

Part II

Foundations

3 Goals

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- ▶ Learn how to analyze and judge the efficiency of algorithms.
- ▶ Learn how to design efficient algorithms.

4 Modelling Issues

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- ▶ Implementing and testing on representative inputs
 - ▶ How do you choose your inputs?
 - ▶ May be very time-consuming.
 - ▶ Very reliable results if done correctly.
 - ▶ Results only hold for a specific machine and for a specific set of inputs.

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- ▶ Theoretical analysis in a specific **model of computation**.
 - ▶ Gives **asymptotic bounds** like “this algorithm always runs in time $\mathcal{O}(n^2)$ ”.
 - ▶ Typically focuses on the **worst case**.
 - ▶ Can give lower bounds like “any comparison-based sorting algorithm needs at least $\Omega(n \log n)$ comparisons in the worst case”.

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Example 1

Suppose n numbers from the interval $\{1, \dots, N\}$ have to be sorted. In this case we usually say that the input length is n instead of e.g. $n \log N$, which would be the number of bits required to encode the input.

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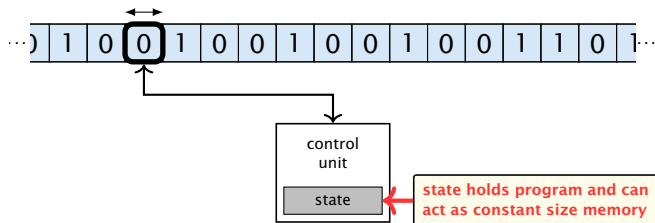
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Version 2. is often easier, but focusing on one type of operation makes it more difficult to obtain meaningful results.

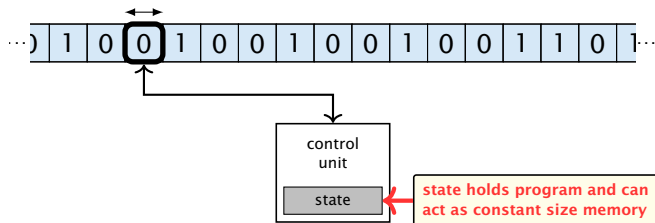
Turing Machine

- ▶ Very simple model of computation.



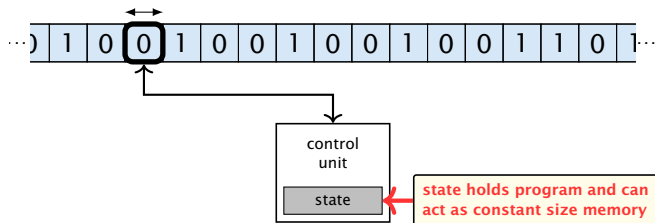
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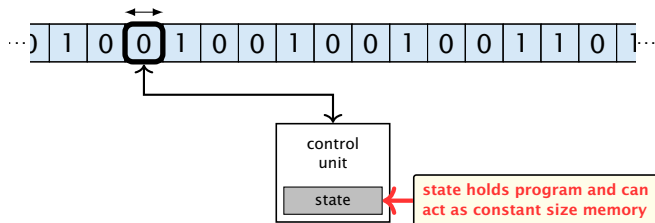
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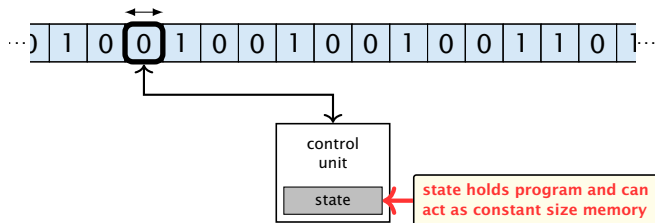
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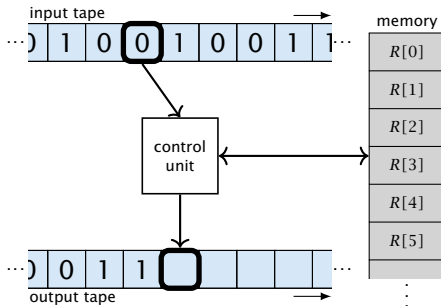
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⇒ **Not a good model for developing efficient algorithms.**



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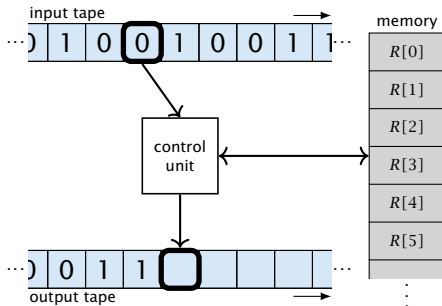
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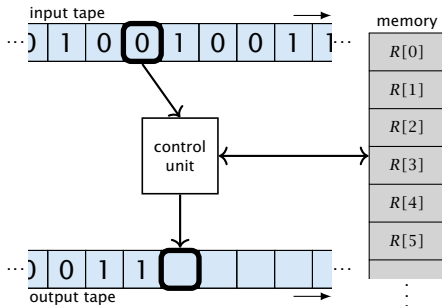
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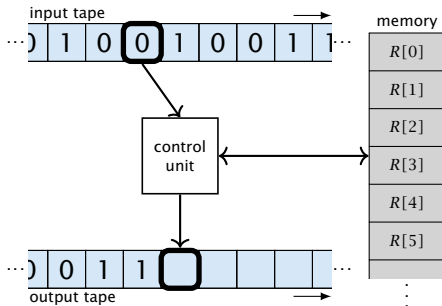
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 - ▶ $R[i] := R[j] + R[k];$
 $R[i] := -R[k];$

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Model of Computation

- ▶ **uniform** cost model
Every operation takes time 1.

