7.6 van Emde Boas Trees

Dynamic Set Data Structure S:

- $S.\mathrm{insert}(x)$
- $S.\mathrm{delete}(x)$
- $S.\mathrm{search}(x)$
- $S.\mathrm{min}()$
- $S.\mathrm{max}()$
- $S.\mathrm{succ}(x)$
- $S.\mathrm{pred}(x)$

7.6 van Emde Boas Trees

For this chapter we ignore the problem of storing satellite data:

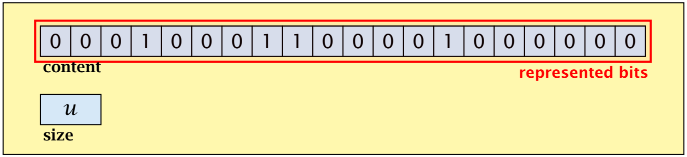
- $\boldsymbol{S}.\,\textbf{insert}(\boldsymbol{x})$: Inserts x into S.
- $\boldsymbol{S}.\,\textbf{delete}(\boldsymbol{x})$: Deletes x from S. Usually assumes that $x \in S$.
- $\boldsymbol{S}.\,\textbf{member}(\boldsymbol{x})$: Returns 1 if $x \in S$ and 0 otw.
- $\boldsymbol{S}.\,\textbf{min}()$: Returns the value of the minimum element in S.
- $\boldsymbol{S}.\,\textbf{max}()$: Returns the value of the maximum element in S.
- $\boldsymbol{S}.\,\textbf{succ}(\boldsymbol{x})$: Returns successor of x in S. Returns null if x is maximum or larger than any element in S. Note that x needs not to be in S.
- $\boldsymbol{S}.\,\textbf{pred}(\boldsymbol{x})$: Returns the predecessor of x in S. Returns null if x is minimum or smaller than any element in S. Note that x needs not to be in S.

7.6 van Emde Boas Trees

Can we improve the existing algorithms when the keys are from a restricted set?

In the following we assume that the keys are from $\{0, 1, \ldots, u - 1\}$, where u denotes the size of the universe.

Implementation 1: Array



one array of u bits

Use an array that encodes the indicator function of the dynamic set.

Implementation 1: Array

Algorithm 1 array.insert(x)

```
1: content[$x$] $\leftarrow$ 1;
```

Algorithm 2 array.delete(x)

```
1: content[$x$] $\leftarrow$ 0;
```

Algorithm 3 array.member(x)

```
1: **return** content[$x$];
```

- Note that we assume that x is valid, i.e., it falls within the array boundaries.
- Obviously(?) the running time is constant.

Implementation 1: Array

Algorithm 4 array.max()

```
for ($i = \text{size} - 1$; $i \geq 0$; $i$--) do
    if content[$i$] $= 1$ then return $i$;
return null;
```

Implementation 1: Array

Algorithm 4 array.max()

```
for ($i$ = size $-1$; $i \ge 0$; $i$--) do
    if content[$i$] $= 1$ then return $i$;
return null;
```

Algorithm 5 array.min()

```
for ($i = 0$; $i <$ size; $i$++) do
    if content[$i$] $= 1$ then return $i$;
return null;
```

Implementation 1: Array

Algorithm 4 array.max()

```
for ($i$ = size $-1$; $i \geq 0$; $i$--) do
    if content[$i$] $= 1$ then return $i$;
return null;
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Algorithm 5 array.min()

