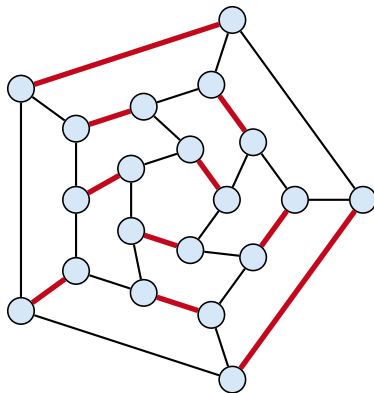


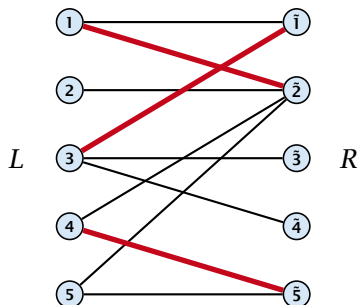
Matching

- ▶ Input: undirected graph $G = (V, E)$.
- ▶ $M \subseteq E$ is a **matching** if each node appears in at most one edge in M .
- ▶ Maximum Matching: find a matching of maximum cardinality



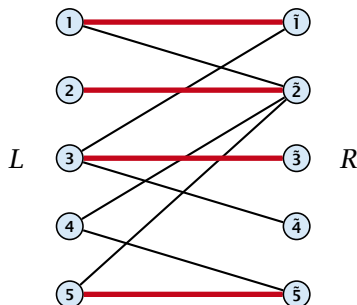
Bipartite Matching

- ▶ Input: undirected, **bipartite** graph $G = (L \uplus R, E)$.
- ▶ $M \subseteq E$ is a **matching** if each node appears in at most one edge in M .
- ▶ Maximum Matching: find a matching of maximum cardinality



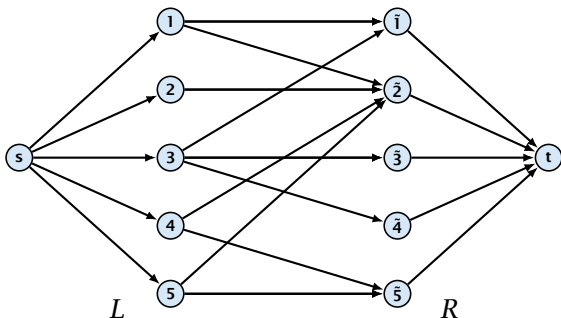
Bipartite Matching

- ▶ Input: undirected, **bipartite** graph $G = (L \uplus R, E)$.
- ▶ $M \subseteq E$ is a **matching** if each node appears in at most one edge in M .
- ▶ Maximum Matching: find a matching of maximum cardinality



Maxflow Formulation

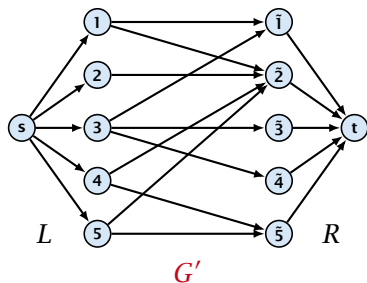
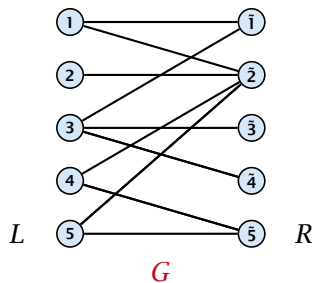
- ▶ Input: undirected, bipartite graph $G = (L \uplus R \uplus \{s, t\}, E')$.
- ▶ Direct all edges from L to R .
- ▶ Add source s and connect it to all nodes on the left.
- ▶ Add t and connect all nodes on the right to t .
- ▶ All edges have unit capacity.



Proof

Max cardinality matching in $G \leq$ value of maxflow in G'

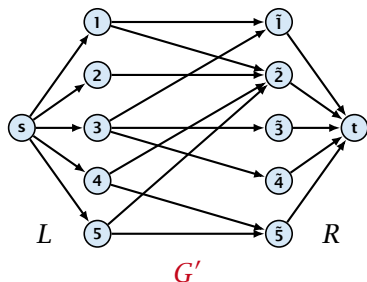
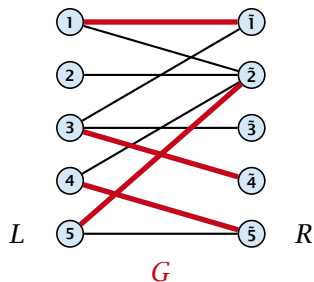
- ▶ Given a maximum matching M of cardinality k .
- ▶ Consider flow f that sends one unit along each of k paths.
- ▶ f is a flow and has cardinality k .



