

WS 2024/25

Efficient Algorithms

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TU München

<https://www.moodle.tum.de/course/view.php?id=100478>

Winter Term 2024/25

Part I

Organizational Matters

Part I

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- ▶ Modul: IN2003
- ▶ Name: “Efficient Algorithms and Data Structures”
“Effiziente Algorithmen und Datenstrukturen”
- ▶ ECTS: 8 Credit points
- ▶ Lectures:
 - ▶ 4 SWS
Mon 10:00–12:00 (Room Interim2)
Fri 10:00–12:00 (Room Interim2)
- ▶ Webpage:
<https://www.moodle.tum.de/course/view.php?id=100478>

- ▶ Required knowledge:
 - ▶ IN0001, IN0003
“**Introduction to Informatics 1/2**”
“Einführung in die Informatik 1/2”
 - ▶ IN0007
“**Fundamentals of Algorithms and Data Structures**”
“Grundlagen: Algorithmen und Datenstrukturen” (GAD)
 - ▶ IN0011
“**Basic Theoretic Informatics**”
“Einführung in die Theoretische Informatik” (THEO)
 - ▶ IN0015
“**Discrete Structures**”
“Diskrete Strukturen” (DS)
 - ▶ IN0018
“**Discrete Probability Theory**”
“Diskrete Wahrscheinlichkeitstheorie” (DWT)

The Lecturer

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1 Contents

- ▶ Foundations
 - ▶ Machine models
 - ▶ Efficiency measures
 - ▶ Asymptotic notation
 - ▶ Recursion

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- ▶ Matchings

2 Literatur

-  Alfred V. Aho, John E. Hopcroft, Jeffrey D. Ullman:
The design and analysis of computer algorithms,
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-  Thomas H. Cormen, Charles E. Leiserson, Ron L. Rivest,
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-  Ronald L. Graham, Donald E. Knuth, Oren Patashnik:
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The Algorithm Design Manual,

Springer, 1998

Part II

Foundations

3 Goals

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- ▶ Learn how to design efficient algorithms.

4 Modelling Issues

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- ▶ Implementing and testing on representative inputs
 - ▶ How do you choose your inputs?
 - ▶ May be very time-consuming.
 - ▶ Very reliable results if done correctly.
 - ▶ Results only hold for a specific machine and for a specific set of inputs.

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How do you measure?

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 - ▶ May be very time-consuming.
 - ▶ Very reliable results if done correctly.
 - ▶ Results only hold for a specific machine and for a specific set of inputs.
- ▶ Theoretical analysis in a specific **model of computation**.
 - ▶ Gives **asymptotic bounds** like “this algorithm always runs in time $\mathcal{O}(n^2)$ ”.
 - ▶ Typically focuses on the **worst case**.
 - ▶ Can give lower bounds like “any comparison-based sorting algorithm needs at least $\Omega(n \log n)$ comparisons in the worst case”.

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Example 1

Suppose n numbers from the interval $\{1, \dots, N\}$ have to be sorted. In this case we usually say that the input length is n instead of e.g. $n \log N$, which would be the number of bits required to encode the input.

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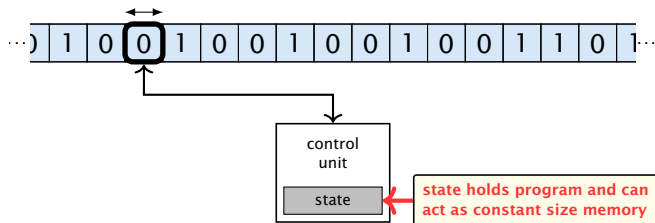
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Version 2. is often easier, but focusing on one type of operation makes it more difficult to obtain meaningful results.

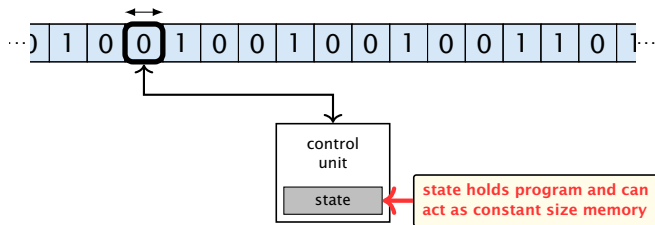
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- ▶ Very simple model of computation.



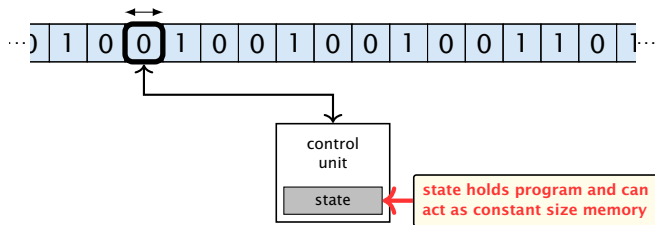
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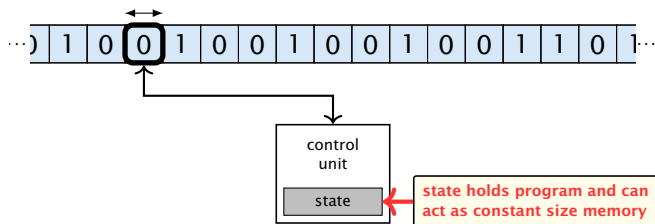
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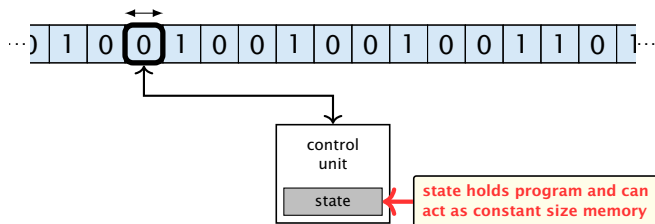
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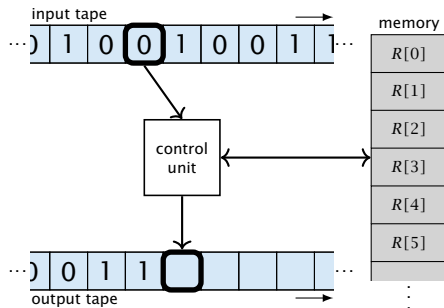
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⇒ **Not a good model for developing efficient algorithms.**



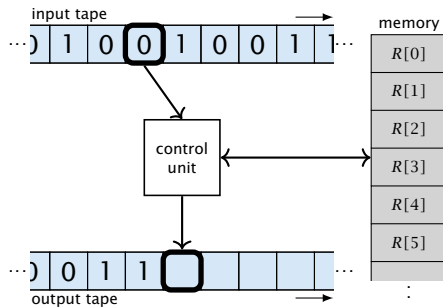
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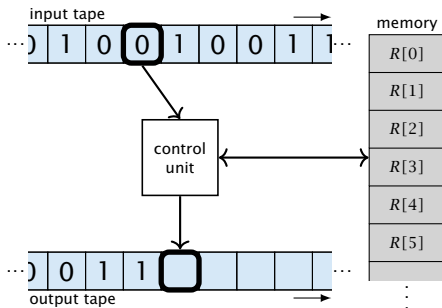
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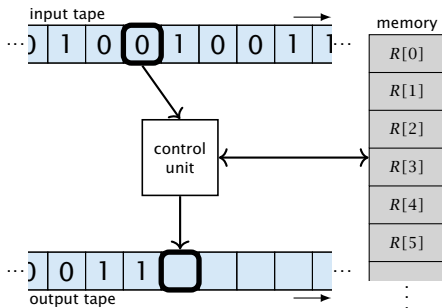
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 - ▶ $R[i] := R[j] + R[k];$
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Bounded word RAM model: cost is uniform but the largest value stored in a register may not exceed 2^w , where usually $w = \log_2 n$.

4 Modelling Issues

Example 2

Algorithm 1 RepeatedSquaring(n)

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more general: probability measure μ

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▶ **randomized** complexity:

The algorithm may use random bits. Expected running time (over all possible choices of random bits) for a fixed input x . Then take the worst-case over all x with $|x| = n$.

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- ▶ Running time should be expressed by simple functions.

Asymptotic Notation

Formal Definition

Let f, g denote functions from \mathbb{N} to \mathbb{R}^+ .

- ▶ $\mathcal{O}(f) = \{g \mid \exists c > 0 \exists n_0 \in \mathbb{N}_0 \forall n \geq n_0 : [g(n) \leq c \cdot f(n)]\}$
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There is an equivalent definition using limes notation (**assuming that the respective limes exists**). f and g are functions from \mathbb{N}_0 to \mathbb{R}_0^+ .

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3. People write e.g. $h(n) = f(n) + o(g(n))$ when they mean that there exists a function $z : \mathbb{N} \rightarrow \mathbb{R}^+, n \mapsto z(n), z \in o(g)$ such that $h(n) = f(n) + z(n)$.

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4. People write $\mathcal{O}(f(n)) = \mathcal{O}(g(n))$, when they mean $\mathcal{O}(f(n)) \subseteq \mathcal{O}(g(n))$. Again this is not an equality.

Asymptotic Notation in Equations

How do we interpret an expression like:

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Note that $\Theta(n)$ is on the right hand side, otw. this interpretation is wrong.

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How do we interpret an expression like:

$$2n^2 + \mathcal{O}(n) = \Theta(n^2)$$

Regardless of how we choose the anonymous function $f(n) \in \mathcal{O}(n)$ there is an anonymous function $g(n) \in \Theta(n^2)$ that makes the expression true.

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Careful!

“It is understood” that every occurrence of an Θ -symbol (or $\Theta, \Omega, o, \omega$) on the left represents **one anonymous function**.

Hence, the left side is **not** equal to

$$\Theta(1) + \Theta(2) + \dots + \Theta(n-1) + \Theta(n)$$

Asymptotic Notation in Equations

We can view an expression containing asymptotic notation as generating a set:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n)$$

represents

$$\left\{ f : \mathbb{N} \rightarrow \mathbb{R}^+ \mid f(n) = n^2 \cdot g(n) + h(n) \right. \\ \left. \text{with } g(n) \in \mathcal{O}(n) \text{ and } h(n) \in \mathcal{O}(\log n) \right\}$$

Asymptotic Notation in Equations

Then an asymptotic equation can be interpreted as containment btw. two sets:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) = \Theta(n^2)$$

represents

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) \subseteq \Theta(n^2)$$

Asymptotic Notation

Lemma 3

Let f, g be functions with the property

$\exists n_0 > 0 \forall n \geq n_0 : f(n) > 0$ (the same for g). Then

- ▶ $c \cdot f(n) \in \Theta(f(n))$ for any constant c

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The expressions also hold for Ω . Note that this means that $f(n) + g(n) \in \Theta(\max\{f(n), g(n)\})$.

Asymptotic Notation

Comments

- ▶ Do not use asymptotic notation within induction proofs.

Asymptotic Notation

Comments

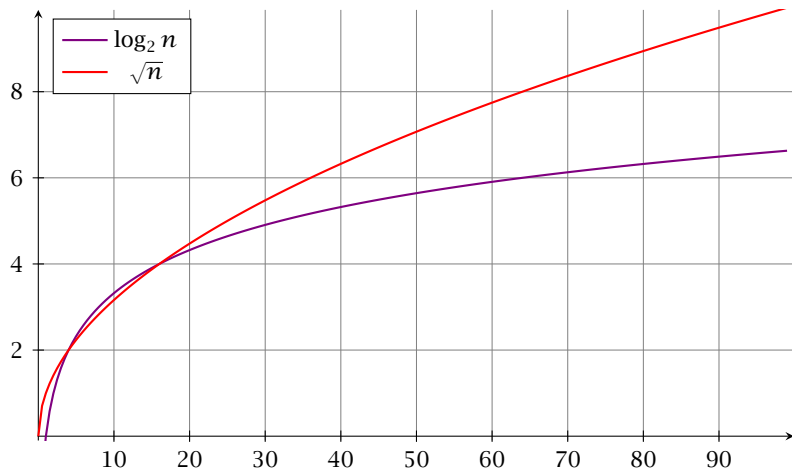
- ▶ Do not use asymptotic notation within induction proofs.
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Therefore, we will usually ignore the base of a logarithm within asymptotic notation.

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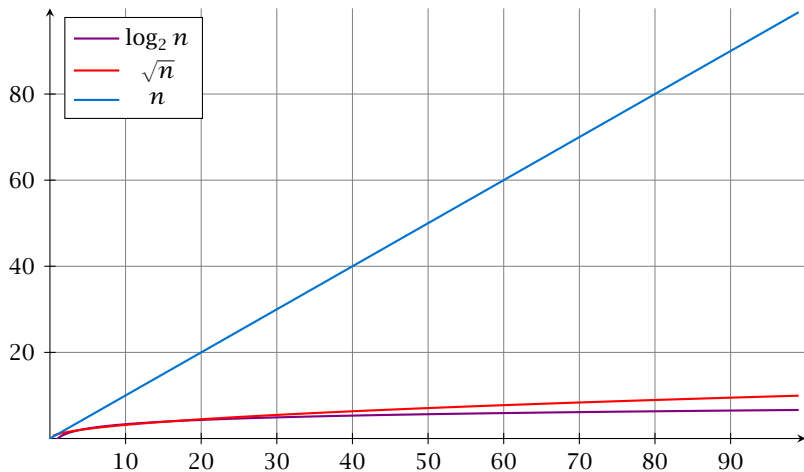
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- ▶ In general $\log n = \log_2 n$, i.e., we use 2 as the default base for the logarithm.

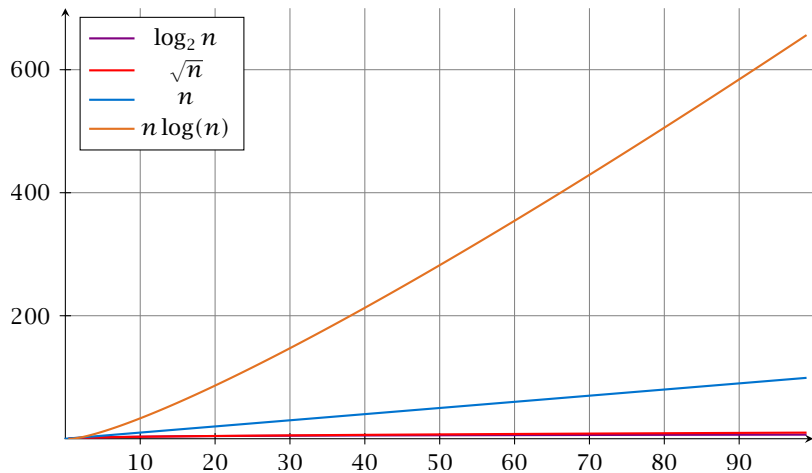
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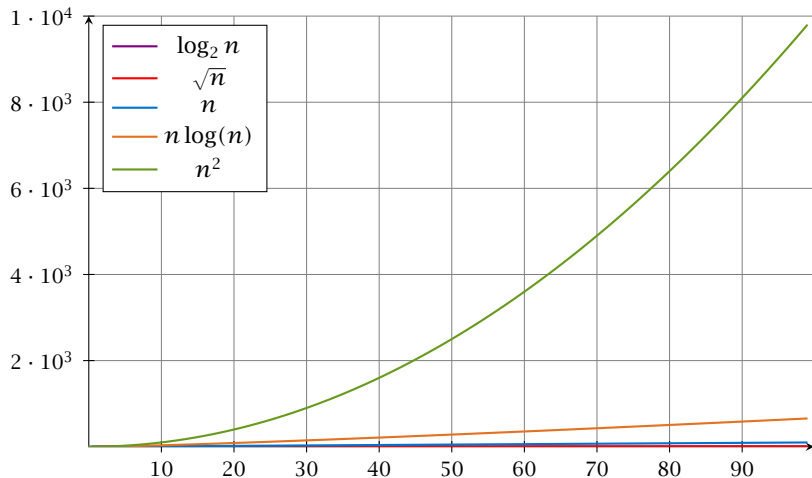
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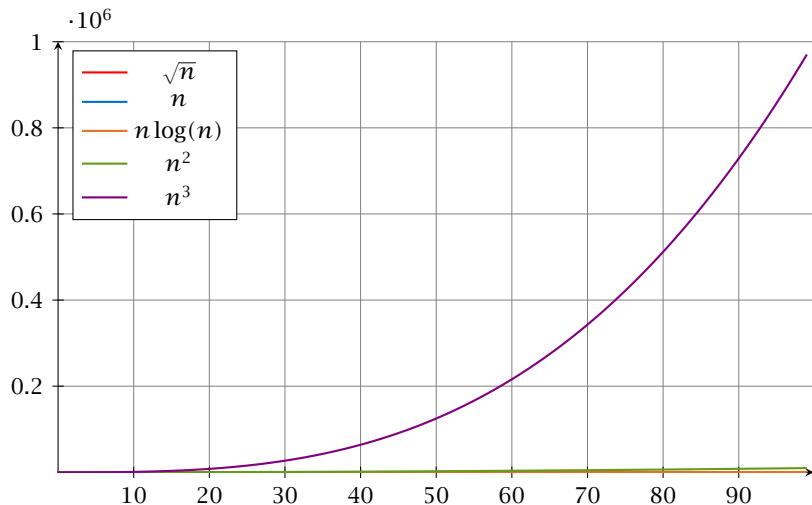
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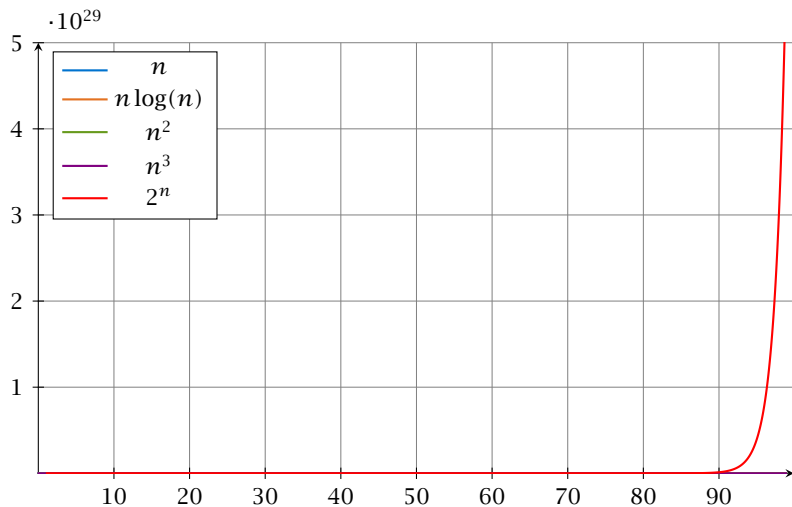
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Funktionen



Laufzeiten

Funktion	Eingabelänge n							
	10	10^2	10^3	10^4	10^5	10^6	10^7	10^8
$\log n$	33ns	66ns	0.1 μ s	0.1 μ s	0.2 μ s	0.2 μ s	0.2 μ s	0.3 μ s
\sqrt{n}	32ns	0.1 μ s	0.3 μ s	1 μ s	3.1 μ s	10 μ s	31 μ s	0.1ms
n	100ns	1 μ s	10 μ s	0.1ms	1ms	10ms	0.1s	1s
$n \log n$	0.3 μ s	6.6 μ s	0.1ms	1.3ms	16ms	0.2s	2.3s	27s
$n^{3/2}$	0.3 μ s	10 μ s	0.3ms	10ms	0.3s	10s	5.2min	2.7h
n^2	1 μ s	0.1ms	10ms	1s	1.7min	2.8h	11d	3.2y
n^3	10 μ s	10ms	10s	2.8h	115d	317y	$3.2 \cdot 10^5$ y	
1.1^n	26ns	0.1ms	$7.8 \cdot 10^{25}$ y					
2^n	10 μ s	$4 \cdot 10^{14}$ y						
$n!$	36ms	$3 \cdot 10^{142}$ y						

1 Operation = 10ns; 100MHz

Alter des Universums: ca. $13.8 \cdot 10^9$ y

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In general asymptotic classification of running times is a good measure for comparing algorithms:

- ▶ If the running time analysis is tight and actually occurs in practise (i.e., the asymptotic bound is not a purely theoretical worst-case bound), then the algorithm that has better asymptotic running time will always outperform a weaker algorithm for large enough values of n .

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Clearly $f = o(g)$. However, as long as $\log n \leq 1000$ Algorithm B will be more efficient.

Multiple Variables in Asymptotic Notation

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6 Recurrences

Algorithm 2 mergesort(list L)

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1:  $n \leftarrow \text{size}(L)$ 
2: if  $n \leq 1$  return  $L$ 
3:  $L_1 \leftarrow L[1 \cdots \lfloor \frac{n}{2} \rfloor]$ 
4:  $L_2 \leftarrow L[\lfloor \frac{n}{2} \rfloor + 1 \cdots n]$ 
5: mergesort( $L_1$ )
6: mergesort( $L_2$ )
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This algorithm requires

$$T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \mathcal{O}(n) \leq 2T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \mathcal{O}(n)$$

comparisons when $n > 1$ and 0 comparisons when $n \leq 1$.

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How do we bring the expression for the number of comparisons (\approx running time) into a **closed form**?

For this we need to **solve** the recurrence.

Methods for Solving Recurrences

1. Guessing+Induction

Guess the right solution and prove that it is correct via induction. It needs experience to make the right guess.

2. Master Theorem

For a lot of recurrences that appear in the analysis of algorithms this theorem can be used to obtain tight asymptotic bounds. It does not provide exact solutions.

3. Characteristic Polynomial

Linear homogenous recurrences can be solved via this method.

4. Generating Functions

A more general technique that allows to solve certain types of linear inhomogenous relations and also sometimes non-linear recurrence relations.

5. Transformation of the Recurrence

Sometimes one can transform the given recurrence relations so that it e.g. becomes linear and can therefore be solved with one of the other techniques.

6.1 Guessing+Induction

First we need to get rid of the \mathcal{O} -notation in our recurrence:

$$T(n) \leq \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

Informal way:

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Informal way:

Assume that instead we have

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

One way of solving such a recurrence is to **guess** a solution, and check that it is correct by plugging it in.

6.1 Guessing+Induction

Suppose we guess $T(n) \leq dn \log n$ for a constant d .

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Suppose we guess $T(n) \leq dn \log n$ for a constant d . Then

$$\begin{aligned}T(n) &\leq 2T\left(\frac{n}{2}\right) + cn \\ &\leq 2\left(d\frac{n}{2}\log\frac{n}{2}\right) + cn\end{aligned}$$

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$$\begin{aligned}T(n) &\leq 2T\left(\frac{n}{2}\right) + cn \\&\leq 2\left(d\frac{n}{2}\log\frac{n}{2}\right) + cn \\&= dn(\log n - 1) + cn\end{aligned}$$

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if we choose $d \geq c$.

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if we choose $d \geq c$.

Formally, this is not correct if n is not a power of 2. Also even in this case one would need to do an induction proof.

6.1 Guessing+Induction

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

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Proof. (by induction)

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- ▶ **base case** ($2 \leq n < 16$):

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- ▶ **base case** ($2 \leq n < 16$): **true** if we choose $d \geq b$.
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Let $n = 2^k \geq 16$. Suppose statem. is true for $n' = n/2$. We prove it for n :

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Hence, statement is **true** if we choose $d \geq c$.

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$$T(n) \leq \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 16 \\ b & \text{otherwise} \end{cases}$$

Note that we can do this as for constant-sized inputs the running time is always some constant (b in the above case).

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$$\boxed{\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1} \leq 2(d(n/2 + 1) \log(n/2 + 1)) + cn$$

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We also make a guess of $T(n) \leq dn \log n$ and get

$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

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$$\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1$$

$$\leq 2\left(d\left(\frac{n}{2} + 1\right) \log\left(\frac{n}{2} + 1\right)\right) + cn$$

$$\frac{n}{2} + 1 \leq \frac{9}{16}n$$

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$$\log \frac{9}{16}n = \log n + (\log 9 - 4)$$

$$= dn \log n + (\log 9 - 4)dn + 2d \log n + cn$$

$$\log n \leq \frac{n}{4}$$

$$\leq dn \log n + (\log 9 - 3.5)dn + cn$$

6.1 Guessing+Induction

We also make a guess of $T(n) \leq dn \log n$ and get

$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1$$

$$\leq 2\left(d\left(\frac{n}{2} + 1\right) \log\left(\frac{n}{2} + 1\right)\right) + cn$$

$$\frac{n}{2} + 1 \leq \frac{9}{16}n$$

$$\leq dn \log\left(\frac{9}{16}n\right) + 2d \log n + cn$$

$$\log \frac{9}{16}n = \log n + (\log 9 - 4)$$

$$= dn \log n + (\log 9 - 4)dn + 2d \log n + cn$$

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$$\leq dn \log n - 0.33dn + cn$$

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We also make a guess of $T(n) \leq dn \log n$ and get

$$\begin{aligned}T(n) &\leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn \\&\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn \\&\leq 2\left(d\left(\frac{n}{2} + 1\right) \log\left(\frac{n}{2} + 1\right)\right) + cn \\&\leq dn \log\left(\frac{9}{16}n\right) + 2d \log n + cn \\&= dn \log n + (\log 9 - 4)dn + 2d \log n + cn \\&\leq dn \log n + (\log 9 - 3.5)dn + cn \\&\leq dn \log n - 0.33dn + cn \\&\leq dn \log n\end{aligned}$$

$$\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1$$

$$\frac{n}{2} + 1 \leq \frac{9}{16}n$$

$$\log \frac{9}{16}n = \log n + (\log 9 - 4)$$

$$\log n \leq \frac{n}{4}$$

for a suitable choice of d .

6.2 Master Theorem

Lemma 5

Let $a \geq 1$, $b > 1$ and $\epsilon > 0$ denote constants. Consider the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) .$$

Case 1.

If $f(n) = \mathcal{O}(n^{\log_b(a)-\epsilon})$ then $T(n) = \Theta(n^{\log_b a})$.

Case 2.

If $f(n) = \Theta(n^{\log_b(a)} \log^k n)$ then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$,
 $k \geq 0$.

Case 3.

If $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ and for sufficiently large n
 $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$ then $T(n) = \Theta(f(n))$.

6.2 Master Theorem

We prove the Master Theorem for the case that n is of the form b^ℓ , and we assume that the non-recursive case occurs for problem size 1 and incurs cost 1.

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

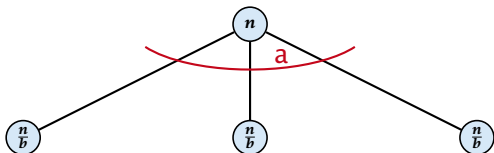
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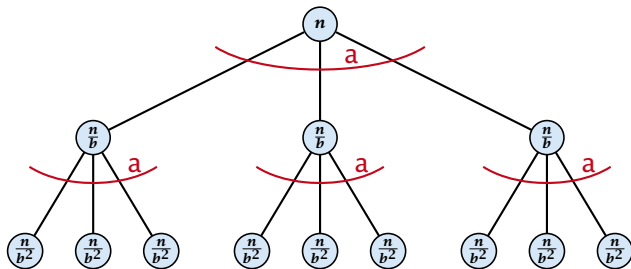
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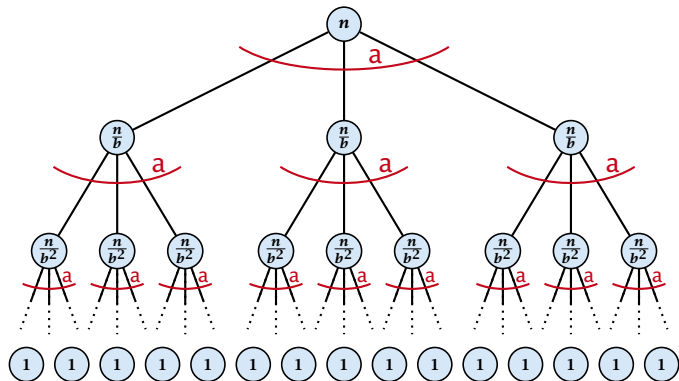
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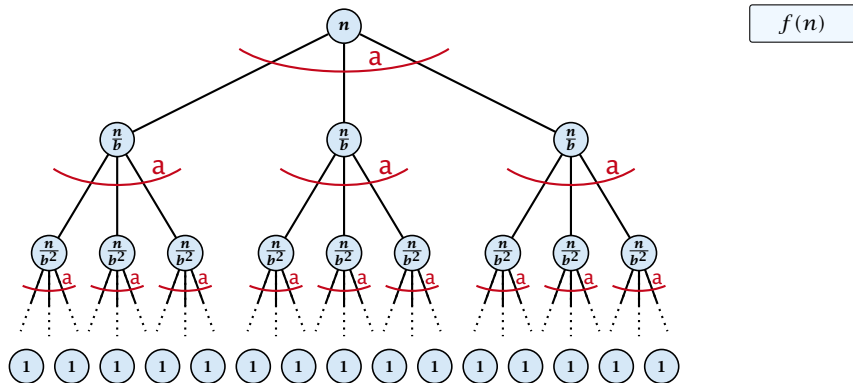
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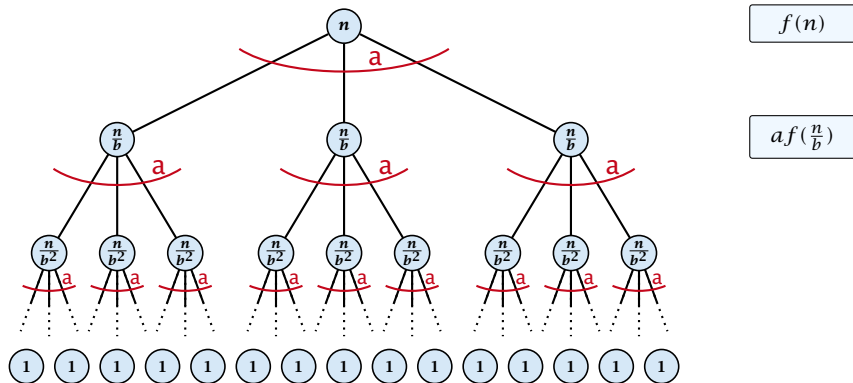
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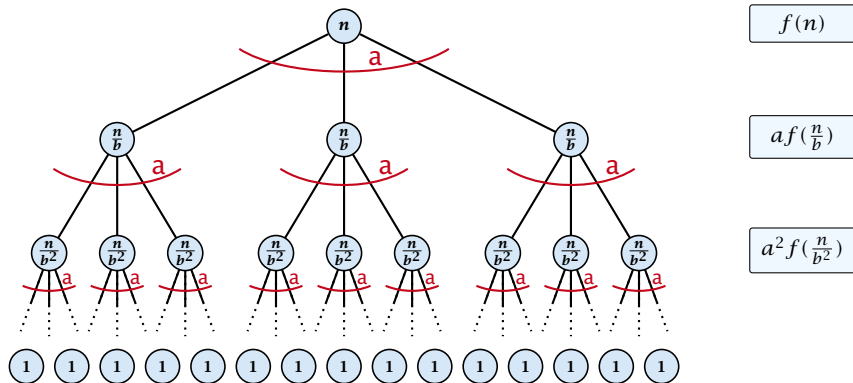
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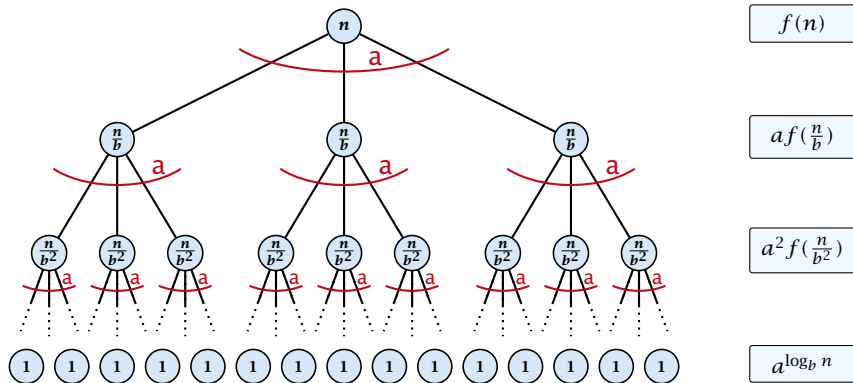
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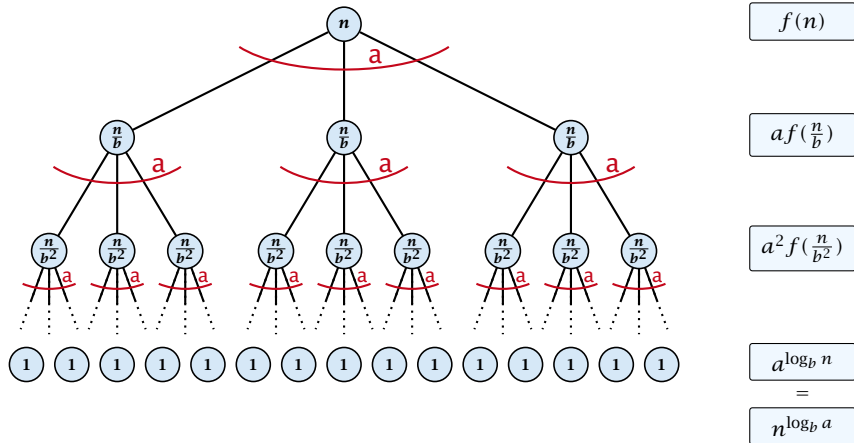
The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) .$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \end{aligned}$$

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$$b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}$$

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Hence,

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$$\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n).$$

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Hence,

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Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1 \\ \hline \end{array}$$

The diagram shows two 9-bit integers, A and B , aligned for addition. Integer A is represented by the red bits 1 1 0 1 1 0 1 0 1, and integer B is represented by the blue bits 1 0 0 0 1 0 0 1 1. A horizontal line is drawn under the bits of B . A vertical light blue box highlights the rightmost bit of A (the least significant bit) and the bit of B directly below it, indicating the first step of the addition process.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
								1	
								0	

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & & & & 1 & & \\ & & & & & & & & 0 & \end{array}$$

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>								0	0

The diagram illustrates the addition of two 10-bit integers, A and B. The bits of A are 1 1 0 1 1 0 1 0 1 and the bits of B are 1 0 0 0 1 0 0 1 1. A horizontal line is drawn under the 7th bit of B. A vertical box highlights the 8th and 9th bits of both numbers, which are 0 and 1 in A, and 1 and 1 in B. Below the line, the result of the addition for these two bits is shown as 0 and 0.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
						1	1	0	0

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>							0	0	0

The diagram illustrates the addition of two integers, A and B, using a register of constant size. The integers are represented as binary strings: A = 110110101 and B = 100010011. The addition is performed bit-by-bit, with the result shown below a horizontal line. The result is 000. The bits of the result are shown in a light blue box, indicating that the register can only hold a constant number of bits.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
					1	1	1		
						0	0	0	

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
					0	1	1	1	
					1	0	0	0	

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
				0	1	1	1		
					1	0	0	0	

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & 0 & 1 & 0 & 0 & 0 & \end{array}$$

The diagram illustrates the addition of two 9-bit integers, A and B. The bits of A are 1 1 0 1 1 0 1 0 1 and the bits of B are 1 0 0 0 1 0 0 1 1. A horizontal line is drawn under the bits of B. The result of the addition is shown below the line, with the bits 0 1 0 0 0. A vertical box highlights the 5th bit of the result, which is 0. This bit is the result of adding the 5th bits of A and B (1 + 1) and the carry-in from the 4th bit (1). The carry-out from the 5th bit is 1, which is the carry-in for the 6th bit.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
			1	0	1	1	1		
			0	1	0	0	0		

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & 0 & 0 & 1 & 0 & 0 & 0 & \end{array}$$

The diagram illustrates the addition of two 9-bit integers, A and B, to produce a 9-bit result. The numbers are aligned to the right. A vertical light blue box highlights the 4th bit position (from the right). Below the horizontal line, the 4th bit of the result is 0. Small subscripts are placed below the 4th bit of B (0) and the 4th bit of the result (0), and the 5th bit of the result (1).

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
			0	0	1	0	0	0	

The diagram illustrates the addition of two 8-bit integers, A and B. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1 and the bits of B are 1, 0, 0, 0, 1, 0, 0, 1, 1. A vertical box highlights the carry propagation from the 3rd bit to the 8th bit. The carry bits are 1, 1, 0, 1, 1, 1, 1. The result of the addition is 0, 0, 1, 0, 0, 0, 0.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & B \\ \hline & & 1 & 0 & 0 & 1 & 0 & 0 & 0 & \\ \end{array}$$

The diagram illustrates the addition of two 9-bit integers, A and B, using a ripple carry adder. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1 and the bits of B are 1, 0, 0, 0, 1, 0, 0, 0, 1. A vertical box highlights the carry propagation from the second bit to the third bit. Small subscripts below the bits indicate the carry-in for each bit position: 0 for the first bit, and 1 for the second, third, fourth, fifth, sixth, seventh, and eighth bits. The resulting sum is 1, 0, 0, 1, 0, 0, 0, 0.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
		1	0	0	1	0	0	0	

0 1 1 0 1 1 1

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

	1	1	0	1	1	0	1	0	1	A
	1	0	0	0	1	0	0	1	1	B
	0	0	1	1	0	1	1	1		
	1	1	0	0	1	0	0	0		

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

	1	1	0	1	1	0	1	0	1	A
	1	0	0	0	1	0	0	1	1	B
	0	0	1	1	0	1	1	1		
		1	1	0	0	1	0	0	0	

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

	1	1	0	1	1	0	1	0	1	A
	1	0	0	0	1	0	0	1	1	B
1	0	0	1	1	0	1	1	1		
	0	1	1	0	0	1	0	0	0	

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

	1	1	0	1	1	0	1	0	1	A
	1	0	0	0	1	0	0	1	1	B
1	0	0	1	1	0	1	1	1		
	0	1	1	0	0	1	0	0	0	

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

	1	1	0	1	1	0	1	0	1	A
	1	0	0	0	1	0	0	1	1	B
	<hr/>									
	1	0	1	1	0	0	1	0	0	0

The diagram illustrates the addition of two integers A and B. A light blue vertical box on the left contains a '1' at the bottom, representing the carry-in. The numbers A and B are aligned to the right. A horizontal line is drawn under the digits of B. The result of the addition is shown in the bottom row.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0 \end{array}$$

This gives that two n -bit integers can be added in time $\mathcal{O}(n)$.

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 10001 \\ \times 1011 \\ \hline \end{array}$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 10001 \times 1011 \\ \hline \end{array}$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 10001 \times 1011 \\ \hline 10001 \end{array}$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \end{array}$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 0 \\ 0 \\ 0 \end{array}$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \end{array}$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \end{array}$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0 \end{array}$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \end{array}$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \end{array}$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0 \end{array}$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \end{array}$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline \end{array}$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1 \end{array}$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1 \end{array}$$

Time requirement:

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1 \end{array}$$

Time requirement:

- ▶ Computing intermediate results: $\mathcal{O}(nm)$.

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1 \end{array}$$

Time requirement:

- ▶ Computing intermediate results: $\mathcal{O}(nm)$.
- ▶ Adding m numbers of length $\leq 2n$: $\mathcal{O}((m+n)m) = \mathcal{O}(nm)$.

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .

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Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .

$$\boxed{b_{n-1} \quad \dots \quad b_0} \times \boxed{a_{n-1} \quad \dots \quad a_0}$$

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .

$$\boxed{b_{n-1} \quad \cdots \quad b_{\frac{n}{2}} \quad b_{\frac{n}{2}-1} \quad \cdots \quad b_0} \times \boxed{a_{n-1} \quad \cdots \quad a_{\frac{n}{2}} \quad a_{\frac{n}{2}-1} \quad \cdots \quad a_0}$$

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .



Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .



Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .



Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 B_0$$

Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

```
1: if  $|A| = |B| = 1$  then  
2:   return  $a_0 \cdot b_0$   
3: split  $A$  into  $A_0$  and  $A_1$   
4: split  $B$  into  $B_0$  and  $B_1$   
5:  $Z_2 \leftarrow \text{mult}(A_1, B_1)$   
6:  $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$   
7:  $Z_0 \leftarrow \text{mult}(A_0, B_0)$   
8: return  $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$ 
```

Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

```
1: if  $|A| = |B| = 1$  then  
2:     return  $a_0 \cdot b_0$   
3: split  $A$  into  $A_0$  and  $A_1$   
4: split  $B$  into  $B_0$  and  $B_1$   
5:  $Z_2 \leftarrow \text{mult}(A_1, B_1)$   
6:  $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$   
7:  $Z_0 \leftarrow \text{mult}(A_0, B_0)$   
8: return  $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$ 
```

$\mathcal{O}(1)$

Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

1: **if** $|A| = |B| = 1$ **then**

$\mathcal{O}(1)$

2: **return** $a_0 \cdot b_0$

$\mathcal{O}(1)$

3: **split** A into A_0 and A_1

4: **split** B into B_0 and B_1

5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$

6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$

7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$

8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

1: **if** $|A| = |B| = 1$ **then**

$\mathcal{O}(1)$

2: **return** $a_0 \cdot b_0$

$\mathcal{O}(1)$

3: **split** A into A_0 and A_1

$\mathcal{O}(n)$

4: **split** B into B_0 and B_1

5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$

6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$

7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$

8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

1: if $ A = B = 1$ then	$\mathcal{O}(1)$
2: return $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split A into A_0 and A_1	$\mathcal{O}(n)$
4: split B into B_0 and B_1	$\mathcal{O}(n)$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$	
6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$	
7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$	
8: return $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	

Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

1: **if** $|A| = |B| = 1$ **then**

2: **return** $a_0 \cdot b_0$

3: **split** A into A_0 and A_1

4: **split** B into B_0 and B_1

5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$

6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$

7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$

8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

$\mathcal{O}(1)$

$\mathcal{O}(1)$

$\mathcal{O}(n)$

$\mathcal{O}(n)$

$T(\frac{n}{2})$

Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

1: **if** $|A| = |B| = 1$ **then**

$\mathcal{O}(1)$

2: **return** $a_0 \cdot b_0$

$\mathcal{O}(1)$

3: **split** A into A_0 and A_1

$\mathcal{O}(n)$

4: **split** B into B_0 and B_1

$\mathcal{O}(n)$

5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$

$T(\frac{n}{2})$

6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$

$2T(\frac{n}{2}) + \mathcal{O}(n)$

7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$

8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

1: **if** $|A| = |B| = 1$ **then**

$\mathcal{O}(1)$

2: **return** $a_0 \cdot b_0$

$\mathcal{O}(1)$

3: split A into A_0 and A_1

$\mathcal{O}(n)$

4: split B into B_0 and B_1

$\mathcal{O}(n)$

5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$

$T(\frac{n}{2})$

6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$

$2T(\frac{n}{2}) + \mathcal{O}(n)$

7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$

$T(\frac{n}{2})$

8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

1: if $ A = B = 1$ then	$\mathcal{O}(1)$
2: return $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split A into A_0 and A_1	$\mathcal{O}(n)$
4: split B into B_0 and B_1	$\mathcal{O}(n)$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$	$T(\frac{n}{2})$
6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$	$2T(\frac{n}{2}) + \mathcal{O}(n)$
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Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

1: if $ A = B = 1$ then	$\mathcal{O}(1)$
2: return $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split A into A_0 and A_1	$\mathcal{O}(n)$
4: split B into B_0 and B_1	$\mathcal{O}(n)$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$	$T\left(\frac{n}{2}\right)$
6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$	$2T\left(\frac{n}{2}\right) + \mathcal{O}(n)$
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We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) .$$

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Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- ▶ Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $T(n) = \Theta(n^{\log_b a})$
- ▶ Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
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In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$.

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⇒ Not better than the “school method”.

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Hence,

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1: if  $|A| = |B| = 1$  then  
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3: split  $A$  into  $A_0$  and  $A_1$   
4: split  $B$  into  $B_0$  and  $B_1$   
5:  $Z_2 \leftarrow \text{mult}(A_1, B_1)$   
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A huge improvement over the “school method”.

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Consider the recurrence relation:

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Note that we ignore **boundary conditions** for the moment.

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- ▶ First consider the homogenous case.

The Homogenous Case

The solution space

$$S = \{ \mathcal{T} = T[1], T[2], T[3], \dots \mid \mathcal{T} \text{ fulfills recurrence relation} \}$$

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How do we find a non-trivial solution?

We guess that the solution is of the form λ^n , $\lambda \neq 0$, and see what happens. In order for this guess to fulfill the recurrence we need

$$c_0\lambda^n + c_1\lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \dots + c_k \cdot \lambda^{n-k} = 0$$

for all $n \geq k$.

The Homogenous Case

Dividing by λ^{n-k} gives that all these constraints are identical to

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This means that if λ_i is a root (Nullstelle) of $P[\lambda]$ then $T[n] = \lambda_i^n$ is a solution to the recurrence relation.

Let $\lambda_1, \dots, \lambda_k$ be the k (complex) roots of $P[\lambda]$. Then, because of the vector space property

$$\alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \dots + \alpha_k\lambda_k^n$$

is a solution for arbitrary values α_i .

The Homogenous Case

Lemma 6

Assume that the characteristic polynomial has k *distinct* roots $\lambda_1, \dots, \lambda_k$. Then *all* solutions to the recurrence relation are of the form

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Proof.

There is one solution for every possible choice of boundary conditions for $T[1], \dots, T[k]$.

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Proof.

There is one solution for every possible choice of boundary conditions for $T[1], \dots, T[k]$.

We show that the above set of solutions contains one solution for every choice of boundary conditions.

The Homogenous Case

Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the α'_i 's such that these conditions are met:

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We show that the column vectors are linearly independent. Then the above equation has a solution.

Computing the Determinant

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} =$$

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$$= \prod_{i=1}^k \lambda_i \cdot \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{k-2} & \lambda_1^{k-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{k-2} & \lambda_2^{k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-2} & \lambda_k^{k-1} \end{vmatrix}$$

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$$\begin{vmatrix} 1 & \lambda_1 - \lambda_1 \cdot 1 & \cdots & \lambda_1^{k-2} - \lambda_1 \cdot \lambda_1^{k-3} & \lambda_1^{k-1} - \lambda_1 \cdot \lambda_1^{k-2} \\ 1 & \lambda_2 - \lambda_1 \cdot 1 & \cdots & \lambda_2^{k-2} - \lambda_1 \cdot \lambda_2^{k-3} & \lambda_2^{k-1} - \lambda_1 \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k - \lambda_1 \cdot 1 & \cdots & \lambda_k^{k-2} - \lambda_1 \cdot \lambda_k^{k-3} & \lambda_k^{k-1} - \lambda_1 \cdot \lambda_k^{k-2} \end{vmatrix}$$

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$$\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2} \end{vmatrix}$$

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$$\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & (\lambda_2 - \lambda_1) \cdot \mathbf{1} & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & (\lambda_k - \lambda_1) \cdot \mathbf{1} & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2} \end{vmatrix} =$$

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$$\prod_{i=2}^k (\lambda_i - \lambda_1) \cdot \begin{vmatrix} 1 & \lambda_2 & \cdots & \lambda_2^{k-3} & \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-3} & \lambda_k^{k-2} \end{vmatrix}$$

Computing the Determinant

Repeating the above steps gives:

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^k \lambda_i \cdot \prod_{i>\ell} (\lambda_i - \lambda_\ell)$$

Hence, if all λ_i 's are different, then the determinant is non-zero.

The Homogeneous Case

What happens if the roots are not all distinct?

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Suppose we have a root λ_i with multiplicity (**Vielfachheit**) at least 2. Then not only is λ_i^n a solution to the recurrence but also $n\lambda_i^{n-1}$.

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Suppose we have a root λ_i with multiplicity (**Vielfachheit**) at least 2. Then not only is λ_i^n a solution to the recurrence but also $n\lambda_i^{n-1}$.

To see this consider the polynomial

$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \dots + c_k\lambda^{n-k}$$

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Since λ_i is a root we can write this as $Q[\lambda] \cdot (\lambda - \lambda_i)^2$. Calculating the derivative gives a polynomial that still has root λ_i .

This means

$$c_0 n \lambda_i^{n-1} + c_1 (n-1) \lambda_i^{n-2} + \dots + c_k (n-k) \lambda_i^{n-k-1} = 0$$

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Hence,

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Hence,

$$\underbrace{c_0 n \lambda_i^n}_{T[n]} + \underbrace{c_1 (n-1) \lambda_i^{n-1}}_{T[n-1]} + \dots + \underbrace{c_k (n-k) \lambda_i^{n-k}}_{T[n-k]} = 0$$

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We can continue $j-1$ times.

Hence, $n^\ell \lambda_i^n$ is a solution for $\ell \in 0, \dots, j-1$.

The Homogeneous Case

Lemma 7

Let $P[\lambda]$ denote the characteristic polynomial to the recurrence

$$c_0T[n] + c_1T[n-1] + \dots + c_kT[n-k] = 0$$

Let $\lambda_i, i = 1, \dots, m$ be the (complex) roots of $P[\lambda]$ with multiplicities ℓ_i . Then the general solution to the recurrence is given by

$$T[n] = \sum_{i=1}^m \sum_{j=0}^{\ell_i-1} \alpha_{ij} \cdot (n^j \lambda_i^n) .$$

The full proof is omitted. We have only shown that any choice of α_{ij} 's is a solution to the recurrence.

Example: Fibonacci Sequence

$$T[0] = 0$$

$$T[1] = 1$$

$$T[n] = T[n - 1] + T[n - 2] \text{ for } n \geq 2$$

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Finding the roots, gives

$$\lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2} (1 \pm \sqrt{5})$$

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Hence, the solution is of the form

$$\alpha \left(\frac{1 + \sqrt{5}}{2} \right)^n + \beta \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

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$$\alpha \left(\frac{1 + \sqrt{5}}{2} \right) + \beta \left(\frac{1 - \sqrt{5}}{2} \right) = 1 \implies \alpha - \beta = \frac{2}{\sqrt{5}}$$

Example: Fibonacci Sequence

Hence, the solution is

$$\frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right]$$

The Inhomogeneous Case

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \cdots + c_kT(n-k) = f(n)$$

with $f(n) \neq 0$.

While we have a fairly general technique for solving **homogeneous**, linear recurrence relations the inhomogeneous case is different.

The Inhomogeneous Case

The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n) ,$$

where T_h is **any** solution to the homogeneous equation, and T_p is **one** particular solution to the inhomogeneous equation.

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There is no general method to find a particular solution.

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$$T[n] = T[n - 1] + 1 \quad T[0] = 1$$

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Subtracting the first from the second equation gives,

$$T[n] - T[n - 1] = T[n - 1] - T[n - 2] \quad (n \geq 2)$$

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I get a completely determined recurrence if I add $T[0] = 1$ and $T[1] = 2$.

The Inhomogeneous Case

Example: Characteristic polynomial:

$$\lambda^2 - 2\lambda + 1 = 0$$

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$T[0] = 1$ gives $\alpha = 1$.

$T[1] = 2$ gives $1 + \beta = 2 \Rightarrow \beta = 1$.

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If $f(n)$ is a polynomial of degree r this method can be applied $r + 1$ times to obtain a homogeneous equation:

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Shift:

$$T[n - 1] = T[n - 2] + (n - 1)^2 = T[n - 2] + n^2 - 2n + 1$$

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$$T[n] = T[n - 1] + n^2$$

Shift:

$$T[n - 1] = T[n - 2] + (n - 1)^2 = T[n - 2] + n^2 - 2n + 1$$

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and so on...

6.4 Generating Functions

Definition 8 (Generating Function)

Let $(a_n)_{n \geq 0}$ be a sequence. The corresponding

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- ▶ **exponential generating function** (**exponentielle Erzeugendenfunktion**) is

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There are no convergence issues here.

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Then, it is important to think about convergence/convergence radius etc.

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It means that the power series $1 - z$ and the power series $\sum_{n \geq 0} z^n$ are invers, i.e.,

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This is well-defined.

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Hence, the generating function of the sequence $a_n = n + 1$ is $1/(1-z)^2$.

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Hence, the generating function of the sequence

$$a_n = (n+1)(n+2) \text{ is } \frac{2}{(1-z)^3} .$$

6.4 Generating Functions

Computing the k -th derivative of $\sum z^n$.

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The generating function of the sequence $a_n = \binom{n+k}{k}$ is $\frac{1}{(1-z)^{k+1}}$.

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Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

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Hence, $a_n = n + 1$.

Some Generating Functions

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$\frac{1}{n!}$	e^z

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Solving Recursions with Generating Functions

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Techniques:

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4. Solving for $A(z)$ gives an equation of the form $A(z) = f(z)$, where hopefully $f(z)$ is a simple function.
5. Write $f(z)$ as a formal power series.
Techniques:
 - ▶ partial fraction decomposition (**Partialbruchzerlegung**)
 - ▶ lookup in tables
6. The coefficients of the resulting power series are the a_n .

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

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$$A + B + C = 1$$

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This leads to the following conditions:

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which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

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6. This means $a_n = \frac{7}{4}3^n - \frac{1}{2}n - \frac{3}{4}$.

6.5 Transformation of the Recurrence

Example 10

$$f_0 = 1$$

$$f_1 = 2$$

$$f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 .$$

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$$g_n := \log f_n .$$

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$$f_n = 2^{F_n}$$

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Then:

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6 Recurrences

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6 Recurrences

We get

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6 Recurrences

We get

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We get

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$$\begin{aligned}g_k &= 3 [g_{k-1}] + 2^k \\&= 3 [3g_{k-2} + 2^{k-1}] + 2^k \\&= 3^2 [g_{k-2}] + 3 \cdot 2^{k-1} + 2^k \\&= 3^2 [3g_{k-3} + 2^{k-2}] + 3 \cdot 2^{k-1} + 2^k \\&= 3^3 g_{k-3} + 3^2 2^{k-2} + 3 \cdot 2^{k-1} + 2^k \\&= 2^k \cdot \sum_{i=0}^k \left(\frac{3}{2}\right)^i \\&= 2^k \cdot \frac{\left(\frac{3}{2}\right)^{k+1} - 1}{1/2}\end{aligned}$$

6 Recurrences

We get

$$\begin{aligned}g_k &= 3 [g_{k-1}] + 2^k \\&= 3 [3g_{k-2} + 2^{k-1}] + 2^k \\&= 3^2 [g_{k-2}] + 3 \cdot 2^{k-1} + 2^k \\&= 3^2 [3g_{k-3} + 2^{k-2}] + 3 \cdot 2^{k-1} + 2^k \\&= 3^3 g_{k-3} + 3^2 2^{k-2} + 3 \cdot 2^{k-1} + 2^k \\&= 2^k \cdot \sum_{i=0}^k \left(\frac{3}{2}\right)^i \\&= 2^k \cdot \frac{\left(\frac{3}{2}\right)^{k+1} - 1}{1/2} = 3^{k+1} - 2^{k+1}\end{aligned}$$

6 Recurrences

Let $n = 2^k$:

$$g_k = 3^{k+1} - 2^{k+1}, \text{ hence}$$

$$f_n = 3 \cdot 3^k - 2 \cdot 2^k$$

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6 Recurrences

Let $n = 2^k$:

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Part III

Data Structures

Abstract Data Type

An abstract data type (ADT) is defined by an interface of operations or methods that can be performed and that have a defined behavior.

The data types in this lecture all operate on objects that are represented by a **[key, value]** pair.

- ▶ The **key** comes from a totally ordered set, and we assume that there is an efficient comparison function.
- ▶ The **value** can be anything; it usually carries satellite information important for the application that uses the ADT.

Dynamic Set Operations

- ▶ **S .search(k)**: Returns pointer to object x from S with $\text{key}[x] = k$ or **null**.

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- ▶ **S . insert(x)**: Inserts object x into set S . $\text{key}[x]$ must not currently exist in the data-structure.
- ▶ **S . delete(x)**: Given pointer to object x from S , delete x from the set.

Dynamic Set Operations

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Dynamic Set Operations

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 $S := \{x \in S \mid \text{key}[x] \leq k\}$, $S' := \{x \in S \mid \text{key}[x] > k\}$.

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 $S := \{x \in S \mid \text{key}[x] \leq k\}$, $S' := \{x \in S \mid \text{key}[x] > k\}$.
- ▶ **S . concatenate(S'):** $S := S \cup S'$.
Requires $\text{key}[S.\text{maximum}()] \leq \text{key}[S'.\text{minimum}()]$.

Dynamic Set Operations

- ▶ **S. union(S'):** Sets $S := S \cup S'$. The set S' is destroyed.
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- ▶ **S. concatenate(S'):** $S := S \cup S'$.
Requires $\text{key}[S.\text{maximum}()] \leq \text{key}[S'.\text{minimum}()]$.
- ▶ **S. decrease-key(x, k):** Replace $\text{key}[x]$ by $k \leq \text{key}[x]$.

Examples of ADTs

Stack:

- ▶ $S.$ **push**(x): Insert an element.
- ▶ $S.$ **pop**(): Return the element from S that was inserted most recently; delete it from S .
- ▶ $S.$ **empty**(): Tell if S contains any object.

Examples of ADTs

Stack:

- ▶ **$S.$ push(x)**: Insert an element.
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Queue:

- ▶ **$S.$ enqueue(x)**: Insert an element.
- ▶ **$S.$ dequeue()**: Return the element that is longest in the structure; delete it from S .
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Examples of ADTs

Stack:

- ▶ ***S.push(x)***: Insert an element.
- ▶ ***S.pop()***: Return the element from *S* that was inserted most recently; delete it from *S*.
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- ▶ ***S.enqueue(x)***: Insert an element.
- ▶ ***S.dequeue()***: Return the element that is longest in the structure; delete it from *S*.
- ▶ ***S.empty()***: Tell if *S* contains any object.

Priority-Queue:

- ▶ ***S.insert(x)***: Insert an element.
- ▶ ***S.delete-min()***: Return the element with lowest key-value; delete it from *S*.

7 Dictionary

Dictionary:

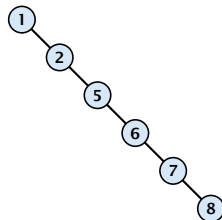
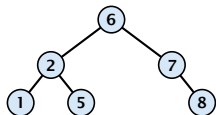
- ▶ **S . insert(x)**: Insert an element x .
- ▶ **S . delete(x)**: Delete the element pointed to by x .
- ▶ **S . search(k)**: Return a pointer to an element e with $\text{key}[e] = k$ in S if it exists; otherwise return **null**.

7.1 Binary Search Trees

An (**internal**) **binary search tree** stores the elements in a binary tree. Each tree-node corresponds to an element. All elements in the left sub-tree of a node v have a smaller key-value than $\text{key}[v]$ and elements in the right sub-tree have a larger-key value. We assume that all key-values are different.

(**External** Search Trees store objects only at leaf-vertices)

Examples:

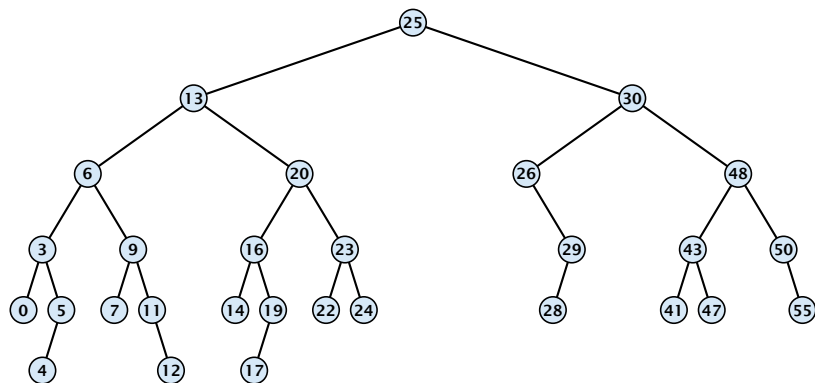


7.1 Binary Search Trees

We consider the following operations on binary search trees. Note that this is a super-set of the dictionary-operations.

- ▶ $T.\text{insert}(x)$
- ▶ $T.\text{delete}(x)$
- ▶ $T.\text{search}(k)$
- ▶ $T.\text{successor}(x)$
- ▶ $T.\text{predecessor}(x)$
- ▶ $T.\text{minimum}()$
- ▶ $T.\text{maximum}()$

Binary Search Trees: Searching

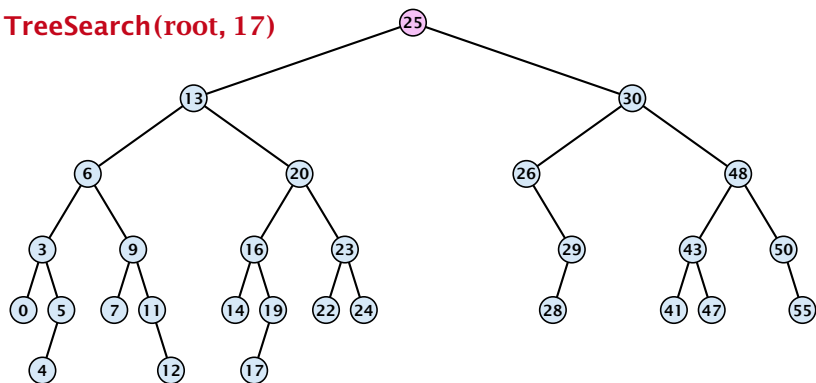


Algorithm 1 $\text{TreeSearch}(x, k)$

- 1: **if** $x = \text{null}$ **or** $k = \text{key}[x]$ **return** x
- 2: **if** $k < \text{key}[x]$ **return** $\text{TreeSearch}(\text{left}[x], k)$
- 3: **else return** $\text{TreeSearch}(\text{right}[x], k)$

Binary Search Trees: Searching

TreeSearch(root, 17)

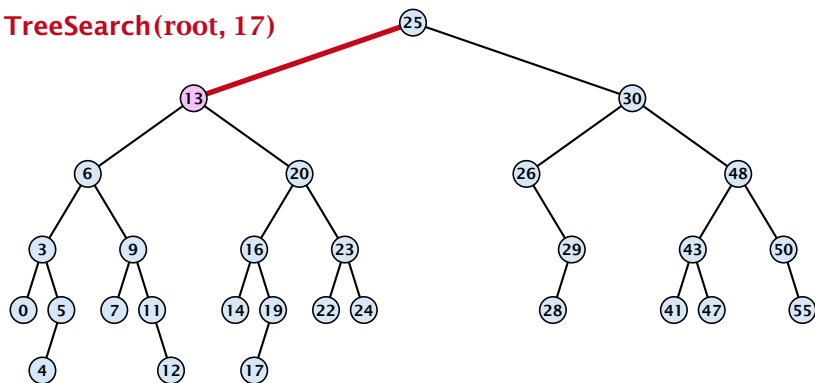


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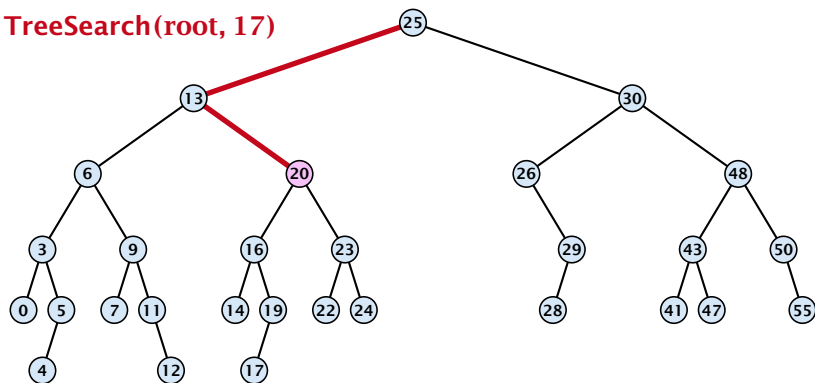


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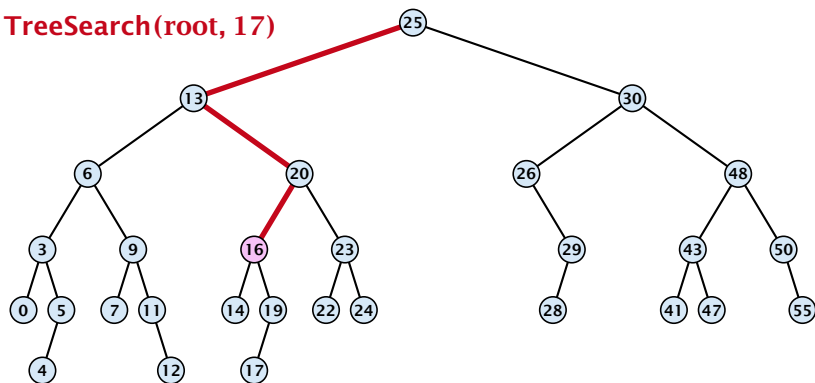


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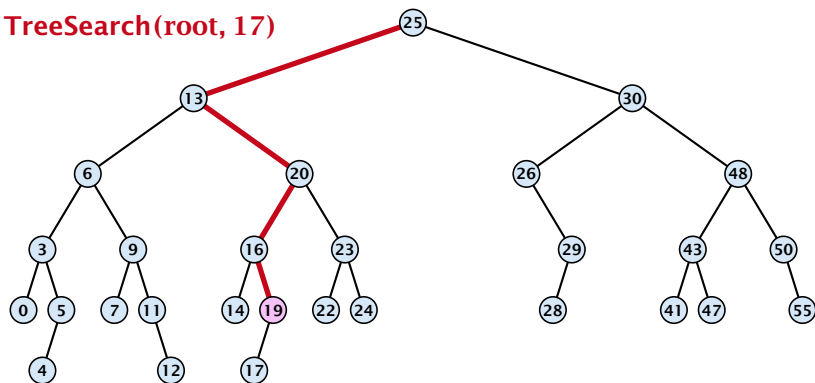


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Binary Search Trees: Searching

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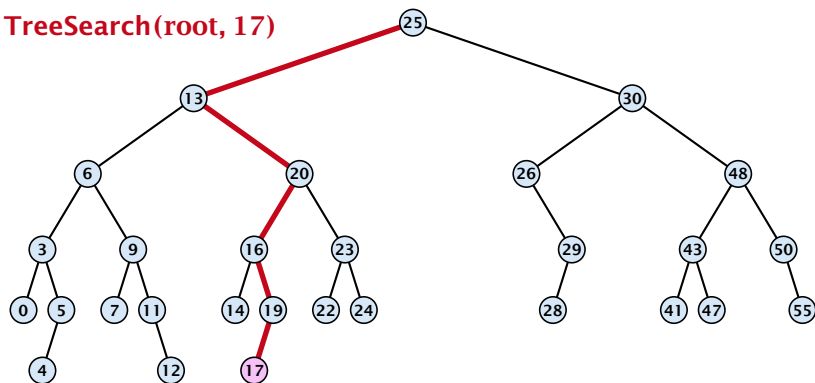


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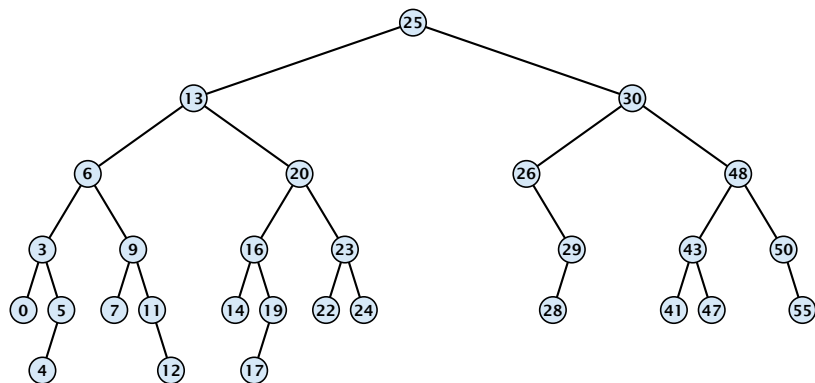
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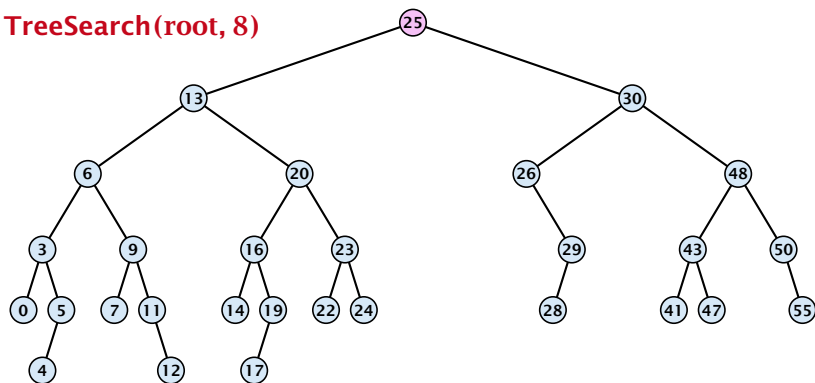


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Binary Search Trees: Searching

TreeSearch(root, 8)

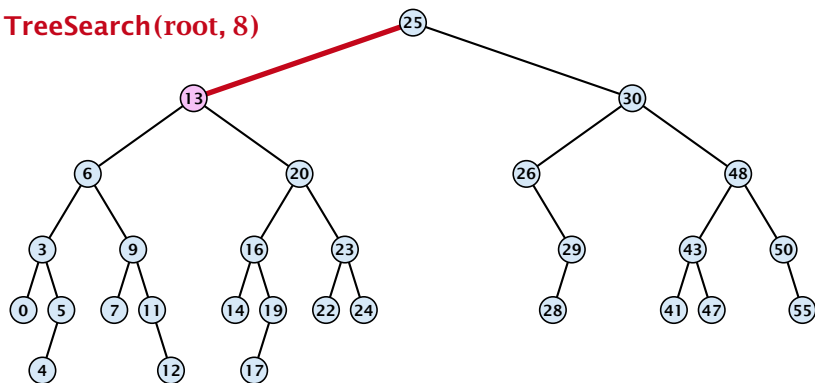


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Binary Search Trees: Searching

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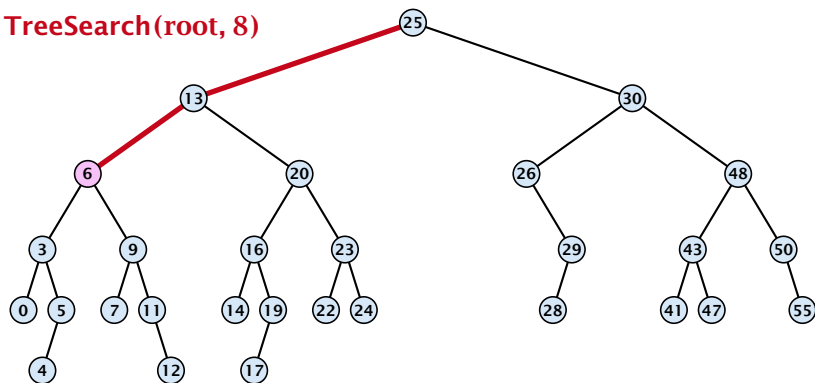


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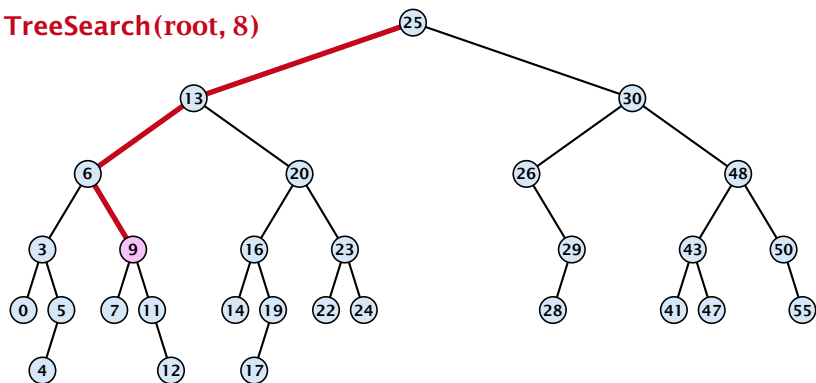


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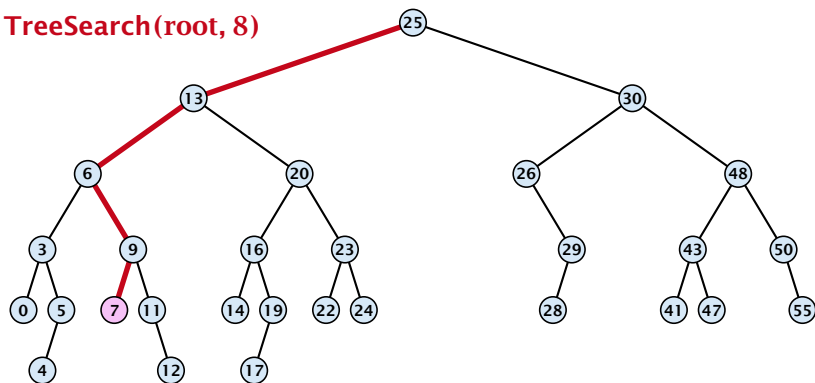


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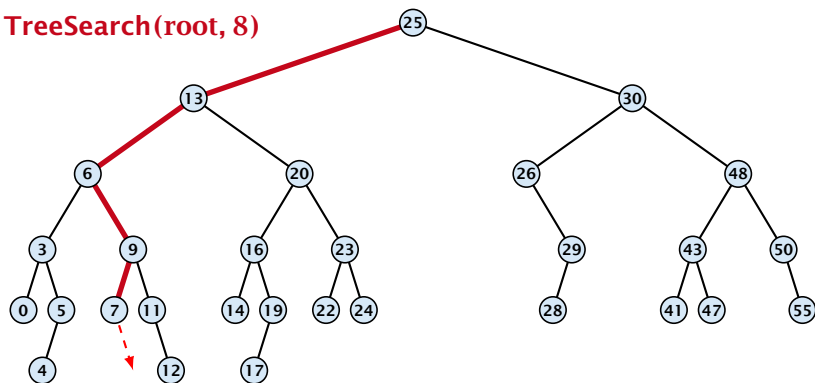


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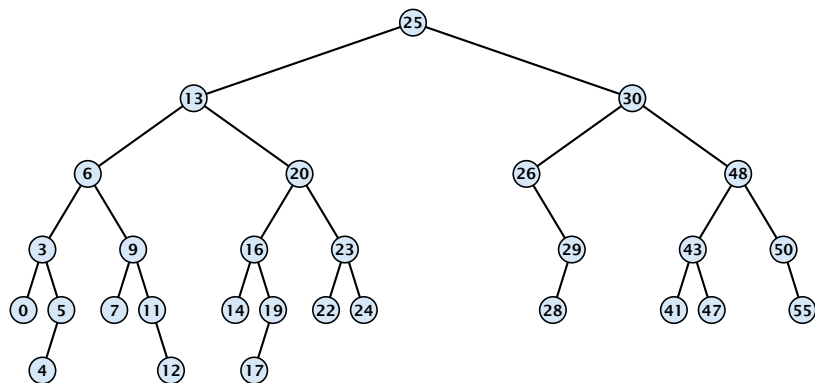
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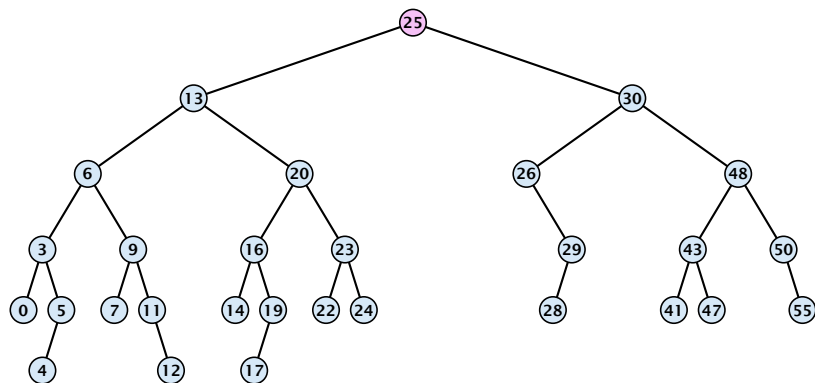
Binary Search Trees: Minimum



Algorithm 2 TreeMin(x)

- 1: **if** $x = \text{null}$ **or** $\text{left}[x] = \text{null}$ **return** x
- 2: **return** $\text{TreeMin}(\text{left}[x])$

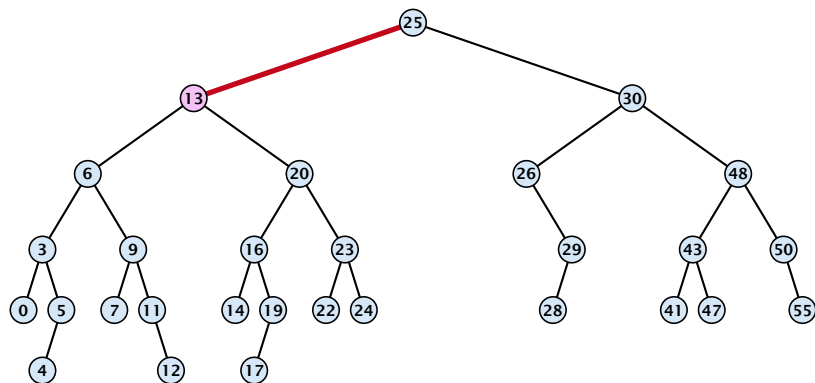
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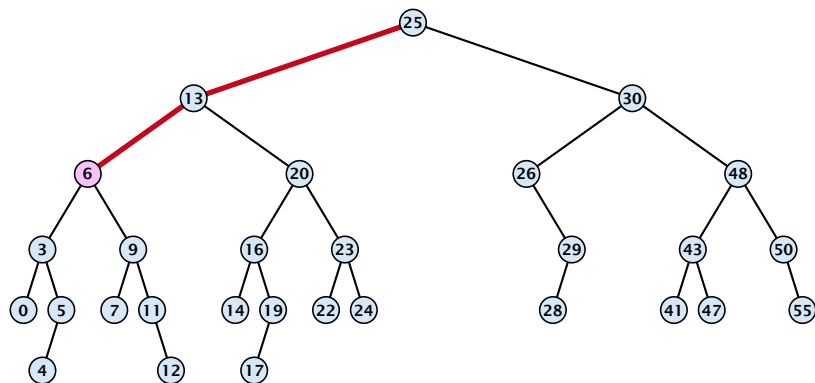
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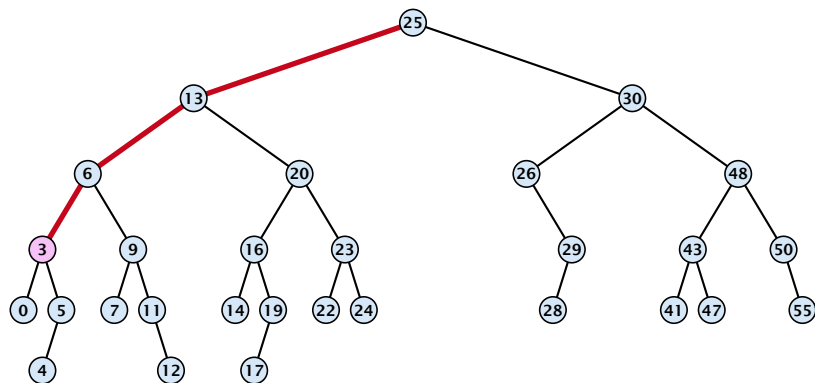
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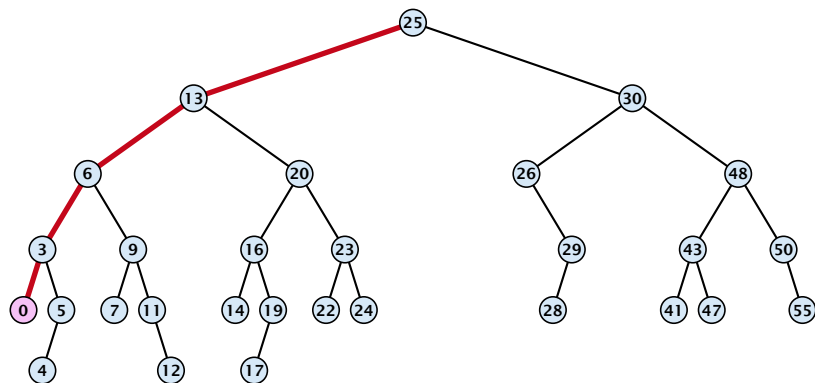
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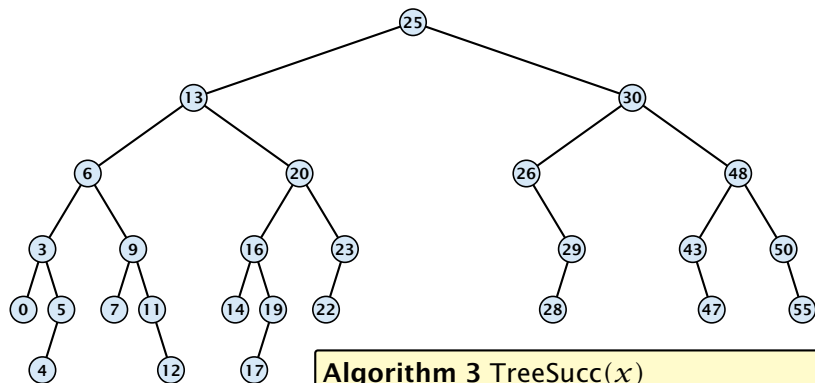
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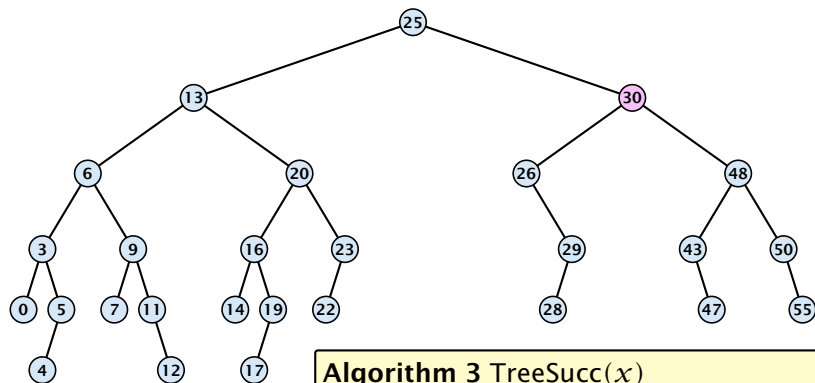
Binary Search Trees: Successor



Algorithm 3 TreeSucc(x)

- 1: **if** right[x] \neq null **return** TreeMin(right[x])
- 2: $y \leftarrow$ parent[x]
- 3: **while** $y \neq$ null **and** $x =$ right[y] **do**
- 4: $x \leftarrow y$; $y \leftarrow$ parent[x]
- 5: **return** y ;

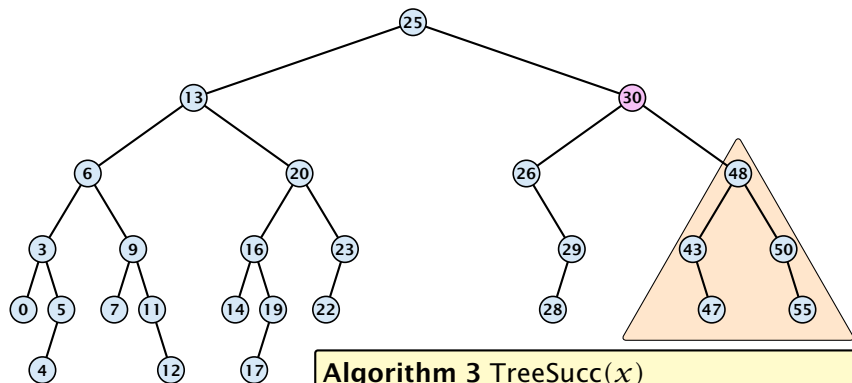
Binary Search Trees: Successor



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- 5: **return** y ;

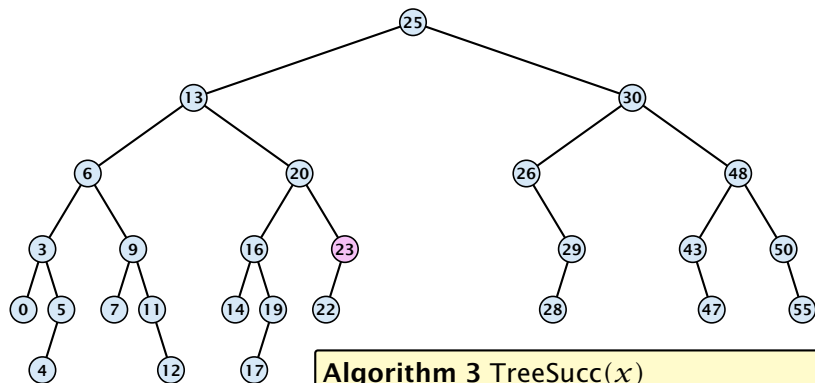
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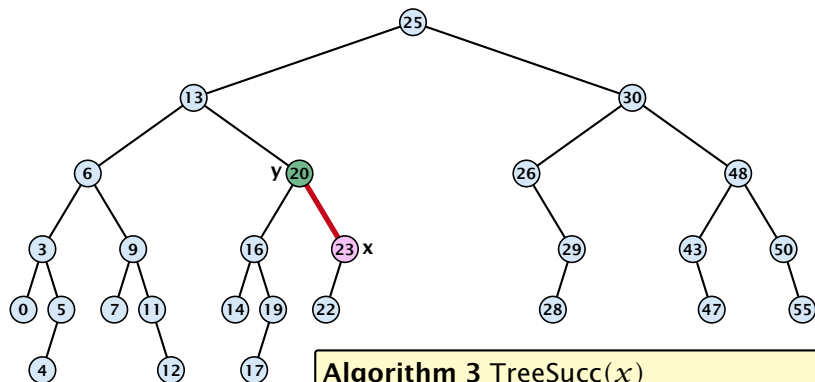
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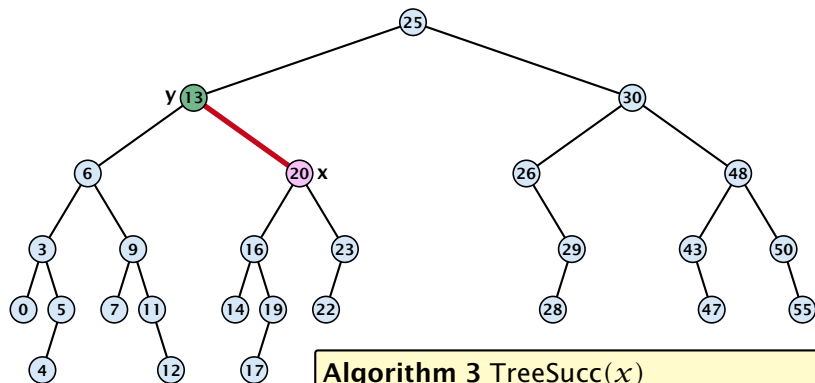
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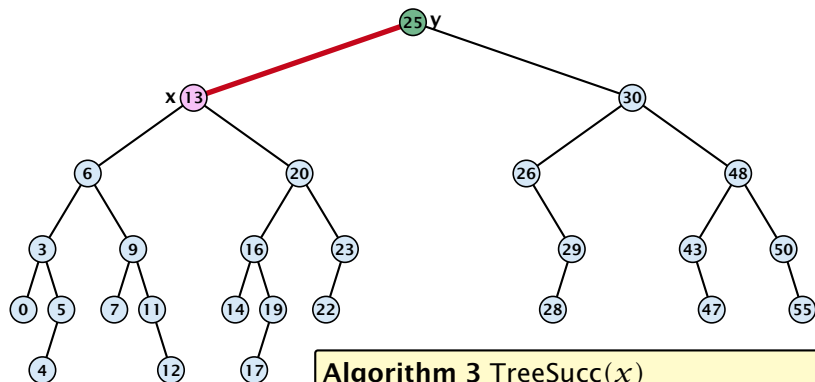
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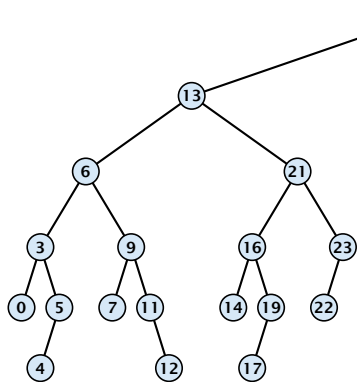
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Binary Search Trees: Insert

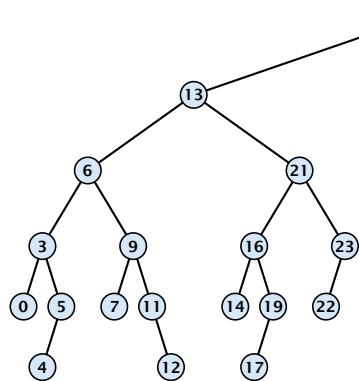


Algorithm 4 TreeInsert(x, z)

```
1: if  $x = \text{null}$  then
2:      $\text{root}[T] \leftarrow z$ ;  $\text{parent}[z] \leftarrow \text{null}$ ;
3:     return;
4: if  $\text{key}[x] > \text{key}[z]$  then
5:     if  $\text{left}[x] = \text{null}$  then
6:          $\text{left}[x] \leftarrow z$ ;  $\text{parent}[z] \leftarrow x$ ;
7:     else TreeInsert( $\text{left}[x], z$ );
8: else
9:     if  $\text{right}[x] = \text{null}$  then
10:         $\text{right}[x] \leftarrow z$ ;  $\text{parent}[z] \leftarrow x$ ;
11:    else TreeInsert( $\text{right}[x], z$ );
```

Binary Search Trees: Insert

Insert element **not** in the tree.