```
for ($i = 0$; $i <$ size; $i$++) do
    if content[$i$] $= 1$ then return $i$;
return null;
```

- Running time is $\mathcal{O}(u)$ in the worst case.

Implementation 1: Array

Algorithm 6 array.succ(x)

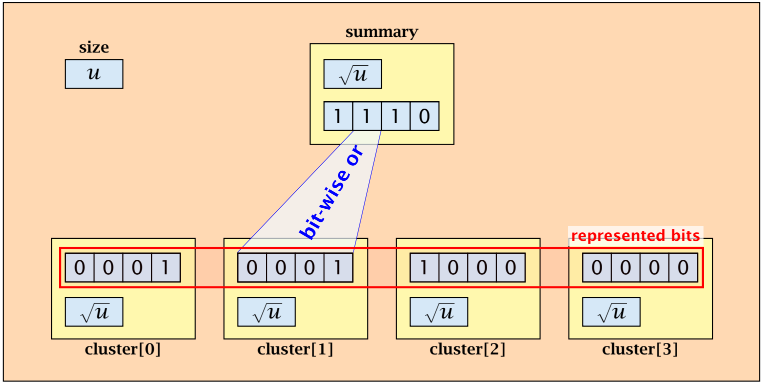
```
for ($i = x + 1$; $i <$ size; $i$++) do
    if content[$i$] $= 1$ then return $i$;
return null;
```

Algorithm 7 array.pred(x)

```
for ($i = x - 1$; $i \geq 0$; $i$--) do
    if content[$i$] $= 1$ then return $i$;
return null;
```

- Running time is $\mathcal{O}(u)$ in the worst case.

Implementation 2: Summary Array



- $\sqrt{u}$ cluster-arrays of $\sqrt{u}$ bits.
- One summary-array of $\sqrt{u}$ bits. The i-th bit in the summary array stores the bit-wise or of the bits in the i-th cluster.

Implementation 2: Summary Array

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The bit for a key x is contained in cluster number $\left\lfloor \frac{x}{\sqrt{u}} \right\rfloor$.

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Within the cluster-array the bit is at position $x \bmod \sqrt{u}$.

For simplicity we assume that $u = 2^{2^k}$ for some $k \geq 1$. Then we can compute the cluster-number for an entry x as $\mathrm{high}(x)$ (the upper half of the dual representation of x) and the position of x within its cluster as $\mathrm{low}(x)$ (the lower half of the dual representation).

Implementation 2: Summary Array

Algorithm 8 member(x)

```
1: return cluster[high($x$)].member(low($x$));
```

Implementation 2: Summary Array

Algorithm 8 member(x)

```
return cluster[high($x$)].member(low($x$));
```

Algorithm 9 insert(x)

```
cluster[high($x$)].insert(low($x$));
summary.insert(high($x$));
```

Implementation 2: Summary Array

Algorithm 8 member(x)

```
1: return $\text{cluster}[\text{high}(x)].\text{member}(\text{low}(x))$;
```

Algorithm 9 insert(x)

```
1: $\text{cluster}[\text{high}(x)].\text{insert}(\text{low}(x))$;
2: $\text{summary}.\text{insert}(\text{high}(x))$;
```

- The running times are constant, because the corresponding array-functions have constant running times.

Implementation 2: Summary Array

Algorithm 10 delete(x)

```
1: cluster[high($x$)].delete(low($x$));
2: if cluster[high($x$)].min() = null then
3:     summary.delete(high($x$));
```

Implementation 2: Summary Array

Algorithm 10 delete(x)

```
1: cluster[high($x$)].delete(low($x$));
2: if cluster[high($x$)].min() = null then
3:     summary.delete(high($x$));
```

- The running time is dominated by the cost of a minimum computation on an array of size $\sqrt{u}$. Hence, $\mathcal{O}(\sqrt{u})$.

Implementation 2: Summary Array

Algorithm 11 max()

```
1: $\mathit{maxcluster} \leftarrow \text{summary}.\text{max}()$;
2: if $\mathit{maxcluster} = \text{null}$ return null;
3: $\mathit{offs} \leftarrow \text{cluster}[\mathit{maxcluster}].\text{max}()$
4: return $\mathit{maxcluster} \circ \mathit{offs}$;
```

Implementation 2: Summary Array

Algorithm 11 max()

```
$maxcluster \leftarrow$ summary.max();
if $maxcluster = $ null return null;
$offs \leftarrow$ cluster[$maxcluster$].max()
return $maxcluster \circ offs$;
```

Algorithm 12 min()

```
$mincluster \leftarrow$ summary.min();
if $mincluster = $ null return null;
$offs \leftarrow$ cluster[$mincluster$].min();
return $mincluster \circ offs$;
```

Implementation 2: Summary Array

Algorithm 11 max()