The latter model is quite realistic as the word-size of a standard computer that handles a problem of size n must be at least $\log_2 n$ as otherwise the computer could either not store the problem instance or not address all its memory.

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Bounded word RAM model: cost is uniform but the largest value stored in a register may not exceed 2^w , where usually $w = \log_2 n$.

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Example 2

Algorithm 1 RepeatedSquaring(n)

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Usually easy to analyze, but not very meaningful.

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more general: probability measure μ

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▶ **randomized** complexity:

The algorithm may use random bits. Expected running time (over all possible choices of random bits) for a fixed input x . Then take the worst-case over all x with $|x| = n$.

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- ▶ Running time should be expressed by simple functions.

Asymptotic Notation

Formal Definition

Let f, g denote functions from \mathbb{N} to \mathbb{R}^+ .

- ▶ $\mathcal{O}(f) = \{g \mid \exists c > 0 \exists n_0 \in \mathbb{N}_0 \forall n \geq n_0 : [g(n) \leq c \cdot f(n)]\}$
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- ▶ $\Theta(f) = \Omega(f) \cap \mathcal{O}(f)$
(functions that asymptotically have **the same growth** as f)

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Formal Definition

Let f, g denote functions from \mathbb{N} to \mathbb{R}^+ .

- ▶ $\mathcal{O}(f) = \{g \mid \exists c > 0 \exists n_0 \in \mathbb{N}_0 \forall n \geq n_0 : [g(n) \leq c \cdot f(n)]\}$
(set of functions that asymptotically grow **not faster** than f)
- ▶ $\Omega(f) = \{g \mid \exists c > 0 \exists n_0 \in \mathbb{N}_0 \forall n \geq n_0 : [g(n) \geq c \cdot f(n)]\}$
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- ▶ $\Theta(f) = \Omega(f) \cap \mathcal{O}(f)$
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There is an equivalent definition using limes notation (**assuming that the respective limes exists**). f and g are functions from \mathbb{N}_0 to \mathbb{R}_0^+ .

$$\blacktriangleright g \in \mathcal{O}(f): 0 \leq \lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} < \infty$$

- Note that for the version of the Landau notation defined here, we assume that f and g are positive functions.
- There also exist versions for arbitrary functions, and for the case that the limes is not infinity.

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Abuse of notation

1. People write $f = \mathcal{O}(g)$, when they mean $f \in \mathcal{O}(g)$. This is **not** an equality (how could a function be equal to a set of functions).

2. In this context $f(n)$ does **not** mean the function f evaluated at n , but instead it is a shorthand for the function itself (leaving out domain and codomain and only giving the rule of correspondence of the function).

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4. People write $\mathcal{O}(f(n)) = \mathcal{O}(g(n))$, when they mean $\mathcal{O}(f(n)) \subseteq \mathcal{O}(g(n))$. Again this is not an equality.

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How do we interpret an expression like:

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Note that $\Theta(n)$ is on the right hand side, otw. this interpretation is wrong.

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How do we interpret an expression like:

$$2n^2 + \mathcal{O}(n) = \Theta(n^2)$$

Regardless of how we choose the anonymous function $f(n) \in \mathcal{O}(n)$ there is an anonymous function $g(n) \in \Theta(n^2)$ that makes the expression true.

Asymptotic Notation in Equations

How do we interpret an expression like:

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Careful!

Asymptotic Notation in Equations

The $\Theta(i)$ -symbol on the left represents **one** anonymous function $f : \mathbb{N} \rightarrow \mathbb{R}^+$, and then $\sum_i f(i)$ is computed.

How do we interpret an expression like:

$$\sum_{i=1}^n \Theta(i) = \Theta(n^2)$$

Careful!

“It is understood” that every occurrence of an Θ -symbol (or Ω, o, ω) on the left represents **one anonymous function**.

Hence, the left side is **not** equal to

$$\Theta(1) + \Theta(2) + \dots + \Theta(n-1) + \Theta(n)$$

$\Theta(1) + \Theta(2) + \dots + \Theta(n-1) + \Theta(n)$ does not really have a reasonable interpretation.

Asymptotic Notation in Equations

We can view an expression containing asymptotic notation as generating a set:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n)$$

represents

$$\{f : \mathbb{N} \rightarrow \mathbb{R}^+ \mid f(n) = n^2 \cdot g(n) + h(n)\}$$

with $g(n) \in \mathcal{O}(n)$ and $h(n) \in \mathcal{O}(\log n)$

Recall that according to the previous slide e.g. the expressions $\sum_{i=1}^n \mathcal{O}(i)$ and $\sum_{i=1}^{n/2} \mathcal{O}(i) + \sum_{i=n/2+1}^n \mathcal{O}(i)$ generate different sets.

