Part II

Foundations

3 Goals

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- Learn how to analyze and judge the efficiency of algorithms.
- Learn how to design efficient algorithms.

4 Modelling Issues

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- Memory requirement

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- Implementing and testing on representative inputs
  - How do you choose your inputs?
  - May be very time-consuming.
  - Very reliable results if done correctly.
  - Results only hold for a specific machine and for a specific set of inputs.

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- Theoretical analysis in a specific model of computation.
  - Gives asymptotic bounds like “this algorithm always runs in time $\mathcal{O}(n^2)$”.
  - Typically focuses on the worst case.
  - Can give lower bounds like “any comparison-based sorting algorithm needs at least $\Omega(n \log n)$ comparisons in the worst case”.

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Example 1

Suppose n numbers from the interval $\{1, \dots, N\}$ have to be sorted. In this case we usually say that the input length is n instead of e.g. $n \log N$, which would be the number of bits required to encode the input.

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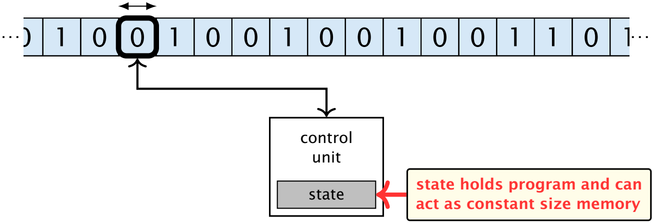
Version 2. is often easier, but focusing on one type of operation makes it more difficult to obtain meaningful results.

Turing Machine

- Very simple model of computation.



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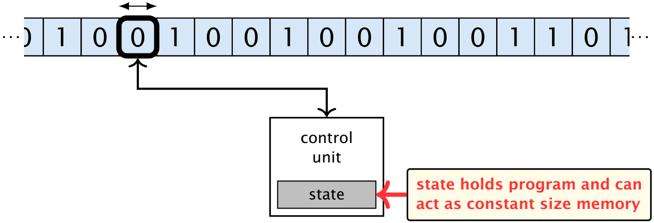

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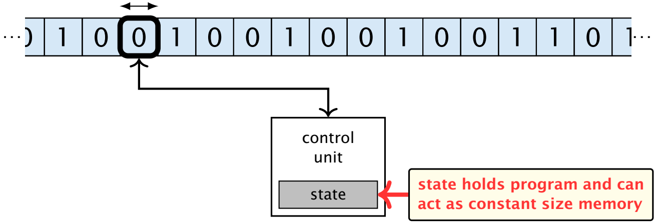

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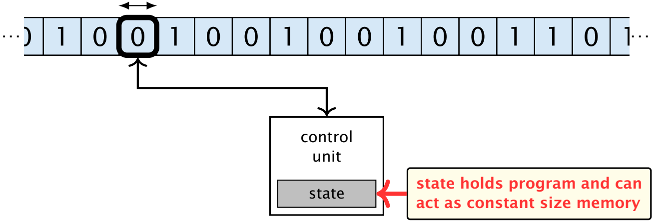

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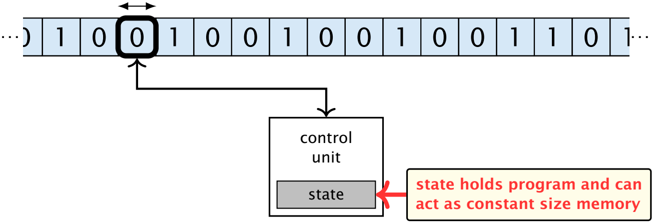
$\Longrightarrow$ **Not a good model for developing efficient algorithms.**



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Random Access Machine (RAM)

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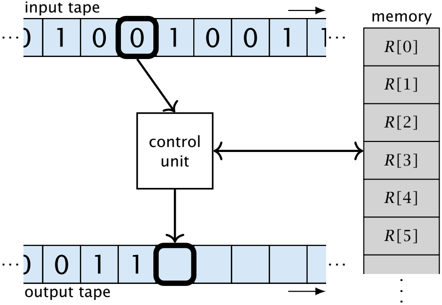

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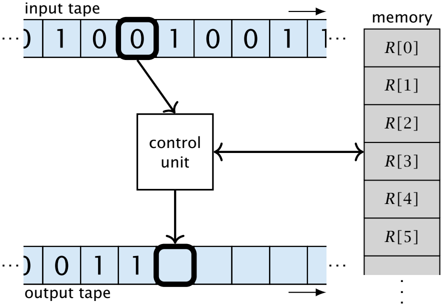

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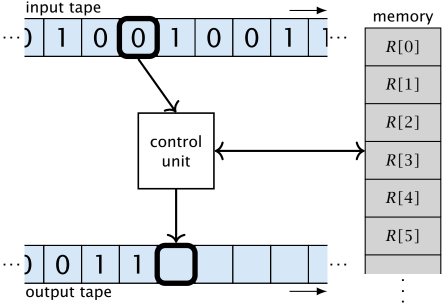

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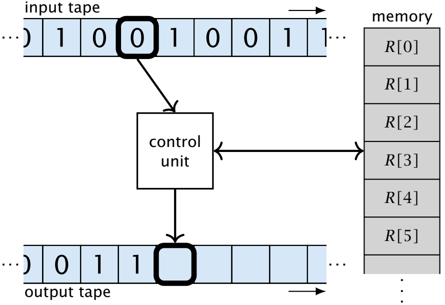

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  - $R[i]$ `:=` $R[j]$ `+` $R[k]$;
    $R[i]$ `:=` `-`$R[k]$;

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Model of Computation

- uniform cost model
  Every operation takes time 1.

The latter model is quite realistic as the word-size of a standard computer that handles a problem of size n must be at least $\log_2 n$ as otherwise the computer could either not store the problem instance or not address all its memory.

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Bounded word RAM model: cost is uniform but the largest value stored in a register may not exceed 2^w, where usually $w = \log_2 n$.

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4 Modelling Issues

Example 2

Algorithm 1 RepeatedSquaring(n)

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$r \leftarrow 2$;
for $i = 1 \rightarrow n$ do
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There are different types of complexity bounds:

- best-case complexity:

$$C_{\text{bc}}(n) := \min\{C(x) \mid |x| = n\}$$

Usually easy to analyze, but not very meaningful.

μ is a probability distribution over inputs of length n.

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  more general: probability measure μ

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- amortized complexity:
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- randomized complexity:
  The algorithm may use random bits. Expected running time (over all possible choices of random bits) for a fixed input x. Then take the worst-case over all x with $|x| = n$.

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- Running time should be expressed by simple functions.

Asymptotic Notation

Formal Definition

Let f, g denote functions from $\mathbb{N}$ to $\mathbb{R}^+$.

- $\mathcal{O}(f) = \{g \mid \exists c > 0\ \exists n_0 \in \mathbb{N}_0\ \forall n \geq n_0 \colon [g(n) \leq c \cdot f(n)]\}$
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There is an equivalent definition using limes notation (assuming that the respective limes exists). f and g are functions from $\mathbb{N}_0$ to $\mathbb{R}_0^+$.