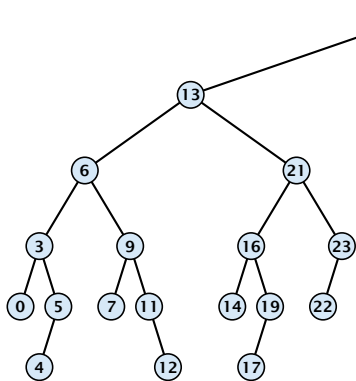


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- 1: **if** $x = \text{null}$ **then**
- 2: $\text{root}[T] \leftarrow z$; $\text{parent}[z] \leftarrow \text{null}$;
- 3: **return**;
- 4: **if** $\text{key}[x] > \text{key}[z]$ **then**
- 5: **if** $\text{left}[x] = \text{null}$ **then**
- 6: $\text{left}[x] \leftarrow z$; $\text{parent}[z] \leftarrow x$;
- 7: **else** TreeInsert($\text{left}[x], z$);
- 8: **else**
- 9: **if** $\text{right}[x] = \text{null}$ **then**
- 10: $\text{right}[x] \leftarrow z$; $\text{parent}[z] \leftarrow x$;
- 11: **else** TreeInsert($\text{right}[x], z$);

Binary Search Trees: Insert

Insert element **not** in the tree.



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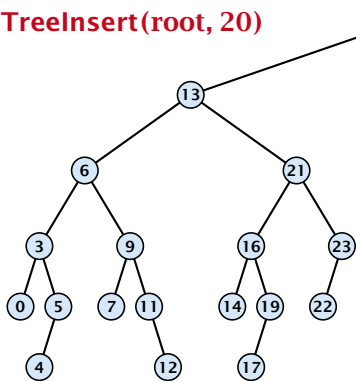
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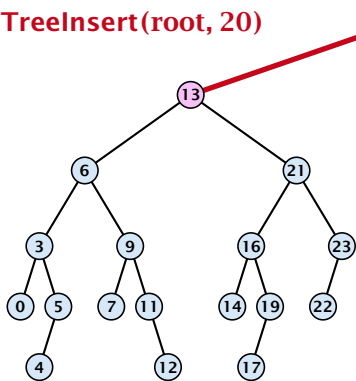
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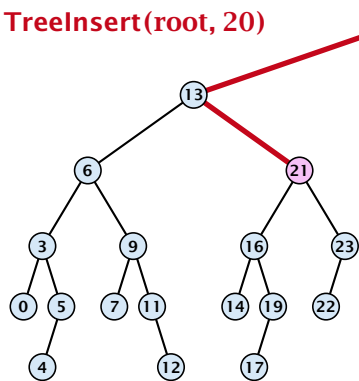
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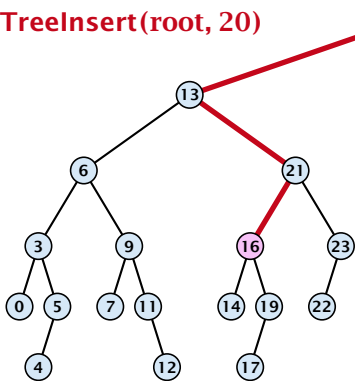
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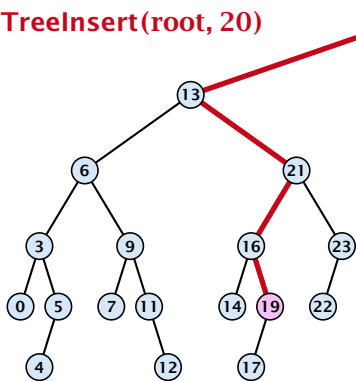
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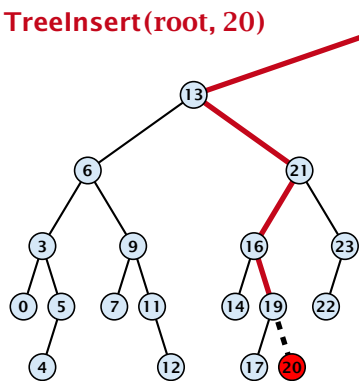
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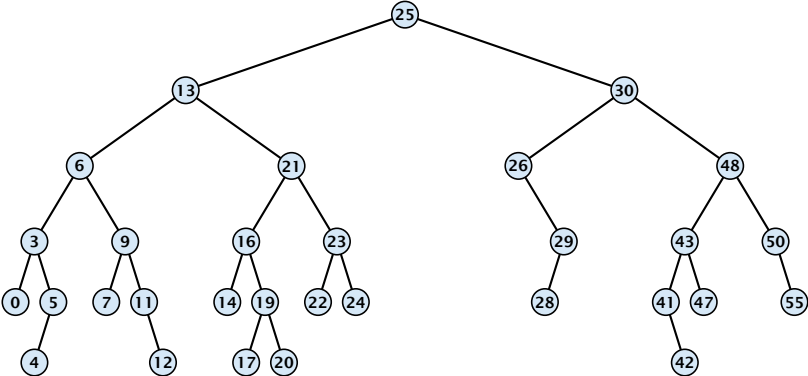


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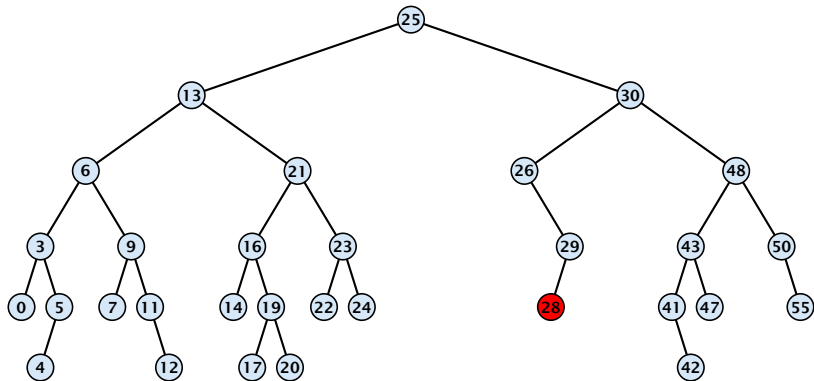
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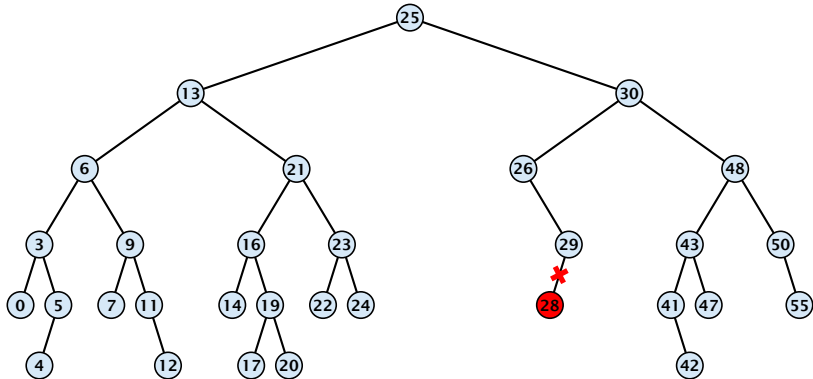


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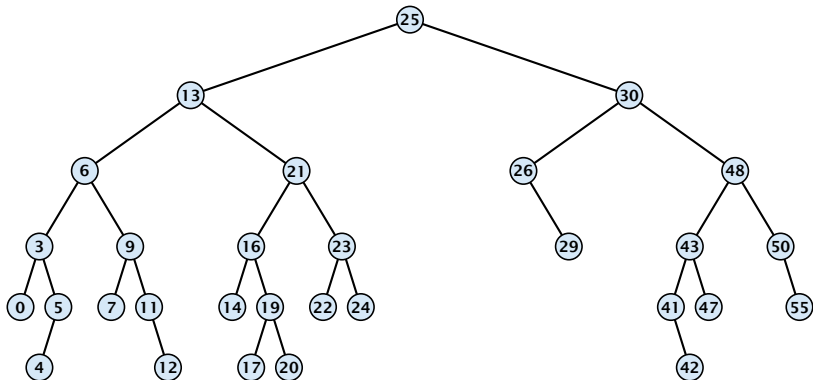


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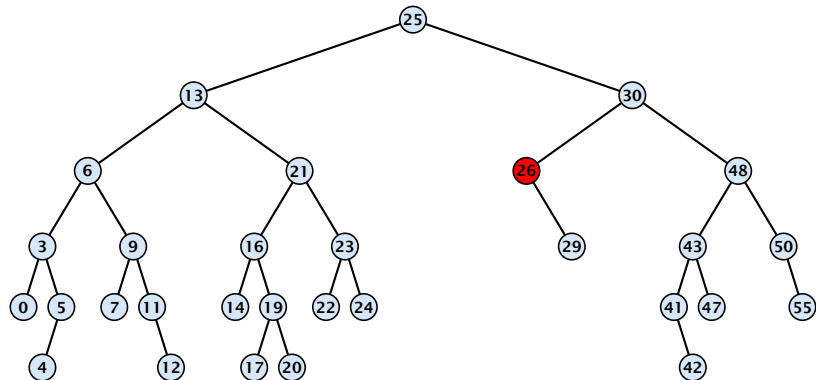


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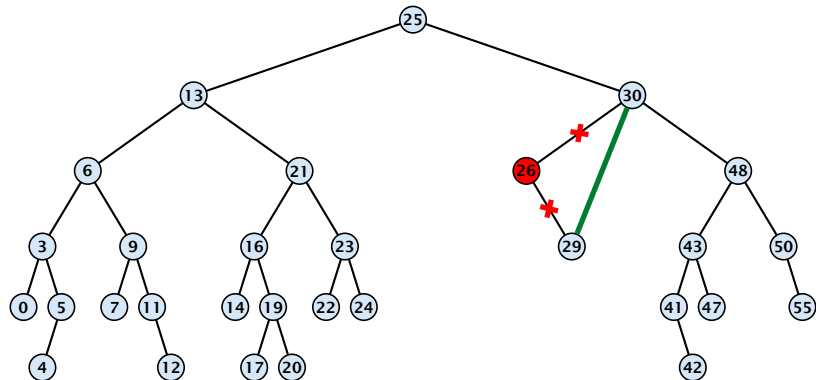


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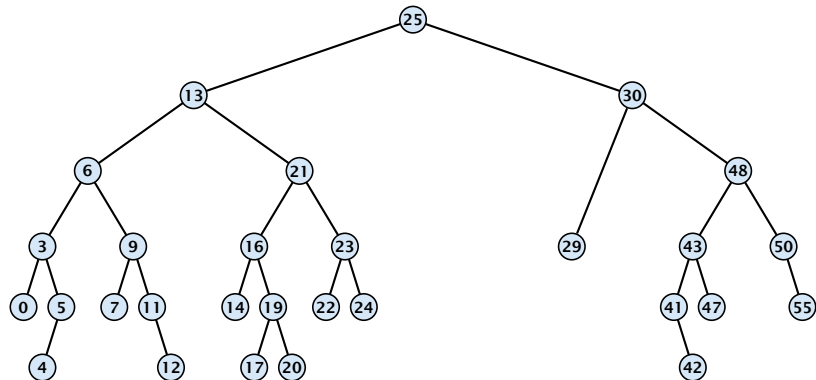


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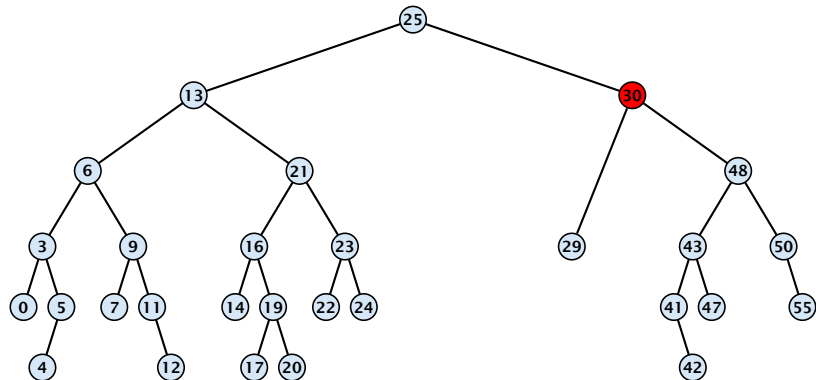


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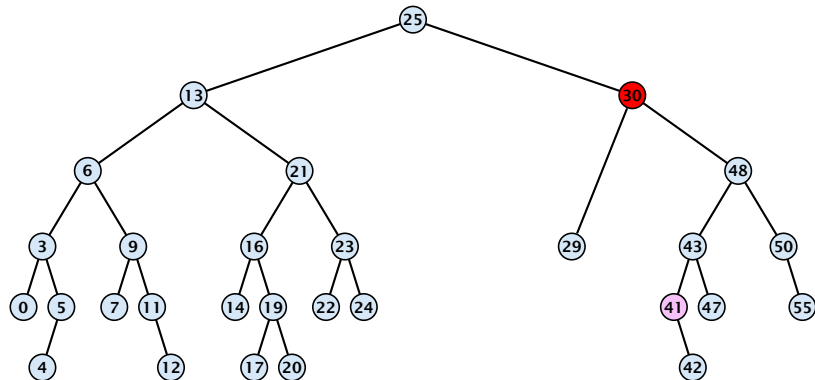


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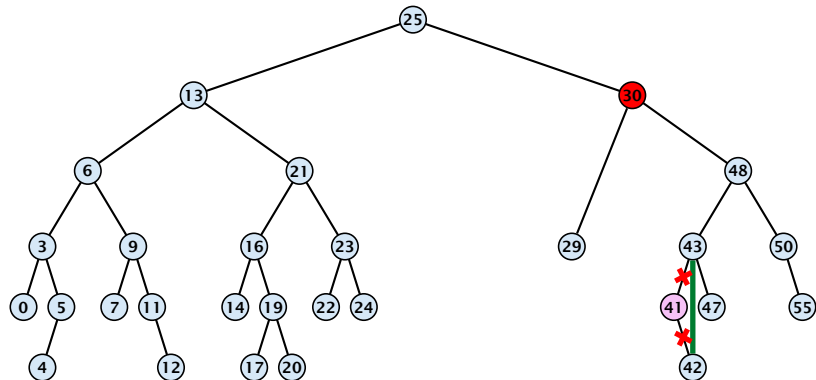


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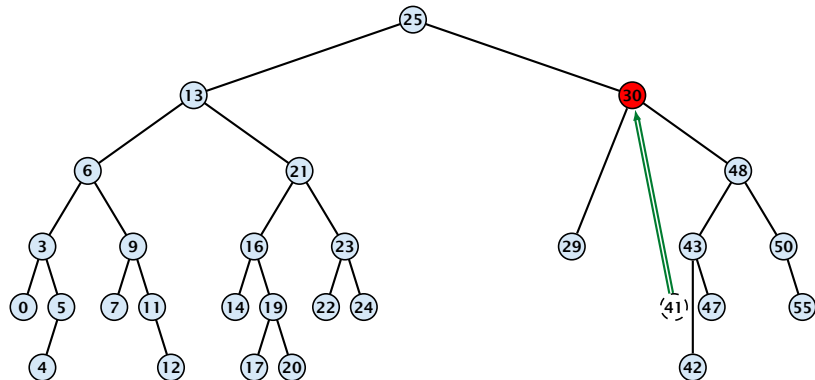


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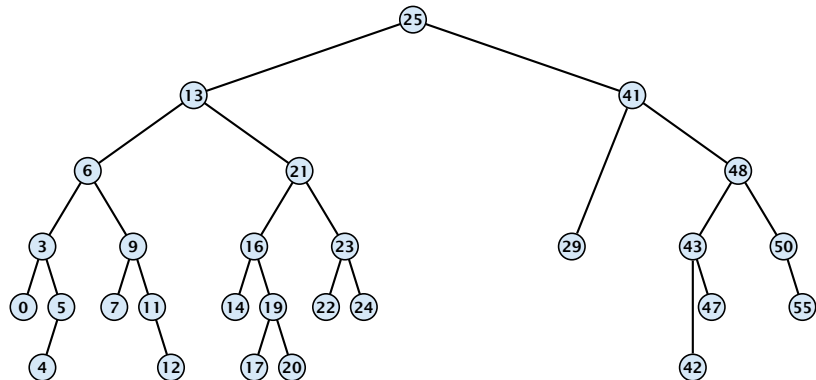


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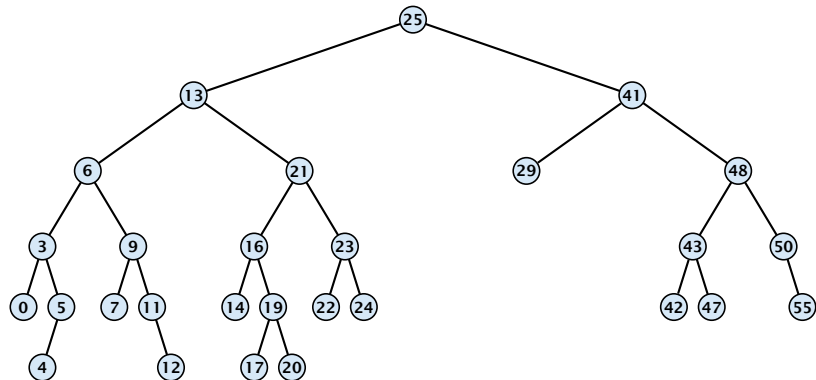


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Algorithm 9 TreeDelete(z)

```
1: if left[ $z$ ] = null or right[ $z$ ] = null
2:   then  $y \leftarrow z$  else  $y \leftarrow \text{TreeSucc}(z)$ ;   select  $y$  to splice out
3:   if left[ $y$ ]  $\neq$  null
4:     then  $x \leftarrow \text{left}[y]$  else  $x \leftarrow \text{right}[y]$ ;  $x$  is child of  $y$  (or null)
5:   if  $x \neq \text{null}$  then parent[ $x$ ]  $\leftarrow$  parent[ $y$ ];   parent[ $x$ ] is correct
6:   if parent[ $y$ ] = null then
7:     root[ $T$ ]  $\leftarrow x$ 
8:   else
9:     if  $y = \text{left}[\text{parent}[y]]$  then
10:      left[parent[ $y$ ]]  $\leftarrow x$ 
11:    else
12:      right[parent[ $y$ ]]  $\leftarrow x$ 
13:   if  $y \neq z$  then copy  $y$ -data to  $z$ 
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} fix pointer to x

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AVL-trees, Red-black trees, Scapegoat trees, 2-3 trees, B-trees, AA trees, Treaps

similar: SPLAY trees.

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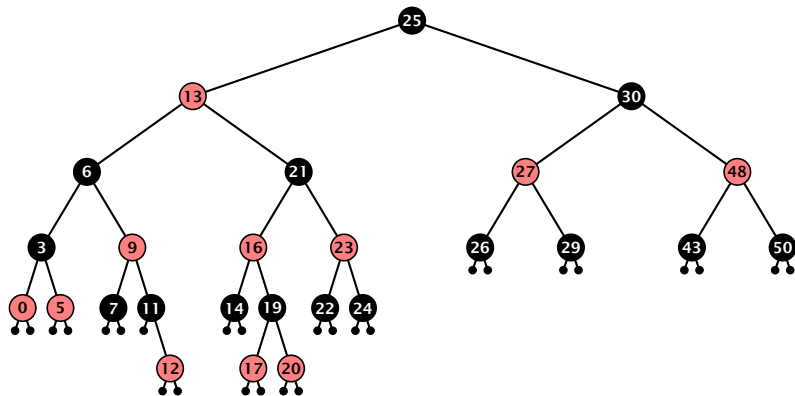
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Red Black Trees: Example



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We first show:

Lemma 15

A sub-tree of black height $\text{bh}(v)$ in a red black tree contains at least $2^{\text{bh}(v)} - 1$ internal vertices.

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- ▶ The sub-tree rooted at v contains $0 = 2^{\text{bh}(v)} - 1$ inner vertices.

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- ▶ Then T_v contains at least $2(2^{\text{bh}(v)-1} - 1) + 1 \geq 2^{\text{bh}(v)} - 1$ vertices.



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Hence, $h \leq 2 \log(n + 1) = \mathcal{O}(\log n)$. □

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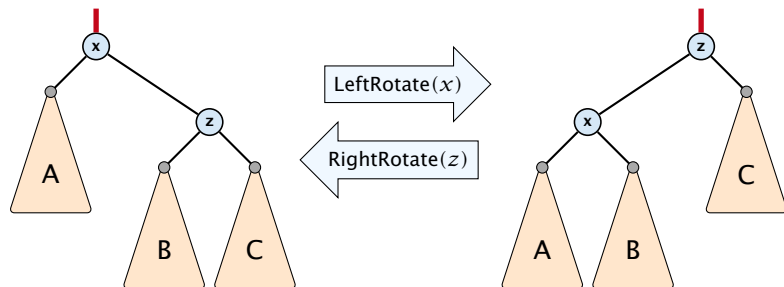
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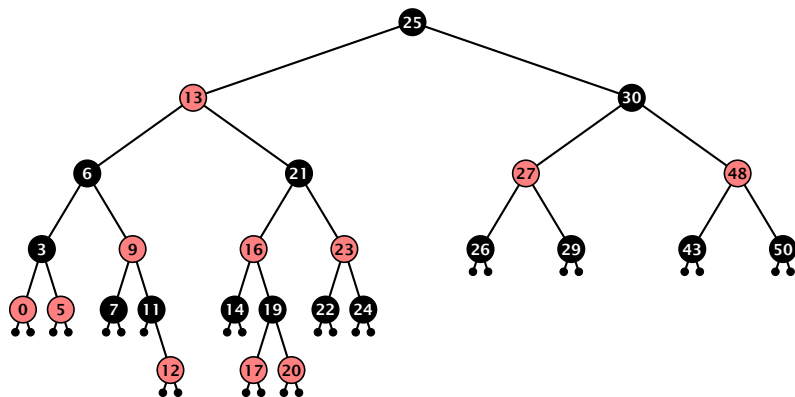
We need to adapt the insert and delete operations so that the red black properties are maintained.

Rotations

The properties will be maintained through rotations:



Red Black Trees: Insert

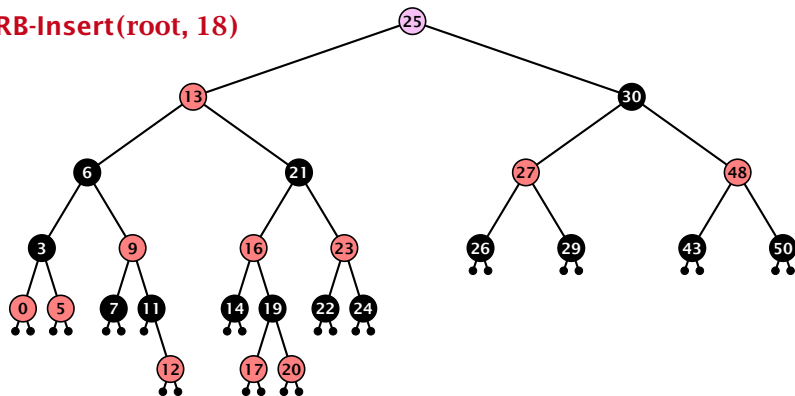


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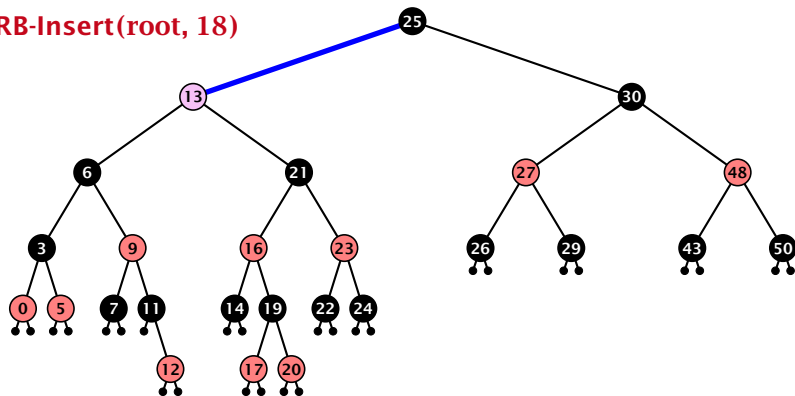


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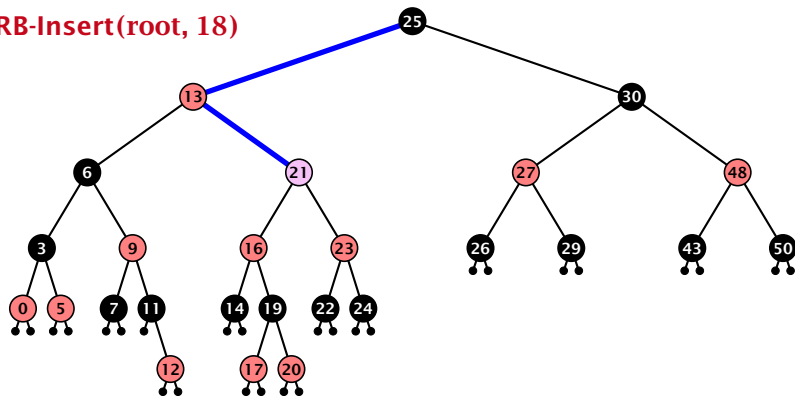


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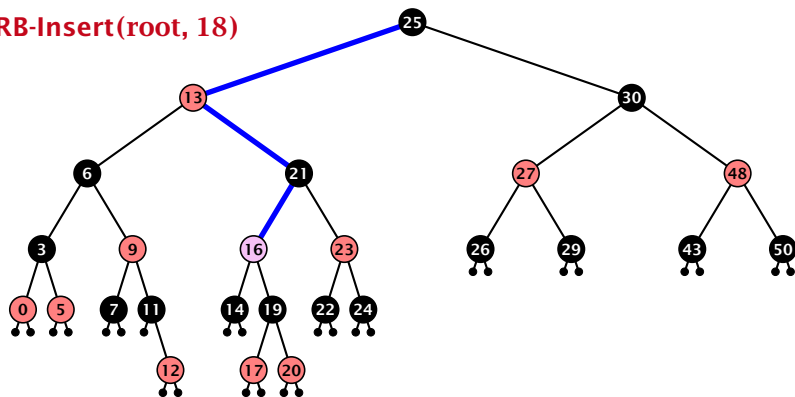


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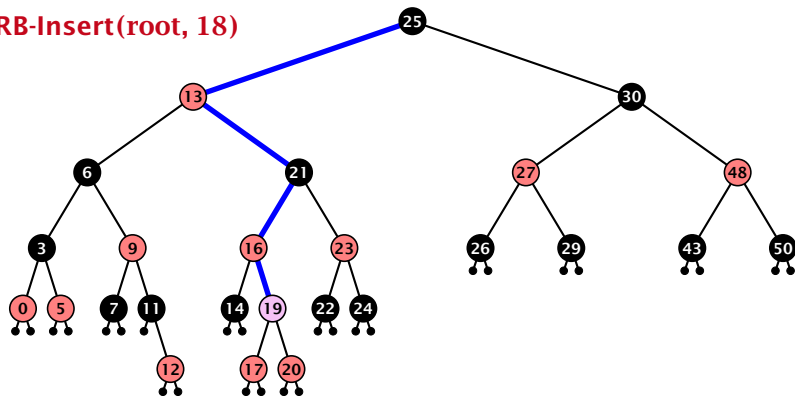


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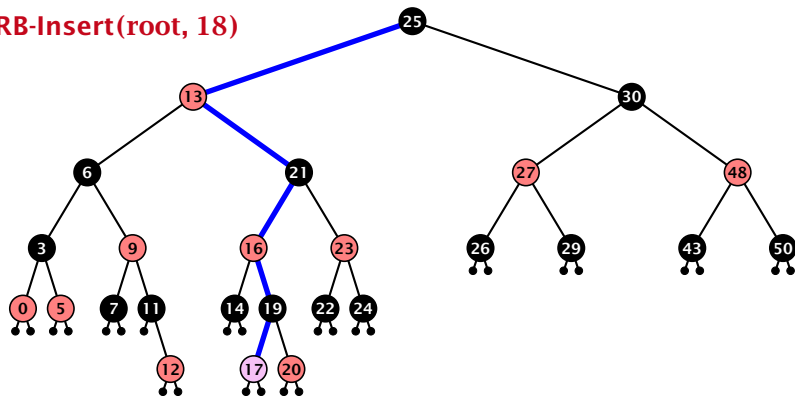


Insert:

- ▶ first make a normal insert into a binary search tree
- ▶ then fix red-black properties

Red Black Trees: Insert

RB-Insert(root, 18)

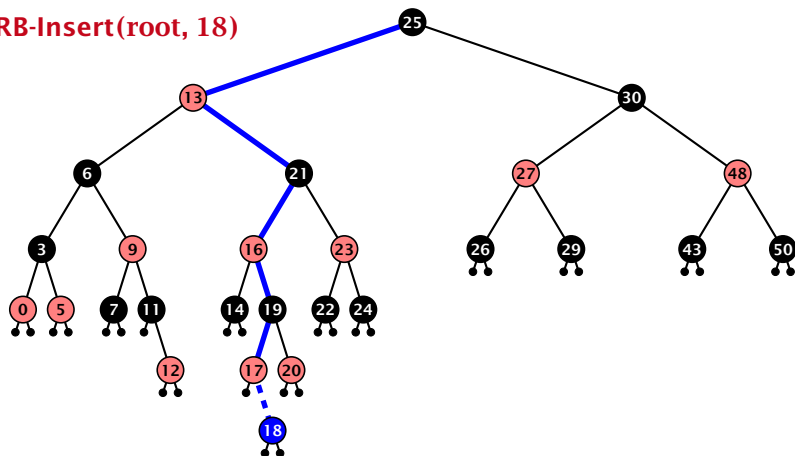


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Red Black Trees: Insert

RB-Insert(root, 18)

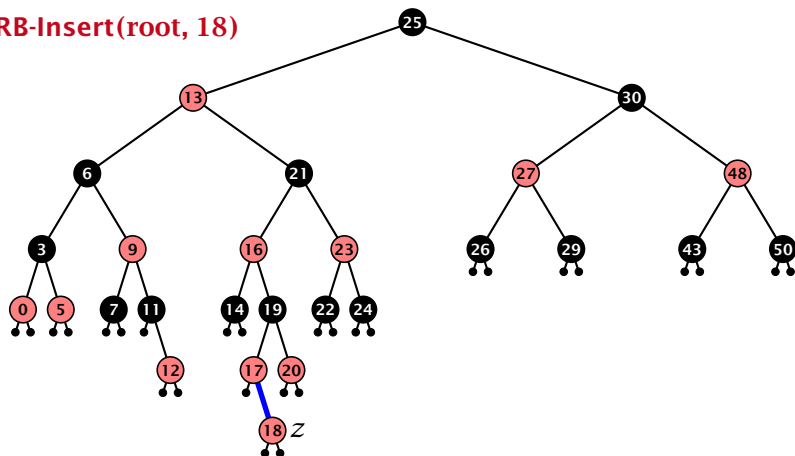


Insert:

- ▶ first make a normal insert into a binary search tree
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Red Black Trees: Insert

RB-Insert(root, 18)



Insert:

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- ▶ then fix red-black properties

Red Black Trees: Insert

Invariant of the fix-up algorithm:

- ▶ z is a red node

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- ▶ the only violation of red-black properties occurs at z and $\text{parent}[z]$

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Invariant of the fix-up algorithm:

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(most important case)

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 - ▶ or the parent does not exist
(violation since root must be black)

Red Black Trees: Insert

Invariant of the fix-up algorithm:

- ▶ z is a red node
- ▶ the black-height property is fulfilled at every node
- ▶ the only violation of red-black properties occurs at z and $\text{parent}[z]$
 - ▶ either both of them are red
(most important case)
 - ▶ or the parent does not exist
(violation since root must be black)

If z has a parent but no grand-parent we could simply color the parent/root black; however this case never happens.

Red Black Trees: Insert

Algorithm 10 InsertFix(z)

```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
2:   if parent[ $z$ ] = left[gp[ $z$ ]] then
3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
7:     else
8:       if  $z$  = right[parent[ $z$ ]] then
9:          $z \leftarrow$  p[ $z$ ]; LeftRotate( $z$ );
10:      col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red;
11:      RightRotate(gp[ $z$ ]);
12:     else same as then-clause but right and left exchanged
13: col(root[ $T$ ])  $\leftarrow$  black;
```

Red Black Trees: Insert

Algorithm 10 InsertFix(z)

```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
2:   if parent[ $z$ ] = left[gp[ $z$ ]] then  $z$  in left subtree of grandparent
3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
7:     else
8:       if  $z$  = right[parent[ $z$ ]] then
9:          $z \leftarrow$  p[ $z$ ]; LeftRotate( $z$ );
10:      col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red;
11:      RightRotate(gp[ $z$ ]);
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Algorithm 10 InsertFix(z)

```
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2:   if parent[ $z$ ] = left[gp[ $z$ ]] then
3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then Case 1: uncle red
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
7:     else
8:       if  $z$  = right[parent[ $z$ ]] then
9:          $z \leftarrow$  p[ $z$ ]; LeftRotate( $z$ );
10:      col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red;
11:      RightRotate(gp[ $z$ ]);
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```

Red Black Trees: Insert

Algorithm 10 InsertFix(z)

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1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
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4:     if col[ $uncle$ ] = red then
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
7:   else Case 2: uncle black
8:     if  $z$  = right[parent[ $z$ ]] then
9:        $z \leftarrow$  p[ $z$ ]; LeftRotate( $z$ );
10:    col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red;
11:    RightRotate(gp[ $z$ ]);
12:   else same as then-clause but right and left exchanged
13: col(root[ $T$ ])  $\leftarrow$  black;
```

Red Black Trees: Insert

Algorithm 10 InsertFix(z)

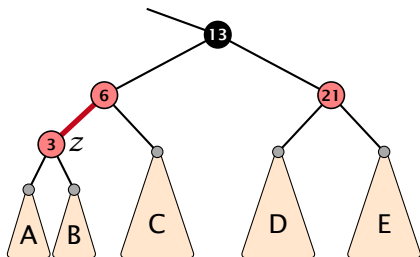
```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
2:   if parent[ $z$ ] = left[gp[ $z$ ]] then
3:     uncle  $\leftarrow$  right[grandparent[ $z$ ]]
4:     if col[uncle] = red then
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[u]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
7:     else
8:       if  $z$  = right[parent[ $z$ ]] then 2a:  $z$  right child
9:          $z \leftarrow$  p[ $z$ ]; LeftRotate( $z$ );
10:        col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red;
11:        RightRotate(gp[ $z$ ]);
12:       else same as then-clause but right and left exchanged
13: col(root[ $T$ ])  $\leftarrow$  black;
```


Red Black Trees: Insert

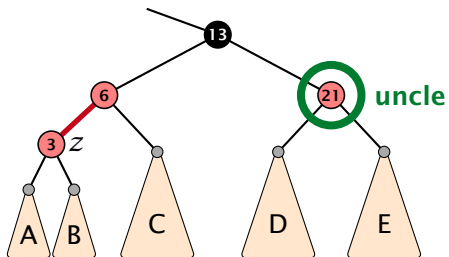
Algorithm 10 InsertFix(z)

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9:          $z \leftarrow$  p[ $z$ ]; LeftRotate( $z$ );
10:      col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red; 2b:  $z$  left child
11:      RightRotate(gp[ $z$ ]);
12:     else same as then-clause but right and left exchanged
13: col(root[ $T$ ])  $\leftarrow$  black;
```

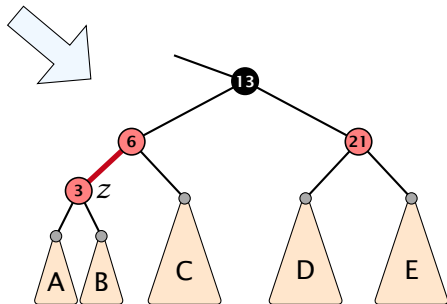
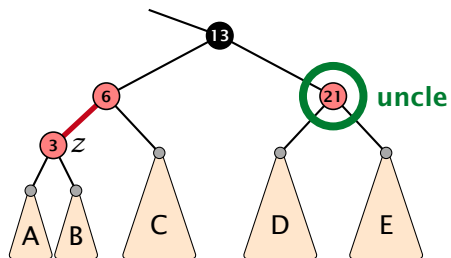
Case 1: Red Uncle



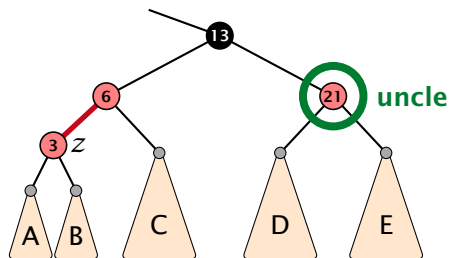
Case 1: Red Uncle



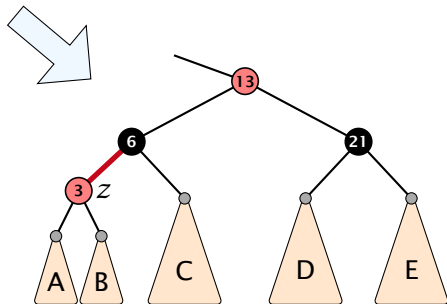
Case 1: Red Uncle



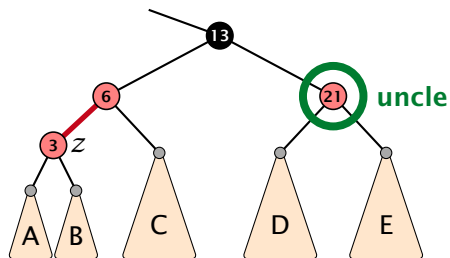
Case 1: Red Uncle



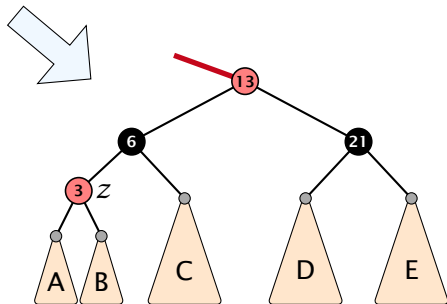
1. recolour



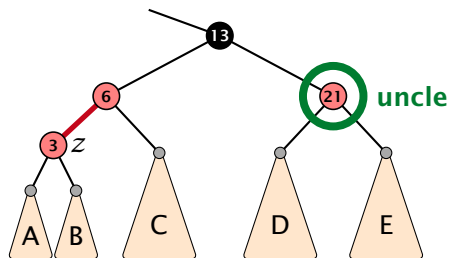
Case 1: Red Uncle



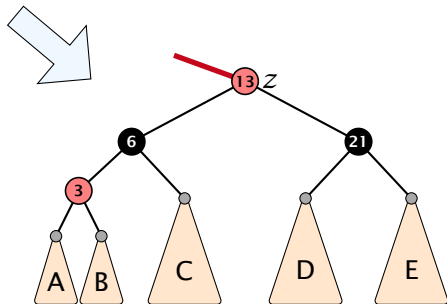
1. recolour



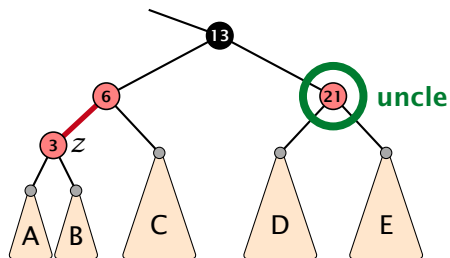
Case 1: Red Uncle



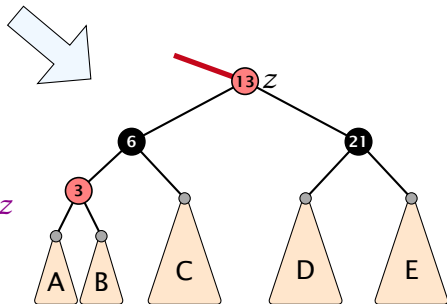
1. recolour
2. move z to grand-parent



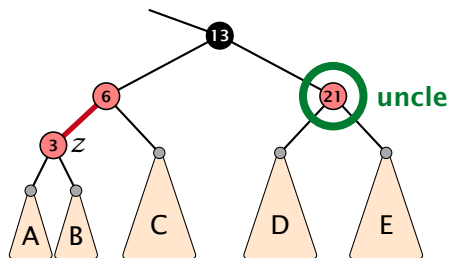
Case 1: Red Uncle



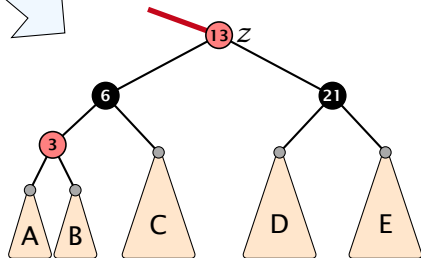
1. recolour
2. move z to grand-parent
3. invariant is fulfilled for new z



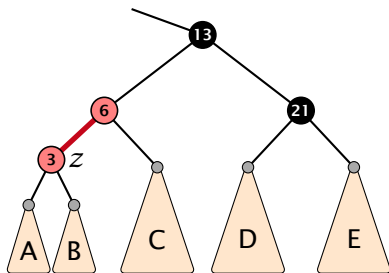
Case 1: Red Uncle



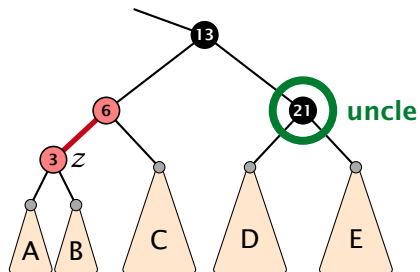
1. recolor
2. move z to grand-parent
3. invariant is fulfilled for new z
4. you made progress



Case 2b: Black uncle and z is left child

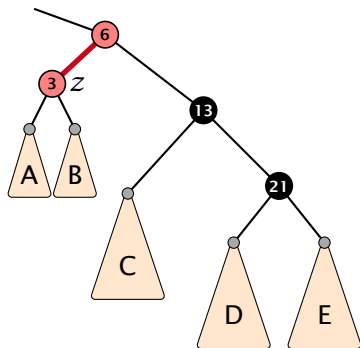
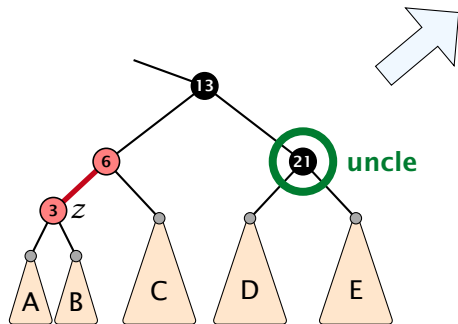


Case 2b: Black uncle and z is left child



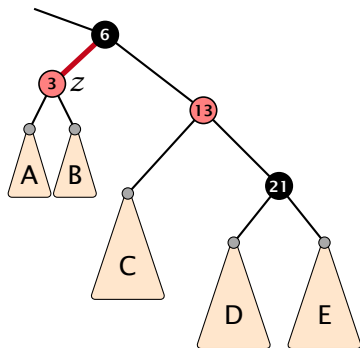
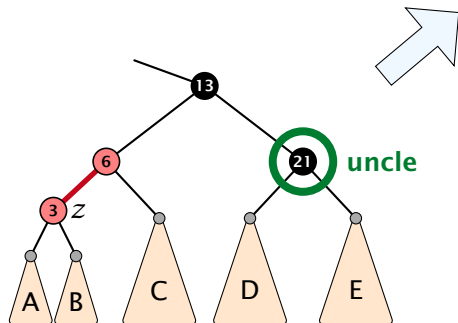
Case 2b: Black uncle and z is left child

1. rotate around grandparent



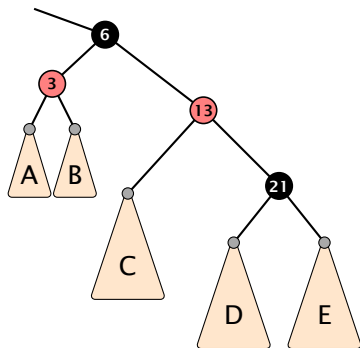
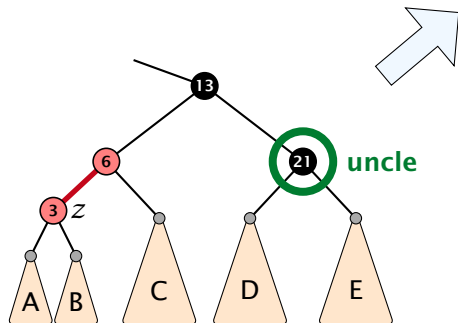
Case 2b: Black uncle and z is left child

1. rotate around grandparent
2. re-colour to ensure that black height property holds

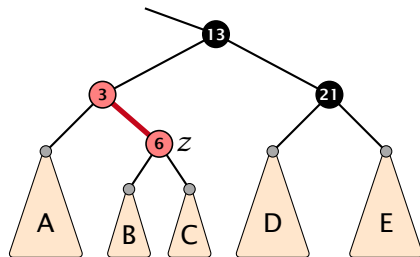


Case 2b: Black uncle and z is left child

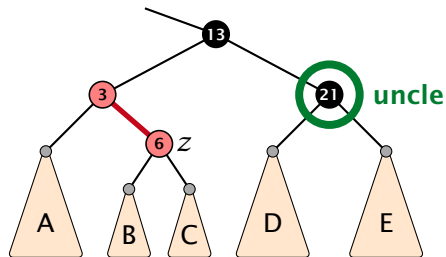
1. rotate around grandparent
2. re-colour to ensure that black height property holds
3. you have a red black tree



Case 2a: Black uncle and z is right child

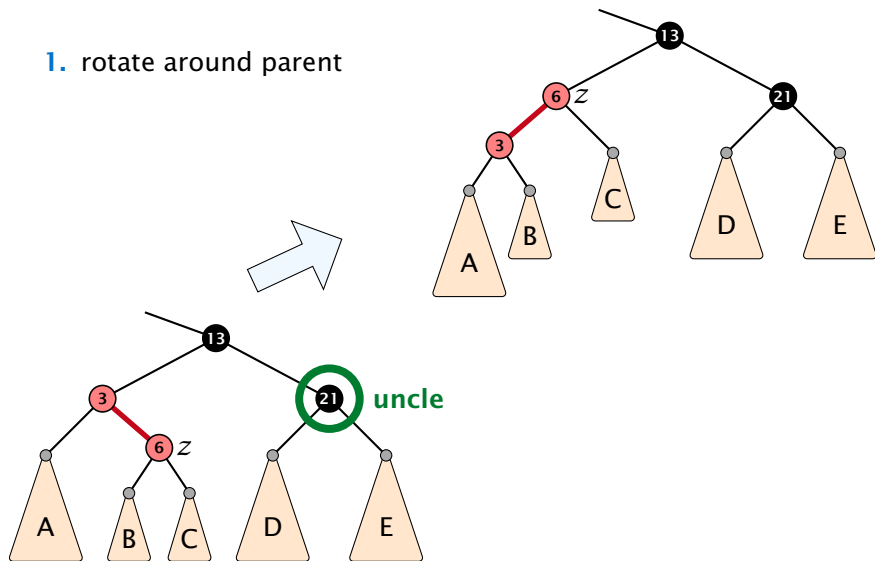


Case 2a: Black uncle and z is right child



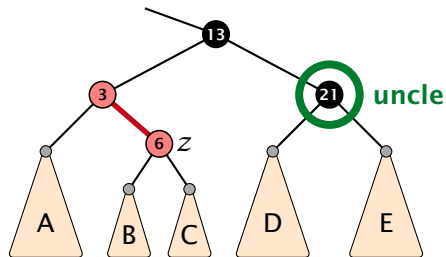
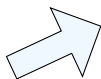
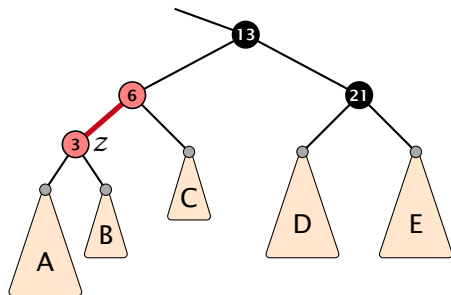
Case 2a: Black uncle and z is right child

1. rotate around parent



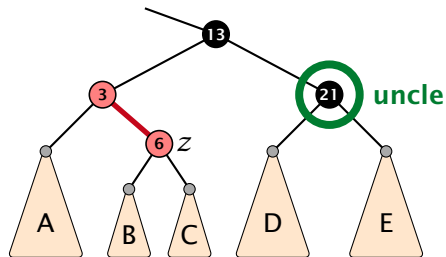
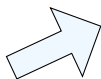
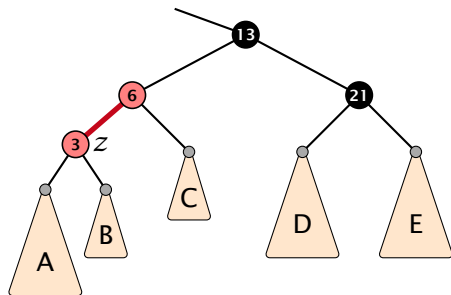
Case 2a: Black uncle and z is right child

1. rotate around parent
2. move z downwards



Case 2a: Black uncle and z is right child

1. rotate around parent
2. move z downwards
3. you have Case 2b.



Red Black Trees: Insert

Running time:

- ▶ Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.

Red Black Trees: Insert

Running time:

- ▶ Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- ▶ Case 2a \rightarrow Case 2b \rightarrow red-black tree

Red Black Trees: Insert

Running time:

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- ▶ Case 2a → Case 2b → red-black tree
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Red Black Trees: Insert

Running time:

- ▶ Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- ▶ Case 2a \rightarrow Case 2b \rightarrow red-black tree
- ▶ Case 2b \rightarrow red-black tree

Performing Case 1 at most $\mathcal{O}(\log n)$ times and every other case at most once, we get a red-black tree. Hence $\mathcal{O}(\log n)$ re-colorings and at most 2 rotations.

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First do a standard delete.

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If the spliced out node x was red everything is fine.

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- ▶ Parent and child of x were red; two adjacent red vertices.

Red Black Trees: Delete

First do a standard delete.

If the spliced out node x was red everything is fine.

If it was black there may be the following problems.

- ▶ Parent and child of x were red; two adjacent red vertices.
- ▶ If you delete the root, the root may now be red.

Red Black Trees: Delete

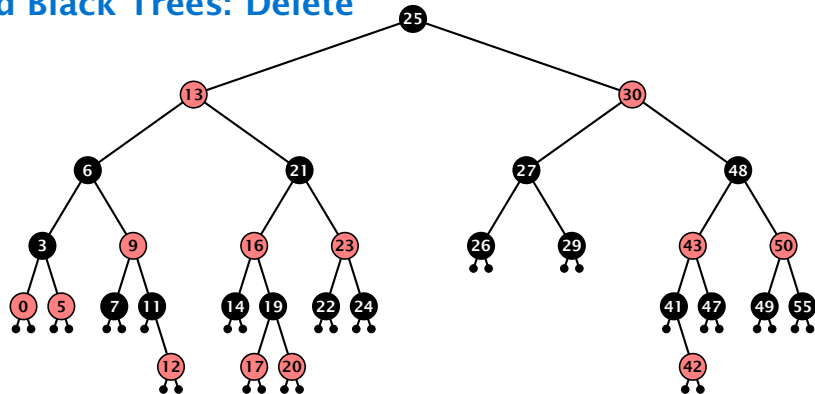
First do a standard delete.

If the spliced out node x was red everything is fine.

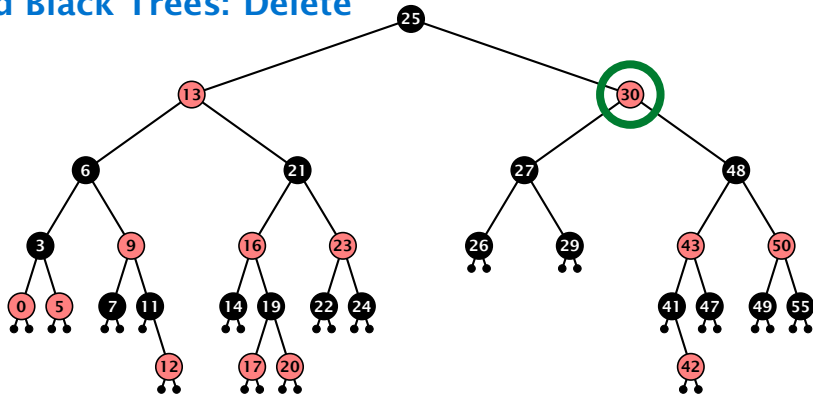
If it was black there may be the following problems.

- ▶ Parent and child of x were red; two adjacent red vertices.
- ▶ If you delete the root, the root may now be red.
- ▶ Every path from an ancestor of x to a descendant leaf of x changes the number of black nodes. Black height property might be violated.

Red Black Trees: Delete



Red Black Trees: Delete

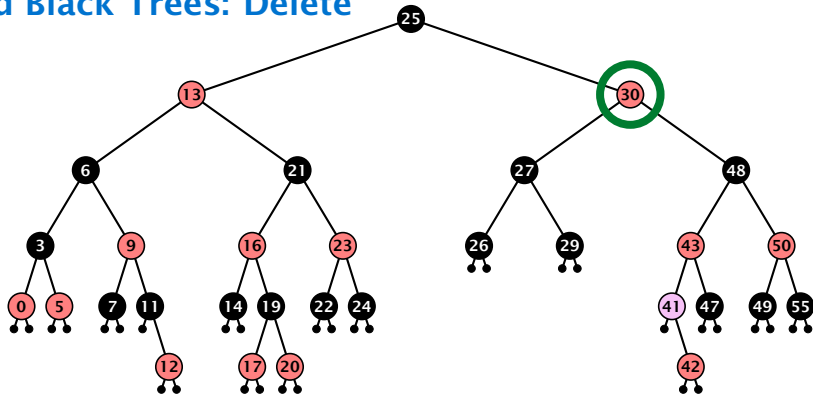


Case 3:

Element has two children

- ▶ do normal delete
- ▶ when replacing content by content of successor, don't change color of node

Red Black Trees: Delete

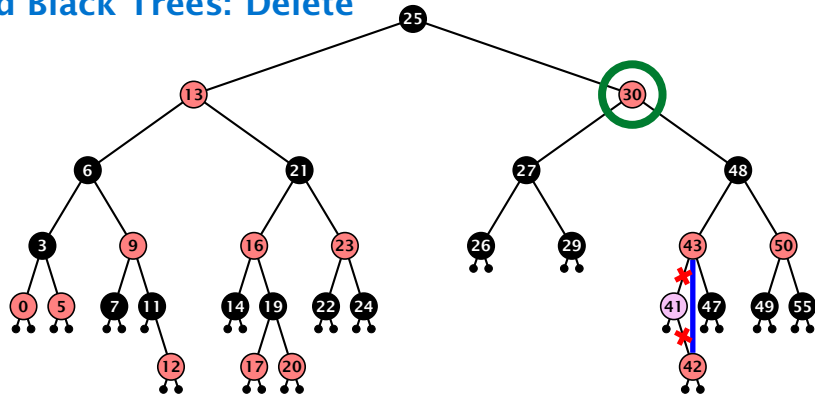


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Red Black Trees: Delete

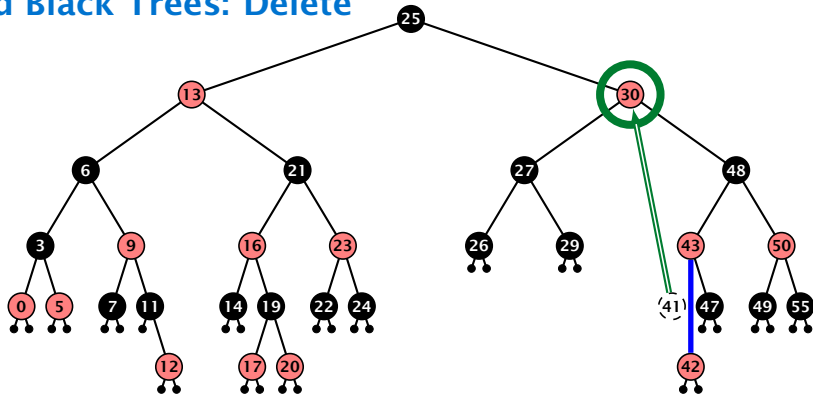


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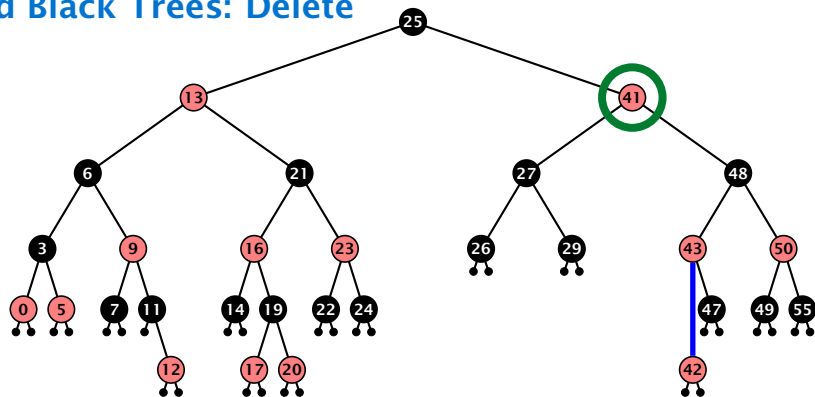


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Red Black Trees: Delete

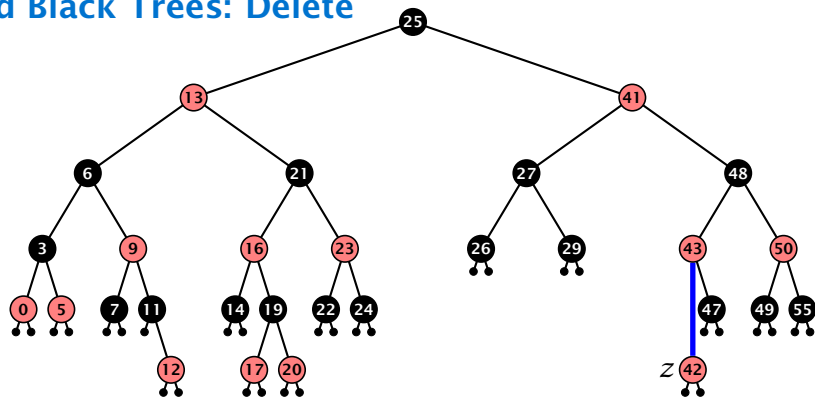


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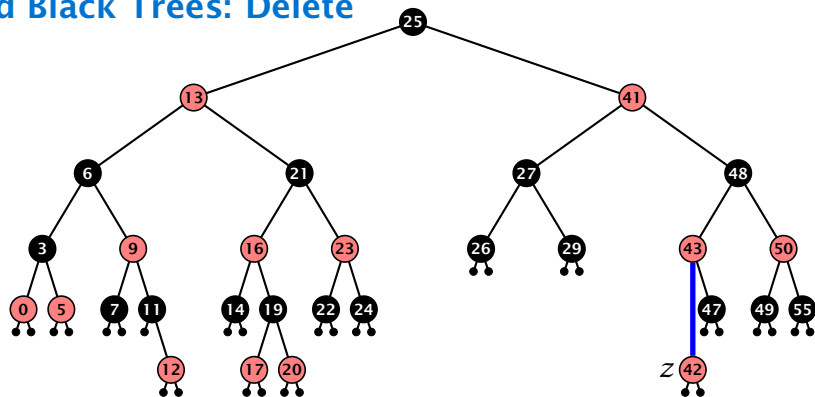
Red Black Trees: Delete



Delete:

- ▶ deleting black node messes up black-height property

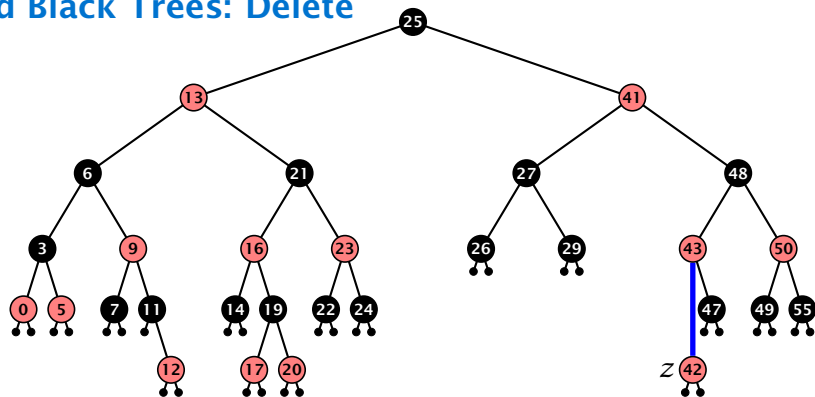
Red Black Trees: Delete



Delete:

- ▶ deleting black node messes up black-height property
- ▶ if z is red, we can simply color it black and everything is fine

Red Black Trees: Delete



Delete:

- ▶ deleting black node messes up black-height property
- ▶ if z is red, we can simply color it black and everything is fine
- ▶ the problem is if z is black (e.g. a dummy-leaf); we call a fix-up procedure to fix the problem.

Red Black Trees: Delete

Invariant of the fix-up algorithm

- ▶ the node z is black

Red Black Trees: Delete

Invariant of the fix-up algorithm

- ▶ the node z is black
- ▶ if we “assign” a fake black unit to the edge from z to its parent then the black-height property is fulfilled

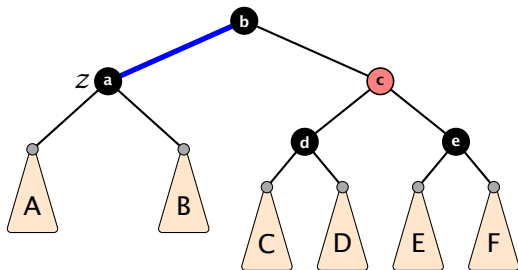
Red Black Trees: Delete

Invariant of the fix-up algorithm

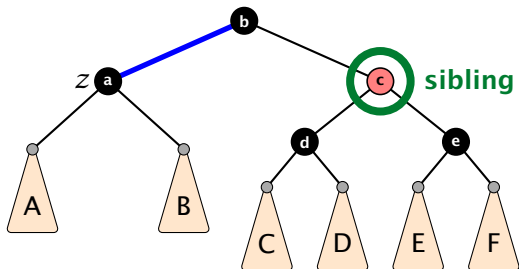
- ▶ the node z is black
- ▶ if we “assign” a fake black unit to the edge from z to its parent then the black-height property is fulfilled

Goal: make rotations in such a way that you at some point can remove the fake black unit from the edge.

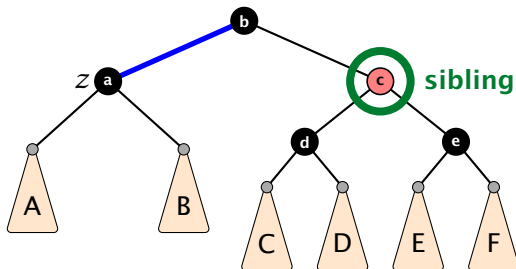
Case 1: Sibling of z is red



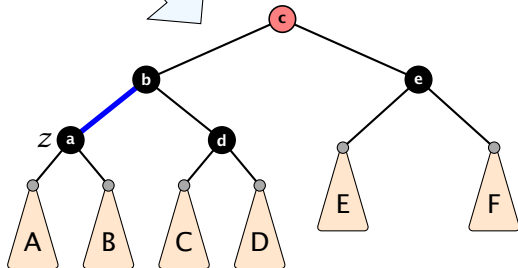
Case 1: Sibling of z is red



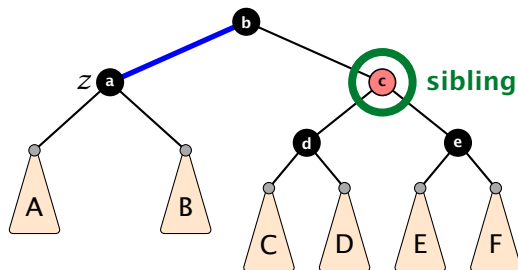
Case 1: Sibling of z is red



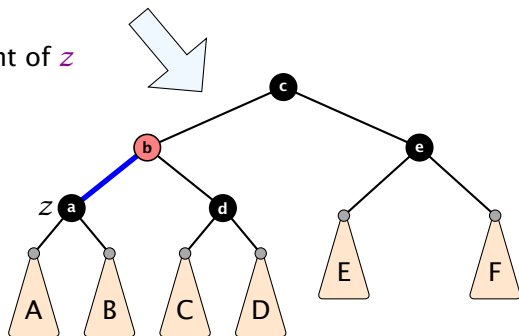
1. left-rotate around parent of z



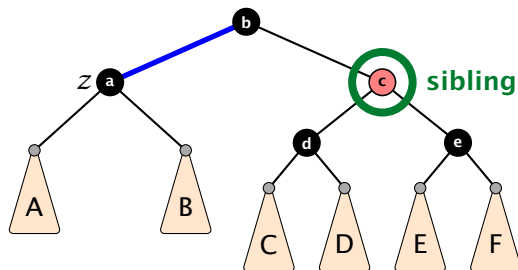
Case 1: Sibling of z is red



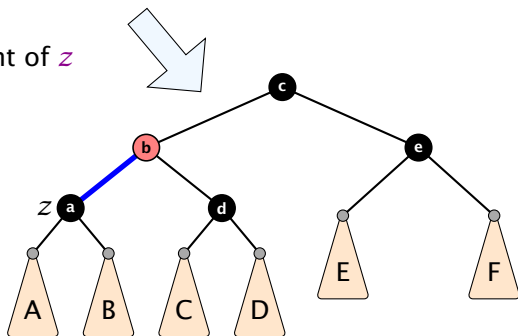
1. left-rotate around parent of z
2. recolor nodes b and c



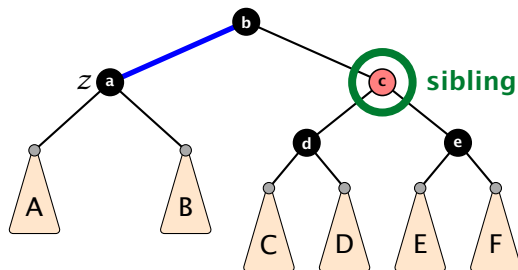
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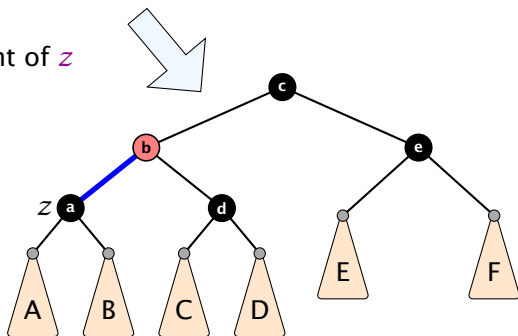
1. left-rotate around parent of z
2. recolor nodes b and c
3. the new sibling is black (and parent of z is red)



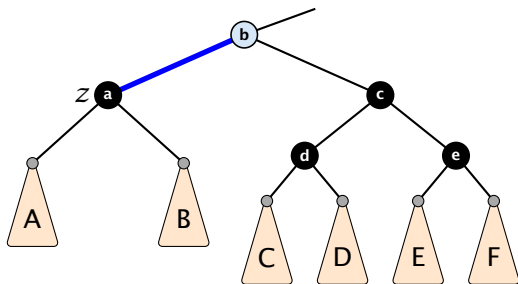
Case 1: Sibling of z is red



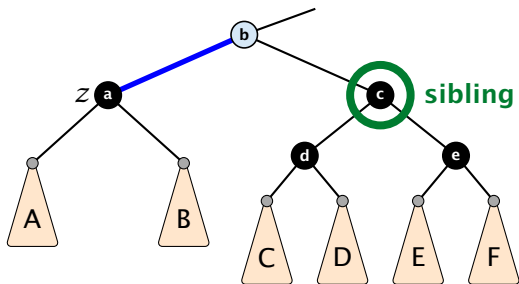
1. left-rotate around parent of z
2. recolor nodes b and c
3. the new sibling is black (and parent of z is red)
4. Case 2 (special), or Case 3, or Case 4



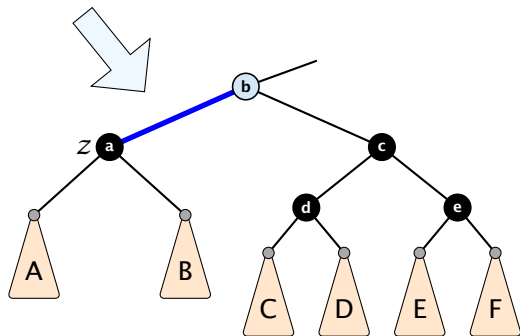
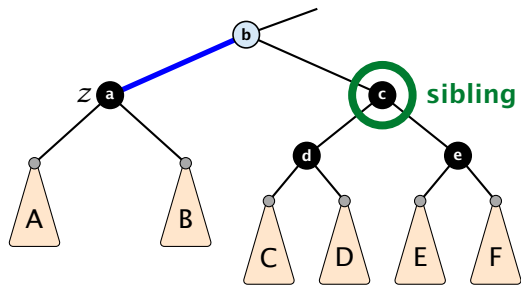
Case 2: Sibling is black with two black children



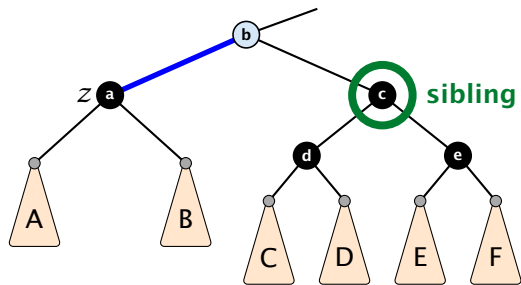
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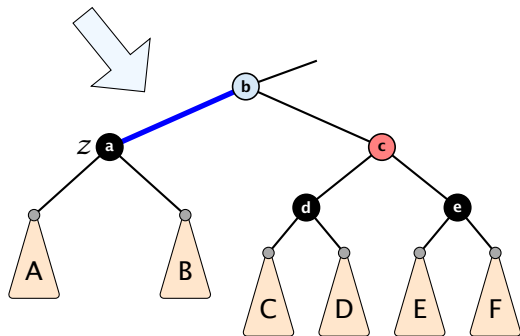
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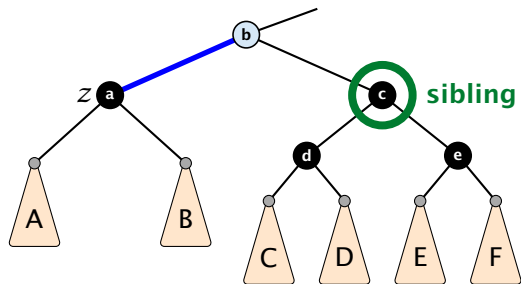
Case 2: Sibling is black with two black children



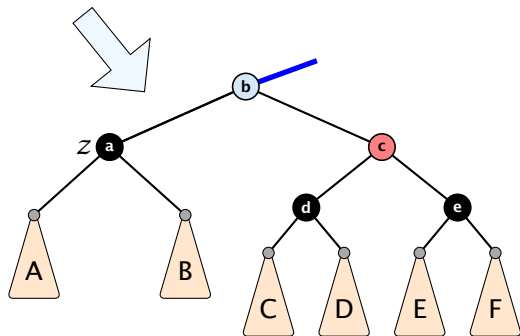
1. re-color node **c**



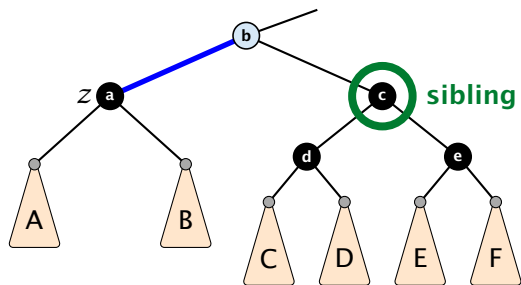
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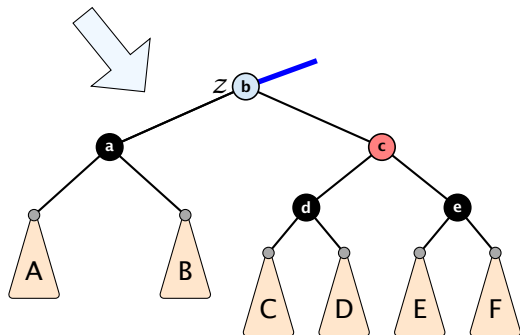
1. re-color node *c*
2. move fake black unit upwards



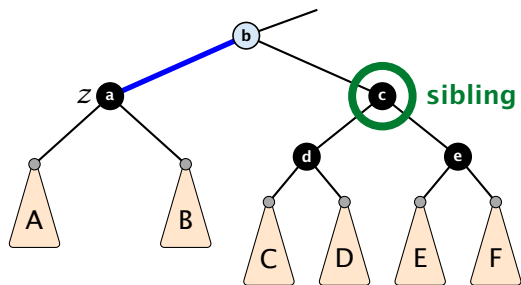
Case 2: Sibling is black with two black children



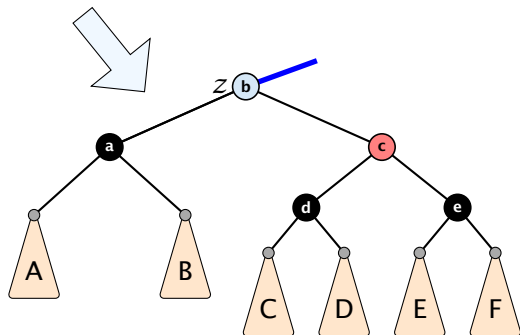
1. re-color node **c**
2. move fake black unit upwards
3. move **z** upwards



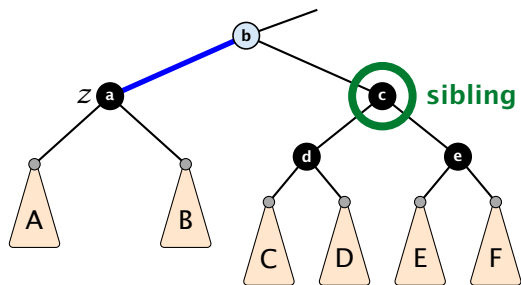
Case 2: Sibling is black with two black children



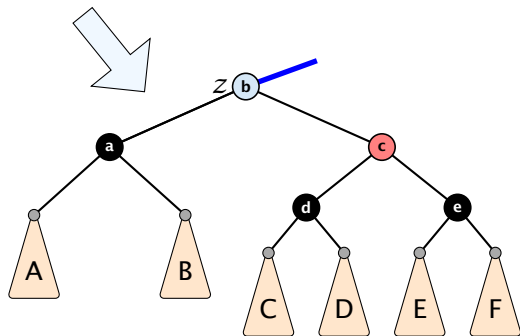
1. re-color node **c**
2. move fake black unit upwards
3. move **z** upwards
4. we made progress



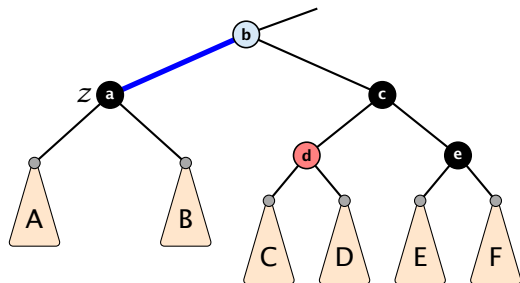
Case 2: Sibling is black with two black children



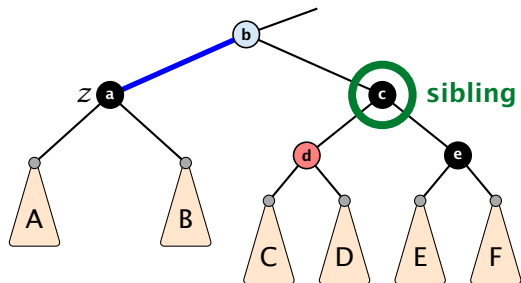
1. re-color node c
2. move fake black unit upwards
3. move z upwards
4. we made progress
5. if b is red we color it black and are done



Case 3: Sibling black with one black child to the right

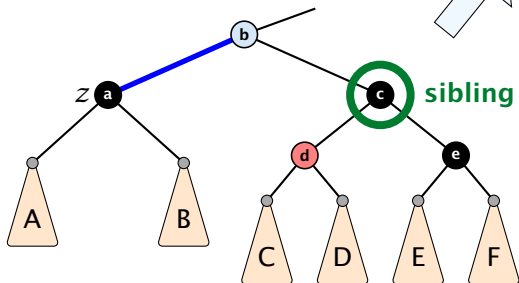
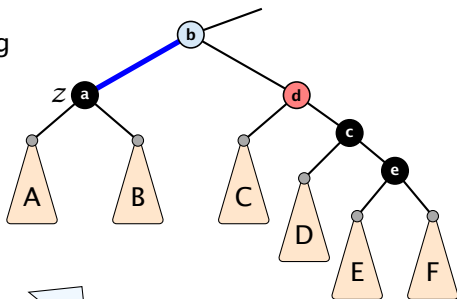


Case 3: Sibling black with one black child to the right



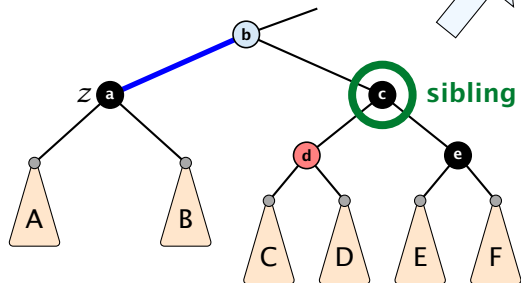
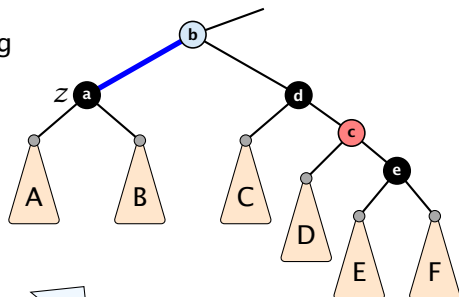
Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling



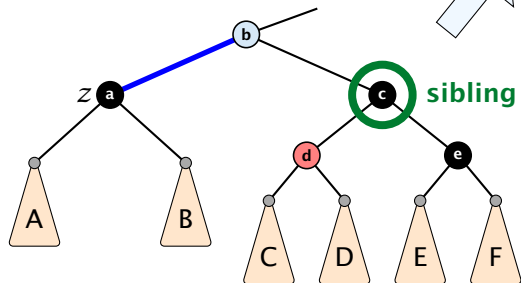
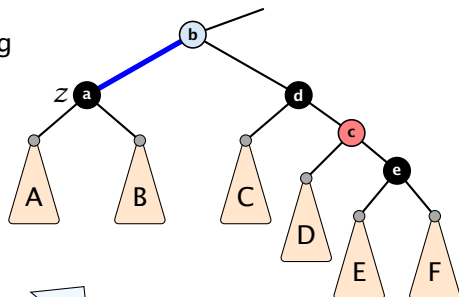
Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling
2. recolor c and d

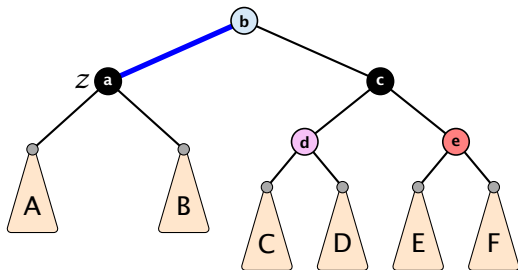


Case 3: Sibling black with one black child to the right

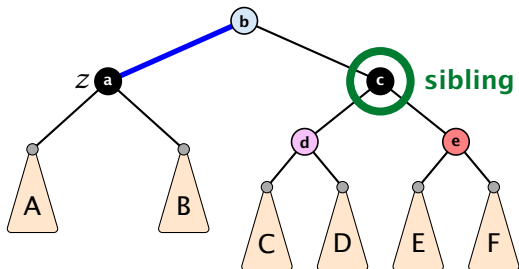
1. do a right-rotation at sibling
2. recolor c and d
3. new sibling is black with red right child (Case 4)



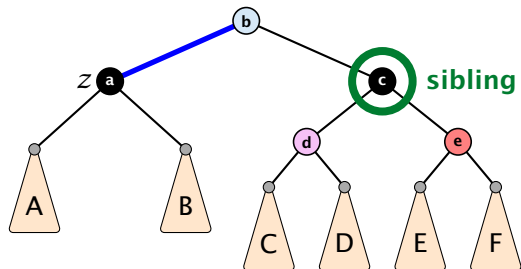
Case 4: Sibling is black with red right child



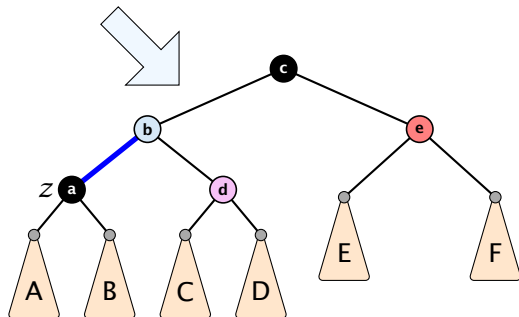
Case 4: Sibling is black with red right child



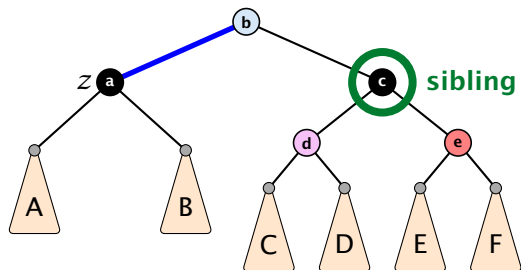
Case 4: Sibling is black with red right child



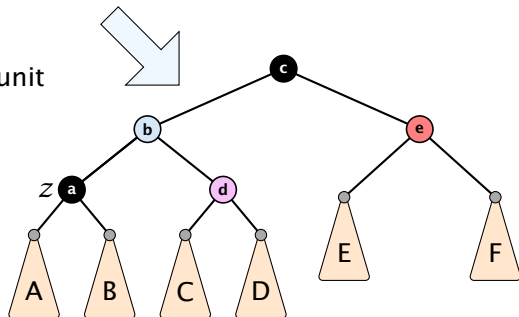
1. left-rotate around *b*



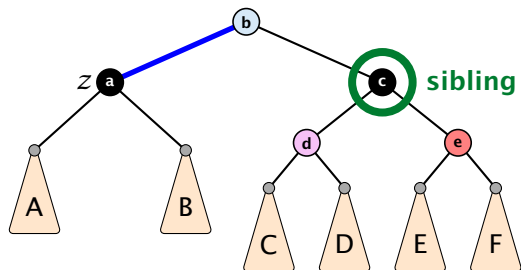
Case 4: Sibling is black with red right child



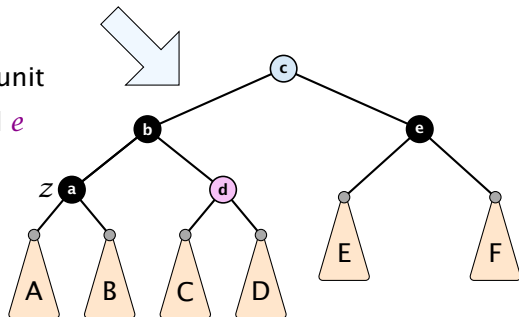
1. left-rotate around *b*
2. remove the fake black unit



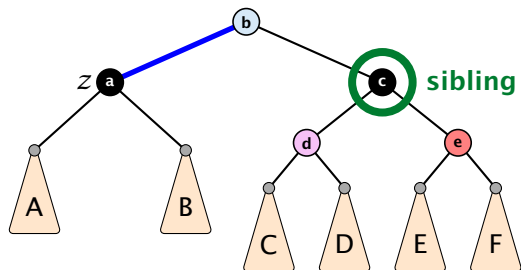
Case 4: Sibling is black with red right child



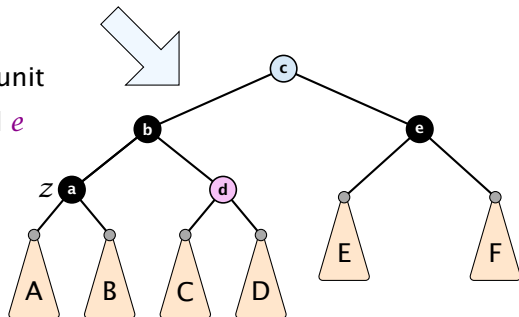
1. left-rotate around b
2. remove the fake black unit
3. recolor nodes b , c , and e



Case 4: Sibling is black with red right child



1. left-rotate around *b*
2. remove the fake black unit
3. recolor nodes *b*, *c*, and *e*
4. you have a valid red black tree



Running time:

- ▶ only Case 2 can repeat; but only h many steps, where h is the height of the tree

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Case 1 → Case 3 → Case 4 → red black tree
Case 1 → Case 4 → red black tree
- ▶ Case 3 → Case 4 → red black tree
- ▶ Case 4 → red black tree

Performing Case 2 at most $\mathcal{O}(\log n)$ times and every other step at most once, we get a red black tree. Hence, $\mathcal{O}(\log n)$ re-colorings and at most 3 rotations.

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Splay Trees:

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repeated accesses are faster
- only amortized guarantee
- read-operations change the tree

Splay Trees

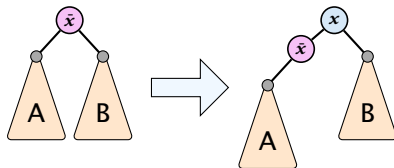
find(x)

- ▶ search for x according to a search tree
- ▶ let \tilde{x} be last element on search-path
- ▶ $\text{splay}(\tilde{x})$

Splay Trees

insert(x)

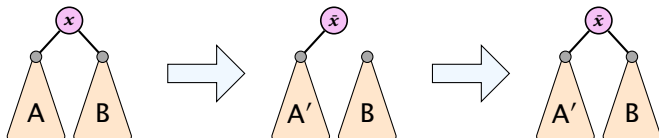
- ▶ search for x ; \bar{x} is last visited element during search (successor or predecessor of x)
- ▶ splay(\bar{x}) moves \bar{x} to the root
- ▶ insert x as new root



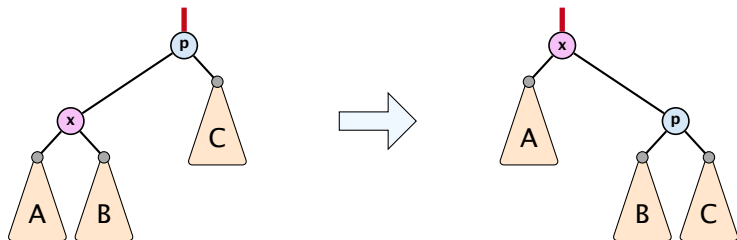
Splay Trees

delete(x)

- ▶ search for x ; splay(x); remove x
- ▶ search largest element \bar{x} in A
- ▶ splay(\bar{x}) (on subtree A)
- ▶ connect root of B as right child of \bar{x}



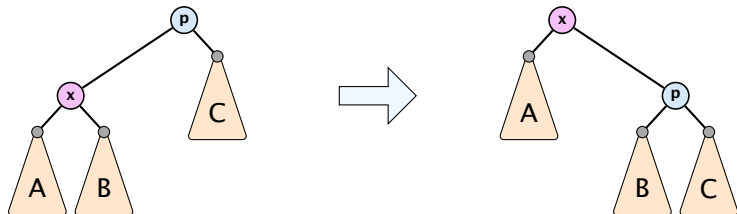
Move to Root



How to bring element to root?

- ▶ one (bad) option: `moveToRoot(x)`
- ▶ iteratively do rotation around parent of x until x is root
- ▶ if x is left child do right rotation otw. left rotation

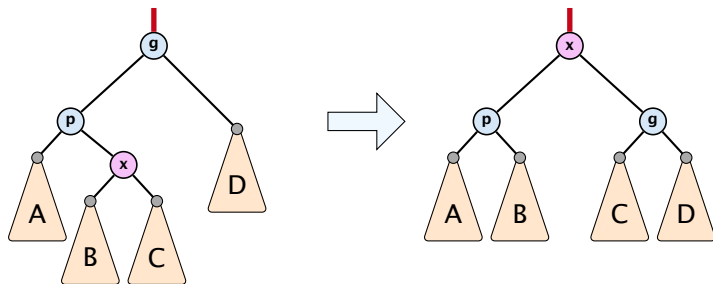
Splay: Zig Case



better option splay(x):

- ▶ zig case: if x is child of root do left rotation or right rotation around parent

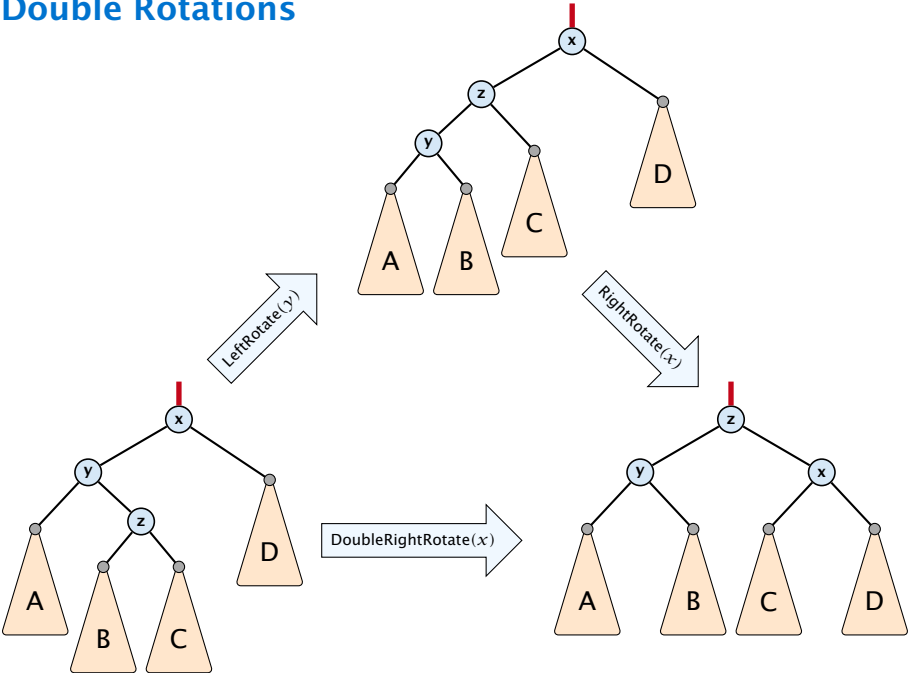
Splay: Zigzag Case



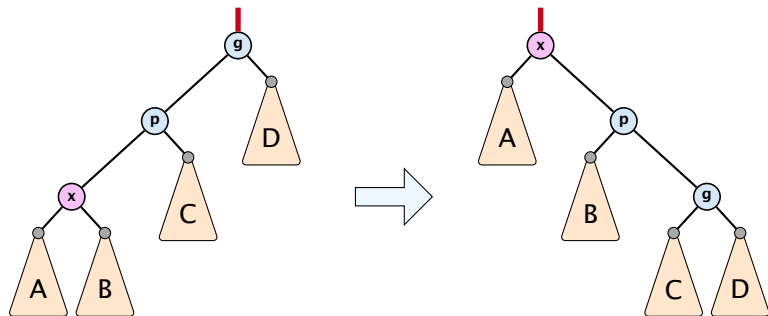
better option $\text{splay}(x)$:

- ▶ zigzag case: if x is right child and parent of x is left child (or x left child parent of x right child)
- ▶ do double right rotation around grand-parent (resp. double left rotation)

Double Rotations



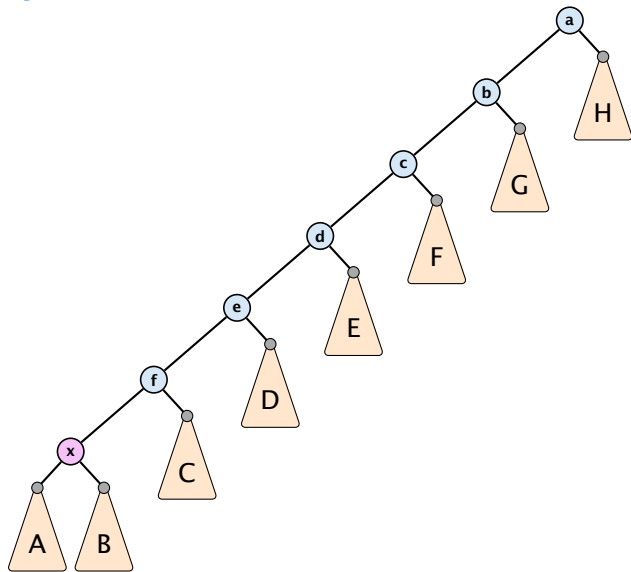
Splay: Zigzig Case



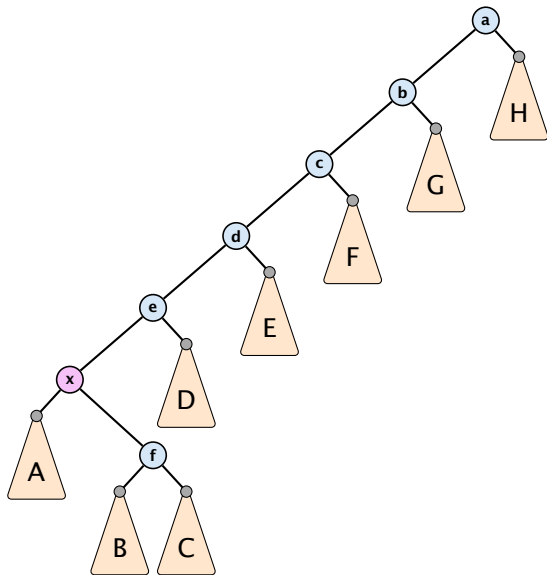
better option $\text{splay}(x)$:

- ▶ zigzig case: if x is left child and parent of x is left child (or x right child, parent of x right child)
- ▶ do right rotation around grand-parent followed by right rotation around parent (resp. left rotations)

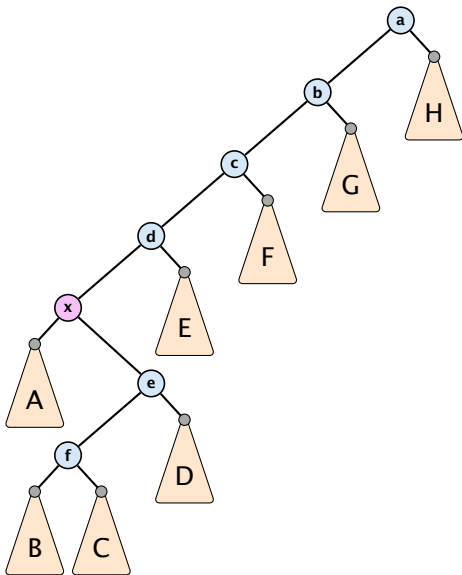
Splay vs. Move to Root



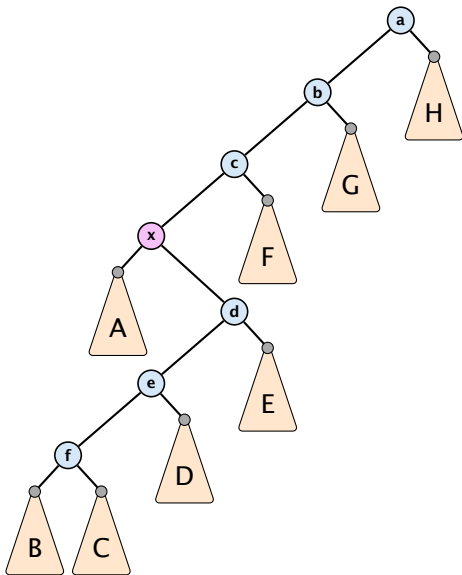
Splay vs. Move to Root



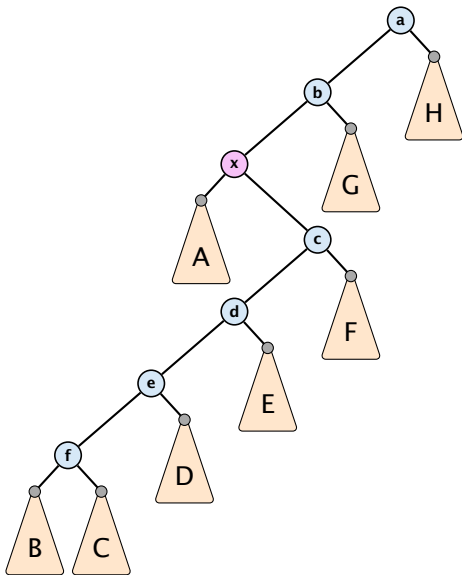
Splay vs. Move to Root



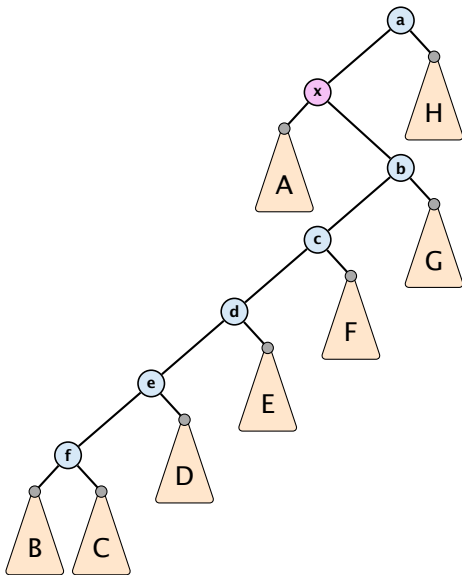
Splay vs. Move to Root



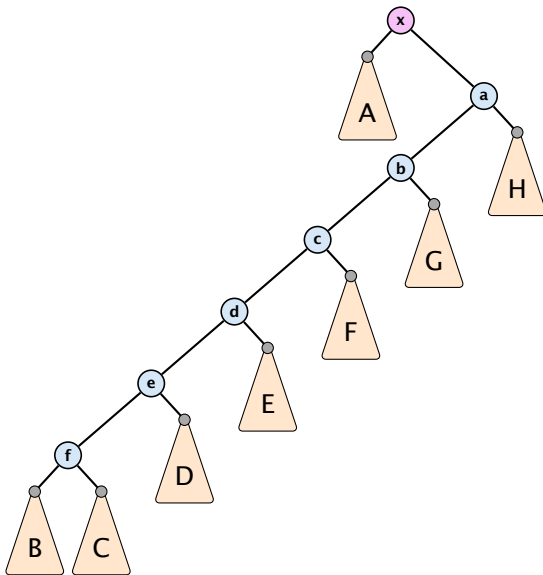
Splay vs. Move to Root



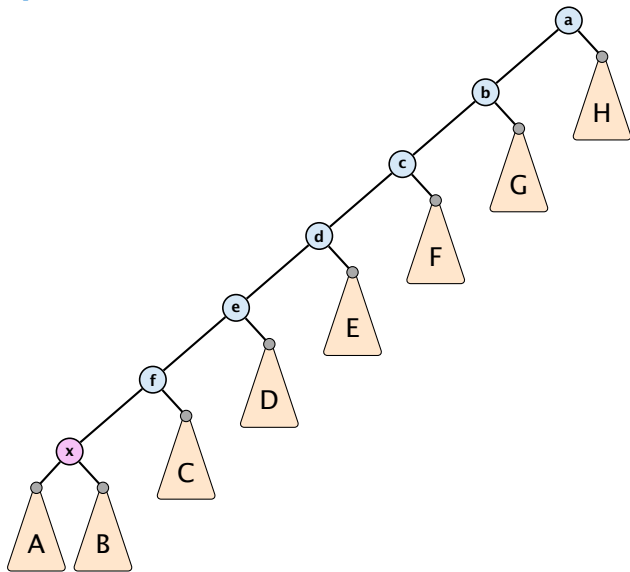
Splay vs. Move to Root



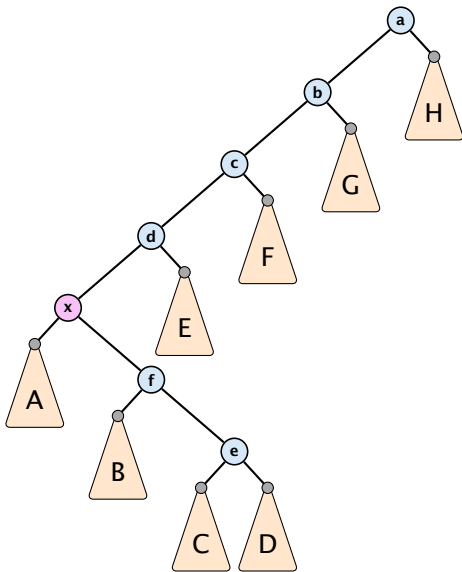
Splay vs. Move to Root



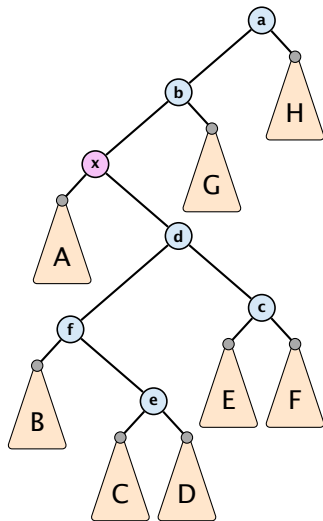
Splay vs. Move to Root



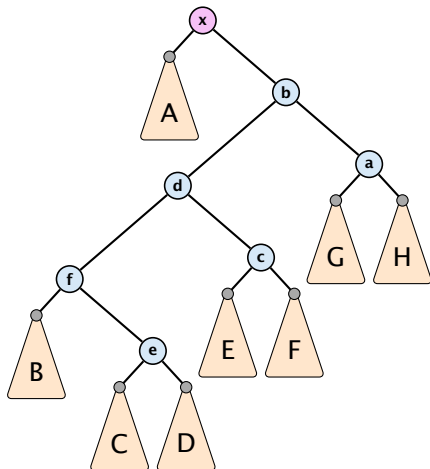
Splay vs. Move to Root



Splay vs. Move to Root



Splay vs. Move to Root



Static Optimality

Suppose we have a sequence of m find-operations. $\text{find}(x)$ appears h_x times in this sequence.

The cost of a **static** search tree T is:

$$\text{cost}(T) = m + \sum_x h_x \text{depth}_T(x)$$

The total cost for processing the sequence on a splay-tree is $\mathcal{O}(\text{cost}(T_{\min}))$, where T_{\min} is an **optimal static search tree**.

Dynamic Optimality

Let S be a sequence with m find-operations.

Let A be a data-structure based on a search tree:

- ▶ the cost for accessing element x is $1 + \text{depth}(x)$;
- ▶ after accessing x the tree may be re-arranged through rotations;

Conjecture:

A splay tree that only contains elements from S has cost $\mathcal{O}(\text{cost}(A, S))$, for processing S .

Lemma 16

*Splay Trees have an **amortized** running time of $\mathcal{O}(\log n)$ for all operations.*

Amortized Analysis

Definition 17

A data structure with operations $\text{op}_1(), \dots, \text{op}_k()$ has amortized running times t_1, \dots, t_k for these operations if the following holds.

Suppose you are given a sequence of operations (**starting with an empty data-structure**) that operate on at most n elements, and let k_i denote the number of occurrences of $\text{op}_i()$ within this sequence. Then the actual running time must be at most $\sum_i k_i \cdot t_i(n)$.

Potential Method

Introduce a potential for the data structure.

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$$\sum_{i=1}^k c_i$$

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- ▶ Show that $\Phi(D_i) \geq \Phi(D_0)$.

Then

$$\sum_{i=1}^k c_i \leq \sum_{i=1}^k c_i + \Phi(D_k) - \Phi(D_0)$$

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- ▶ Show that $\Phi(D_i) \geq \Phi(D_0)$.

Then

$$\sum_{i=1}^k c_i \leq \sum_{i=1}^k c_i + \Phi(D_k) - \Phi(D_0) = \sum_{i=1}^k \hat{c}_i$$

This means the amortized costs can be used to derive a bound on the total cost.

Example: Stack

Stack

- ▶ $S.$ push()
- ▶ $S.$ pop()
- ▶ $S.$ multipop(k): removes k items from the stack. If the stack currently contains less than k items it empties the stack.
- ▶ The user has to ensure that pop and multipop do not generate an underflow.

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Actual cost:

- ▶ $S.$ push(): cost 1.
- ▶ $S.$ pop(): cost 1.
- ▶ $S.$ multipop(k): cost $\min\{\text{size}, k\} = k$.

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$$\hat{C}_{\text{mp}} = C_{\text{mp}} + \Delta\Phi = \min\{\text{size}, k\} - \min\{\text{size}, k\} \leq 0 .$$

Example: Binary Counter

Incrementing a binary counter:

Consider a computational model where each bit-operation costs one time-unit.

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Actual cost:

- ▶ Changing bit from 0 to 1: cost 1.
- ▶ Changing bit from 1 to 0: cost 1.
- ▶ **Increment**: cost is $k + 1$, where k is the number of consecutive ones in the least significant bit-positions (e.g, 001101 has $k = 1$).

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- ▶ Changing bit from 1 to 0:

$$\hat{C}_{1 \rightarrow 0} = C_{1 \rightarrow 0} + \Delta\Phi = 1 - 1 \leq 0 .$$

- ▶ **Increment:** Let k denotes the number of consecutive ones in the least significant bit-positions. An increment involves k $(1 \rightarrow 0)$ -operations, and one $(0 \rightarrow 1)$ -operation.

Hence, the amortized cost is $k\hat{C}_{1 \rightarrow 0} + \hat{C}_{0 \rightarrow 1} \leq 2$.

Splay Trees

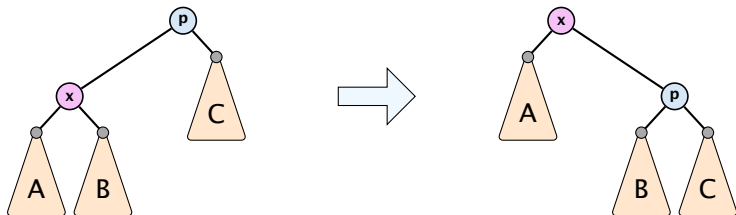
potential function for splay trees:

- ▶ size $s(x) = |T_x|$
- ▶ rank $r(x) = \log_2(s(x))$
- ▶ $\Phi(T) = \sum_{v \in T} r(v)$

amortized cost = real cost + potential change

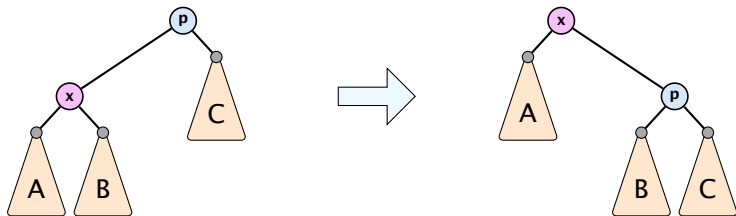
The cost is essentially the cost of the splay-operation, which is 1 plus the number of rotations.

Splay: Zig Case



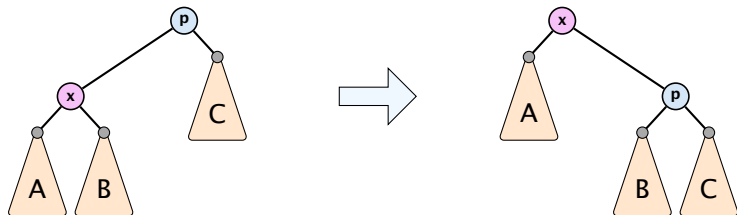
$$\Delta\Phi =$$

Splay: Zig Case



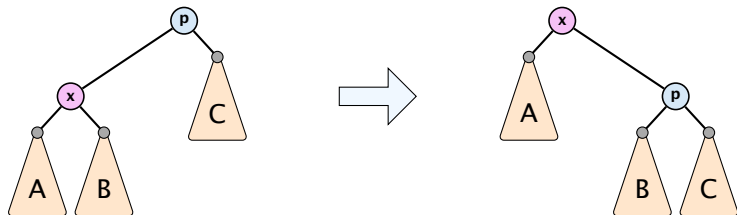
$$\Delta\Phi = r'(x) + r'(p) - r(x) - r(p)$$

Splay: Zig Case



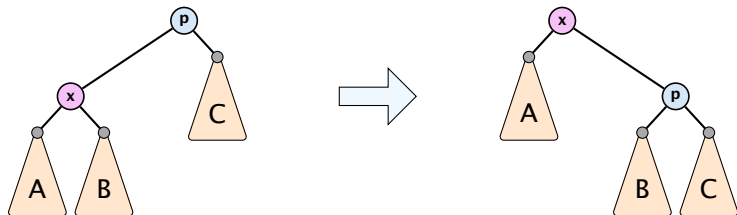
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p) \\ &= r'(p) - r(x)\end{aligned}$$

Splay: Zig Case



$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p) \\ &= r'(p) - r(x) \\ &\leq r'(x) - r(x)\end{aligned}$$

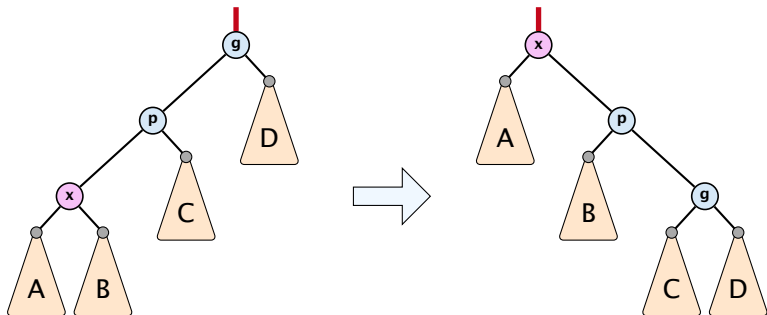
Splay: Zig Case



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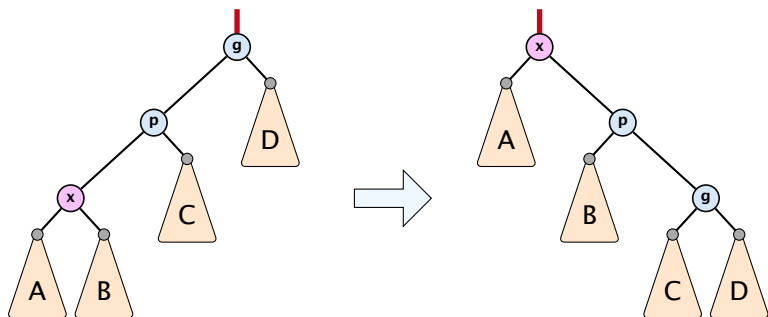
$$\text{cost}_{\text{zig}} \leq 1 + 3(r'(x) - r(x))$$

Splay: Zigzig Case



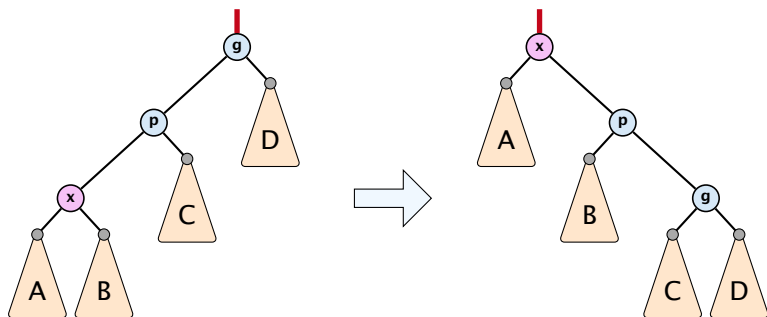
$$\Delta\Phi =$$

Splay: Zigzig Case



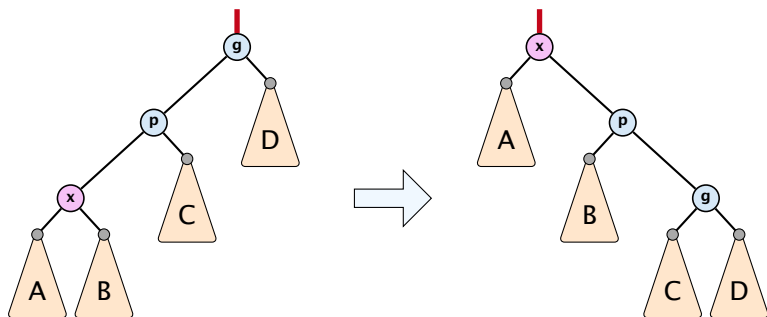
$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$

Splay: Zigzig Case



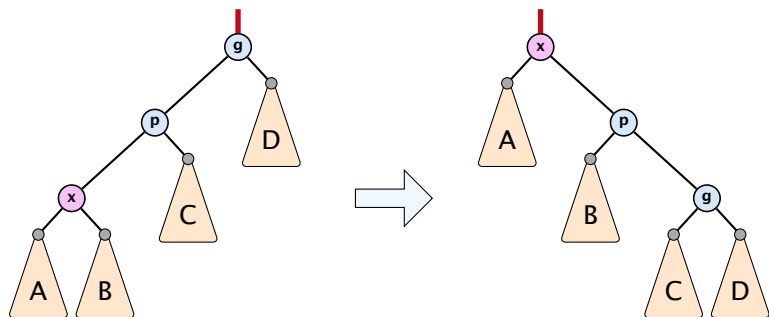
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p)\end{aligned}$$

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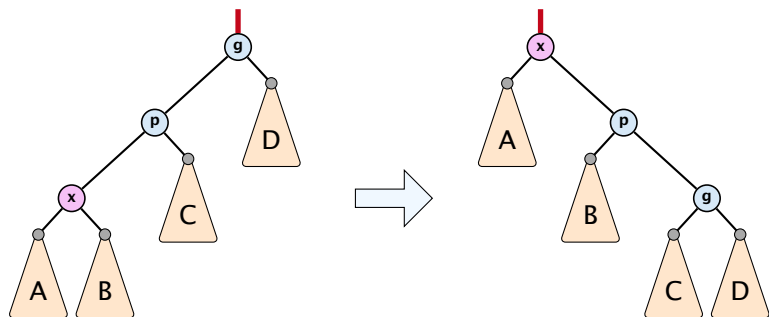
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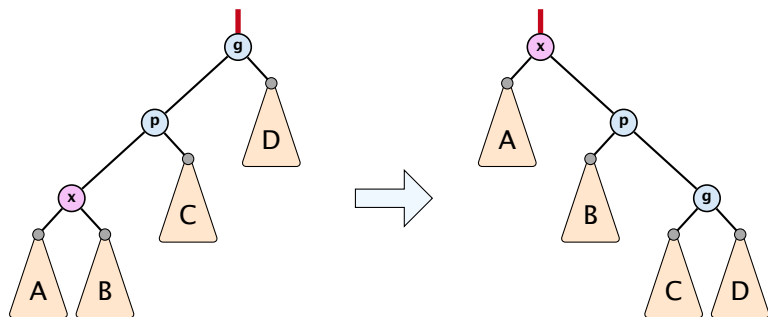
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(x) + r'(g) - r(x) - r(x) \\ &= r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x)\end{aligned}$$

Splay: Zigzig Case



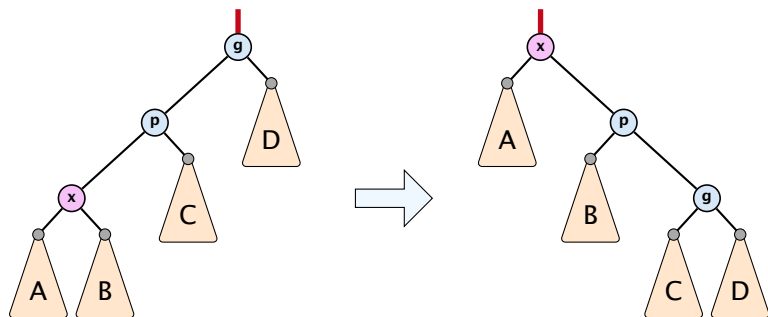
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(x) + r'(g) - r(x) - r(x) \\ &= r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x) \\ &= -2r'(x) + r'(g) + r(x) + 3(r'(x) - r(x))\end{aligned}$$

Splay: Zigzig Case



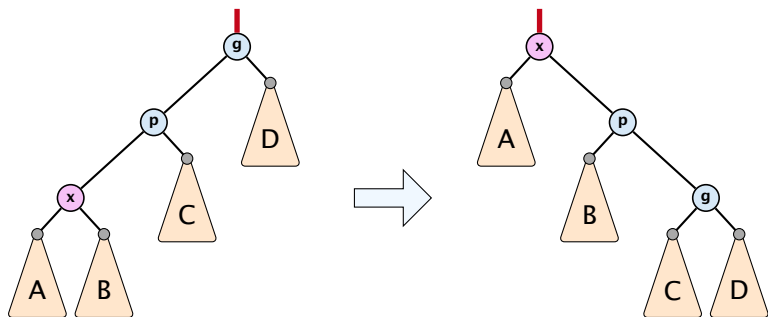
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Splay: Zigzig Case



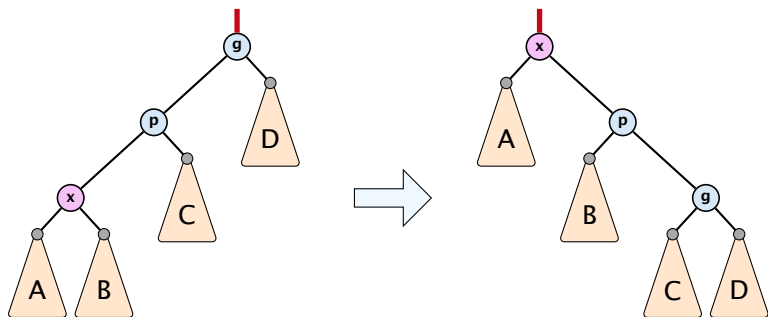
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Splay: Zigzig Case



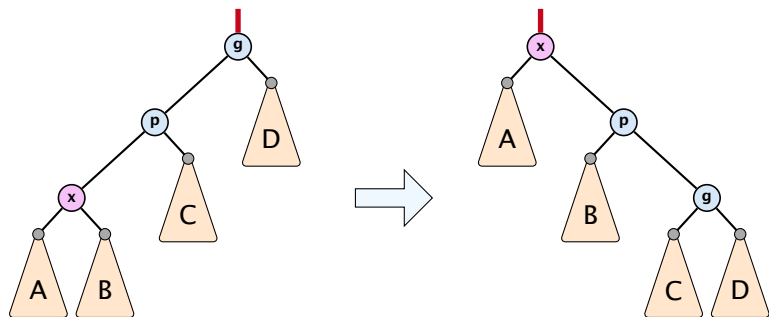
$$\frac{1}{2}(r(x) + r'(g) - 2r'(x))$$

Splay: Zigzig Case



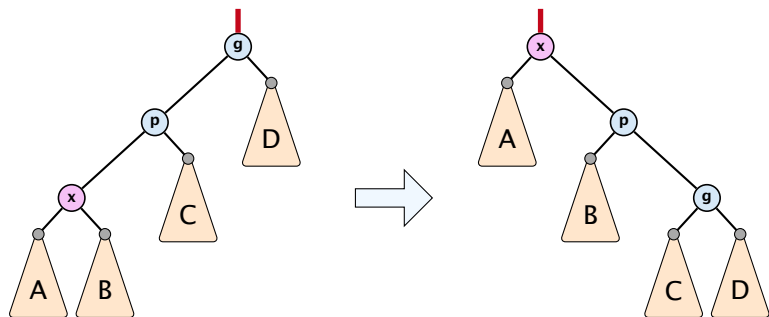
$$\begin{aligned} & \frac{1}{2} (r(x) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} (\log(s(x)) + \log(s'(g)) - 2\log(s'(x))) \end{aligned}$$

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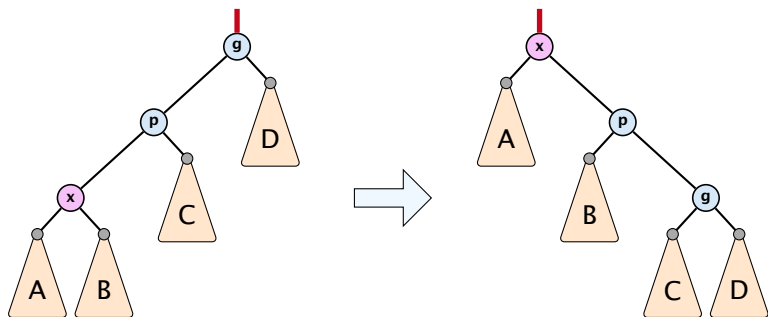
$$\begin{aligned} & \frac{1}{2} (r(x) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} (\log(s(x)) + \log(s'(g)) - 2\log(s'(x))) \\ &= \frac{1}{2} \log\left(\frac{s(x)}{s'(x)}\right) + \frac{1}{2} \log\left(\frac{s'(g)}{s'(x)}\right) \end{aligned}$$

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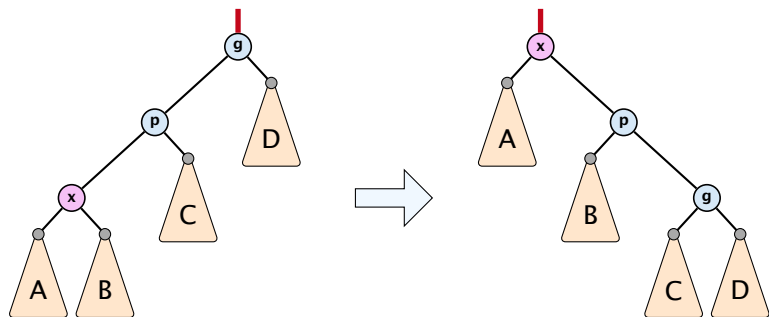
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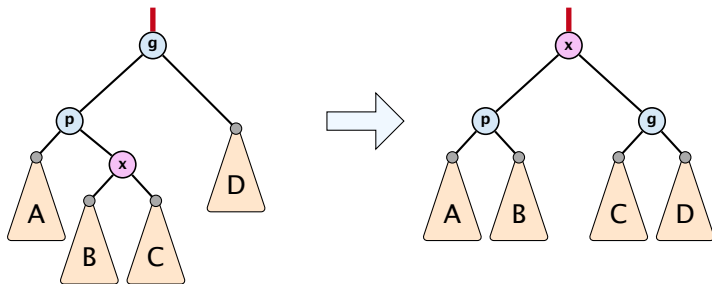
$$\begin{aligned} & \frac{1}{2} (r(x) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} \left(\log(s(x)) + \log(s'(g)) - 2 \log(s'(x)) \right) \\ &= \frac{1}{2} \log \left(\frac{s(x)}{s'(x)} \right) + \frac{1}{2} \log \left(\frac{s'(g)}{s'(x)} \right) \\ &\leq \log \left(\frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)} \right) \leq \log \left(\frac{1}{2} \right) \end{aligned}$$

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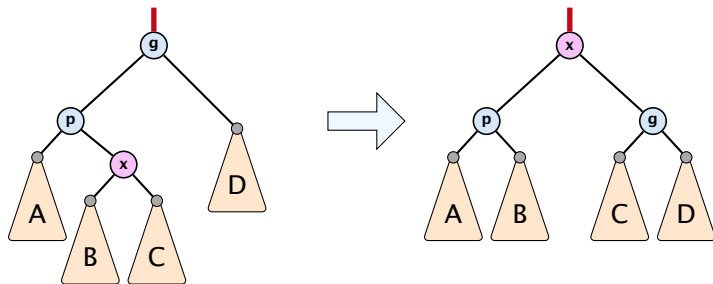
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Splay: Zigzag Case



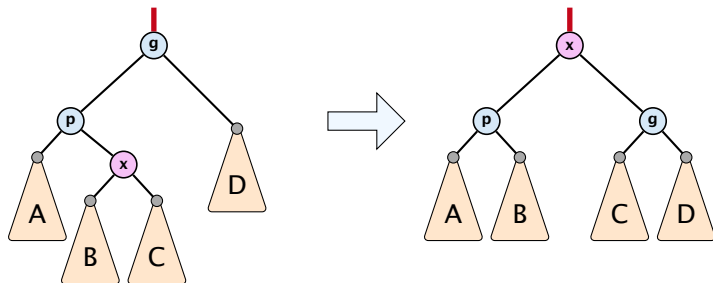
$\Delta\Phi =$

Splay: Zigzag Case



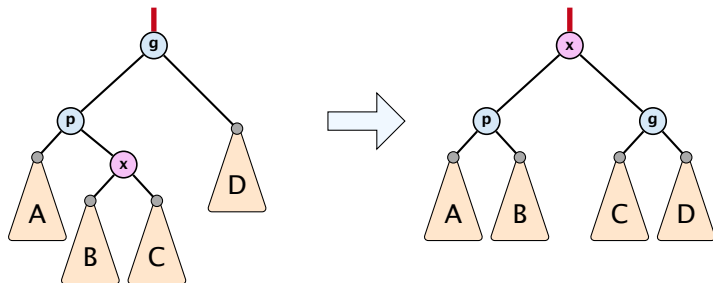
$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$

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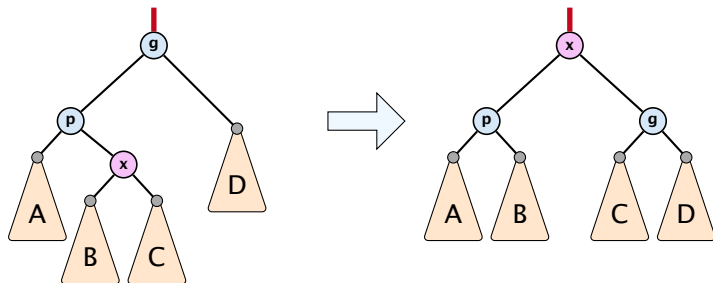
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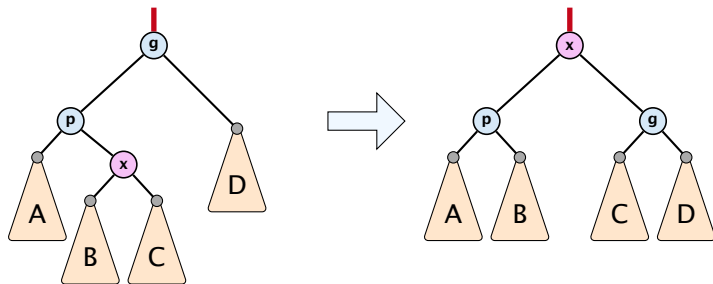
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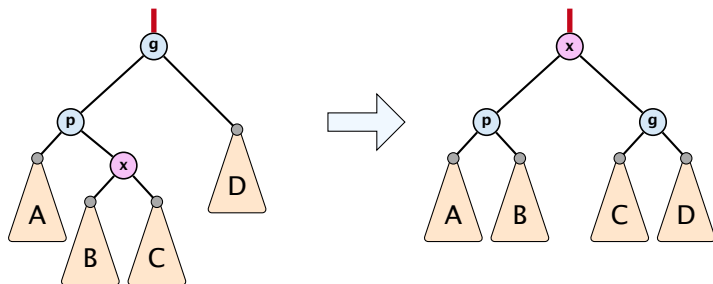
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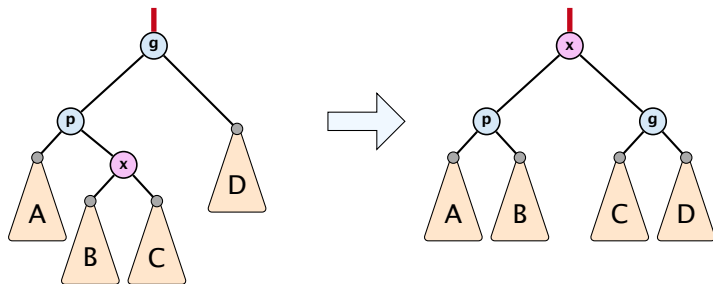
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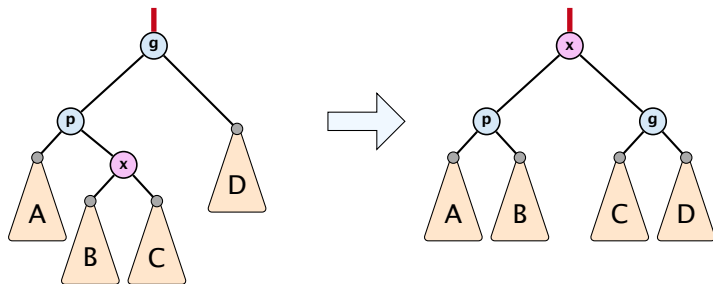
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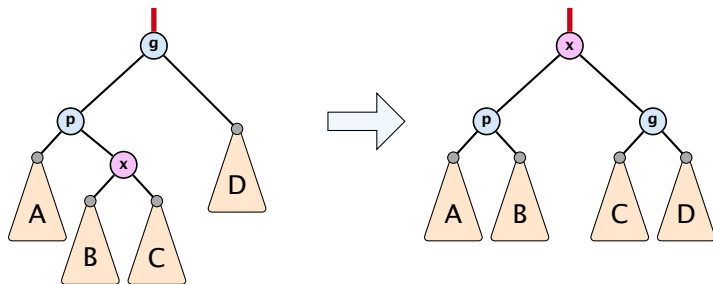
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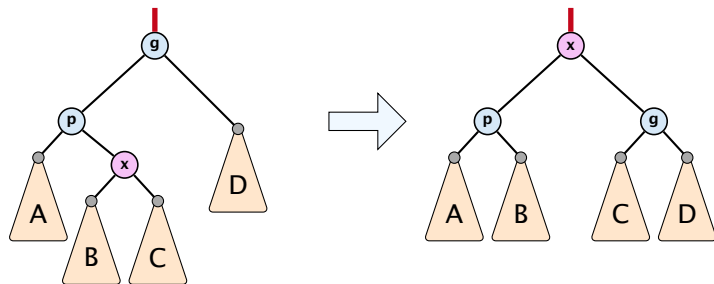
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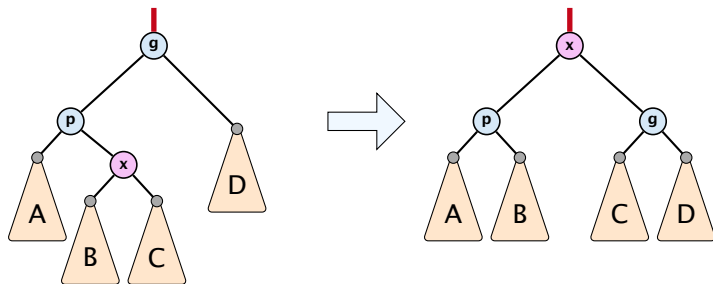
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Amortized cost of the whole splay operation:

$$\begin{aligned} &\leq 1 + 1 + \sum_{\text{steps } t} 3(r_t(x) - r_{t-1}(x)) \\ &= 2 + 3(r(\text{root}) - r_0(x)) \\ &\leq \mathcal{O}(\log n) \end{aligned}$$

7.4 Augmenting Data Structures

Suppose you want to develop a data structure with:

- ▶ **Insert(x)**: insert element x .
- ▶ **Search(k)**: search for element with key k .
- ▶ **Delete(x)**: delete element referenced by pointer x .
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Augment an existing data-structure instead of developing a new one.

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4. develop the new operations

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Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

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1. We choose a red-black tree as the underlying data-structure.
2. We store in each node v the size of the sub-tree rooted at v .
3. We need to be able to update the size-field in each node without asymptotically affecting the running time of insert, delete, and search. We come back to this step later...

7.4 Augmenting Data Structures

Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

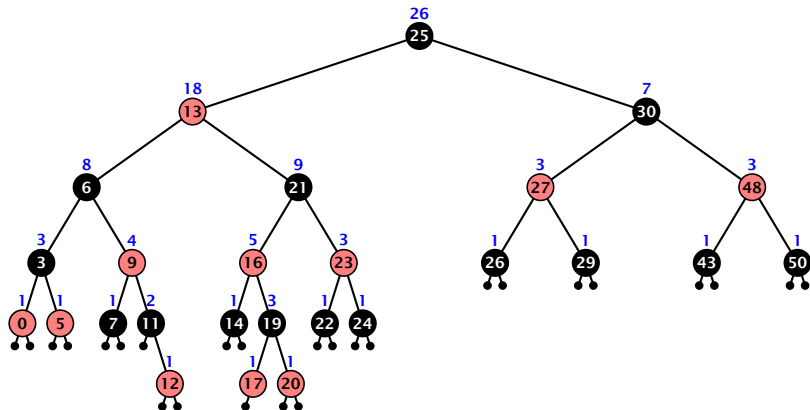
4. How does find-by-rank work?

Find-by-rank(k) := Select(root, k) with

Algorithm 1 Select(x, i)