Asymptotic Notation in Equations

Then an asymptotic equation can be interpreted as containment btw. two sets:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) = \Theta(n^2)$$

represents

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) \subseteq \Theta(n^2)$$

Note that the equation does not hold.

Asymptotic Notation

Lemma 3

Let f, g be functions with the property

$\exists n_0 > 0 \forall n \geq n_0 : f(n) > 0$ (the same for g). Then

- ▶ $c \cdot f(n) \in \Theta(f(n))$ for any constant c

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The expressions also hold for Ω . Note that this means that $f(n) + g(n) \in \Theta(\max\{f(n), g(n)\})$.

Comments

- ▶ Do not use asymptotic notation within induction proofs.

Comments

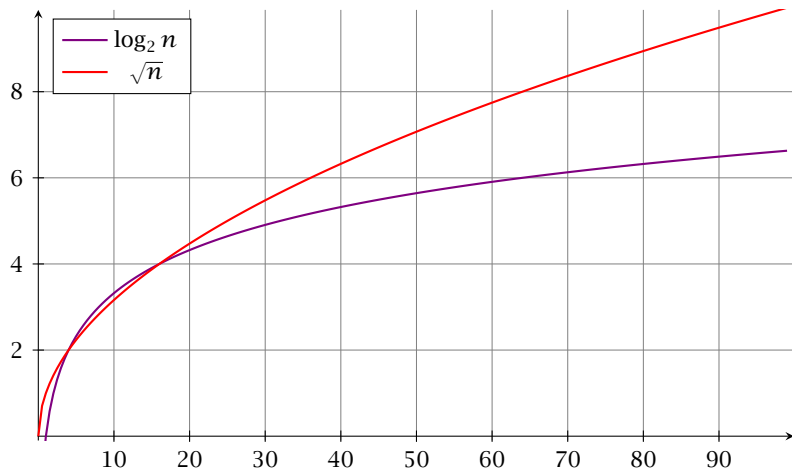
- ▶ Do not use asymptotic notation within induction proofs.
- ▶ For any constants a, b we have $\log_a n = \Theta(\log_b n)$.
Therefore, we will usually ignore the base of a logarithm within asymptotic notation.

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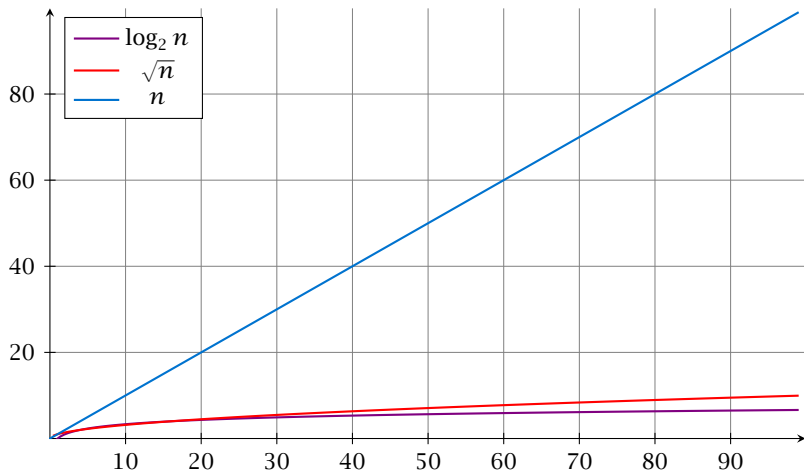
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Therefore, we will usually ignore the base of a logarithm within asymptotic notation.
- ▶ In general $\log n = \log_2 n$, i.e., we use 2 as the default base for the logarithm.