- $g \in \mathcal{O}(f)$: $\quad 0 \leq \lim_{n\to\infty} \frac{g(n)}{f(n)} < \infty$

- Note that for the version of the Landau notation defined here, we assume that f and g are positive functions.
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Abuse of notation

1. People write $f = \mathcal{O}(g)$, when they mean $f \in \mathcal{O}(g)$. This is **not** an equality (how could a function be equal to a set of functions).

2. In this context $f(n)$ does **not** mean the function f evaluated at n, but instead it is a shorthand for the function itself (leaving out domain and codomain and only giving the rule of correspondence of the function).

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4. People write $\mathcal{O}(f(n)) = \mathcal{O}(g(n))$, when they mean $\mathcal{O}(f(n)) \subseteq \mathcal{O}(g(n))$. Again this is not an equality.

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Asymptotic Notation in Equations

How do we interpret an expression like:

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Note that $\Theta(n)$ is on the right hand side, otw. this interpretation is wrong.

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Regardless of how we choose the anonymous function $f(n) \in \mathcal{O}(n)$ there is an anonymous function $g(n) \in \Theta(n^2)$ that makes the expression true.

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How do we interpret an expression like:

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Careful!

Asymptotic Notation in Equations

The $\Theta(i)$-symbol on the left represents one anonymous function $f : \mathbb{N} \to \mathbb{R}^+$, and then $\sum_i f(i)$ is computed.

How do we interpret an expression like:

$$\sum_{i=1}^{n} \Theta(i) = \Theta(n^2)$$

Careful!

"It is understood" that every occurence of an $\mathcal{O}$-symbol (or $\Theta, \Omega, o, \omega$) on the left represents one anonymous function.

Hence, the left side is **not** equal to

$$\Theta(1) + \Theta(2) + \cdots + \Theta(n-1) + \Theta(n)$$

$\Theta(1) + \Theta(2) + \cdots + \Theta(n-1) + \Theta(n)$ does not really have a reasonable interpretation.

Asymptotic Notation in Equations

We can view an expression containing asymptotic notation as generating a set:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n)$$

represents

$$\Big\{f : \mathbb{N} \to \mathbb{R}^+ \mid f(n) = n^2 \cdot g(n) + h(n)$$

$$\text{with } g(n) \in \mathcal{O}(n) \text{ and } h(n) \in \mathcal{O}(\log n)\Big\}$$

Recall that according to the previous slide e.g. the expressions $\sum_{i=1}^{n} \mathcal{O}(i)$ and $\sum_{i=1}^{n/2} \mathcal{O}(i) + \sum_{i=n/2+1}^{n} \mathcal{O}(i)$ generate different sets.

Asymptotic Notation in Equations

Then an asymptotic equation can be interpreted as containement btw. two sets:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) = \Theta(n^2)$$

represents

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) \subseteq \Theta(n^2)$$

Note that the equation does not hold.

Asymptotic Notation

Lemma 3

Let f, g *be functions with the property* $\exists n_0 > 0\,\forall n \geq n_0 : f(n) > 0$ *(the same for* g*). Then*

- $c \cdot f(n) \in \Theta(f(n))$ *for any constant* c

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The expressions also hold for Ω. Note that this means that $f(n) + g(n) \in \Theta(\max\{f(n), g(n)\})$.

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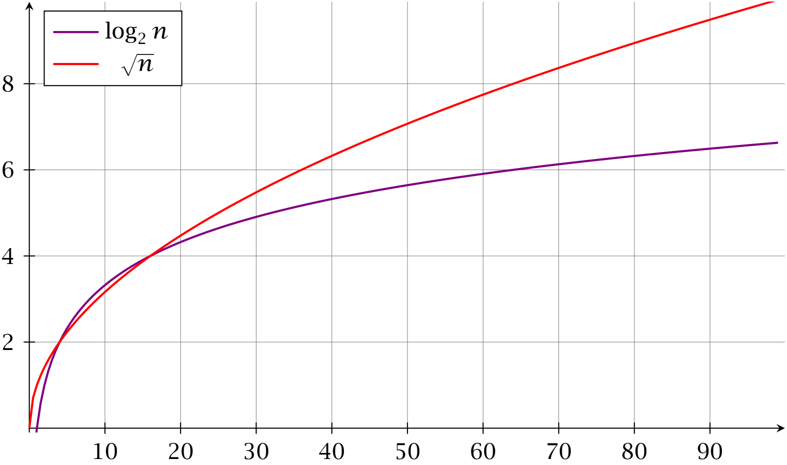
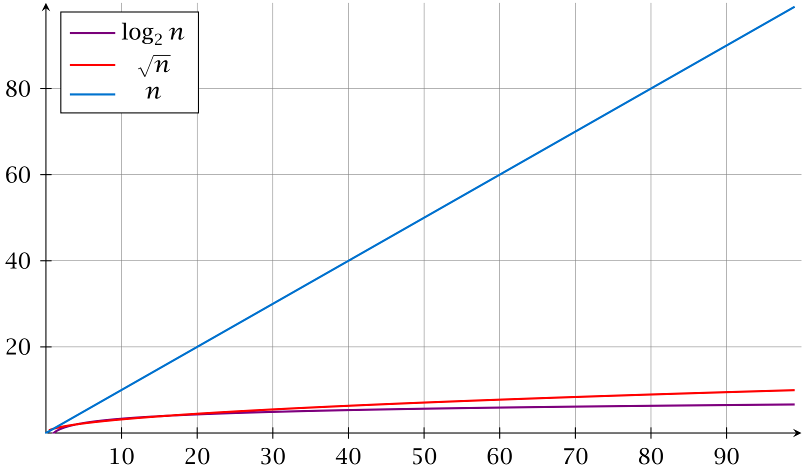
- For any constants a, b we have $\log_a n = \Theta(\log_b n)$. Therefore, we will usually ignore the base of a logarithm within asymptotic notation.
- In general $\log n = \log_2 n$, i.e., we use 2 as the default base for the logarithm.

Funktionen



Funktionen



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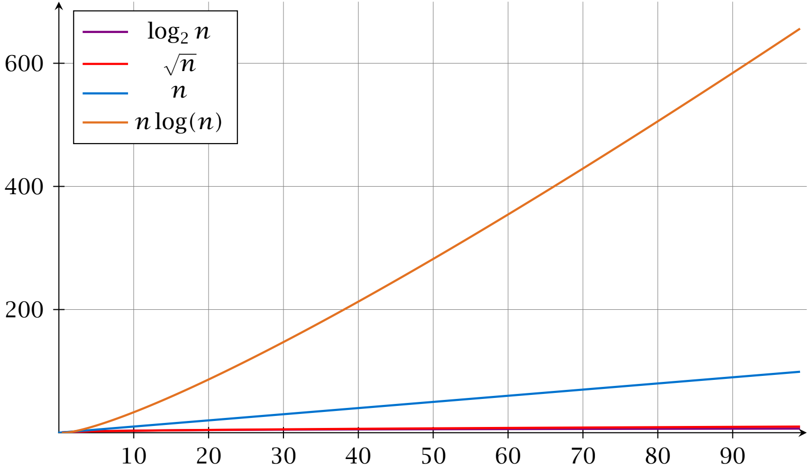