Proof

Max cardinality matching in $G \leq$ value of maxflow in G'

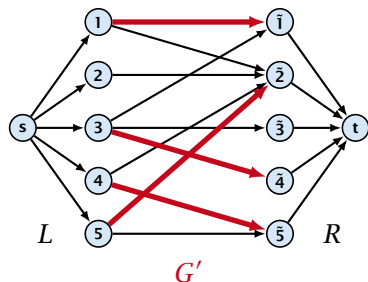
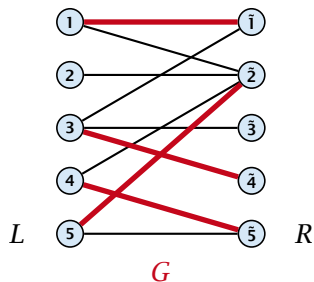
- ▶ Given a maximum matching M of cardinality k .
- ▶ Consider flow f that sends one unit along each of k paths.
- ▶ f is a flow and has cardinality k .



Proof

Max cardinality matching in $G \leq$ value of maxflow in G'

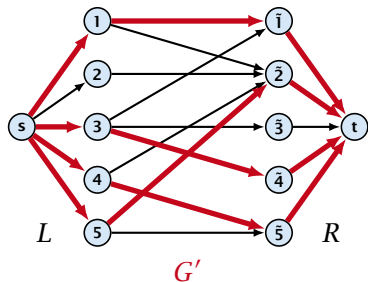
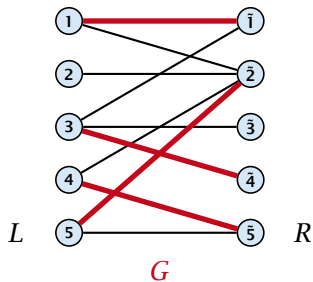
- ▶ Given a maximum matching M of cardinality k .
- ▶ Consider flow f that sends one unit along each of k paths.
- ▶ f is a flow and has cardinality k .



Proof

Max cardinality matching in $G \leq$ value of maxflow in G'

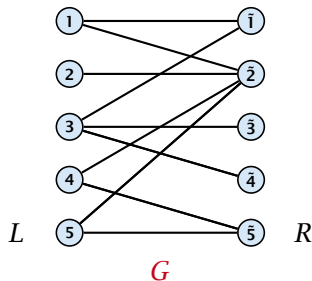
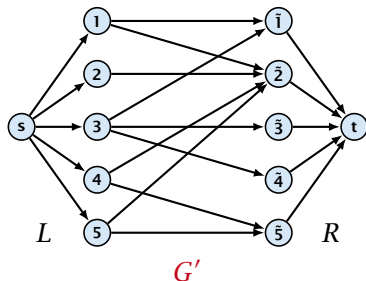
- ▶ Given a maximum matching M of cardinality k .
- ▶ Consider flow f that sends one unit along each of k paths.
- ▶ f is a flow and has cardinality k .



Proof

Max cardinality matching in $G \geq$ value of maxflow in G'

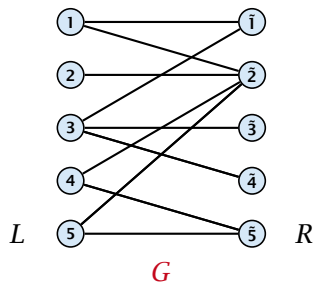
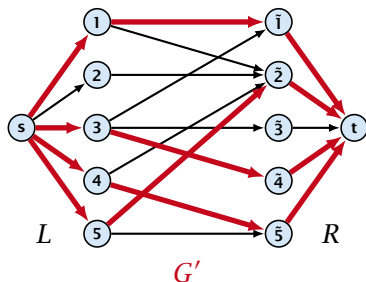
- ▶ Let f be a maxflow in G' of value k
- ▶ Integrality theorem $\Rightarrow k$ integral; we can assume f is 0/1.
- ▶ Consider $M =$ set of edges from L to R with $f(e) = 1$.
- ▶ Each node in L and R participates in at most one edge in M .
- ▶ $|M| = k$, as the flow must use at least k middle edges.