```
1: if  $x = \text{null}$  then return error
2: if  $\text{left}[x] \neq \text{null}$  then  $r \leftarrow \text{left}[x].\text{size} + 1$  else  $r \leftarrow 1$ 
3: if  $i = r$  then return  $x$ 
4: if  $i < r$  then
5:     return Select( $\text{left}[x], i$ )
6: else
7:     return Select( $\text{right}[x], i - r$ )
```

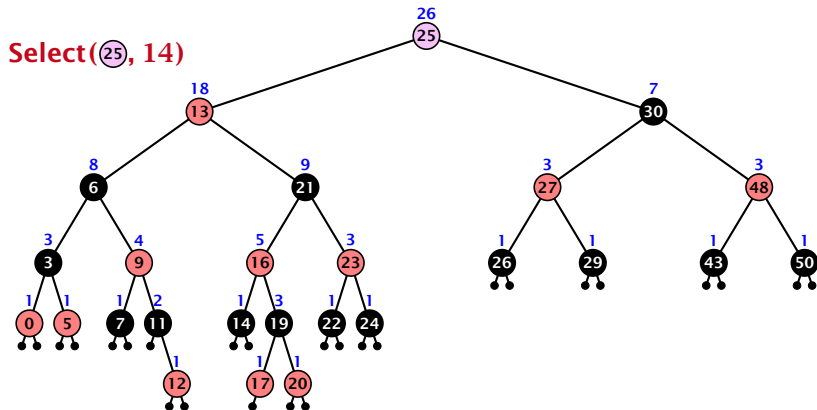
Select(x, i)



Find-by-rank:

- ▶ decide whether you have to proceed into the left or right sub-tree
- ▶ adjust the rank that you are searching for if you go right

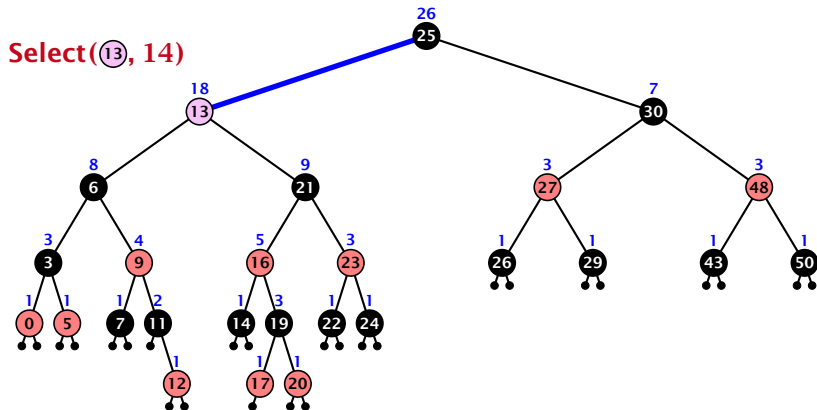
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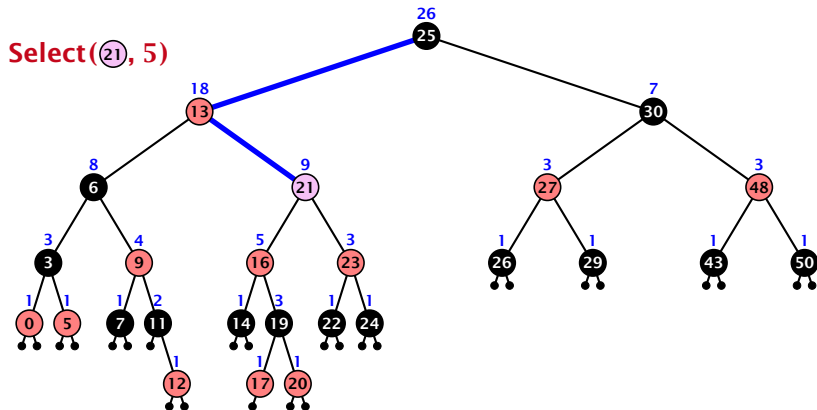
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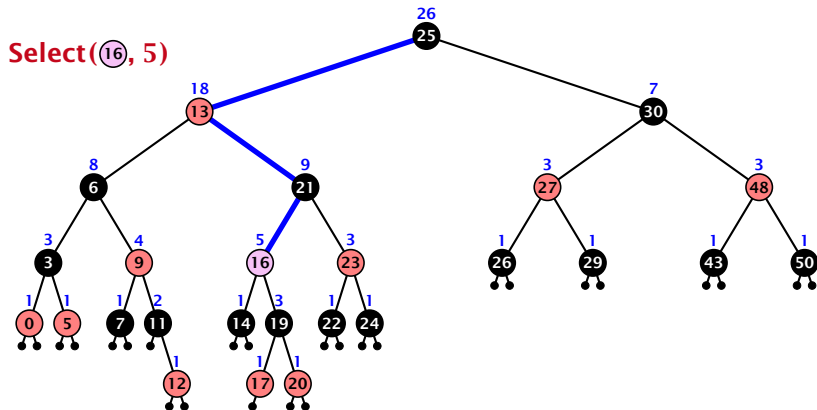
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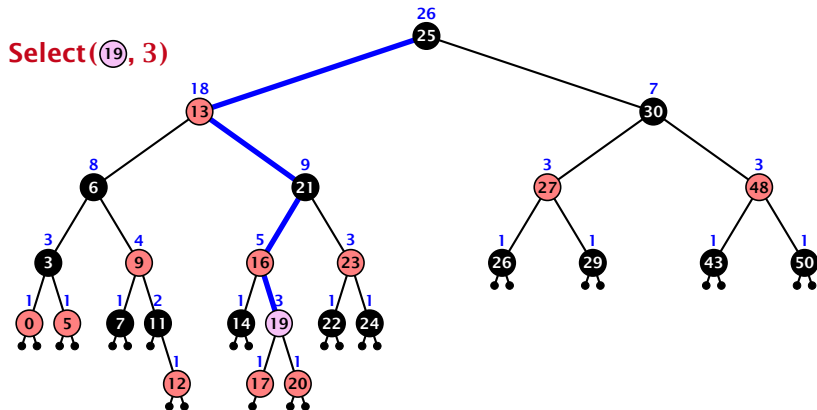
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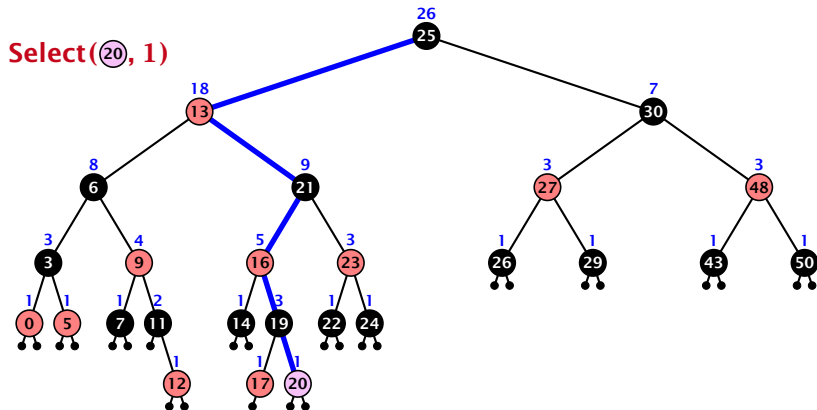
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Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

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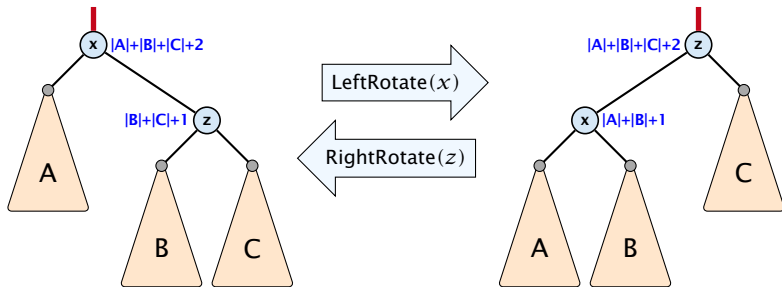
Search(k): Nothing to do.

Insert(x): When going down the search path increase the size field for each visited node. **Maintain the size field during rotations.**

Delete(x): Directly after splicing out a node traverse the path from the spliced out node upwards, and decrease the size counter on every node on this path. **Maintain the size field during rotations.**

Rotations

The only operation during the fix-up procedure that alters the tree and requires an update of the size-field:



The nodes x and z are the only nodes changing their size-fields.

The new size-fields can be computed **locally** from the size-fields of the children.

7.5 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

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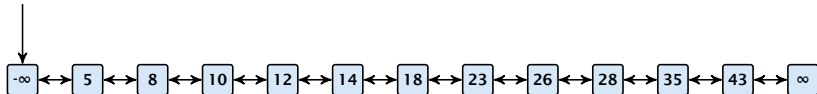
Why do we not use a list for implementing the ADT Dynamic Set?

- ▶ time for search $\Theta(n)$
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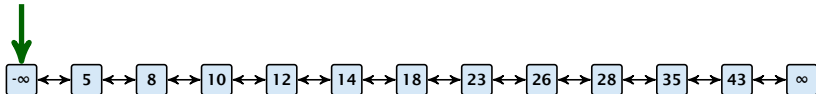
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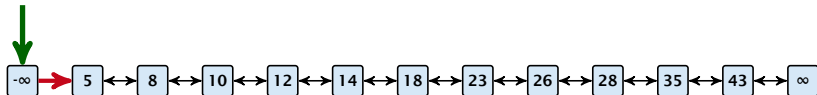
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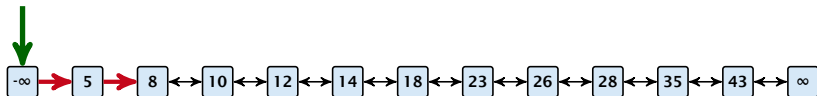
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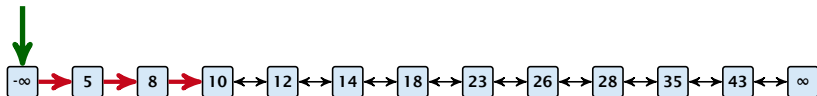
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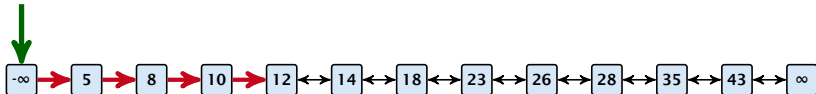
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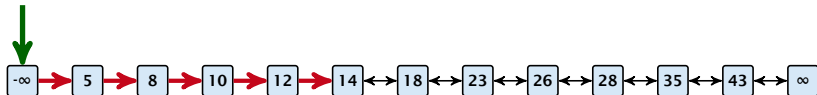
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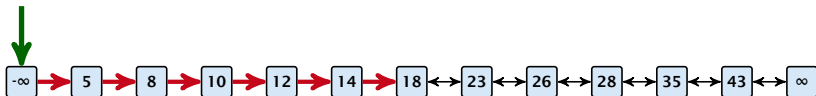
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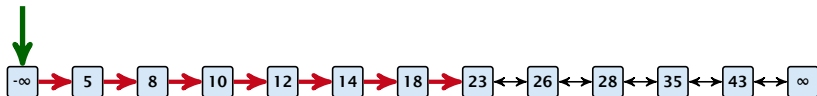
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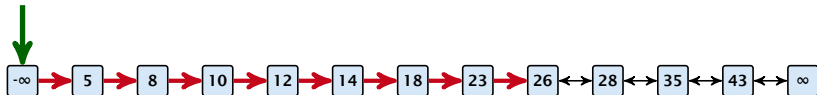
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How can we improve the search-operation?

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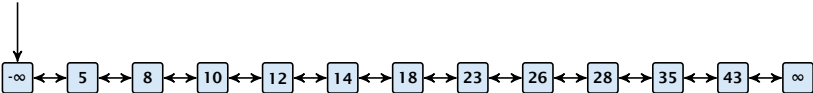
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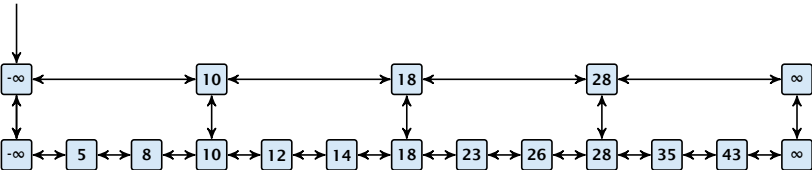
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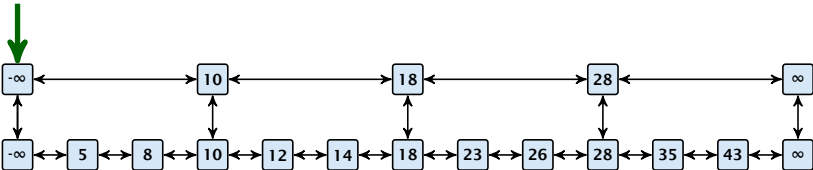
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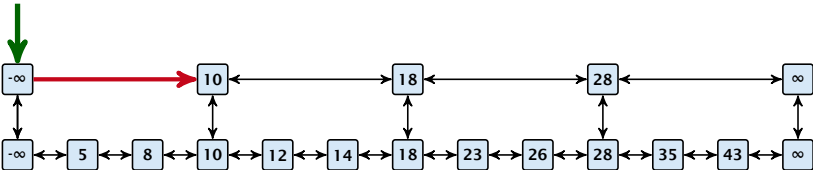
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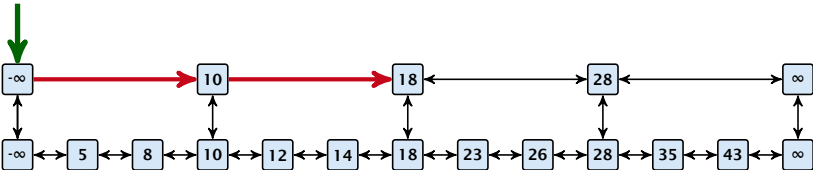
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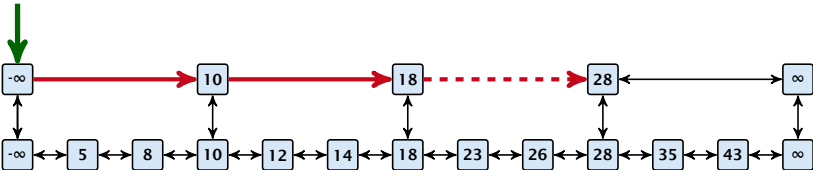
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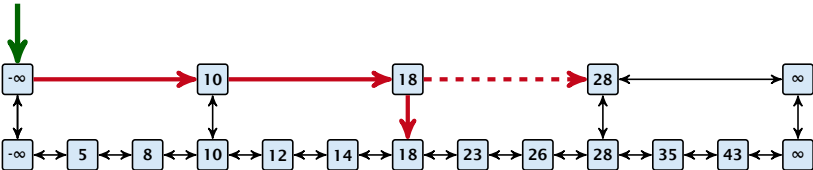
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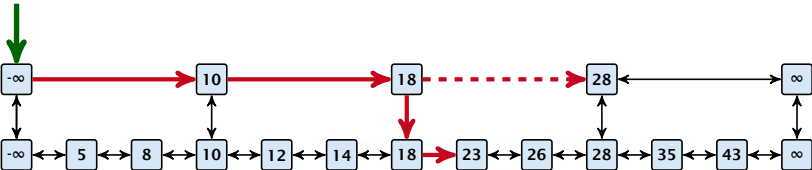
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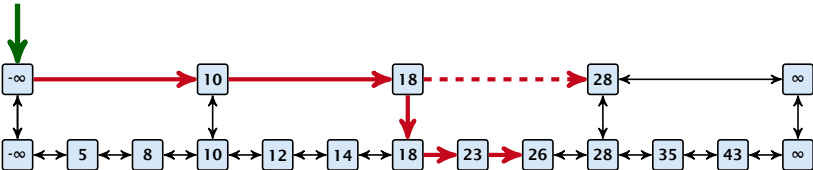
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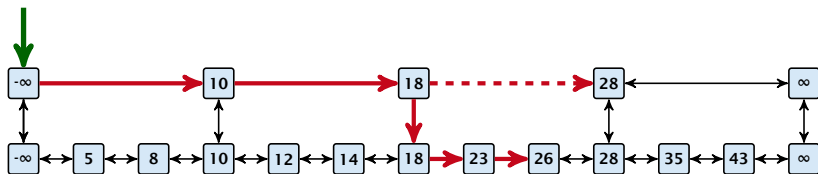
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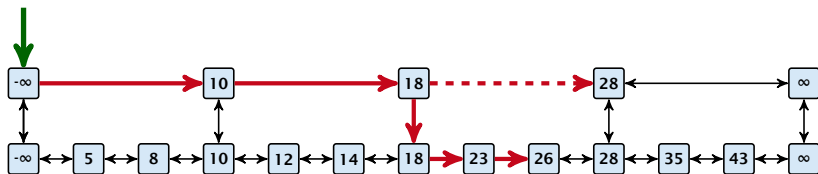


Let $|L_1|$ denote the number of elements in the “express lane”, and $|L_0| = n$ the number of all elements (ignoring dummy elements).

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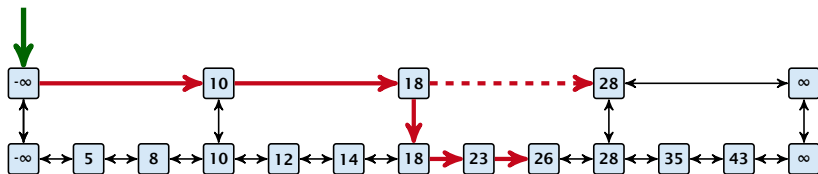
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Choose $|L_1| = \sqrt{n}$. Then search time $\Theta(\sqrt{n})$.

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Add more express lanes. Lane L_i contains roughly every $\frac{L_{i-1}}{L_i}$ -th item from list L_{i-1} .

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- ▶ At most $|L_k| + \sum_{i=1}^k \frac{L_{i-1}}{L_i} + 3(k + 1)$ steps.

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Choosing $k = \Theta(\log n)$ gives a logarithmic running time.

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Use randomization instead!

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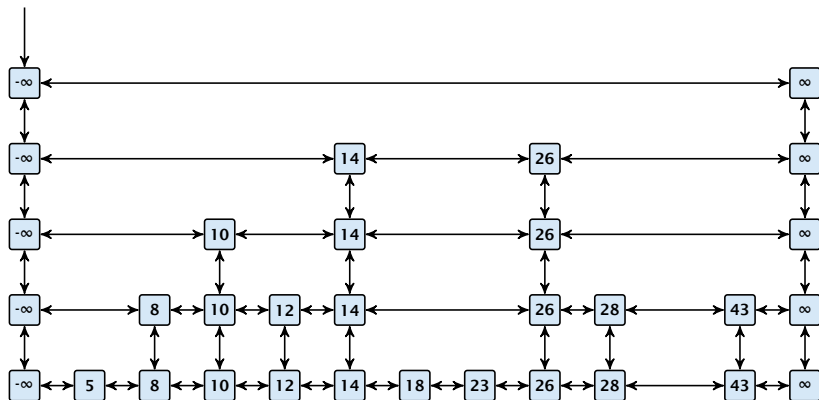
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The time for both operations is dominated by the search time.

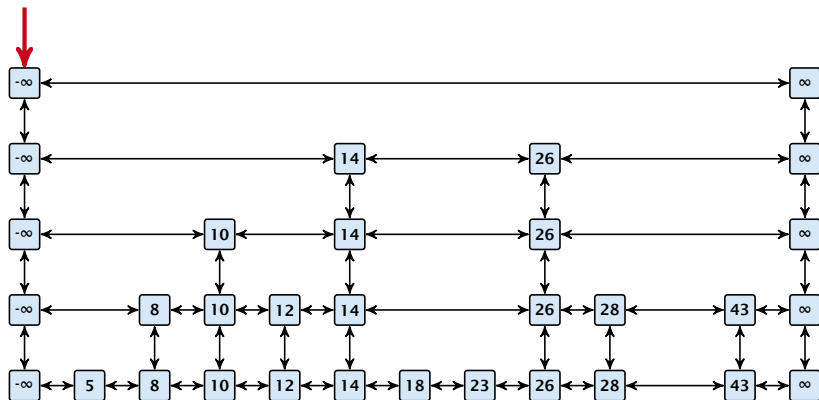
7.5 Skip Lists

Insert (35):



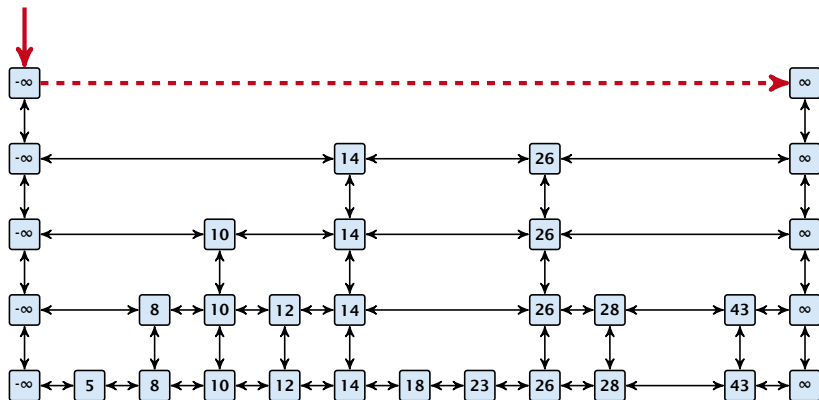
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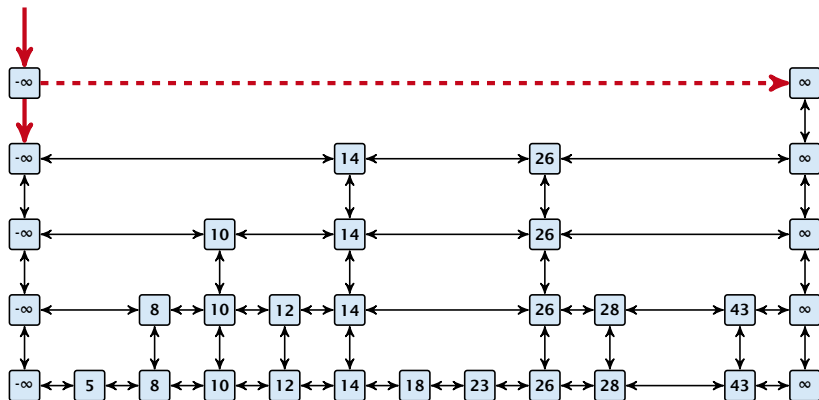
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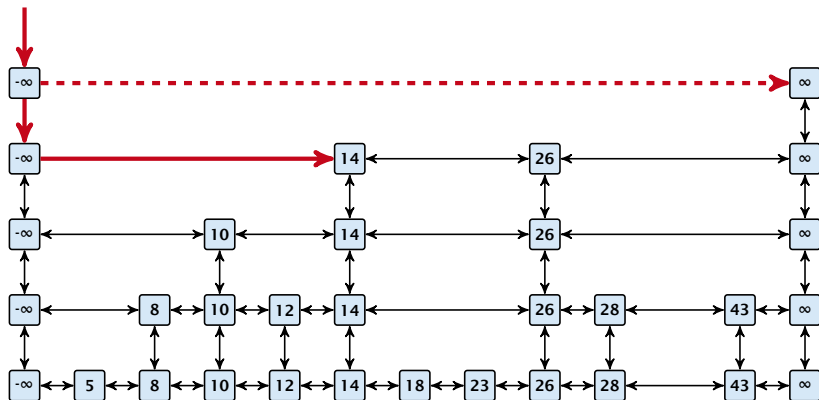
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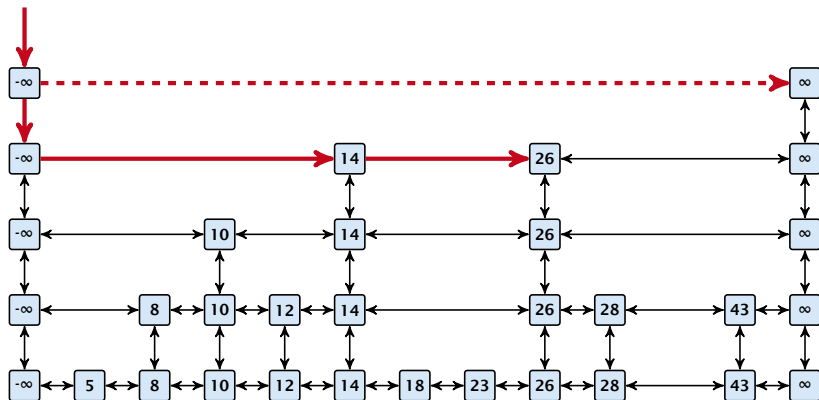
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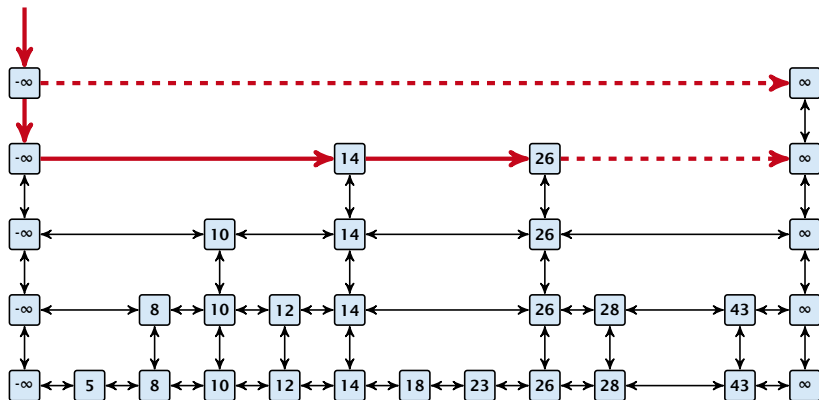
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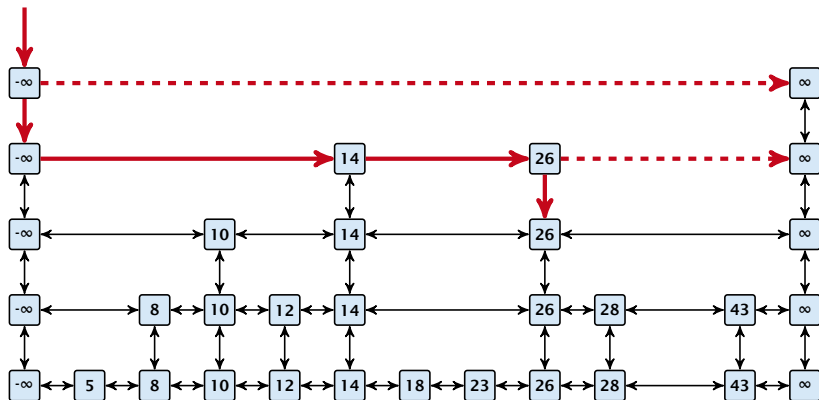
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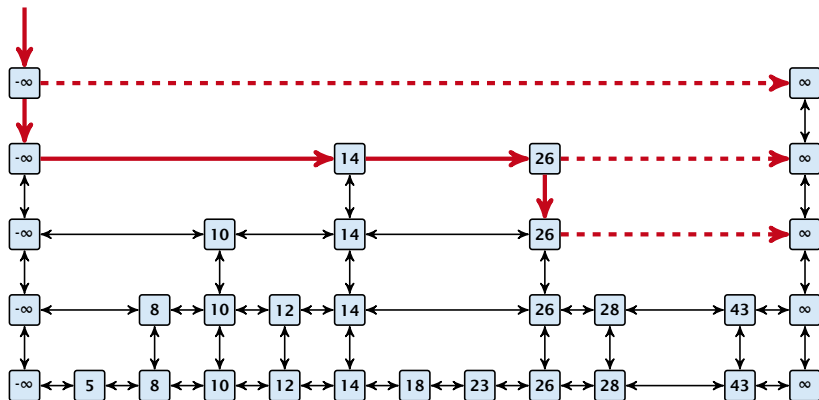
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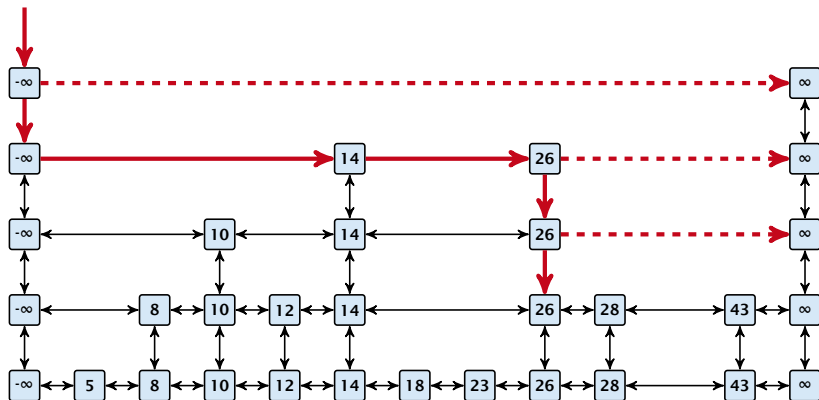
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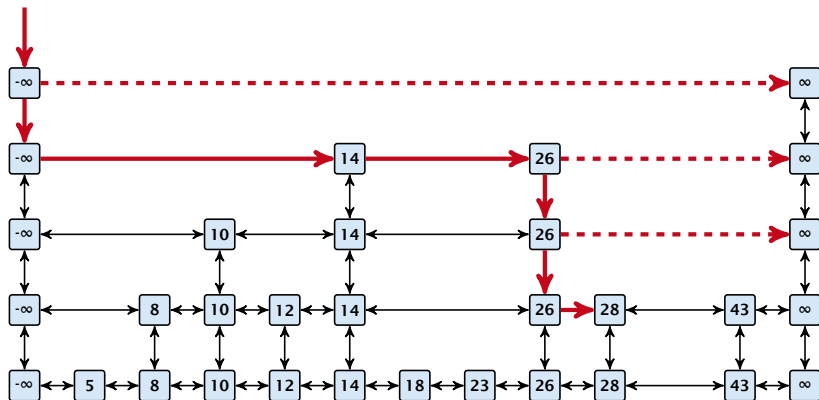
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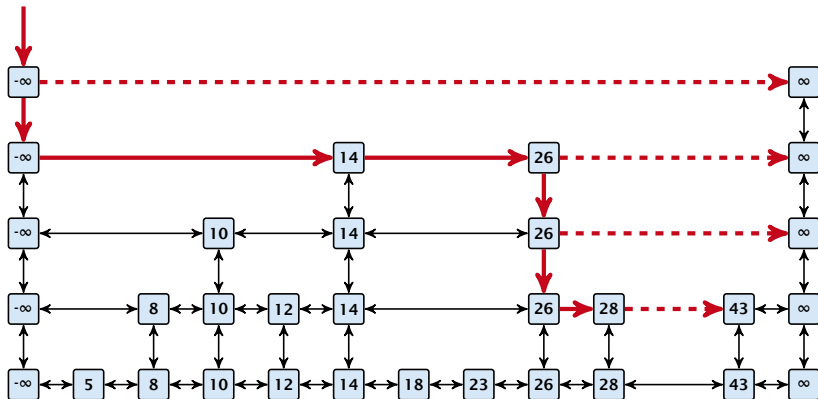
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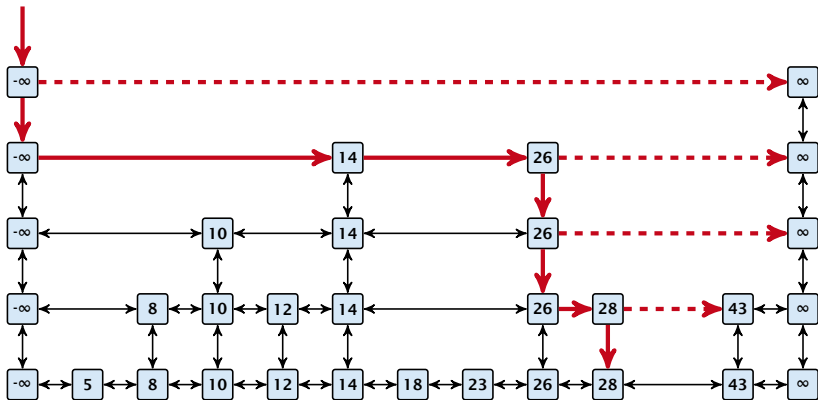
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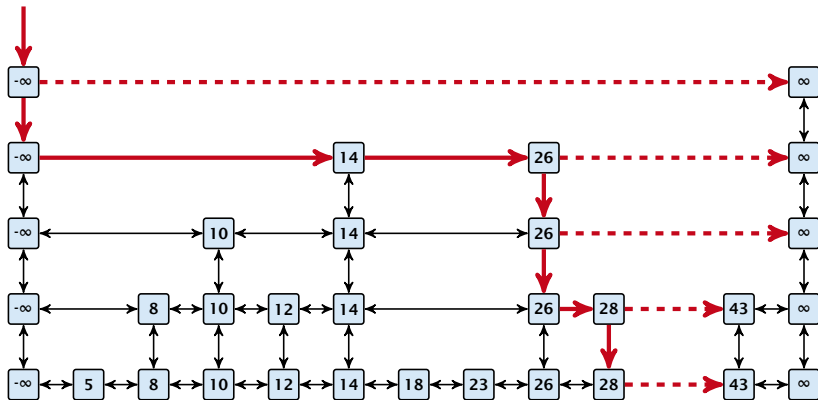
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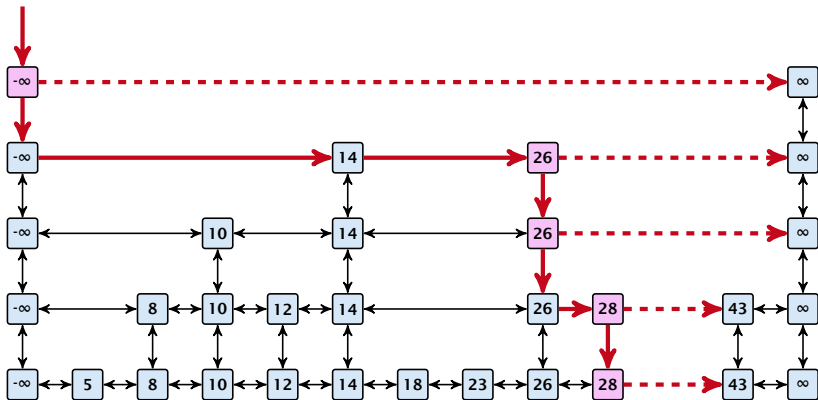
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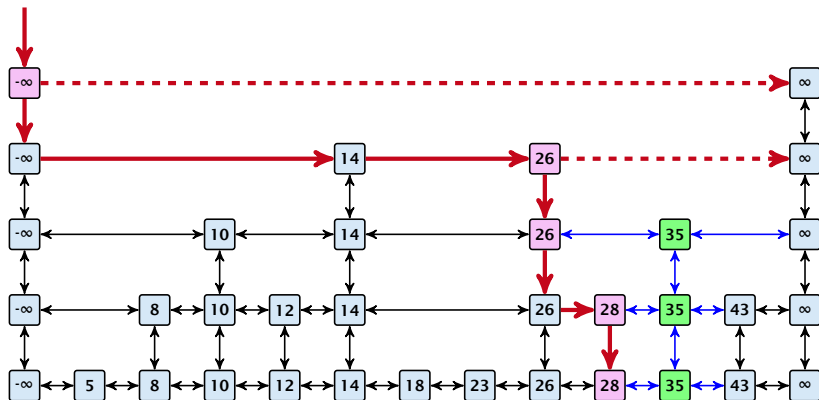
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High Probability

Definition 18 (High Probability)

We say a **randomized** algorithm has running time $\mathcal{O}(\log n)$ with **high probability** if for any constant α the running time is at most $\mathcal{O}(\log n)$ with probability at least $1 - \frac{1}{n^\alpha}$.

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Here the \mathcal{O} -notation hides a constant that may depend on α .

High Probability

Suppose there are **polynomially** many events E_1, E_2, \dots, E_ℓ , $\ell = n^c$ each holding with high probability (e.g. E_i may be the event that the i -th search in a skip list takes time at most $\mathcal{O}(\log n)$).

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Then the probability that all E_i hold is at least

$$\Pr[E_1 \wedge \dots \wedge E_\ell]$$

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This means $E_1 \wedge \dots \wedge E_\ell$ holds with high probability.

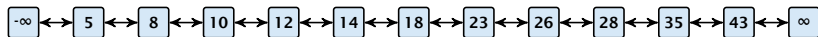
7.5 Skip Lists

Lemma 19

A search (and, hence, also insert and delete) in a skip list with n elements takes time $\mathcal{O}(\log n)$ with high probability (w. h. p.).

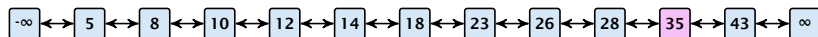
7.5 Skip Lists

Backward analysis:



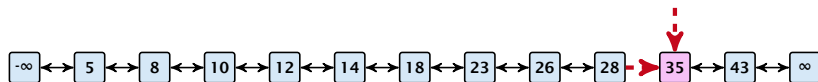
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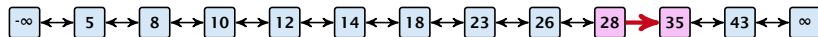
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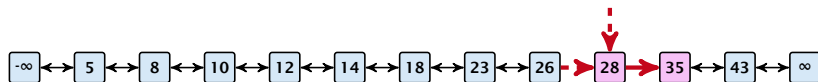
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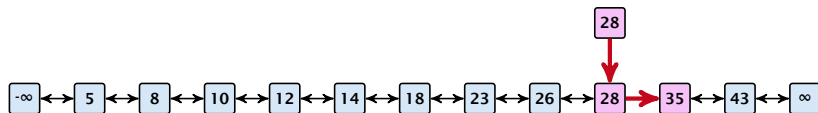
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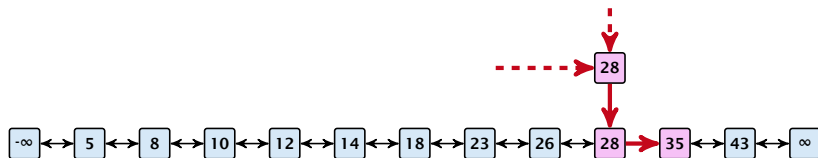
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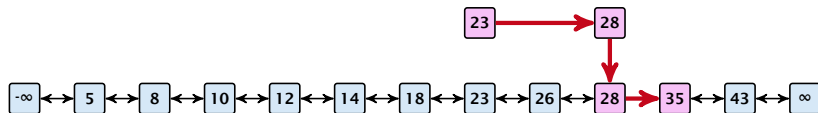
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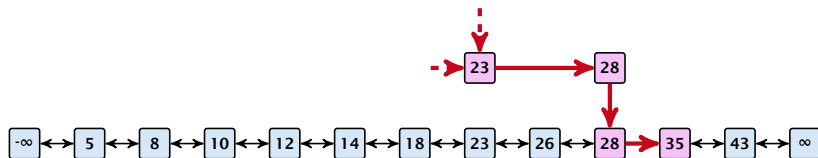
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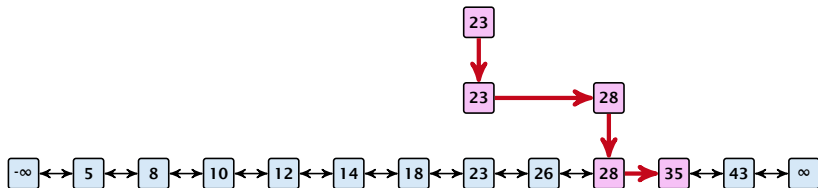
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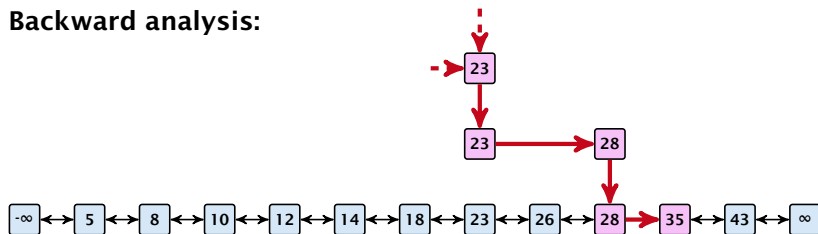
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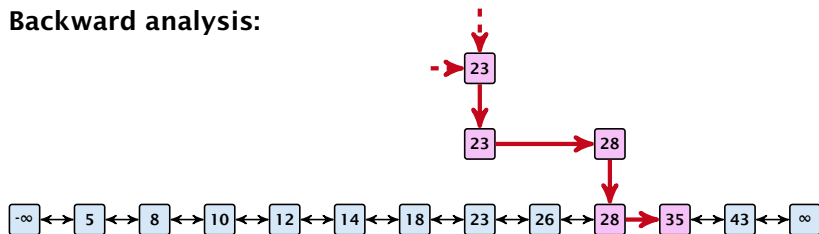
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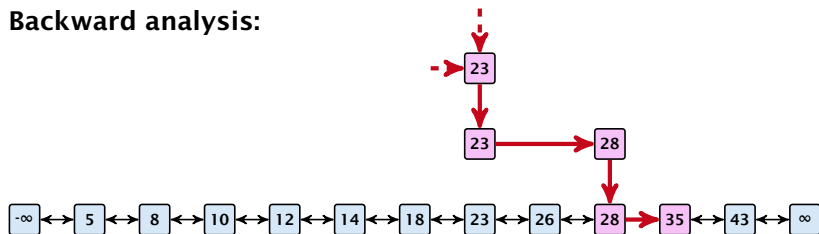
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At each point the path goes up with probability $1/2$ and left with probability $1/2$.

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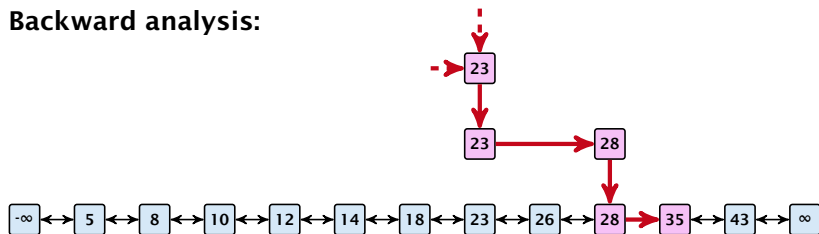
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We show that w.h.p:

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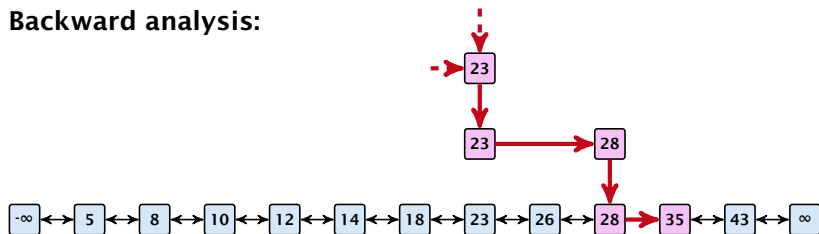
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We show that w.h.p:

- ▶ A “long” search path must also go very high.
- ▶ There are no elements in high lists.

From this it follows that w.h.p. there are no long paths.

7.5 Skip Lists

Estimation for Binomial Coefficients

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

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In particular, this means that during the construction in the backward analysis we see at most k heads (i.e., coin flips that tell you to go up) in z trials.

7.5 Skip Lists

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choosing $k = \gamma \log n$ with $\gamma \geq 1$ and $z = (\beta + \alpha)\gamma \log n$

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Hence,

$$\begin{aligned} \Pr[\text{search requires } z \text{ steps}] &\leq \Pr[E_{z,k}] + \Pr[A_{k+1}] \\ &\leq n^{-\alpha} + n^{-(\gamma-1)} \end{aligned}$$

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This means that a search path of length $\Omega(\log n)$ visits a list on a level $\Omega(\log n)$, w.h.p.

Let A_{k+1} denote the event that the list L_{k+1} is non-empty. Then

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For the search to take at least $z = 7\alpha\gamma \log n$ steps either the event $E_{z,k}$ or the event A_{k+1} must hold.

Hence,

$$\begin{aligned} \Pr[\text{search requires } z \text{ steps}] &\leq \Pr[E_{z,k}] + \Pr[A_{k+1}] \\ &\leq n^{-\alpha} + n^{-(\gamma-1)} \end{aligned}$$

This means, the search requires at most z steps, w. h. p.

7.6 van Emde Boas Trees

Dynamic Set Data Structure S :

- ▶ $S.insert(x)$
- ▶ $S.delete(x)$
- ▶ $S.search(x)$
- ▶ $S.min()$
- ▶ $S.max()$
- ▶ $S.succ(x)$
- ▶ $S.pred(x)$

7.6 van Emde Boas Trees

For this chapter we ignore the problem of storing satellite data:

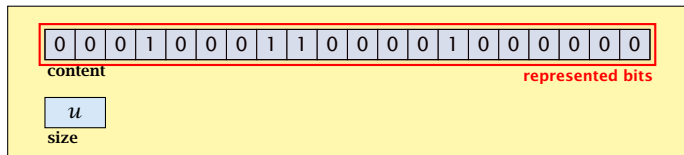
- ▶ **S . insert(x):** Inserts x into S .
- ▶ **S . delete(x):** Deletes x from S . Usually assumes that $x \in S$.
- ▶ **S . member(x):** Returns 1 if $x \in S$ and 0 otherwise.
- ▶ **S . min():** Returns the value of the minimum element in S .
- ▶ **S . max():** Returns the value of the maximum element in S .
- ▶ **S . succ(x):** Returns successor of x in S . Returns **null** if x is maximum or larger than any element in S . Note that x needs not to be in S .
- ▶ **S . pred(x):** Returns the predecessor of x in S . Returns **null** if x is minimum or smaller than any element in S . Note that x needs not to be in S .

7.6 van Emde Boas Trees

Can we improve the existing algorithms when the keys are from a restricted set?

In the following we assume that the keys are from $\{0, 1, \dots, u - 1\}$, where u denotes the size of the universe.

Implementation 1: Array



one array of u bits

Use an array that encodes the indicator function of the dynamic set.

Implementation 1: Array

Algorithm 1 `array.insert(x)`

1: `content[x] ← 1;`

Algorithm 2 `array.delete(x)`

1: `content[x] ← 0;`

Algorithm 3 `array.member(x)`

1: **return** `content[x];`

- ▶ Note that we assume that x is valid, i.e., it falls within the array boundaries.
- ▶ Obviously(?) the running time is constant.

Implementation 1: Array

Algorithm 4 `array.max()`

```
1: for ( $i = \text{size} - 1; i \geq 0; i--$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
3: return null;
```

Implementation 1: Array

Algorithm 4 `array.max()`

```
1: for ( $i = \text{size} - 1; i \geq 0; i--$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
3: return null;
```

Algorithm 5 `array.min()`

```
1: for ( $i = 0; i < \text{size}; i++$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
3: return null;
```

Implementation 1: Array

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Algorithm 5 `array.min()`

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1: for ( $i = 0; i < \text{size}; i++$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
3: return null;
```

- ▶ Running time is $\mathcal{O}(u)$ in the worst case.