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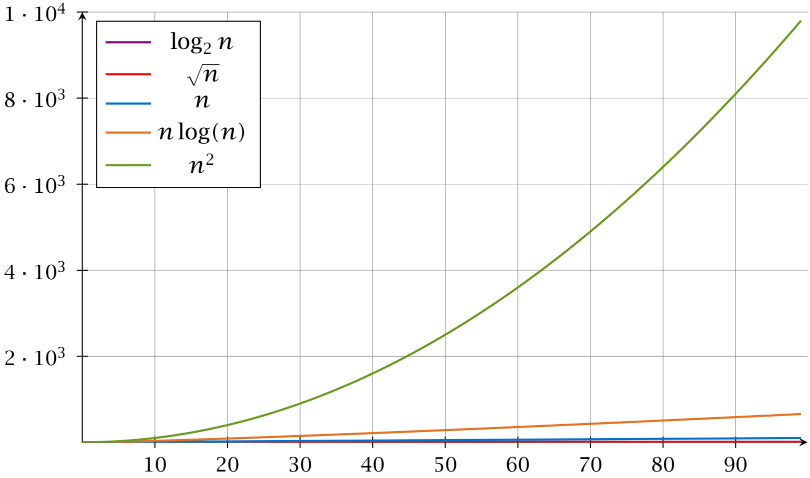
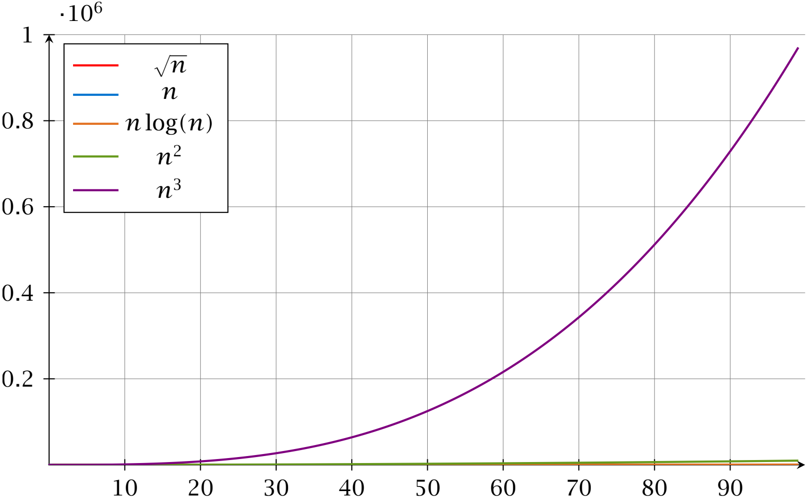

$\log_2 n$
$\sqrt{n}$
n
$n\log(n)$
n^2
$1 \cdot 10^4$
$8 \cdot 10^3$
$6 \cdot 10^3$
$4 \cdot 10^3$
$2 \cdot 10^3$
10
20
30
40
50
60
70
80
90

Funktionen



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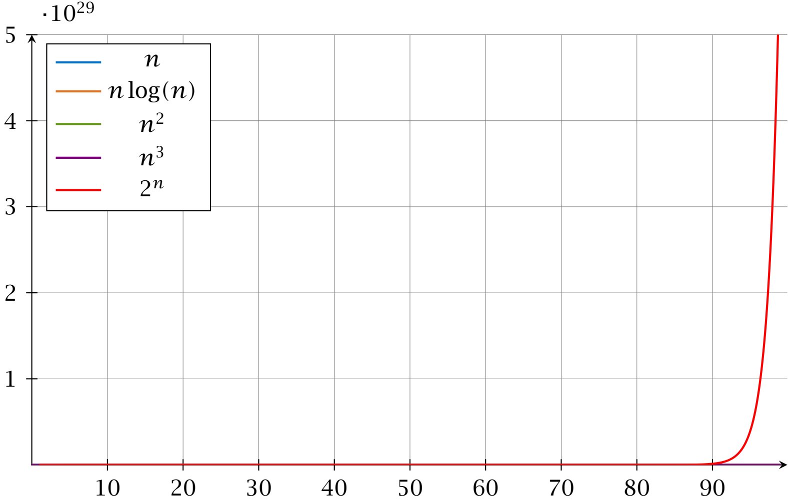

Laufzeiten

Funktion	***Eingabelänge n***							
$\boldsymbol{f(n)}$	**10**	$\mathbf{10^2}$	$\mathbf{10^3}$	$\mathbf{10^4}$	$\mathbf{10^5}$	$\mathbf{10^6}$	$\mathbf{10^7}$	$\mathbf{10^8}$
$\log n$	33ns	66ns	0.1μs	0.1μs	0.2μs	0.2μs	0.2μs	0.3μs
$\sqrt{n}$	32ns	0.1μs	0.3μs	1μs	3.1μs	10μs	31μs	0.1ms
n	100ns	1μs	10μs	0.1ms	1ms	10ms	0.1s	1s
$n \log n$	0.3μs	6.6μs	0.1ms	1.3ms	16ms	0.2s	2.3s	27s
$n^{3/2}$	0.3μs	10μs	0.3ms	10ms	0.3s	10s	5.2min	2.7h
n^2	1μs	0.1ms	10ms	1s	1.7min	2.8h	11d	3.2y
n^3	10μs	10ms	10s	2.8h	115d	317y	$3.2 \cdot 10^5$y	
1.1^n	26ns	0.1ms	$7.8 \cdot 10^{25}$y					
2^n	10μs	$4 \cdot 10^{14}$y						
$n!$	36ms	$3 \cdot 10^{142}$y						

1 Operation = 10ns; 100MHz

Alter des Universums: ca. $13.8 \cdot 10^9$y

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In general asymptotic classification of running times is a good measure for comparing algorithms:

- If the running time analysis is tight and actually occurs in practise (i.e., the asymptotic bound is not a purely theoretical worst-case bound), then the algorithm that has better asymptotic running time will always outperform a weaker algorithm for large enough values of n.

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  Clearly $f = o(g)$. However, as long as $\log n \leq 1000$ Algorithm B will be more efficient.

Multiple Variables in Asymptotic Notation

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Example 4

- $f : \mathbb{N} \to \mathbb{R}_0^+$, $f(n,m) = 1$ und $g : \mathbb{N} \to \mathbb{R}_0^+$, $g(n,m) = n - 1$

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6 Recurrences

Algorithm 2 mergesort(list L)

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$n \leftarrow \text{size}(L)$
if $n \le 1$ return $L$
$L_1 \leftarrow L[1 \cdots \lfloor \frac{n}{2} \rfloor]$
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```

This algorithm requires

$$T(n) = T\left(\left\lceil \frac{n}{2} \right\rceil\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \mathcal{O}(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + \mathcal{O}(n)$$

comparisons when $n > 1$ and 0 comparisons when $n \leq 1$.

Recurrences

How do we bring the expression for the number of comparisons ($\approx$ running time) into a closed form?

Recurrences

How do we bring the expression for the number of comparisons ($\approx$ running time) into a closed form?

For this we need to solve the recurrence.

Methods for Solving Recurrences

1. **Guessing+Induction**
   Guess the right solution and prove that it is correct via induction. It needs experience to make the right guess.
2. **Master Theorem**
   For a lot of recurrences that appear in the analysis of algorithms this theorem can be used to obtain tight asymptotic bounds. It does not provide exact solutions.
3. **Characteristic Polynomial**
   Linear homogenous recurrences can be solved via this method.