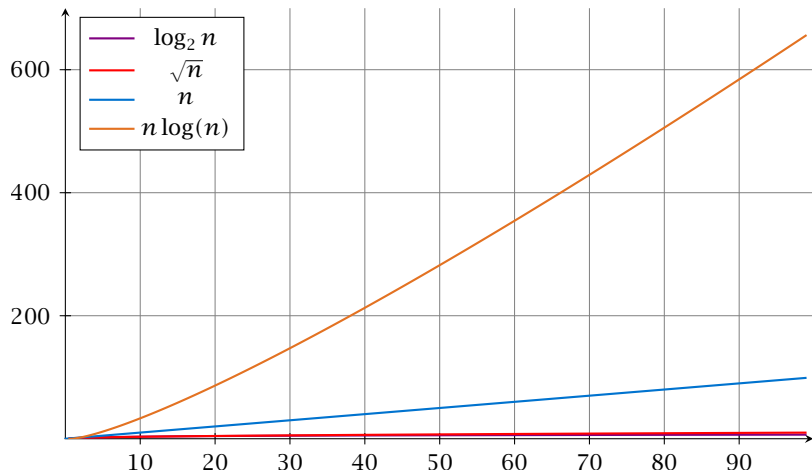
Funktionen



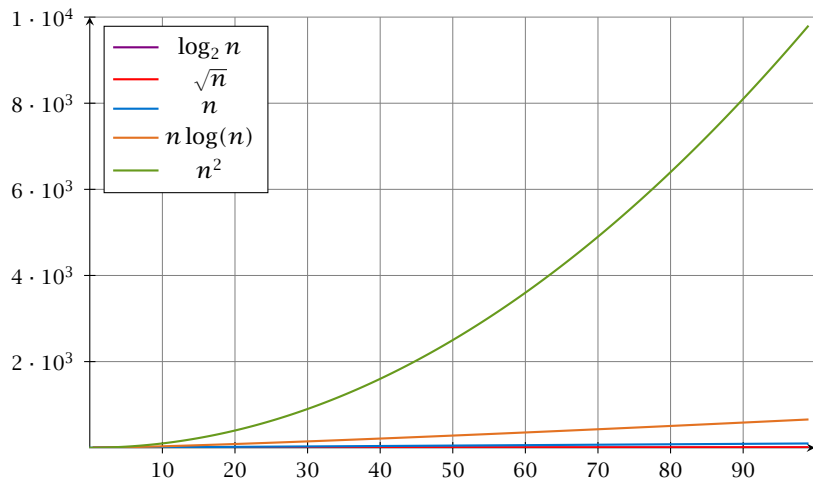
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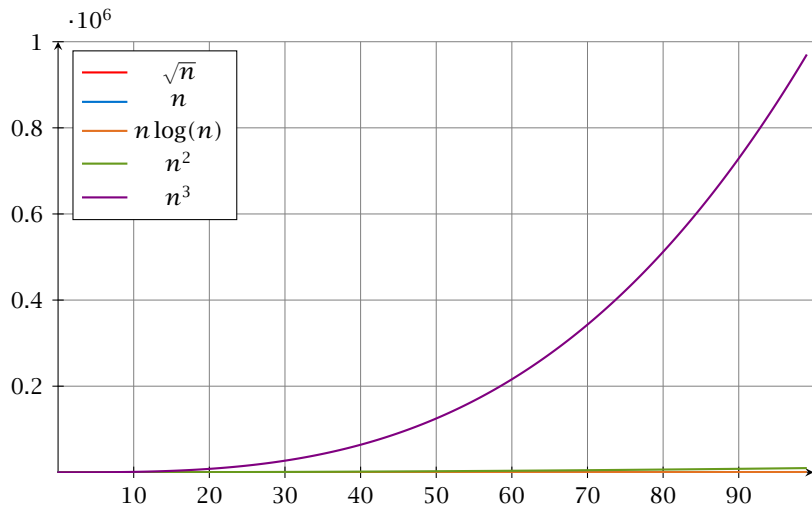
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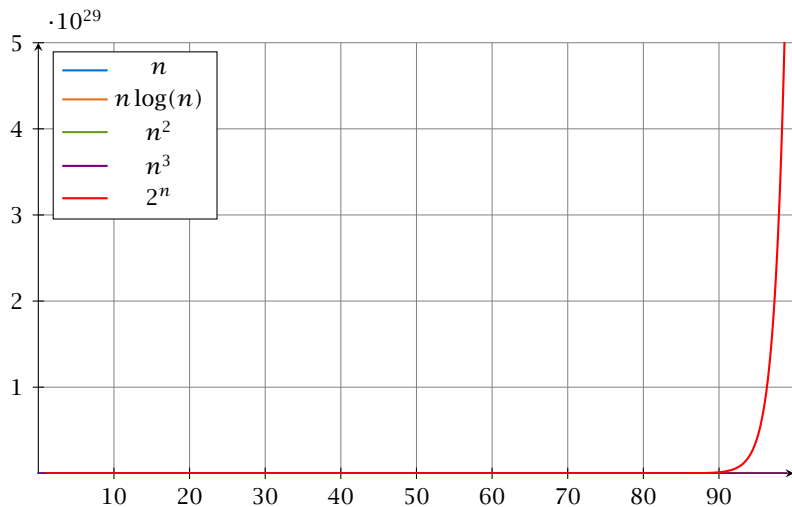
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Funktionen



Laufzeiten

Funktion	Eingabelänge n							
	10	10^2	10^3	10^4	10^5	10^6	10^7	10^8
$\log n$	33ns	66ns	0.1 μ s	0.1 μ s	0.2 μ s	0.2 μ s	0.2 μ s	0.3 μ s
\sqrt{n}	32ns	0.1 μ s	0.3 μ s	1 μ s	3.1 μ s	10 μ s	31 μ s	0.1ms
n	100ns	1 μ s	10 μ s	0.1ms	1ms	10ms	0.1s	1s
$n \log n$	0.3 μ s	6.6 μ s	0.1ms	1.3ms	16ms	0.2s	2.3s	27s
$n^{3/2}$	0.3 μ s	10 μ s	0.3ms	10ms	0.3s	10s	5.2min	2.7h
n^2	1 μ s	0.1ms	10ms	1s	1.7min	2.8h	11d	3.2y
n^3	10 μ s	10ms	10s	2.8h	115d	317y	$3.2 \cdot 10^5$ y	
1.1^n	26ns	0.1ms	$7.8 \cdot 10^{25}$ y					
2^n	10 μ s	$4 \cdot 10^{14}$ y						
$n!$	36ms	$3 \cdot 10^{142}$ y						