```
1: $maxcluster \leftarrow \text{summary.max}()$;
2: if $maxcluster = \text{null}$ return null;
3: $offs \leftarrow \text{cluster}[maxcluster].\text{max}()$
4: return $maxcluster \circ offs$;
```

Algorithm 12 min()

```
1: $mincluster \leftarrow \text{summary.min}()$;
2: if $mincluster = \text{null}$ return null;
3: $offs \leftarrow \text{cluster}[mincluster].\text{min}()$;
4: return $mincluster \circ offs$;
```

The operator $\circ$ stands for the concatenation of two bitstrings. This means if $x = 0111_2$ and $y = 0001_2$ then $x \circ y = 01110001_2$.

- Running time is roughly $2\sqrt{u} = \mathcal{O}(\sqrt{u})$ in the worst case.

Implementation 2: Summary Array

Algorithm 13 succ(x)

```
1: $m \leftarrow \text{cluster}[\text{high}(\text{x})].\text{succ}(\text{low}(x))$
2: if $m \neq \text{null}$ then return $\text{high}(x) \circ m$;
3: $\textit{succcluster} \leftarrow \text{summary}.\text{succ}(\text{high}(x))$;
4: if $\textit{succcluster} \neq \text{null}$ then
5:         $\textit{offs} \leftarrow \text{cluster}[\textit{succcluster}].\text{min}()$;
6:         return $\textit{succcluster} \circ \textit{offs}$;
7: return null;
```

Implementation 2: Summary Array

```
Algorithm 13 succ($x$)
1: $m \leftarrow \text{cluster}[\text{high}(\text{x})].\text{succ}(\text{low}(x))$
2: if $m \neq$ null then return $\text{high}(x) \circ m$;
3: $\textit{succcluster} \leftarrow \text{summary}.\text{succ}(\text{high}(x))$;
4: if $\textit{succcluster} \neq$ null then
5:         $\textit{offs} \leftarrow \text{cluster}[\textit{succcluster}].\text{min}()$;
6:         return $\textit{succcluster} \circ \textit{offs}$;
7: return null;
```

- Running time is roughly $3\sqrt{u} = \mathcal{O}(\sqrt{u})$ in the worst case.

Implementation 2: Summary Array

Algorithm 14 pred(x)

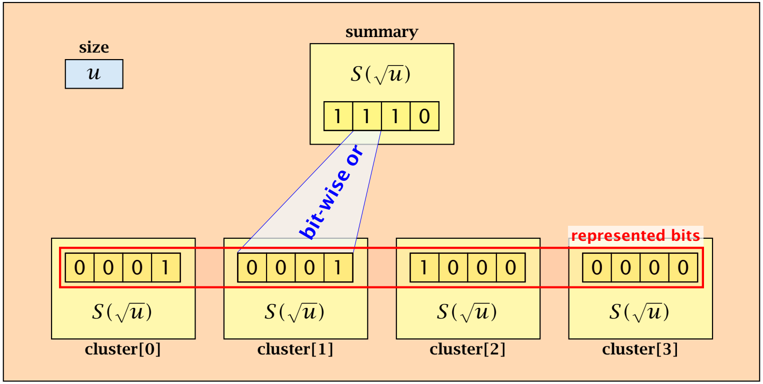
```
1: $m \leftarrow \text{cluster}[\text{high}(x)].\text{pred}(\text{low}(x))$
2: **if** $m \neq \text{null}$ **then return** $\text{high}(x) \circ m$;
3: $\textit{predcluster} \leftarrow \text{summary}.\text{pred}(\text{high}(x))$;
4: **if** $\textit{predcluster} \neq \text{null}$ **then**
5:         $\textit{offs} \leftarrow \text{cluster}[\textit{predcluster}].\text{max}()$;
6:         **return** $\textit{predcluster} \circ \textit{offs}$;
7: **return** null;
```

- Running time is roughly $3\sqrt{u} = \mathcal{O}(\sqrt{u})$ in the worst case.

Implementation 3: Recursion

Instead of using sub-arrays, we build a recursive data-structure.

$S(u)$ is a dynamic set data-structure representing u bits:



Implementation 3: Recursion

We assume that $u = 2^{2^k}$ for some k.

The data-structure $S(2)$ is defined as an array of 2-bits (end of the recursion).

Implementation 3: Recursion

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The code from Implementation 2 can be used unchanged. We only need to redo the analysis of the running time.

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Note that in the code we do not need to specifically address the non-recursive case. This is achieved by the fact that an $S(4)$ will contain $S(2)$'s as sub-datastructures, which are arrays. Hence, a call like $\text{cluster}[1].\text{min}()$ from within the data-structure $S(4)$ is not a recursive call as it will call the function $\text{array}.\text{min}()$.

Implementation 3: Recursion

The code from Implementation 2 can be used unchanged. We only need to redo the analysis of the running time.

Note that in the code we do not need to specifically address the non-recursive case. This is achieved by the fact that an $S(4)$ will contain $S(2)$'s as sub-datastructures, which are arrays. Hence, a call like $\text{cluster}[1].\text{min}()$ from within the data-structure $S(4)$ is not a recursive call as it will call the function $\text{array}.\text{min}()$.

This means that the non-recursive case is been dealt with while initializing the data-structure.

Implementation 3: Recursion

Algorithm 15 member(x)