Methods for Solving Recurrences

4. **Generating Functions**
   A more general technique that allows to solve certain types of linear inhomogenous relations and also sometimes non-linear recurrence relations.
5. **Transformation of the Recurrence**
   Sometimes one can transform the given recurrence relations so that it e.g. becomes linear and can therefore be solved with one of the other techniques.

6.1 Guessing+Induction

First we need to get rid of the $\mathcal{O}$-notation in our recurrence:

$$T(n) \leq \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

Informal way:

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Informal way:

Assume that instead we have

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

One way of solving such a recurrence is to guess a solution, and check that it is correct by plugging it in.

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Suppose we guess $T(n) \leq dn\log n$ for a constant d.

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if we choose $d \ge c$.

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if we choose $d \ge c$.

Formally, this is not correct if n is not a power of 2. Also even in this case one would need to do an induction proof.

6.1 Guessing+Induction

$$T(n) \le \begin{cases} 2T(\frac{n}{2}) + cn & n \ge 16 \\ b & \text{otw.} \end{cases}$$

- Note that this proves the statement for $n = 2^k$, $k \in \mathbb{N}_{\ge 1}$, as the statement is wrong for $n = 1$.
- The base case is usually omitted, as it is the same for different recurrences.

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Hence, statement is true if we choose $d \ge c$.

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We consider the following recurrence instead of the original one:

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Note that we can do this as for constant-sized inputs the running time is always some constant (b in the above case).

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We also make a guess of $T(n) \leq dn\log n$ and get

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\boxed{\log n \le \tfrac{n}{4}} \quad &\le dn\log n + (\log 9 - 3.5)dn + cn
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\boxed{\log \tfrac{9}{16}n = \log n + (\log 9 - 4)} \quad & = dn \log n + (\log 9 - 4)dn + 2d \log n + cn \\
\boxed{\log n \leq \tfrac{n}{4}} \quad & \leq dn \log n + (\log 9 - 3.5)dn + cn \\
 & \leq dn \log n - 0.33dn + cn
\end{aligned}
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&\le dn\log n - 0.33dn + cn\\
&\le dn\log n
\end{aligned}$$

for a suitable choice of d.

6.2 Master Theorem

Note that the cases do not cover all possibilities.

Lemma 5

Let $a \geq 1, b > 1$ *and* $\epsilon > 0$ *denote constants. Consider the recurrence*

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) \ .$$

Case 1.

If $f(n) = \mathcal{O}(n^{\log_b(a)-\epsilon})$ *then* $T(n) = \Theta(n^{\log_b a})$.

Case 2.

If $f(n) = \Theta(n^{\log_b(a)} \log^k n)$ *then* $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$, $k \geq 0$.

Case 3.

If $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ *and for sufficiently large* n $af(\frac{n}{b}) \leq cf(n)$ *for some constant* $c < 1$ *then* $T(n) = \Theta(f(n))$.

6.2 Master Theorem

We prove the Master Theorem for the case that n is of the form b^ℓ, and we assume that the non-recursive case occurs for problem size 1 and incurs cost 1.

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

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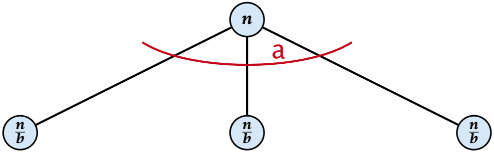

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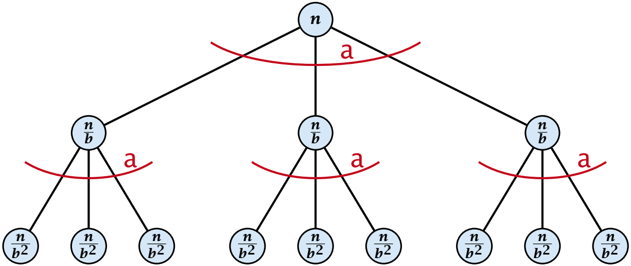
The running time of a recursive algorithm can be visualized by a recursion tree:



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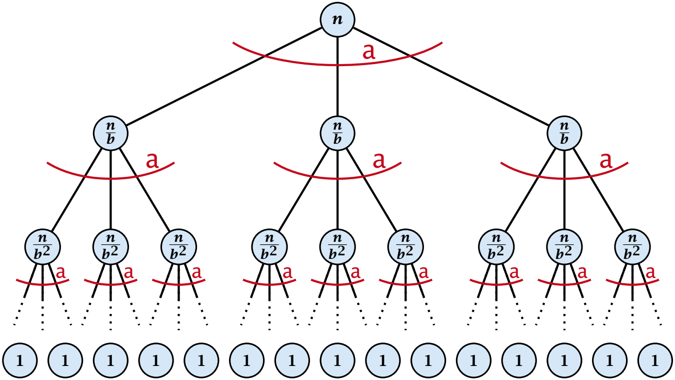

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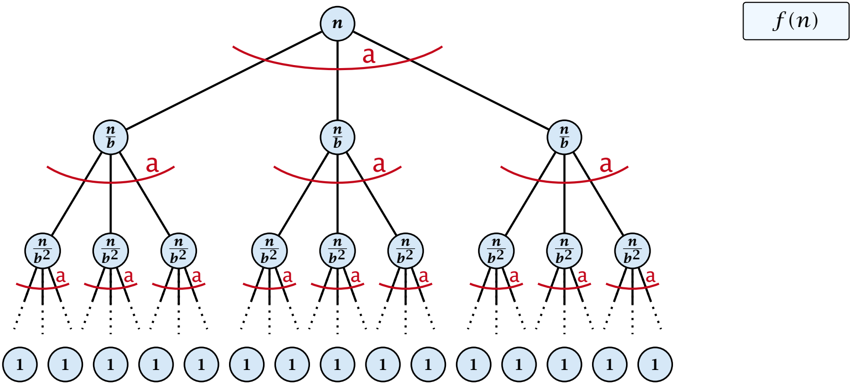

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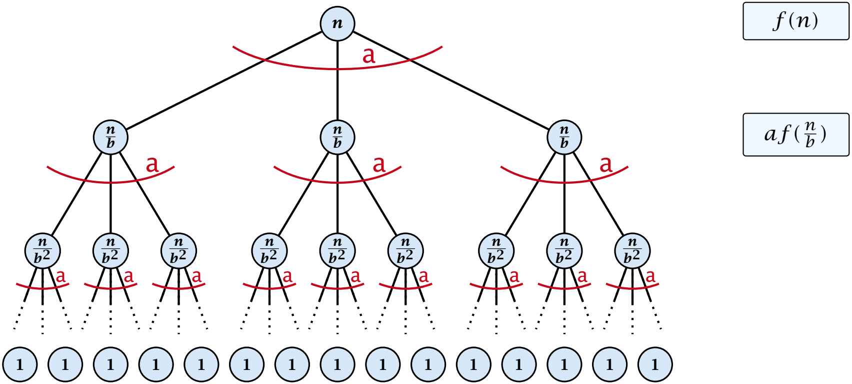