Proof

Max cardinality matching in $G \geq$ value of maxflow in G'

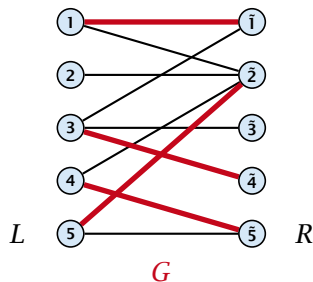
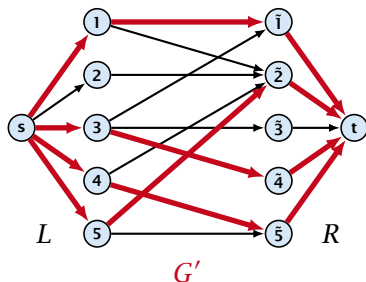
- ▶ Let f be a maxflow in G' of value k
- ▶ Integrality theorem $\Rightarrow k$ integral; we can assume f is 0/1.
- ▶ Consider $M =$ set of edges from L to R with $f(e) = 1$.
- ▶ Each node in L and R participates in at most one edge in M .
- ▶ $|M| = k$, as the flow must use at least k middle edges.



Proof

Max cardinality matching in $G \geq$ value of maxflow in G'

- ▶ Let f be a maxflow in G' of value k
- ▶ Integrality theorem $\Rightarrow k$ integral; we can assume f is 0/1.
- ▶ Consider $M =$ set of edges from L to R with $f(e) = 1$.
- ▶ Each node in L and R participates in at most one edge in M .
- ▶ $|M| = k$, as the flow must use at least k middle edges.



12.1 Matching

Which flow algorithm to use?

- ▶ Generic augmenting path: $\mathcal{O}(m \text{val}(f^*)) = \mathcal{O}(mn)$.
- ▶ Capacity scaling: $\mathcal{O}(m^2 \log C) = \mathcal{O}(m^2)$.
- ▶ Shortest augmenting path: $\mathcal{O}(mn^2)$.

For **unit capacity simple graphs** shortest augmenting path can be implemented in time $\mathcal{O}(m\sqrt{n})$.

Baseball Elimination

<i>team</i> <i>i</i>	<i>wins</i> w_i	<i>losses</i> ℓ_i	<i>remaining games</i>			
			<i>Atl</i>	<i>Phi</i>	<i>NY</i>	<i>Mon</i>
Atlanta	83	71	–	1	6	1
Philadelphia	80	79	1	–	0	2
New York	78	78	6	0	–	0
Montreal	77	82	1	2	0	–

Which team can end the season with most wins?

- ▶ Montreal is eliminated, since even after winning all remaining games there are only 80 wins.
- ▶ But also Philadelphia is eliminated. Why?

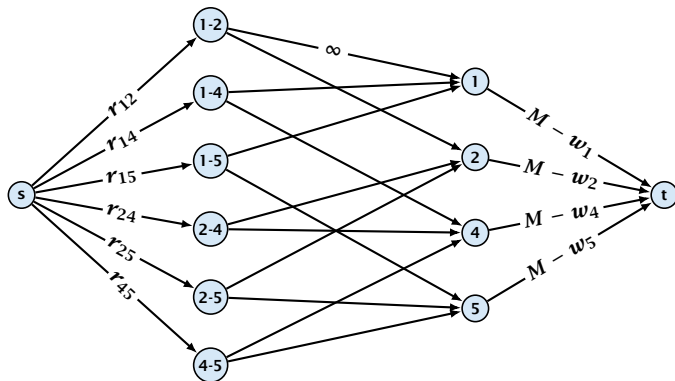
Baseball Elimination

Formal definition of the problem:

- ▶ Given a set S of teams, and one specific team $z \in S$.
- ▶ Team x has already won w_x games.
- ▶ Team x still has to play team y , r_{xy} times.
- ▶ Does team z still have a chance to finish with the most number of wins.

Baseball Elimination

Flow network for $z = 3$. M is number of wins Team 3 can still obtain.