```
1: return cluster[high($x$)].member(low($x$));
```

- $T_{\mathrm{mem}}(u) = T_{\mathrm{mem}}(\sqrt{u}) + 1$.

Implementation 3: Recursion

Algorithm 16 insert(x)

```
1: cluster[high($x$)].insert(low($x$));
2: summary.insert(high($x$));
```

- $T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1$.

Implementation 3: Recursion

Algorithm 17 delete(x)

```
1: cluster[high$(x)$].delete(low$(x)$);
2: if cluster[high$(x)$].min() = null then
3:     summary.delete(high$(x)$);
```

- $T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1$.

Implementation 3: Recursion

Algorithm 18 min()

```
$mincluster \leftarrow \text{summary.min}();$
if $mincluster = \text{null}$ return null;
$offs \leftarrow \text{cluster}[mincluster].\text{min}();$
return $mincluster \circ offs;$
```

- $T_{\min}(u) = 2T_{\min}(\sqrt{u}) + 1$.

Implementation 3: Recursion

Algorithm 19 succ(x)

```
1: $m \leftarrow$ cluster[high(x)].succ(low($x$))
2: if $m \neq$ null then return high($x$) $\circ$ $m$;
3: $succcluster \leftarrow$ summary.succ(high($x$));
4: if $succcluster \neq$ null then
5:     $offs \leftarrow$ cluster[$succcluster$].min();
6:     return $succcluster \circ offs$;
7: return null;
```

- $T_{\text{succ}}(u) = 2T_{\text{succ}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1$.

Implementation 3: Recursion

$$T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1:$$

Implementation 3: Recursion

$\boldsymbol{T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1}$**:**

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^{\ell})$.

Implementation 3: Recursion

$\boldsymbol{T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1}$**:**

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Implementation 3: Recursion

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$$X(\ell)$$

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$$X(\ell) = T_{\mathrm{mem}}(2^\ell) = T_{\mathrm{mem}}(u)$$

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$$\begin{aligned} X(\ell) = T_{\mathrm{mem}}(2^{\ell}) = T_{\mathrm{mem}}(u) &= T_{\mathrm{mem}}(\sqrt{u}) + 1 \\ &= T_{\mathrm{mem}}(2^{\frac{\ell}{2}}) + 1 \end{aligned}$$

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Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^\ell)$.Then

$$\begin{aligned} X(\ell) = T_{\text{mem}}(2^\ell) &= T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1 \\ &= T_{\text{mem}}(2^{\frac{\ell}{2}}) + 1 = X(\tfrac{\ell}{2}) + 1 \enspace . \end{aligned}$$

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Using Master theorem gives $X(\ell) = \mathcal{O}(\log \ell)$, and hence $T_{\mathrm{mem}}(u) = \mathcal{O}(\log\log u)$.

Implementation 3: Recursion

$$T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$$

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$$X(\ell) = T_{\text{ins}}(2^\ell) = T_{\text{ins}}(u)$$

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Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$. Then

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Using Master theorem gives $X(\ell) = \mathcal{O}(\ell)$, and hence $T_{\mathrm{ins}}(u) = \mathcal{O}(\log u)$.

Implementation 3: Recursion