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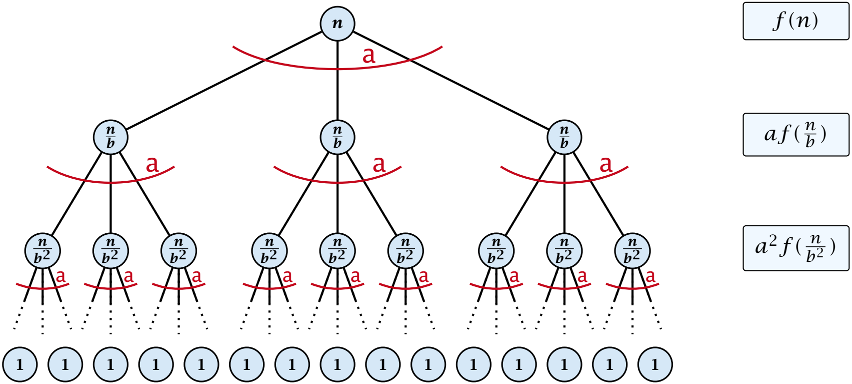

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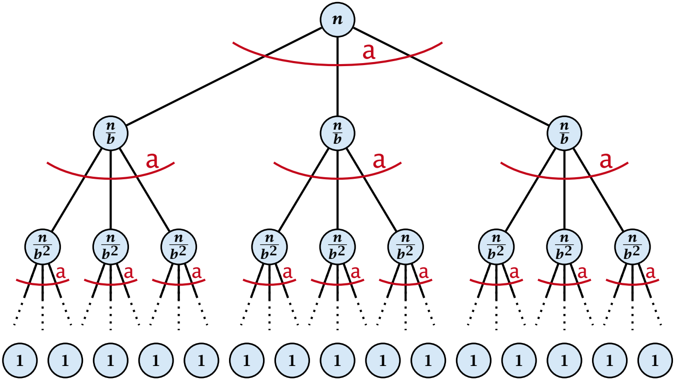

$f(n)$

$af(\frac{n}{b})$

$a^2f(\frac{n}{b^2})$

$a^{\log_b n}$

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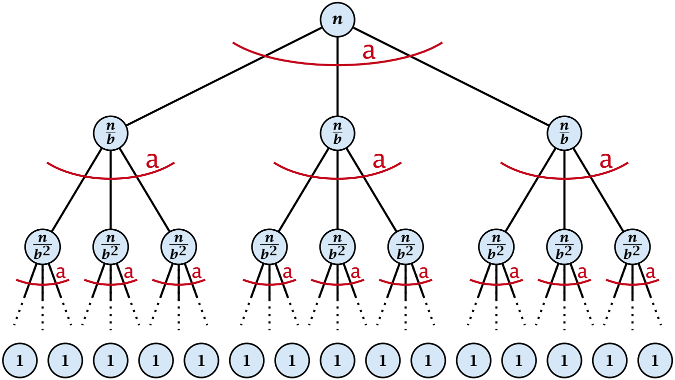

$f(n)$

$af(\frac{n}{b})$

$a^2f(\frac{n}{b^2})$

$a^{\log_b n}$

$=$

$n^{\log_b a}$

6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \ .$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

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$$T(n) - n^{\log_b a}$$

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a-\epsilon}$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a-\epsilon} \end{aligned}$$

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$$\boxed{b^{-i(\log_b a-\epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i}a^{-i}}$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a-\epsilon}$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a-\epsilon} \\ &= cn^{\log_b a-\epsilon} \sum_{i=0}^{\log_b n-1} \left(b^\epsilon\right)^i \end{aligned}$$

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$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)\\
&\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a-\epsilon}\\
&= cn^{\log_b a-\epsilon} \sum_{i=0}^{\log_b n-1} \left(b^{\epsilon}\right)^i
\end{aligned}
$$

$$\boxed{b^{-i(\log_b a-\epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i}a^{-i}}$$

$$\boxed{\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}}$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\ &\le c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a-\epsilon} \\ &= cn^{\log_b a-\epsilon} \sum_{i=0}^{\log_b n-1} (b^{\epsilon})^i \\ &= cn^{\log_b a-\epsilon}(b^{\epsilon \log_b n} - 1)/(b^{\epsilon} - 1) \end{aligned}$$

Case 1. Now suppose that $f(n) \le cn^{\log_b a-\epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)$$

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$$\boxed{b^{-i(\log_b a-\epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i}a^{-i}} \quad = cn^{\log_b a-\epsilon} \sum_{i=0}^{\log_b n-1} (b^{\epsilon})^i$$

$$\boxed{\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}} \quad = cn^{\log_b a-\epsilon}(b^{\epsilon \log_b n} - 1)/(b^{\epsilon} - 1)$$

$$= cn^{\log_b a-\epsilon}(n^{\epsilon} - 1)/(b^{\epsilon} - 1)$$

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$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \\
&= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} \left(b^\epsilon\right)^i \\
&= cn^{\log_b a - \epsilon}(b^{\epsilon \log_b n} - 1)/(b^\epsilon - 1) \\
&= cn^{\log_b a - \epsilon}(n^\epsilon - 1)/(b^\epsilon - 1) \\
&= \frac{c}{b^\epsilon - 1} n^{\log_b a}(n^\epsilon - 1)/(n^\epsilon)
\end{aligned}
$$

$$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}}$$

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\boxed{b^{-i(\log_b a-\epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i}a^{-i}} \quad &= cn^{\log_b a-\epsilon} \sum_{i=0}^{\log_b n-1} (b^\epsilon)^i \\
\boxed{\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}} \quad &= cn^{\log_b a-\epsilon}(b^{\epsilon \log_b n} - 1)/(b^\epsilon - 1) \\
&= cn^{\log_b a-\epsilon}(n^\epsilon - 1)/(b^\epsilon - 1) \\
&= \frac{c}{b^\epsilon - 1} n^{\log_b a}(n^\epsilon - 1)/(n^\epsilon)
\end{aligned}$$

Hence,

$$T(n) \leq \left(\frac{c}{b^\epsilon - 1} + 1\right) n^{\log_b(a)}$$

Case 1. Now suppose that $f(n) \le cn^{\log_b a - \epsilon}$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \end{aligned}$$

$$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} \quad = cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^{\epsilon})^i$$

$$\boxed{\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}} \quad = cn^{\log_b a - \epsilon}(b^{\epsilon \log_b n} - 1)/(b^{\epsilon} - 1)$$

$$\begin{aligned} &= cn^{\log_b a - \epsilon}(n^{\epsilon} - 1)/(b^{\epsilon} - 1) \\ &= \frac{c}{b^{\epsilon} - 1} n^{\log_b a}(n^{\epsilon} - 1)/(n^{\epsilon}) \end{aligned}$$

Hence,

$$T(n) \le \left(\frac{c}{b^{\epsilon} - 1} + 1\right) n^{\log_b (a)} \qquad \boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a}).}$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}$.

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

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Hence,

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&\geq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}
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Hence,

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Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n)$$

$$\boxed{\Rightarrow T(n) = \Omega(n^{\log_b a} \log n).}$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}(\log_b(n))^k$.