Implementation 1: Array

Algorithm 6 `array.succ(x)`

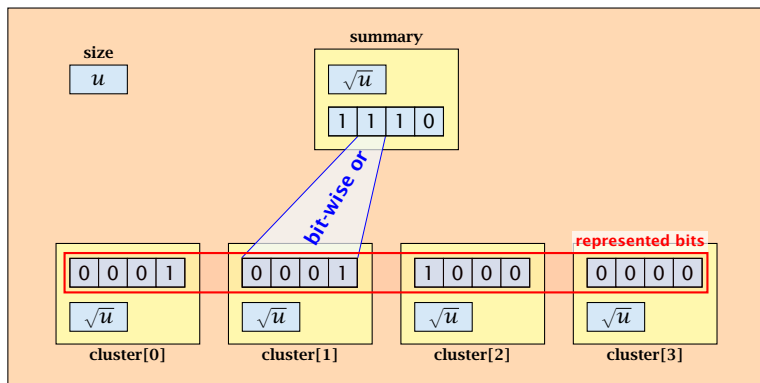
```
1: for ( $i = x + 1$ ;  $i < \text{size}$ ;  $i++$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
3: return null;
```

Algorithm 7 `array.pred(x)`

```
1: for ( $i = x - 1$ ;  $i \geq 0$ ;  $i--$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
3: return null;
```

- ▶ Running time is $\mathcal{O}(u)$ in the worst case.

Implementation 2: Summary Array



- ▶ \sqrt{u} cluster-arrays of \sqrt{u} bits.
- ▶ One summary-array of \sqrt{u} bits. The i -th bit in the summary array stores the bit-wise or of the bits in the i -th cluster.

Implementation 2: Summary Array

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The bit for a key x is contained in cluster number $\left\lfloor \frac{x}{\sqrt{u}} \right\rfloor$.

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Within the cluster-array the bit is at position $x \bmod \sqrt{u}$.

For simplicity we assume that $u = 2^{2k}$ for some $k \geq 1$. Then we can compute the cluster-number for an entry x as $\text{high}(x)$ (the upper half of the dual representation of x) and the position of x within its cluster as $\text{low}(x)$ (the lower half of the dual representation).

Implementation 2: Summary Array

Algorithm 8 $\text{member}(x)$

1: **return** $\text{cluster}[\text{high}(x)].\text{member}(\text{low}(x));$

Implementation 2: Summary Array

Algorithm 8 $\text{member}(x)$

1: **return** $\text{cluster}[\text{high}(x)].\text{member}(\text{low}(x));$

Algorithm 9 $\text{insert}(x)$

1: $\text{cluster}[\text{high}(x)].\text{insert}(\text{low}(x));$

2: $\text{summary}.\text{insert}(\text{high}(x));$

Implementation 2: Summary Array

Algorithm 8 $\text{member}(x)$

```
1: return cluster[high(x)].member(low(x));
```

Algorithm 9 $\text{insert}(x)$

```
1: cluster[high(x)].insert(low(x));  
2: summary.insert(high(x));
```

- ▶ The running times are constant, because the corresponding array-functions have constant running times.

Implementation 2: Summary Array

Algorithm 10 delete(x)

- 1: cluster[high(x)].delete(low(x));
- 2: **if** cluster[high(x)].min() = null **then**
- 3: summary.delete(high(x));

Implementation 2: Summary Array

Algorithm 10 delete(x)

```
1: cluster[high( $x$ )].delete(low( $x$ ));  
2: if cluster[high( $x$ )].min() = null then  
3:     summary.delete(high( $x$ ));
```

- ▶ The running time is dominated by the cost of a minimum computation on an array of size \sqrt{u} . Hence, $\mathcal{O}(\sqrt{u})$.

Implementation 2: Summary Array

Algorithm 11 $\text{max}()$

- 1: $\text{maxcluster} \leftarrow \text{summary.max}();$
- 2: **if** $\text{maxcluster} = \text{null}$ **return** $\text{null};$
- 3: $\text{offs} \leftarrow \text{cluster}[\text{maxcluster}].\text{max}();$
- 4: **return** $\text{maxcluster} \circ \text{offs};$

Implementation 2: Summary Array

Algorithm 11 $\text{max}()$

```
1:  $\text{maxcluster} \leftarrow \text{summary.max}();$   
2: if  $\text{maxcluster} = \text{null}$  return  $\text{null}$ ;  
3:  $\text{offs} \leftarrow \text{cluster}[\text{maxcluster}].\text{max}();$   
4: return  $\text{maxcluster} \circ \text{offs};$ 
```

Algorithm 12 $\text{min}()$

```
1:  $\text{mincluster} \leftarrow \text{summary.min}();$   
2: if  $\text{mincluster} = \text{null}$  return  $\text{null}$ ;  
3:  $\text{offs} \leftarrow \text{cluster}[\text{mincluster}].\text{min}();$   
4: return  $\text{mincluster} \circ \text{offs};$ 
```

Implementation 2: Summary Array

Algorithm 11 $\text{max}()$

```
1:  $\text{maxcluster} \leftarrow \text{summary.max}();$   
2: if  $\text{maxcluster} = \text{null}$  return  $\text{null}$ ;  
3:  $\text{offs} \leftarrow \text{cluster}[\text{maxcluster}].\text{max}();$   
4: return  $\text{maxcluster} \circ \text{offs}$ ;
```

Algorithm 12 $\text{min}()$

```
1:  $\text{mincluster} \leftarrow \text{summary.min}();$   
2: if  $\text{mincluster} = \text{null}$  return  $\text{null}$ ;  
3:  $\text{offs} \leftarrow \text{cluster}[\text{mincluster}].\text{min}();$   
4: return  $\text{mincluster} \circ \text{offs}$ ;
```

The operator \circ stands for the concatenation of two bitstrings.

This means if $x = 0111_2$ and $y = 0001_2$ then $x \circ y = 01110001_2$.

- ▶ Running time is roughly $2\sqrt{u} = \mathcal{O}(\sqrt{u})$ in the worst case.

Implementation 2: Summary Array

Algorithm 13 $\text{succ}(x)$

```
1:  $m \leftarrow \text{cluster}[\text{high}(x)].\text{succ}(\text{low}(x))$ 
2: if  $m \neq \text{null}$  then return  $\text{high}(x) \circ m$ ;
3:  $\text{succcluster} \leftarrow \text{summary}.\text{succ}(\text{high}(x))$ ;
4: if  $\text{succcluster} \neq \text{null}$  then
5:      $\text{offs} \leftarrow \text{cluster}[\text{succcluster}].\text{min}()$ ;
6:     return  $\text{succcluster} \circ \text{offs}$ ;
7: return  $\text{null}$ ;
```

Implementation 2: Summary Array

Algorithm 13 $\text{succ}(x)$

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1:  $m \leftarrow \text{cluster}[\text{high}(x)].\text{succ}(\text{low}(x))$ 
2: if  $m \neq \text{null}$  then return  $\text{high}(x) \circ m$ ;
3:  $\text{succcluster} \leftarrow \text{summary}.\text{succ}(\text{high}(x))$ ;
4: if  $\text{succcluster} \neq \text{null}$  then
5:      $\text{offs} \leftarrow \text{cluster}[\text{succcluster}].\text{min}()$ ;
6:     return  $\text{succcluster} \circ \text{offs}$ ;
7: return  $\text{null}$ ;
```

- ▶ Running time is roughly $3\sqrt{u} = \mathcal{O}(\sqrt{u})$ in the worst case.

Implementation 2: Summary Array

Algorithm 14 $\text{pred}(x)$

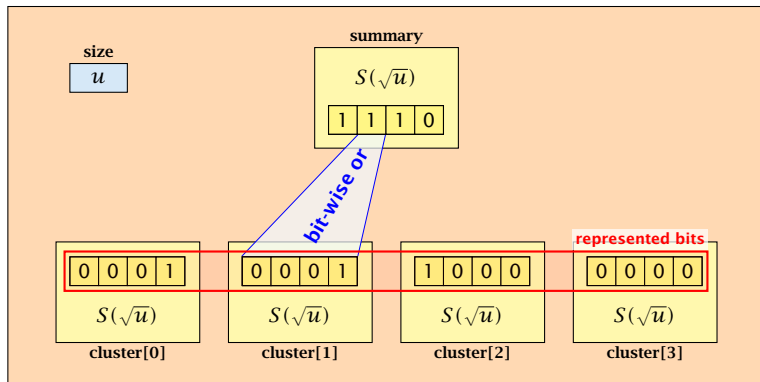
```
1:  $m \leftarrow \text{cluster}[\text{high}(x)].\text{pred}(\text{low}(x))$ 
2: if  $m \neq \text{null}$  then return  $\text{high}(x) \circ m$ ;
3:  $\text{predcluster} \leftarrow \text{summary}.\text{pred}(\text{high}(x))$ ;
4: if  $\text{predcluster} \neq \text{null}$  then
5:      $\text{offs} \leftarrow \text{cluster}[\text{predcluster}].\text{max}()$ ;
6:     return  $\text{predcluster} \circ \text{offs}$ ;
7: return  $\text{null}$ ;
```

- ▶ Running time is roughly $3\sqrt{u} = \mathcal{O}(\sqrt{u})$ in the worst case.

Implementation 3: Recursion

Instead of using sub-arrays, we build a recursive data-structure.

$S(u)$ is a dynamic set data-structure representing u bits:



Implementation 3: Recursion

We assume that $u = 2^{2^k}$ for some k .

The data-structure $S(2)$ is defined as an array of 2-bits (end of the recursion).

Implementation 3: Recursion

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The code from Implementation 2 can be used **unchanged**. We only need to redo the analysis of the running time.

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Note that in the code we do not need to specifically address the non-recursive case. This is achieved by the fact that an $S(4)$ will contain $S(2)$'s as sub-datastructures, which are **arrays**. Hence, a call like `cluster[1].min()` from within the data-structure $S(4)$ is **not** a recursive call as it will call the function `array.min()`.

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This means that the non-recursive case is been dealt with while initializing the data-structure.

Implementation 3: Recursion

Algorithm 15 $\text{member}(x)$

1: **return** $\text{cluster}[\text{high}(x)].\text{member}(\text{low}(x));$

- ▶ $T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1.$

Implementation 3: Recursion

Algorithm 16 insert(x)

```
1: cluster[high( $x$ )].insert(low( $x$ ));  
2: summary.insert(high( $x$ ));
```

► $T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$

Implementation 3: Recursion

Algorithm 17 delete(x)

```
1: cluster[high( $x$ )].delete(low( $x$ ));  
2: if cluster[high( $x$ )].min() = null then  
3:     summary.delete(high( $x$ ));
```

► $T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + T_{\text{min}}(\sqrt{u}) + 1.$

Implementation 3: Recursion

Algorithm 18 $\text{min}()$

```
1: mincluster  $\leftarrow$  summary.min();  
2: if mincluster = null return null;  
3: offs  $\leftarrow$  cluster[mincluster].min();  
4: return mincluster  $\circ$  offs;
```

► $T_{\min}(u) = 2T_{\min}(\sqrt{u}) + 1.$

Implementation 3: Recursion

Algorithm 19 $\text{succ}(x)$

```
1:  $m \leftarrow \text{cluster}[\text{high}(x)].\text{succ}(\text{low}(x))$ 
2: if  $m \neq \text{null}$  then return  $\text{high}(x) \circ m$ ;
3:  $\text{succcluster} \leftarrow \text{summary}.\text{succ}(\text{high}(x))$ ;
4: if  $\text{succcluster} \neq \text{null}$  then
5:      $\text{offs} \leftarrow \text{cluster}[\text{succcluster}].\text{min}()$ ;
6:     return  $\text{succcluster} \circ \text{offs}$ ;
7: return  $\text{null}$ ;
```

► $T_{\text{succ}}(u) = 2T_{\text{succ}}(\sqrt{u}) + T_{\text{min}}(\sqrt{u}) + 1.$

Implementation 3: Recursion

$$T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + 1:$$

Implementation 3: Recursion

$$T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + \mathbf{1}:$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^\ell)$.

Implementation 3: Recursion

$$T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + 1:$$

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$$X(\ell) = T_{\text{mem}}(2^\ell) = T_{\text{mem}}(u)$$

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$$\begin{aligned} X(\ell) = T_{\text{mem}}(2^\ell) &= T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + 1 \\ &= T_{\text{mem}}(2^{\frac{\ell}{2}}) + 1 \end{aligned}$$

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Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^\ell)$. Then

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Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^\ell)$. Then

$$\begin{aligned} X(\ell) &= T_{\text{mem}}(2^\ell) = T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1 \\ &= T_{\text{mem}}(2^{\frac{\ell}{2}}) + 1 = X\left(\frac{\ell}{2}\right) + 1 . \end{aligned}$$

Using Master theorem gives $X(\ell) = \mathcal{O}(\log \ell)$, and hence $T_{\text{mem}}(u) = \mathcal{O}(\log \log u)$.

Implementation 3: Recursion

$$T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$$

Implementation 3: Recursion

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Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$.

Implementation 3: Recursion

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Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$. Then

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Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$. Then

$$X(\ell)$$

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Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$. Then

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Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$. Then

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Using Master theorem gives $X(\ell) = \mathcal{O}(\ell)$, and hence $T_{\text{ins}}(\mathbf{u}) = \mathcal{O}(\log u)$.

Implementation 3: Recursion

$$T_{\text{ins}}(\mathbf{u}) = 2T_{\text{ins}}(\sqrt{\mathbf{u}}) + 1.$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$. Then

$$\begin{aligned} X(\ell) &= T_{\text{ins}}(2^\ell) = T_{\text{ins}}(\mathbf{u}) = 2T_{\text{ins}}(\sqrt{\mathbf{u}}) + 1 \\ &= 2T_{\text{ins}}(2^{\frac{\ell}{2}}) + 1 = 2X(\frac{\ell}{2}) + 1 . \end{aligned}$$

Using Master theorem gives $X(\ell) = \mathcal{O}(\ell)$, and hence $T_{\text{ins}}(\mathbf{u}) = \mathcal{O}(\log u)$.

The same holds for $T_{\text{max}}(\mathbf{u})$ and $T_{\text{min}}(\mathbf{u})$.

Implementation 3: Recursion

$$T_{\text{del}}(\mathbf{u}) = 2T_{\text{del}}(\sqrt{\mathbf{u}}) + T_{\text{min}}(\sqrt{\mathbf{u}}) + 1 \leq 2T_{\text{del}}(\sqrt{\mathbf{u}}) + c \log(\mathbf{u}).$$

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$$X(\ell)$$

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Using Master theorem gives $X(\ell) = \Theta(\ell \log \ell)$, and hence $T_{\text{del}}(u) = \mathcal{O}(\log u \log \log u)$.

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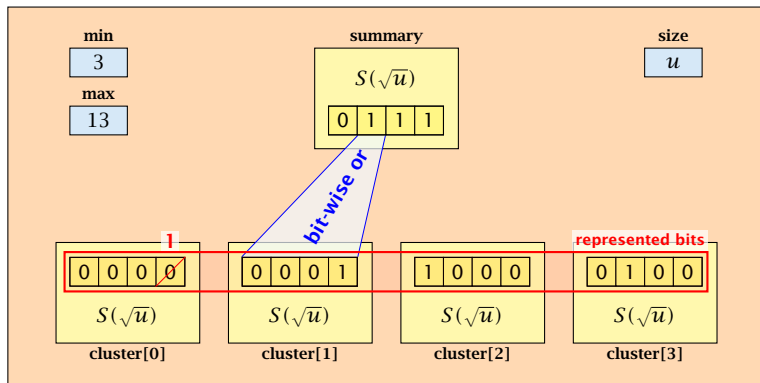
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Using Master theorem gives $X(\ell) = \Theta(\ell \log \ell)$, and hence $T_{\text{del}}(u) = \mathcal{O}(\log u \log \log u)$.

The same holds for $T_{\text{pred}}(u)$ and $T_{\text{succ}}(u)$.

Implementation 4: van Emde Boas Trees



- ▶ The bit referenced by **min** is **not** set within sub-datastructures.
- ▶ The bit referenced by **max** is set within sub-datastructures (if $\text{max} \neq \text{min}$).

Implementation 4: van Emde Boas Trees

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- ▶ **min = max \neq null** means that the data-structure contains exactly one element.
- ▶ We can insert into an empty datastructure in constant time by only setting **min = max = x** .
- ▶ We can delete from a data-structure that just contains one element in constant time by setting **min = max = null**.

Implementation 4: van Emde Boas Trees

Algorithm 20 max()

1: **return** max;

Algorithm 21 min()

1: **return** min;

- ▶ Constant time.

Implementation 4: van Emde Boas Trees

Algorithm 22 `member(x)`

```
1: if  $x = \min$  then return 1; // TRUE  
2: return cluster[high(x)].member(low(x));
```

- ▶ $T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1 \Rightarrow T(u) = \mathcal{O}(\log \log u)$.

Implementation 4: van Emde Boas Trees

Algorithm 23 $\text{succ}(x)$

```
1: if  $\text{min} \neq \text{null} \wedge x < \text{min}$  then return  $\text{min}$ ;  
2:  $\text{maxincluster} \leftarrow \text{cluster}[\text{high}(x)].\text{max}()$ ;  
3: if  $\text{maxincluster} \neq \text{null} \wedge \text{low}(x) < \text{maxincluster}$  then  
4:    $\text{offs} \leftarrow \text{cluster}[\text{high}(x)].\text{succ}(\text{low}(x))$ ;  
5:   return  $\text{high}(x) \circ \text{offs}$ ;  
6: else  
7:    $\text{succcluster} \leftarrow \text{summary}.\text{succ}(\text{high}(x))$ ;  
8:   if  $\text{succcluster} = \text{null}$  then return  $\text{null}$ ;  
9:    $\text{offs} \leftarrow \text{cluster}[\text{succcluster}].\text{min}()$ ;  
10:  return  $\text{succcluster} \circ \text{offs}$ ;
```

► $T_{\text{succ}}(u) = T_{\text{succ}}(\sqrt{u}) + 1 \implies T_{\text{succ}}(u) = \mathcal{O}(\log \log u)$.

Implementation 4: van Emde Boas Trees

Algorithm 35 insert(x)

```
1: if min = null then
2:     min =  $x$ ; max =  $x$ ;
3: else
4:     if  $x < \text{min}$  then exchange  $x$  and min;
5:     if  $x > \text{max}$  then max =  $x$ ;
6:     if cluster[high( $x$ )].min = null; then
7:         summary.insert(high( $x$ ));
8:         cluster[high( $x$ )].insert(low( $x$ ));
9:     else
10:        cluster[high( $x$ )].insert(low( $x$ ));
```

► $T_{\text{ins}}(u) = T_{\text{ins}}(\sqrt{u}) + 1 \Rightarrow T_{\text{ins}}(u) = \mathcal{O}(\log \log u)$.

Implementation 4: van Emde Boas Trees

Note that the recursive call in Line 8 takes constant time as the if-condition in Line 6 ensures that we are inserting in an empty sub-tree.

The only non-constant recursive calls are the call in Line 7 and in Line 10. These are mutually exclusive, i.e., only one of these calls will actually occur.

From this we get that $T_{\text{ins}}(u) = T_{\text{ins}}(\sqrt{u}) + 1$.

Implementation 4: van Emde Boas Trees

- ▶ Assumes that x is contained in the structure.

Algorithm 36 delete(x)

```
1: if min = max then
2:     min = max = null;
3: else
4:     if  $x = \text{min}$  then
5:          $\text{firstcluster} \leftarrow \text{summary.min}()$ ;
6:          $\text{offs} \leftarrow \text{cluster}[\text{firstcluster}].\text{min}()$ ;
7:          $x \leftarrow \text{firstcluster} \circ \text{offs}$ ;
8:         min  $\leftarrow x$ ;
9:     cluster[high( $x$ )].delete(low( $x$ ));
```

continued...

Implementation 4: van Emde Boas Trees

- ▶ **Assumes that x is contained in the structure.**

Algorithm 36 delete(x)

```
1: if min = max then
2:     min = max = null;
3: else
4:     if  $x = \text{min}$  then find new minimum
5:          $\text{firstcluster} \leftarrow \text{summary.min}()$ ;
6:          $\text{offs} \leftarrow \text{cluster}[\text{firstcluster}].\text{min}()$ ;
7:          $x \leftarrow \text{firstcluster} \circ \text{offs}$ ;
8:         min  $\leftarrow x$ ;
9:         cluster[high( $x$ )].delete(low( $x$ ));
continued...
```

Implementation 4: van Emde Boas Trees

- ▶ Assumes that x is contained in the structure.

Algorithm 36 delete(x)

```
1: if min = max then  
2:     min = max = null;  
3: else  
4:     if  $x = \text{min}$  then  
5:          $\text{firstcluster} \leftarrow \text{summary.min}()$ ;  
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9:     cluster[high( $x$ )].delete(low( $x$ ));
```

delete

continued...

Implementation 4: van Emde Boas Trees

Algorithm 36 delete(x)

...continued

```
10:   if cluster[high( $x$ )].min() = null then
11:       summary.delete(high( $x$ ));
12:   if  $x$  = max then
13:       summax  $\leftarrow$  summary.max();
14:       if summax = null then max  $\leftarrow$  min;
15:       else
16:           offs  $\leftarrow$  cluster[summax].max();
17:           max  $\leftarrow$  summax  $\circ$  offs
18:   else
19:       if  $x$  = max then
20:           offs  $\leftarrow$  cluster[high( $x$ )].max();
21:           max  $\leftarrow$  high( $x$ )  $\circ$  offs;
```


Implementation 4: van Emde Boas Trees

Algorithm 36 delete(x)

...continued

fix maximum

```
10:   if cluster[high( $x$ )].min() = null then
11:       summary.delete(high( $x$ ));
12:       if  $x$  = max then
13:           summax  $\leftarrow$  summary.max();
14:           if summax = null then max  $\leftarrow$  min;
15:           else
16:               offs  $\leftarrow$  cluster[summax].max();
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20:               offs  $\leftarrow$  cluster[high( $x$ )].max();
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```

Implementation 4: van Emde Boas Trees

Note that only one of the possible recursive calls in Line 9 and Line 11 in the deletion-algorithm may take non-constant time.

To see this observe that the call in Line 11 only occurs if the cluster where x was deleted is now empty. But this means that the call in Line 9 deleted the last element in $\text{cluster}[\text{high}(x)]$. Such a call only takes constant time.

Hence, we get a recurrence of the form

$$T_{\text{del}}(u) = T_{\text{del}}(\sqrt{u}) + c .$$

This gives $T_{\text{del}}(u) = \mathcal{O}(\log \log u)$.

7.6 van Emde Boas Trees

Space requirements:

- ▶ The space requirement fulfills the recurrence

$$S(u) = (\sqrt{u} + 1)S(\sqrt{u}) + \mathcal{O}(\sqrt{u}) .$$

- ▶ Note that we cannot solve this recurrence by the Master theorem as the branching factor is not constant.
- ▶ One can show by induction that the space requirement is $S(u) = \mathcal{O}(u)$. Exercise.

- ▶ Let the “real” recurrence relation be

$$S(k^2) = (k + 1)S(k) + c_1 \cdot k; S(4) = c_2$$

- ▶ Replacing $S(k)$ by $R(k) := S(k)/c_2$ gives the recurrence

$$R(k^2) = (k + 1)R(k) + ck; R(4) = 1$$

where $c = c_1/c_2 < 1$.

- ▶ Now, we show $R(k^2) \leq k^2 - 2$ for $k^2 \geq 4$.
 - ▶ Obviously, this holds for $k^2 = 4$.
 - ▶ For $k^2 > 4$ we have

$$\begin{aligned} R(k^2) &= (1 + k)R(k) + ck \\ &\leq (1 + k)(k - 2) + k \leq k^2 - 2 \end{aligned}$$

- ▶ This shows that $R(k)$ and, hence, $S(k)$ grows linearly.

7.7 Hashing

Dictionary:

- ▶ **$S.insert(x)$** : Insert an element x .
- ▶ **$S.delete(x)$** : Delete the element pointed to by x .
- ▶ **$S.search(k)$** : Return a pointer to an element e with $key[e] = k$ in S if it exists; otherwise return **null**.

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So far we have implemented the search for a key by carefully choosing split-elements.

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So far we have implemented the search for a key by carefully choosing split-elements.

Then the memory location of an object x with key k is determined by successively comparing k to split-elements.

Hashing tries to **directly** compute the memory location from the given key. The goal is to have constant search time.

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Definitions:

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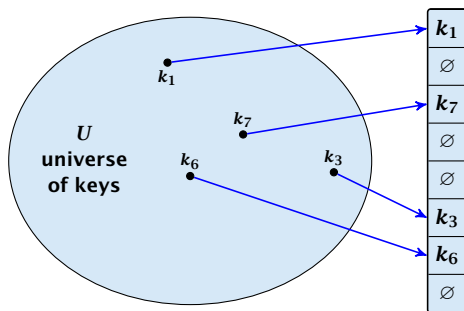
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The hash-function h should fulfill:

- ▶ Fast to evaluate.
- ▶ Small storage requirement.
- ▶ Good distribution of elements over the whole table.

Direct Addressing

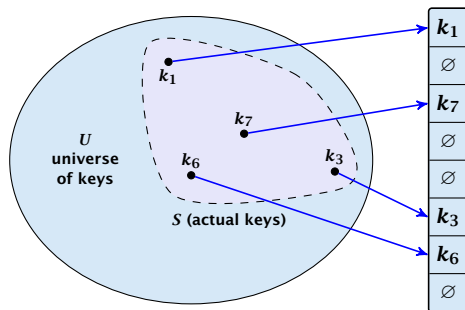
Ideally the hash function maps **all** keys to different memory locations.



This special case is known as **Direct Addressing**. It is usually very unrealistic as the universe of keys typically is quite large, and in particular larger than the available memory.

Perfect Hashing

Suppose that we **know** the set S of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



Such a hash function h is called a **perfect hash function** for set S .

Collisions

If we do not know the keys in advance, the best we can hope for is that the hash function distributes keys evenly across the table.

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Usually the universe U is much larger than the table-size n .

Hence, there may be two elements k_1, k_2 from the set S that map to the same memory location (i.e., $h(k_1) = h(k_2)$). This is called a **collision**.

Collisions

Typically, collisions do not appear once the size of the set S of actual keys gets close to n , but already when $|S| \geq \omega(\sqrt{n})$.

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The probability of having a collision when hashing m elements into a table of size n under uniform hashing is at least

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Uniform hashing:

Choose a hash function uniformly at random from all functions $f : U \rightarrow [0, \dots, n-1]$.

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Proof.

Let $A_{m,n}$ denote the event that inserting m keys into a table of size n does **not** generate a collision. Then

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$$\Pr[A_{m,n}] = \prod_{\ell=1}^m \frac{n - \ell + 1}{n} = \prod_{j=0}^{m-1} \left(1 - \frac{j}{n}\right)$$

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$$\begin{aligned}\Pr[A_{m,n}] &= \prod_{\ell=1}^m \frac{n - \ell + 1}{n} = \prod_{j=0}^{m-1} \left(1 - \frac{j}{n}\right) \\ &\leq \prod_{j=0}^{m-1} e^{-j/n}\end{aligned}$$

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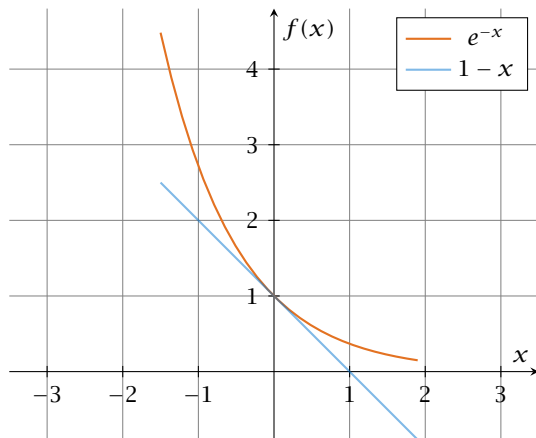
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Here the first equality follows since the ℓ -th element that is hashed has a probability of $\frac{n-\ell+1}{n}$ to not generate a collision under the condition that the previous elements did not induce collisions. □

Collisions



The inequality $1 - x \leq e^{-x}$ is derived by stopping the Taylor-expansion of e^{-x} after the second term.

Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

- ▶ **open addressing**, aka. closed hashing
- ▶ **hashing with chaining**, aka. closed addressing, open hashing.

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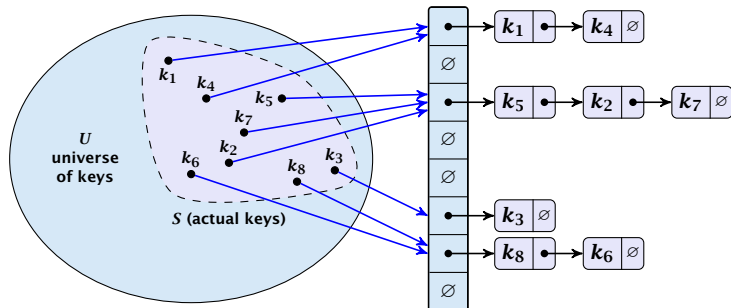
- ▶ **open addressing**, aka. closed hashing
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There are applications e.g. computer chess where you do not resolve collisions at all.

Hashing with Chaining

Arrange elements that map to the same position in a linear list.

- ▶ Access: compute $h(x)$ and search list for $\text{key}[x]$.
- ▶ Insert: insert at the front of the list.



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We assume **uniform hashing** for the following analysis.

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$$A^- = 1 + \alpha .$$

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For a successful search observe that we do **not** choose a list at random, but we consider a random key k in the hash-table and ask for the search-time for k .

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Hashing with Chaining

$$\begin{aligned} E \left[\frac{1}{m} \sum_{i=1}^m \left(1 + \sum_{j=i+1}^m X_{ij} \right) \right] &= \frac{1}{m} \sum_{i=1}^m \left(1 + \sum_{j=i+1}^m E[X_{ij}] \right) \\ &= \frac{1}{m} \sum_{i=1}^m \left(1 + \sum_{j=i+1}^m \frac{1}{n} \right) \end{aligned}$$

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Hence, the expected cost for a successful search is $A^+ \leq 1 + \frac{\alpha}{2}$.

Hashing with Chaining

Disadvantages:

- ▶ pointers increase memory requirements
- ▶ pointers may lead to bad cache efficiency

Advantages:

- ▶ no à priori limit on the number of elements
- ▶ deletion can be implemented efficiently
- ▶ by using balanced trees instead of linked list one can also obtain worst-case guarantees.

Open Addressing

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All objects are stored in the table itself.

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Define a function $h(k, j)$ that determines the table-position to be examined in the j -th step. The values $h(k, 0), \dots, h(k, n - 1)$ must form a permutation of $0, \dots, n - 1$.

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Insert(x): Search until you find an empty slot; insert your element there. If your search reaches $h(k, n - 1)$, and this slot is non-empty then your table is full.

Open Addressing

Choices for $h(k, j)$:

- ▶ **Linear probing:**

$$h(k, i) = h(k) + i \bmod n$$

(sometimes: $h(k, i) = h(k) + ci \bmod n$).

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For quadratic probing and double hashing one has to ensure that the search covers all positions in the table (i.e., for double hashing $h_2(k)$ must be relatively prime to n (**teilerfremd**); for quadratic probing c_1 and c_2 have to be chosen carefully).

Linear Probing

- ▶ Advantage: **Cache-efficiency**. The new probe position is very likely to be in the cache.

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Lemma 21

Let L be the method of linear probing for resolving collisions:

$$L^+ \approx \frac{1}{2} \left(1 + \frac{1}{1 - \alpha} \right)$$

$$L^- \approx \frac{1}{2} \left(1 + \frac{1}{(1 - \alpha)^2} \right)$$

Quadratic Probing

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Lemma 22

Let Q be the method of quadratic probing for resolving collisions:

$$Q^+ \approx 1 + \ln\left(\frac{1}{1-\alpha}\right) - \frac{\alpha}{2}$$

$$Q^- \approx \frac{1}{1-\alpha} + \ln\left(\frac{1}{1-\alpha}\right) - \alpha$$

Double Hashing

- ▶ Any probe into the hash-table usually creates a cache-miss.

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Lemma 23

Let D be the method of double hashing for resolving collisions:

$$D^+ \approx \frac{1}{\alpha} \ln \left(\frac{1}{1 - \alpha} \right)$$

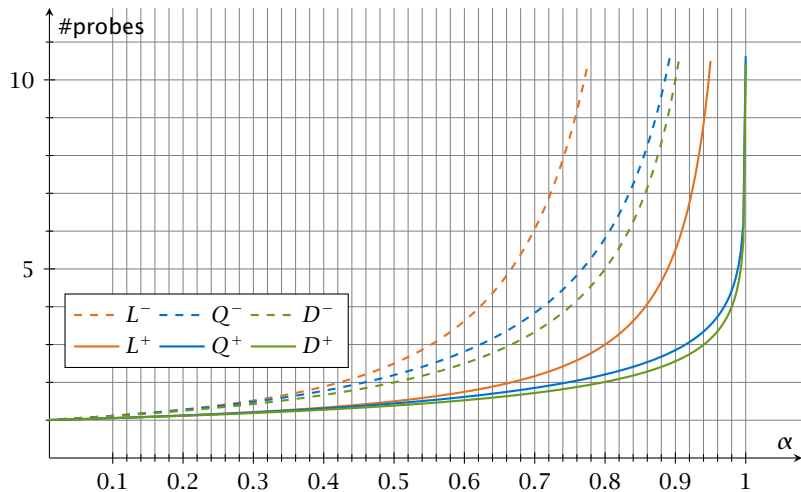
$$D^- \approx \frac{1}{1 - \alpha}$$

Open Addressing

Some values:

α	<i>Linear Probing</i>		<i>Quadratic Probing</i>		<i>Double Hashing</i>	
	L^+	L^-	Q^+	Q^-	D^+	D^-
0.5	1.5	2.5	1.44	2.19	1.39	2
0.9	5.5	50.5	2.85	11.40	2.55	10
0.95	10.5	200.5	3.52	22.05	3.15	20

Open Addressing



Analysis of Idealized Open Address Hashing

We analyze the time for a search in a very idealized Open Addressing scheme.

- ▶ The probe sequence $h(k, 0), h(k, 1), h(k, 2), \dots$ is equally likely to be any permutation of $\langle 0, 1, \dots, n - 1 \rangle$.

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$$\Pr[X \geq i] = \frac{m}{n} \cdot \frac{m-1}{n-1} \cdot \frac{m-2}{n-2} \cdot \dots \cdot \frac{m-i+2}{n-i+2}$$

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$E[X]$

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Analysis of Idealized Open Address Hashing

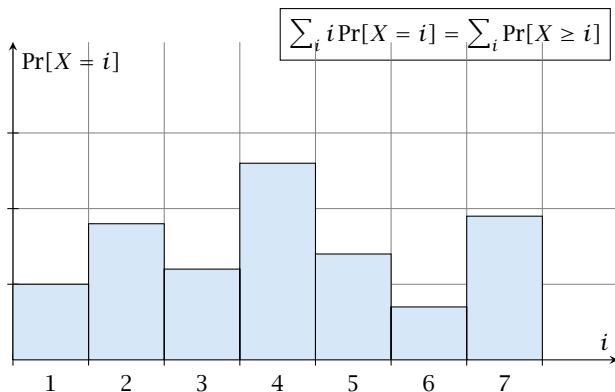
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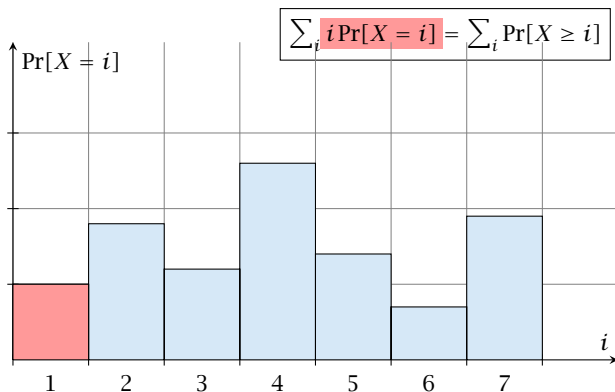
$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \dots$$

Analysis of Idealized Open Address Hashing



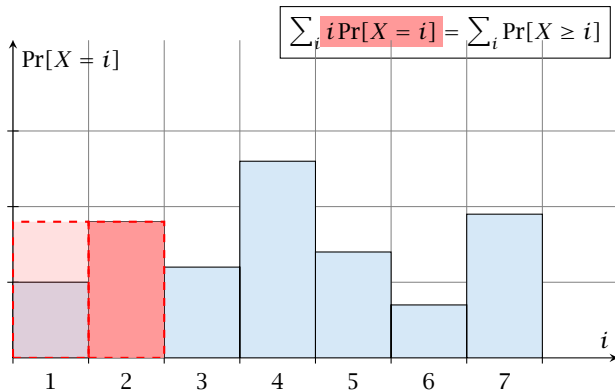
Analysis of Idealized Open Address Hashing

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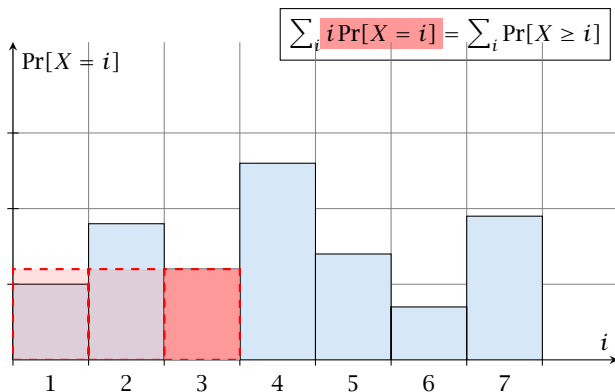
Analysis of Idealized Open Address Hashing

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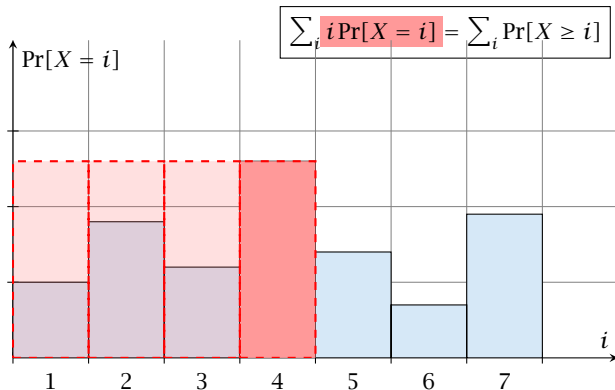
Analysis of Idealized Open Address Hashing

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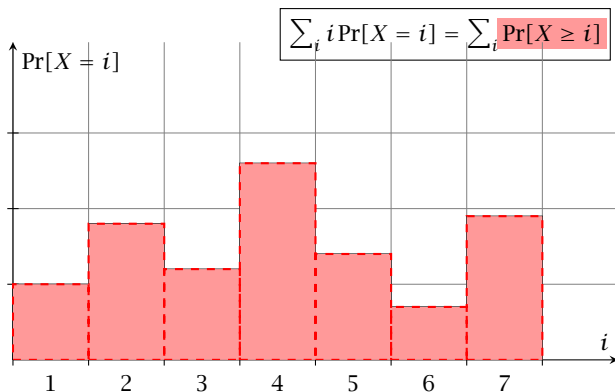
Analysis of Idealized Open Address Hashing

$i = 4$



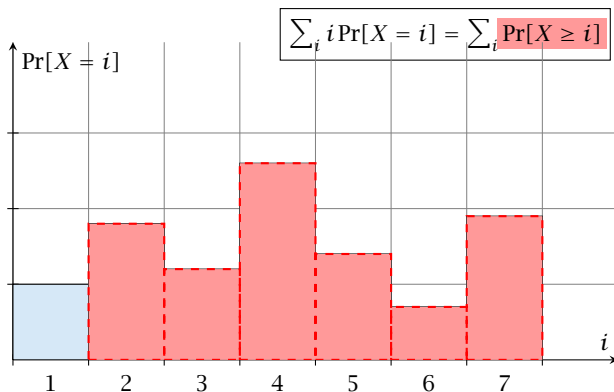
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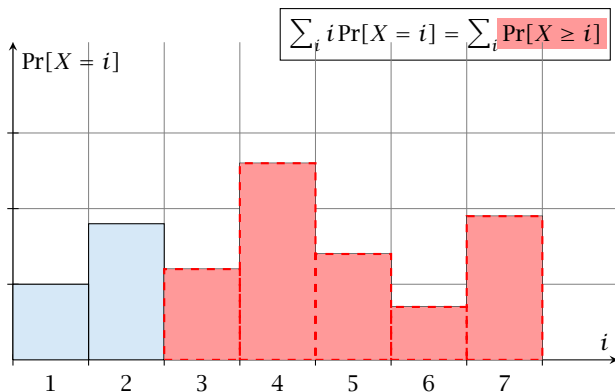
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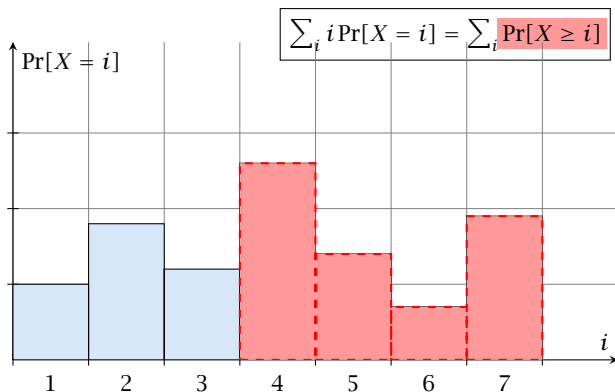
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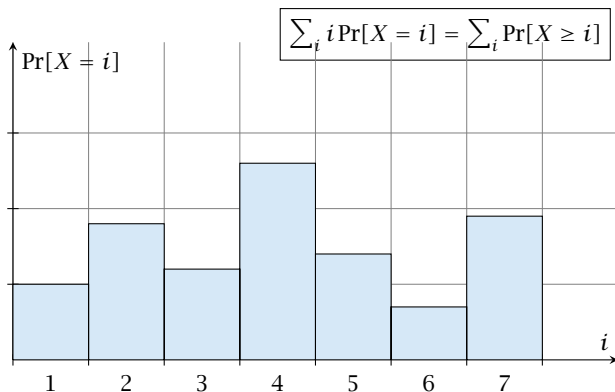


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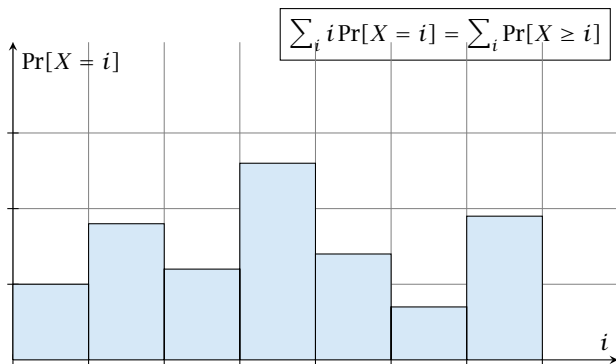
$i = 4$



Analysis of Idealized Open Address Hashing



Analysis of Idealized Open Address Hashing



The j -th rectangle appears in both sums j times. (j times in the first due to multiplication with j ; and j times in the second for summands $i = 1, 2, \dots, j$)

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$$\frac{1}{m} \sum_{i=0}^{m-1} \frac{n}{n-i}$$

Analysis of Idealized Open Address Hashing

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$$\frac{1}{m} \sum_{i=0}^{m-1} \frac{n}{n-i} = \frac{n}{m} \sum_{i=0}^{m-1} \frac{1}{n-i} = \frac{1}{\alpha} \sum_{k=n-m+1}^n \frac{1}{k}$$

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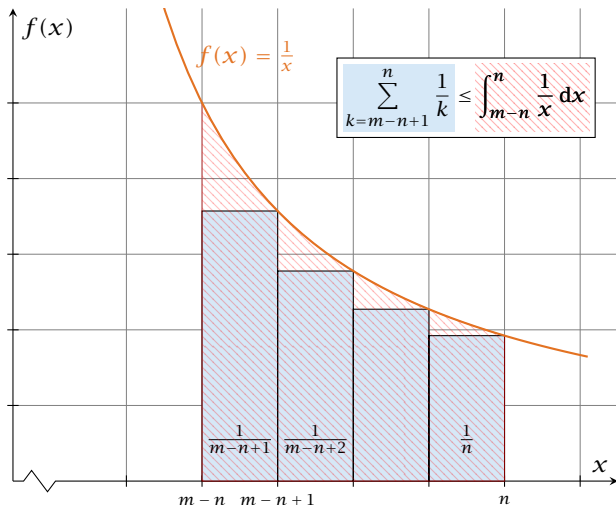
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Analysis of Idealized Open Address Hashing



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- ▶ For open addressing this is difficult.

Deletions in Hashtables

- ▶ Simply removing a key might interrupt the probe sequence of other keys which then cannot be found anymore.

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 - ▶ During a search a **deleted**-marker must not be used to terminate the probe sequence.
- ▶ The table could fill up with **deleted**-markers leading to bad performance.
- ▶ If a table contains many deleted-markers (linear fraction of the keys) one can rehash the whole table and amortize the cost for this rehash against the cost for the deletions.

Deletions for Linear Probing

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- ▶ Upon a deletion elements that are further down in the probe-sequence may be moved to guarantee that they are still found during a search.

Deletions for Linear Probing

Algorithm 37 delete(p)

```
1:  $T[p] \leftarrow \text{null}$ 
2:  $p \leftarrow \text{succ}(p)$ 
3: while  $T[p] \neq \text{null}$  do
4:    $y \leftarrow T[p]$ 
5:    $T[p] \leftarrow \text{null}$ 
6:    $p \leftarrow \text{succ}(p)$ 
7:    $\text{insert}(y)$ 
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p is the index into the table-cell that contains the object to be deleted.

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Pointers into the hash-table become invalid.

Universal Hashing

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However, the assumption of uniform hashing that h is chosen randomly from all functions $f : U \rightarrow [0, \dots, n - 1]$ is clearly unrealistic as there are $n^{|U|}$ such functions. Even writing down such a function would take $|U| \log n$ bits.

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Universal hashing tries to define a set \mathcal{H} of functions that is much smaller but still leads to good average case behaviour when selecting a hash-function uniformly at random from \mathcal{H} .

Universal Hashing

Definition 24

A class \mathcal{H} of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called **universal** if for all $u_1, u_2 \in U$ with $u_1 \neq u_2$

$$\Pr[h(u_1) = h(u_2)] \leq \frac{1}{n} ,$$

where the probability is w. r. t. the choice of a random hash-function from set \mathcal{H} .

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Note that this means that the probability of a collision between two arbitrary elements is at most $\frac{1}{n}$.

Universal Hashing

Definition 25

A class \mathcal{H} of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called **2-independent** (pairwise independent) if the following two conditions hold

- ▶ For any key $u \in U$, and $t \in \{0, \dots, n-1\}$ $\Pr[h(u) = t] = \frac{1}{n}$, i.e., a key is distributed uniformly within the hash-table.
- ▶ For all $u_1, u_2 \in U$ with $u_1 \neq u_2$, and for any two hash-positions t_1, t_2 :

$$\Pr[h(u_1) = t_1 \wedge h(u_2) = t_2] \leq \frac{1}{n^2} .$$

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This requirement clearly implies a universal hash-function.

Definition 26

A class \mathcal{H} of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called **k -independent** if for any choice of $\ell \leq k$ distinct keys $u_1, \dots, u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions t_1, \dots, t_ℓ :

$$\Pr[h(u_1) = t_1 \wedge \dots \wedge h(u_\ell) = t_\ell] \leq \frac{1}{n^\ell} ,$$

where the probability is w. r. t. the choice of a random hash-function from set \mathcal{H} .

Universal Hashing

Definition 27

A class \mathcal{H} of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called (μ, k) -independent if for any choice of $\ell \leq k$ distinct keys $u_1, \dots, u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions t_1, \dots, t_ℓ :

$$\Pr[h(u_1) = t_1 \wedge \dots \wedge h(u_\ell) = t_\ell] \leq \frac{\mu}{n^\ell},$$

where the probability is w. r. t. the choice of a random hash-function from set \mathcal{H} .

Universal Hashing

Universal Hashing

Let $U := \{0, \dots, p - 1\}$ for a prime p . Let $\mathbb{Z}_p := \{0, \dots, p - 1\}$, and let $\mathbb{Z}_p^* := \{1, \dots, p - 1\}$ denote the set of invertible elements in \mathbb{Z}_p .

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Lemma 28

The class

$$\mathcal{H} = \{h_{a,b} \mid a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$$

is a universal class of hash-functions from U to $\{0, \dots, n-1\}$.

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Proof.

Let $x, y \in U$ be two distinct keys. We have to show that the probability of a collision is only $1/n$.

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where we use that \mathbb{Z}_p is a field (**Körper**) and, hence, has no zero divisors (**nullteilerfrei**).

Universal Hashing

- ▶ The hash-function does not generate collisions before the $(\text{mod } n)$ -operation. Furthermore, every choice (a, b) is mapped to a different pair (t_x, t_y) with $t_x := ax + b$ and $t_y := ay + b$.

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$$t_y \equiv ay + b \pmod{p}$$

$$a \equiv (t_x - t_y)(x - y)^{-1} \pmod{p}$$

$$b \equiv t_y - ay \pmod{p}$$

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There is a one-to-one correspondence between hash-functions (pairs (a, b) , $a \neq 0$) and pairs (t_x, t_y) , $t_x \neq t_y$.

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What happens when we do the $\text{mod } n$ operation?

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From the range $0, \dots, p - 1$ the values $t_x, t_x + n, t_x + 2n, \dots$ map to t_x after the modulo-operation. These are at most $\lceil p/n \rceil$ values.

Universal Hashing

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possibilities for choosing t_y such that the final hash-value creates a collision.

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This happens with probability at most $\frac{1}{n}$.

Universal Hashing

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It is also possible to show that \mathcal{H} is an (almost) pairwise independent class of hash-functions.

$$\Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[\begin{array}{l} t_x \bmod n = h_1 \\ t_y \bmod n = h_2 \end{array} \right]$$

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$$\frac{\left\lfloor \frac{p}{n} \right\rfloor^2}{p(p-1)} \leq \Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[\begin{array}{l} t_x \bmod n = h_1 \\ t_y \bmod n = h_2 \end{array} \right] \leq \frac{\left\lfloor \frac{p}{n} \right\rfloor^2}{p(p-1)}$$

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Note that the middle is the probability that $h(x) = h_1$ and $h(y) = h_2$. The total number of choices for (t_x, t_y) is $p(p-1)$. The number of choices for t_x (t_y) such that $t_x \bmod n = h_1$ ($t_y \bmod n = h_2$) lies between $\lfloor \frac{p}{n} \rfloor$ and $\lceil \frac{p}{n} \rceil$.

Universal Hashing

Definition 29

Let $d \in \mathbb{N}$; $q \geq (d + 1)n$ be a prime; and let $\bar{a} \in \{0, \dots, q - 1\}^{d+1}$. Define for $x \in \{0, \dots, q - 1\}$

$$h_{\bar{a}}(x) := \left(\sum_{i=0}^d a_i x^i \bmod q \right) \bmod n .$$

Let $\mathcal{H}_n^d := \{h_{\bar{a}} \mid \bar{a} \in \{0, \dots, q - 1\}^{d+1}\}$. The class \mathcal{H}_n^d is $(e, d + 1)$ -independent.

Note that in the previous case we had $d = 1$ and chose $a_d \neq 0$.

Universal Hashing

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For the coefficients $\bar{a} \in \{0, \dots, q-1\}^{d+1}$ let $f_{\bar{a}}$ denote the polynomial

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The polynomial is defined by $d+1$ distinct points.

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$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

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We first fix the values for inputs x_1, \dots, x_ℓ .

Universal Hashing

Fix $\ell \leq d + 1$; let $x_1, \dots, x_\ell \in \{0, \dots, q - 1\}$ be keys, and let t_1, \dots, t_ℓ denote the corresponding hash-function values.

Let $A^\ell = \{h_{\bar{a}} \in \mathcal{H} \mid h_{\bar{a}}(x_i) = t_i \text{ for all } i \in \{1, \dots, \ell\}\}$

Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

$$f_{\bar{a}}(x_i) \in \underbrace{\{t_i + \alpha \cdot n \mid \alpha \in \{0, \dots, \lceil \frac{q}{n} \rceil - 1\}\}}_{=: B_i}$$

In order to obtain the cardinality of A^ℓ we choose our polynomial by fixing $d + 1$ points.

We first fix the values for inputs x_1, \dots, x_ℓ .

We have

$$|B_1| \cdot \dots \cdot |B_\ell|$$

possibilities to do this (so that $h_{\bar{a}}(x_i) = t_i$).

Universal Hashing

Now, we choose $d - \ell + 1$ other inputs and choose their value arbitrarily. We have $q^{d-\ell+1}$ possibilities to do this.

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Therefore we have

$$|B_1| \cdot \dots \cdot |B_\ell| \cdot q^{d-\ell+1} \leq \left\lceil \frac{q}{n} \right\rceil^\ell \cdot q^{d-\ell+1}$$

possibilities to choose \bar{a} such that $h_{\bar{a}} \in A_\ell$.

Universal Hashing

Therefore the probability of choosing $h_{\bar{a}}$ from A_ℓ is only

$$\frac{\lceil \frac{q}{n} \rceil^\ell \cdot q^{d-\ell+1}}{q^{d+1}}$$

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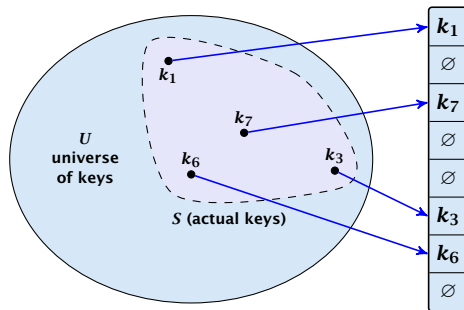
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This shows that the \mathcal{H} is $(e, d+1)$ -universal.

The last step followed from $q \geq (d+1)n$, and $\ell \leq d+1$.

Perfect Hashing

Suppose that we **know** the set S of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



Perfect Hashing

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Let $m = |S|$. We could simply choose the hash-table size very large so that we don't get any collisions.

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The probability of having **1** or more collisions can be at most $\frac{1}{2}$ as otherwise the expectation would be larger than $\frac{1}{2}$.

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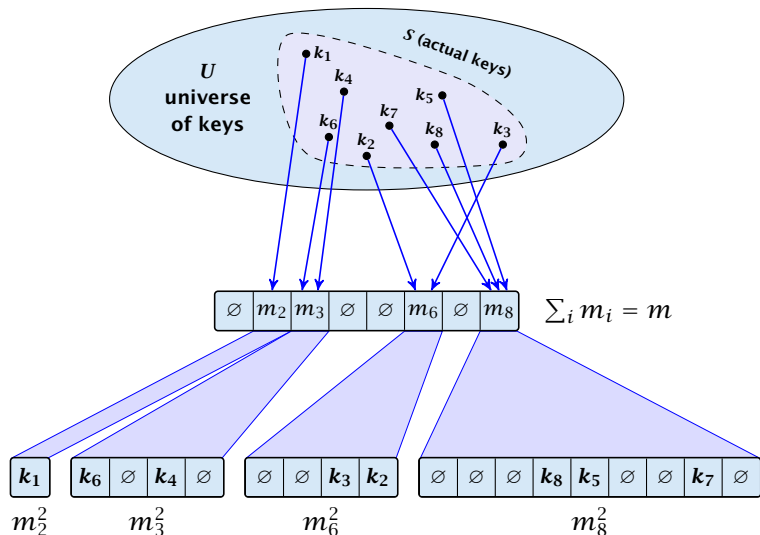
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However, a hash-table size of $n = m^2$ is very very high.

We construct a two-level scheme. We first use a hash-function that maps elements from S to m buckets.

Let m_j denote the number of items that are hashed to the j -th bucket. For each bucket we choose a second hash-function that maps the elements of the bucket into a table of size m_j^2 . The second function can be chosen such that all elements are mapped to different locations.

Perfect Hashing



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The total memory that is required by all hash-tables is $\mathcal{O}(\sum_j m_j^2)$.
Note that m_j is a random variable.

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The first expectation is simply the expected number of collisions, for the first level. Since we use universal hashing we have

$$= 2 \binom{m}{2} \frac{1}{m} + m = 2m - 1 .$$

Perfect Hashing

We need only $\mathcal{O}(m)$ time to construct a hash-function h with $\sum_j m_j^2 = \mathcal{O}(4m)$, because with probability at least $1/2$ a random function from a universal family will have this property.

Then we construct a hash-table h_j for every bucket. This takes expected time $\mathcal{O}(m_j)$ for every bucket. A random function h_j is collision-free with probability at least $1/2$. We need $\mathcal{O}(m_j)$ to test this.

We only need that the hash-functions are chosen from a universal family!!!

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Cuckoo Hashing

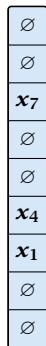
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- ▶ An object x is either stored at location $T_1[h_1(x)]$ or $T_2[h_2(x)]$.
- ▶ A search clearly takes constant time if the above constraint is met.

Cuckoo Hashing

Insert:



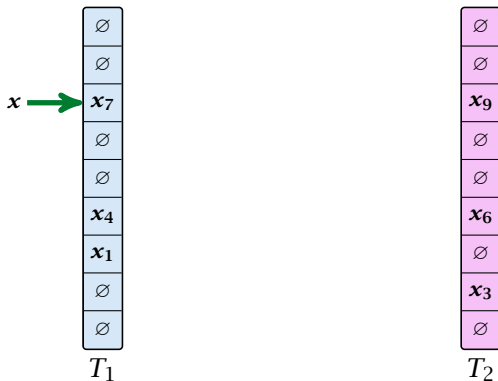
T_1



T_2

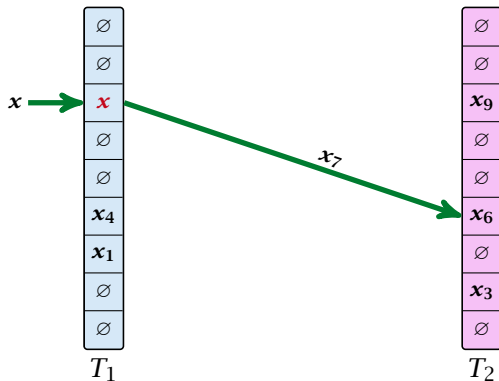
Cuckoo Hashing

Insert:



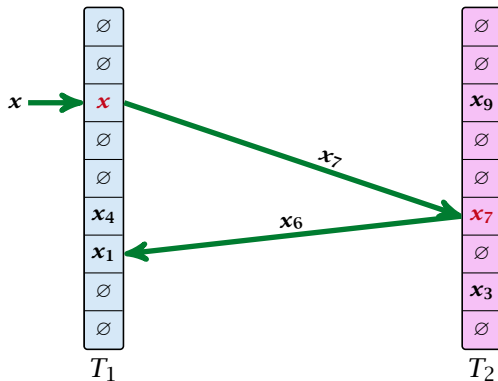
Cuckoo Hashing

Insert:



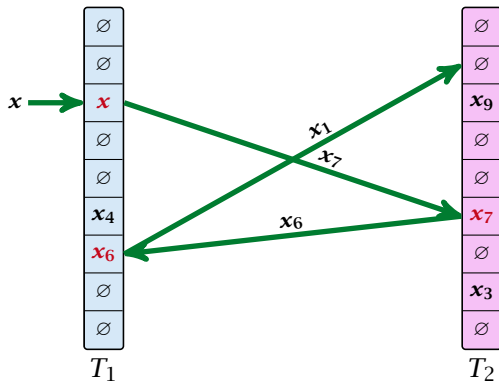
Cuckoo Hashing

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Algorithm 38 Cuckoo-Insert(x)

```
1: if  $T_1[h_1(x)] = x \vee T_2[h_2(x)] = x$  then return  
2: steps  $\leftarrow 1$   
3: while steps  $\leq$  maxsteps do  
4:     exchange  $x$  and  $T_1[h_1(x)]$   
5:     if  $x = \text{null}$  then return  
6:     exchange  $x$  and  $T_2[h_2(x)]$   
7:     if  $x = \text{null}$  then return  
8:     steps  $\leftarrow$  steps + 1  
9: rehash() // change hash-functions; rehash everything  
10: Cuckoo-Insert( $x$ )
```

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- ▶ We call one iteration through the while-loop a **step** of the algorithm.

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- ▶ We call one iteration through the while-loop a **step** of the algorithm.
- ▶ We call a sequence of iterations through the while-loop without the termination condition becoming true a **phase** of the algorithm.
- ▶ We say a phase is **successful** if it is not terminated by the **maxstep**-condition, but the while loop is left because $x = \text{null}$.

Cuckoo Hashing

What is the expected time for an insert-operation?

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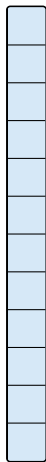
We first analyze the probability that we end-up in an infinite loop (that is then terminated after **maxsteps** steps).

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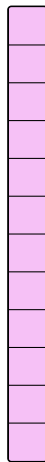
We first analyze the probability that we end-up in an infinite loop (that is then terminated after **maxsteps** steps).

Formally what is the probability to enter an infinite loop that touches s different keys?

Cuckoo Hashing: Insert

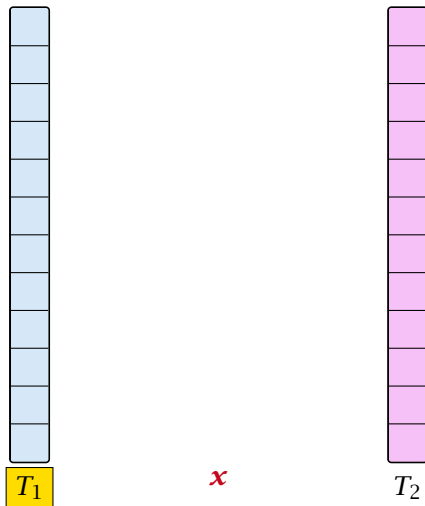


T_1

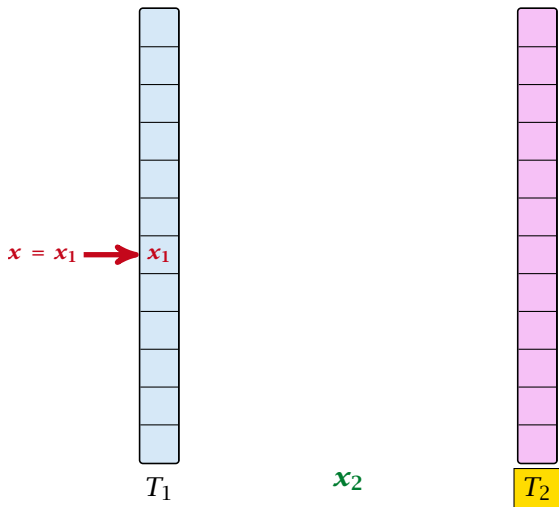


T_2

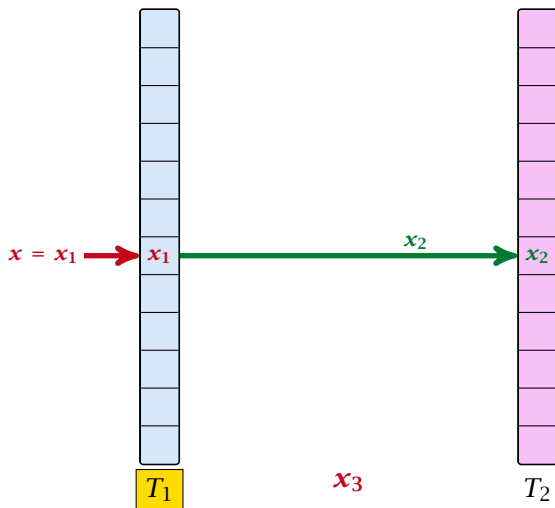
Cuckoo Hashing: Insert



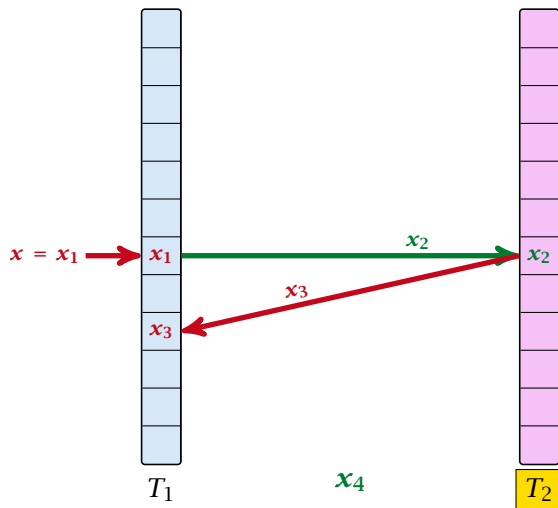
Cuckoo Hashing: Insert



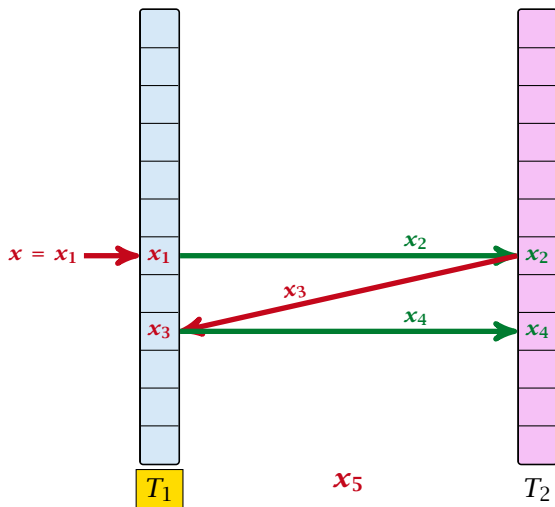
Cuckoo Hashing: Insert



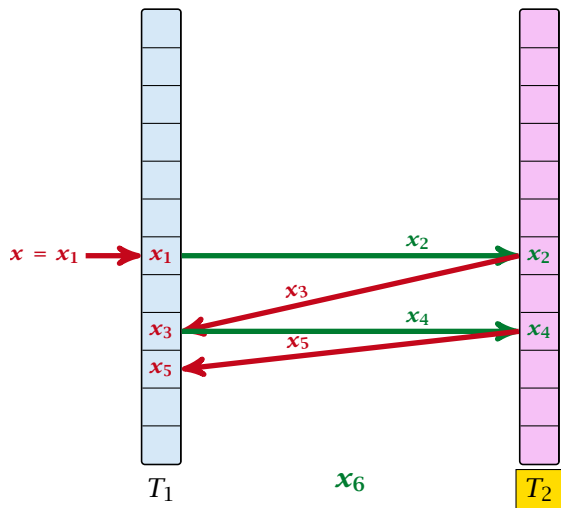
Cuckoo Hashing: Insert



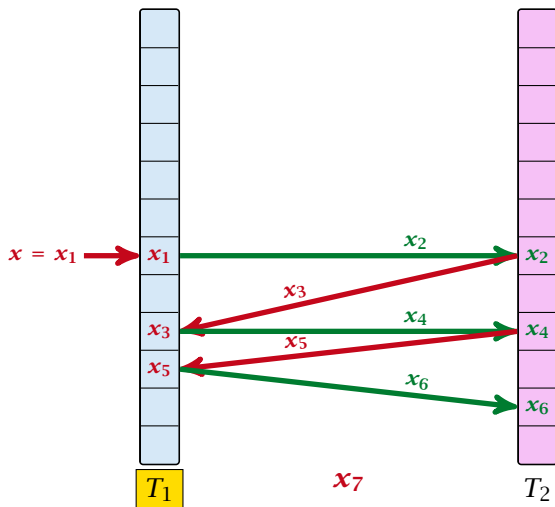
Cuckoo Hashing: Insert



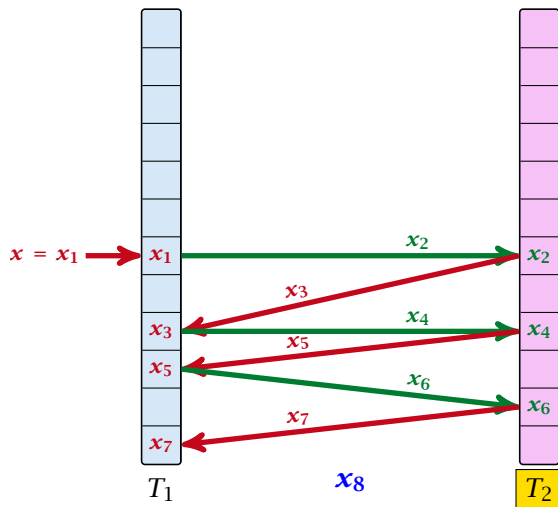
Cuckoo Hashing: Insert



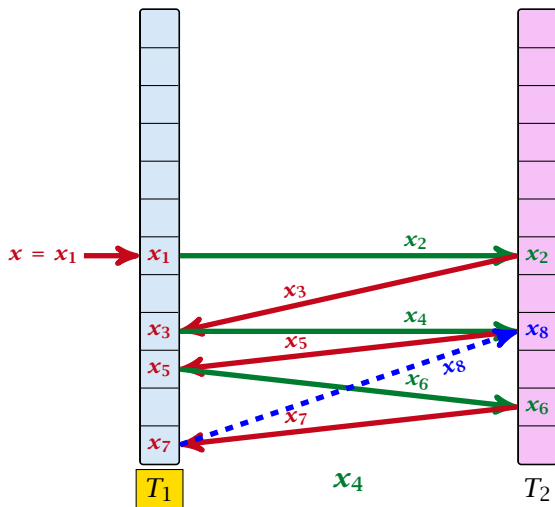
Cuckoo Hashing: Insert



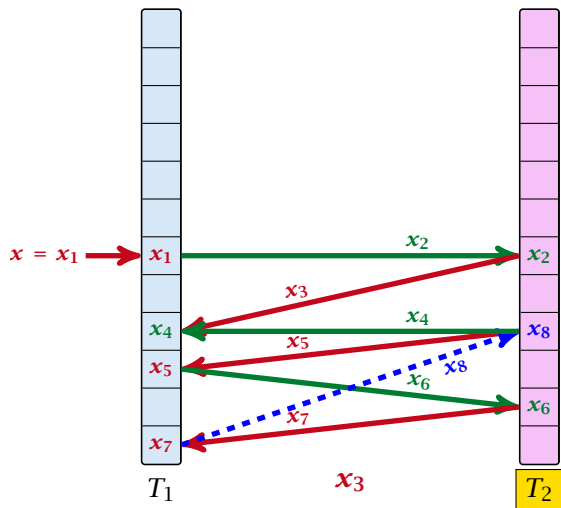
Cuckoo Hashing: Insert



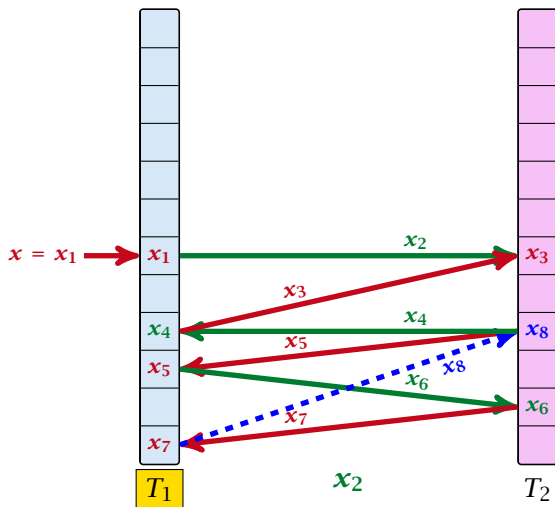
Cuckoo Hashing: Insert



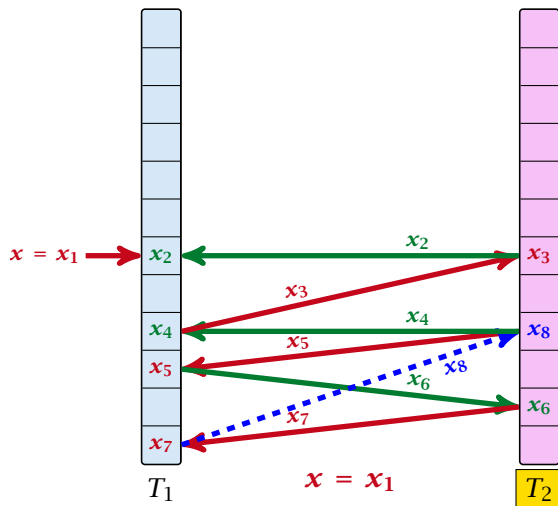
Cuckoo Hashing: Insert



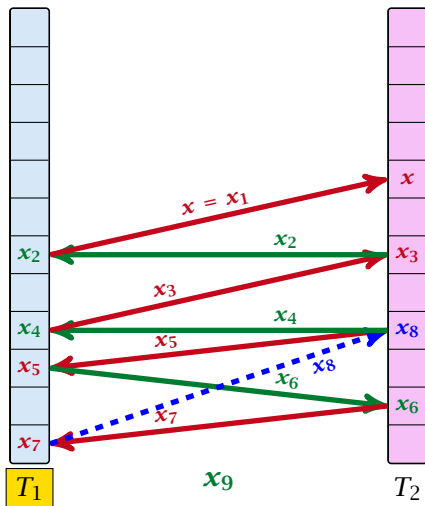
Cuckoo Hashing: Insert



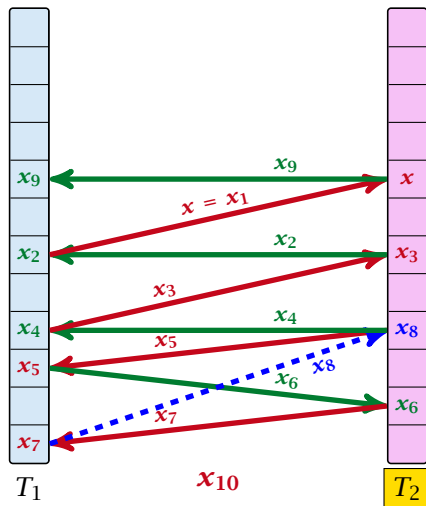
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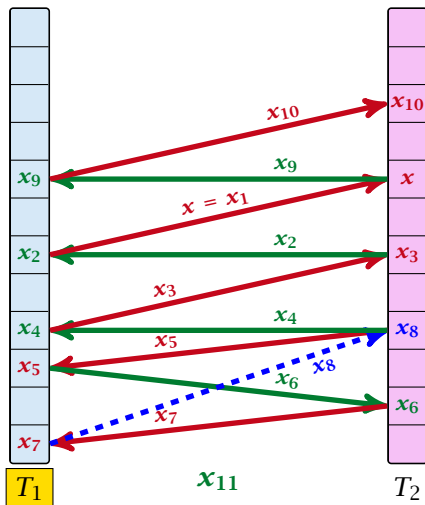
Cuckoo Hashing: Insert



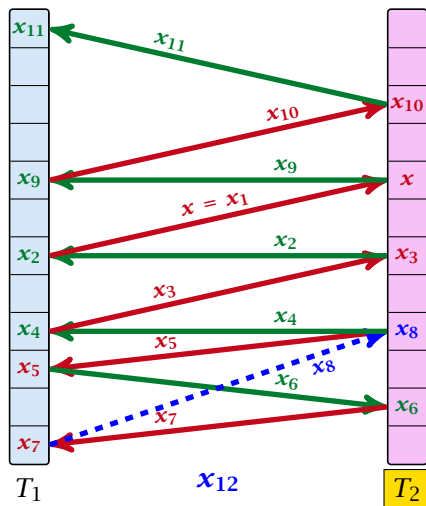
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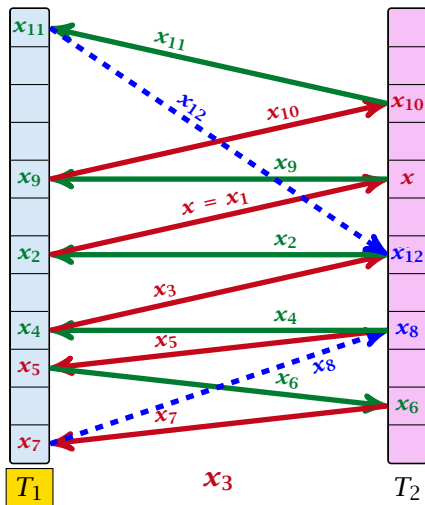
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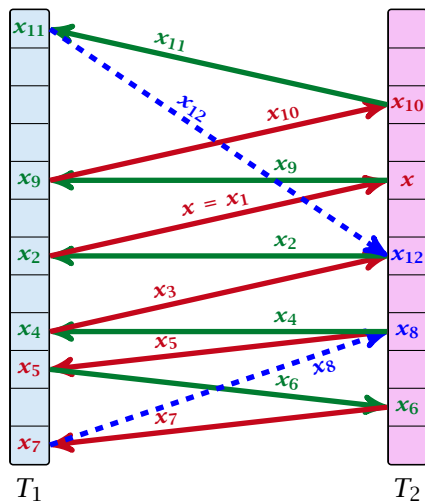
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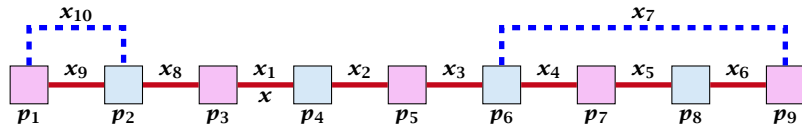
Cuckoo Hashing: Insert



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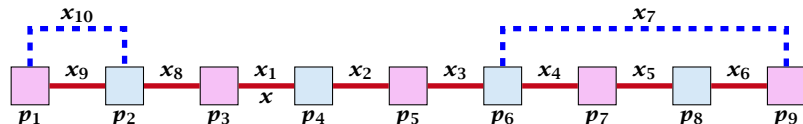


Cuckoo Hashing



A cycle-structure of size s is defined by

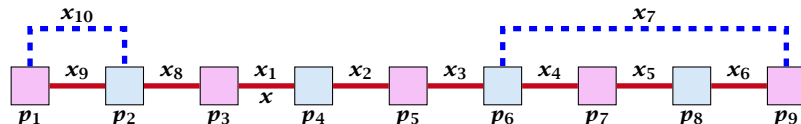
Cuckoo Hashing



A cycle-structure of size s is defined by

- ▶ $s - 1$ different cells (alternating btw. cells from T_1 and T_2).

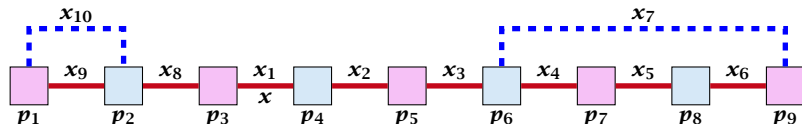
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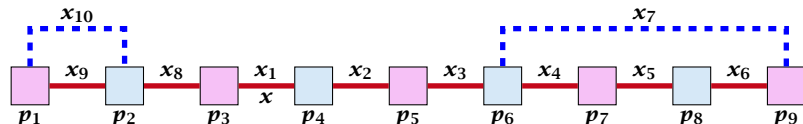
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- ▶ The leftmost cell is “linked forward” to some cell on the right.

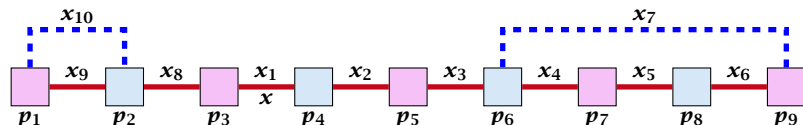
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- ▶ The rightmost cell is “linked backward” to a cell on the left.
- ▶ One link represents key x ; this is where the counting starts.

Cuckoo Hashing

A cycle-structure is **active** if for every key x_ℓ (linking a cell p_i from T_1 and a cell p_j from T_2) we have

$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

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Observation:

If during a phase the insert-procedure runs into a cycle there must exist an active cycle structure of size $s \geq 3$.

Cuckoo Hashing

What is the probability that all keys in a cycle-structure of size s correctly map into their T_1 -cell?

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What is the probability that all keys in the cycle-structure of size s correctly map into their T_2 -cell?

This probability is at most $\frac{\mu}{n^s}$ since h_2 is a (μ, s) -independent hash-function.

These events are independent.

Cuckoo Hashing

The probability that a given cycle-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

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The probability that a given cycle-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

What is the probability that **there exists** an active cycle structure of size s ?

Cuckoo Hashing

The number of cycle-structures of size s is at most

$$s^3 \cdot n^{s-1} \cdot m^{s-1} .$$

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- ▶ There are at most s possibilities to choose where to place key x .
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- ▶ There are n^{s-1} possibilities to choose the cells.

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} = \frac{\mu^2}{nm} \sum_{s=3}^{\infty} s^3 \left(\frac{m}{n}\right)^s$$

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Here we used the fact that $(1 + \epsilon)m \leq n$.

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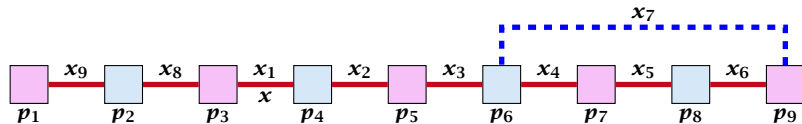
Hence,

$$\Pr[\text{cycle}] = \mathcal{O}\left(\frac{1}{m^2}\right).$$

Cuckoo Hashing

Now, we analyze the probability that a phase is not successful without running into a closed cycle.

Cuckoo Hashing



Sequence of visited keys:

$x = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_3, x_2, x_1 = x, x_8, x_9, \dots$

Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with x in the order that they are visited during the phase.

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Lemma 30

*If the sequence is of length p then there exists a sub-sequence of at least $\frac{p+2}{3}$ keys starting with x of *distinct* keys.*

Cuckoo Hashing

Proof.

Let i be the number of keys (including x) that we see before the first repeated key. Let j denote the total number of distinct keys.

The sequence is of the form:

$$x = x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i \rightarrow x_r \rightarrow x_{r-1} \rightarrow \dots \rightarrow x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$$

As $r \leq i - 1$ the length p of the sequence is

$$p = i + r + (j - i) \leq i + j - 1 .$$

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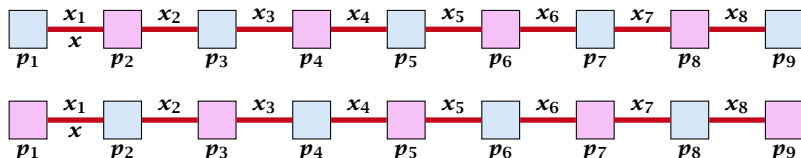
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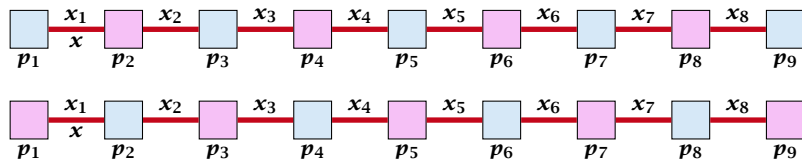
Either sub-sequence $x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i$ or sub-sequence $x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$ has at least $\frac{p+2}{3}$ elements. □

Cuckoo Hashing



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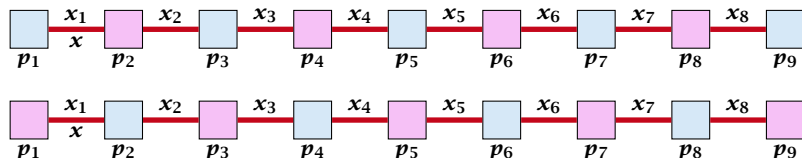
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A path-structure of size s is defined by

- ▶ $s + 1$ different cells (alternating btw. cells from T_1 and T_2).

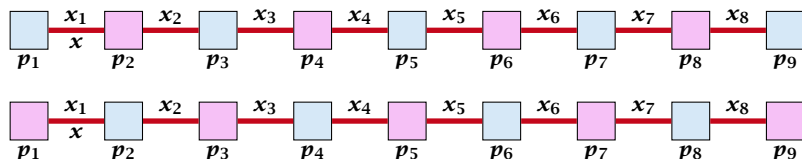
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- ▶ The leftmost cell is either from T_1 or T_2 .

Cuckoo Hashing

A path-structure is **active** if for every key x_ℓ (linking a cell p_i from T_1 and a cell p_j from T_2) we have

$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

Observation:

If a phase takes at least t steps without running into a cycle there must exist an active path-structure of size $(2t + 2)/3$.

Cuckoo Hashing

The probability that a given path-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

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This gives $\text{maxsteps} = \Theta(\log m)$.

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So far we estimated

$$\Pr[\text{cycle}] \leq \mathcal{O}\left(\frac{1}{m^2}\right)$$

and

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for a suitable constant $c > 0$.

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This means the expected cost for a successful phase is constant (even after accounting for the cost of the incomplete step that finishes the phase).

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A rehash try requires m insertions and takes expected constant time per insertion. It fails with probability $p := \mathcal{O}(1/m)$.

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The probability that a phase is not successful is $q = \mathcal{O}(1/m^2)$ (probability $\mathcal{O}(1/m^2)$ of running into a cycle and probability $\mathcal{O}(1/m^2)$ of reaching `maxsteps` without running into a cycle).

A rehash try requires m insertions and takes expected constant time per insertion. It fails with probability $p := \mathcal{O}(1/m)$.

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Therefore the expected cost for re-hashes is $\mathcal{O}(m) \cdot \mathcal{O}(p) = \mathcal{O}(1)$.

Formal Proof

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Therefore, it is sufficient to have $(\mu, \Theta(\log m))$ -independent hash-functions.

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- ▶ Whenever m drops below $\alpha n/4$ we divide n by 2 and do a rehash (**table-shrink**).
- ▶ Note that right after a change in table-size we have $m = \alpha n/2$. In order for a table-expand to occur at least $\alpha n/2$ insertions are required. Similar, for a table-shrink at least $\alpha n/4$ deletions must occur.
- ▶ Therefore we can amortize the rehash cost after a change in table-size against the cost for insertions and deletions.

Cuckoo Hashing

Lemma 31

Cuckoo Hashing has an expected constant insert-time and a worst-case constant search-time.

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Note that the above lemma only holds if the fill-factor (number of keys/total number of hash-table slots) is at most $\frac{1}{2(1+\epsilon)}$.

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Sometimes we also have

- ▶ **S . merge(S')**: $S := S \cup S'$; $S' := \emptyset$.

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- ▶ **S . delete(h)**: Deletes element specified through handle h .
- ▶ **S . decrease-key(h, k)**: Decreases the key of the element specified by handle h to k . Assumes that the key is at least k before the operation.

Dijkstra's Shortest Path Algorithm

Algorithm 39 Shortest-Path($G = (V, E, d), s \in V$)

```
1: Input: weighted graph  $G = (V, E, d)$ ; start vertex  $s$ ;  
2: Output: key-field of every node contains distance from  $s$ ;  
3:  $S.build()$ ; // build empty priority queue  
4: for all  $v \in V \setminus \{s\}$  do  
5:      $v.key \leftarrow \infty$ ;  
6:      $h_v \leftarrow S.insert(v)$ ;  
7:  $s.key \leftarrow 0$ ;  $S.insert(s)$ ;  
8: while  $S.is-empty() = false$  do  
9:      $v \leftarrow S.delete-min()$ ;  
10:    for all  $x \in V$  s.t.  $(v, x) \in E$  do  
11:        if  $x.key > v.key + d(v, x)$  then  
12:             $S.decrease-key(h_x, v.key + d(v, x))$ ;  
13:             $x.key \leftarrow v.key + d(v, x)$ ;
```

Prim's Minimum Spanning Tree Algorithm

Algorithm 40 Prim-MST($G = (V, E, d), s \in V$)

```
1: Input: weighted graph  $G = (V, E, d)$ ; start vertex  $s$ ;  
2: Output: pred-fields encode MST;  
3:  $S.build()$ ; // build empty priority queue  
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14:             $x.pred \leftarrow v$ ;
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Analysis of Dijkstra and Prim

Both algorithms require:

- ▶ 1 build() operation
- ▶ $|V|$ insert() operations
- ▶ $|V|$ delete-min() operations
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How good a running time can we obtain?