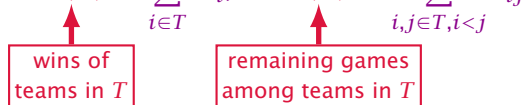


Idea. Distribute the results of remaining games in such a way that no team gets too many wins.

Certificate of Elimination

Let $T \subseteq S$ be a subset of teams. Define

$$w(T) := \sum_{i \in T} w_i, \quad r(T) := \sum_{i, j \in T, i < j} r_{ij}$$



If $\frac{w(T)+r(T)}{|T|} > M$ then one of the teams in T will have more than M wins in the end. A team that can win at most M games is therefore eliminated.

Theorem 63

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{i,j \in S \setminus \{z\}, i < j} r_{ij}$.

Theorem 63

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{i \in S \setminus \{z\}, i < j} r_{ij}$.

Proof (\Leftarrow)

- ▶ Consider the mincut A in the flow network. Let T be the set of team-nodes in A .

Theorem 63

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{ij \in S \setminus \{z\}, i < j} r_{ij}$.

Proof (\Leftarrow)

- ▶ Consider the mincut A in the flow network. Let T be the set of team-nodes in A .
- ▶ If for node $x-y$ not both team-nodes x and y are in T , then $x-y \notin A$ as otw. the cut would cut an infinite capacity edge.

Theorem 63

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{i,j \in S \setminus \{z\}, i < j} r_{ij}$.

Proof (\Leftarrow)

- ▶ Consider the mincut A in the flow network. Let T be the set of team-nodes in A .
- ▶ If for node $x-y$ not both team-nodes x and y are in T , then $x-y \notin A$ as otw. the cut would cut an infinite capacity edge.
- ▶ We don't find a flow that saturates all source edges:

$$r(S \setminus \{z\})$$

Theorem 63

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{ij \in S \setminus \{z\}, i < j} r_{ij}$.

Proof (\Leftarrow)

- ▶ Consider the mincut A in the flow network. Let T be the set of team-nodes in A .
- ▶ If for node $x-y$ not both team-nodes x and y are in T , then $x-y \notin A$ as otw. the cut would cut an infinite capacity edge.
- ▶ We don't find a flow that saturates all source edges:

$$r(S \setminus \{z\}) > \text{cap}(A, V \setminus A)$$

Theorem 63

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{i \in S \setminus \{z\}, i < j} r_{ij}$.

Proof (\Leftarrow)

- ▶ Consider the mincut A in the flow network. Let T be the set of team-nodes in A .
- ▶ If for node $x-y$ not both team-nodes x and y are in T , then $x-y \notin A$ as otw. the cut would cut an infinite capacity edge.
- ▶ We don't find a flow that saturates all source edges:

$$\begin{aligned} r(S \setminus \{z\}) &> \text{cap}(A, V \setminus A) \\ &\geq \sum_{i < j: i \notin T \vee j \notin T} r_{ij} + \sum_{i \in T} (M - w_i) \end{aligned}$$

Theorem 63

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{i \in S \setminus \{z\}, i < j} r_{ij}$.

Proof (\Leftarrow)

- ▶ Consider the mincut A in the flow network. Let T be the set of team-nodes in A .
- ▶ If for node $x-y$ not both team-nodes x and y are in T , then $x-y \notin A$ as otw. the cut would cut an infinite capacity edge.
- ▶ We don't find a flow that saturates all source edges:

$$\begin{aligned}r(S \setminus \{z\}) &> \text{cap}(A, V \setminus A) \\ &\geq \sum_{i < j: i \notin T \vee j \notin T} r_{ij} + \sum_{i \in T} (M - w_i) \\ &\geq r(S \setminus \{z\}) - r(T) + |T|M - w(T)\end{aligned}$$

Theorem 63

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{i \in S \setminus \{z\}, i < j} r_{ij}$.

Proof (\Leftarrow)

- ▶ Consider the mincut A in the flow network. Let T be the set of team-nodes in A .
- ▶ If for node $x-y$ not both team-nodes x and y are in T , then $x-y \notin A$ as otw. the cut would cut an infinite capacity edge.
- ▶ We don't find a flow that saturates all source edges:

$$\begin{aligned}r(S \setminus \{z\}) &> \text{cap}(A, V \setminus A) \\ &\geq \sum_{i < j: i \notin T \vee j \notin T} r_{ij} + \sum_{i \in T} (M - w_i) \\ &\geq r(S \setminus \{z\}) - r(T) + |T|M - w(T)\end{aligned}$$

- ▶ This gives $M < (w(T) + r(T))/|T|$, i.e., z is eliminated.