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$$\boxed{n = b^\ell \Rightarrow \ell = \log_b n}$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\ &\le c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \\ \boxed{n = b^\ell \Rightarrow \ell = \log_b n} \quad &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \end{aligned}$$

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T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \\
&= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\
&= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k
\end{aligned}$$

$\boxed{n = b^\ell \Rightarrow \ell = \log_b n}$

Case 2. Now suppose that $f(n) \le cn^{\log_b a}(\log_b(n))^k$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)$$

$$\le c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k$$

$$\boxed{n = b^\ell \Rightarrow \ell = \log_b n}$$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k$$

$$= cn^{\log_b a} \sum_{i=1}^{\ell} i^k$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}(\log_b(n))^k$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \\ &= cn^{\log_b a} \sum_{i=0}^{\ell - 1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\ &= cn^{\log_b a} \sum_{i=0}^{\ell - 1} (\ell - i)^k \\ &= cn^{\log_b a} \boxed{\sum_{i=1}^{\ell} i^k} \approx \tfrac{1}{k}\ell^{k+1} \end{aligned}$$

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Case 2. Now suppose that $f(n) \leq cn^{\log_b a}(\log_b(n))^k$.

$$\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \\
\boxed{n = b^\ell \Rightarrow \ell = \log_b n} \quad &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\
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\end{aligned}$$

Case 2. Now suppose that $f(n) \le cn^{\log_b a}(\log_b(n))^k$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
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&\approx \frac{c}{k} n^{\log_b a} \ell^{k+1}
\end{aligned}
$$

$$\boxed{n = b^\ell \Rightarrow \ell = \log_b n}$$

$$\boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n).}$$

Case 3. Now suppose that $f(n) \geq dn^{\log_b a+\epsilon}$, and that for sufficiently large n: $af(n/b) \leq cf(n)$, for $c < 1$.

Where did we use $f(n) \geq \Omega(n^{\log_b a+\epsilon})$?

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n-1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

Where did we use $f(n) \geq \Omega(n^{\log_b a+\epsilon})$?

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$q < 1 : \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$

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Hence,

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq \sum_{i=0}^{\log_b n-1} c^i f(n) + \mathcal{O}(n^{\log_b a})$$

$$\boxed{q < 1 : \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}} \qquad \leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a})$$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

$$\boxed{\Rightarrow T(n) = \Theta(f(n)).}$$

Where did we use $f(n) \geq \Omega(n^{\log_b a+\epsilon})$?

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

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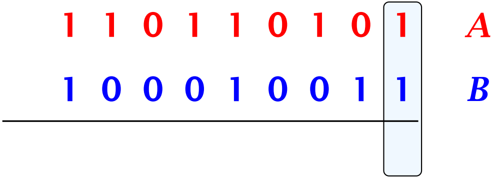

$$\begin{array}{ccccccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & \boxed{1} & & A \\
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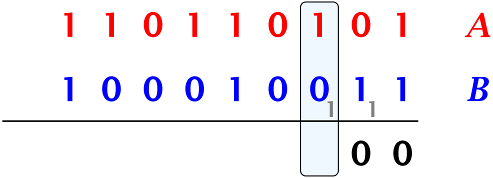

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
						1	1		
							0	0	

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\hline
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				0	1	1	1		
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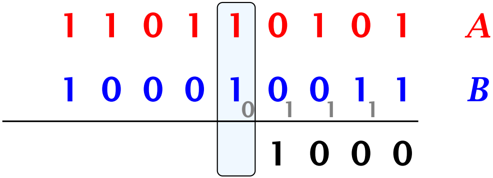

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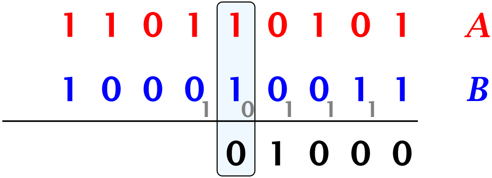

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 & & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
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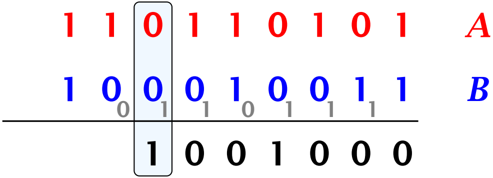

1	1	0	1	1	0	1	0	1	A
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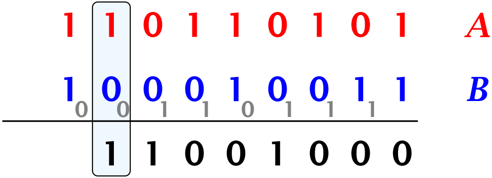

1	1	0	1	1	0	1	0	1	A
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 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\
\scriptstyle 1 & \scriptstyle 0 & \scriptstyle 0 & \scriptstyle 1 & \scriptstyle 1 & \scriptstyle 0 & \scriptstyle 1 & \scriptstyle 1 & \scriptstyle 1 & & \\
\hline
 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 &
\end{array}$$

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 {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
\hline
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & &
\end{array}$$

This gives that two n-bit integers can be added in time $\mathcal{O}(n)$.

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

- This is also nown as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \le 2n$ bits.

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Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

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\hline
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 & & & & & 1 & 0 & 0 & 0 & 1 \\
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\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
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 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0
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$$\begin{array}{rrrrrrrrrr}
 & 1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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\hline
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Time requirement:

Example: Multiplying Two Integers

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$$\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
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- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

Time requirement:

- Computing intermediate results: $\mathcal{O}(nm)$.

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
 & & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1
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- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

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- Computing intermediate results: $\mathcal{O}(nm)$.
- Adding m numbers of length $\leq 2n$: $\mathcal{O}((m+n)m) = \mathcal{O}(nm)$.

Example: Multiplying Two Integers

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Suppose that integers A and B are of length $n = 2^k$, for some k.

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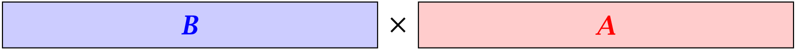

$B \times A$

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Suppose that integers A and B are of length $n = 2^k$, for some k.

$$b_{n-1} \quad \cdots \quad b_0 \times a_{n-1} \quad \cdots \quad a_0$$

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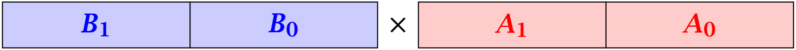

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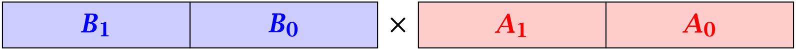

B_1 B_0 × A_1 A_0

Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

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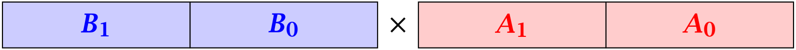

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Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 B_0$$

Example: Multiplying Two Integers

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if $|A| = |B| = 1$ then
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We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) \ .$$

Example: Multiplying Two Integers

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- Case 1: $f(n) = \mathcal{O}(n^{\log_b a-\epsilon})$ $\quad T(n) = \Theta(n^{\log_b a})$
- Case 2: $f(n) = \Theta(n^{\log_b a}\log^k n)$ $\quad T(n) = \Theta(n^{\log_b a}\log^{k+1} n)$
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In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a-\epsilon})$.

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$\Longrightarrow$ Not better then the “school method”.

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We can use the following identity to compute Z_1:

A more precise (correct) analysis would say that computing Z_1 needs time $T(\frac{n}{2}+1)+\mathcal{O}(n)$.