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<i>Operation</i>	<i>Binary Heap</i>	<i>BST</i>	<i>Binomial Heap</i>	<i>Fibonacci Heap*</i>
build	n	$n \log n$	$n \log n$	n
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n^{**}$	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
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Note that most applications use **build()** only to create an empty heap which then costs time 1.

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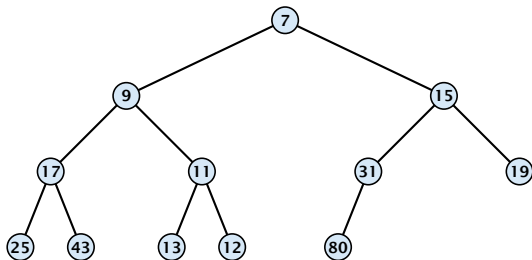
Fibonacci heaps only give an **amortized** guarantee.

8 Priority Queues

Using Binary Heaps, Prim and Dijkstra run in time $\mathcal{O}((|V| + |E|) \log |V|)$.

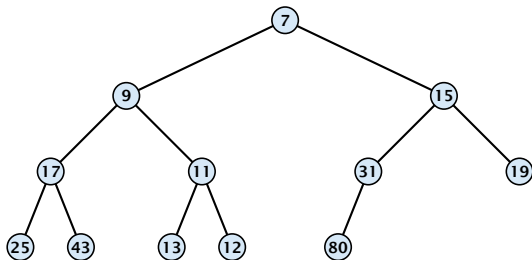
Using Fibonacci Heaps, Prim and Dijkstra run in time $\mathcal{O}(|V| \log |V| + |E|)$.

8.1 Binary Heaps



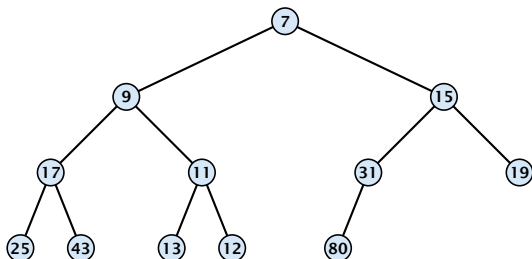
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- ▶ Nearly complete binary tree; only the last level is not full, and this one is filled from left to right.
- ▶ **Heap property:** A node's key is not larger than the key of one of its children.



Operations:

Binary Heaps

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- ▶ **minimum()**: return the root-element. Time $\mathcal{O}(1)$.

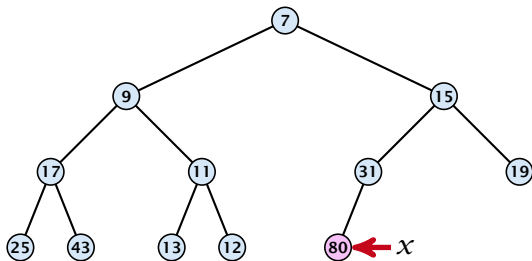
Binary Heaps

Operations:

- ▶ **minimum()**: return the root-element. Time $\mathcal{O}(1)$.
- ▶ **is-empty()**: check whether root-pointer is **null**. Time $\mathcal{O}(1)$.

8.1 Binary Heaps

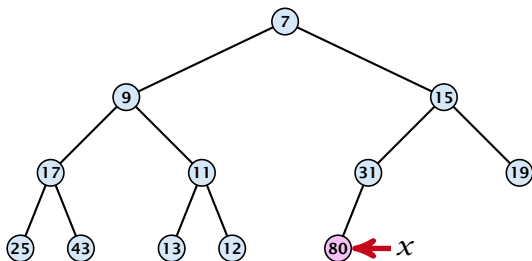
Maintain a pointer to the **last element** x .



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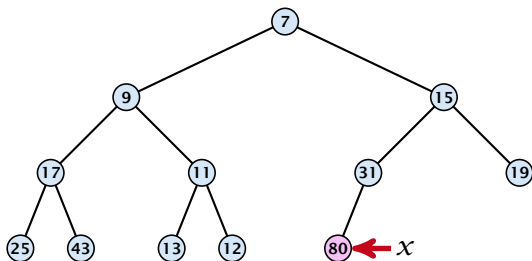
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go up until the last edge used was a right edge.

go left; go right until you reach a leaf



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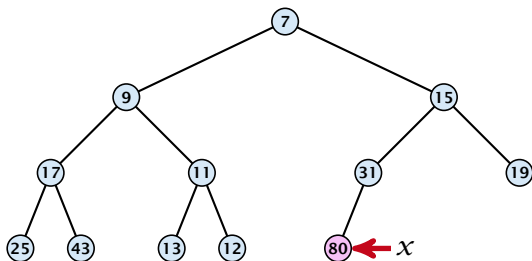
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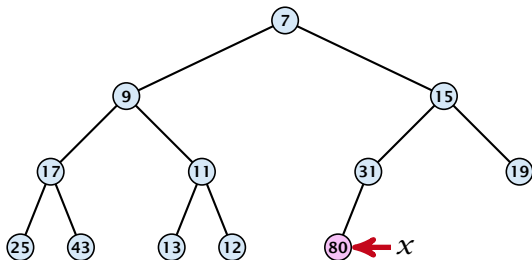
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if you hit the root on the way up, go to the rightmost element



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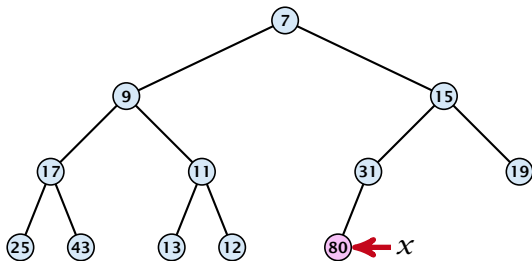
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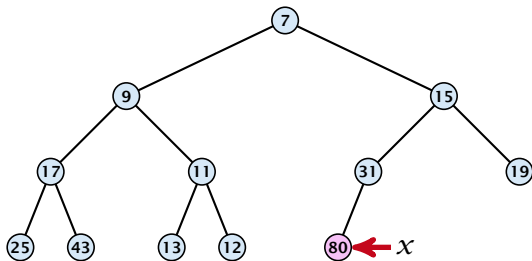
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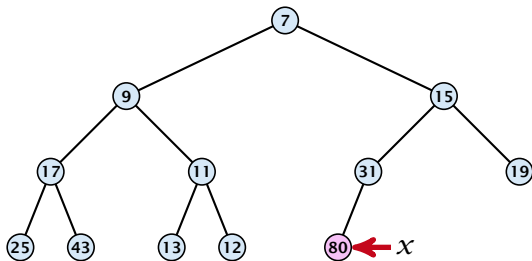
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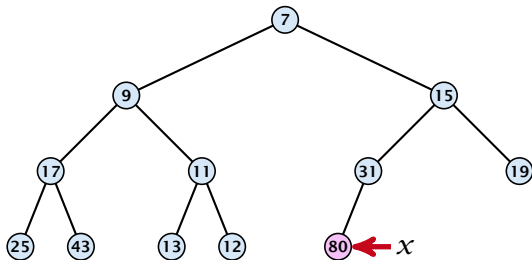
if you hit the root on the way up, go to the leftmost element;

insert a new element as a left child;



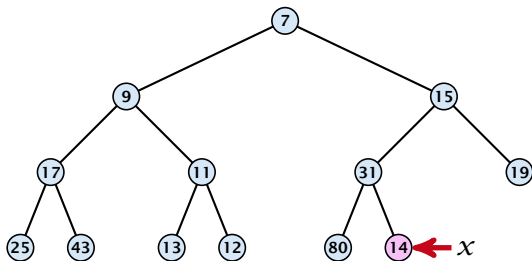
Insert

1. Insert element at successor of x .



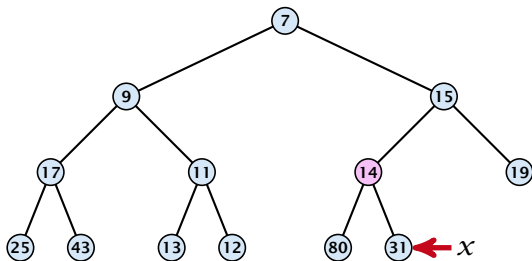
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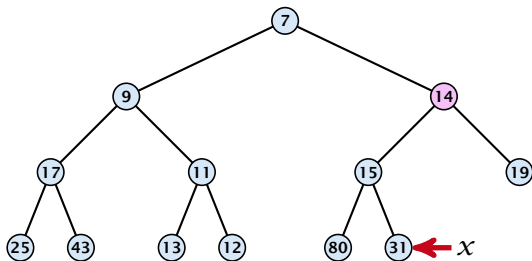
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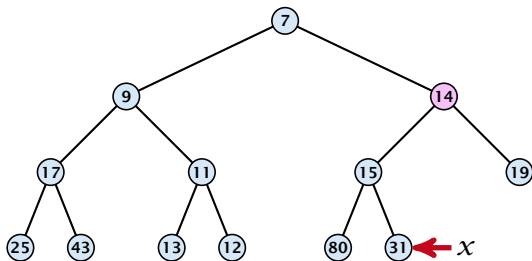
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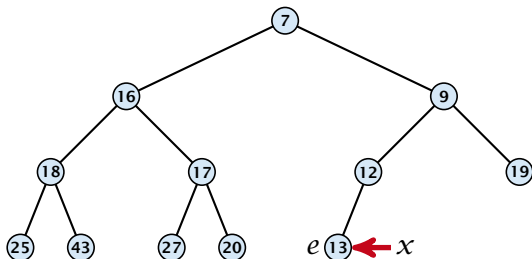
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Note that an exchange can either be done by moving the data or by changing pointers. The latter method leads to an addressable priority queue.

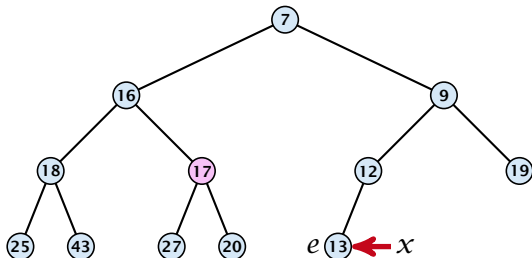
Delete

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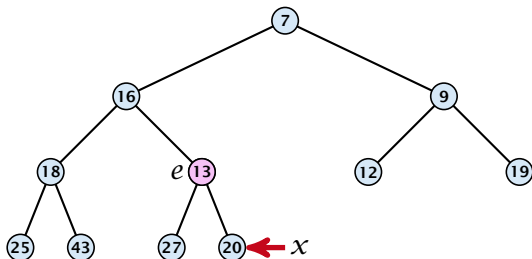
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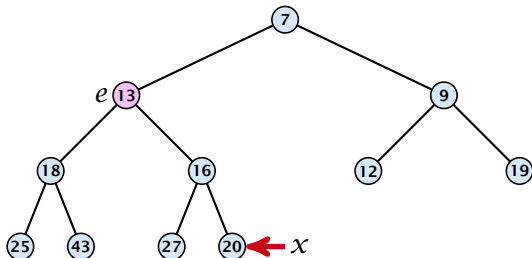
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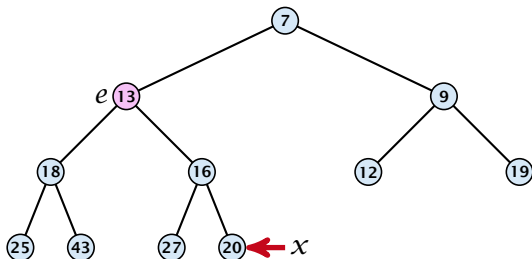
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At its new position e may either travel up or down in the tree (but not both directions).

Binary Heaps

Operations:

- ▶ **minimum()**: return the root-element. Time $\mathcal{O}(1)$.
- ▶ **is-empty()**: check whether root-pointer is **null**. Time $\mathcal{O}(1)$.
- ▶ **insert(k)**: insert at successor of x and bubble up. Time $\mathcal{O}(\log n)$.
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- ▶ **delete(h)**: Swap with x and bubble up or sift-down. Time $\mathcal{O}(\log n)$.
- ▶ **build(x_1, \dots, x_n)**: Insert elements arbitrarily; then do sift-down operations starting with the lowest layer in the tree. Time $\mathcal{O}(n)$.

Binary Heaps

Binary Heaps

The standard implementation of binary heaps is via arrays. Let $A[0, \dots, n - 1]$ be an array

- ▶ The parent of i -th element is at position $\lfloor \frac{i-1}{2} \rfloor$.
- ▶ The left child of i -th element is at position $2i + 1$.
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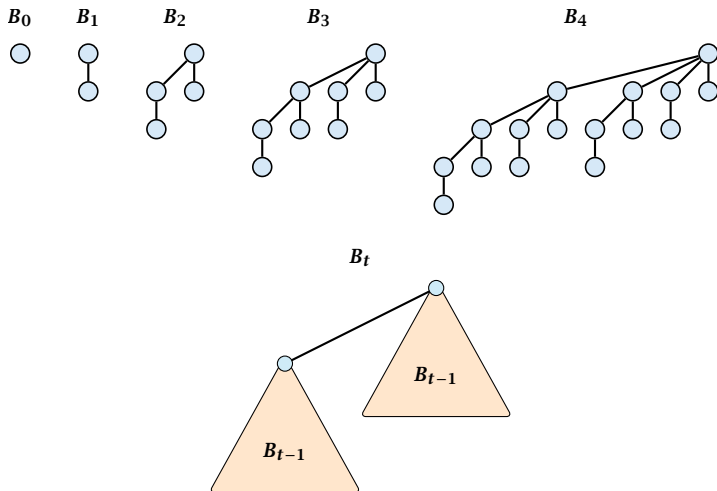
Finding the successor of x is much easier than in the description on the previous slide. Simply increase or decrease x .

The resulting binary heap is not addressable. The elements don't maintain their positions and therefore there are no stable handles.

8.2 Binomial Heaps

<i>Operation</i>	<i>Binary Heap</i>	<i>BST</i>	<i>Binomial Heap</i>	<i>Fibonacci Heap*</i>
build	n	$n \log n$	$n \log n$	n
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n^{**}$	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	n	$n \log n$	$\log n$	1

Binomial Trees



Properties of Binomial Trees

- ▶ B_k has 2^k nodes.

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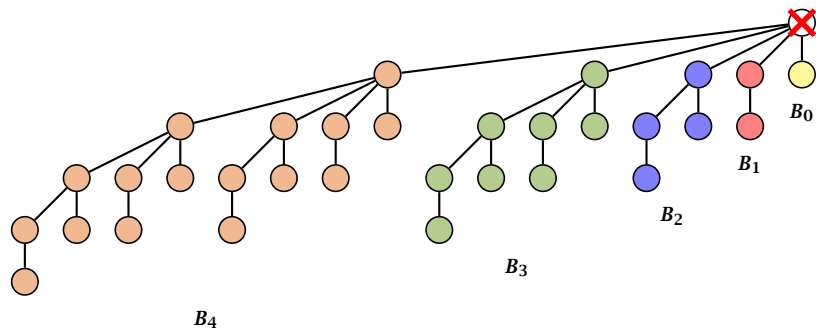
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Properties of Binomial Trees

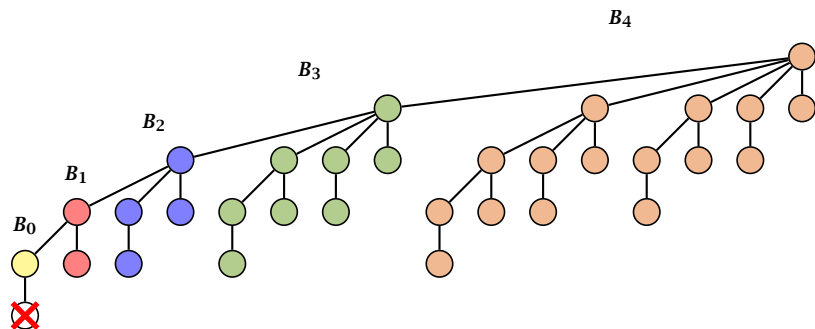
- ▶ B_k has 2^k nodes.
- ▶ B_k has height k .
- ▶ The root of B_k has degree k .
- ▶ B_k has $\binom{k}{\ell}$ nodes on level ℓ .
- ▶ Deleting the root of B_k gives trees B_0, B_1, \dots, B_{k-1} .

Binomial Trees



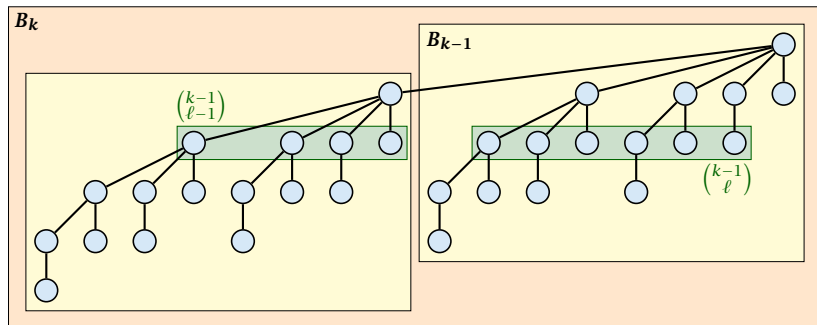
Deleting the root of B_5 leaves sub-trees B_4 , B_3 , B_2 , B_1 , and B_0 .

Binomial Trees



Deleting the leaf furthest from the root (in B_5) leaves a path that connects the roots of sub-trees B_4 , B_3 , B_2 , B_1 , and B_0 .

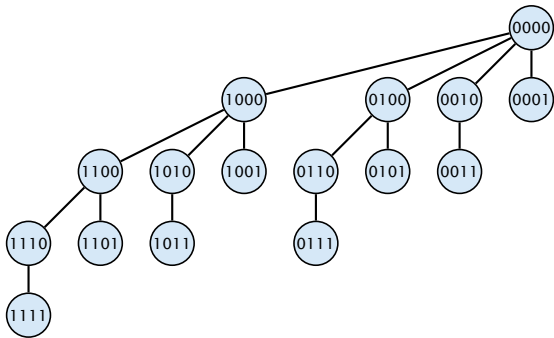
Binomial Trees



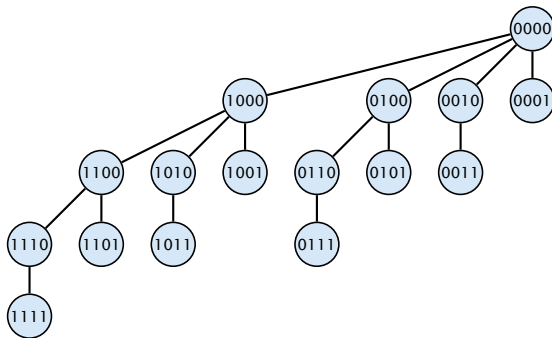
The number of nodes on level ℓ in tree B_k is therefore

$$\binom{k-1}{\ell-1} + \binom{k-1}{\ell} = \binom{k}{\ell}$$

Binomial Trees

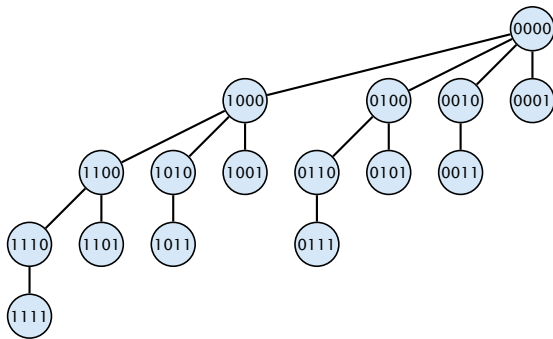


Binomial Trees



The binomial tree B_k is a sub-graph of the hypercube H_k .

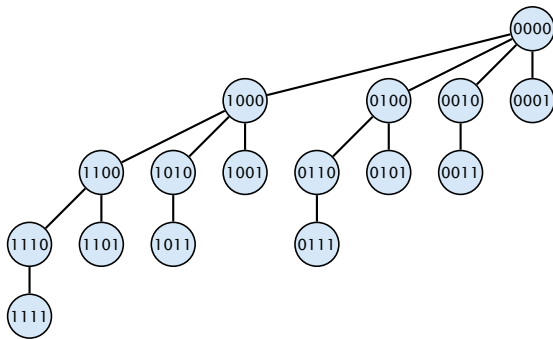
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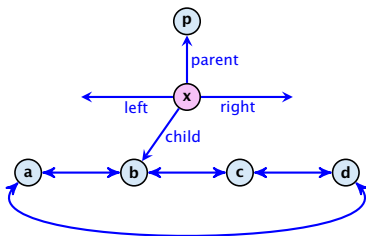
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The ℓ -th level contains nodes that have ℓ 1's in their label.

8.2 Binomial Heaps

How do we implement trees with non-constant degree?

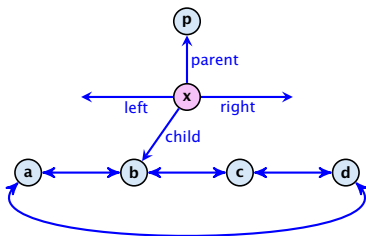
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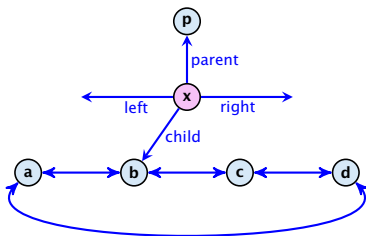
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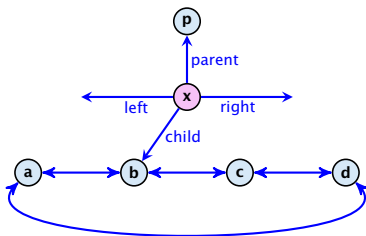
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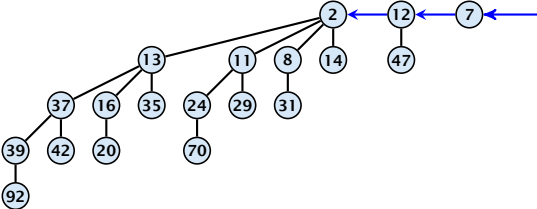
- ▶ The children of a node are arranged in a **circular linked list**.
- ▶ A child-pointer points to an arbitrary node within the list.
- ▶ A parent-pointer points to the parent node.
- ▶ Pointers $x.left$ and $x.right$ point to the left and right sibling of x (if x does not have siblings then $x.left = x.right = x$).



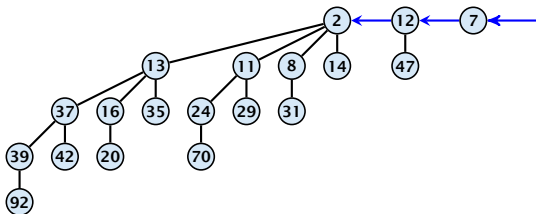
8.2 Binomial Heaps

- ▶ Given a pointer to a node x we can splice out the sub-tree rooted at x in constant time.
- ▶ We can add a child-tree T to a node x in constant time if we are given a pointer to x and a pointer to the root of T .

Binomial Heap

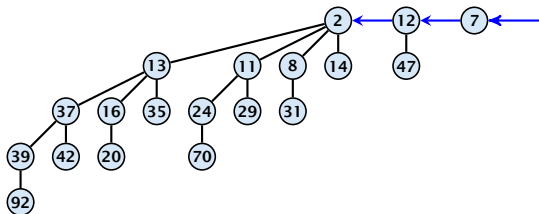


Binomial Heap



In a binomial heap the keys are arranged in a collection of binomial trees.

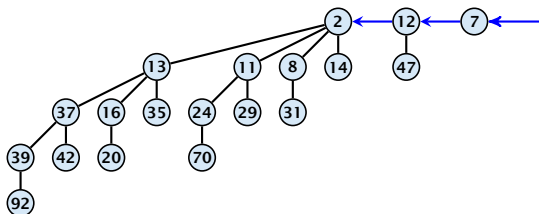
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There is at most one tree for every dimension/order. For example the above heap contains trees B_0 , B_1 , and B_4 .

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Let $B_{k_1}, B_{k_2}, B_{k_3}, k_i < k_{i+1}$ denote the binomial trees in the collection and recall that every tree may be contained at most once.

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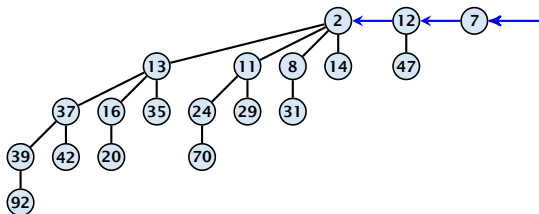
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Then $n = \sum_i 2^{k_i}$ must hold. But since the k_i are all distinct this means that the k_i define the non-zero bit-positions in the binary representation of n .

Binomial Heap

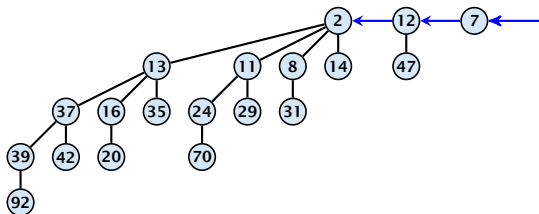
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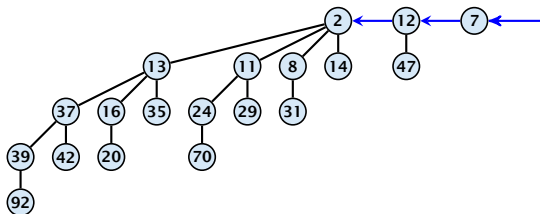
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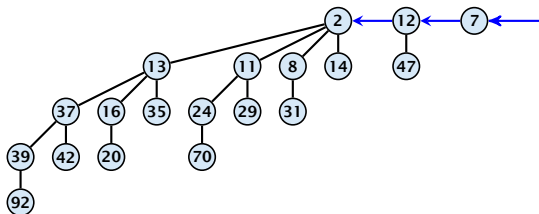
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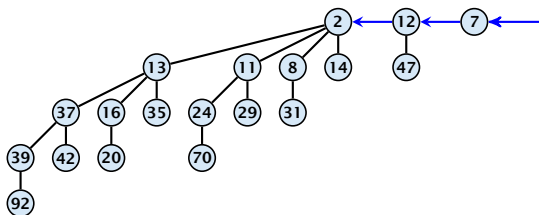
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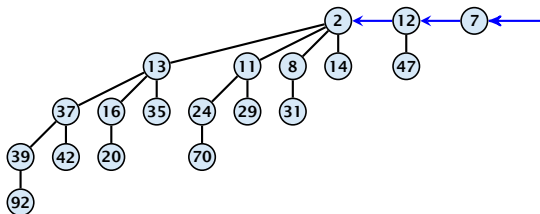
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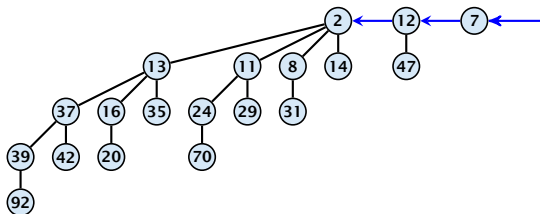
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- ▶ The height of the largest tree is at most $\lfloor \log n \rfloor$.
- ▶ The trees are stored in a single-linked list; ordered by dimension/size.



Binomial Heap: Merge

The merge-operation is instrumental for binomial heaps.

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Otherwise, we cannot do this because the merged heap is not allowed to contain two trees of the same order.

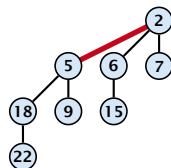
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Merging two trees of the same size: Add the tree with larger root-value as a child to the other tree.



Binomial Heap: Merge

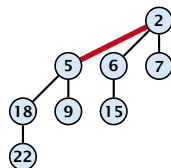
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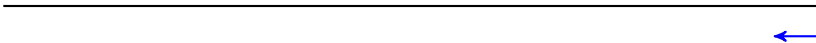
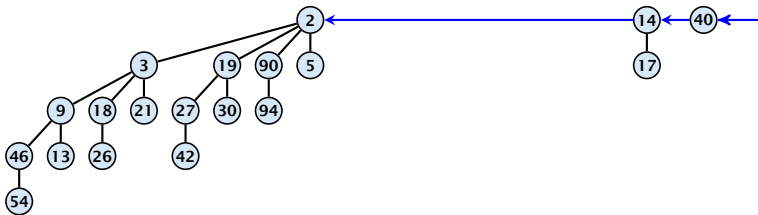
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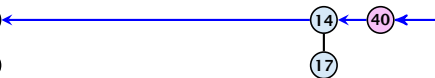
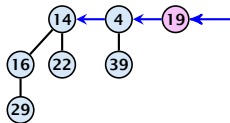
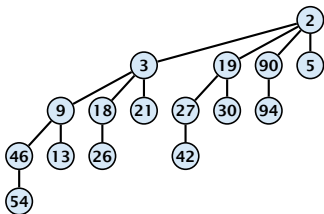
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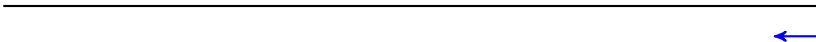
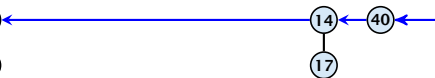
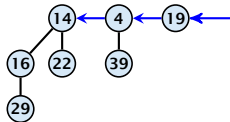
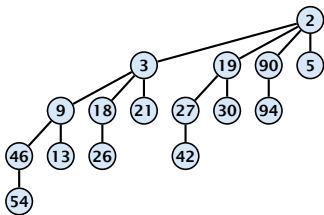
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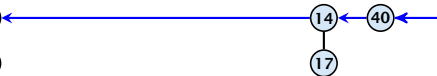
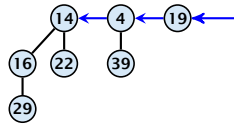
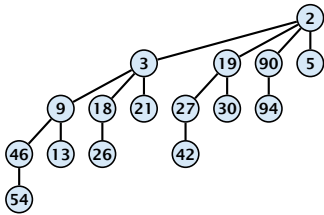
For more trees the technique is analogous to binary addition.

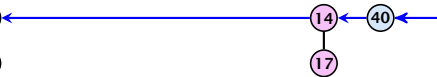
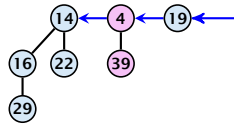
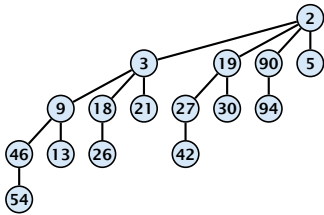


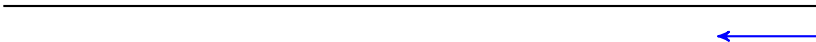
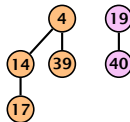
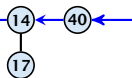
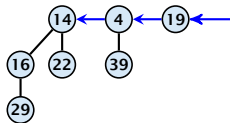
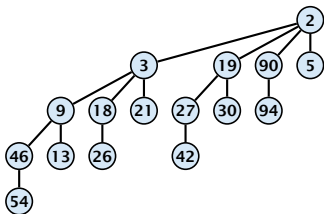


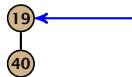
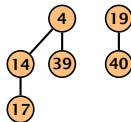
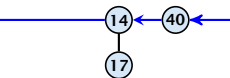
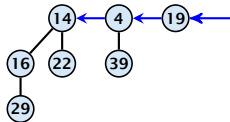
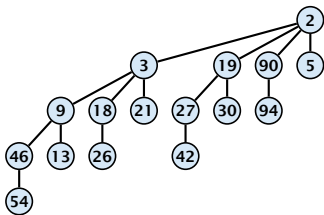


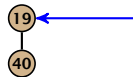
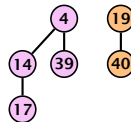
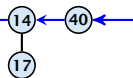
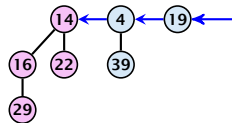
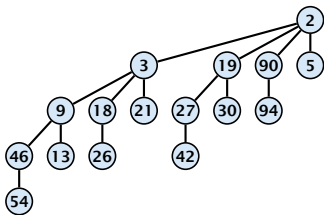


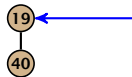
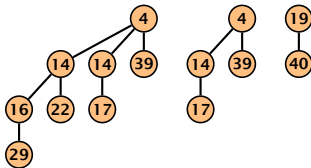
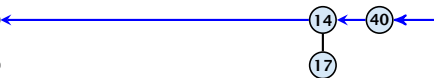
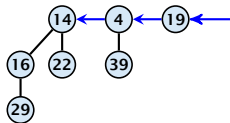
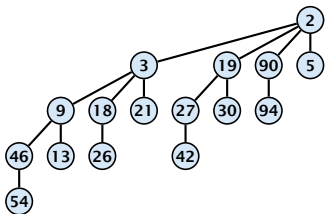


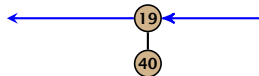
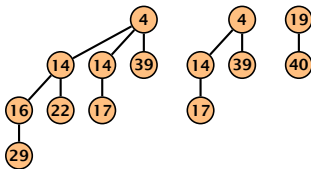
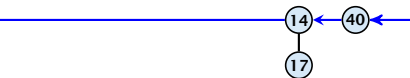
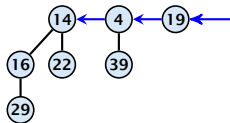
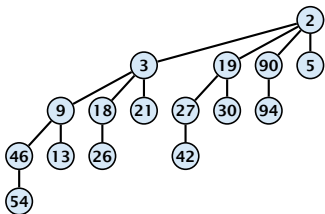


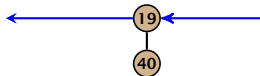
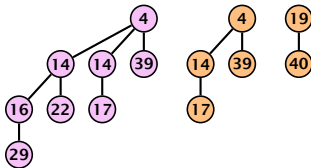
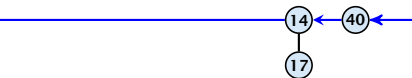
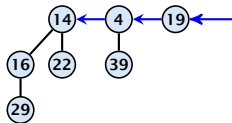
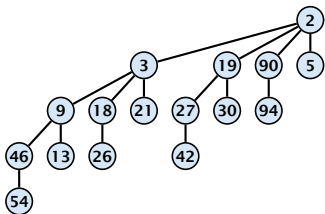




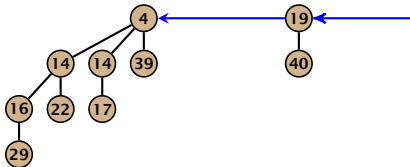
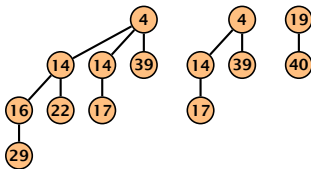
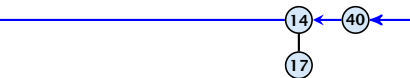
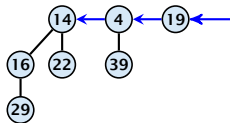
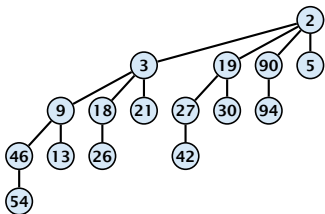




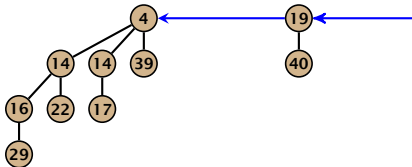
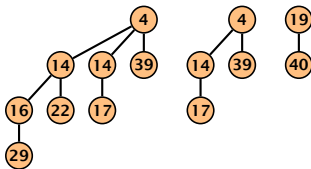
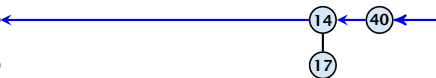
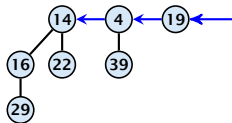
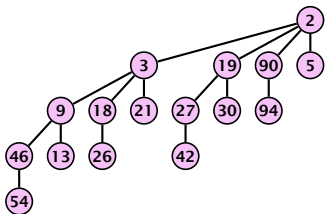


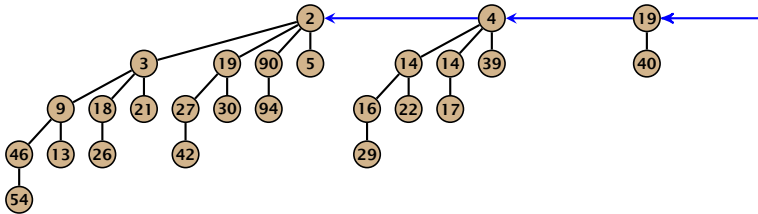
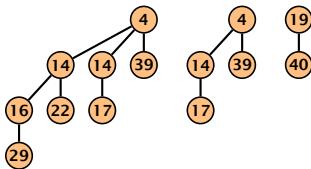
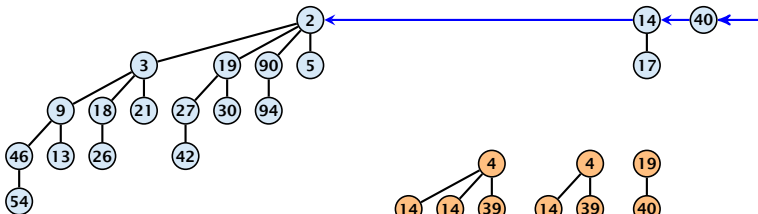


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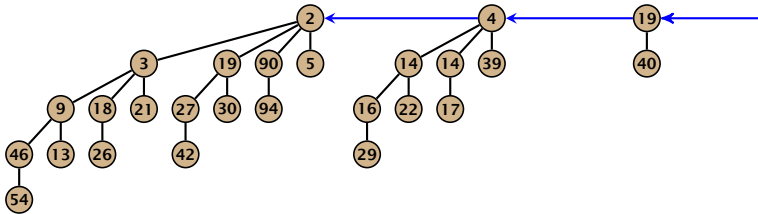
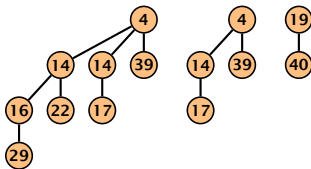
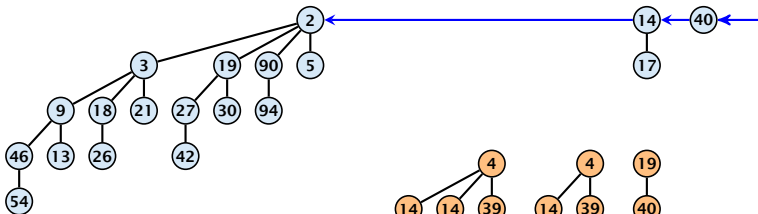


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8.2 Binomial Heaps

S_1 . merge(S_2):

- ▶ Analogous to binary addition.

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- ▶ Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

S. minimum():

- ▶ Find the minimum key-value among all roots.
- ▶ Time: $\mathcal{O}(\log n)$.

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- ▶ Decrease the key of the element pointed to by h .
- ▶ Bubble the element up in the tree until the heap property is fulfilled.
- ▶ Time: $\mathcal{O}(\log n)$ since the trees have height $\mathcal{O}(\log n)$.

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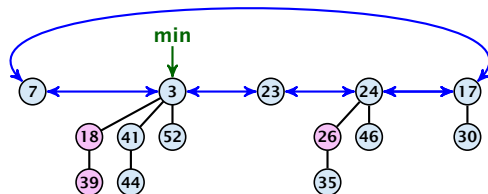
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8.3 Fibonacci Heaps

Collection of trees that fulfill the heap property.

Structure is much more relaxed than binomial heaps.



8.3 Fibonacci Heaps

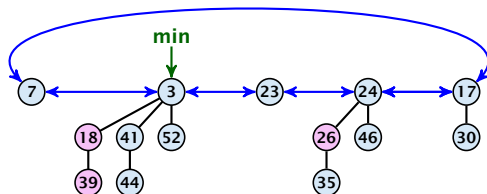
Additional implementation details:

- ▶ Every node x stores its degree in a field $x.degree$. Note that this can be updated in constant time when adding a child to x .
- ▶ Every node stores a boolean value $x.marked$ that specifies whether x is **marked** or not.

8.3 Fibonacci Heaps

The potential function:

- ▶ $t(S)$ denotes the number of trees in the heap.
- ▶ $m(S)$ denotes the number of marked nodes.
- ▶ We use the potential function $\Phi(S) = t(S) + 2m(S)$.



The potential is $\Phi(S) = 5 + 2 \cdot 3 = 11$.

8.3 Fibonacci Heaps

We assume that one unit of potential can pay for a constant amount of work, where the constant is chosen “big enough” (to take care of the constants that occur).

To make this more explicit we use c to denote the amount of work that a unit of potential can pay for.

8.3 Fibonacci Heaps

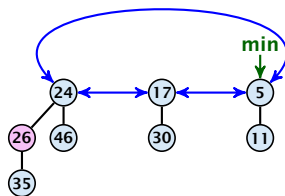
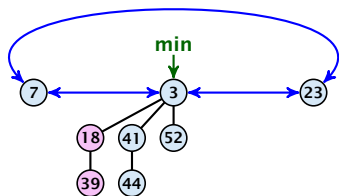
S. minimum()

- ▶ Access through the min-pointer.
- ▶ Actual cost $\mathcal{O}(1)$.
- ▶ No change in potential.
- ▶ Amortized cost $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

S . merge(S')

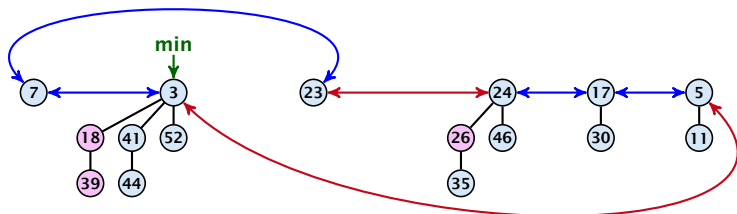
- ▶ Merge the root lists.
- ▶ Adjust the min-pointer



8.3 Fibonacci Heaps

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- ▶ Adjust the min-pointer



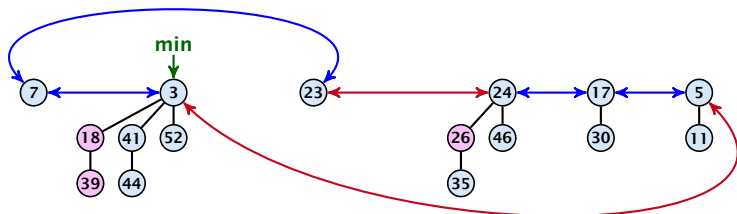
Running time:

- ▶ Actual cost $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

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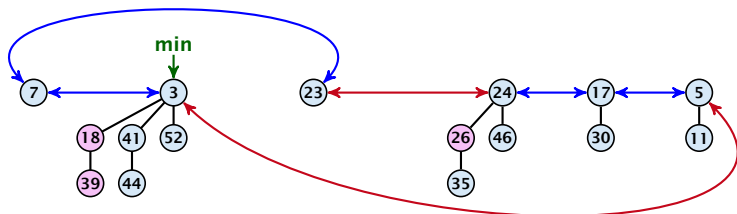
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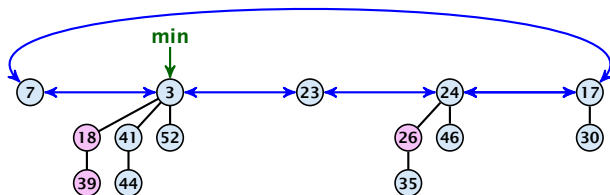
Running time:

- ▶ Actual cost $\mathcal{O}(1)$.
- ▶ No change in potential.
- ▶ Hence, amortized cost is $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

S. insert(x)

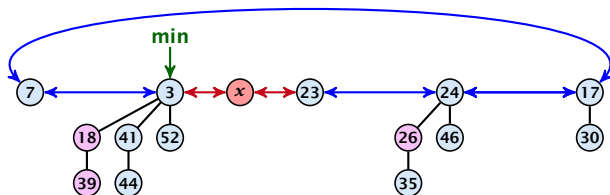
- ▶ Create a new tree containing x .
- ▶ Insert x into the root-list.
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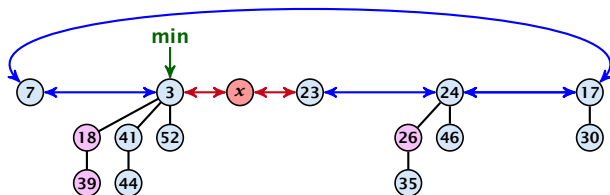
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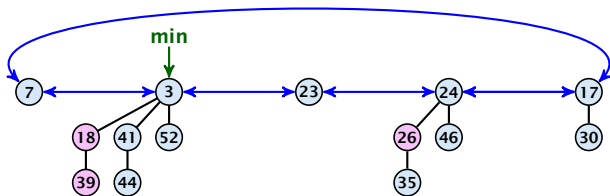


Running time:

- ▶ Actual cost $\mathcal{O}(1)$.
- ▶ Change in potential is $+1$.
- ▶ Amortized cost is $c + \mathcal{O}(1) = \mathcal{O}(1)$.

8.3 Fibonacci Heaps

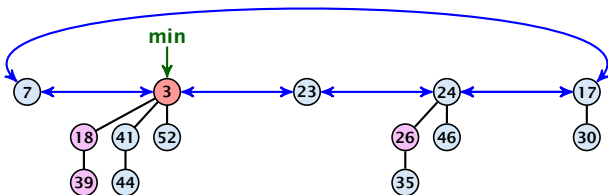
S. delete-min(x)



8.3 Fibonacci Heaps

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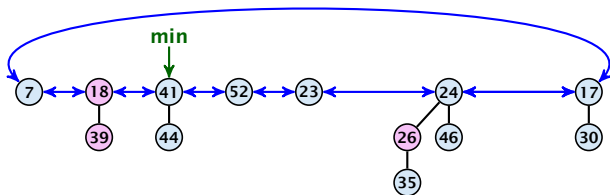
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time: $D(\min) \cdot \mathcal{O}(1)$.



8.3 Fibonacci Heaps

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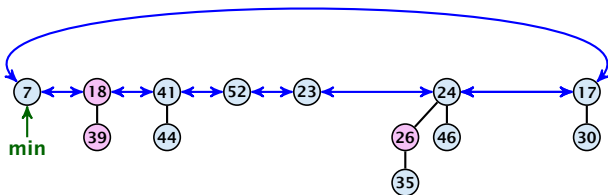
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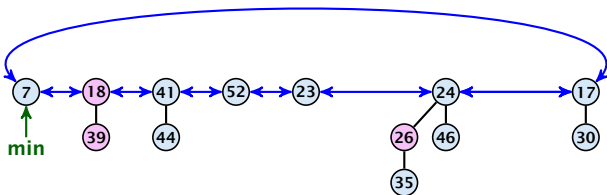
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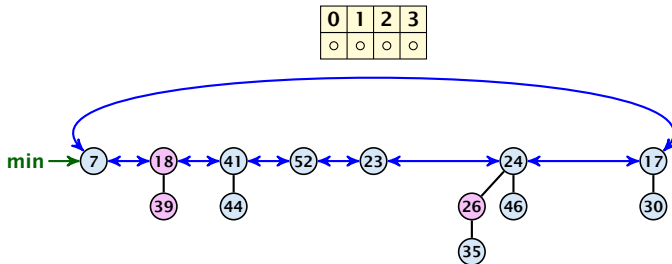
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- ▶ Consolidate root-list so that no roots have the same degree. Time $t \cdot \mathcal{O}(1)$ (see next slide).

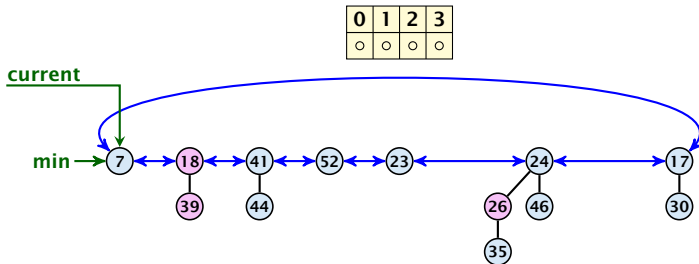
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Consolidate:



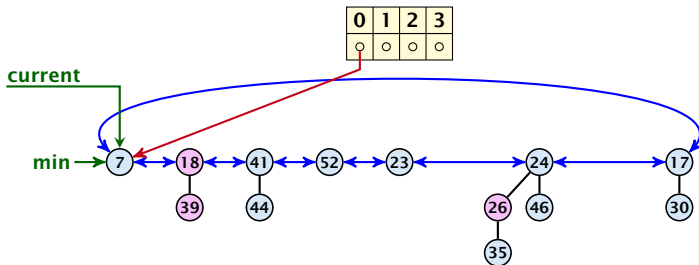
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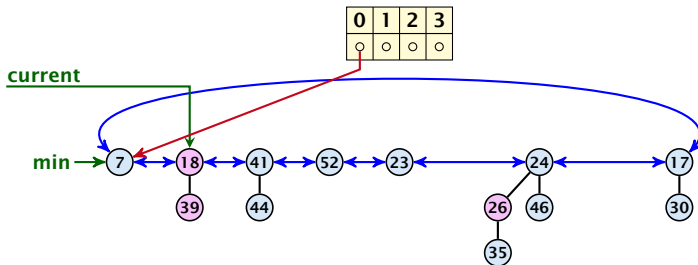
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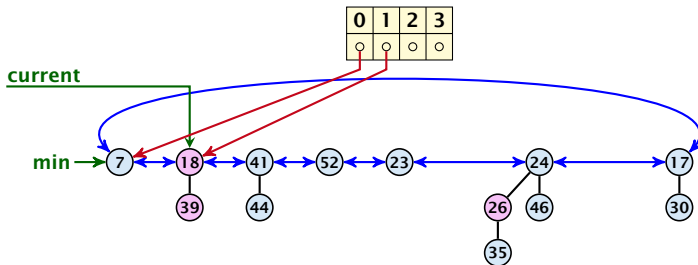
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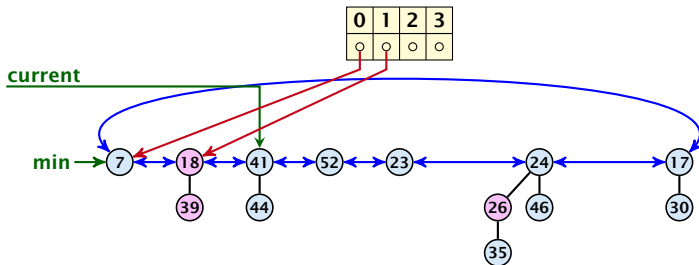
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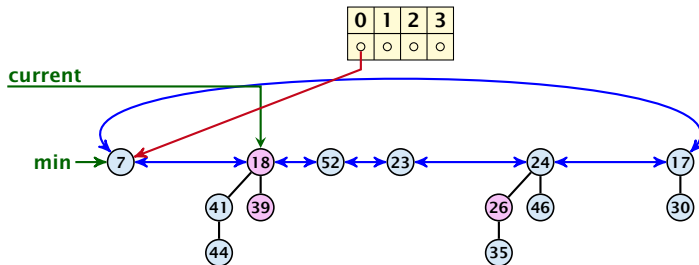
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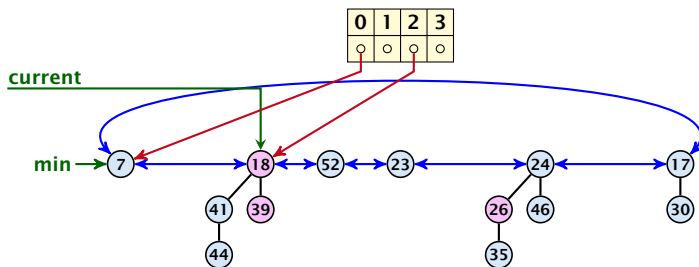
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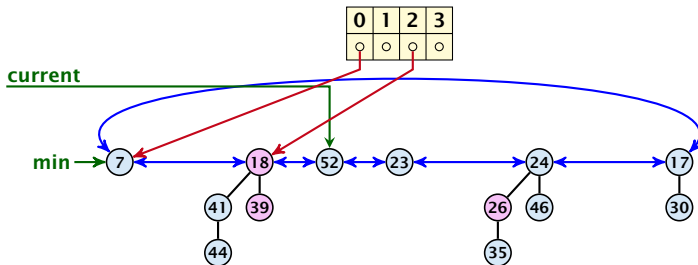
8.3 Fibonacci Heaps

Consolidate:



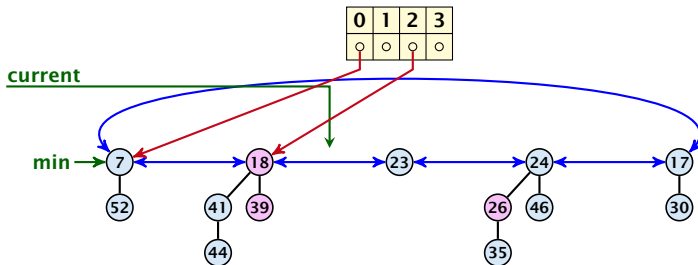
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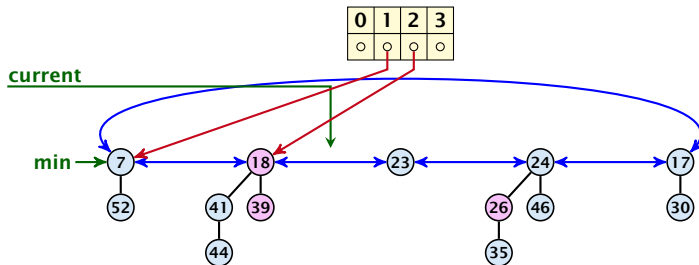
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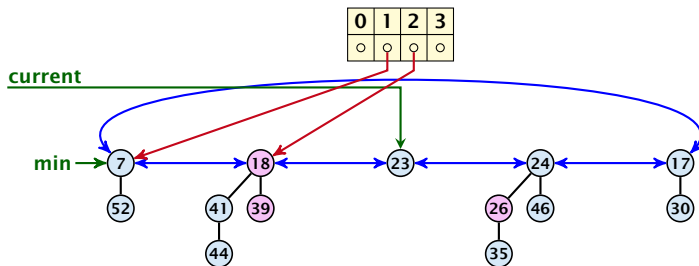
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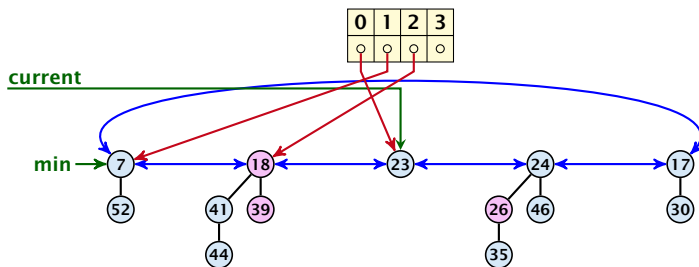
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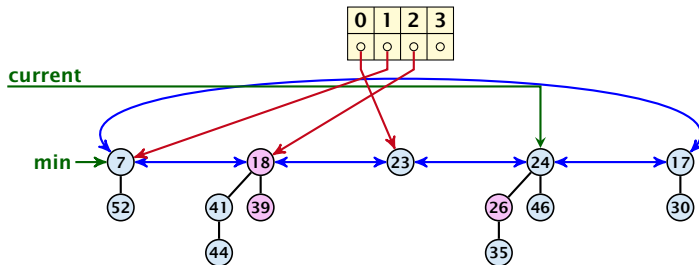
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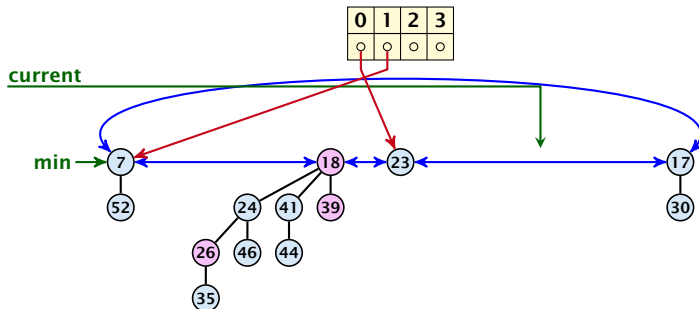
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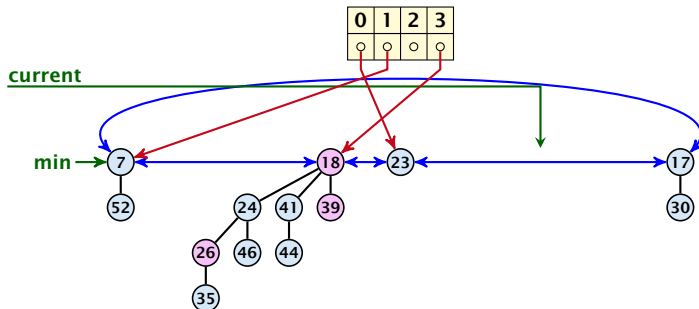
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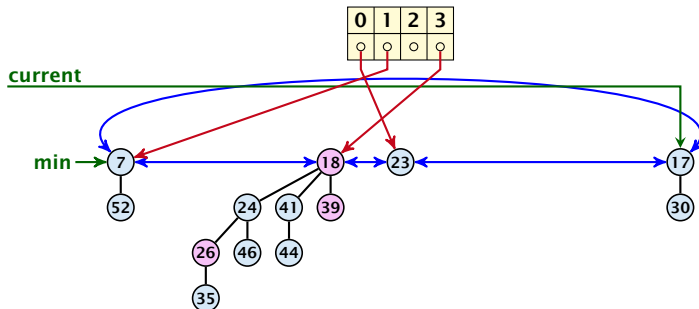
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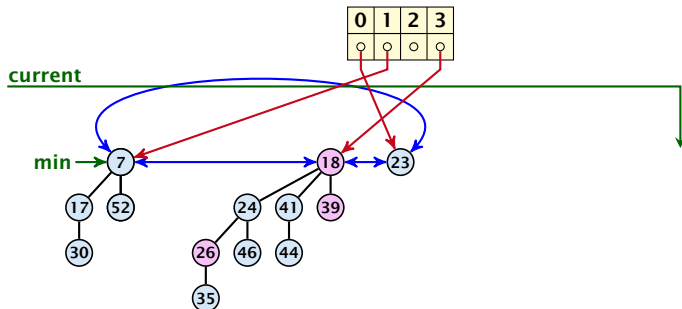
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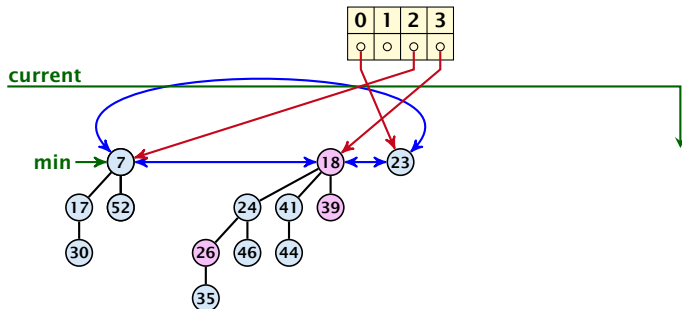
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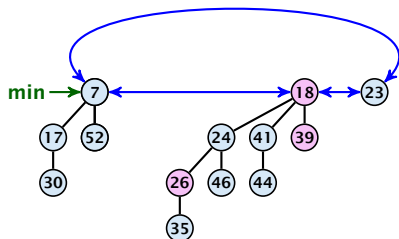
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for $c \geq c_1$.

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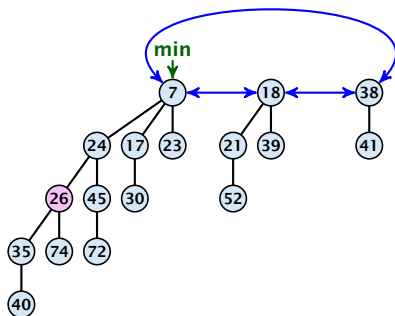
If the input trees of the consolidation procedure are binomial trees (for example only singleton vertices) then the output will be a set of distinct binomial trees, and, hence, the Fibonacci heap will be (more or less) a Binomial heap right after the consolidation.

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If we do not have delete or decrease-key operations then
 $D_n \leq \log n$.

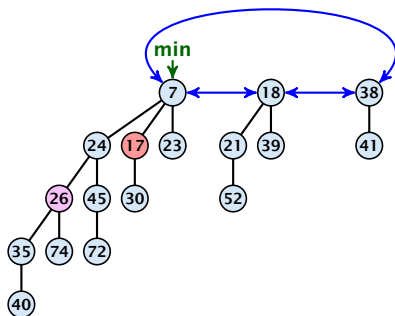
Fibonacci Heaps: decrease-key(handle h, v)



Case 1: decrease-key does not violate heap-property

- ▶ Just decrease the key-value of element referenced by h . Nothing else to do.

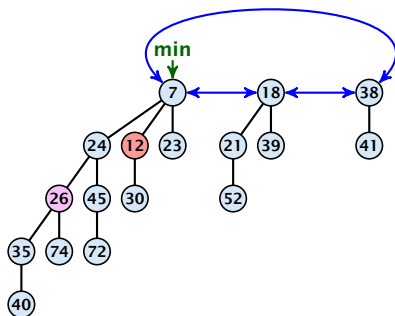
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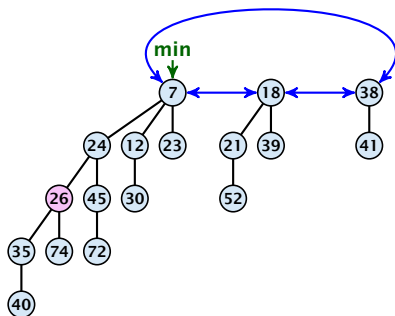
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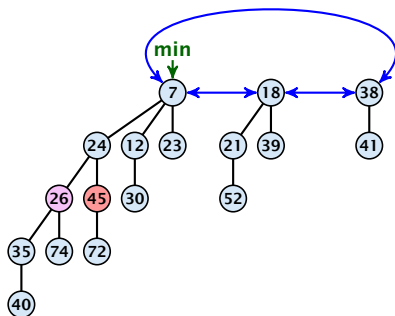
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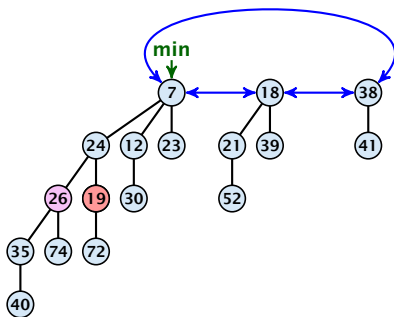
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Case 2: heap-property is violated, but parent is not marked

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- ▶ Adjust min-pointers, if necessary.
- ▶ Mark the (previous) parent of x (unless it's a root).

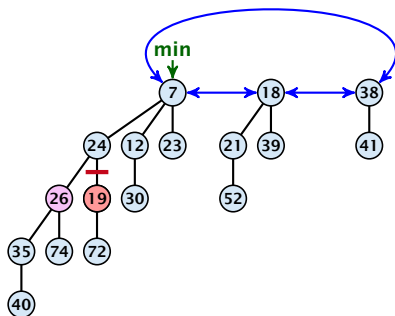
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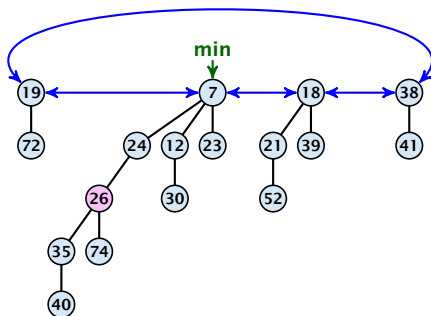
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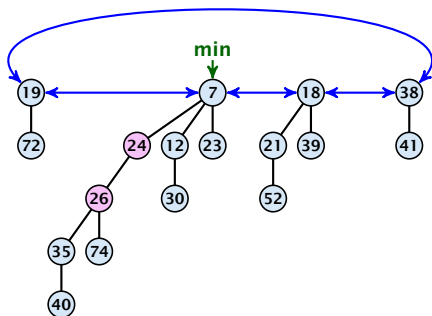
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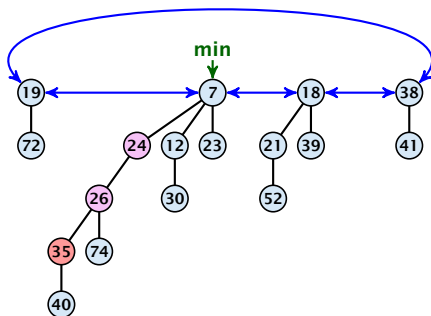
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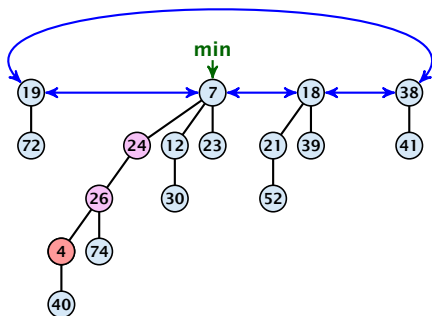
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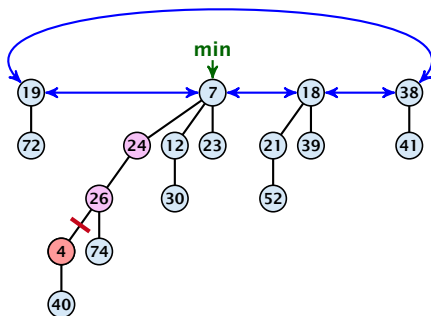
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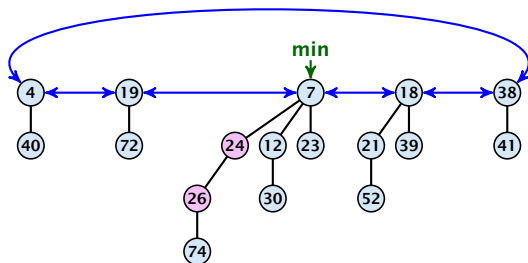
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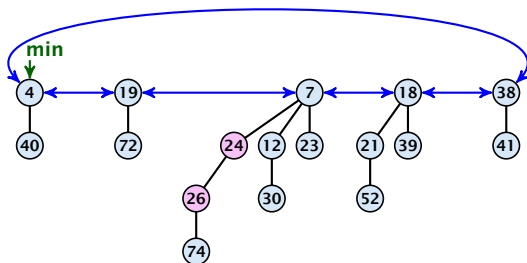
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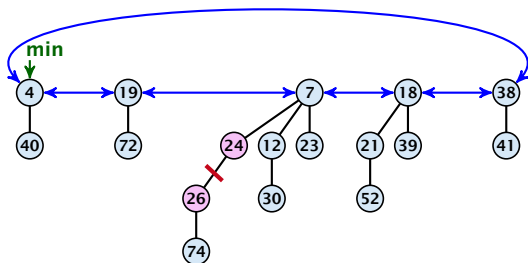
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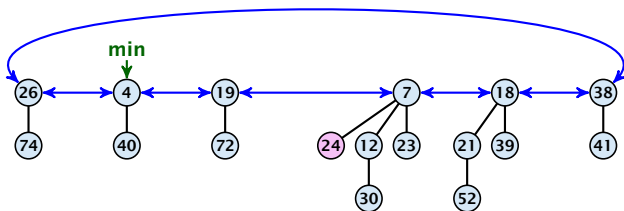
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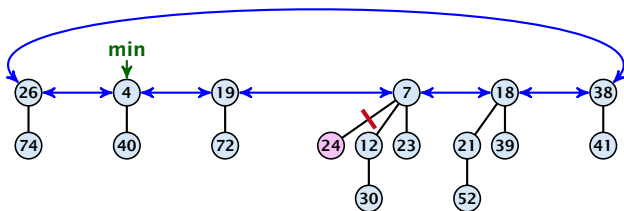
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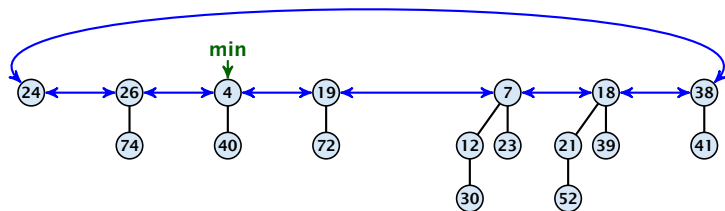
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- ▶ Adjust min-pointers, if necessary.
- ▶ Execute the following:

```
 $p \leftarrow \text{parent}[x];$   
while ( $p$  is marked)  
     $pp \leftarrow \text{parent}[p];$   
    cut of  $p$ ; make it into a root; unmark it;  
     $p \leftarrow pp;$   
if  $p$  is unmarked and not a root mark it;
```

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

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- ▶ $t' = t + \ell$, as every cut creates one new root.

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- ▶ $t' = t + \ell$, as every cut creates one new root.
- ▶ $m' \leq m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.

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$$c_2(\ell + 1) + c(4 - \ell) \leq (c_2 - c)\ell + 4c + c_2$$

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- ▶ Constant cost for decreasing the value.
- ▶ Constant cost for each of ℓ cuts.
- ▶ Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2 .

Amortized cost:

- ▶ $t' = t + \ell$, as every cut creates one new root.
- ▶ $m' \leq m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.
- ▶ $\Delta\Phi \leq \ell + 2(-\ell + 2) = 4 - \ell$
- ▶ Amortized cost is at most
$$c_2(\ell + 1) + c(4 - \ell) \leq (c_2 - c)\ell + 4c + c_2 = \mathcal{O}(1),$$
if $c \geq c_2$.

Delete node

H. delete(x):

- ▶ decrease value of x to $-\infty$.
- ▶ delete-min.

Amortized cost: $\mathcal{O}(D_n)$

- ▶ $\mathcal{O}(1)$ for decrease-key.
- ▶ $\mathcal{O}(D_n)$ for delete-min.

8.3 Fibonacci Heaps

Lemma 32

Let x be a node with degree k and let y_1, \dots, y_k denote the children of x in the order that they were linked to x . Then

$$\text{degree}(y_i) \geq \begin{cases} 0 & \text{if } i = 1 \\ i - 2 & \text{if } i > 1 \end{cases}$$

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- ▶ Since, then y_i has lost at most one child.
- ▶ Therefore, $\text{degree}(y_i) \geq i - 2$.

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$$s_k = 2 + \sum_{i=2}^k \text{size}(y_i)$$

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$$\begin{aligned} s_k &= 2 + \sum_{i=2}^k \text{size}(y_i) \\ &\geq 2 + \sum_{i=2}^k s_{i-2} \\ &= 2 + \sum_{i=0}^{k-2} s_i \end{aligned}$$

8.3 Fibonacci Heaps

Definition 33

Consider the following non-standard Fibonacci type sequence:

$$F_k = \begin{cases} 1 & \text{if } k = 0 \\ 2 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

Facts:

1. $F_k \geq \phi^k$.
2. For $k \geq 2$: $F_k = 2 + \sum_{i=0}^{k-2} F_i$.

The above facts can be easily proved by induction. From this it follows that $s_k \geq F_k \geq \phi^k$, which gives that the maximum degree in a Fibonacci heap is logarithmic.

$$k=0: \quad 1 = F_0 \geq \Phi^0 = 1$$

$$k=1: \quad 2 = F_1 \geq \Phi^1 \approx 1.61$$

$$k-2, k-1 \rightarrow k: \quad F_k = F_{k-1} + F_{k-2} \geq \Phi^{k-1} + \Phi^{k-2} = \Phi^{k-2} \underbrace{(\Phi + 1)}_{\Phi^2} = \Phi^k$$

$$k=2: \quad 3 = F_2 = 2 + 1 = 2 + F_0$$

$$k-1 \rightarrow k: \quad F_k = F_{k-1} + F_{k-2} = 2 + \sum_{i=0}^{k-3} F_i + F_{k-2} = 2 + \sum_{i=0}^{k-2} F_i$$

9 Union Find

Union Find Data Structure \mathcal{P} : Maintains a partition of **disjoint** sets over elements.

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- ▶ **\mathcal{P} . find(x):** Given a handle for an element x ; find the set that contains x . Returns a representative/identifier for this set.
- ▶ **\mathcal{P} . union(x, y):** Given two elements x , and y that are currently in sets S_x and S_y , respectively, the function replaces S_x and S_y by $S_x \cup S_y$ and returns an identifier for the new set.

9 Union Find

Applications:

- ▶ Keep track of the connected components of a dynamic graph that changes due to insertion of nodes and edges.

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- ▶ Keep track of the connected components of a dynamic graph that changes due to insertion of nodes and edges.
- ▶ Kruskals Minimum Spanning Tree Algorithm

9 Union Find

Algorithm 1 Kruskal-MST($G = (V, E), w$)

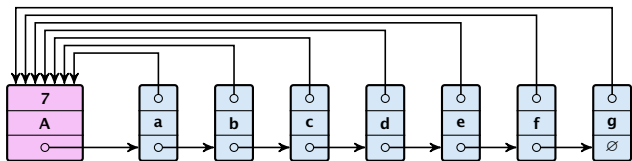
```
1:  $A \leftarrow \emptyset$ ;  
2: for all  $v \in V$  do  
3:    $v.\text{set} \leftarrow \mathcal{P}.\text{makeset}(v.\text{label})$   
4: sort edges in non-decreasing order of weight  $w$   
5: for all  $(u, v) \in E$  in non-decreasing order do  
6:   if  $\mathcal{P}.\text{find}(u.\text{set}) \neq \mathcal{P}.\text{find}(v.\text{set})$  then  
7:      $A \leftarrow A \cup \{(u, v)\}$   
8:      $\mathcal{P}.\text{union}(u.\text{set}, v.\text{set})$ 
```

List Implementation

- ▶ The elements of a set are stored in a list; each node has a backward pointer to the head.

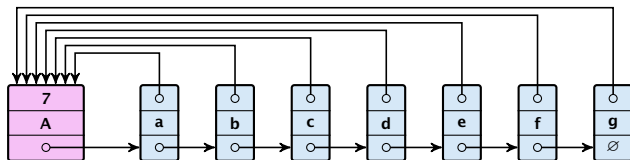
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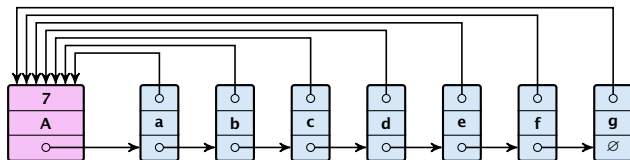
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- ▶ **makeset**(x) can be performed in constant time.
- ▶ **find**(x) can be performed in constant time.

List Implementation

union(x, y)

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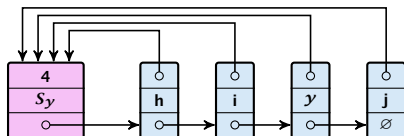
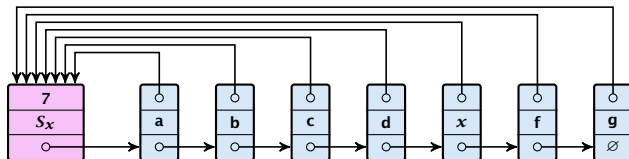
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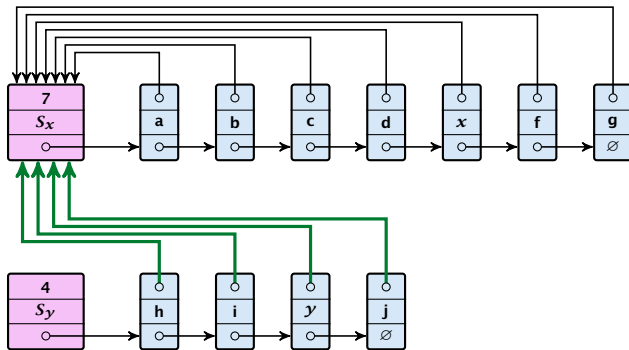
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- ▶ Time: $\min\{|S_x|, |S_y|\}$.

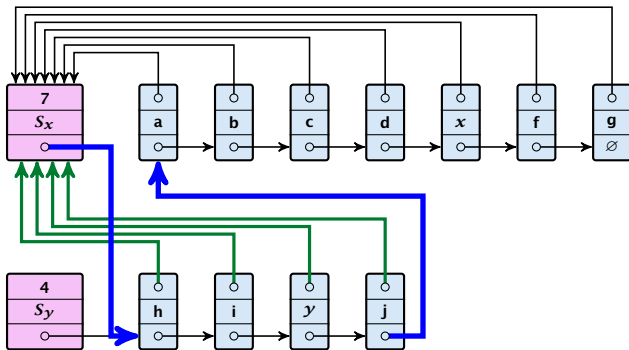
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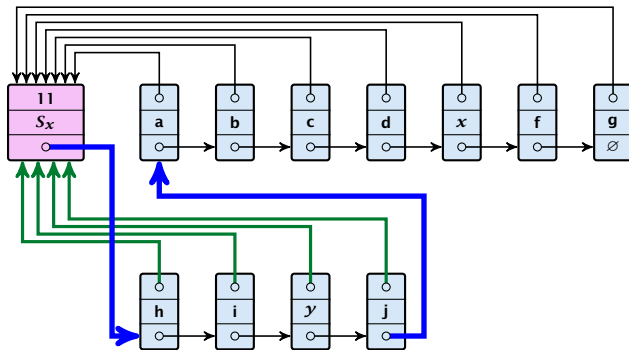
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List Implementation



List Implementation

Running times:

- ▶ $\text{find}(x)$: constant
- ▶ $\text{makeset}(x)$: constant
- ▶ $\text{union}(x, y)$: $\mathcal{O}(n)$, where n denotes the number of elements contained in the set system.

List Implementation

Lemma 34

The list implementation for the ADT union find fulfills the following amortized time bounds:

- ▶ $\text{find}(x): \mathcal{O}(1)$.
- ▶ $\text{makeset}(x): \mathcal{O}(\log n)$.
- ▶ $\text{union}(x, y): \mathcal{O}(1)$.

