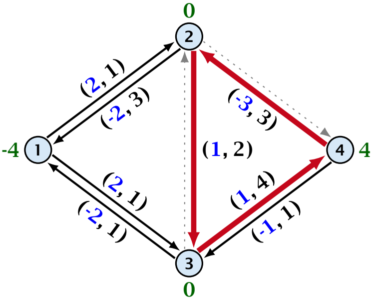

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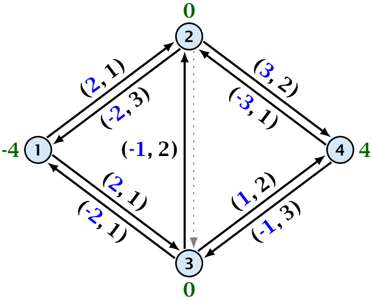

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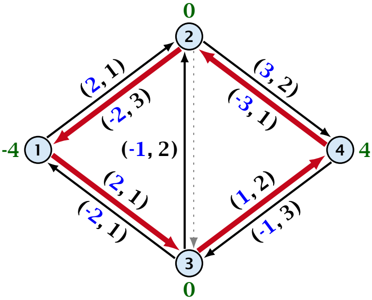

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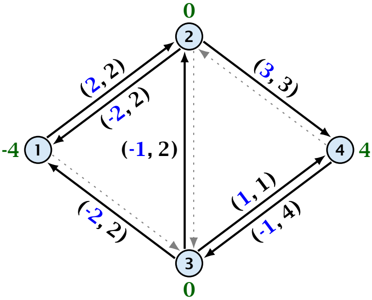

14 Mincost Flow

Lemma 87

The improving cycle algorithm runs in time $\mathcal{O}(nm^2CU)$, for integer capacities and costs, when for all edges e, $|c(e)| \le C$ and $|u(e)| \le U$.

- Running time of Bellman-Ford is $\mathcal{O}(mn)$.
- Pushing flow along the cycle can be done in time $\mathcal{O}(n)$.
- Each iteration decreases the total cost by at least 1.
- The true optimum cost must lie in the interval $[-mCU, \dots, +mCU]$.

Note that this lemma is weak since it does not allow for edges with infinite capacity.

14 Mincost Flow

A general mincost flow problem is of the following form:

$$\begin{array}{ll} \text{min} & \sum_e c(e)f(e) \\ \text{s.t.} & \forall e\in E:\ \ \ell(e)\le f(e)\le u(e) \\ & \forall v\in V:\ \ a(v)\le f(v)\le b(v) \end{array}$$

where $a:V\to\mathbb{R}$, $b:V\to\mathbb{R}$; $\ell:E\to\mathbb{R}\cup\{-\infty\}$, $u:E\to\mathbb{R}\cup\{\infty\}$ $c:E\to\mathbb{R}$;

Lemma 88 (without proof)

A general mincost flow problem can be solved in polynomial time.