1 Operation = 10ns; 100MHz

Alter des Universums: ca. $13.8 \cdot 10^9$ y

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In general asymptotic classification of running times is a good measure for comparing algorithms:

- ▶ If the running time analysis is tight and actually occurs in practise (i.e., the asymptotic bound is not a purely theoretical worst-case bound), then the algorithm that has better asymptotic running time will always outperform a weaker algorithm for large enough values of n .

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Clearly $f = o(g)$. However, as long as $\log n \leq 1000$ Algorithm B will be more efficient.

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6 Recurrences

Algorithm 2 mergesort(list L)

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1:  $n \leftarrow \text{size}(L)$ 
2: if  $n \leq 1$  return  $L$ 
3:  $L_1 \leftarrow L[1 \cdots \lfloor \frac{n}{2} \rfloor]$ 
4:  $L_2 \leftarrow L[\lfloor \frac{n}{2} \rfloor + 1 \cdots n]$ 
5: mergesort( $L_1$ )
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```

This algorithm requires

$$T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \mathcal{O}(n) \leq 2T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \mathcal{O}(n)$$

comparisons when $n > 1$ and 0 comparisons when $n \leq 1$.

Recurrences

How do we bring the expression for the number of comparisons (\approx running time) into a **closed form**?

Recurrences

How do we bring the expression for the number of comparisons (\approx running time) into a **closed form**?

For this we need to **solve** the recurrence.

Methods for Solving Recurrences

1. Guessing+Induction

Guess the right solution and prove that it is correct via induction. It needs experience to make the right guess.

2. Master Theorem

For a lot of recurrences that appear in the analysis of algorithms this theorem can be used to obtain tight asymptotic bounds. It does not provide exact solutions.

3. Characteristic Polynomial

Linear homogenous recurrences can be solved via this method.

4. Generating Functions

A more general technique that allows to solve certain types of linear inhomogenous relations and also sometimes non-linear recurrence relations.

5. Transformation of the Recurrence

Sometimes one can transform the given recurrence relations so that it e.g. becomes linear and can therefore be solved with one of the other techniques.

6.1 Guessing+Induction

First we need to get rid of the \mathcal{O} -notation in our recurrence:

$$T(n) \leq \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

Informal way:

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Informal way:

Assume that instead we have

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

One way of solving such a recurrence is to **guess** a solution, and check that it is correct by plugging it in.

6.1 Guessing+Induction

Suppose we guess $T(n) \leq dn \log n$ for a constant d .

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if we choose $d \geq c$.

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if we choose $d \geq c$.

Formally, this is not correct if n is not a power of 2. Also even in this case one would need to do an induction proof.

6.1 Guessing+Induction

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

- Note that this proves the statement for $n = 2^k$, $k \in \mathbb{N}_{\geq 1}$, as the statement is wrong for $n = 1$.
- The base case is usually omitted, as it is the same for different recurrences.

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Hence, statement is **true** if we choose $d \geq c$.

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We consider the following recurrence instead of the original one:

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Note that we can do this as for constant-sized inputs the running time is always some constant (b in the above case).

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We also make a guess of $T(n) \leq dn \log n$ and get

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$$\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1$$

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$$\boxed{\log \frac{9}{16}n = \log n + (\log 9 - 4)} = dn \log n + (\log 9 - 4)dn + 2d \log n + cn$$

$$\boxed{\log n \leq \frac{n}{4}} \leq dn \log n + (\log 9 - 3.5)dn + cn$$

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$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

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$$\log \frac{9}{16}n = \log n + (\log 9 - 4)$$

$$= dn \log n + (\log 9 - 4)dn + 2d \log n + cn$$

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$$\leq dn \log n + (\log 9 - 3.5)dn + cn$$

$$\leq dn \log n - 0.33dn + cn$$

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$$\leq dn \log n + (\log 9 - 3.5)dn + cn$$

$$\leq dn \log n - 0.33dn + cn$$

$$\leq dn \log n$$

for a suitable choice of d .

6.2 Master Theorem

Note that the cases do not cover all possibilities.

Lemma 5

Let $a \geq 1$, $b \geq 1$ and $\epsilon > 0$ denote constants. Consider the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) .$$

Case 1.

If $f(n) = \mathcal{O}(n^{\log_b(a)-\epsilon})$ then $T(n) = \Theta(n^{\log_b a})$.

Case 2.

If $f(n) = \Theta(n^{\log_b(a)} \log^k n)$ then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$,
 $k \geq 0$.

Case 3.

If $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ and for sufficiently large n
 $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$ then $T(n) = \Theta(f(n))$.