Baseball Elimination

Proof (\Rightarrow)

- ▶ Suppose we have a flow that saturates all source edges.

Baseball Elimination

Proof (\Rightarrow)

- ▶ Suppose we have a flow that saturates all source edges.
- ▶ We can assume that this flow is **integral**.

Baseball Elimination

Proof (\Rightarrow)

- ▶ Suppose we have a flow that saturates all source edges.
- ▶ We can assume that this flow is **integral**.
- ▶ For every pairing x - y it defines how many games team x and team y should win.

Baseball Elimination

Proof (\Rightarrow)

- ▶ Suppose we have a flow that saturates all source edges.
- ▶ We can assume that this flow is **integral**.
- ▶ For every pairing x - y it defines how many games team x and team y should win.
- ▶ The flow leaving the team-node x can be interpreted as the additional number of wins that team x will obtain.

Baseball Elimination

Proof (\Rightarrow)

- ▶ Suppose we have a flow that saturates all source edges.
- ▶ We can assume that this flow is **integral**.
- ▶ For every pairing x - y it defines how many games team x and team y should win.
- ▶ The flow leaving the team-node x can be interpreted as the additional number of wins that team x will obtain.
- ▶ This is less than $M - w_x$ because of capacity constraints.

Baseball Elimination

Proof (\Rightarrow)

- ▶ Suppose we have a flow that saturates all source edges.
- ▶ We can assume that this flow is **integral**.
- ▶ For every pairing x - y it defines how many games team x and team y should win.
- ▶ The flow leaving the team-node x can be interpreted as the additional number of wins that team x will obtain.
- ▶ This is less than $M - w_x$ because of capacity constraints.
- ▶ Hence, we found a set of results for the remaining games, such that no team obtains more than M wins in total.

Baseball Elimination

Proof (\Rightarrow)

- ▶ Suppose we have a flow that saturates all source edges.
- ▶ We can assume that this flow is **integral**.
- ▶ For every pairing x - y it defines how many games team x and team y should win.
- ▶ The flow leaving the team-node x can be interpreted as the additional number of wins that team x will obtain.
- ▶ This is less than $M - w_x$ because of capacity constraints.
- ▶ Hence, we found a set of results for the remaining games, such that no team obtains more than M wins in total.
- ▶ Hence, team z is not eliminated.

Project Selection

Project selection problem:

- ▶ Set P of possible projects. Project v has an associated profit p_v (can be positive or negative).

Project Selection

Project selection problem:

- ▶ Set P of possible projects. Project v has an associated profit p_v (can be positive or negative).
- ▶ Some projects have requirements (taking course EA2 requires course EA1).

Project Selection

Project selection problem:

- ▶ Set P of possible projects. Project v has an associated profit p_v (can be positive or negative).
- ▶ Some projects have requirements (taking course EA2 requires course EA1).
- ▶ Dependencies are modelled in a graph. Edge (u, v) means “can’t do project u without also doing project v .”

Project Selection

Project selection problem:

- ▶ Set P of possible projects. Project v has an associated profit p_v (can be positive or negative).
- ▶ Some projects have requirements (taking course EA2 requires course EA1).
- ▶ Dependencies are modelled in a graph. Edge (u, v) means “can’t do project u without also doing project v .”
- ▶ A subset A of projects is **feasible** if the prerequisites of every project in A also belong to A .

Project Selection

Project selection problem:

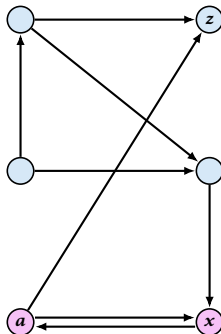
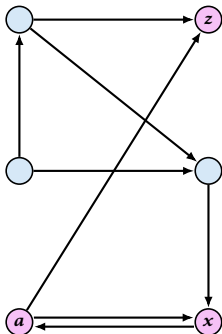
- ▶ Set P of possible projects. Project v has an associated profit p_v (can be positive or negative).
- ▶ Some projects have requirements (taking course EA2 requires course EA1).
- ▶ Dependencies are modelled in a graph. Edge (u, v) means “can’t do project u without also doing project v .”
- ▶ A subset A of projects is **feasible** if the prerequisites of every project in A also belong to A .