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$Z_0 \leftarrow \text{mult}(A_0, B_0)$                   $T(\frac{n}{2})$
$Z_1 \leftarrow \text{mult}(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$   $T(\frac{n}{2}) + \mathcal{O}(n)$
return $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$
```

A more precise (correct) analysis would say that computing Z_1 needs time $T(\frac{n}{2} + 1) + \mathcal{O}(n)$.

Example: Multiplying Two Integers

We can use the following identity to compute Z_1:

$$\begin{aligned} Z_1 &= A_1B_0 + A_0B_1 \\ &= (A_0 + A_1)\cdot(B_0 + B_1) - \overbrace{A_1B_1}^{=\,Z_2} - \overbrace{A_0B_0}^{=\,Z_0} \end{aligned}$$

Hence,

Algorithm 4 mult(A, B)

```
if $|A| = |B| = 1$ then                                  $\mathcal{O}(1)$
    return $a_0 \cdot b_0$                               $\mathcal{O}(1)$
split $A$ into $A_0$ and $A_1$                           $\mathcal{O}(n)$
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- Case 1: $f(n) = \mathcal{O}(n^{\log_b a-\epsilon})$ $\quad T(n) = \Theta(n^{\log_b a})$
- Case 2: $f(n) = \Theta(n^{\log_b a}\log^k n)$ $\quad T(n) = \Theta(n^{\log_b a}\log^{k+1} n)$
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Again we are in Case 1. We get a running time of $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$.

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Again we are in Case 1. We get a running time of $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$.

A huge improvement over the “school method”.

6.3 The Characteristic Polynomial

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \cdots + c_kT(n-k) = f(n)$$

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Note that we ignore boundary conditions for the moment.

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Approach:

- First determine all solutions that satisfy recurrence relation.
- Then pick the right one by analyzing boundary conditions.
- First consider the homogenous case.

The Homogenous Case

The solution space

$$S = \Big\{ \mathcal{T} = T[1], T[2], T[3], \ldots \;\Big|\; \mathcal{T} \text{ fulfills recurrence relation} \Big\}$$

is a vector space.

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How do we find a non-trivial solution?

We guess that the solution is of the form λ^n, $\lambda \neq 0$, and see what happens. In order for this guess to fulfill the recurrence we need

$$c_0\lambda^n + c_1\lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \cdots + c_k \cdot \lambda^{n-k} = 0$$

for all $n \geq k$.

The Homogenous Case

Dividing by λ^{n-k} gives that all these constraints are identical to

$$c_0\lambda^k + c_1\lambda^{k-1} + c_2 \cdot \lambda^{k-2} + \cdots + c_k = 0$$

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Let $\lambda_1, \dots, \lambda_k$ be the k (complex) roots of $P[\lambda]$. Then, because of the vector space property

$$\alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \cdots + \alpha_k\lambda_k^n$$

is a solution for arbitrary values α_i.

The Homogenous Case

Lemma 6

Assume that the characteristic polynomial has k distinct roots $\lambda_1,\ldots,\lambda_k$. Then all solutions to the recurrence relation are of the form

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Proof.

There is one solution for every possible choice of boundary conditions for $T[1], \dots, T[k]$.

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There is one solution for every possible choice of boundary conditions for $T[1], \dots, T[k]$.

We show that the above set of solutions contains one solution for every choice of boundary conditions.

The Homogenous Case

Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i's$ such that these conditions are met:

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$$\alpha_1 \cdot \lambda_1 \quad + \quad \alpha_2 \cdot \lambda_2 \quad + \quad \cdots \quad + \quad \alpha_k \cdot \lambda_k \quad = \quad T[1]$$

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Proof (cont.).

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 & & & & \vdots & & & & \\
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We show that the column vectors are linearly independent. Then the above equation has a solution.

Computing the Determinant

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} =$$

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$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^{k} \lambda_i \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_{k-1}^{k-1} & \lambda_k^{k-1} \end{vmatrix}$$

$$= \prod_{i=1}^{k} \lambda_i \cdot \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{k-2} & \lambda_1^{k-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{k-2} & \lambda_2^{k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-2} & \lambda_k^{k-1} \end{vmatrix}$$

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$$\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2} \end{vmatrix}$$

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$$\prod_{i=2}^{k}(\lambda_i - \lambda_1) \cdot \begin{vmatrix} 1 & \lambda_2 & \cdots & \lambda_2^{k-3} & \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-3} & \lambda_k^{k-2} \end{vmatrix}$$

Computing the Determinant

Repeating the above steps gives:

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^{k} \lambda_i \cdot \prod_{i>\ell} (\lambda_i - \lambda_\ell)$$

Hence, if all λ_i's are different, then the determinant is non-zero.

The Homogeneous Case

What happens if the roots are not all distinct?

The Homogeneous Case

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To see this consider the polynomial

$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_k\lambda^{n-k}$$

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Since λ_i is a root we can write this as $Q[\lambda] \cdot (\lambda - \lambda_i)^2$. Calculating the derivative gives a polynomial that still has root λ_i.

This means

$$c_0 n\lambda_i^{n-1} + c_1(n-1)\lambda_i^{n-2} + \cdots + c_k(n-k)\lambda_i^{n-k-1} = 0$$

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Hence,

$$c_0 \underbrace{n\lambda_i^{n}}_{T[n]} + c_1 \underbrace{(n-1)\lambda_i^{n-1}}_{T[n-1]} + \cdots + c_k \underbrace{(n-k)\lambda_i^{n-k}}_{T[n-k]} = 0$$

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We can continue $j-1$ times.

Hence, $n^\ell\lambda_i^n$ is a solution for $\ell \in 0,\ldots,j-1$.

The Homogeneous Case

Lemma 7

Let $P[\lambda]$ denote the characteristic polynomial to the recurrence

$$c_0T[n] + c_1T[n-1] + \cdots + c_kT[n-k] = 0$$

Let λ_i, $i = 1, \dots, m$ be the (complex) roots of $P[\lambda]$ with multiplicities ℓ_i. Then the general solution to the recurrence is given by

$$T[n] = \sum_{i=1}^{m} \sum_{j=0}^{\ell_i - 1} \alpha_{ij} \cdot (n^j \lambda_i^n) \enspace .$$

The full proof is omitted. We have only shown that any choice of α_{ij}'s is a solution to the recurrence.

Example: Fibonacci Sequence

$$T[0] = 0$$

$$T[1] = 1$$

$$T[n] = T[n-1] + T[n-2] \text{ for } n \geq 2$$

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Finding the roots, gives

$$\lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$$

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Hence, the solution is of the form

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$$\alpha\left(\frac{1+\sqrt{5}}{2}\right)+\beta\left(\frac{1-\sqrt{5}}{2}\right)=1\Longrightarrow\alpha-\beta=\frac{2}{\sqrt{5}}$$

Example: Fibonacci Sequence

Hence, the solution is

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

The Inhomogeneous Case

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \cdots + c_kT(n-k) = f(n)$$

with $f(n) \neq 0$.

While we have a fairly general technique for solving homogeneous, linear recurrence relations the inhomogeneous case is different.

The Inhomogeneous Case

The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n) \ ,$$

where T_h is any solution to the homogeneous equation, and T_p is one particular solution to the inhomogeneous equation.

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There is no general method to find a particular solution.

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$$T[n] = T[n-1] + 1 \qquad T[0] = 1$$

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I get a completely determined recurrence if I add $T[0] = 1$ and $T[1] = 2$.

The Inhomogeneous Case

Example: Characteristic polynomial:

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$T[0] = 1$ gives $\alpha = 1$.