The Accounting Method for Amortized Time Bounds

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- ▶ If we can find a charging scheme that guarantees that balances always stay positive the amortized time bounds are proven.

List Implementation

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- ▶ Later operations charge the account but the balance never drops below zero.

List Implementation

makeset(x): The actual cost is $\mathcal{O}(1)$. Due to the cost inflation the amortized cost is $\mathcal{O}(\log n)$.

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- ▶ Charge c to every element in set S_x .

List Implementation

Lemma 35

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Proof.

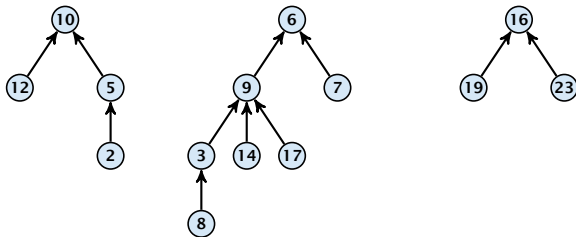
Whenever an element x is charged the number of elements in x 's set doubles. This can happen at most $\lceil \log n \rceil$ times. \square

Implementation via Trees

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- ▶ Example:



Set system $\{2, 5, 10, 12\}$, $\{3, 6, 7, 8, 9, 14, 17\}$, $\{16, 19, 23\}$.

Implementation via Trees

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- ▶ Time: $\mathcal{O}(\text{level}(x))$, where $\text{level}(x)$ is the distance of element x to the root in its tree. **Not constant.**

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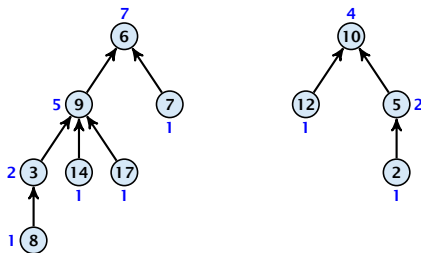
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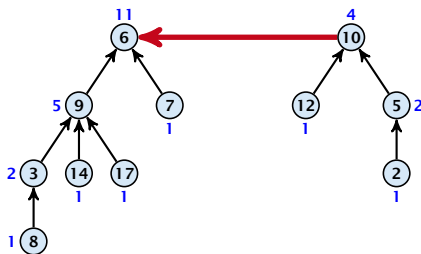


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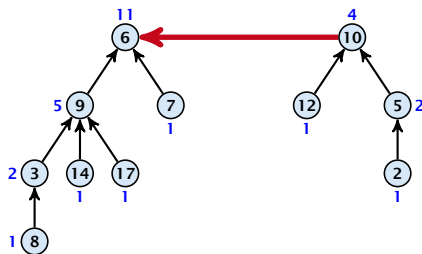


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- ▶ Time: constant for $\text{link}(a, b)$ plus two find-operations.

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find(x):

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- ▶ Re-attach all visited nodes as children of the root.

Path Compression

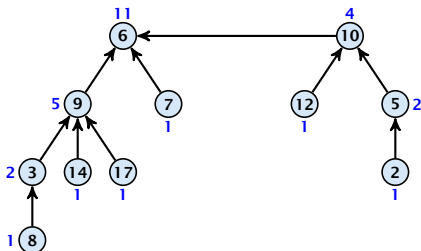
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- ▶ Re-attach all visited nodes as children of the root.
- ▶ Speeds up successive find-operations.

Path Compression

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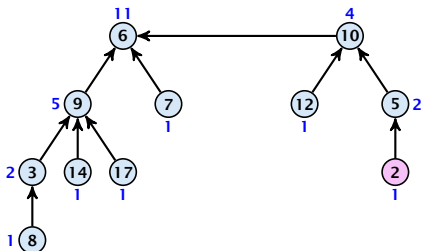
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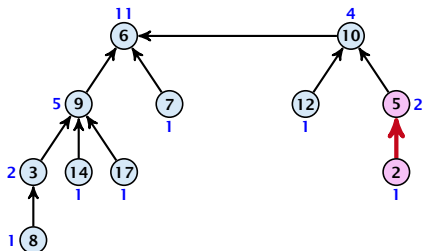
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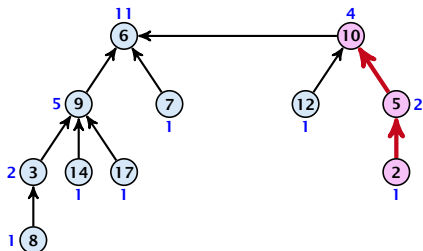
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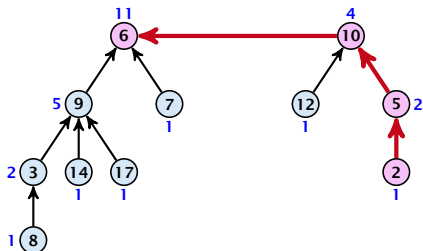
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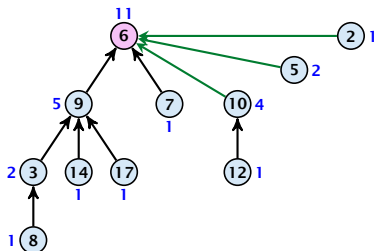
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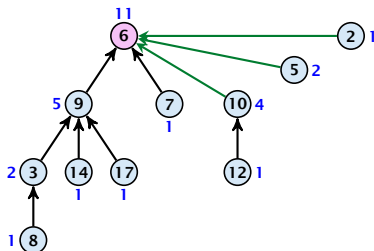
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- ▶ Note that the size-fields now only give an upper bound on the size of a sub-tree.

Path Compression

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However, for a worst-case analysis there is no improvement on the running time. It can still happen that a find-operation takes time $\mathcal{O}(\log n)$.

Amortized Analysis

Definitions:

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- ▶ $\text{size}(v)$:= the number of nodes that were in the sub-tree rooted at v when v became the child of another node (or the number of nodes if v is the root).

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- ▶ $\text{rank}(v) := \lfloor \log(\text{size}(v)) \rfloor$.
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Lemma 37

The rank of a parent must be strictly larger than the rank of a child.

Amortized Analysis

Lemma 38

There are at most $n/2^s$ nodes of rank s .

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- ▶ Let's say a node v sees node x if v is in x 's sub-tree at the time that x becomes a child.



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- ▶ Let's say a node v sees node x if v is in x 's sub-tree at the time that x becomes a child.
- ▶ A node v sees at most one node of rank s during the running time of the algorithm.



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- ▶ This holds because the rank-sequence of the roots of the different trees that contain v during the running time of the algorithm is a strictly increasing sequence.



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- ▶ A node v sees at most one node of rank s during the running time of the algorithm.
- ▶ This holds because the rank-sequence of the roots of the different trees that contain v during the running time of the algorithm is a strictly increasing sequence.
- ▶ Hence, every node sees at most one rank s node, but every rank s node is seen by at least 2^s different nodes. □

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Theorem 39

Union find with path compression fulfills the following amortized running times:

- ▶ $\text{makeset}(x) : \mathcal{O}(\log^*(n))$
- ▶ $\text{find}(x) : \mathcal{O}(\log^*(n))$
- ▶ $\text{union}(x, y) : \mathcal{O}(\log^*(n))$

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- ▶ The maximum non-empty rank group is $\log^*(\lfloor \log n \rfloor) \leq \log^*(n) - 1$ (which holds for $n \geq 2$).
- ▶ Hence, the total number of rank-groups is at most $\log^* n$.

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- ▶ If the group-number of $\text{rank}(v)$ is the same as that of $\text{rank}(\text{parent}[v])$ (before starting path compression) we charge the cost to the node-account of v .

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- ▶ If the group-number of $\text{rank}(v)$ is the same as that of $\text{rank}(\text{parent}[v])$ (before starting path compression) we charge the cost to the node-account of v .
- ▶ Otherwise we charge the cost to the find-account.

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- ▶ After some charges to v the parent will be in a larger rank-group. $\Rightarrow v$ will **never** be charged again.
- ▶ The total charge made to a node in rank-group g is at most $\text{tow}(g) - \text{tow}(g - 1) - 1 \leq \text{tow}(g)$.

Amortized Analysis

What is the total charge made to nodes?

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- ▶ The total charge is at most

$$\sum_g n(g) \cdot \text{tow}(g) ,$$

where $n(g)$ is the number of nodes in group g .

Amortized Analysis

For $g \geq 1$ we have

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Hence,

$$\sum_g n(g) \text{tow}(g) \leq n(0) \text{tow}(0) + \sum_{g \geq 1} n(g) \text{tow}(g) \leq n \log^*(n)$$

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Without loss of generality we can assume that all **makeset**-operations occur at the start.

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This means if we inflate the cost of **makeset** to $\log^* n$ and add this to the node account of v then the balances of all node accounts will sum up to a positive value (this is sufficient to obtain an amortized bound).

Amortized Analysis

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The analysis is not tight. In fact it has been shown that the amortized time for the union-find data structure with path compression is $\mathcal{O}(\alpha(m, n))$, where $\alpha(m, n)$ is the inverse Ackermann function which grows a lot lot slower than $\log^* n$. (Here, we consider the average running time of m operations on at most n elements).

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There is also a lower bound of $\Omega(\alpha(m, n))$.

Amortized Analysis

$$A(x, y) = \begin{cases} y + 1 & \text{if } x = 0 \\ A(x - 1, 1) & \text{if } y = 0 \\ A(x - 1, A(x, y - 1)) & \text{otw.} \end{cases}$$

$$\alpha(m, n) = \min\{i \geq 1 : A(i, \lfloor m/n \rfloor) \geq \log n\}$$

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- ▶ $A(0, y) = y + 1$
- ▶ $A(1, y) = y + 2$
- ▶ $A(2, y) = 2y + 3$
- ▶ $A(3, y) = 2^{y+3} - 3$
- ▶ $A(4, y) = \underbrace{2^{2^{2^2}}}_{y+3 \text{ times}} - 3$

Part IV

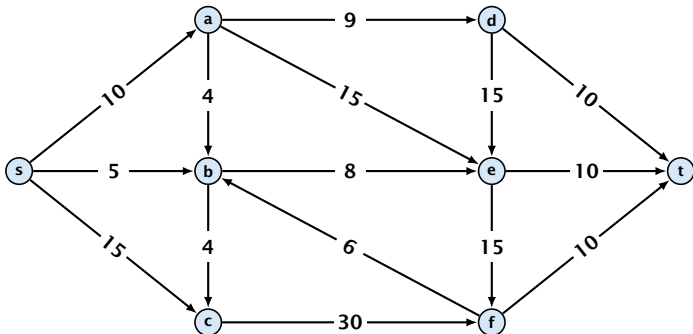
Flows and Cuts

The following slides are partially based on slides by Kevin Wayne.

10 Introduction

Flow Network

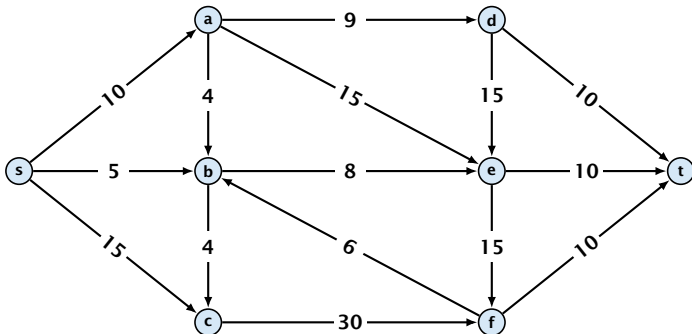
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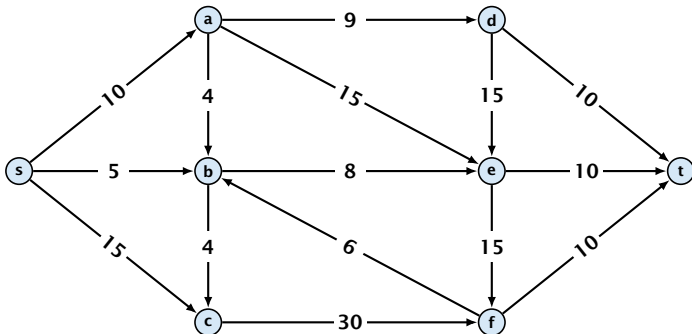
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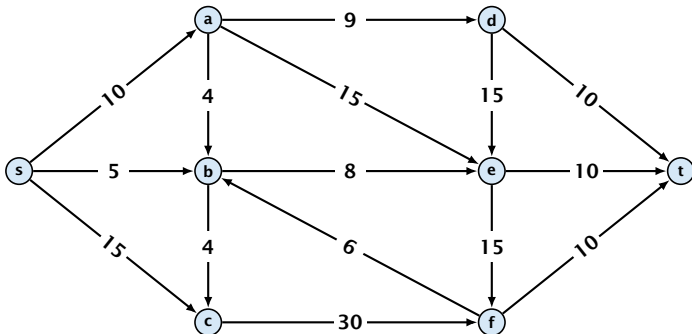
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10 Introduction

Flow Network

- ▶ directed graph $G = (V, E)$; edge capacities $c(e)$
- ▶ two special nodes: source s ; target t ;
- ▶ no edges entering s or leaving t ;
- ▶ at least for now: no parallel edges;



Cuts

Definition 40

An (s, t) -cut in the graph G is given by a set $A \subset V$ with $s \in A$ and $t \in V \setminus A$.

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The **capacity** of a cut A is defined as

$$\text{cap}(A, V \setminus A) := \sum_{e \in \text{out}(A)} c(e) ,$$

where $\text{out}(A)$ denotes the set of edges of the form $A \times V \setminus A$ (i.e. edges leaving A).

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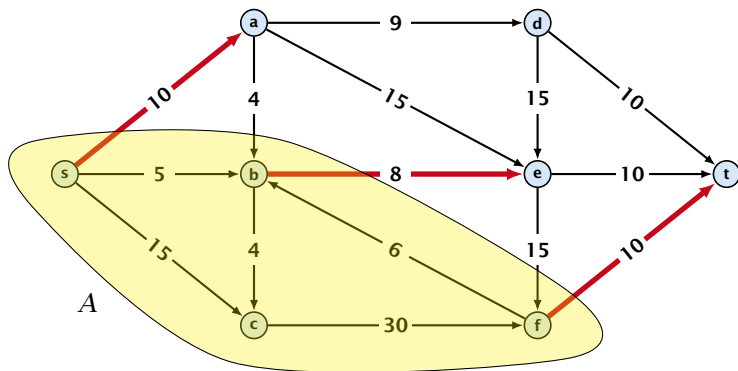
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Minimum Cut Problem: Find an (s, t) -cut with minimum capacity.

Cuts

Example 42



The capacity of the cut is $\text{cap}(A, V \setminus A) = 28$.

Definition 43

An (s, t) -flow is a function $f : E \mapsto \mathbb{R}^+$ that satisfies

1. For each edge e

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(capacity constraints)

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2. For each $v \in V \setminus \{s, t\}$

$$\sum_{e \in \text{out}(v)} f(e) = \sum_{e \in \text{into}(v)} f(e) .$$

(flow conservation constraints)

Definition 44

The value of an (s, t) -flow f is defined as

$$\text{val}(f) = \sum_{e \in \text{out}(s)} f(e) .$$

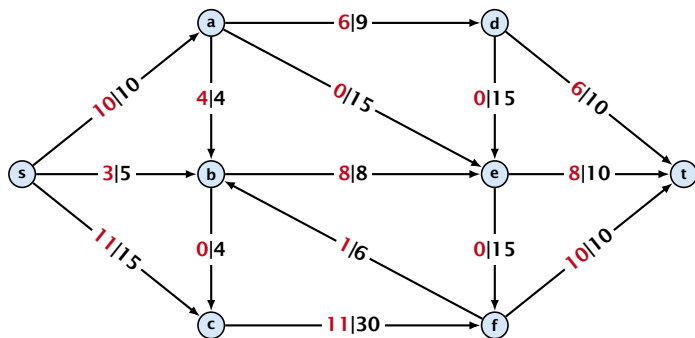
Definition 44

The **value of an (s, t) -flow f** is defined as

$$\text{val}(f) = \sum_{e \in \text{out}(s)} f(e) .$$

Maximum Flow Problem: Find an (s, t) -flow with maximum value.

Example 45



The value of the flow is $\text{val}(f) = 24$.

Lemma 46 (Flow value lemma)

Let f be a flow, and let $A \subseteq V$ be an (s, t) -cut. Then the *net-flow* across the cut is equal to the amount of flow leaving s , i.e.,

$$\text{val}(f) = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e) .$$

Proof.

$\text{val}(f)$

Proof.

$$\text{val}(f) = \sum_{e \in \text{out}(s)} f(e)$$

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$$\begin{aligned}\text{val}(f) &= \sum_{e \in \text{out}(s)} f(e) \\ &= \sum_{e \in \text{out}(s)} f(e) + \sum_{v \in A \setminus \{s\}} \left(\sum_{e \in \text{out}(v)} f(e) - \sum_{e \in \text{in}(v)} f(e) \right)\end{aligned}$$

Proof.

$$\begin{aligned}\text{val}(f) &= \sum_{e \in \text{out}(s)} f(e) && = 0 \\ &= \sum_{e \in \text{out}(s)} f(e) + \sum_{v \in A \setminus \{s\}} \left(\sum_{e \in \text{out}(v)} f(e) - \sum_{e \in \text{in}(v)} f(e) \right)\end{aligned}$$

Proof.

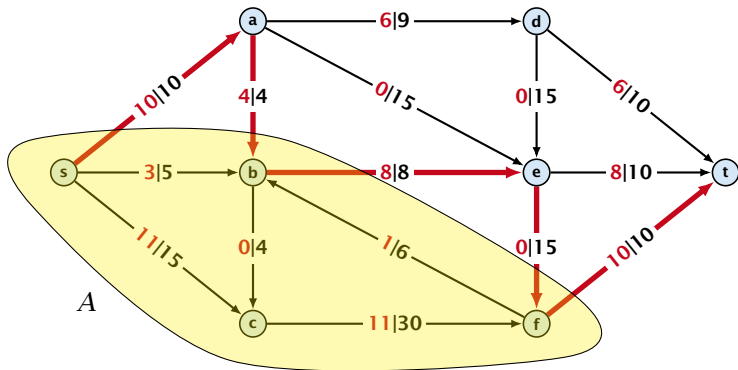
$$\begin{aligned}\text{val}(f) &= \sum_{e \in \text{out}(s)} f(e) \\ &= \sum_{e \in \text{out}(s)} f(e) + \sum_{v \in A \setminus \{s\}} \left(\sum_{e \in \text{out}(v)} f(e) - \sum_{e \in \text{in}(v)} f(e) \right) \\ &= \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e)\end{aligned}$$

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The last equality holds since every edge with both end-points in A contributes negatively as well as positively to the sum in Line 2. The only edges whose contribution doesn't cancel out are edges leaving or entering A . \square

Example 47



The net-flow across the cut is $\text{val}(f) = 24$.

Corollary 48

Let f be an (s, t) -flow and let A be an (s, t) -cut, such that

$$\text{val}(f) = \text{cap}(A, V \setminus A).$$

Then f is a maximum flow.

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Let f be an (s, t) -flow and let A be an (s, t) -cut, such that

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Suppose that there is a flow f' with larger value. Then

$$\begin{aligned} \text{cap}(A, V \setminus A) &< \text{val}(f') \\ &= \sum_{e \in \text{out}(A)} f'(e) - \sum_{e \in \text{into}(A)} f'(e) \end{aligned}$$



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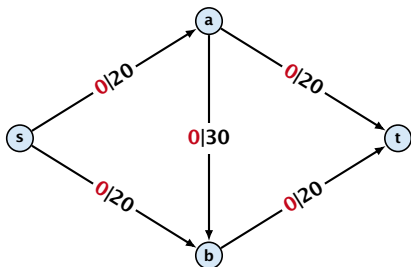
$$\begin{aligned} \text{cap}(A, V \setminus A) &< \text{val}(f') \\ &= \sum_{e \in \text{out}(A)} f'(e) - \sum_{e \in \text{into}(A)} f'(e) \\ &\leq \sum_{e \in \text{out}(A)} f'(e) \\ &\leq \text{cap}(A, V \setminus A) \end{aligned}$$

□

11 Augmenting Path Algorithms

Greedy-algorithm:

- ▶ start with $f(e) = 0$ everywhere
- ▶ find an s - t path with $f(e) < c(e)$ on every edge
- ▶ augment flow along the path
- ▶ repeat as long as possible

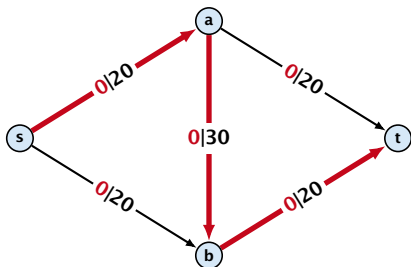


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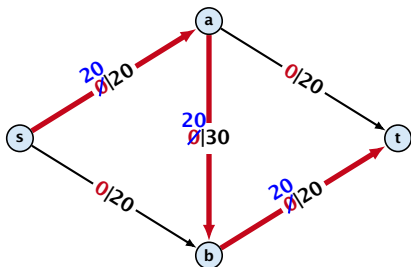


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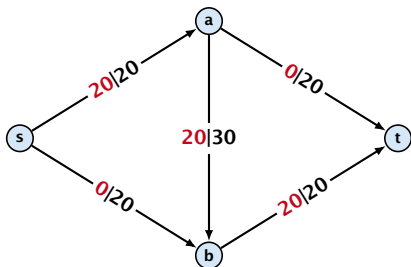


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flow value: 20

The Residual Graph

From the graph $G = (V, E, c)$ and the current flow f we construct an auxiliary graph $G_f = (V, E_f, c_f)$ (the residual graph):

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The Residual Graph

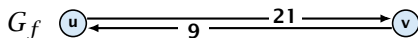
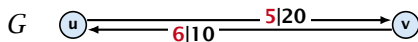
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Augmenting Path Algorithm

Definition 49

An **augmenting path** with respect to flow f , is a path from s to t in the auxiliary graph G_f that contains only edges with non-zero capacity.

Augmenting Path Algorithm

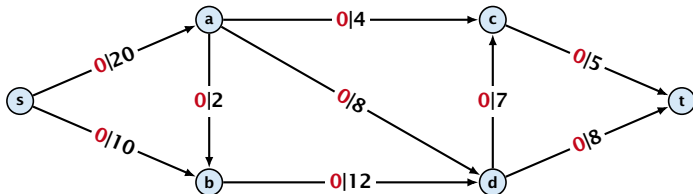
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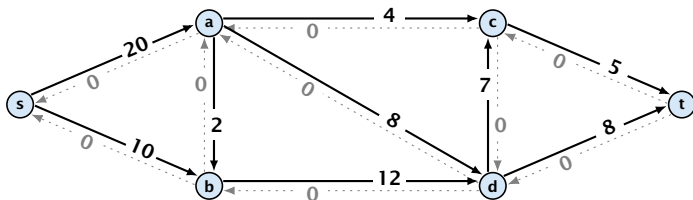
Algorithm 1 FordFulkerson($G = (V, E, c)$)

- 1: Initialize $f(e) \leftarrow 0$ for all edges.
- 2: **while** \exists augmenting path p in G_f **do**
- 3: augment as much flow along p as possible.

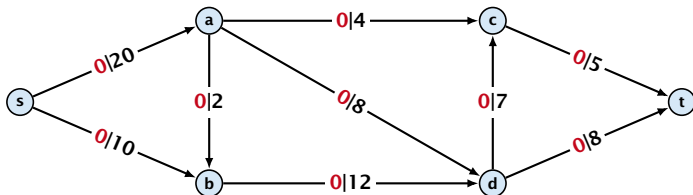
Augmenting Paths



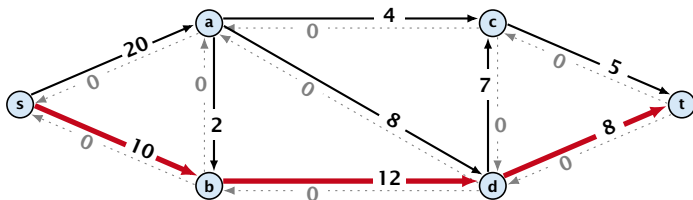
flow value: 0



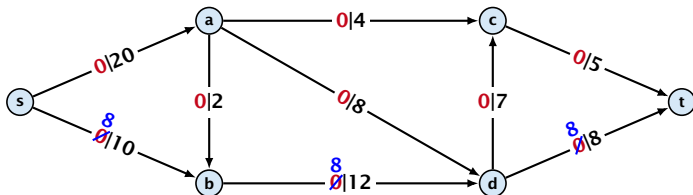
Augmenting Paths



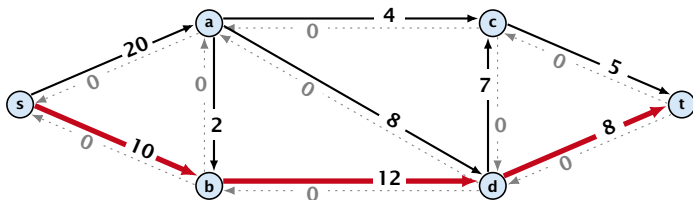
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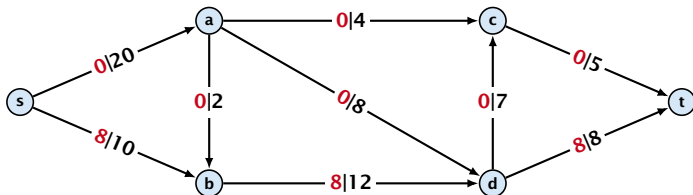
Augmenting Paths



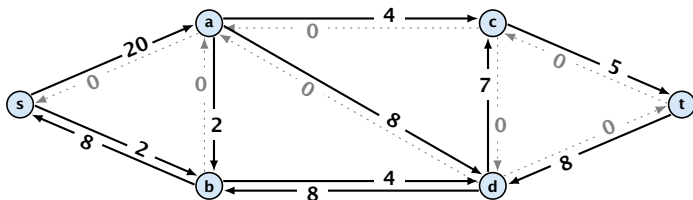
flow value: 0



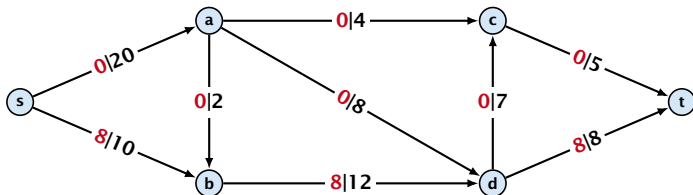
Augmenting Paths



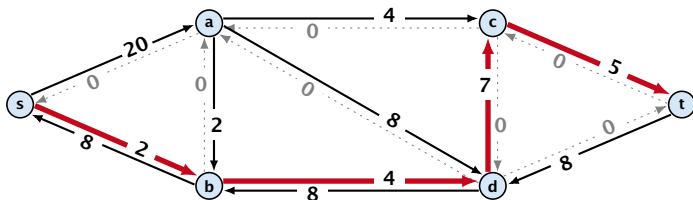
flow value: 8



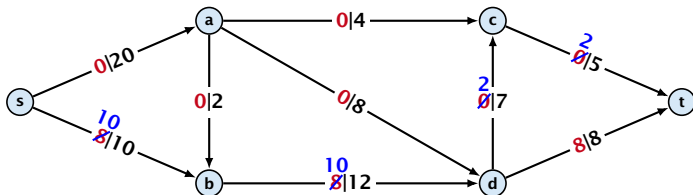
Augmenting Paths



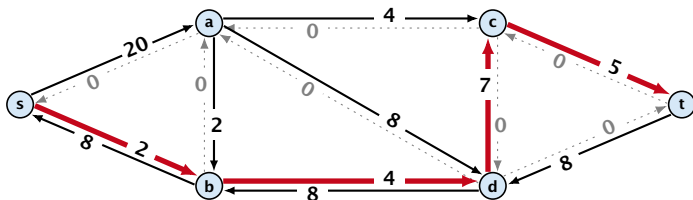
flow value: 8



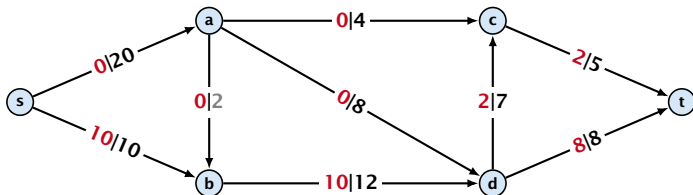
Augmenting Paths



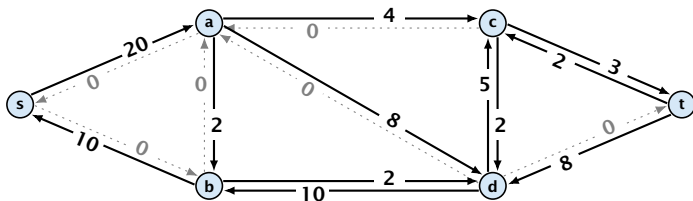
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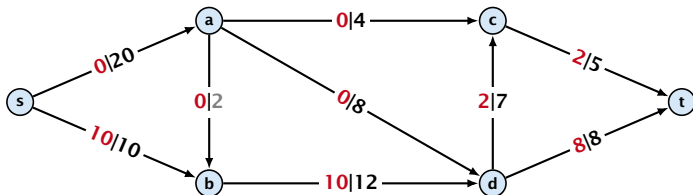
Augmenting Paths



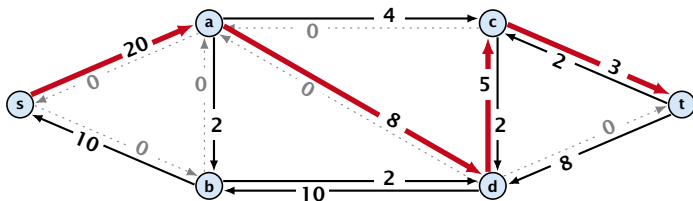
flow value: 10



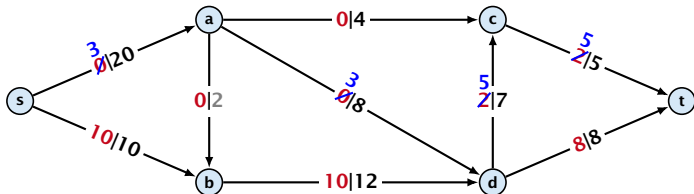
Augmenting Paths



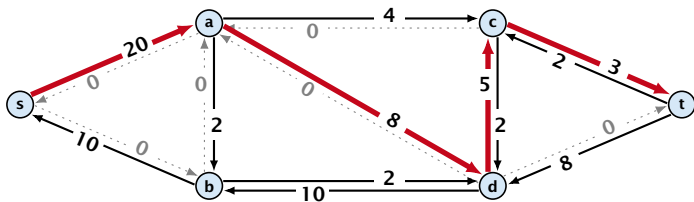
flow value: 10



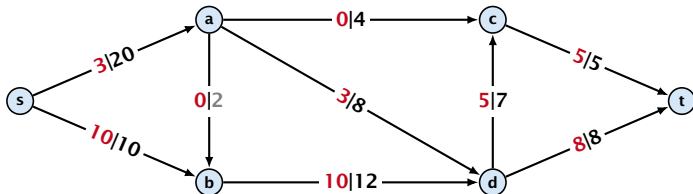
Augmenting Paths



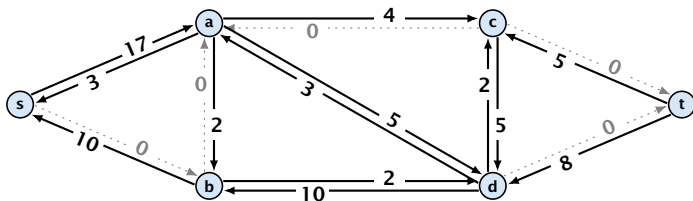
flow value: 10



Augmenting Paths



flow value: 13



Augmenting Path Algorithm

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Theorem 50

A flow f is a maximum flow **iff** there are no augmenting paths.

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The value of a maximum flow is equal to the value of a minimum cut.

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Proof.

Let f be a flow. The following are equivalent:

1. There exists a cut A such that $\text{val}(f) = \text{cap}(A, V \setminus A)$.



Augmenting Path Algorithm

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2. Flow f is a maximum flow.



Augmenting Path Algorithm

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A flow f is a maximum flow **iff** there are no augmenting paths.

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Proof.

Let f be a flow. The following are equivalent:

1. There exists a cut A such that $\text{val}(f) = \text{cap}(A, V \setminus A)$.
2. Flow f is a maximum flow.
3. There is no augmenting path w.r.t. f .



Augmenting Path Algorithm

Augmenting Path Algorithm

1. \Rightarrow 2.

This we already showed.

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If there were an augmenting path, we could improve the flow.

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1. \Rightarrow 2.

This we already showed.

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If there were an augmenting path, we could improve the flow.

Contradiction.

3. \Rightarrow 1.

- ▶ Let f be a flow with no augmenting paths.
- ▶ Let A be the set of vertices reachable from s in the residual graph along non-zero capacity edges.
- ▶ Since there is no augmenting path we have $s \in A$ and $t \notin A$.

Augmenting Path Algorithm

$\text{val}(f)$

Augmenting Path Algorithm

$$\text{val}(f) = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e)$$

Augmenting Path Algorithm

$$\begin{aligned}\text{val}(f) &= \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e) \\ &= \sum_{e \in \text{out}(A)} c(e)\end{aligned}$$

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This finishes the proof.

Here the first equality uses the flow value lemma, and the second exploits the fact that the flow along incoming edges must be 0 as the residual graph does not have edges leaving A .

Assumption:

All capacities are integers between 1 and C .

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Invariant:

Every flow value $f(e)$ and every residual capacity $c_f(e)$ remains integral throughout the algorithm.

Lemma 52

The algorithm terminates in at most $\text{val}(f^*) \leq nC$ iterations, where f^* denotes the maximum flow. Each iteration can be implemented in time $\mathcal{O}(m)$. This gives a total running time of $\mathcal{O}(nmC)$.

Lemma 52

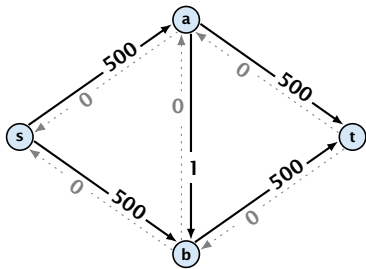
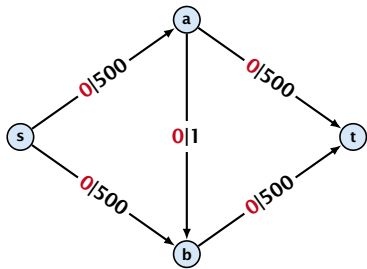
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Theorem 53

If all capacities are integers, then there exists a maximum flow for which every flow value $f(e)$ is integral.

A Bad Input

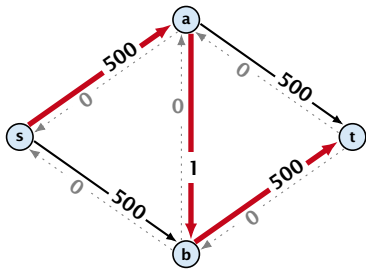
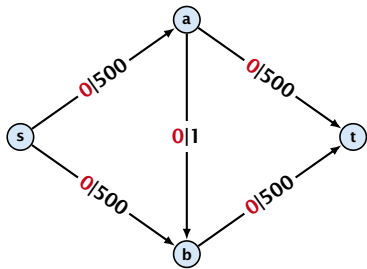
Problem: The running time may not be polynomial



flow value: 0

A Bad Input

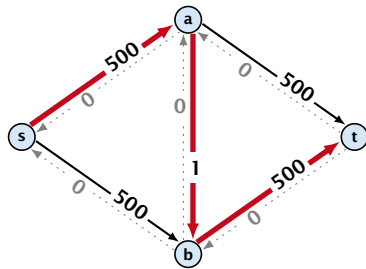
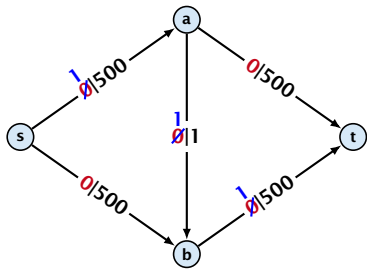
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A Bad Input

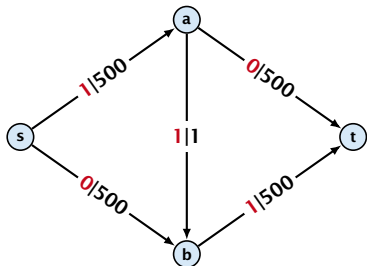
Problem: The running time may not be polynomial



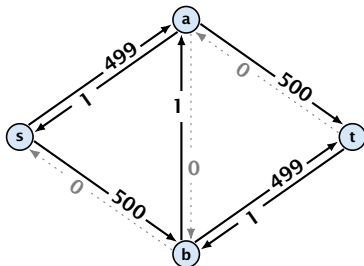
flow value: 0

A Bad Input

Problem: The running time may not be polynomial

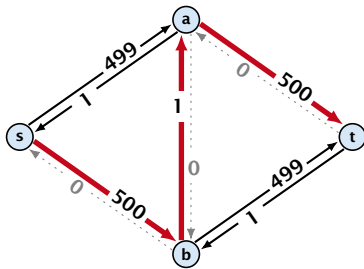
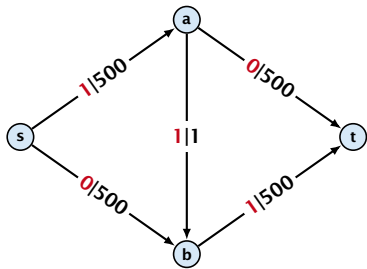


flow value: 1



A Bad Input

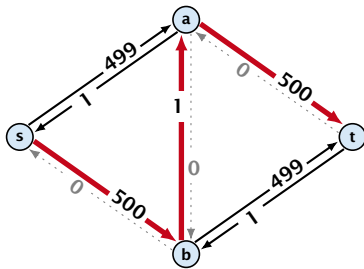
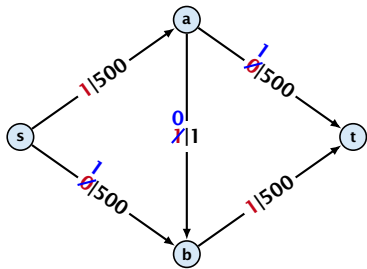
Problem: The running time may not be polynomial



flow value: 1

A Bad Input

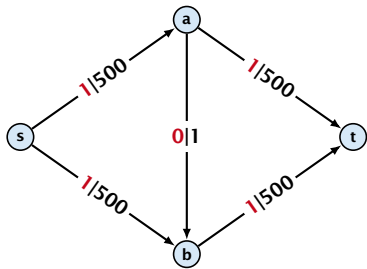
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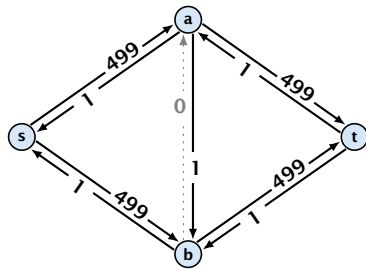
flow value: 1

A Bad Input

Problem: The running time may not be polynomial

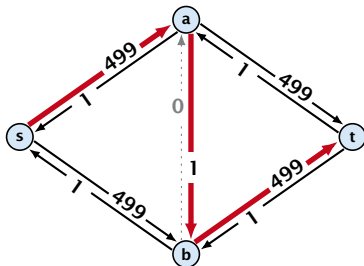
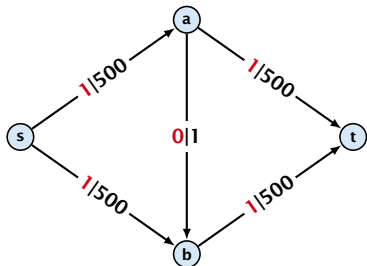


flow value: 2



A Bad Input

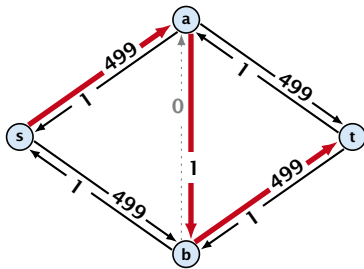
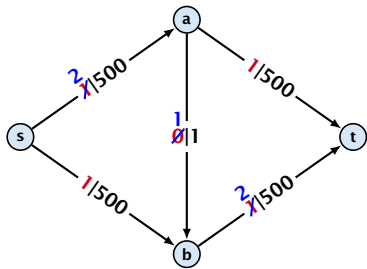
Problem: The running time may not be polynomial



flow value: 2

A Bad Input

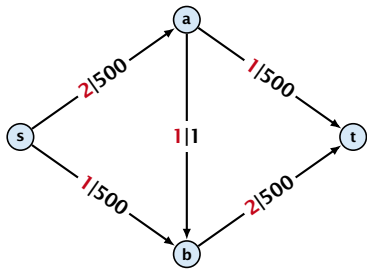
Problem: The running time may not be polynomial



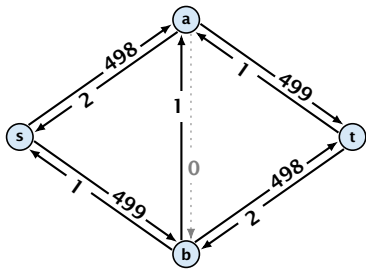
flow value: 2

A Bad Input

Problem: The running time may not be polynomial

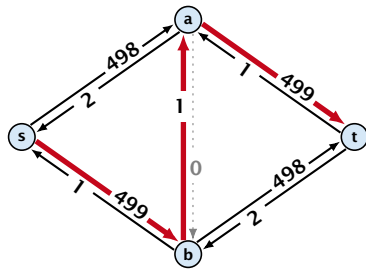
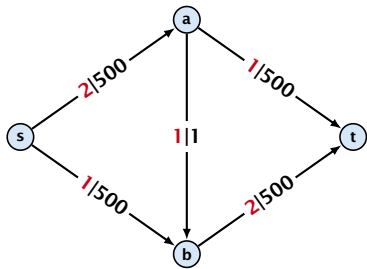


flow value: 3



A Bad Input

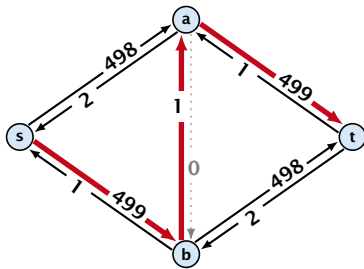
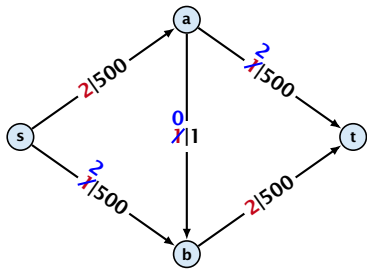
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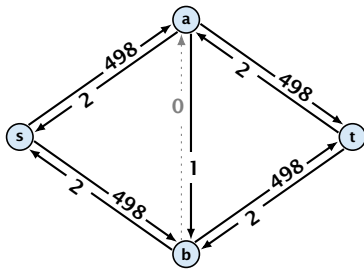
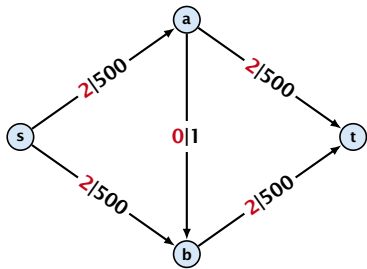
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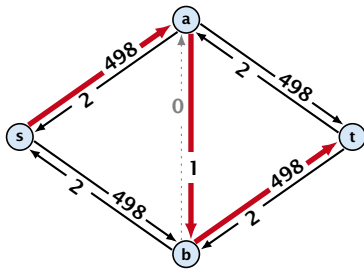
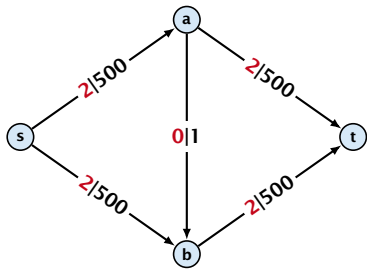
Problem: The running time may not be polynomial



flow value: 4

A Bad Input

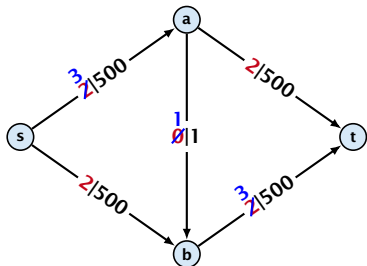
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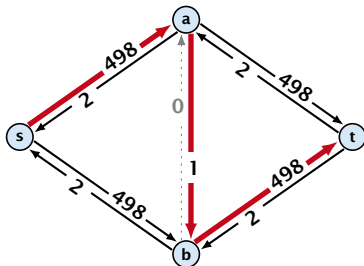
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Problem: The running time may not be polynomial

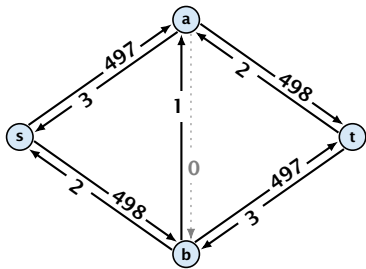
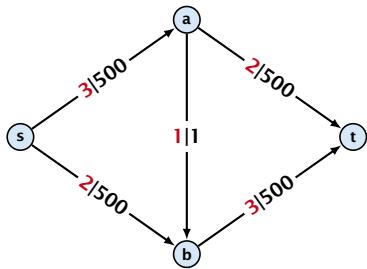


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A Bad Input

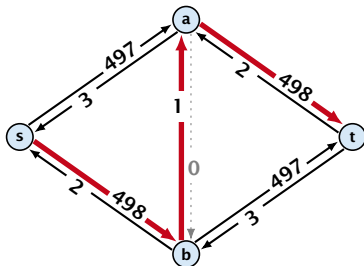
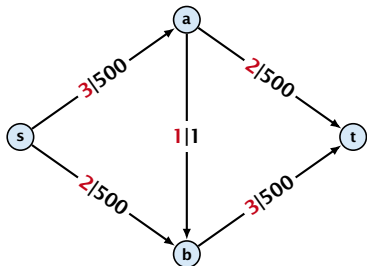
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flow value: 5

A Bad Input

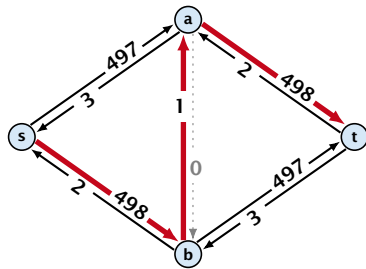
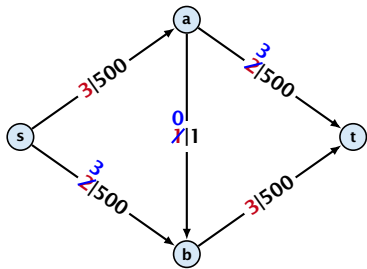
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A Bad Input

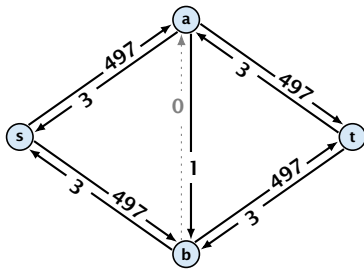
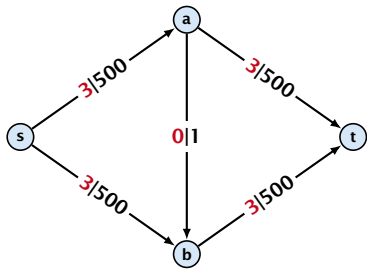
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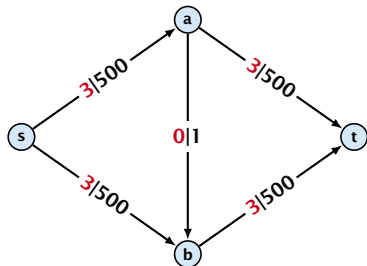
Problem: The running time may not be polynomial



flow value: 6

A Bad Input

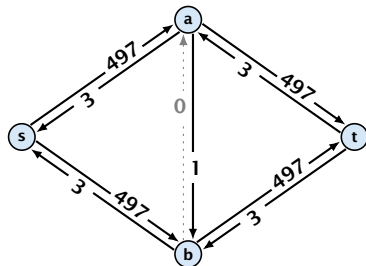
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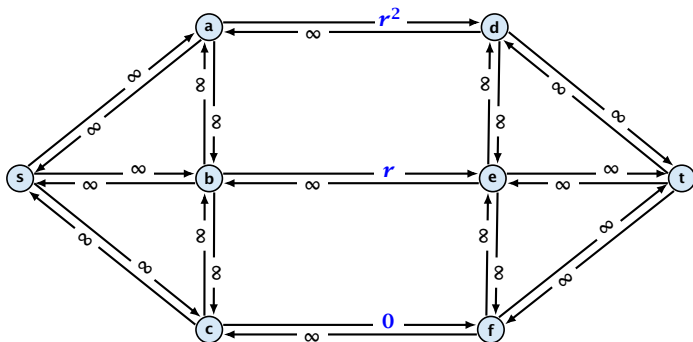
Question:

Can we tweak the algorithm so that the running time is polynomial in the input length?



A Pathological Input

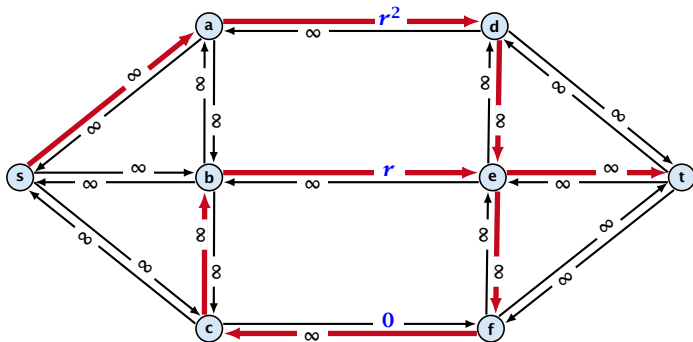
Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.



flow value: 0

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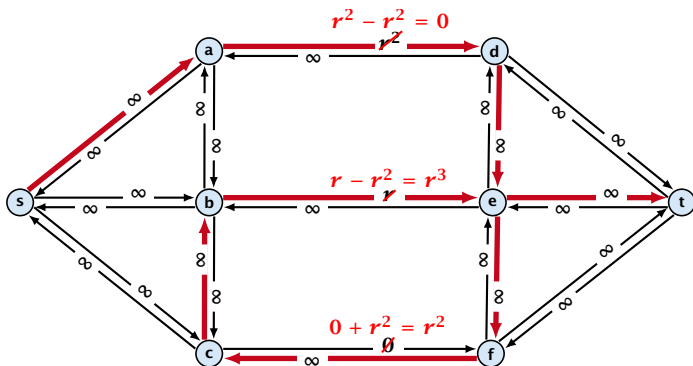
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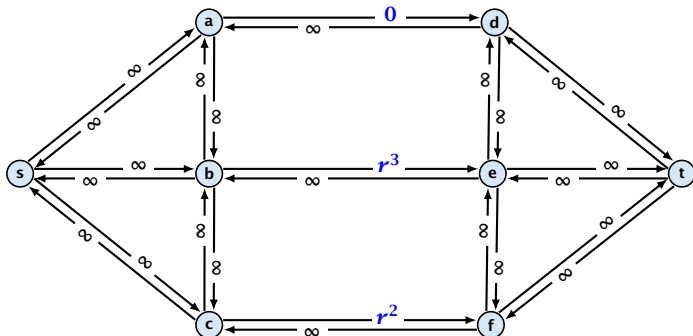
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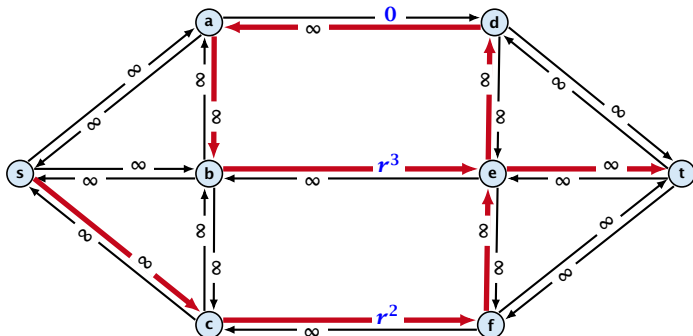
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flow value: r^2

A Pathological Input

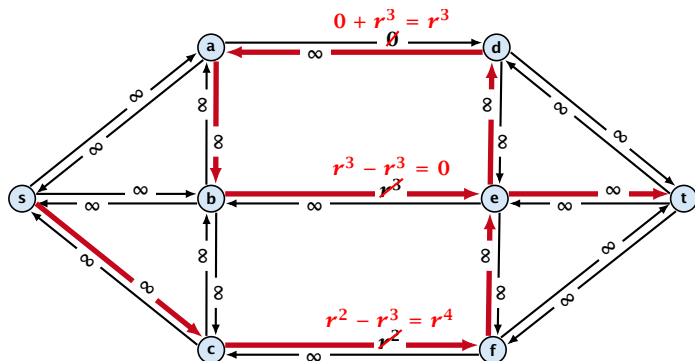
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A Pathological Input

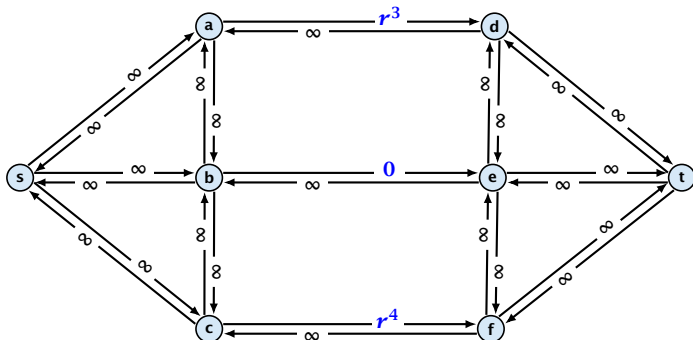
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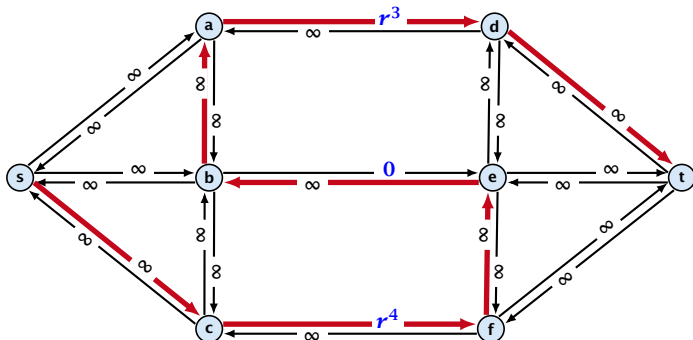
Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.



flow value: $r^2 + r^3$

A Pathological Input

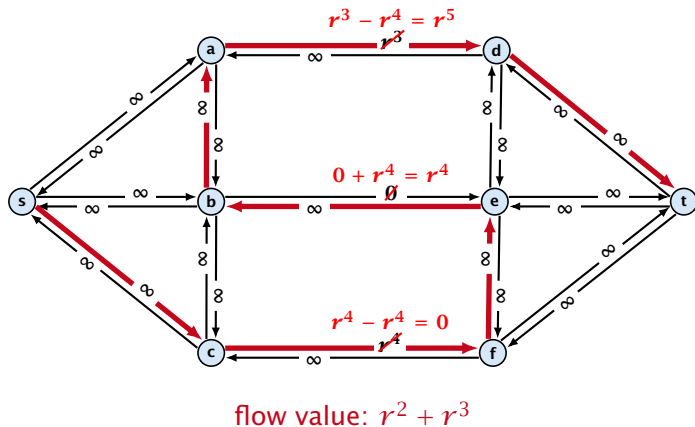
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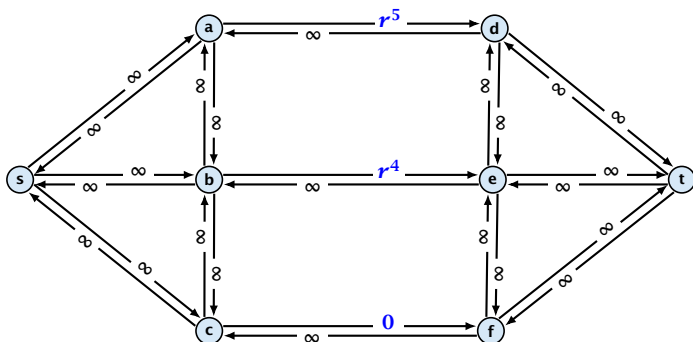
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flow value: $r^2 + r^3 + r^4$

Running time may be infinite!!!

How to choose augmenting paths?

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- ▶ Choose the shortest augmenting path.

Overview: Shortest Augmenting Paths

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Lemma 54

The length of the shortest augmenting path never decreases.

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After at most $\mathcal{O}(m)$ augmentations, the length of the shortest augmenting path strictly increases.

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Proof.

- ▶ We can find the shortest augmenting paths in time $\mathcal{O}(m)$ via BFS.
- ▶ $\mathcal{O}(m)$ augmentations for paths of exactly $k < n$ edges.



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Define the level $\ell(v)$ of a node as the length of the shortest s - v path in G_f (along non-zero edges).

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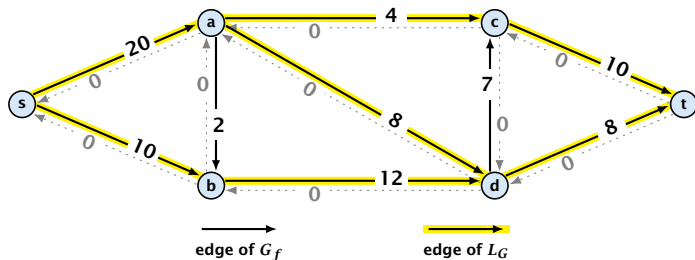
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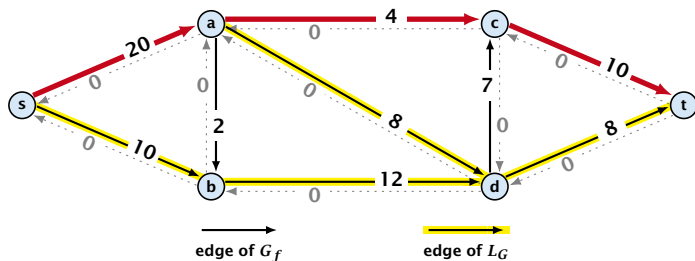


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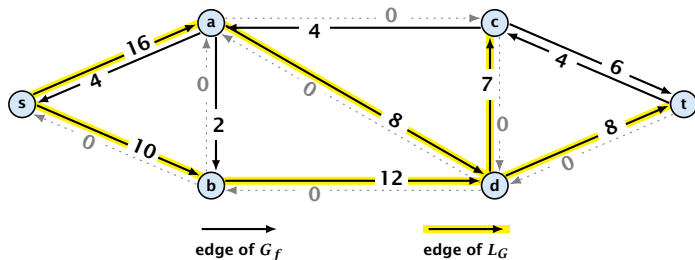


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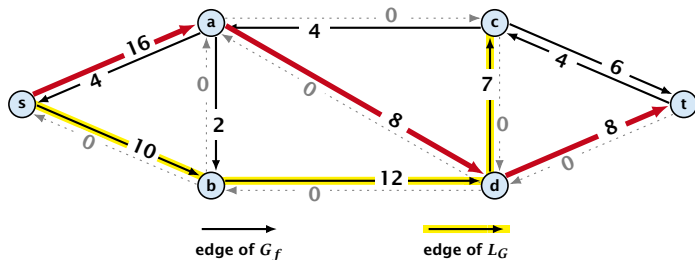


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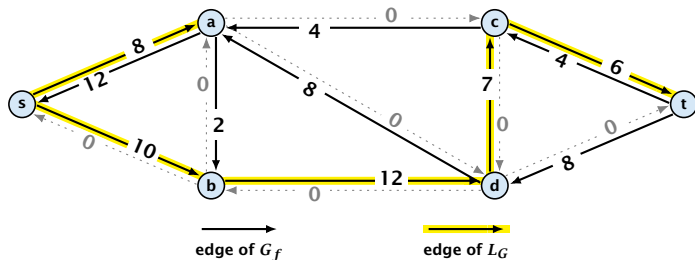


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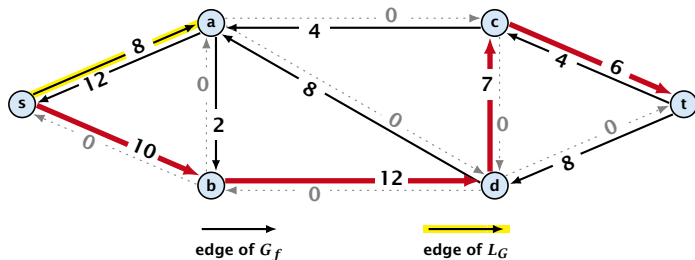


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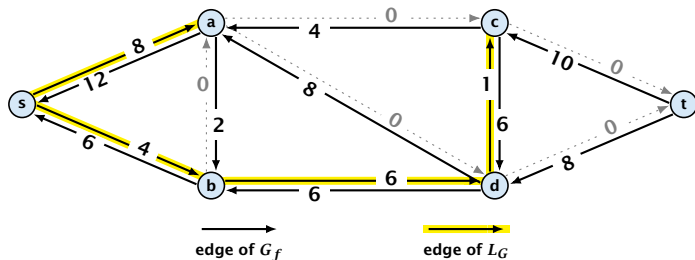


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A path P is a shortest s - t path in G_f **iff** it is an s - t path in L_G .



In the following we assume that the residual graph G_f does not contain zero capacity edges.

This means, we construct it in the usual sense and then delete edges of zero capacity.

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The length of the shortest augmenting path never decreases.

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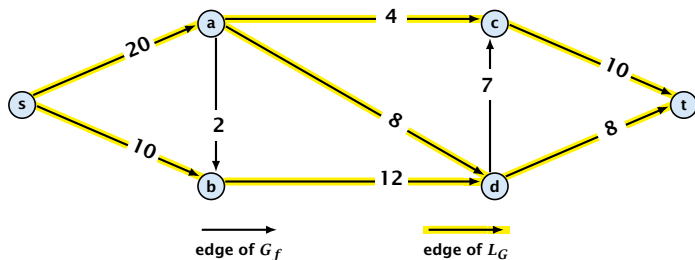
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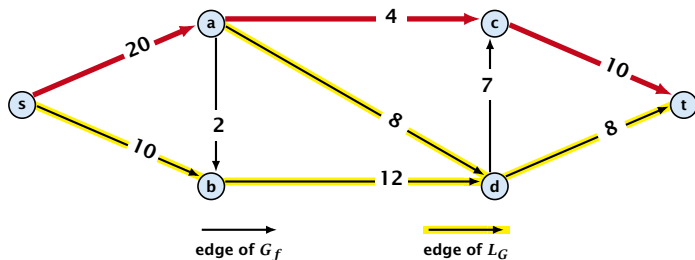
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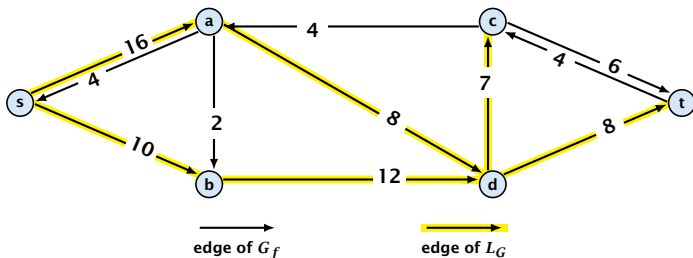
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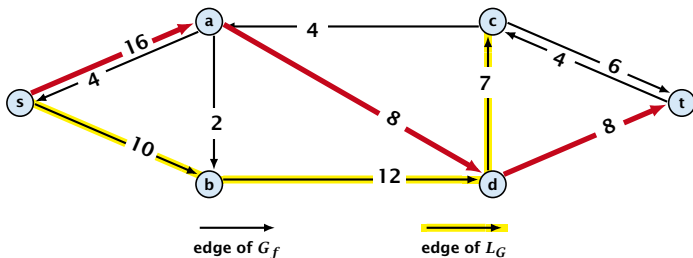
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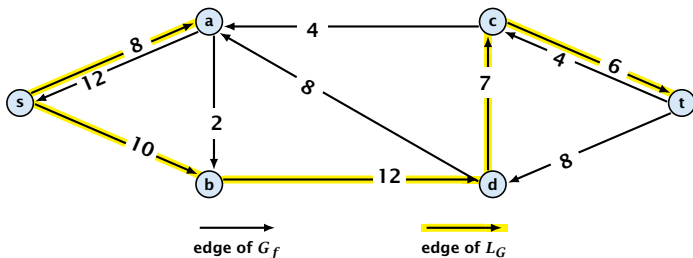
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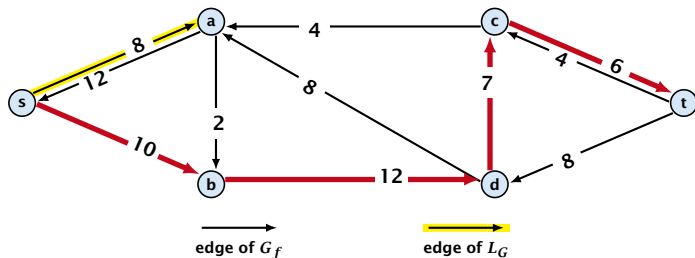
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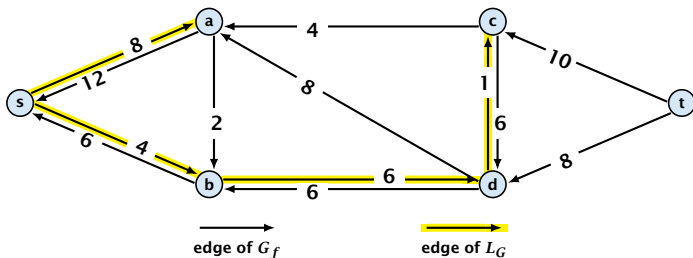
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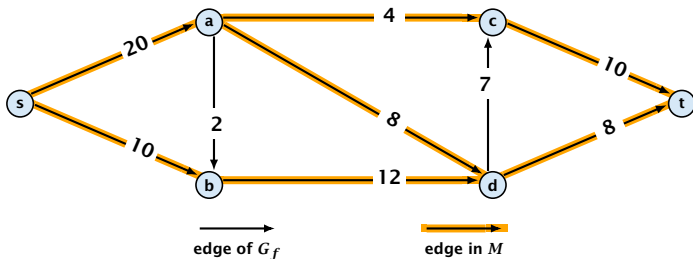
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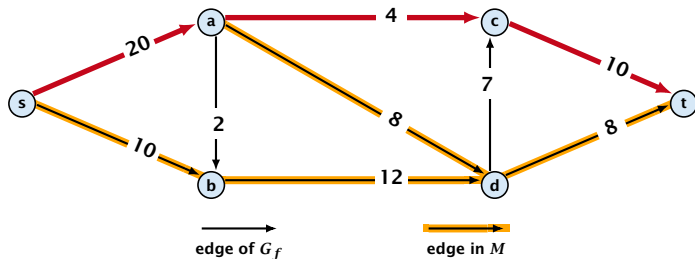
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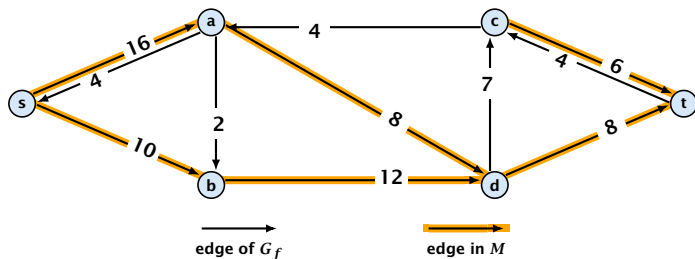
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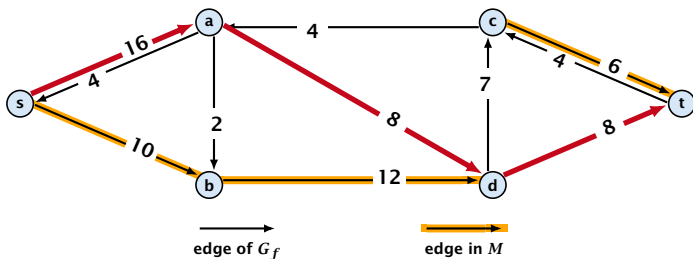
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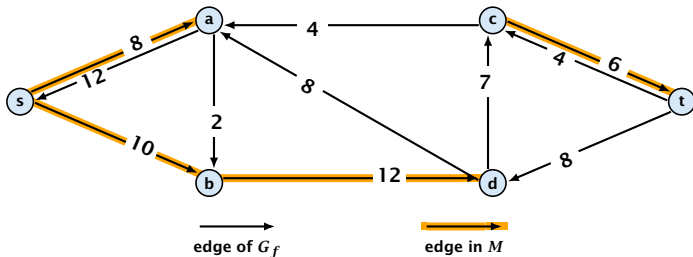
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Let M denote the set of edges in graph L_G at the beginning of a round when the distance between s and t is k .

An s - t path in G_f that uses edges not in M has length larger than k , even when using edges added to G_f during the round.

In each augmentation an edge is deleted from M .



Shortest Augmenting Paths

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Note:

There always exists a set of m augmentations that gives a maximum flow (why?).

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When sticking to shortest augmenting paths we cannot improve (asymptotically) on the number of augmentations.

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However, we can improve the running time to $\mathcal{O}(mn^2)$ by improving the running time for finding an augmenting path (currently we assume $\mathcal{O}(m)$ per augmentation for this).

Shortest Augmenting Paths

We maintain a subset M of the edges of G_f with the guarantee that a shortest s - t path using only edges from M is a shortest augmenting path.

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Note that M is not the set of edges of the level graph but a subset of level-graph edges.

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There are at most n phases. Hence, total cost is $\mathcal{O}(mn^2)$.

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- ▶ Choose path with sufficiently large bottleneck capacity.
- ▶ Choose the shortest augmenting path.

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Intuition:

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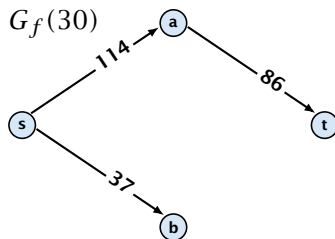
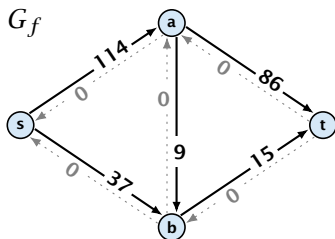
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Capacity Scaling

Algorithm 1 $\text{maxflow}(G, s, t, c)$

```
1: foreach  $e \in E$  do  $f_e \leftarrow 0$ ;  
2:  $\Delta \leftarrow 2^{\lceil \log_2 C \rceil}$   
3: while  $\Delta \geq 1$  do  
4:    $G_f(\Delta) \leftarrow \Delta$ -residual graph  
5:   while there is augmenting path  $P$  in  $G_f(\Delta)$  do  
6:      $f \leftarrow \text{augment}(f, c, P)$   
7:      $\text{update}(G_f(\Delta))$   
8:    $\Delta \leftarrow \Delta/2$   
9: return  $f$ 
```

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- ▶ therefore after the last phase there are no augmenting paths anymore
- ▶ this means we have a maximum flow.

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- ▶ There must exist an s - t cut in $G_f(\Delta)$ of zero capacity.
- ▶ In G_f this cut can have capacity at most $m\Delta$.
- ▶ This gives me an upper bound on the flow that I can still add.

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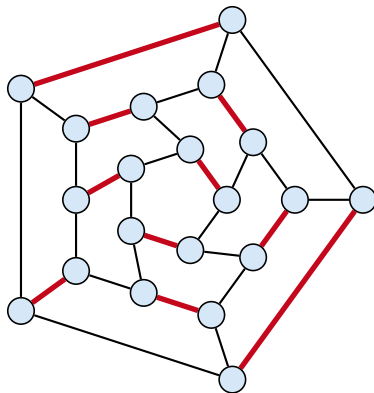
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Theorem 62

We need $\mathcal{O}(m \log C)$ augmentations. The algorithm can be implemented in time $\mathcal{O}(m^2 \log C)$.

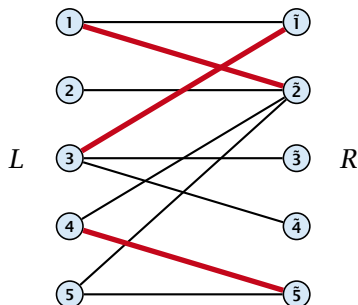
Matching

- ▶ Input: undirected graph $G = (V, E)$.
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- ▶ Maximum Matching: find a matching of maximum cardinality



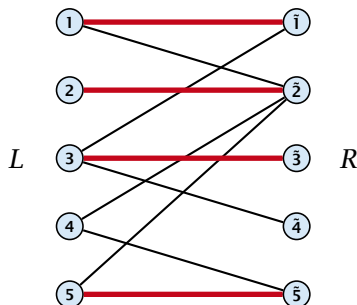
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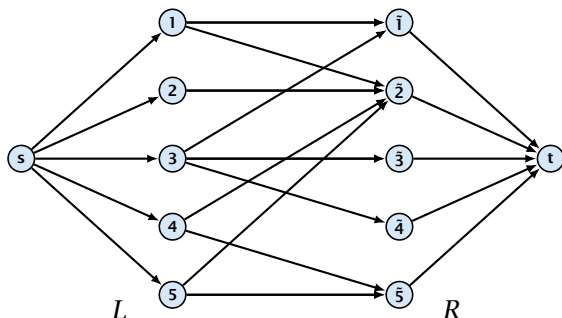
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Maxflow Formulation

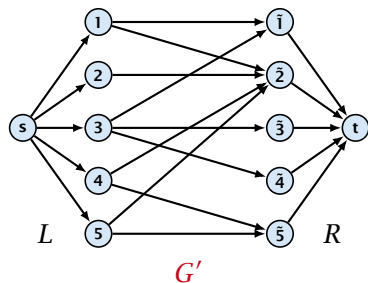
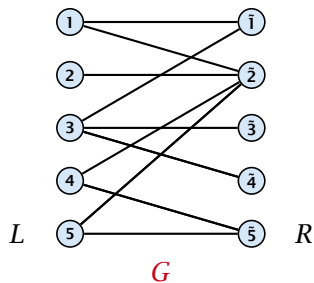
- ▶ Input: undirected, bipartite graph $G = (L \uplus R \uplus \{s, t\}, E')$.
- ▶ Direct all edges from L to R .
- ▶ Add source s and connect it to all nodes on the left.
- ▶ Add t and connect all nodes on the right to t .
- ▶ All edges have unit capacity.



Proof

Max cardinality matching in $G \leq$ value of maxflow in G'

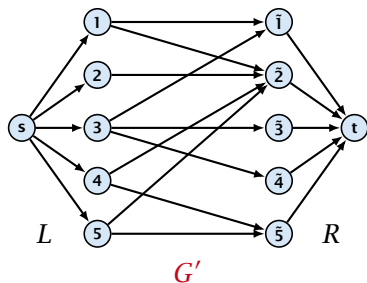
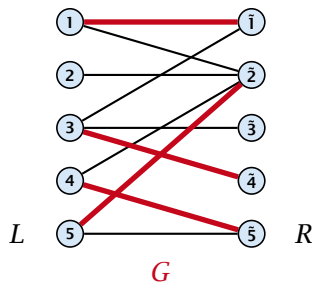
- ▶ Given a maximum matching M of cardinality k .
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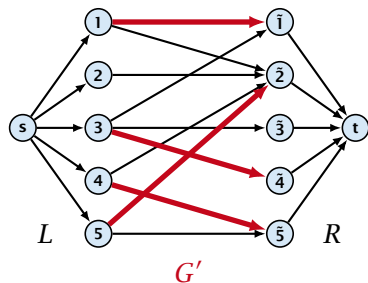
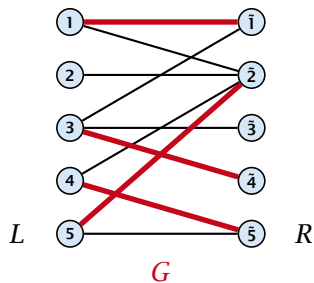
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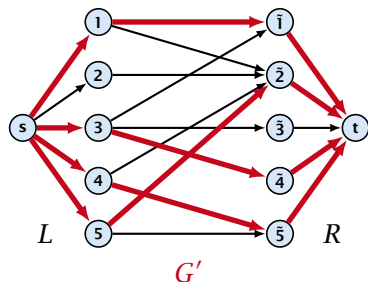
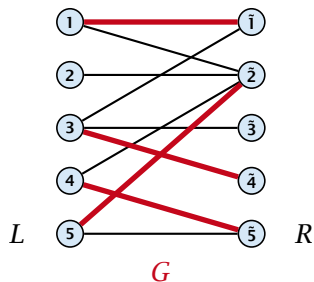
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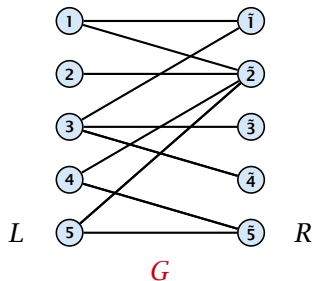
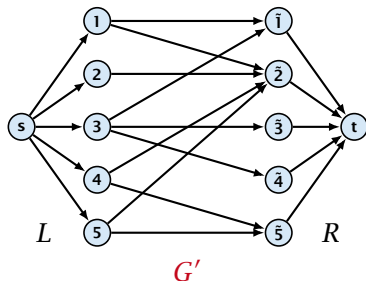
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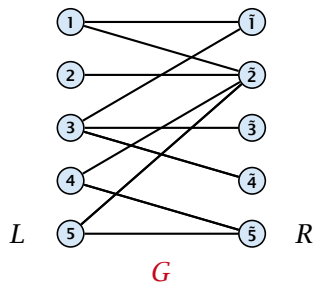
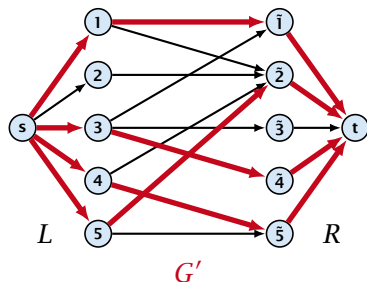
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- ▶ Consider $M =$ set of edges from L to R with $f(e) = 1$.
- ▶ Each node in L and R participates in at most one edge in M .
- ▶ $|M| = k$, as the flow must use at least k middle edges.



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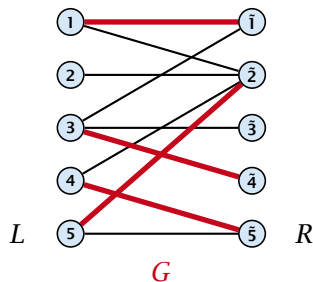
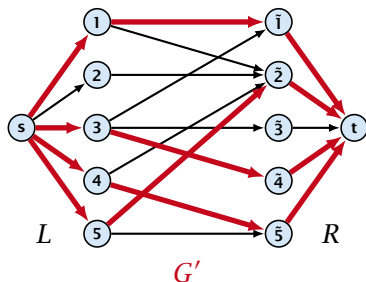
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12.1 Matching

Which flow algorithm to use?

- ▶ Generic augmenting path: $\mathcal{O}(m \text{val}(f^*)) = \mathcal{O}(mn)$.
- ▶ Capacity scaling: $\mathcal{O}(m^2 \log C) = \mathcal{O}(m^2)$.
- ▶ Shortest augmenting path: $\mathcal{O}(mn^2)$.

For **unit capacity simple graphs** shortest augmenting path can be implemented in time $\mathcal{O}(m\sqrt{n})$.

Baseball Elimination

<i>team</i> <i>i</i>	<i>wins</i> w_i	<i>losses</i> ℓ_i	<i>remaining games</i>			
			<i>Atl</i>	<i>Phi</i>	<i>NY</i>	<i>Mon</i>
Atlanta	83	71	–	1	6	1
Philadelphia	80	79	1	–	0	2
New York	78	78	6	0	–	0
Montreal	77	82	1	2	0	–

Which team can end the season with most wins?

- ▶ Montreal is eliminated, since even after winning all remaining games there are only 80 wins.
- ▶ But also Philadelphia is eliminated. Why?

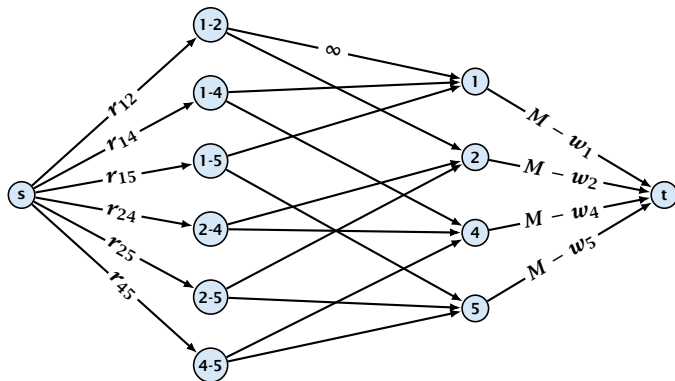
Baseball Elimination

Formal definition of the problem:

- ▶ Given a set S of teams, and one specific team $z \in S$.
- ▶ Team x has already won w_x games.
- ▶ Team x still has to play team y , r_{xy} times.
- ▶ Does team z still have a chance to finish with the most number of wins.

Baseball Elimination

Flow network for $z = 3$. M is number of wins Team 3 can still obtain.

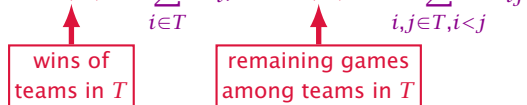


Idea. Distribute the results of remaining games in such a way that no team gets too many wins.

Certificate of Elimination

Let $T \subseteq S$ be a subset of teams. Define

$$w(T) := \sum_{i \in T} w_i, \quad r(T) := \sum_{i, j \in T, i < j} r_{ij}$$



If $\frac{w(T)+r(T)}{|T|} > M$ then one of the teams in T will have more than M wins in the end. A team that can win at most M games is therefore eliminated.

Theorem 63

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{i,j \in S \setminus \{z\}, i < j} r_{ij}$.

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- ▶ Consider the mincut A in the flow network. Let T be the set of team-nodes in A .

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- ▶ We don't find a flow that saturates all source edges:

$$r(S \setminus \{z\})$$

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- ▶ Consider the mincut A in the flow network. Let T be the set of team-nodes in A .
- ▶ If for node $x-y$ not both team-nodes x and y are in T , then $x-y \notin A$ as otw. the cut would cut an infinite capacity edge.
- ▶ We don't find a flow that saturates all source edges:

$$\begin{aligned}r(S \setminus \{z\}) &> \text{cap}(A, V \setminus A) \\ &\geq \sum_{i < j: i \notin T \vee j \notin T} r_{ij} + \sum_{i \in T} (M - w_i) \\ &\geq r(S \setminus \{z\}) - r(T) + |T|M - w(T)\end{aligned}$$

- ▶ This gives $M < (w(T) + r(T))/|T|$, i.e., z is eliminated.

Baseball Elimination

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- ▶ Hence, we found a set of results for the remaining games, such that no team obtains more than M wins in total.
- ▶ Hence, team z is not eliminated.

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Project selection problem:

- ▶ Set P of possible projects. Project v has an associated profit p_v (can be positive or negative).

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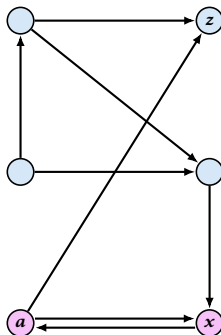
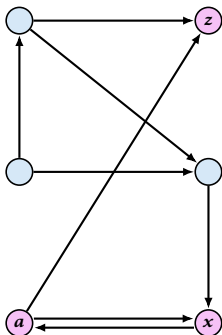
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Goal: Find a feasible set of projects that maximizes the profit.

Project Selection

The prerequisite graph:

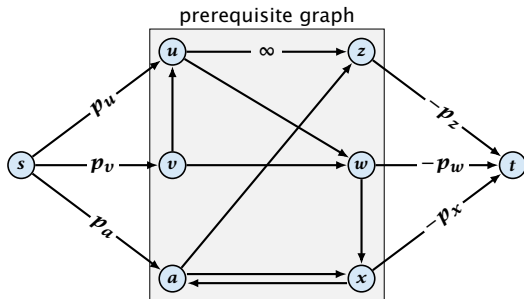
- ▶ $\{x, a, z\}$ is a feasible subset.
- ▶ $\{x, a\}$ is infeasible.



Project Selection

Mincut formulation:

- ▶ Edges in the prerequisite graph get infinite capacity.
- ▶ Add edge (s, v) with capacity p_v for nodes v with positive profit.
- ▶ Create edge (v, t) with capacity $-p_v$ for nodes v with negative profit.



Theorem 64

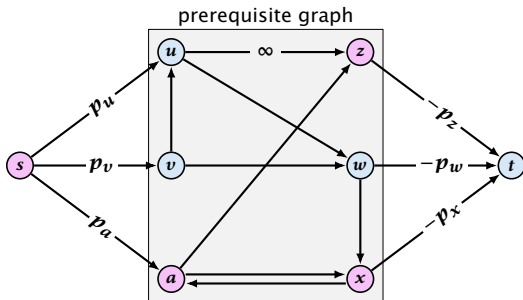
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- ▶ A is feasible because of capacity infinity edges.