Goal: Find a feasible set of projects that maximizes the profit.

Project Selection

The prerequisite graph:

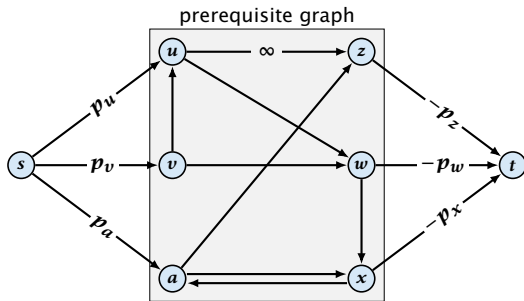
- ▶ $\{x, a, z\}$ is a feasible subset.
- ▶ $\{x, a\}$ is infeasible.



Project Selection

Mincut formulation:

- ▶ Edges in the prerequisite graph get infinite capacity.
- ▶ Add edge (s, v) with capacity p_v for nodes v with positive profit.
- ▶ Create edge (v, t) with capacity $-p_v$ for nodes v with negative profit.



Theorem 64

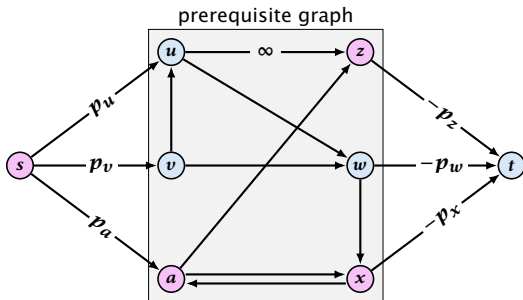
A is a mincut if $A \setminus \{s\}$ is the optimal set of projects.

Theorem 64

A is a mincut if $A \setminus \{s\}$ is the optimal set of projects.

Proof.

- ▶ A is feasible because of capacity infinity edges.

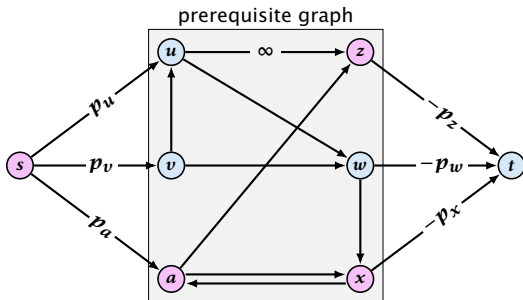


Theorem 64

A is a mincut if $A \setminus \{s\}$ is the optimal set of projects.

Proof.

- ▶ A is feasible because of capacity infinity edges.
- ▶ $\text{cap}(A, V \setminus A)$



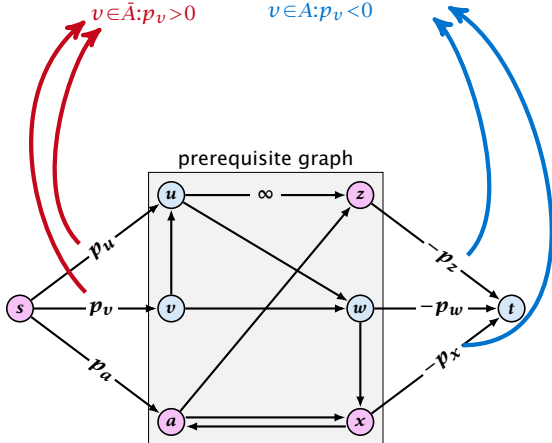
Theorem 64

A is a mincut if $A \setminus \{s\}$ is the optimal set of projects.

Proof.

▶ A is feasible because of capacity infinity edges.

▶ $\text{cap}(A, V \setminus A) = \sum_{v \in \bar{A}: p_v > 0} p_v + \sum_{v \in A: p_v < 0} (-p_v)$



Theorem 64

A is a mincut if $A \setminus \{s\}$ is the optimal set of projects.

Proof.

▶ A is feasible because of capacity infinity edges.

$$\text{cap}(A, V \setminus A) = \sum_{v \in \bar{A}: p_v > 0} p_v + \sum_{v \in A: p_v < 0} (-p_v)$$

$$= \sum_{v: p_v > 0} p_v - \sum_{v \in A} p_v$$