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Using Master theorem gives $X(\ell) = \mathcal{O}(\ell)$, and hence $T_{\text{ins}}(u) = \mathcal{O}(\log u)$.

The same holds for $T_{\max}(u)$ and $T_{\min}(u)$.

Implementation 3: Recursion

$$T_{\mathrm{del}}(u) = 2T_{\mathrm{del}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1 \le 2T_{\mathrm{del}}(\sqrt{u}) + c\log(u).$$

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Set $\ell := \log u$ and $X(\ell) := T_{\text{del}}(2^\ell)$.

Implementation 3: Recursion

$$T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1 \leq 2T_{\text{del}}(\sqrt{u}) + c\log(u).$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{del}}(2^\ell)$. Then

Implementation 3: Recursion

$$T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1 \leq 2T_{\text{del}}(\sqrt{u}) + c\log(u).$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{del}}(2^\ell)$. Then

$$X(\ell)$$

Implementation 3: Recursion

$$T_{\mathrm{del}}(u) = 2T_{\mathrm{del}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1 \leq 2T_{\mathrm{del}}(\sqrt{u}) + c\log(u).$$

Set $\ell := \log u$ and $X(\ell) := T_{\mathrm{del}}(2^\ell)$. Then

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Set $\ell := \log u$ and $X(\ell) := T_{\text{del}}(2^{\ell})$. Then

$$X(\ell) = T_{\text{del}}(2^{\ell}) = T_{\text{del}}(u)$$

Implementation 3: Recursion

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Set $\ell := \log u$ and $X(\ell) := T_{\text{del}}(2^\ell)$. Then

$$\begin{aligned} X(\ell) = T_{\text{del}}(2^\ell) &= T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + c\log u \\ &= 2T_{\text{del}}(2^{\frac{\ell}{2}}) + c\ell \end{aligned}$$

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$$T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1 \le 2T_{\text{del}}(\sqrt{u}) + c\log(u).$$

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Using Master theorem gives $X(\ell) = \Theta(\ell\log\ell)$, and hence $T_{\text{del}}(u) = \mathcal{O}(\log u\log\log u)$.

Implementation 3: Recursion

$$T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1 \leq 2T_{\text{del}}(\sqrt{u}) + c\log(u).$$

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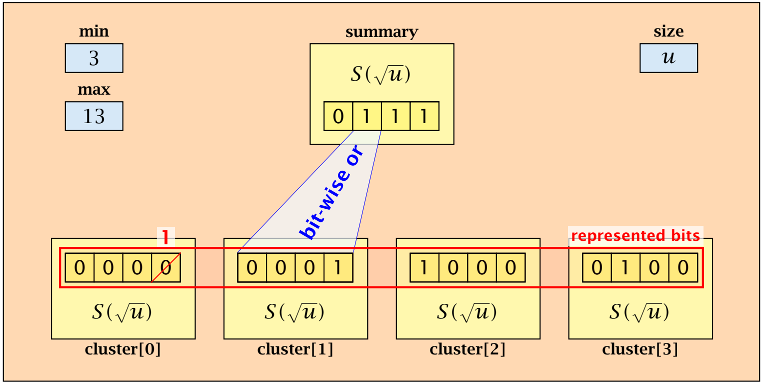
Using Master theorem gives $X(\ell) = \Theta(\ell\log\ell)$, and hence $T_{\text{del}}(u) = \mathcal{O}(\log u\log\log u)$.

The same holds for $T_{\text{pred}}(u)$ and $T_{\text{succ}}(u)$.

Implementation 4: van Emde Boas Trees



- The bit referenced by min is not set within sub-datastructures.
- The bit referenced by max is set within sub-datastructures (if $\max \neq \min$).

Implementation 4: van Emde Boas Trees

Advantages of having max/min pointers:

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- $\mathrm{min} = \mathrm{null}$ means that the data-structure is empty.

Implementation 4: van Emde Boas Trees

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- We can insert into an empty datastructure in constant time by only setting $\mathrm{min} = \mathrm{max} = x$.
- We can delete from a data-structure that just contains one element in constant time by setting $\mathrm{min} = \mathrm{max} = \mathrm{null}$.

Implementation 4: van Emde Boas Trees

Algorithm 20 max()