6.2 Master Theorem

We prove the Master Theorem for the case that n is of the form b^ℓ , and we assume that the non-recursive case occurs for problem size 1 and incurs cost 1.

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

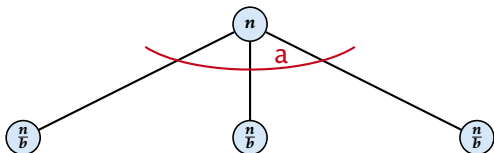
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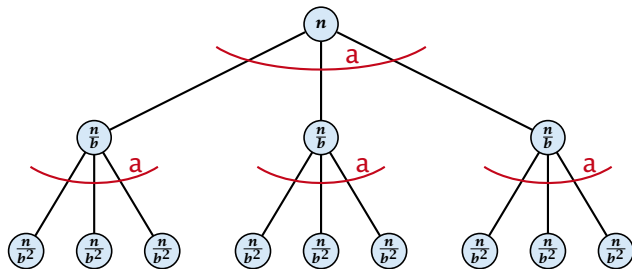
The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



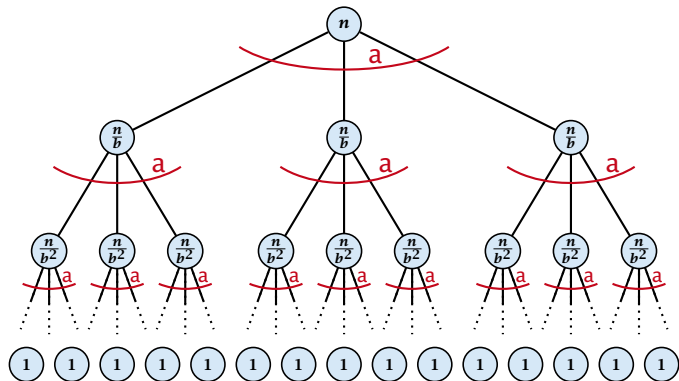
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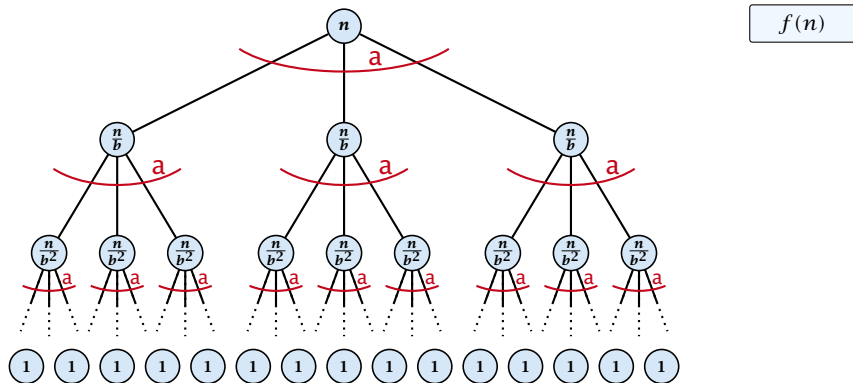
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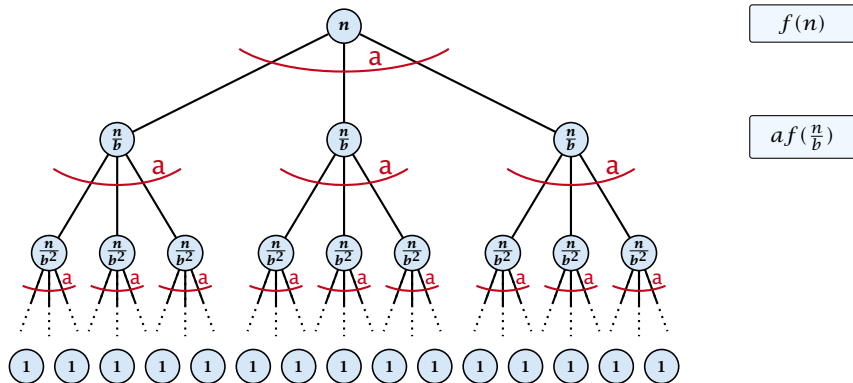
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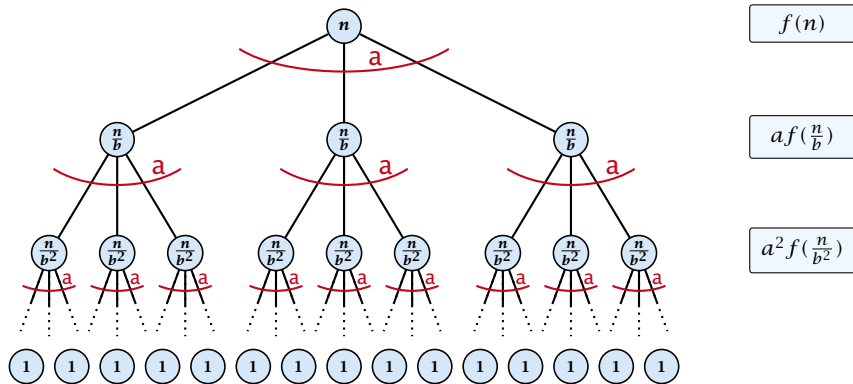
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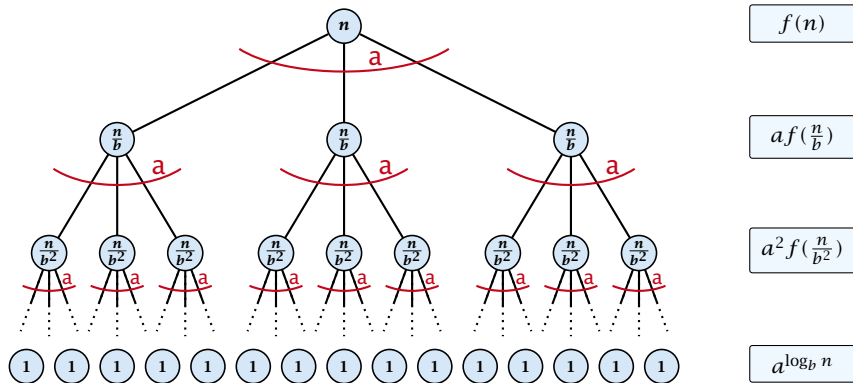
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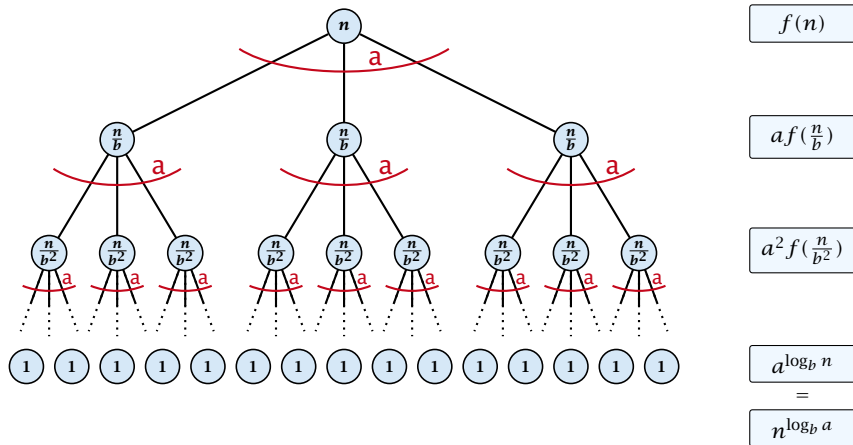
The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) .$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