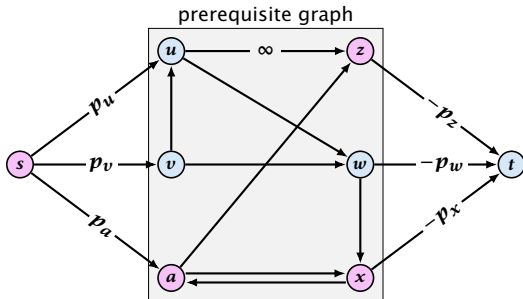


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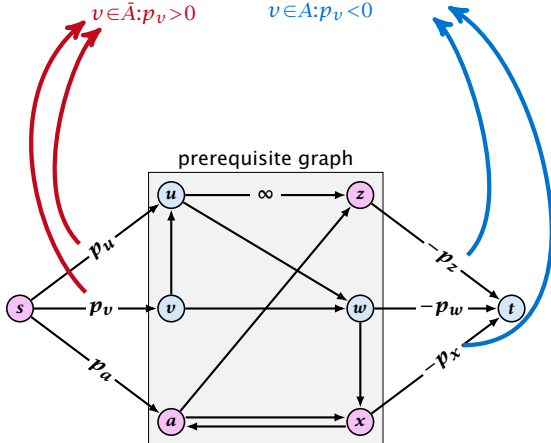
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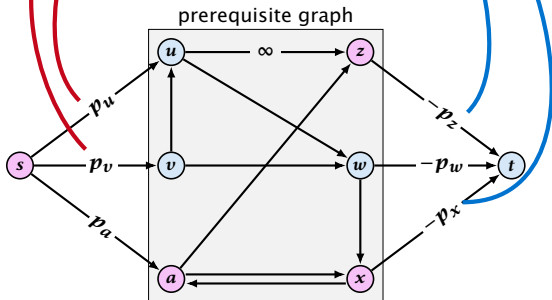
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$$= \sum_{v: p_v > 0} p_v - \sum_{v \in A} p_v$$



Preflows

Definition 65

An (s, t) -preflow is a function $f : E \mapsto \mathbb{R}^+$ that satisfies

1. For each edge e

$$0 \leq f(e) \leq c(e) .$$

(capacity constraints)

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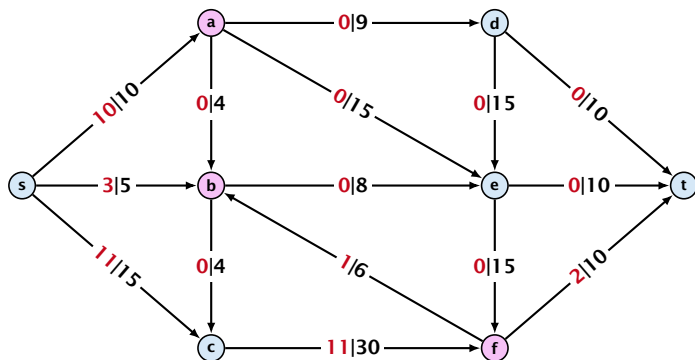
(capacity constraints)

2. For each $v \in V \setminus \{s, t\}$

$$\sum_{e \in \text{out}(v)} f(e) \leq \sum_{e \in \text{into}(v)} f(e) .$$

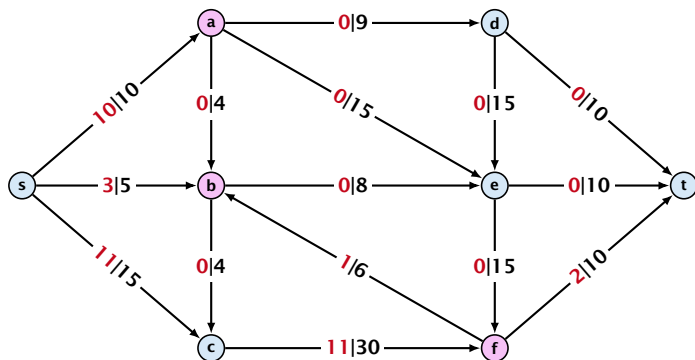
Preflows

Example 66



Preflows

Example 66



A node that has $\sum_{e \in \text{out}(v)} f(e) < \sum_{e \in \text{into}(v)} f(e)$ is called an **active node**.

Preflows

Definition:

A **labelling** is a function $\ell : V \rightarrow \mathbb{N}$. It is **valid** for preflow f if

- ▶ $\ell(u) \leq \ell(v) + 1$ for all edges (u, v) in the residual graph G_f
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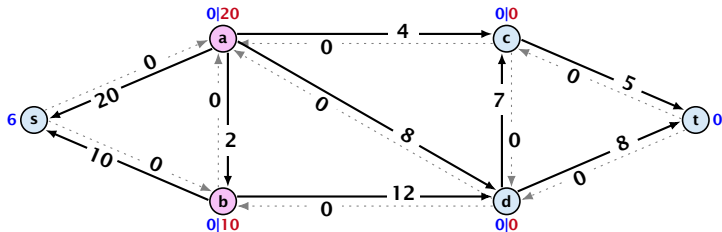
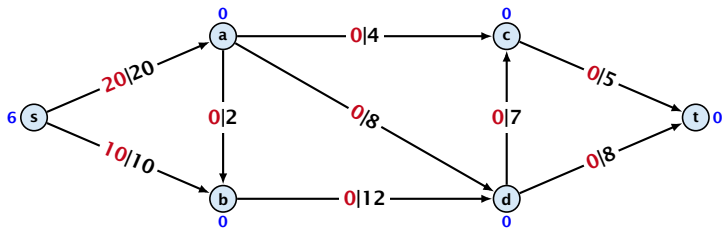
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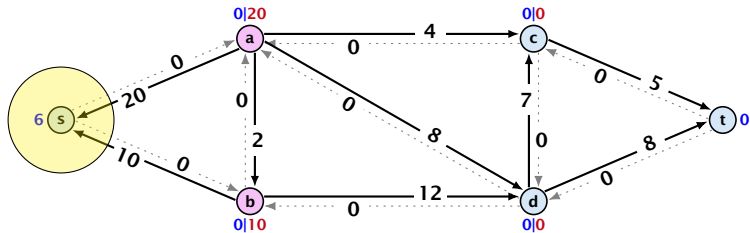
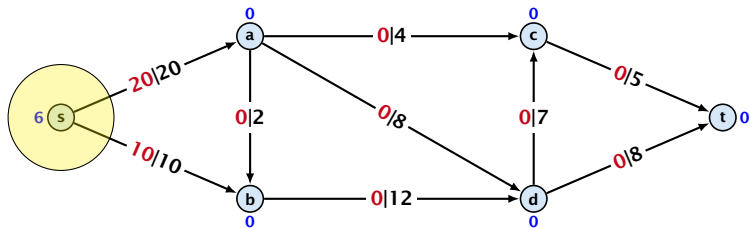
Intuition:

The labelling can be viewed as a height function. Whenever the height from node u to node v decreases by more than 1 (i.e., it goes very steep downhill from u to v), the corresponding edge must be saturated.

Preflows



Preflows



Preflows

Lemma 67

A *preflow* that has a valid labelling saturates a cut.

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Lemma 68

A *flow* that has a valid labelling is a maximum flow.

Push Relabel Algorithms

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Idea:

- ▶ start with some preflow and some valid labelling

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- ▶ start with some preflow and some valid labelling
- ▶ successively change the preflow while maintaining a valid labelling
- ▶ stop when you have a flow (i.e., no more active nodes)

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An arc (u, v) with $c_f(u, v) > 0$ in the residual graph is **admissible** if $\ell(u) = \ell(v) + 1$ (i.e., it goes downwards w.r.t. labelling ℓ).

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The push operation

Consider an active node u with **excess flow**

$f(u) = \sum_{e \in \text{into}(u)} f(e) - \sum_{e \in \text{out}(u)} f(e)$ and suppose $e = (u, v)$ is an admissible arc with residual capacity $c_f(e)$.

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We can send flow $\min\{c_f(e), f(u)\}$ along e and obtain a new preflow. The old labelling is still valid (!!!).

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- ▶ **deactivating push**: $\min\{f(u), c_f(e)\} = f(u)$
the node u becomes inactive

Push Relabel Algorithms

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The relabel operation

Consider an active node u that does not have an outgoing admissible arc.

Increasing the label of u by 1 results in a valid labelling.

- ▶ Edges (w, u) incoming to u still fulfill their constraint $\ell(w) \leq \ell(u) + 1$.
- ▶ An outgoing edge (u, w) had $\ell(u) < \ell(w) + 1$ before since it was not admissible. Now: $\ell(u) \leq \ell(w) + 1$.

Push Relabel Algorithms

Intuition:

We want to send flow downwards, since the source has a height/label of n and the target a height/label of 0 . If we see an active node u with an admissible arc we push the flow at u towards the other end-point that has a lower height/label. If we do not have an admissible arc but excess flow into u it should roughly mean that the level/height/label of u should rise. (If we consider the flow to be water then this would be natural.)

Note that the above intuition is very incorrect as the labels are integral, i.e., they cannot really be seen as the height of a node.

Reminder

- ▶ In a **preflow** nodes may not fulfill conservation constraints; a node may have more incoming flow than outgoing flow.
- ▶ Such a node is called **active**.
- ▶ A labelling is **valid** if for every edge (u, v) in the residual graph $\ell(u) \leq \ell(v) + 1$.
- ▶ An arc (u, v) in residual graph is **admissible** if $\ell(u) = \ell(v) + 1$.
- ▶ A **saturating push** along e pushes an amount of $c(e)$ flow along the edge, thereby saturating the edge (and making it disappear from the residual graph).
- ▶ A **deactivating push** along $e = (u, v)$ pushes a flow of $f(u)$, where $f(u)$ is the **excess flow** of u . This makes u inactive.

Push Relabel Algorithms

Algorithm 1 $\text{maxflow}(G, s, t, c)$

```
1: find initial preflow  $f$ 
2: while there is active node  $u$  do
3:     if there is admiss. arc  $e$  out of  $u$  then
4:          $\text{push}(G, e, f, c)$ 
5:     else
6:          $\text{relabel}(u)$ 
7: return  $f$ 
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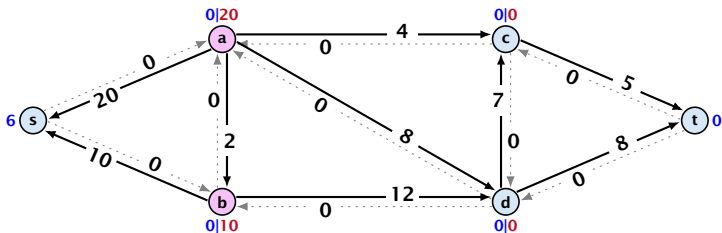
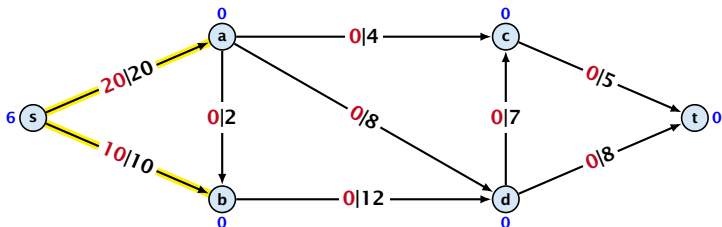
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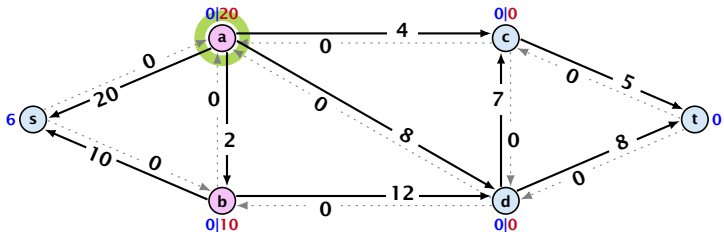
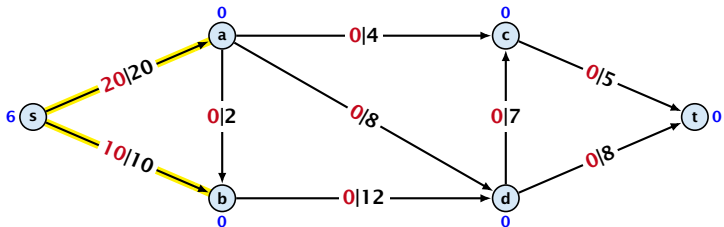
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In the following example we always stick to the same active node u until it becomes inactive but this is not required.

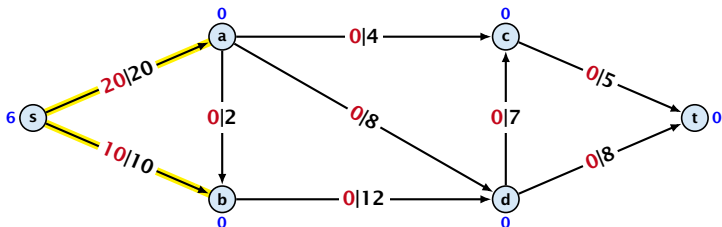
Preflow Push



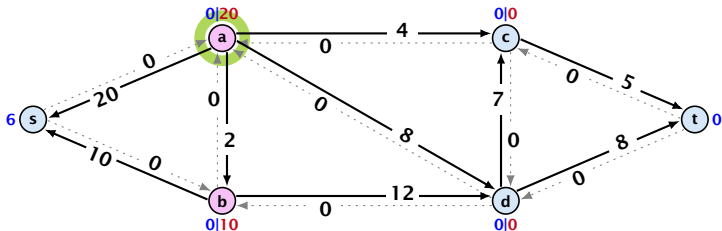
Preflow Push



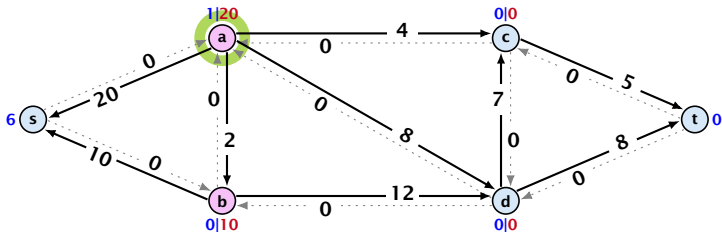
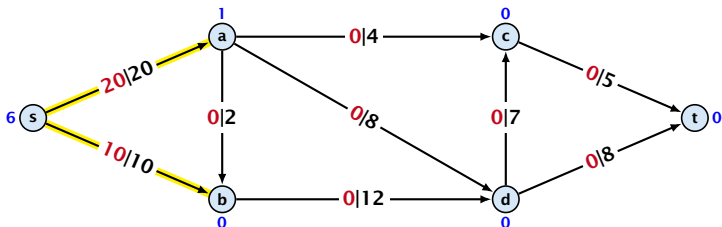
Preflow Push



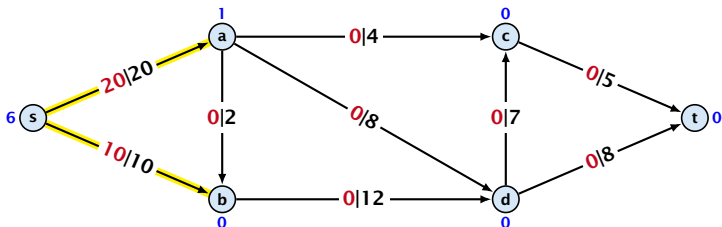
relabel to 1



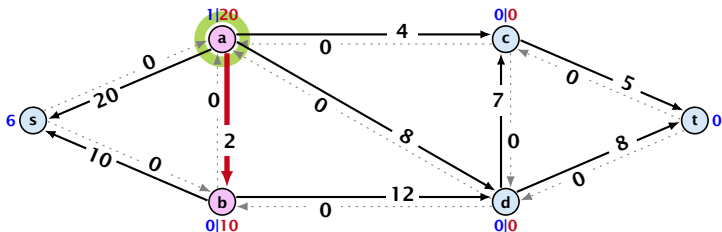
Preflow Push



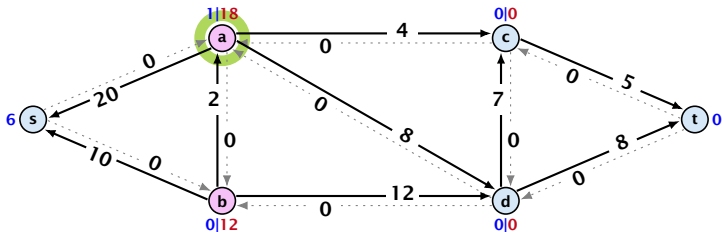
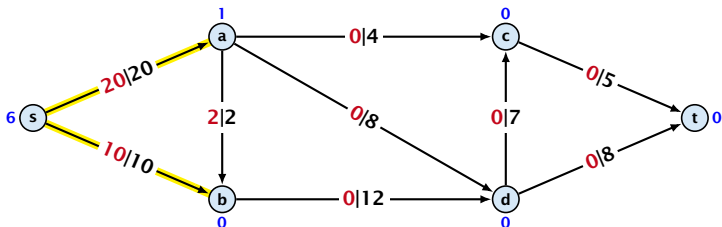
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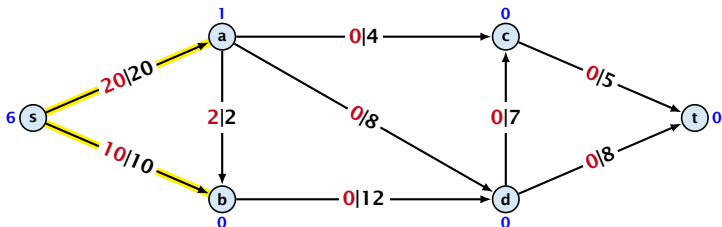
saturation push



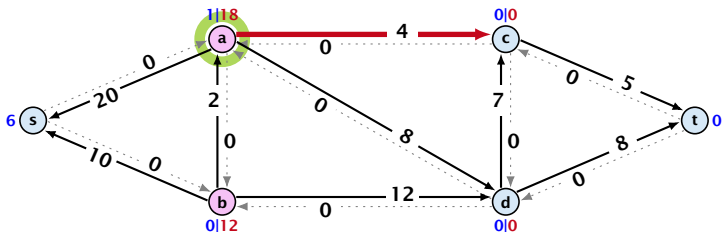
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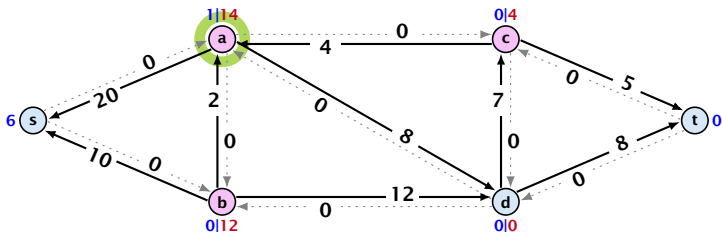
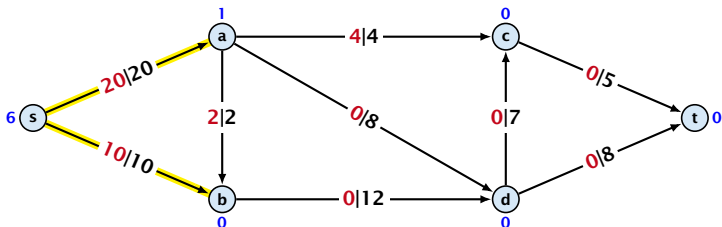
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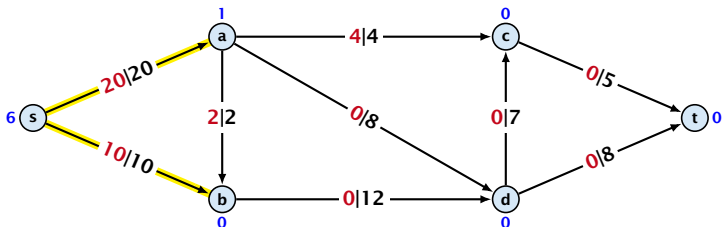
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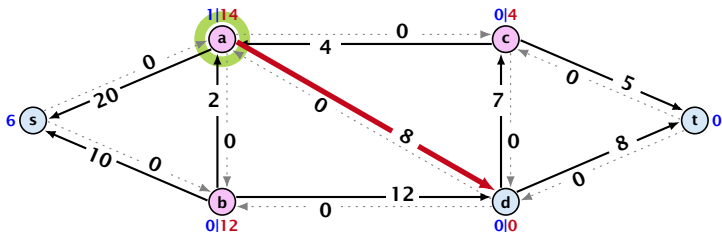
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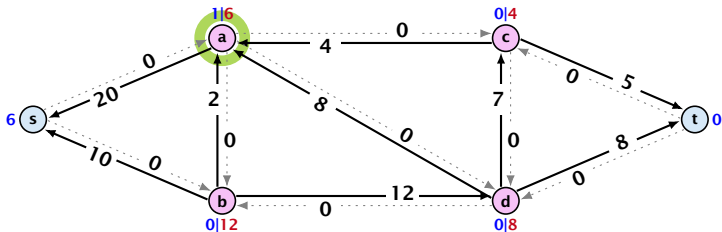
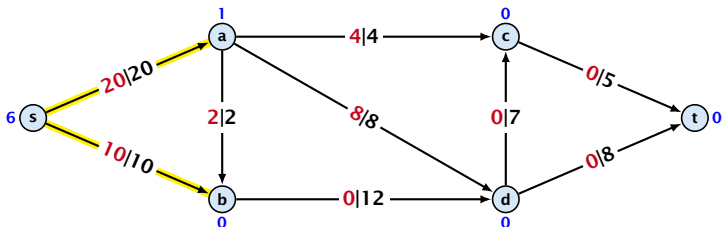
Preflow Push



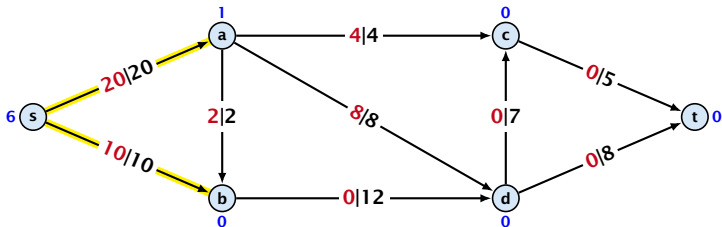
saturation push



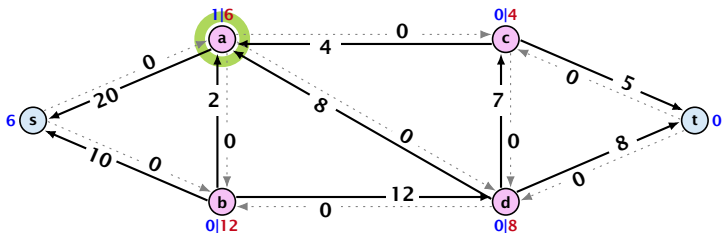
Preflow Push



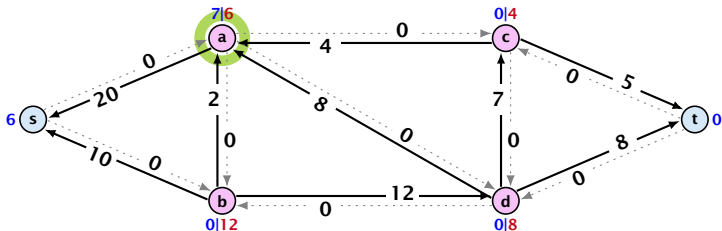
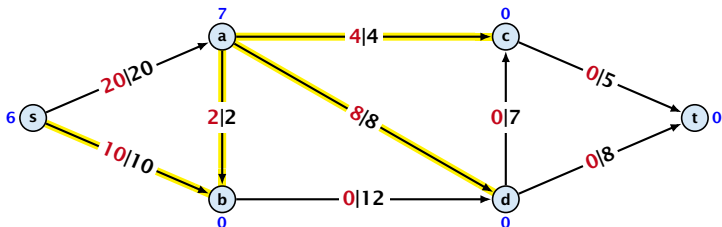
Preflow Push



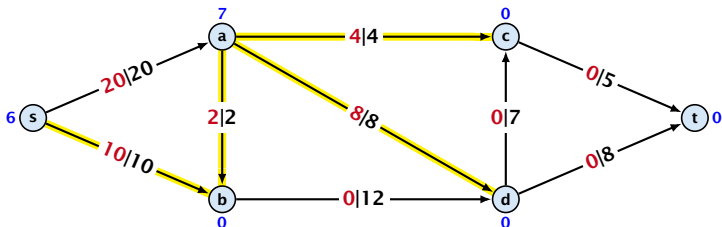
relabel to 7



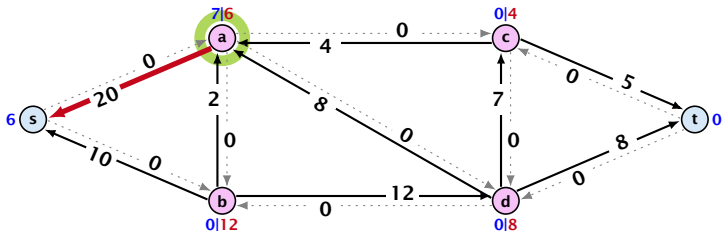
Preflow Push



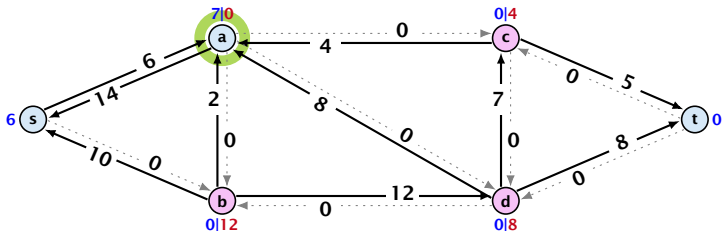
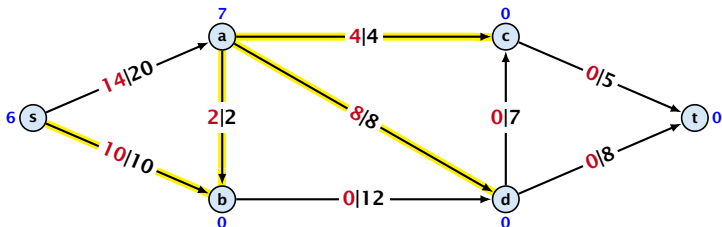
Preflow Push



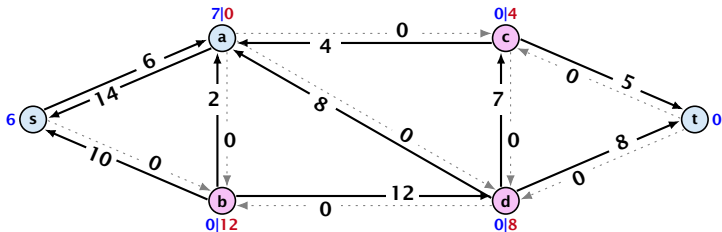
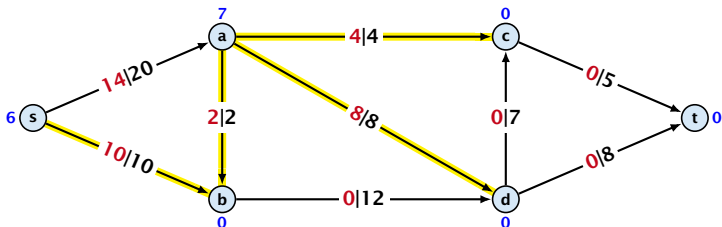
deactivating push



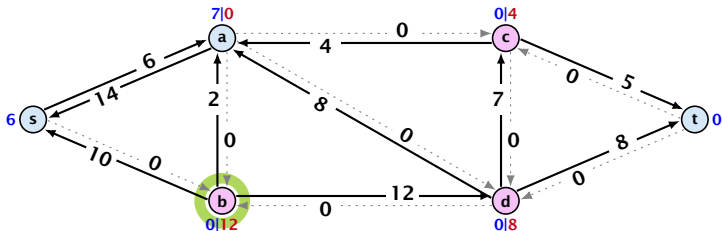
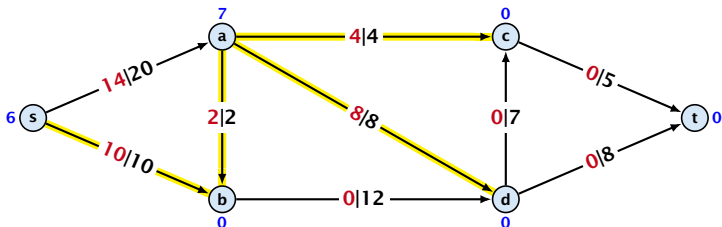
Preflow Push



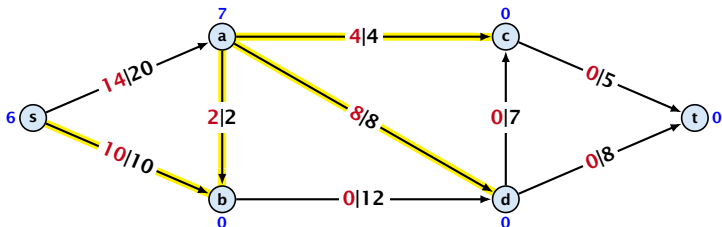
Preflow Push



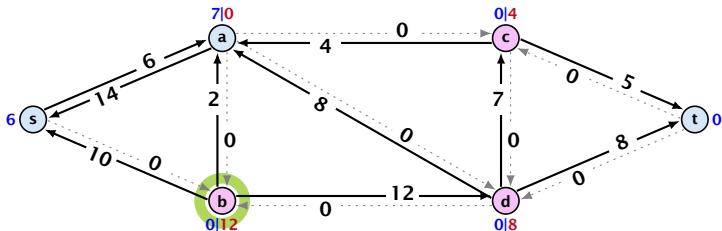
Preflow Push



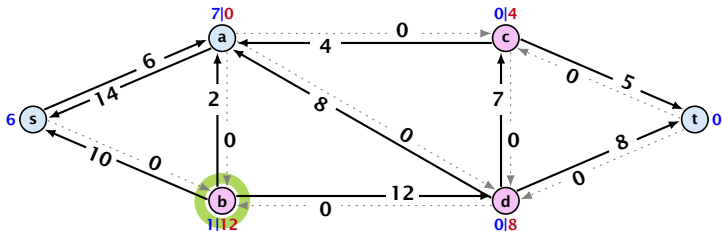
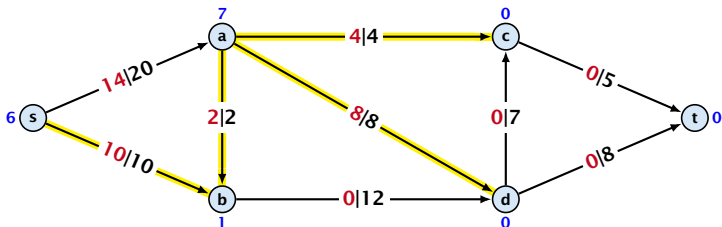
Preflow Push



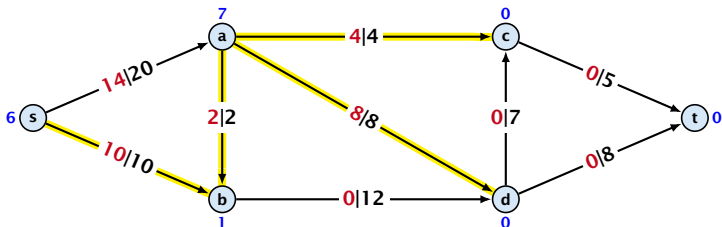
relabel to 1



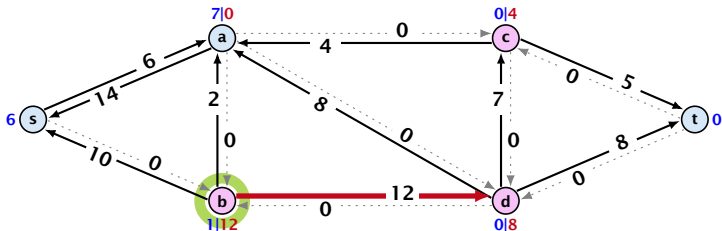
Preflow Push



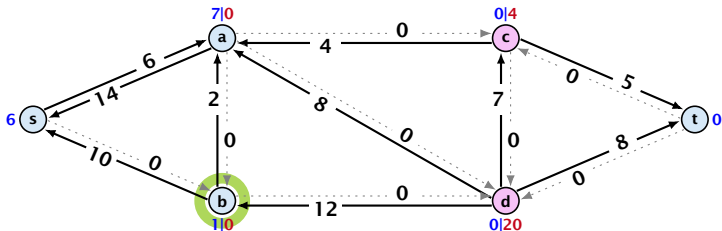
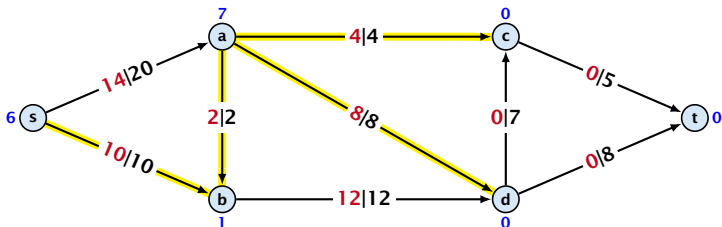
Preflow Push



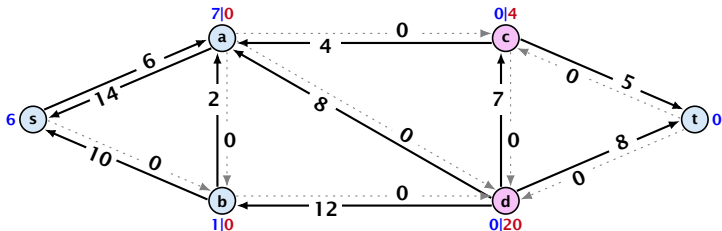
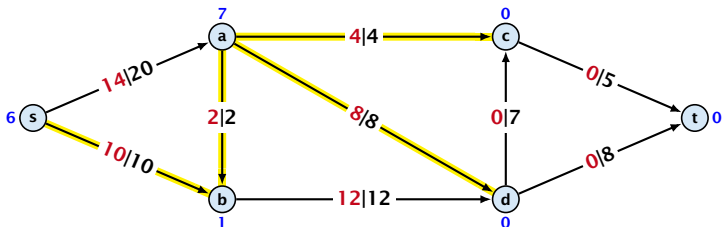
satürating and deactivating push



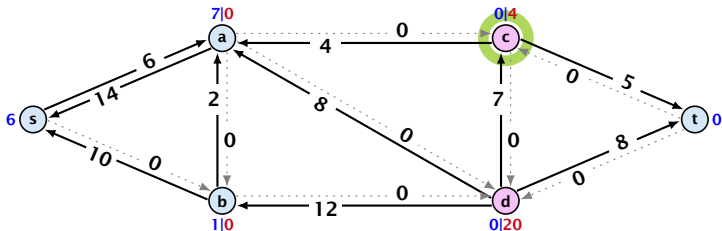
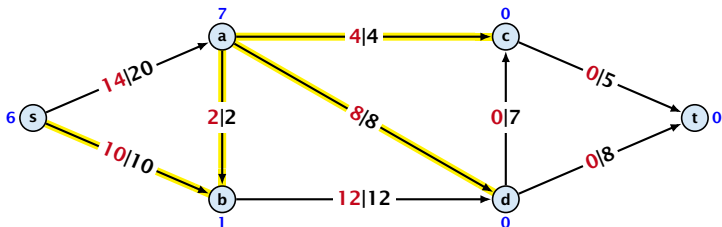
Preflow Push



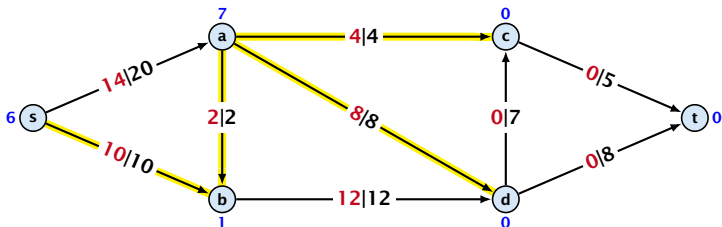
Preflow Push



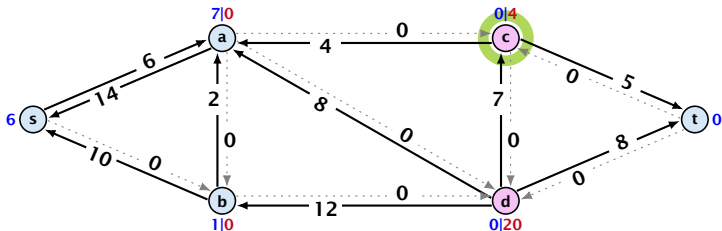
Preflow Push



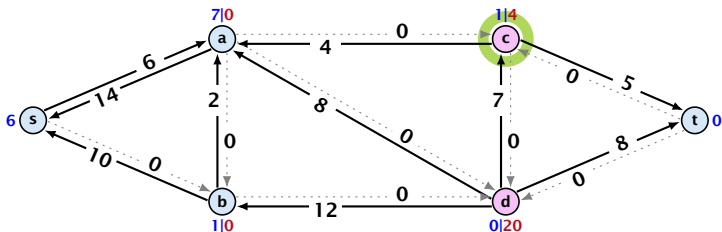
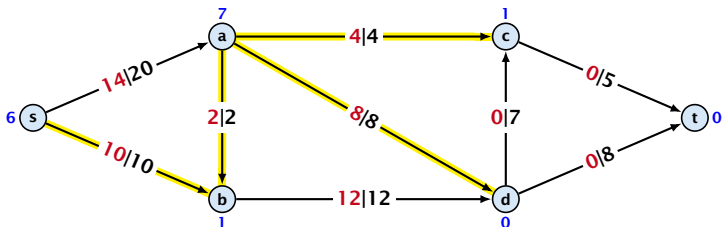
Preflow Push



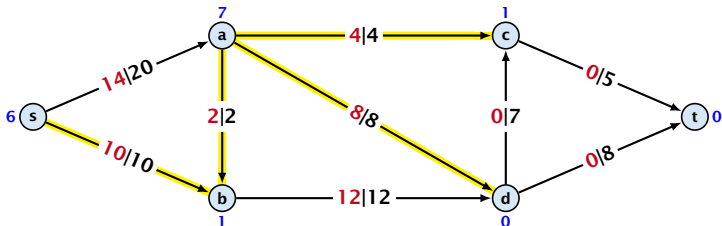
relabel to 1



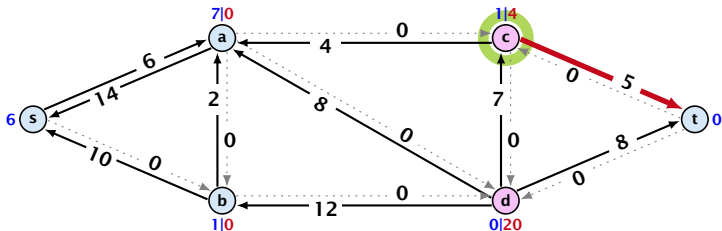
Preflow Push



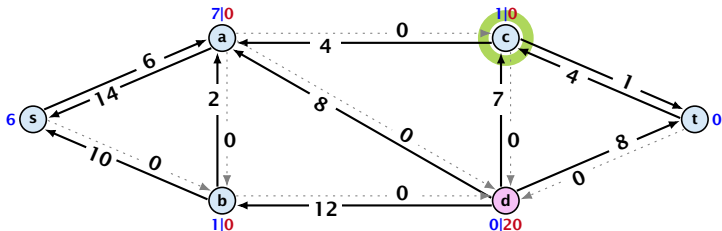
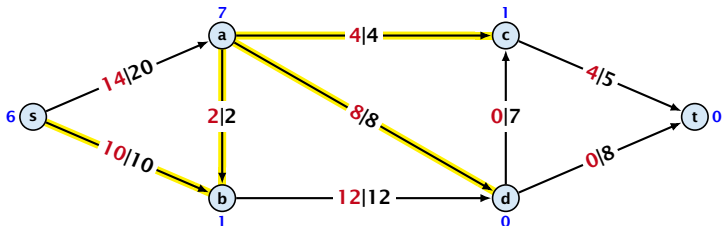
Preflow Push



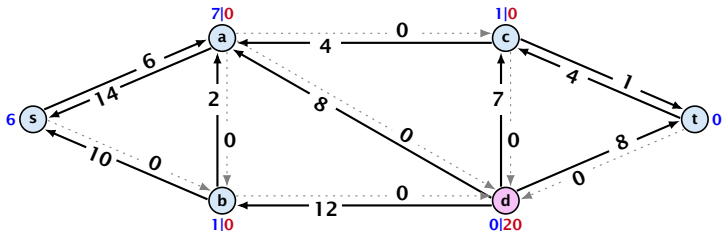
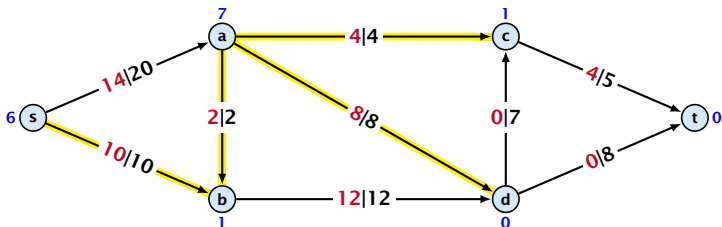
deactivating push



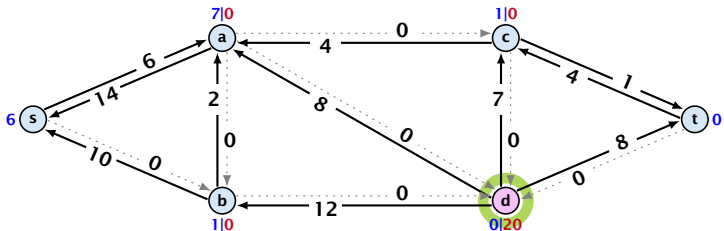
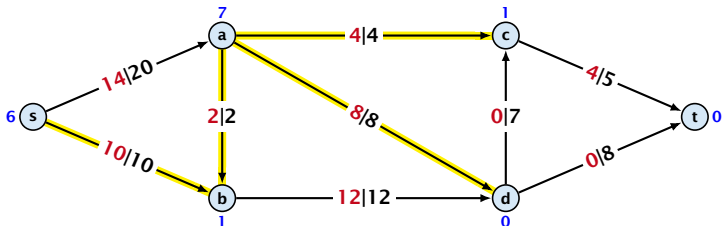
Preflow Push



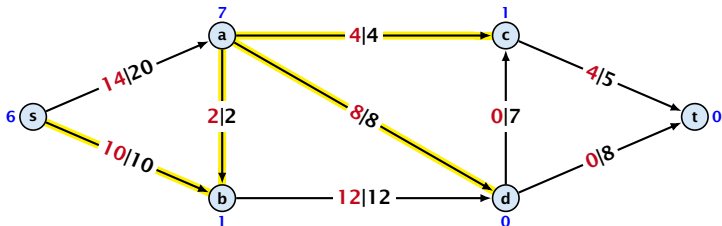
Preflow Push



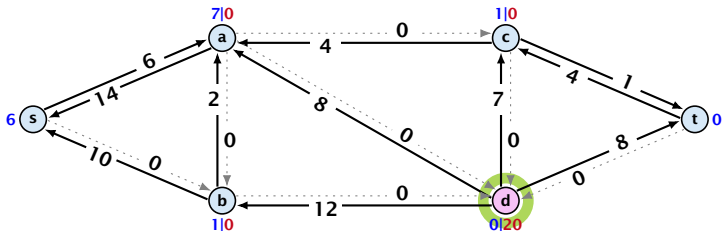
Preflow Push



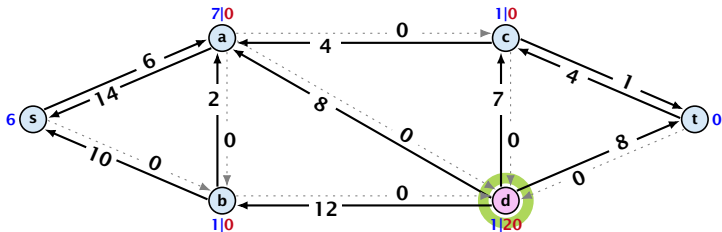
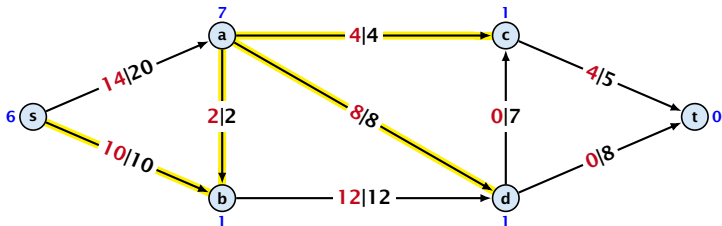
Preflow Push



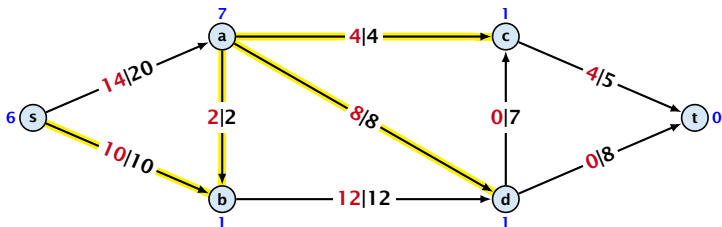
relabel to 1



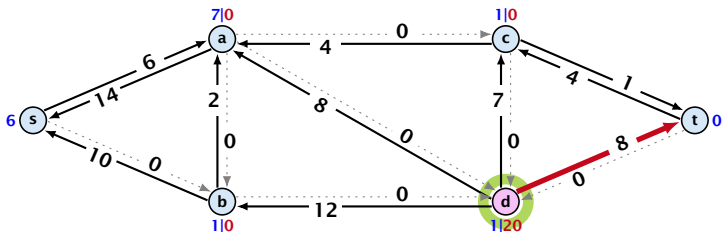
Preflow Push



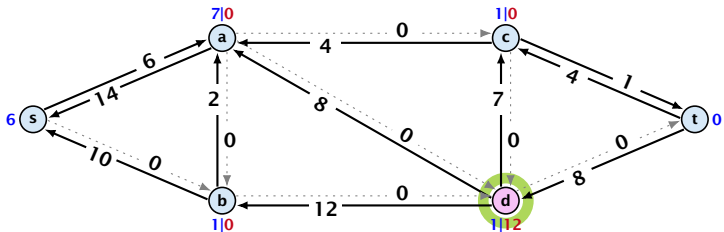
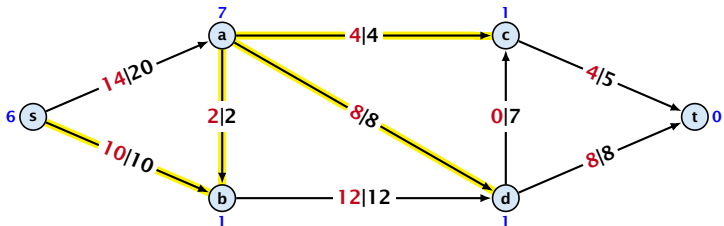
Preflow Push



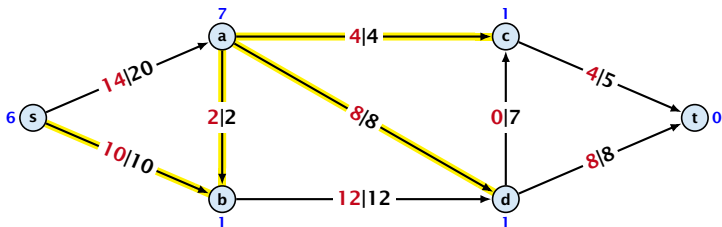
saturating push



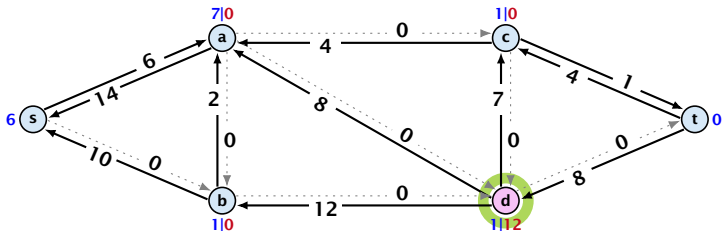
Preflow Push



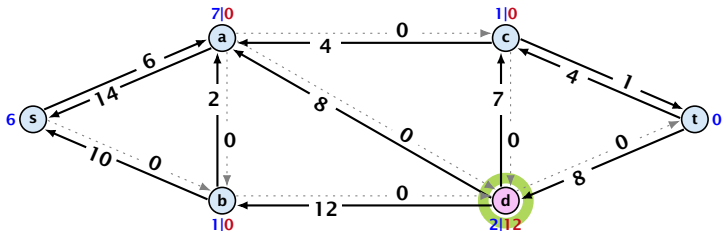
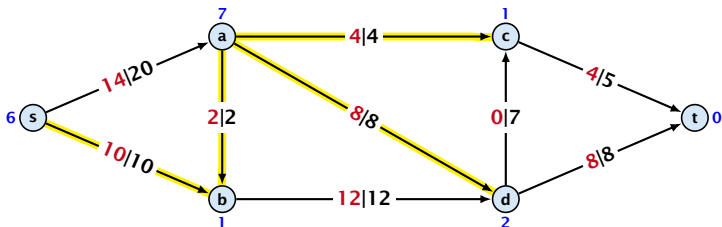
Preflow Push



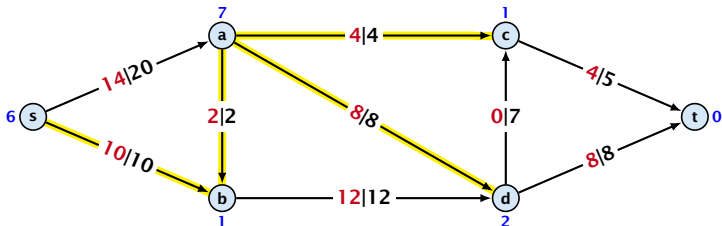
relabel to 2



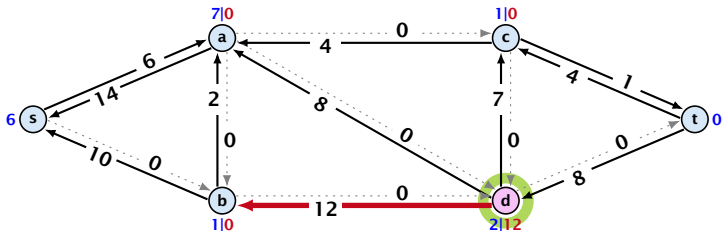
Preflow Push



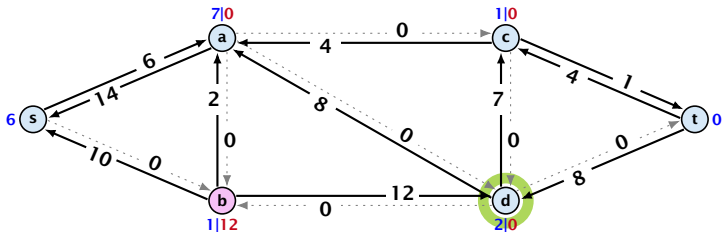
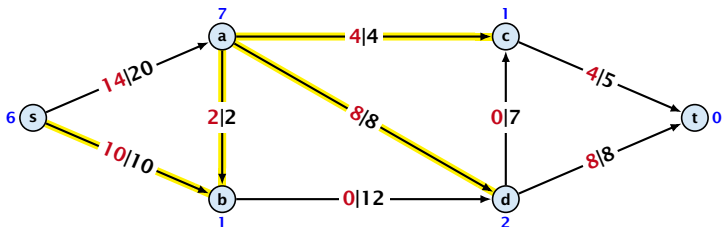
Preflow Push



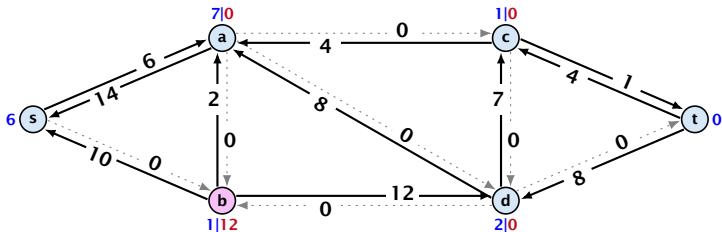
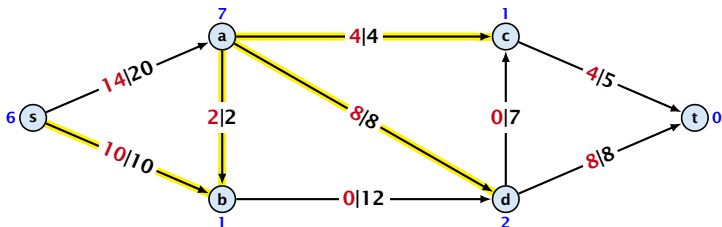
satürating and deactivating push



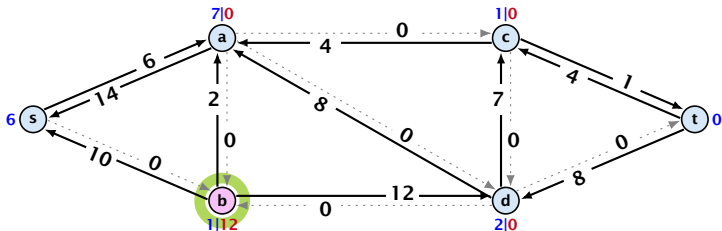
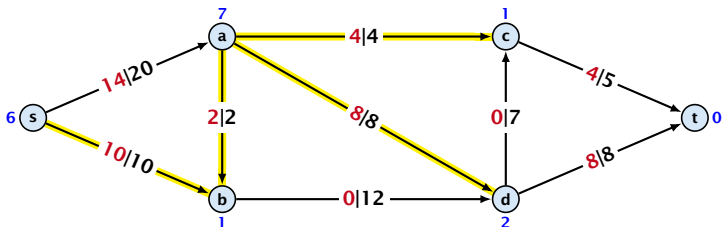
Preflow Push



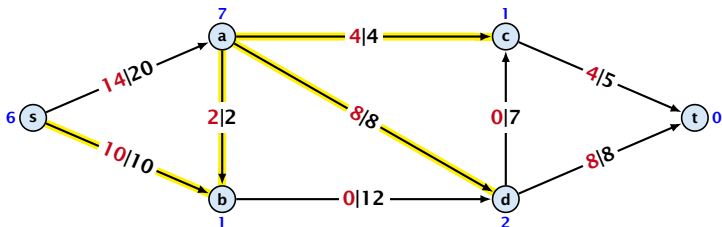
Preflow Push



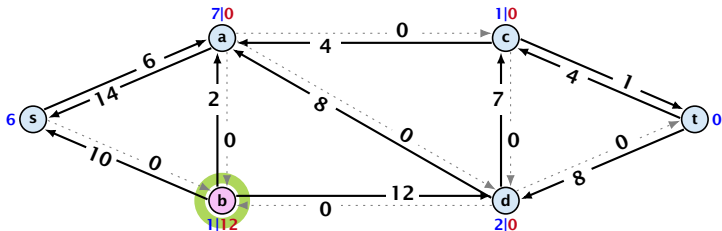
Preflow Push



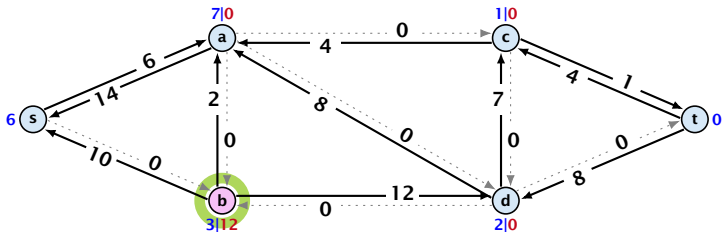
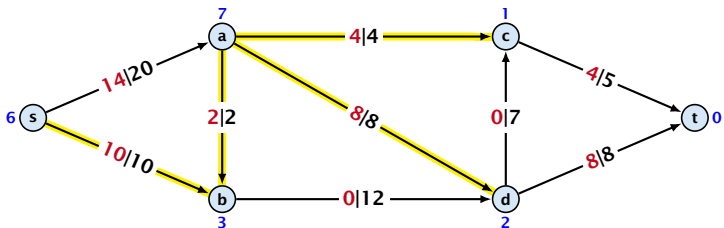
Preflow Push



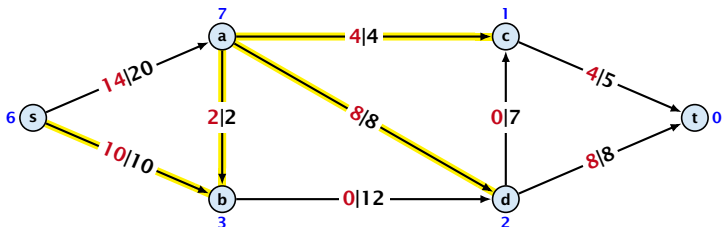
relabel to 3



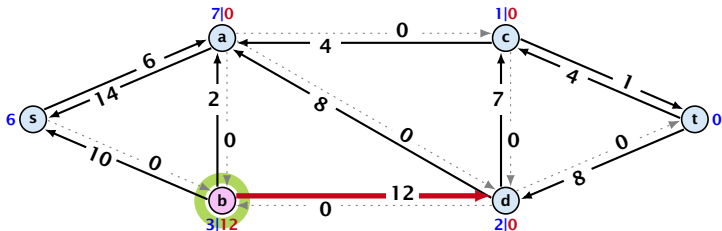
Preflow Push



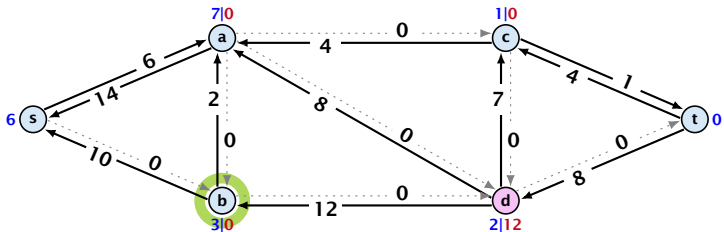
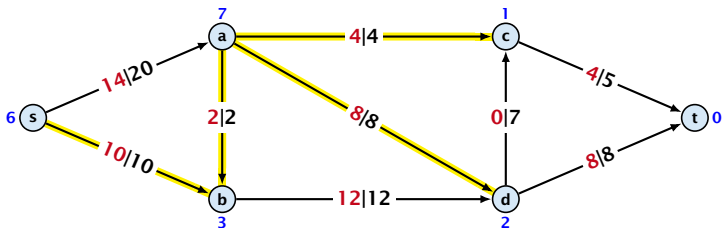
Preflow Push



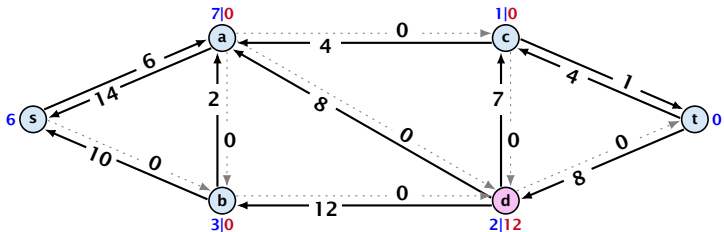
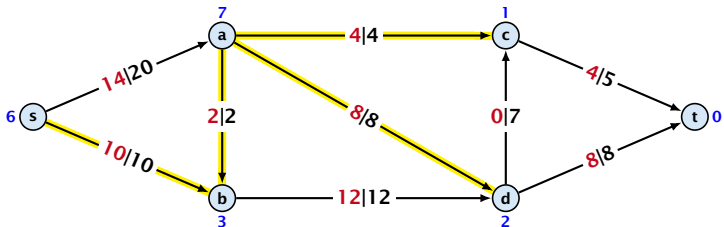
saturation and deactivating push



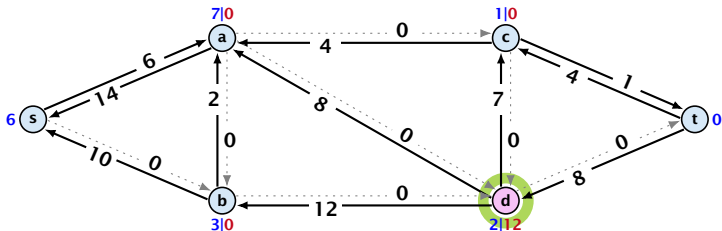
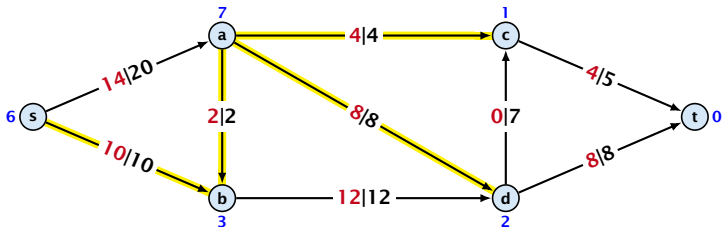
Preflow Push



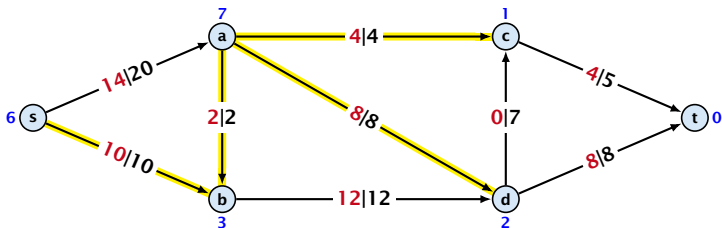
Preflow Push



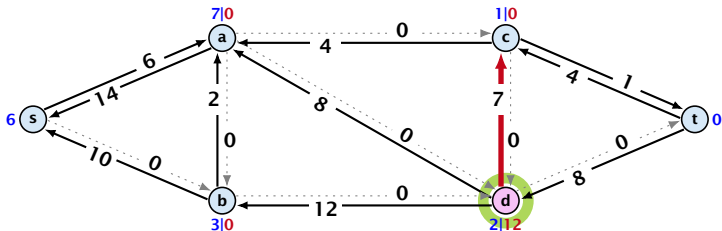
Preflow Push



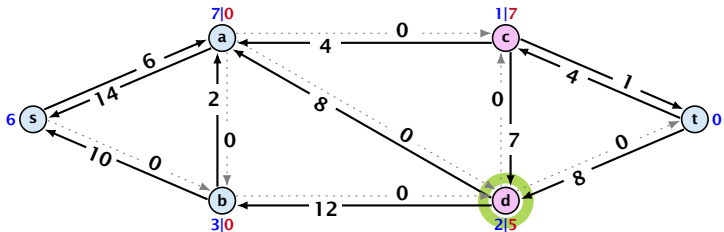
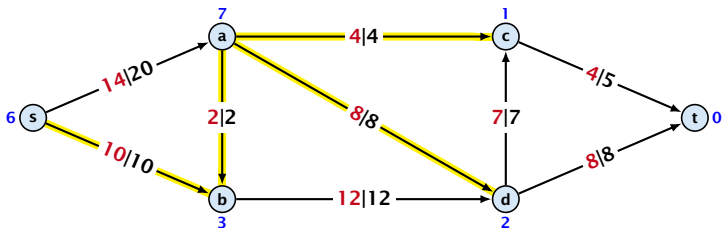
Preflow Push



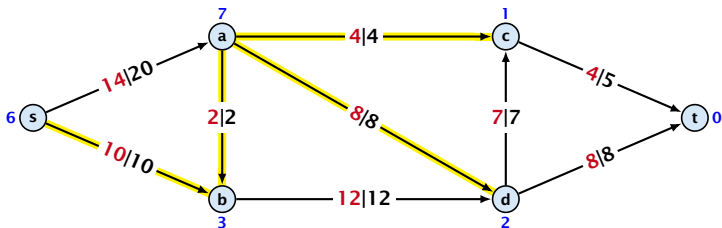
saturating push



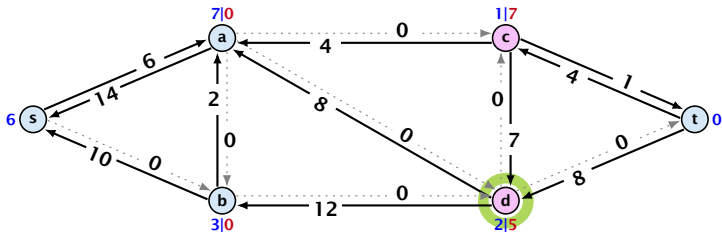
Preflow Push



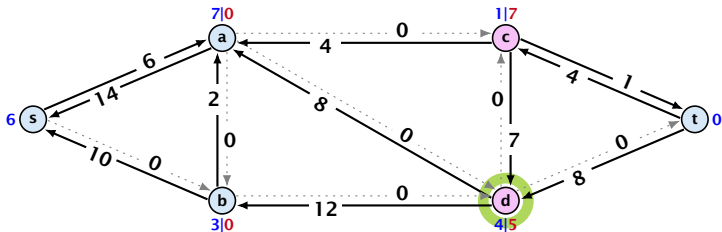
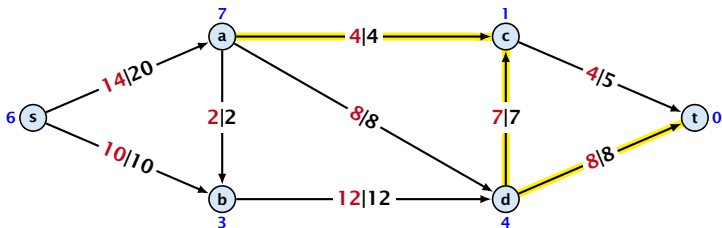
Preflow Push



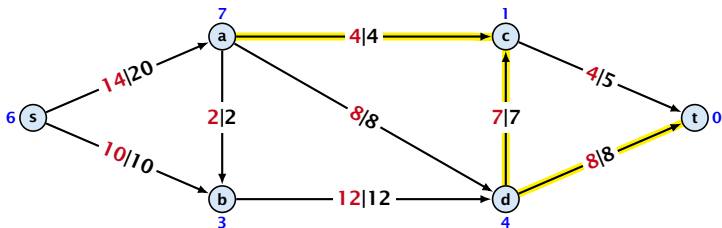
relabel to 4



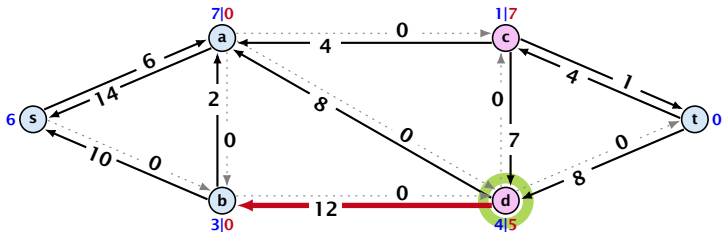
Preflow Push



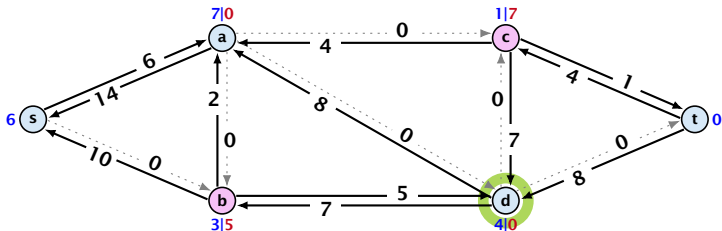
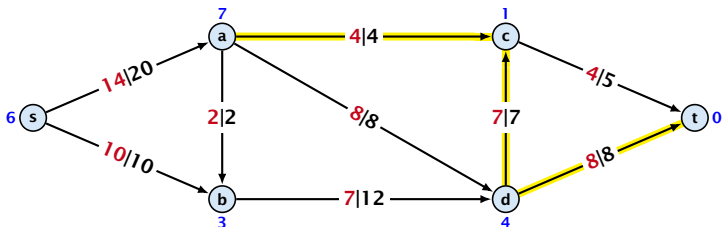
Preflow Push



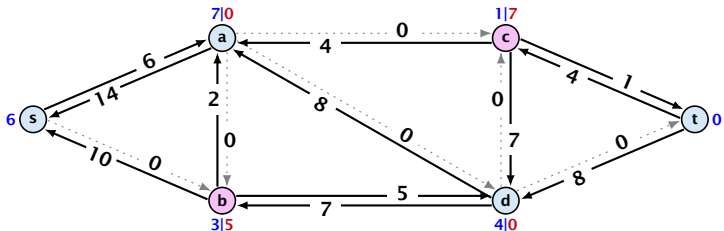
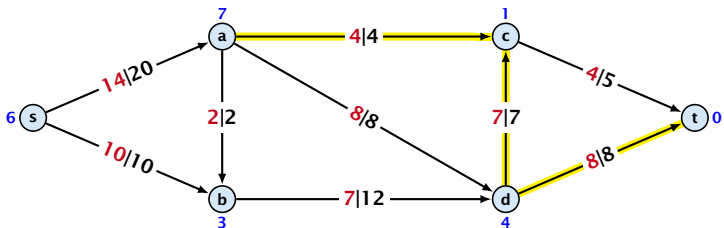
deactivating push



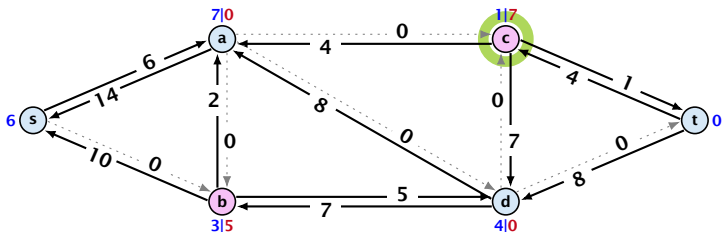
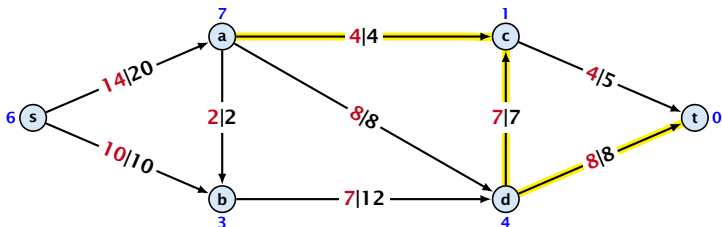
Preflow Push



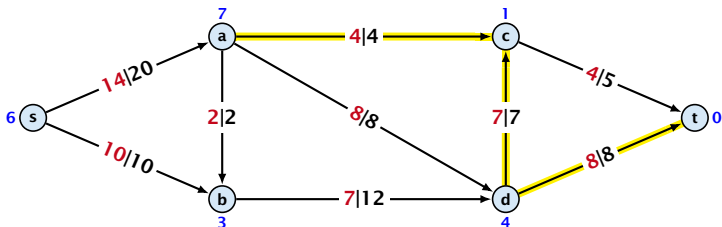
Preflow Push



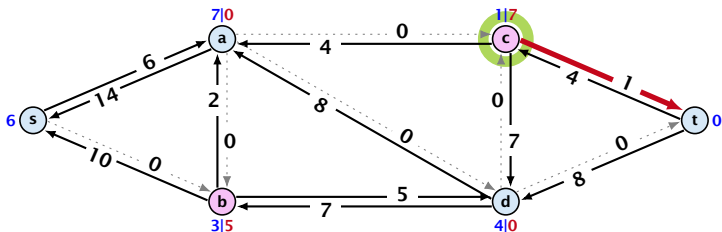
Preflow Push



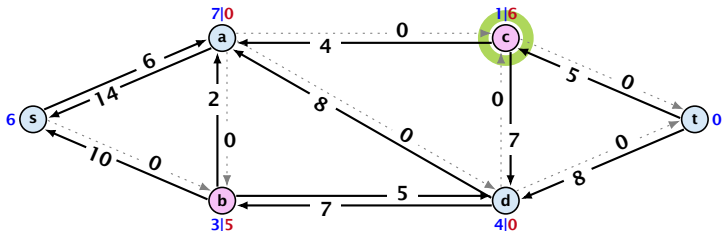
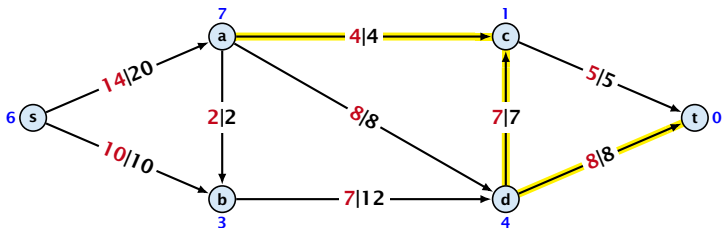
Preflow Push



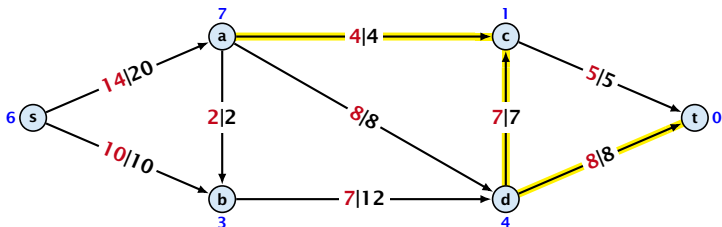
saturation push



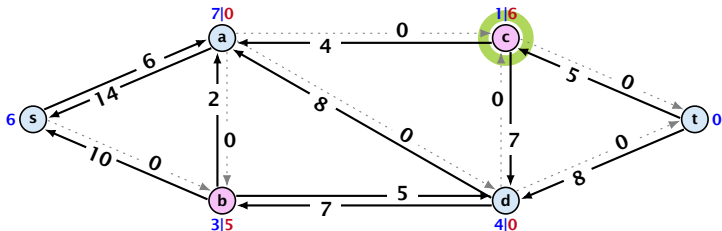
Preflow Push



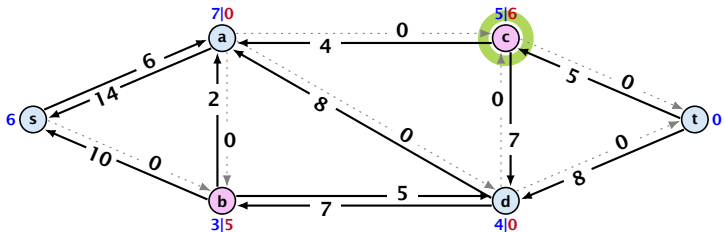
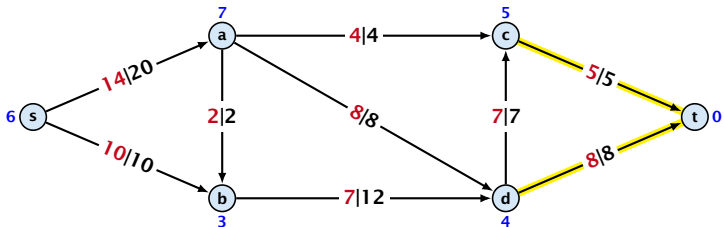
Preflow Push



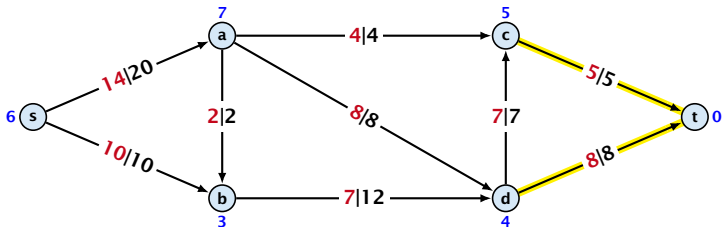
relabel to 5



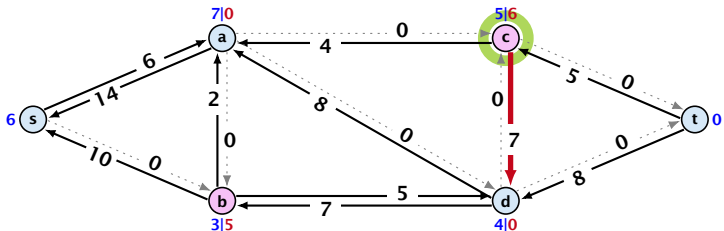
Preflow Push



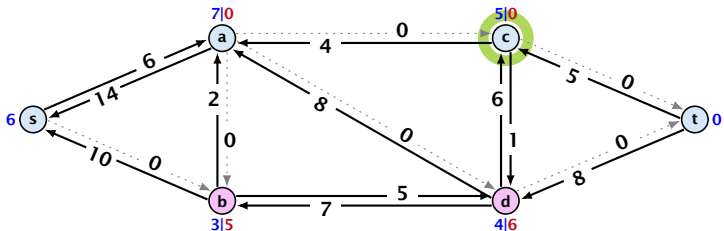
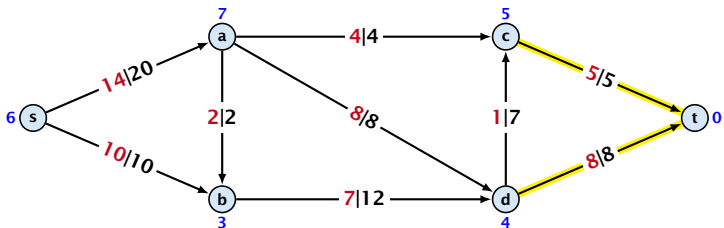
Preflow Push



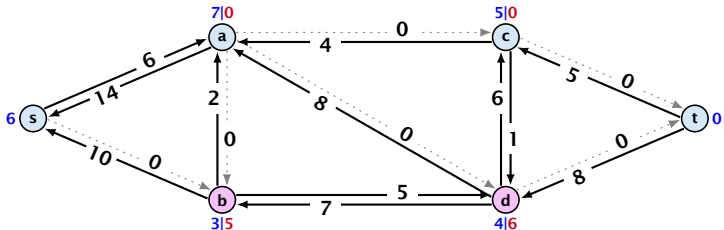
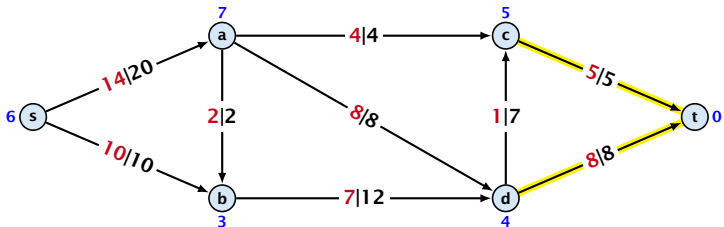
deactivating push



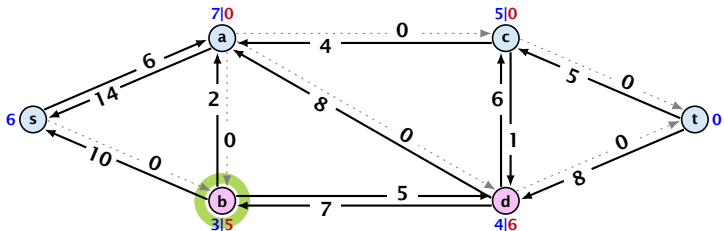
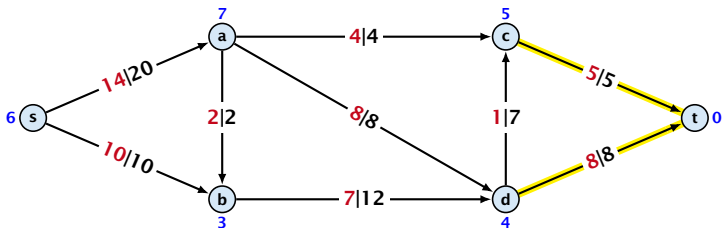
Preflow Push



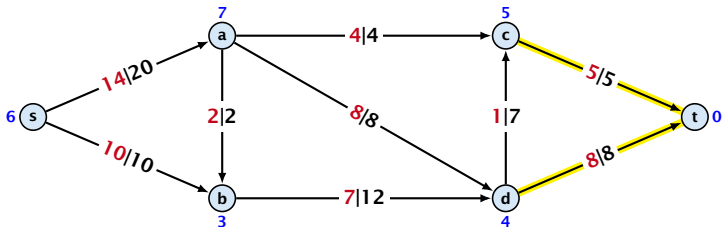
Preflow Push



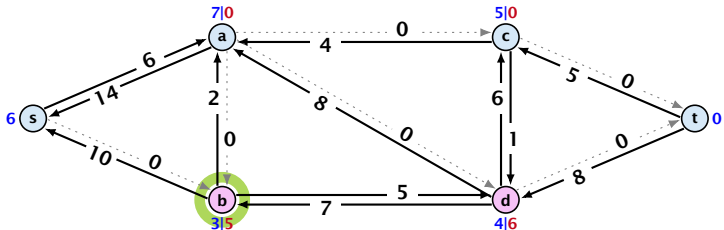
Preflow Push



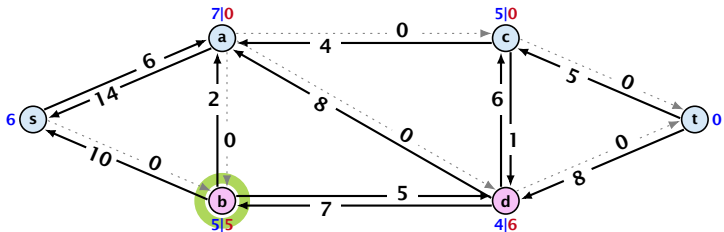
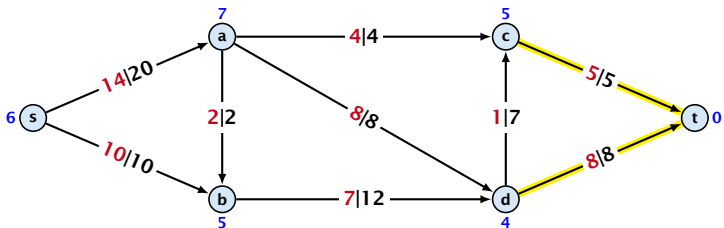
Preflow Push



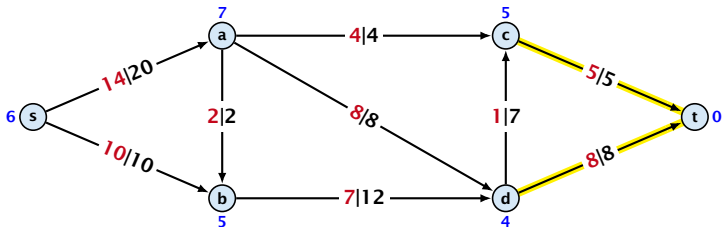
relabel to 5



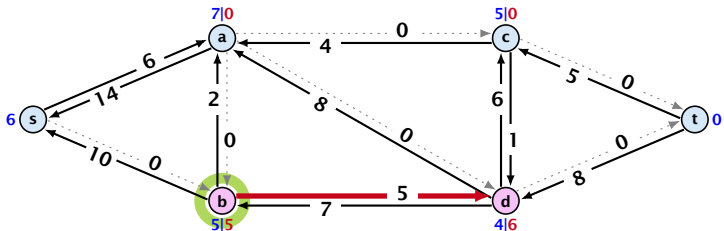
Preflow Push



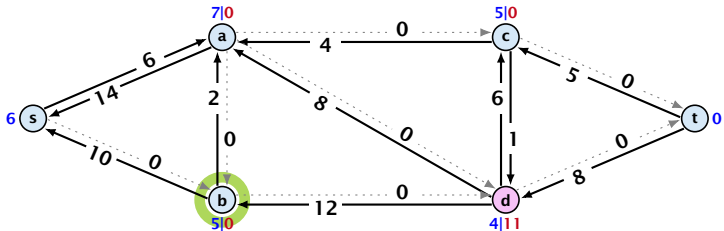
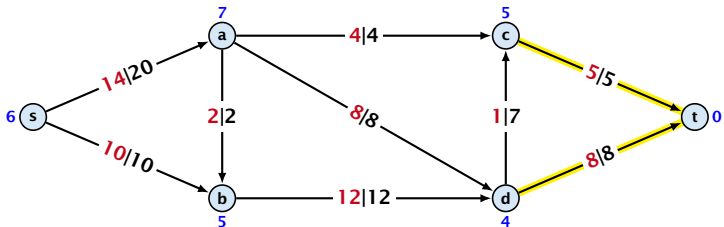
Preflow Push



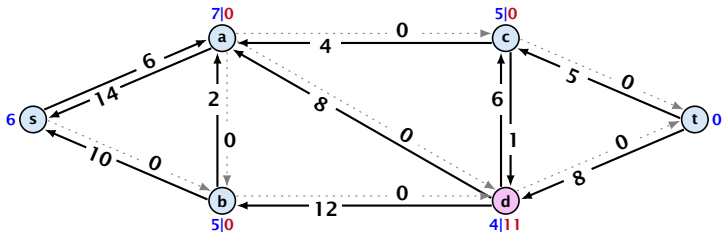
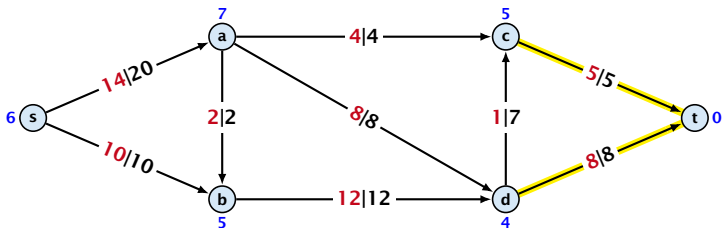
satürating and deactivating push



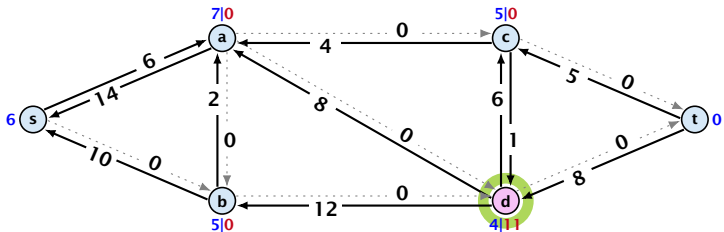
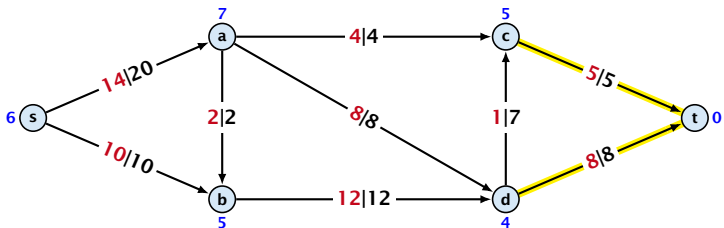
Preflow Push



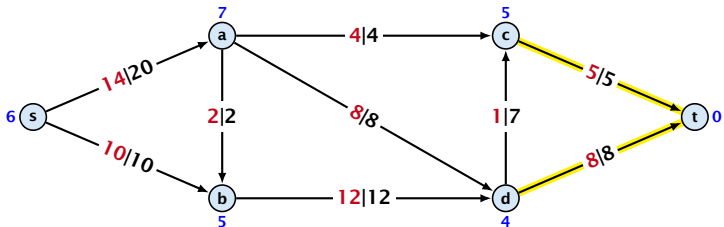
Preflow Push



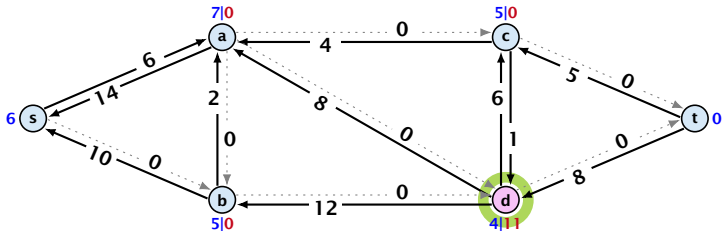
Preflow Push



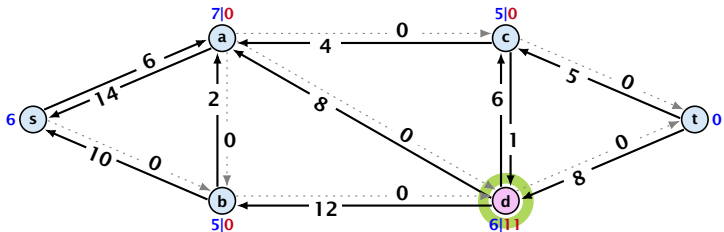
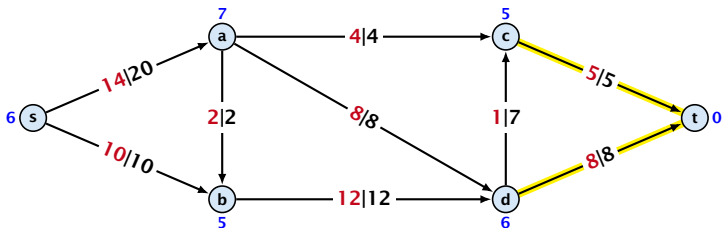
Preflow Push



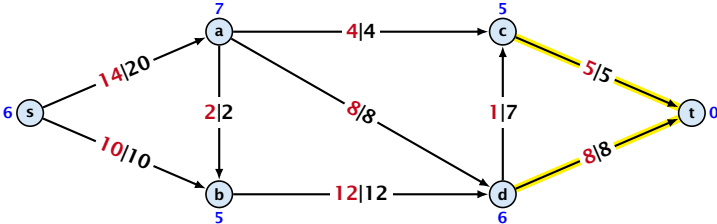
relabel to 6



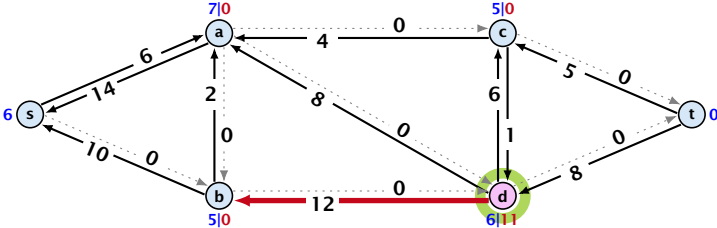
Preflow Push



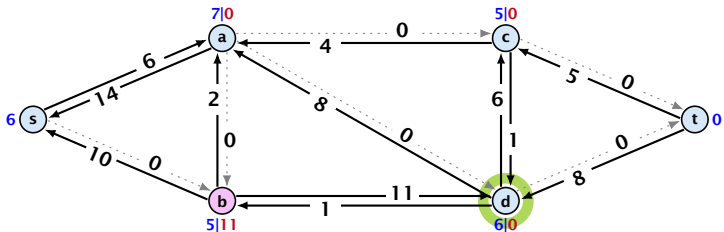
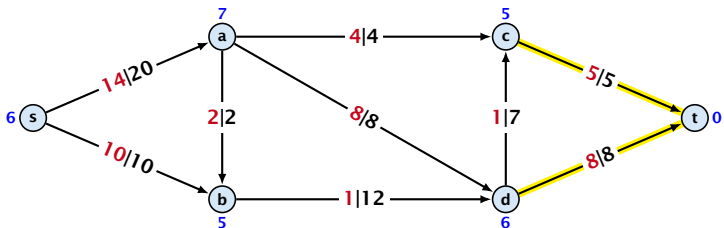
Preflow Push



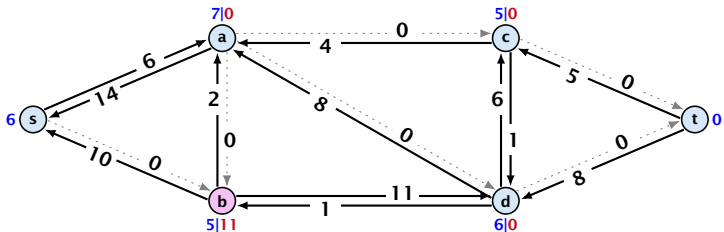
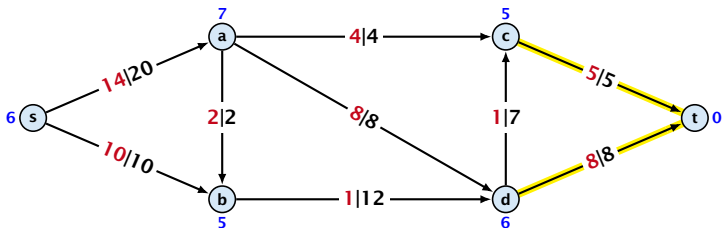
deactivating push



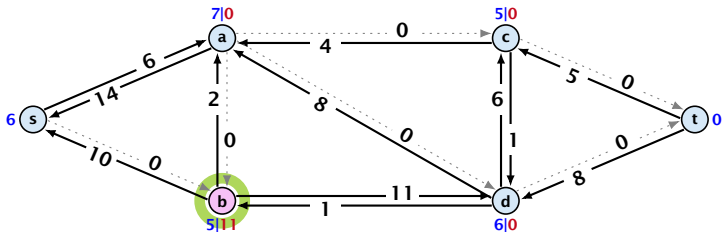
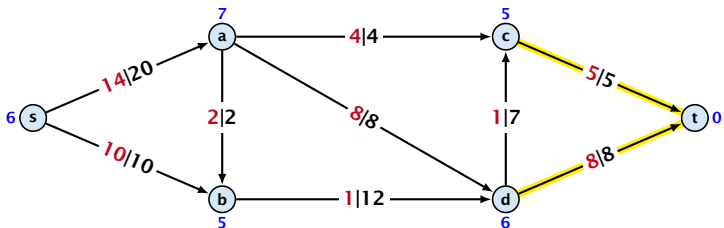
Preflow Push



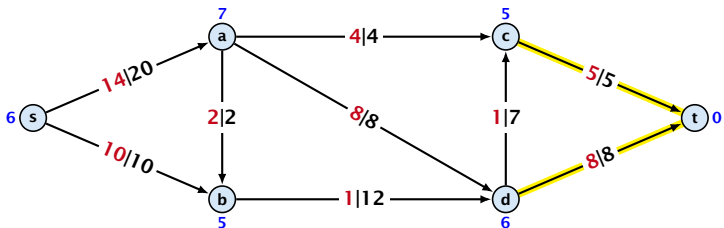
Preflow Push



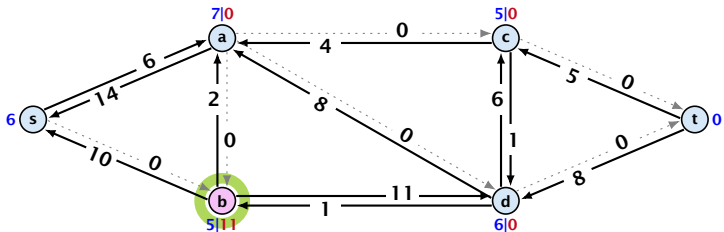
Preflow Push



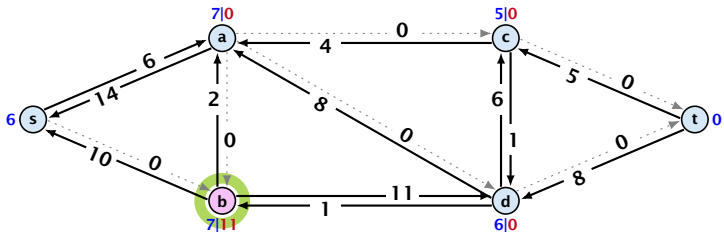
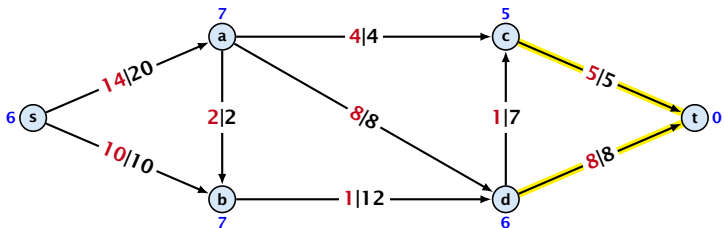
Preflow Push



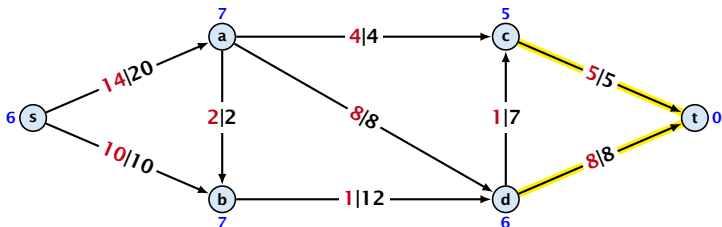
relabel to 7



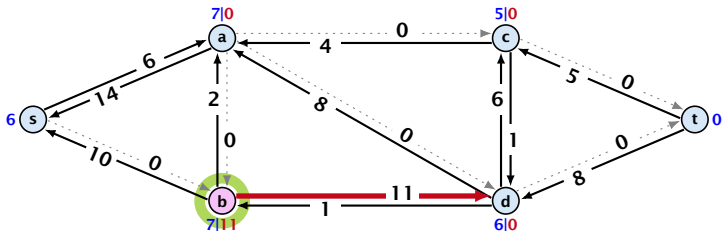
Preflow Push



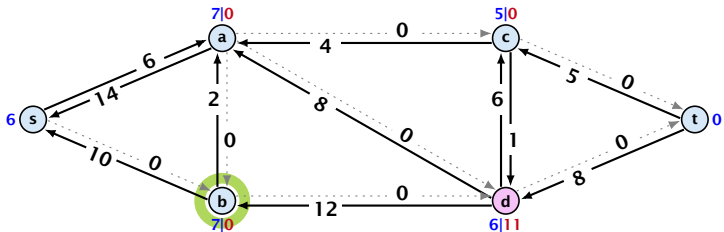
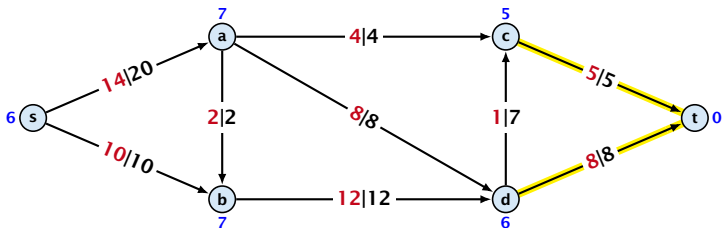
Preflow Push



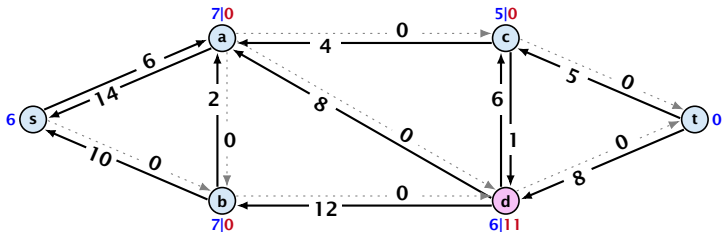
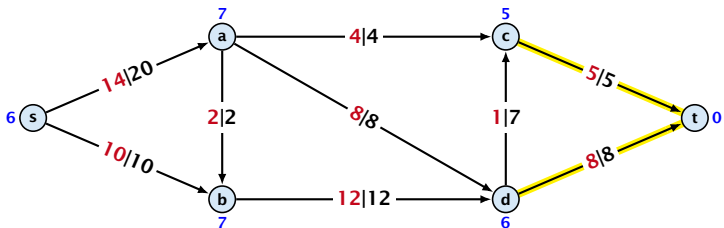
satürating and deactivating push



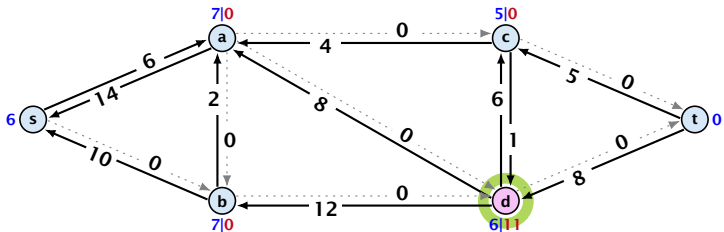
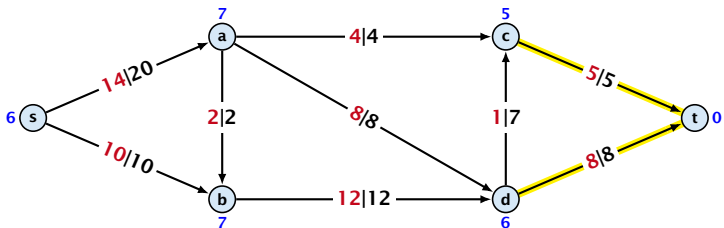
Preflow Push



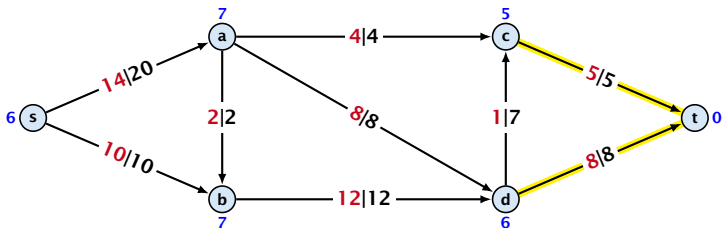
Preflow Push



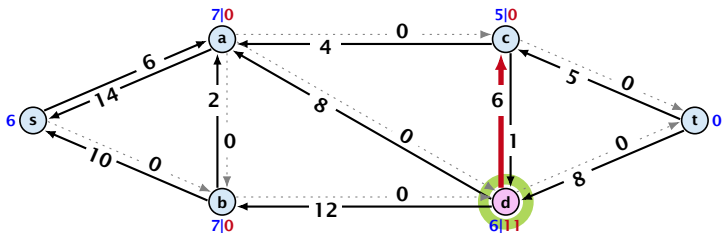
Preflow Push



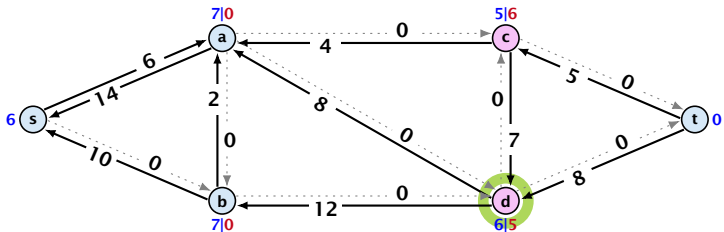
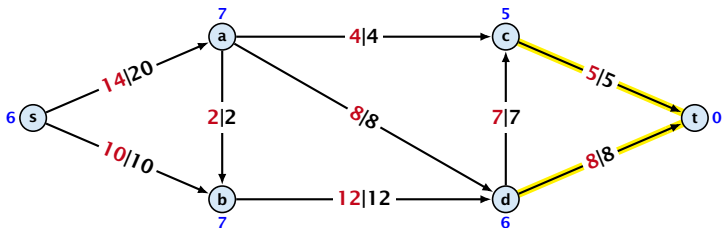
Preflow Push



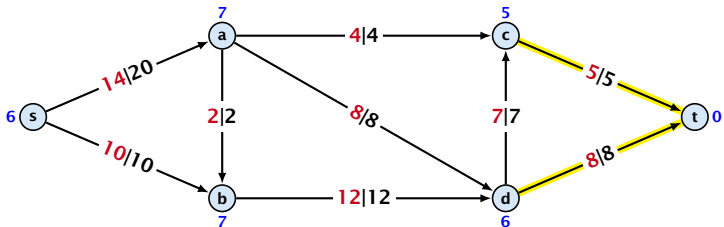
satürating push



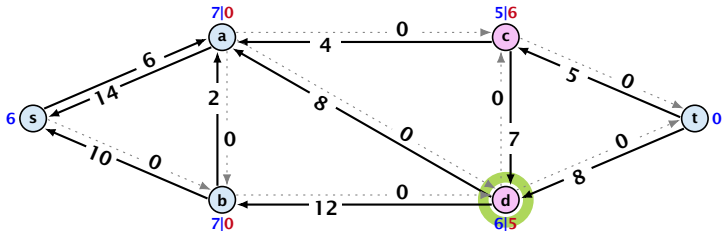
Preflow Push



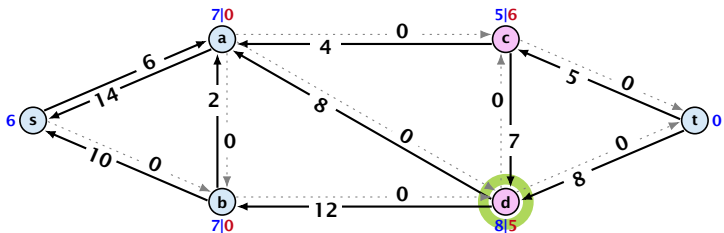
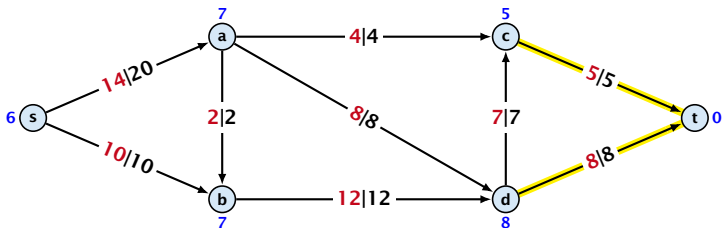
Preflow Push



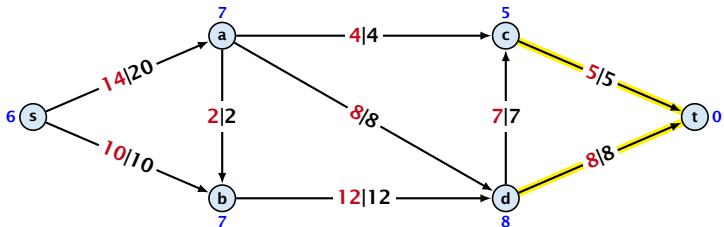
relabel to 8



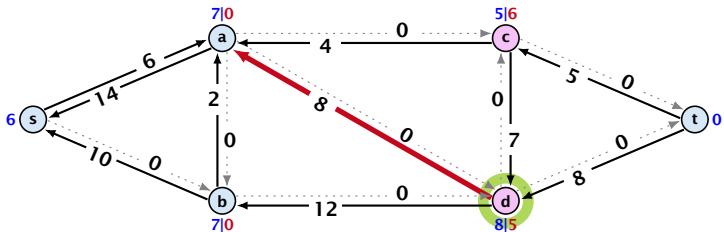
Preflow Push



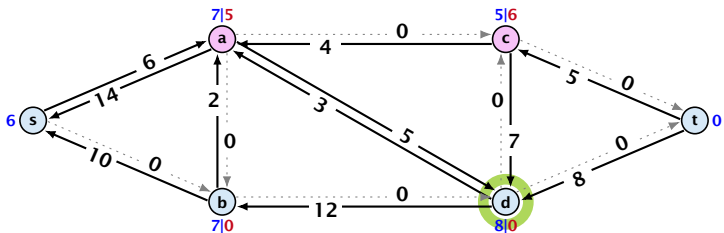
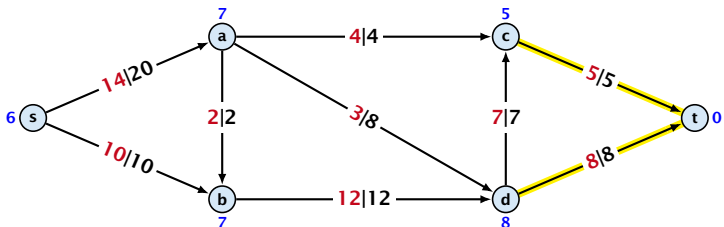
Preflow Push



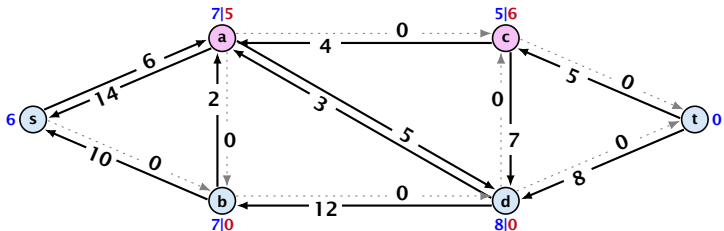
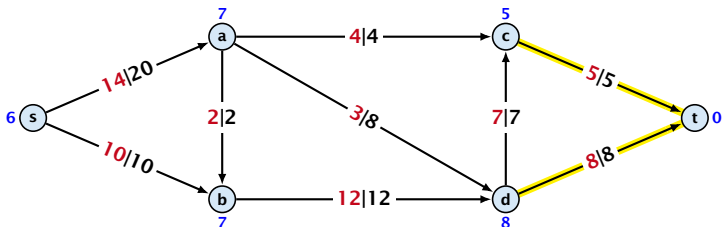
deactivating push



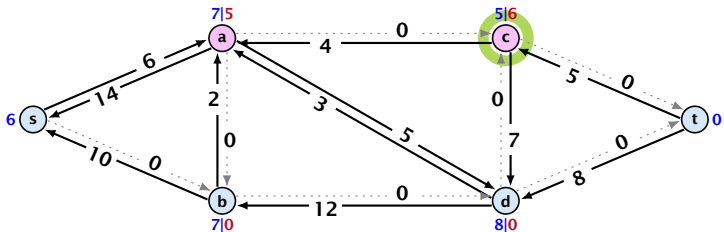
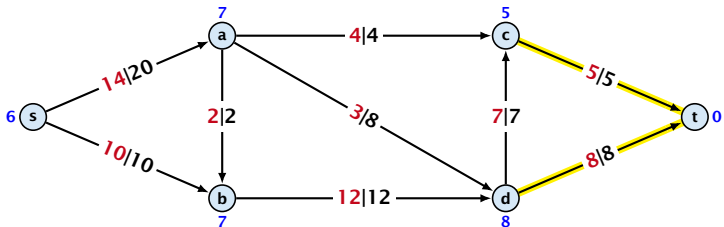
Preflow Push



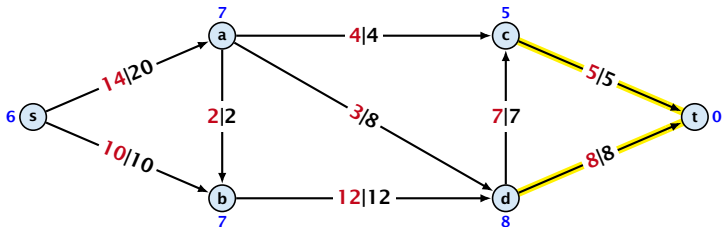
Preflow Push



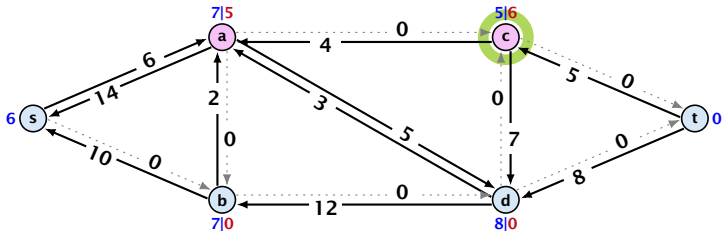
Preflow Push



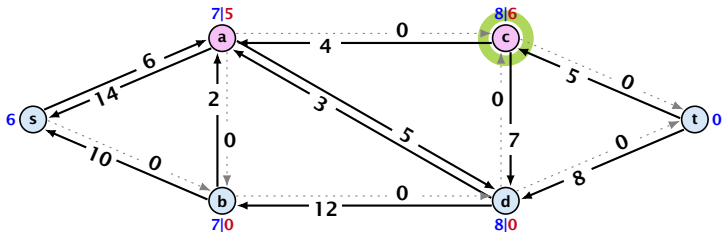
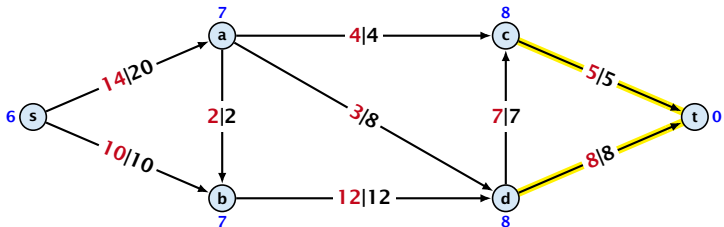
Preflow Push



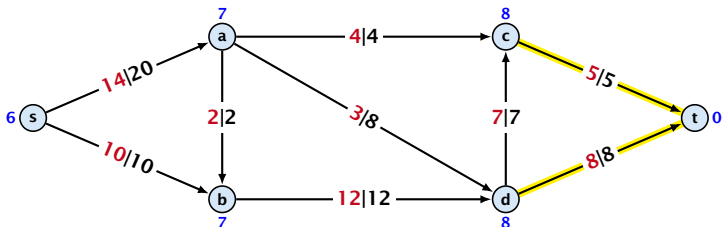
relabel to 8



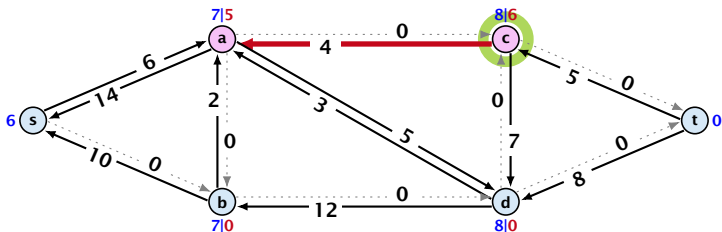
Preflow Push



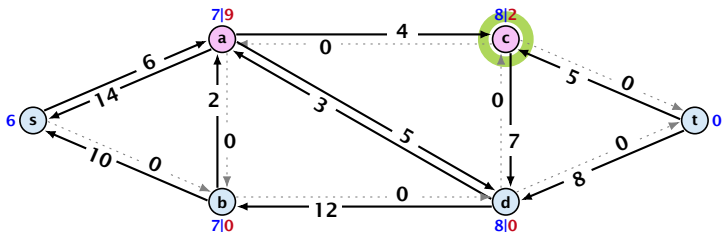
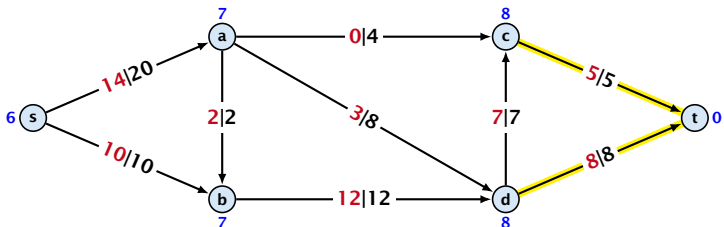
Preflow Push



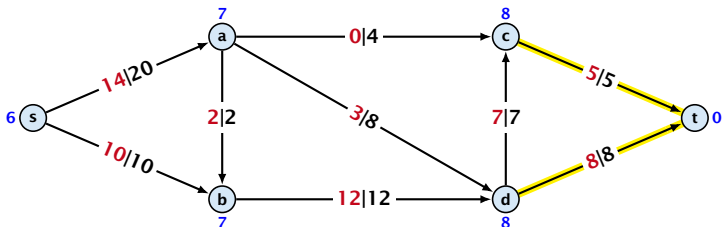
satürating push



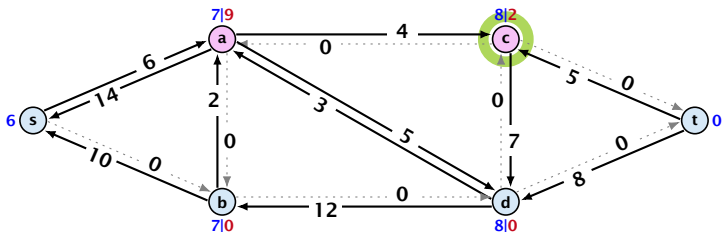
Preflow Push



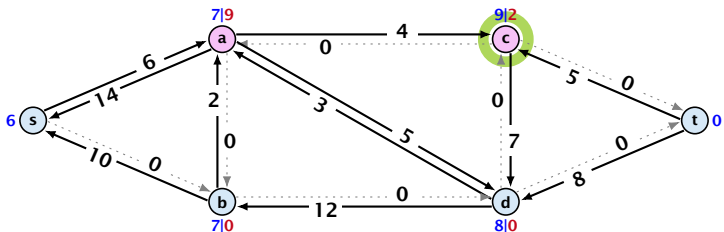
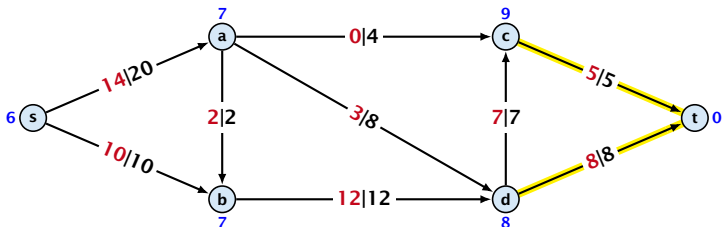
Preflow Push



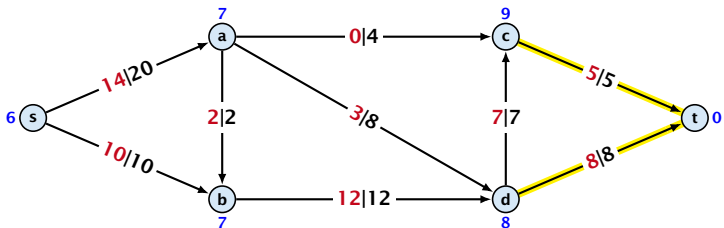
relabel to 9



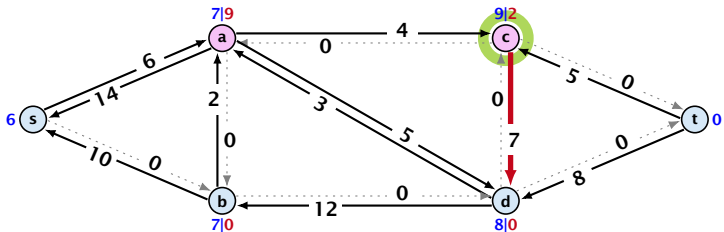
Preflow Push



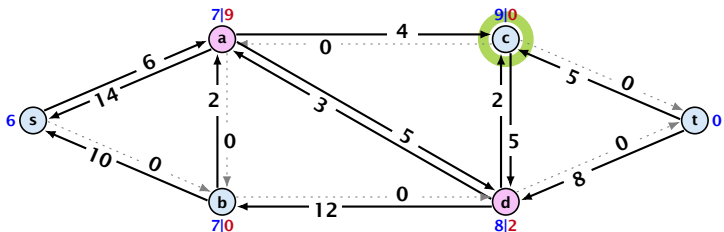
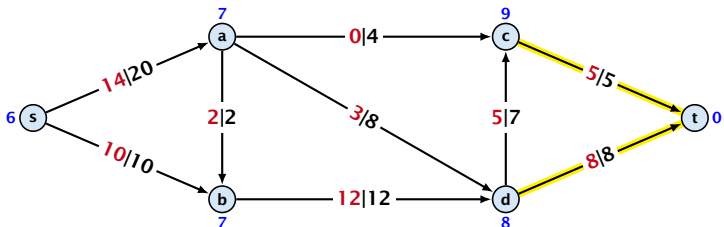
Preflow Push



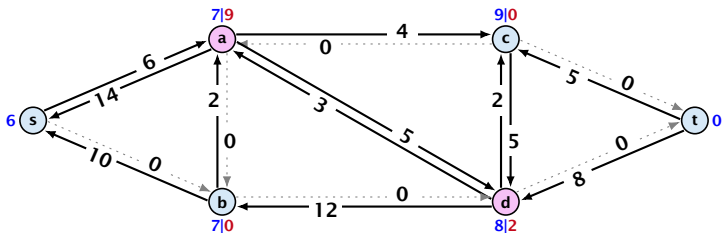
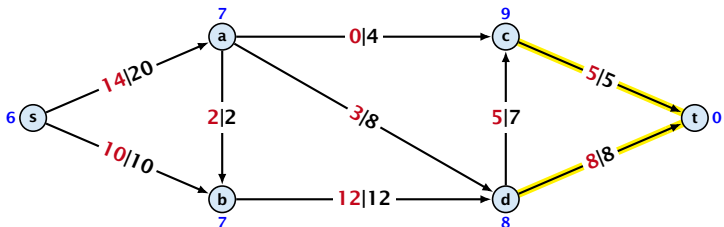
deactivating push



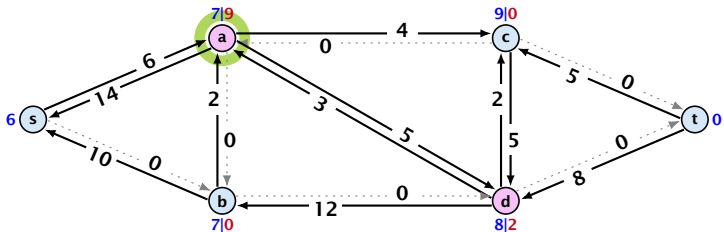
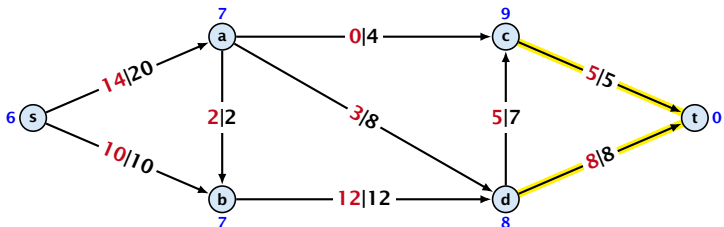
Preflow Push



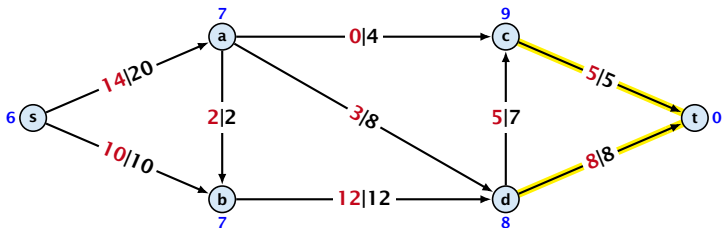
Preflow Push



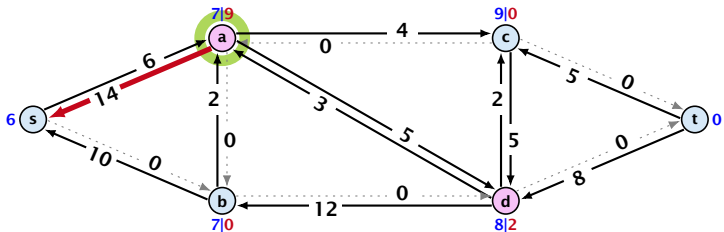
Preflow Push



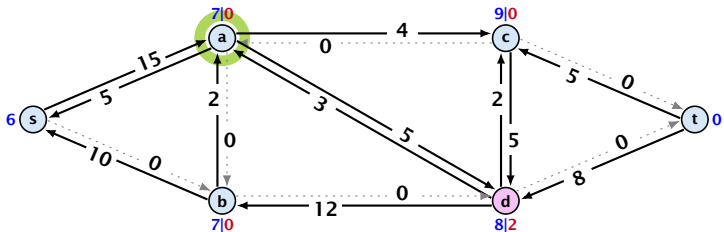
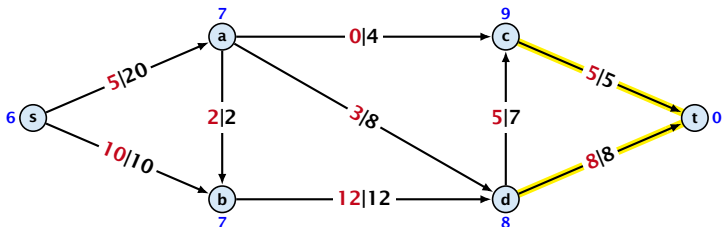
Preflow Push



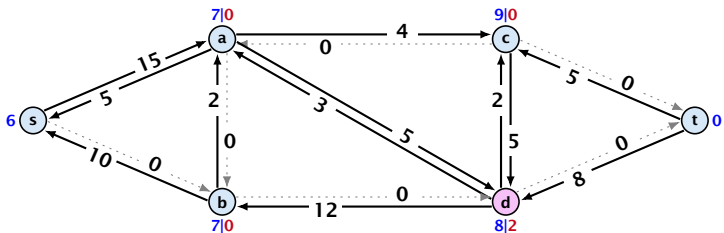
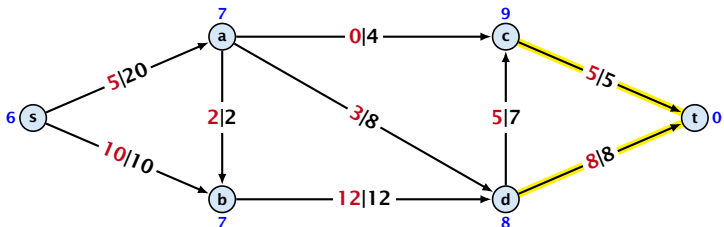
deactivating push



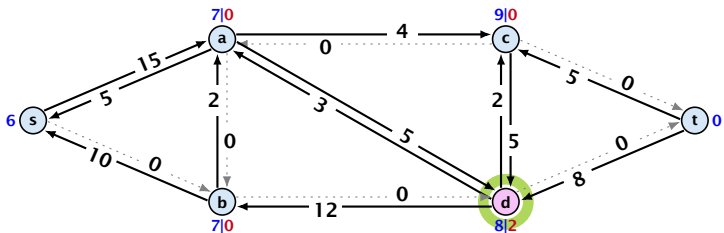
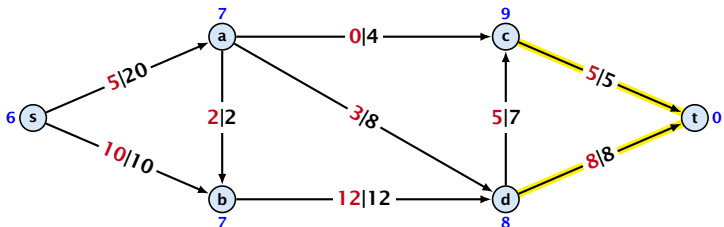
Preflow Push



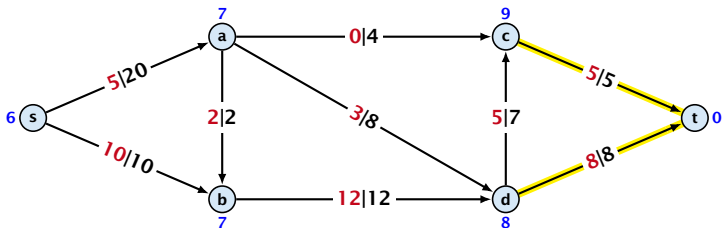
Preflow Push



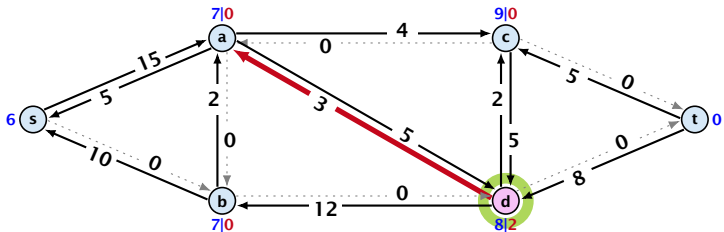
Preflow Push



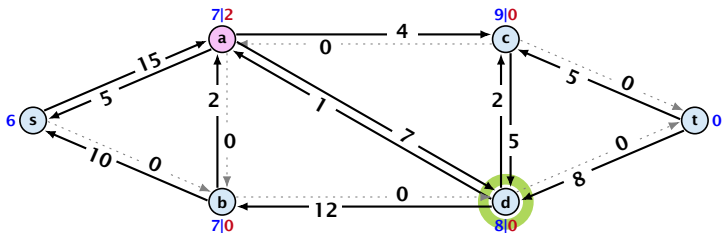
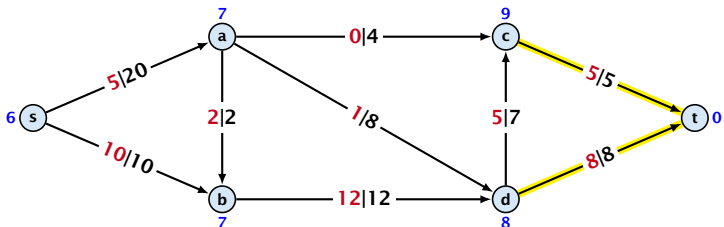
Preflow Push



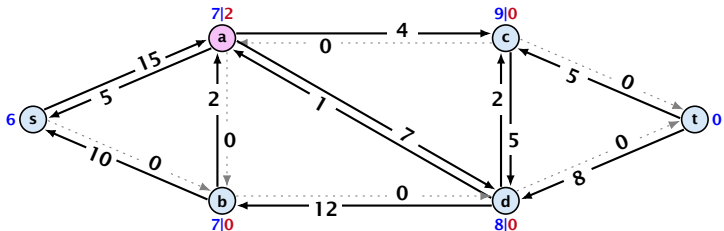
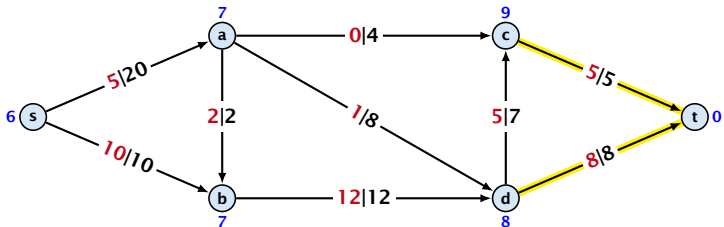
deactivating push



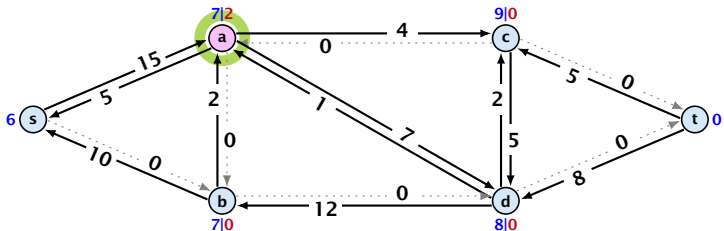
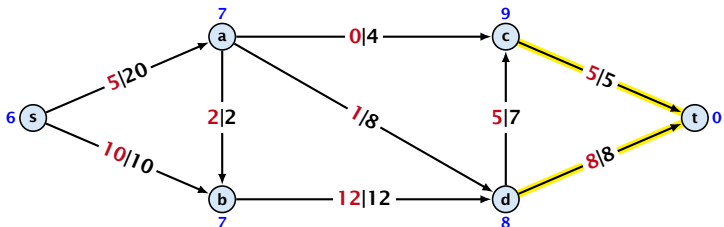
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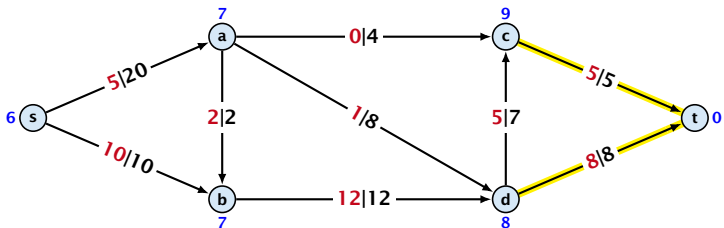
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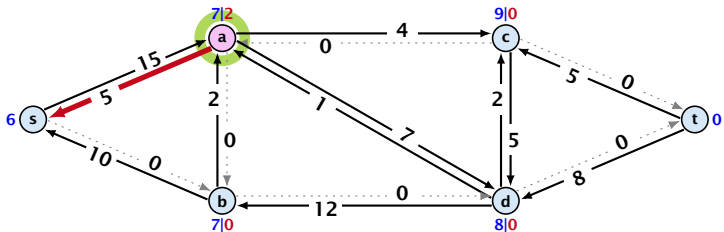
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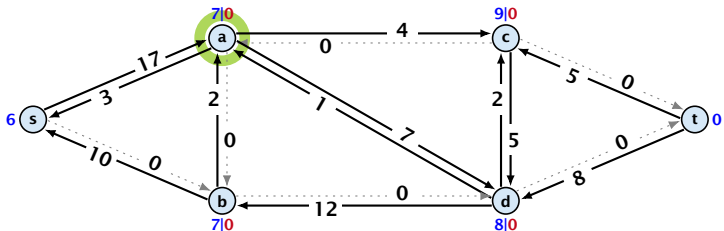
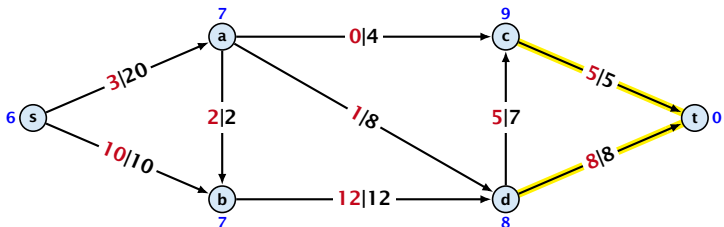
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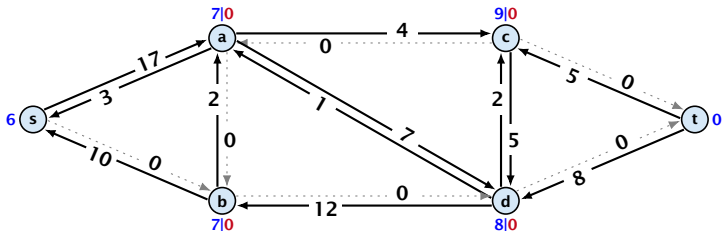
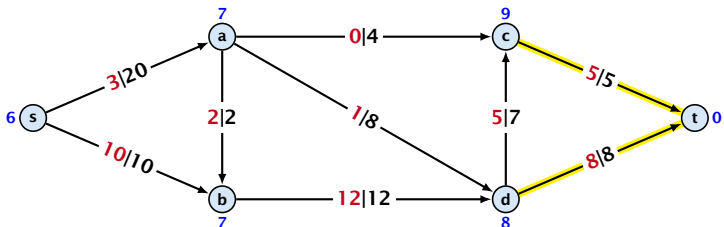
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Preflow Push



Preflow Push



Analysis

Lemma 69

An active node has a path to s in the residual graph.

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Proof.

- ▶ Let A denote the set of nodes that can reach s , and let B denote the remaining nodes. Note that $s \in A$.

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An active node has a path to s in the residual graph.

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- ▶ In the following we show that a node $b \in B$ has excess flow $f(b) = 0$ which gives the lemma.

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- ▶ Let A denote the set of nodes that can reach s , and let B denote the remaining nodes. Note that $s \in A$.
- ▶ In the following we show that a node $b \in B$ has excess flow $f(b) = 0$ which gives the lemma.
- ▶ In the residual graph there are no edges into A , and, hence, no edges leaving A /entering B can carry any flow.

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An active node has a path to s in the residual graph.

Proof.

- ▶ Let A denote the set of nodes that can reach s , and let B denote the remaining nodes. Note that $s \in A$.
- ▶ In the following we show that a node $b \in B$ has excess flow $f(b) = 0$ which gives the lemma.
- ▶ In the residual graph there are no edges into A , and, hence, no edges leaving A /entering B can carry any flow.
- ▶ Let $f(B) = \sum_{v \in B} f(v)$ be the excess flow of all nodes in B .

Let $f : E \rightarrow \mathbb{R}_0^+$ be a preflow. We introduce the notation

$$f(x, y) = \begin{cases} 0 & (x, y) \notin E \\ f((x, y)) & (x, y) \in E \end{cases}$$

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Hence, the excess flow $f(b)$ must be 0 for every node $b \in B$.

Analysis

Lemma 70

The label of a node cannot become larger than $2n - 1$.

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Proof.

- ▶ When increasing the label at a node u there exists a path from u to s of length at most $n - 1$. Along each edge of the path the height/label can at most drop by 1 , and the label of the source is n .

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There are only $\mathcal{O}(n^2)$ relabel operations.

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Lemma 72

The number of *saturating pushes* performed is at most $\mathcal{O}(mn)$.

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- ▶ For a push from v to u the edge (v, u) must become admissible. The label of v must increase by at least 2.
- ▶ Since the label of v is at most $2n - 1$, there are at most n pushes along (u, v) .

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The number of *deactivating pushes* performed is at most $\mathcal{O}(n^2m)$.

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- ▶ A saturating push increases Φ by $\leq 2n$ (when the target node becomes active it may contribute at most $2n$ to the sum).
- ▶ A relabel increases Φ by at most 1 .
- ▶ A deactivating push decreases Φ by at least 1 as the node that is pushed from becomes inactive and has a label that is strictly larger than the target.
- ▶ Hence,

$$\begin{aligned} \#deactivating_pushes &\leq \#relabels + 2n \cdot \#saturating_pushes \\ &\leq \mathcal{O}(n^2m) . \end{aligned}$$

Theorem 74

There is an implementation of the generic push relabel algorithm with running time $\mathcal{O}(n^2m)$.

Analysis

Proof:

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For every node maintain a list of admissible edges starting at that node. Further maintain a list of active nodes.

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A push along an edge (u, v) can be performed in constant time

- ▶ check whether edge (v, u) needs to be added to G_f

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- ▶ check whether (u, v) needs to be deleted (saturating push)

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A relabel at a node u can be performed in time $\mathcal{O}(n)$

- ▶ check for all outgoing edges if they become admissible

Analysis

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A relabel at a node u can be performed in time $\mathcal{O}(n)$

- ▶ check for all outgoing edges if they become admissible
- ▶ check for all incoming edges if they become non-admissible

Analysis

For special variants of push relabel algorithms we organize the neighbours of a node into a linked list (possible neighbours in the residual graph G_f). Then we use the discharge-operation:

Algorithm 2 discharge(u)