```
return $\max$;
```

Algorithm 21 min()

```
return $\min$;
```

- Constant time.

Implementation 4: van Emde Boas Trees

Algorithm 22 member(x)

```
1: if $x = \min$ then return 1; // TRUE
2: return $\text{cluster}[\text{high}(x)].\text{member}(\text{low}(x))$;
```

- $T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1 \Longrightarrow T(u) = \mathcal{O}(\log\log u)$.

Implementation 4: van Emde Boas Trees

Algorithm 23 succ(x)

```
1: if $\min \neq \text{null} \wedge x < \min$ then return min;
2: $maxincluster \leftarrow \text{cluster}[\text{high}(x)].\max()$;
3: if $maxincluster \neq \text{null} \wedge \text{low}(x) < maxincluster$ then
4:     $offs \leftarrow \text{cluster}[\text{high}(x)].\text{succ}(\text{low}(x))$;
5:     return $\text{high}(x) \circ offs$;
6: else
7:     $succcluster \leftarrow \text{summary}.\text{succ}(\text{high}(x))$;
8:     if $succcluster = \text{null}$ then return null;
9:     $offs \leftarrow \text{cluster}[succcluster].\min()$;
10:    return $succcluster \circ offs$;
```

- $T_{\text{succ}}(u) = T_{\text{succ}}(\sqrt{u}) + 1 \Longrightarrow T_{\text{succ}}(u) = \mathcal{O}(\log\log u)$.

Implementation 4: van Emde Boas Trees

Algorithm 35 insert(x)

```
if $\min = \text{null}$ then
    $\min = x$; $\max = x$;
else
    if $x < \min$ then exchange $x$ and min;
    if $x > \max$ then $\max = x$;
    if cluster[high($x$)].min = null; then
        summary.insert(high($x$));
        cluster[high($x$)].insert(low($x$));
    else
        cluster[high($x$)].insert(low($x$));
```

- $T_{\text{ins}}(u) = T_{\text{ins}}(\sqrt{u}) + 1 \Longrightarrow T_{\text{ins}}(u) = \mathcal{O}(\log\log u)$.

Implementation 4: van Emde Boas Trees

Note that the recusive call in Line 8 takes constant time as the if-condition in Line 6 ensures that we are inserting in an empty sub-tree.

The only non-constant recursive calls are the call in Line 7 and in Line 10. These are mutually exclusive, i.e., only one of these calls will actually occur.

From this we get that $T_{\text{ins}}(u) = T_{\text{ins}}(\sqrt{u}) + 1$.

Implementation 4: van Emde Boas Trees

- **Assumes that x is contained in the structure.**

Algorithm 36 delete(x)

```
if $\mathrm{min} = \mathrm{max}$ then
    $\mathrm{min} = \mathrm{max} = \mathrm{null}$;
else
    if $x = \mathrm{min}$ then
        $\mathit{firstcluster} \leftarrow \mathrm{summary.min}()$;
        $\mathit{offs} \leftarrow \mathrm{cluster}[\mathit{firstcluster}].\mathrm{min}()$;
        $x \leftarrow \mathit{firstcluster} \circ \mathit{offs}$;
        $\mathrm{min} \leftarrow x$;
    $\mathrm{cluster}[\mathrm{high}(x)].\mathrm{delete}(\mathrm{low}(x))$;
```

continued...

Implementation 4: van Emde Boas Trees

- **Assumes that x is contained in the structure.**

Algorithm 36 delete(x)