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$$T(n) = n^{\log_b a}$$

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \end{aligned}$$

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$$b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

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$$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} = cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^{\epsilon})^i$$

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$$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} = cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^{\epsilon})^i$$

$$\boxed{\sum_{i=0}^k q^i = \frac{q^{k+1} - 1}{q - 1}}$$

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$$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} = cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^{\epsilon})^i$$

$$\boxed{\sum_{i=0}^k q^i = \frac{q^{k+1} - 1}{q - 1}} = cn^{\log_b a - \epsilon} (b^{\epsilon \log_b n} - 1) / (b^{\epsilon} - 1)$$

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Hence,

$$T(n) \leq \left(\frac{c}{b^{\epsilon} - 1} + 1 \right) n^{\log_b(a)}$$

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Hence,

$$T(n) \leq \left(\frac{c}{b^{\epsilon} - 1} + 1 \right) n^{\log_b(a)} \quad \Rightarrow T(n) = \mathcal{O}(n^{\log_b a}).$$

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Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n)$$

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Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n)$$

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Hence,

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$$\boxed{n = b^\ell \Rightarrow \ell = \log_b n} = cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b \left(\frac{b^\ell}{b^i}\right)\right)^k$$

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$$\begin{aligned} &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\ &= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \end{aligned}$$

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$$n = b^\ell \Rightarrow \ell = \log_b n$$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k$$

$$= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \approx \frac{1}{k} \ell^{k+1}$$

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$$= cn^{\log_b a} \sum_{i=1}^{\ell} i^k$$

$$\approx \frac{c}{k} n^{\log_b a} \ell^{k+1}$$

$$\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n).$$

Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large n : $af(n/b) \leq cf(n)$, for $c < 1$.

Where did we use $f(n) \geq \Omega(n^{\log_b a + \epsilon})$?

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

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From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

$$q < 1 : \sum_{i=0}^n q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$$

Where did we use $f(n) \geq \Omega(n^{\log_b a + \epsilon})$?

Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large n : $af(n/b) \leq cf(n)$, for $c < 1$.

From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\ &\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

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Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

Where did we use $f(n) \geq \Omega(n^{\log_b a + \epsilon})$?

Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large n : $af(n/b) \leq cf(n)$, for $c < 1$.

From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\ &\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

$$q < 1 : \sum_{i=0}^n q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

$$\Rightarrow T(n) = \Theta(f(n)).$$

Where did we use $f(n) \geq \Omega(n^{\log_b a + \epsilon})$?