```
1: while  $u$  is active do  
2:    $v \leftarrow u.current\text{-neighbour}$   
3:   if  $v = \text{null}$  then  
4:     relabel( $u$ )  
5:      $u.current\text{-neighbour} \leftarrow u.neighbour\text{-list-head}$   
6:   else  
7:     if  $(u, v)$  admissible then push( $u, v$ )  
8:     else  $u.current\text{-neighbour} \leftarrow v.next\text{-in-list}$ 
```

Note that $u.current\text{-neighbour}$ is a global variable. It is only changed within the discharge routine, but keeps its value between consecutive calls to discharge.

Lemma 75

If $v = \text{null}$ in Line 3, then there is no outgoing admissible edge from u .

Proof.

- ▶ While pushing from u the current-neighbour pointer is only advanced if the current edge is not admissible.
- ▶ The only thing that could make the edge admissible again would be a relabel at u .
- ▶ If we reach the end of the list ($v = \text{null}$) all edges are not admissible. □

This shows that $\text{discharge}(u)$ is correct, and that we can perform a relabel in Line 4.

13.2 Relabel to Front

Algorithm 1 relabel-to-front(G, s, t)

```
1: initialize preflow
2: initialize node list  $L$  containing  $V \setminus \{s, t\}$  in any order
3: foreach  $u \in V \setminus \{s, t\}$  do
4:    $u.current\text{-neighbour} \leftarrow u.neighbour\text{-list}\text{-head}$ 
5:  $u \leftarrow L.head$ 
6: while  $u \neq \text{null}$  do
7:    $old\text{-height} \leftarrow \ell(u)$ 
8:   discharge( $u$ )
9:   if  $\ell(u) > old\text{-height}$  then // relabel happened
10:    move  $u$  to the front of  $L$ 
11:    $u \leftarrow u.next$ 
```

13.2 Relabel to Front

Lemma 76 (Invariant)

In Line 6 of the relabel-to-front algorithm the following invariant holds.

- 1. The sequence L is topologically sorted w.r.t. the set of admissible edges; this means for an admissible edge (x, y) the node x appears before y in sequence L .*
- 2. No node before u in the list L is active.*

Proof:

▶ Initialization:

1. In the beginning s has label $n \geq 2$, and all other nodes have label 0. Hence, no edge is admissible, which means that any ordering L is permitted.
2. We start with u being the head of the list; hence no node before u can be active

▶ Maintenance:

1.
 - ▶ Pushes do not create any new admissible edges. Therefore, if `discharge()` does not relabel u , L is still topologically sorted.
 - ▶ After relabeling, u cannot have admissible incoming edges as such an edge (x, u) would have had a difference $\ell(x) - \ell(u) \geq 2$ before the re-labeling (such edges do not exist in the residual graph).
Hence, moving u to the front does not violate the sorting property for any edge; however it fixes this property for all admissible edges leaving u that were generated by the relabeling.

13.2 Relabel to Front

Proof:

► Maintenance:

2. If we do a relabel there is nothing to prove because the only node before u' (u in the next iteration) will be the current u ; the discharge(u) operation only terminates when u is not active anymore.

For the case that we do not relabel, observe that the only way a predecessor could be active is that we push flow to it via an admissible arc. However, all admissible arc point to successors of u .

Note that the invariant means that for $u = \text{null}$ we have a preflow with a valid labelling that does not have active nodes. This means we have a maximum flow.

13.2 Relabel to Front

Lemma 77

There are at most $\mathcal{O}(n^3)$ calls to $\text{discharge}(u)$.

Every discharge operation without a relabel advances u (the current node within list L). Hence, if we have n discharge operations without a relabel we have $u = \text{null}$ and the algorithm terminates.

Therefore, the number of calls to discharge is at most $n(\#\text{relabels} + 1) = \mathcal{O}(n^3)$.

13.2 Relabel to Front

Lemma 78

The cost for all relabel-operations is only $\mathcal{O}(n^2)$.

A relabel-operation at a node is constant time (increasing the label and resetting *u .current-neighbour*). In total we have $\mathcal{O}(n^2)$ relabel-operations.

13.2 Relabel to Front

Recall that a saturating push operation ($\min\{c_f(e), f(u)\} = c_f(e)$) can also be a deactivating push operation ($\min\{c_f(e), f(u)\} = f(u)$).

Lemma 79

*The cost for all saturating push-operations that are **not** deactivating is only $\mathcal{O}(mn)$.*

Note that such a push-operation leaves the node u active but makes the edge e disappear from the residual graph. Therefore the push-operation is immediately followed by an increase of the pointer $u.current-neighbour$.

This pointer can traverse the neighbour-list at most $\mathcal{O}(n)$ times (upper bound on number of relabels) and the neighbour-list has only $degree(u) + 1$ many entries (+1 for null-entry).

13.2 Relabel to Front

Lemma 80

The cost for all deactivating push-operations is only $\mathcal{O}(n^3)$.

A deactivating push-operation takes constant time and ends the current call to `discharge()`. Hence, there are only $\mathcal{O}(n^3)$ such operations.

Theorem 81

The push-relabel algorithm with the rule relabel-to-front takes time $\mathcal{O}(n^3)$.

13.3 Highest Label

Algorithm 1 highest-label(G, s, t)

- 1: initialize preflow
- 2: **foreach** $u \in V \setminus \{s, t\}$ **do**
- 3: $u.current-neighbour \leftarrow u.neighbour-list-head$
- 4: **while** \exists active node u **do**
- 5: select active node u with highest label
- 6: discharge(u)

13.3 Highest Label

Lemma 82

When using highest label the number of deactivating pushes is only $\mathcal{O}(n^3)$.

A push from a node on level ℓ can only “activate” nodes on levels strictly less than ℓ .

This means, after a deactivating push from u a relabel is required to make u active again.

Hence, after n deactivating pushes without an intermediate relabel there are no active nodes left.

Therefore, the number of deactivating pushes is at most $n(\#relabels + 1) = \mathcal{O}(n^3)$.

13.3 Highest Label

Since a discharge-operation is terminated by a deactivating push this gives an upper bound of $\mathcal{O}(n^3)$ on the number of discharge-operations.

The cost for relabels and saturating pushes can be estimated in exactly the same way as in the case of the generic push-relabel algorithm.

Question:

How do we find the next node for a discharge operation?

13.3 Highest Label

Maintain lists L_i , $i \in \{0, \dots, 2n\}$, where list L_i contains active nodes with label i (maintaining these lists induces only constant additional cost for every push-operation and for every relabel-operation).

After a discharge operation terminated for a node u with label k , traverse the lists L_k, L_{k-1}, \dots, L_0 , (in that order) until you find a non-empty list.

Unless the last (deactivating) push was to s or t the list $k-1$ must be non-empty (i.e., the search takes constant time).

13.3 Highest Label

Hence, the total time required for searching for active nodes is at most

$$\mathcal{O}(n^3) + n(\#deactivating-pushes-to-s-or-t)$$

Lemma 83

The number of deactivating pushes to s or t is at most $\mathcal{O}(n^2)$.

With this lemma we get

Theorem 84

The push-relabel algorithm with the rule highest-label takes time $\mathcal{O}(n^3)$.

13.3 Highest Label

Proof of the Lemma.

- ▶ We only show that the number of pushes to the source is at most $\mathcal{O}(n^2)$. A similar argument holds for the target.
- ▶ After a node v (which must have $\ell(v) = n + 1$) made a deactivating push to the source there needs to be another node whose label is increased from $\leq n + 1$ to $n + 2$ before v can become active again.
- ▶ This happens for every push that v makes to the source. Since, every node can pass the threshold $n + 2$ at most once, v can make at most n pushes to the source.
- ▶ As this holds for every node the total number of pushes to the source is at most $\mathcal{O}(n^2)$.

Problem Definition:

$$\begin{array}{ll} \min & \sum_e c(e)f(e) \\ \text{s.t.} & \forall e \in E: 0 \leq f(e) \leq u(e) \\ & \forall v \in V: f(v) = b(v) \end{array}$$

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- ▶ $G = (V, E)$ is a **directed graph**.

Mincost Flow

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- ▶ $G = (V, E)$ is a **directed graph**.
- ▶ $u : E \rightarrow \mathbb{R}_0^+ \cup \{\infty\}$ is the **capacity function**.

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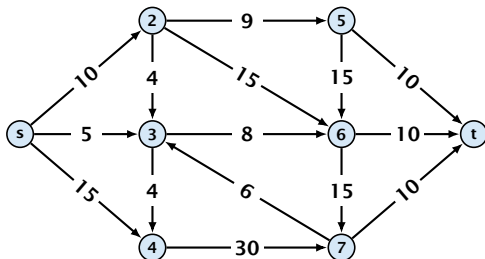
- ▶ $G = (V, E)$ is a **directed graph**.
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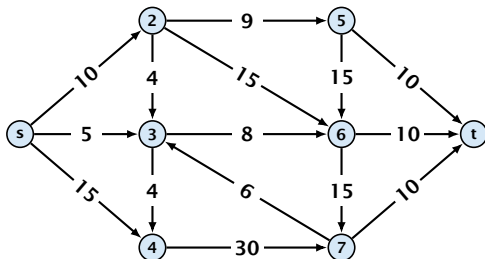
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- ▶ $G = (V, E)$ is a **directed graph**.
- ▶ $u : E \rightarrow \mathbb{R}_0^+ \cup \{\infty\}$ is the **capacity function**.
- ▶ $c : E \rightarrow \mathbb{R}$ is the **cost function**
(note that $c(e)$ may be negative).
- ▶ $b : V \rightarrow \mathbb{R}, \sum_{v \in V} b(v) = 0$ is a **demand function**.

Solve Maxflow Using Mincost Flow

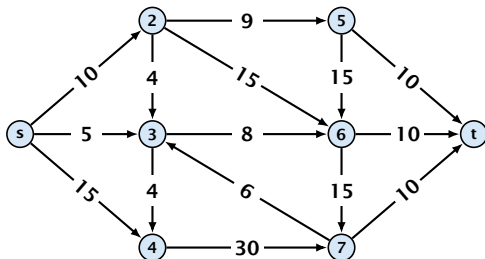


Solve Maxflow Using Mincost Flow



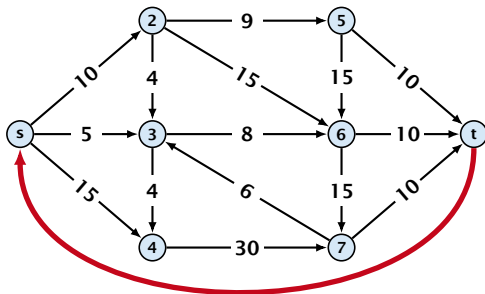
- ▶ Given a flow network for a standard maxflow problem.

Solve Maxflow Using Mincost Flow



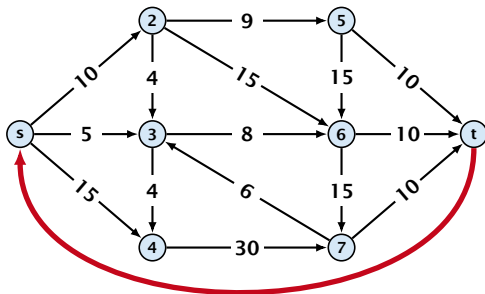
- ▶ Given a flow network for a standard maxflow problem.
- ▶ Set $b(v) = 0$ for every node. Keep the capacity function u for all edges. Set the cost $c(e)$ for every edge to 0.

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- ▶ Given a flow network for a standard maxflow problem.
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- ▶ Add an edge from t to s with infinite capacity and cost -1 .

Solve Maxflow Using Mincost Flow



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- ▶ Set $b(v) = 0$ for every node. Keep the capacity function u for all edges. Set the cost $c(e)$ for every edge to 0.
- ▶ Add an edge from t to s with infinite capacity and cost -1 .
- ▶ Then, $\text{val}(f^*) = -\text{cost}(f_{\min})$, where f^* is a maxflow, and f_{\min} is a mincost-flow.

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Solve decision version of maxflow:

- ▶ Given a flow network for a standard maxflow problem, and a value k .

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- ▶ Set $b(v) = 0$ for every node apart from s or t . Set $b(s) = -k$ and $b(t) = k$.
- ▶ Set edge-costs to zero, and keep the capacities.
- ▶ There exists a maxflow of value at least k if and only if the mincost-flow problem is feasible.

Generalization

Our model:

$$\begin{aligned} \min \quad & \sum_e c(e)f(e) \\ \text{s.t.} \quad & \forall e \in E: 0 \leq f(e) \leq u(e) \\ & \forall v \in V: f(v) = b(v) \end{aligned}$$

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A more general model?

$$\begin{aligned} \min \quad & \sum_e c(e)f(e) \\ \text{s.t.} \quad & \forall e \in E: \ell(e) \leq f(e) \leq u(e) \\ & \forall v \in V: a(v) \leq f(v) \leq b(v) \end{aligned}$$

where $a: V \rightarrow \mathbb{R}$, $b: V \rightarrow \mathbb{R}$; $\ell: E \rightarrow \mathbb{R} \cup \{-\infty\}$, $u: E \rightarrow \mathbb{R} \cup \{\infty\}$
 $c: E \rightarrow \mathbb{R}$;

Differences

- ▶ Flow along an edge e may have non-zero lower bound $\ell(e)$.
- ▶ Flow along e may have negative upper bound $u(e)$.
- ▶ The demand at a node v may have lower bound $a(v)$ and upper bound $b(v)$ instead of just lower bound = upper bound = $b(v)$.

Reduction I

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Reduction I

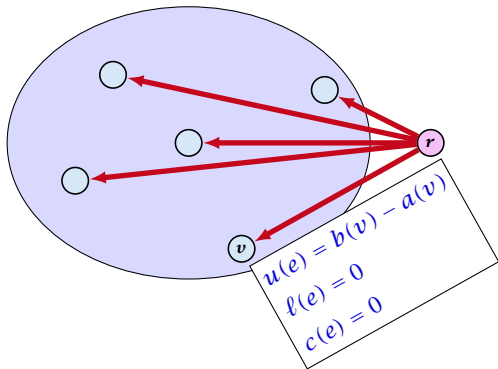
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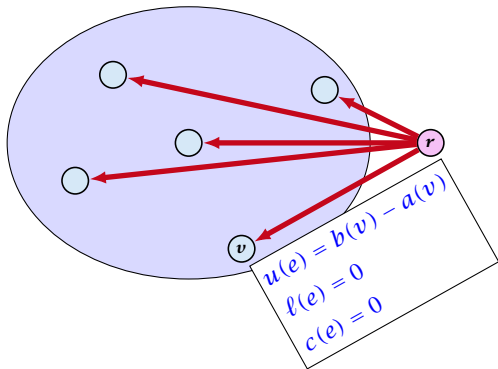


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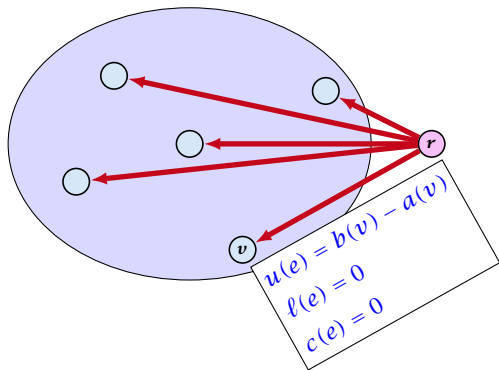
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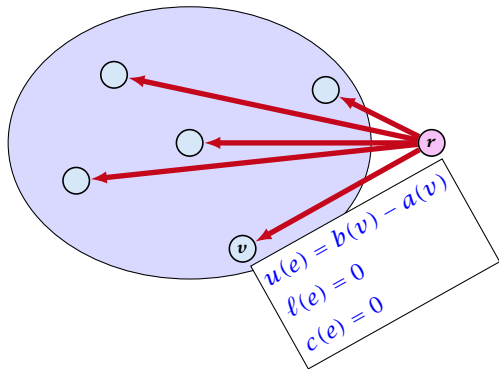
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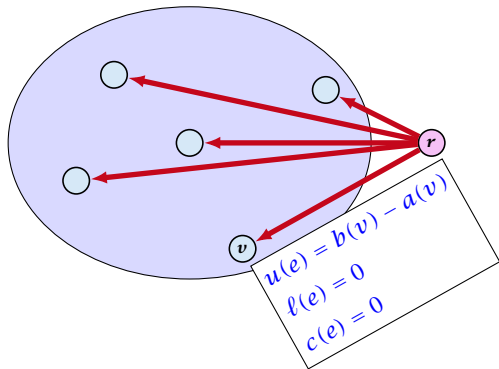
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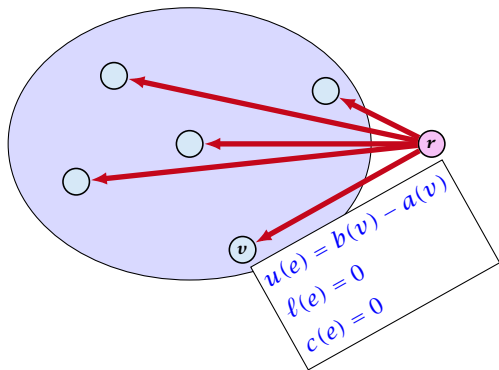
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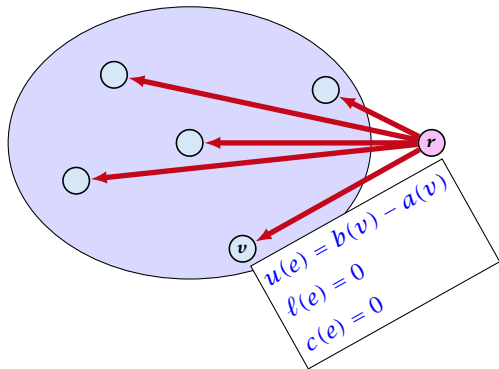
Add edge (r, v) for all $v \in V$.

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Set $b(r) = -\sum_{v \in V} b(v)$.



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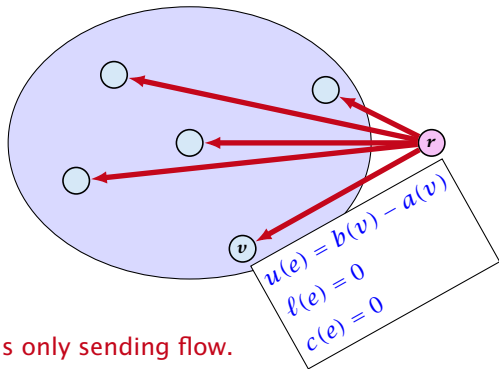
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Set $b(r) = -\sum_{v \in V} b(v)$.

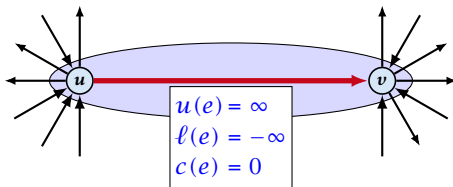
$-\sum_v b(v)$ is negative; hence r is only sending flow.



Reduction II

$$\begin{aligned} \min \quad & \sum_e c(e)f(e) \\ \text{s.t.} \quad & \forall e \in E: \ell(e) \leq f(e) \leq u(e) \\ & \forall v \in V: f(v) = b(v) \end{aligned}$$

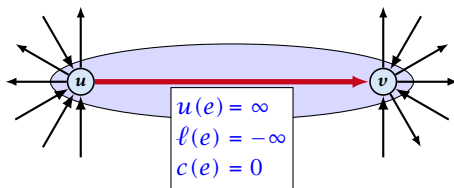
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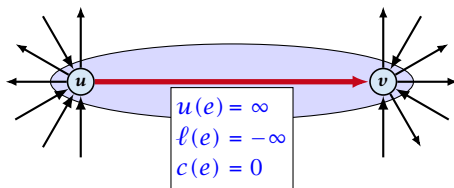


If $c(e) = 0$ we can contract the edge/identify nodes u and v .

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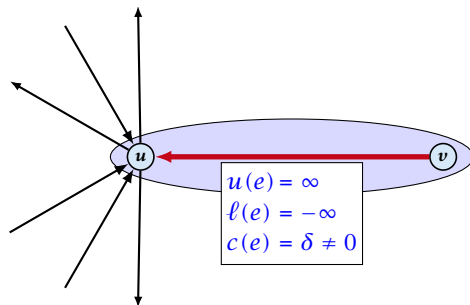


If $c(e) = 0$ we can contract the edge/identify nodes u and v .

If $c(e) \neq 0$ we can transform the graph so that $c(e) = 0$.

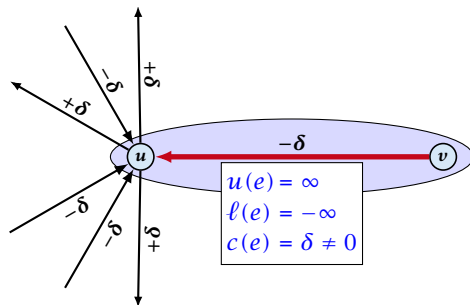
Reduction II

We can transform any network so that a particular edge has cost $c(e) = 0$:



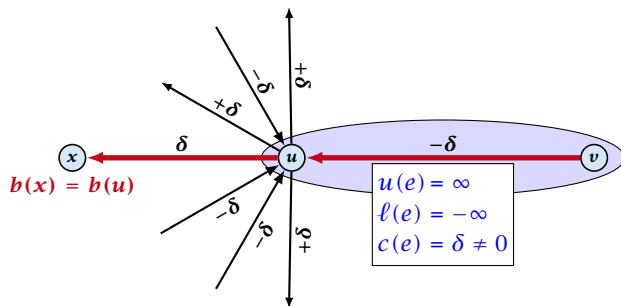
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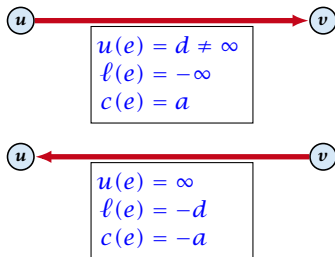


Additionally we set $b(u) = 0$.

Reduction III

$$\begin{aligned} \min \quad & \sum_e c(e)f(e) \\ \text{s.t.} \quad & \forall e \in E: \ell(e) \leq f(e) \leq u(e) \\ & \forall v \in V: f(v) = b(v) \end{aligned}$$

We can assume that $\ell(e) \neq -\infty$:

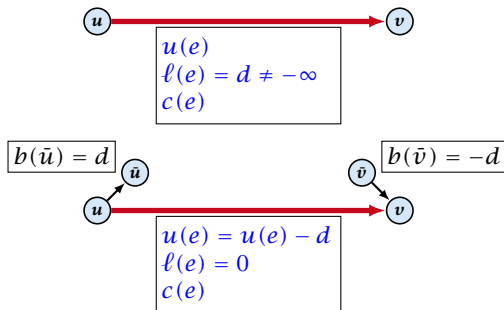


Replace the edge by an edge in opposite direction.

Reduction IV

$$\begin{aligned} \min \quad & \sum_e c(e)f(e) \\ \text{s.t.} \quad & \forall e \in E: \ell(e) \leq f(e) \leq u(e) \\ & \forall v \in V: f(v) = b(v) \end{aligned}$$

We can assume that $\ell(e) = 0$:



The added edges have infinite capacity and cost $c(e)/2$.

Applications

Caterer Problem

- ▶ She needs to supply r_i napkins on N successive days.

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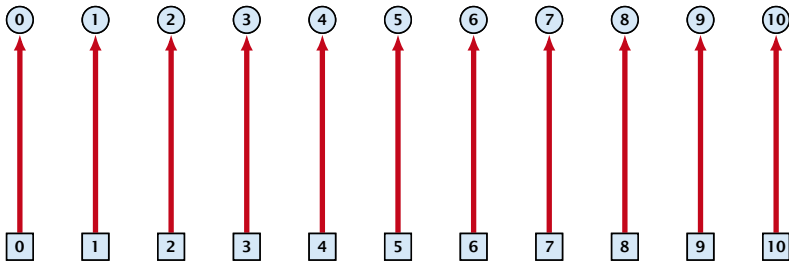
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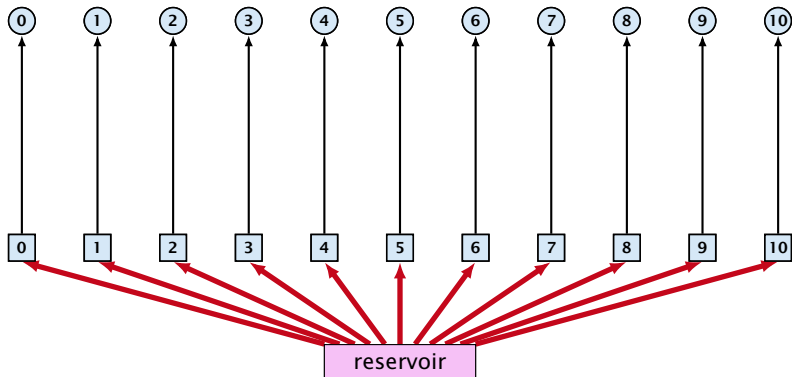
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- ▶ She can use a slow laundry that takes $k > m$ days and costs s cents each.
- ▶ At the end of each day she should determine how many to send to each laundry and how many to buy in order to fulfill demand.
- ▶ Minimize cost.



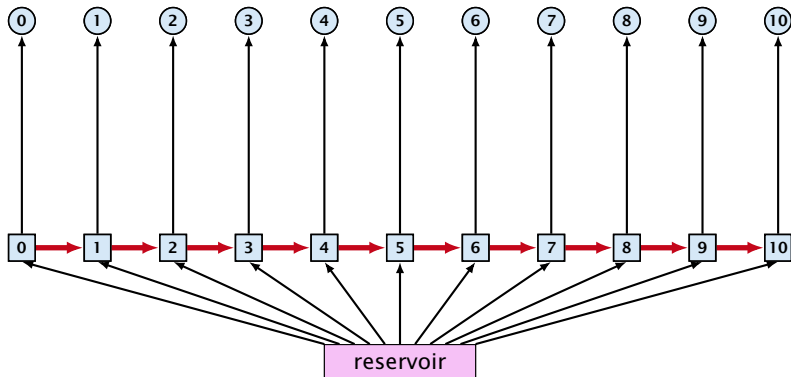
day edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = r_i$;
cost: $c(e) = 0$



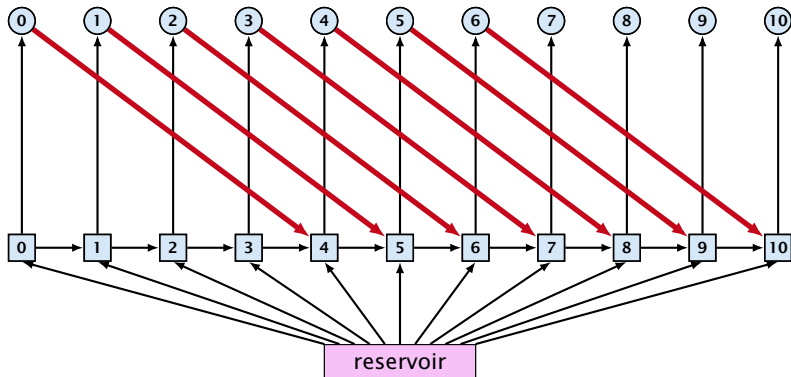
buy edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = p$



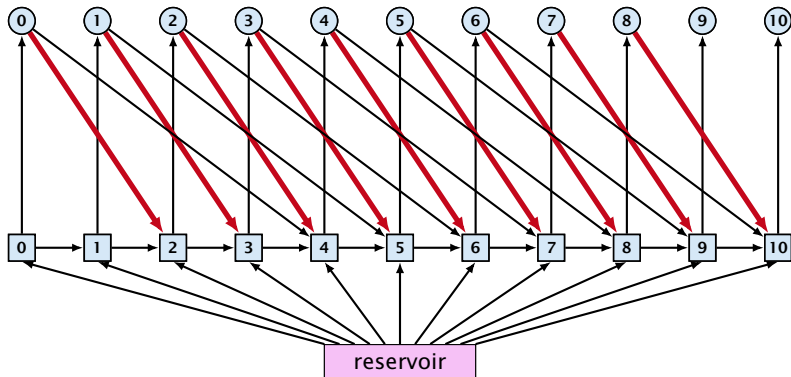
forward edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = 0$



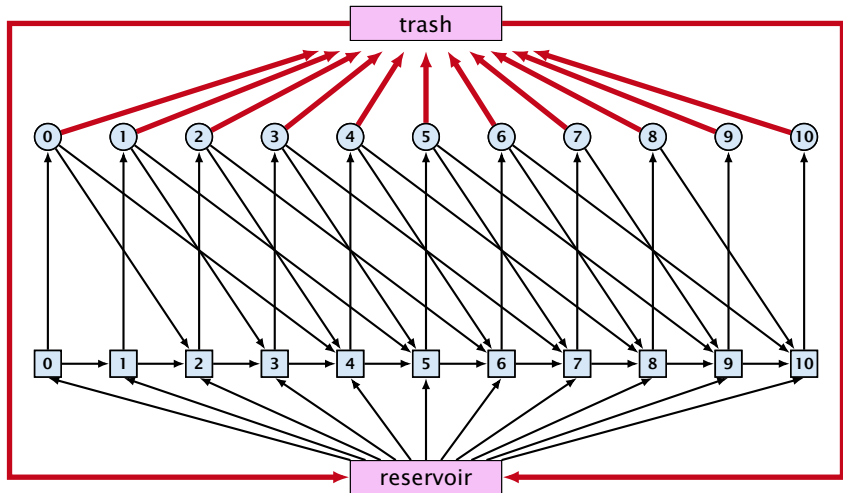
slow edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = s$



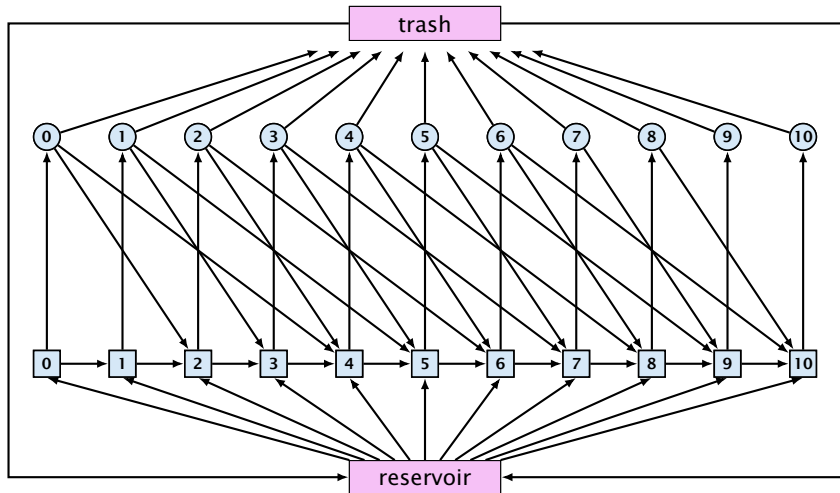
fast edges:

upper bound: $u(e_i) = \infty$;
 lower bound: $\ell(e_i) = 0$;
 cost: $c(e) = f$



trash edges:

upper bound: $u(e_i) = \infty$;
 lower bound: $\ell(e_i) = 0$;
 cost: $c(e) = 0$



Residual Graph

Version A:

The residual graph G' for a mincost flow is just a copy of the graph G .

If we send $f(e)$ along an edge, the corresponding edge e' in the residual graph has its lower and upper bound changed to $\ell(e') = \ell(e) - f(e)$ and $u(e') = u(e) - f(e)$.

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Version B:

The residual graph for a mincost flow is exactly defined as the residual graph for standard flows, with the only exception that one needs to define a cost for the residual edge.

For a flow of z from u to v the residual edge (v, u) has capacity z and a cost of $-c((u, v))$.

14 Mincost Flow

A **circulation** in a graph $G = (V, E)$ is a function $f : E \rightarrow \mathbb{R}^+$ that has an excess flow $f(v) = 0$ for every node $v \in V$.

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A circulation is **feasible** if it fulfills capacity constraints, i.e., $f(e) \leq u(e)$ for every edge of G .

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A given flow is a mincost-flow if and only if the corresponding residual graph G_f does not have a feasible circulation of negative cost.

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⇐ Let f be a non-mincost flow, and let f^* be a min-cost flow. We need to show that the residual graph has a feasible circulation with negative cost.

Clearly $f^* - f$ is a circulation of negative cost. One can also easily see that it is feasible for the residual graph. (after sending $-f$ in the residual graph (pushing all flow back) we arrive at the original graph; for this f^* is clearly feasible)

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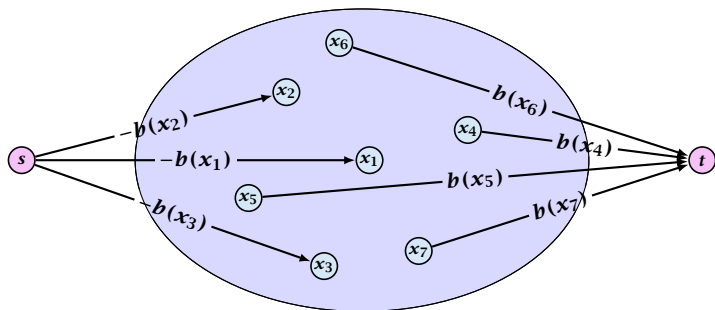
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- ▶ Find directed cycle only using edges that have non-zero flow.
- ▶ If this cycle has negative cost you are done.
- ▶ Otherwise send flow in opposite direction along the cycle until the bottleneck edge(s) does not carry any flow.
- ▶ You still have a circulation with negative cost.
- ▶ Repeat.

14 Mincost Flow

Algorithm 48 CycleCanceling($G = (V, E), c, u, b$)

- 1: establish a feasible flow f in G
- 2: **while** G_f contains negative cycle **do**
- 3: use Bellman-Ford to find a negative circuit Z
- 4: $\delta \leftarrow \min\{u_f(e) \mid e \in Z\}$
- 5: augment δ units along Z and update G_f

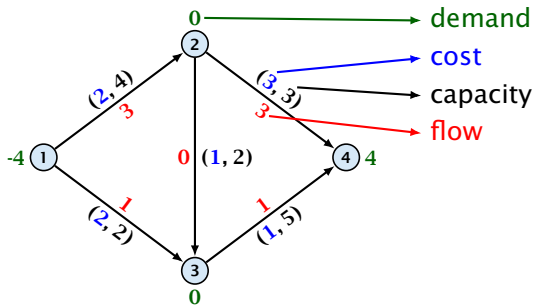
How do we find the initial feasible flow?



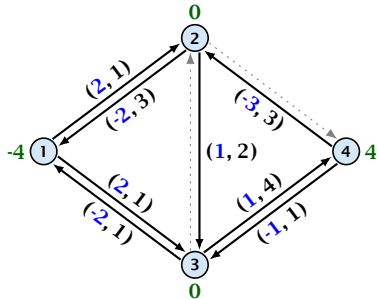
- ▶ Connect new node s to all nodes with negative $b(v)$ -value.
- ▶ Connect nodes with positive $b(v)$ -value to a new node t .
- ▶ There exist a feasible flow in the original graph iff in the resulting graph there exists an s - t flow of value

$$\sum_{v:b(v)<0} (-b(v)) = \sum_{v:b(v)>0} b(v) .$$

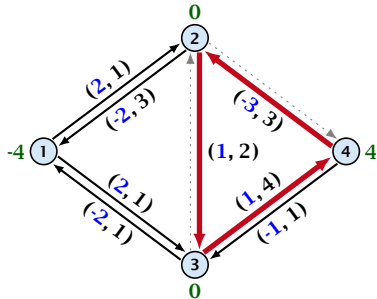
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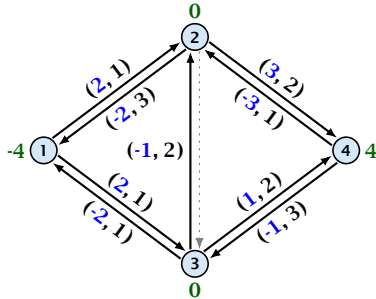
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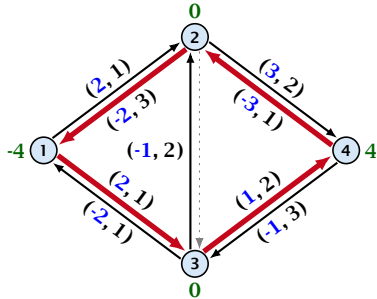
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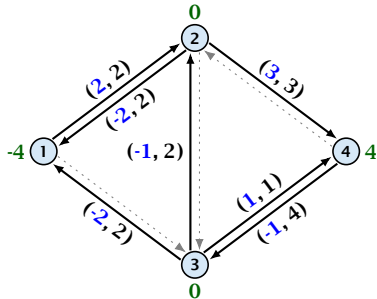
14 Mincost Flow



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14 Mincost Flow

Lemma 87

The improving cycle algorithm runs in time $\mathcal{O}(nm^2CU)$, for integer capacities and costs, when for all edges e , $|c(e)| \leq C$ and $|u(e)| \leq U$.

- ▶ Running time of Bellman-Ford is $\mathcal{O}(mn)$.
- ▶ Pushing flow along the cycle can be done in time $\mathcal{O}(n)$.
- ▶ Each iteration decreases the total cost by at least 1.
- ▶ The true optimum cost must lie in the interval $[-mCU, \dots, +mCU]$.

Note that this lemma is weak since it does not allow for edges with infinite capacity.

14 Mincost Flow

A **general mincost flow problem** is of the following form:

$$\begin{aligned} \min \quad & \sum_e c(e)f(e) \\ \text{s.t.} \quad & \forall e \in E: \ell(e) \leq f(e) \leq u(e) \\ & \forall v \in V: a(v) \leq f(v) \leq b(v) \end{aligned}$$

where $a: V \rightarrow \mathbb{R}$, $b: V \rightarrow \mathbb{R}$; $\ell: E \rightarrow \mathbb{R} \cup \{-\infty\}$, $u: E \rightarrow \mathbb{R} \cup \{\infty\}$
 $c: E \rightarrow \mathbb{R}$;

Lemma 88 (without proof)

A general mincost flow problem can be solved in polynomial time.