```
1: if $\mathrm{min} = \mathrm{max}$ then
2:     $\mathrm{min} = \mathrm{max} = \mathrm{null}$;
3: else
4:     if $x = \mathrm{min}$ then                          find new minimum
5:         $\mathit{firstcluster} \leftarrow \mathrm{summary.min}()$;
6:         $\mathit{offs} \leftarrow \mathrm{cluster}[\mathit{firstcluster}].\mathrm{min}()$;
7:         $x \leftarrow \mathit{firstcluster} \circ \mathit{offs}$;
8:         $\mathrm{min} \leftarrow x$;
9:     $\mathrm{cluster}[\mathrm{high}(x)].\mathrm{delete}(\mathrm{low}(x))$;
                    continued...
```

Implementation 4: van Emde Boas Trees

- **Assumes that x is contained in the structure.**

Algorithm 36 delete(x)

```
if $\min = \max$ then
    $\min = \max = \text{null}$;
else
    if $x = \min$ then
        $\textit{firstcluster} \leftarrow \text{summary.min}()$;
        $\textit{offs} \leftarrow \text{cluster}[\textit{firstcluster}].\text{min}()$;
        $x \leftarrow \textit{firstcluster} \circ \textit{offs}$;
        $\min \leftarrow x$;
    $\text{cluster}[\text{high}(x)].\text{delete}(\text{low}(x))$;                delete
continued...
```

Implementation 4: van Emde Boas Trees

Algorithm 36 delete(x)

...continued

```
10:     if cluster[high($x$)].min() = null then
11:         summary.delete(high($x$));
12:         if $x$ = max then
13:             $summax$ ← summary.max();
14:             if $summax$ = null then max ← min;
15:             else
16:                 $offs$ ← cluster[$summax$].max();
17:                 max ← $summax \circ offs$
18:     else
19:         if $x$ = max then
20:             $offs$ ← cluster[high($x$)].max();
21:             max ← high($x$) $\circ$ $offs$;
```

Implementation 4: van Emde Boas Trees

Algorithm 36 delete(x)

...continued — fix maximum

```
10: if cluster[high($x$)].min() = null then
11:     summary.delete(high($x$));
12:     if $x$ = max then
13:         $summax \leftarrow$ summary.max();
14:         if $summax$ = null then max $\leftarrow$ min;
15:         else
16:             $offs \leftarrow$ cluster[$summax$].max();
17:             max $\leftarrow summax \circ offs$
18: else
19:     if $x$ = max then
20:         $offs \leftarrow$ cluster[high($x$)].max();
21:         max $\leftarrow$ high($x$) $\circ$ $offs$;
```

Implementation 4: van Emde Boas Trees

Note that only one of the possible recusive calls in Line 9 and Line 11 in the deletion-algorithm may take non-constant time.

To see this observe that the call in Line 11 only occurs if the cluster where x was deleted is now empty. But this means that the call in Line 9 deleted the last element in $\text{cluster}[\text{high}(x)]$. Such a call only takes constant time.

Hence, we get a recurrence of the form

$$T_{\text{del}}(u) = T_{\text{del}}(\sqrt{u}) + c \ .$$

This gives $T_{\text{del}}(u) = \mathcal{O}(\log\log u)$.

7.6 van Emde Boas Trees

Space requirements:

- The space requirement fulfills the recurrence

$$S(u) = (\sqrt{u} + 1)S(\sqrt{u}) + \mathcal{O}(\sqrt{u}) \ .$$

- Note that we cannot solve this recurrence by the Master theorem as the branching factor is not constant.
- One can show by induction that the space requirement is $S(u) = \mathcal{O}(u)$. Exercise.

- Let the "real" recurrence relation be

$$S(k^2) = (k+1)S(k) + c_1 \cdot k;\ S(4) = c_2$$

- Replacing $S(k)$ by $R(k) := S(k)/c_2$ gives the recurrence

$$R(k^2) = (k+1)R(k) + ck;\ R(4) = 1$$

  where $c = c_1/c_2 < 1$.
- Now, we show $R(k^2) \leq k^2 - 2$ for $k^2 \geq 4$.
  - Obviously, this holds for $k^2 = 4$.
  - For $k^2 > 4$ we have

$$\begin{aligned} R(k^2) &= (1+k)R(k) + ck \\ &\leq (1+k)(k-2) + k \leq k^2 - 2 \end{aligned}$$

  - This shows that $R(k)$ and, hence, $S(k)$ grows linearly.