$T[1] = 2$ gives $1 + \beta = 2 \Longrightarrow \beta = 1$.

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Shift:

$$\begin{aligned} T[n-1] &= 2T[n-2] - T[n-3] + 2(n-1) - 1 \\ &= 2T[n-2] - T[n-3] + 2n - 3 \end{aligned}$$

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and so on...

6.4 Generating Functions

Definition 8 (Generating Function)

Let $(a_n)_{n\geq 0}$ be a sequence. The corresponding

- generating function (Erzeugendenfunktion) is

$$F(z) := \sum_{n\geq 0} a_n z^n \;;$$

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- exponential generating function (exponentielle Erzeugendenfunktion) is
$$F(z) := \sum_{n\geq 0} \frac{a_n}{n!} z^n \ .$$

6.4 Generating Functions

Example 9

1. The generating function of the sequence $(1, 0, 0, \ldots)$ is

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- **Multiplication:** $f \cdot g := \sum_{n\geq 0} c_n z^n$ with $c_n = \sum_{p=0}^{n} a_p b_{n-p}$.

There are no convergence issues here.

6.4 Generating Functions

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Then, it is important to think about convergence/convergence radius etc.

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It means that the power series $1 - z$ and the power series $\sum_{n\geq 0} z^n$ are invers, i.e.,

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This is well-defined.

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6.4 Generating Functions

Formally the derivative of a formal power series $\sum_{n\geq 0} a_n z^n$ is defined as $\sum_{n\geq 0} n a_n z^{n-1}$.

The known rules for differentiation work for this definition. In particular, e.g. the derivative of $\frac{1}{1-z}$ is $\frac{1}{(1-z)^2}$.

Note that this requires a proof if we consider power series as algebraic objects. However, we did not prove this in the lecture.

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Hence, the generating function of the sequence $a_n = n+1$ is $1/(1-z)^2$.

6.4 Generating Functions

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Hence, the generating function of the sequence $a_n = (n+1)(n+2)$ is $\frac{2}{(1-z)^3}$.

6.4 Generating Functions

Computing the k-th derivative of $\sum z^n$.

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Hence:

$$\sum_{n\geq 0}\binom{n+k}{k}z^n = \frac{1}{(1-z)^{k+1}} \ .$$

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Hence:

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The generating function of the sequence $a_n = \binom{n+k}{k}$ is $\frac{1}{(1-z)^{k+1}}$.

6.4 Generating Functions

$$\sum_{n\geq 0} nz^n = \sum_{n\geq 0}(n+1)z^n - \sum_{n\geq 0} z^n$$

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The generating function of the sequence $a_n = n$ is $\frac{z}{(1-z)^2}$.

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Hence,

$$\sum_{n\geq 0} a^n z^n = \frac{1}{1-az}$$

6.4 Generating Functions

We know

$$\sum_{n\geq 0} y^n = \frac{1}{1-y}$$

Hence,

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The generating function of the sequence $f_n = a^n$ is $\frac{1}{1-az}$.

Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

$$A(z)$$

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&= zA(z) + \sum_{n\geq 0} z^n \\
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Hence, $a_n = n + 1$.

Some Generating Functions

n-th sequence element	*generating function*

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$\frac{1}{n!}$	e^z

Some Generating Functions

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Some Generating Functions

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6. The coefficients of the resulting power series are the a_n.

Example: $a_n = 2a_{n-1}, a_0 = 1$

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&= 1 + 3zA(z) + \frac{z}{(1-z)^2}
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$$\begin{aligned} z^2 - z + 1 &= A(1-z)^2 + B(1-3z)(1-z) + C(1-3z) \\ &= A(1-2z+z^2) + B(1-4z+3z^2) + C(1-3z) \end{aligned}$$

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This gives

$$\begin{aligned} z^2 - z + 1 &= A(1-z)^2 + B(1-3z)(1-z) + C(1-3z) \\ &= A(1-2z+z^2) + B(1-4z+3z^2) + C(1-3z) \\ &= (A+3B)z^2 + (-2A-4B-3C)z + (A+B+C) \end{aligned}$$

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$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

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$$\begin{aligned} A(z) &= \frac{7}{4} \cdot \frac{1}{1-3z} - \frac{1}{4} \cdot \frac{1}{1-z} - \frac{1}{2} \cdot \frac{1}{(1-z)^2} \\ &= \frac{7}{4} \cdot \sum_{n \ge 0} 3^n z^n - \frac{1}{4} \cdot \sum_{n \ge 0} z^n - \frac{1}{2} \cdot \sum_{n \ge 0} (n+1) z^n \\ &= \sum_{n \ge 0} \Big(\frac{7}{4} \cdot 3^n - \frac{1}{4} - \frac{1}{2}(n+1) \Big) z^n \end{aligned}$$

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&= \sum_{n\geq 0} \Big(\frac{7}{4} \cdot 3^n - \frac{1}{2} n - \frac{3}{4}\Big) z^n
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6. This means $a_n = \frac{7}{4}3^n - \frac{1}{2}n - \frac{3}{4}$.

6.5 Transformation of the Recurrence

Example 10

$$
\begin{aligned}
f_0 &= 1 \\
f_1 &= 2 \\
f_n &= f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 \;.
\end{aligned}
$$

6.5 Transformation of the Recurrence

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$$\begin{aligned} f_0 &= 1 \\ f_1 &= 2 \\ f_n &= f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 \ . \end{aligned}$$

Define

$$g_n := \log f_n \ .$$

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g_n &= F_n \ (n\text{-th Fibonacci number})
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$$f_n = 2^{F_n}$$

6.5 Transformation of the Recurrence

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\begin{aligned}
f_1 &= 1 \\
f_n &= 3f_{\frac{n}{2}} + n;\text{ for } n = 2^k,\ k \geq 1\ ;
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6 Recurrences

We get

$$g_k = 3\left[g_{k-1}\right] + 2^k$$

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$$\begin{aligned} g_k &= 3\left[g_{k-1}\right] + 2^k \\ &= 3\left[3g_{k-2} + 2^{k-1}\right] + 2^k \end{aligned}$$

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g_k &= 3\left[g_{k-1}\right] + 2^k \\
&= 3\left[3g_{k-2} + 2^{k-1}\right] + 2^k \\
&= 3^2\left[g_{k-2}\right] + 32^{k-1} + 2^k
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&= 3^2\left[3g_{k-3} + 2^{k-2}\right] + 32^{k-1} + 2^k \\
&= 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k \\
&= 2^k \cdot \sum_{i=0}^{k} \left(\frac{3}{2}\right)^i
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&= 2^k \cdot \sum_{i=0}^{k} \left(\frac{3}{2}\right)^i \\
&= 2^k \cdot \frac{(\frac{3}{2})^{k+1} - 1}{1/2}
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&= 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k \\
&= 2^k \cdot \sum_{i=0}^{k} \left(\frac{3}{2}\right)^i \\
&= 2^k \cdot \frac{(\frac{3}{2})^{k+1} - 1}{1/2} = 3^{k+1} - 2^{k+1}
\end{aligned}$$

6 Recurrences

Let $n = 2^k$:

$$g_k = 3^{k+1} - 2^{k+1}, \text{ hence}$$
$$f_n = 3 \cdot 3^k - 2 \cdot 2^k$$

6 Recurrences

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