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

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$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline \end{array}$$

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For this we first need to be able to add two integers A and B :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1 \\ \hline \end{array}$$

The diagram shows two 9-bit integers, A and B, aligned for addition. Integer A is represented by the red bits 1 1 0 1 1 0 1 0 1, and integer B is represented by the blue bits 1 0 0 0 1 0 0 1 1. A horizontal line is drawn under the bits of B. A vertical light blue box highlights the rightmost bit of A (the least significant bit) and the rightmost bit of B (the least significant bit), which are the bits being added together.

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Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1 \\ \hline 0 \end{array} \begin{array}{l} A \\ B \end{array}$$

The diagram shows the addition of two 9-bit integers, A and B. Integer A is 110110101 and integer B is 100010011. A horizontal line is drawn under the 9th bit of B. A carry bit '1' is shown below the line at the 9th bit position. The result of the addition at the 9th bit position is '0', shown in a box below the line.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1 \\ 0 \end{array}$$

The diagram illustrates the addition of two 10-bit integers, A and B. A vertical box highlights the 8th bit position (from the right), where a carry of 1 is generated from the addition of the 8th bits of A and B. This carry is then added to the 9th bit position of the result.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline 0\ 0 \end{array}$$

The diagram illustrates the addition of two 10-bit integers, A and B. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1 and the bits of B are 1, 0, 0, 0, 1, 0, 0, 1, 1. A horizontal line is drawn under the bits of B. The result of the addition is shown below the line, with the bits 0 and 0 highlighted in a light blue box. The carry bits are indicated by small '1's below the 7th and 8th bits of B.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & & & 1 & 1 & & \\ & & & & & & & 0 & 0 & \end{array}$$

The diagram illustrates the addition of two 9-bit integers, A and B. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1 and the bits of B are 1, 0, 0, 0, 1, 0, 0, 1, 1. A horizontal line is drawn under the numbers. A vertical light blue box highlights the 7th bit of A (1) and the 7th bit of B (0). Below the line, the 7th bit of the sum is 1, and the 8th bit of the sum is 1. Below the 8th bit, the 8th bit of the sum is 0 and the 9th bit of the sum is 0.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & & & 0 & 0 & 0 & \end{array}$$

The diagram illustrates the addition of two 9-bit integers, A and B, using a ripple carry adder. The bits of A are 1 1 0 1 1 0 1 0 1 and the bits of B are 1 0 0 0 1 0 0 1 1. A horizontal line is drawn under the bits of B. The sum is shown below the line as 0 0 0. A vertical box highlights the carry propagation from the 6th bit to the 7th bit, and from the 7th bit to the 8th bit, illustrating the ripple carry effect.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
					1	1	1		
						0	0	0	

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
					0	1	1	1	
					1	0	0	0	

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & 0 & 1 & 1 & 1 & & \\ & & & & & 1 & 0 & 0 & 0 & \end{array}$$

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For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
				1	0	1	1	1	
-----				0	1	0	0	0	

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For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
			1	0	1	1	1		
			0	1	0	0	0		

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For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
			0	0	1	0	0	0	

The diagram illustrates the addition of two 9-bit integers, A and B, using a carry propagation method. The bits of A are 1 1 0 1 1 0 1 0 1 and the bits of B are 1 0 0 0 1 0 0 1 1. A vertical light blue box highlights the carry propagation starting from the 4th bit (index 3 from the right) of A and B, where both are 0, and the carry-in is 1. This results in a carry-out of 0 for the 4th bit position. The carry propagation continues through the 5th and 6th bits, where both A and B are 1, resulting in carry-outs of 0 and 1 respectively. The final carry-out is 0.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & 1 & 1 & 0 & 1 & 1 & 1 & & \\ & & & 0 & 0 & 1 & 0 & 0 & 0 & \end{array}$$

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Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
		1	0	0	1	0	0	0	

The diagram illustrates the addition of two 9-bit integers, A and B. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1. The bits of B are 1, 0, 0, 0, 1, 0, 0, 1, 1. A vertical box highlights the third bit position (index 2 from the left), where both A and B have a 0. Below the horizontal line, the result of the addition is shown as 1, 0, 0, 1, 0, 0, 0. Small subscripts are present below the bits of B: 0 under the first bit, 1 under the second, 1 under the third, 0 under the fourth, 1 under the fifth, 1 under the sixth, and 1 under the seventh.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
		1	0	0	1	0	0	0	

Note: In the original image, a light blue box highlights the first column (bits 1 and 0) of the two numbers, and small subscripts (0, 1, 1, 0, 1, 1, 1) are placed below the second row of bits.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

	1	1	0	1	1	0	1	0	1	A
	1	0	0	0	1	0	0	1	1	B
	0	0	1	1	0	1	1	1		
	1	1	0	0	1	0	0	0		

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	1	1	0	1	1	0	1	0	1	A
	1	0	0	0	1	0	0	1	1	B
	0	0	1	1	0	1	1	1		
		1	1	0	0	1	0	0	0	

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	1	1	0	1	1	0	1	0	1	A
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1	0	0	1	1	0	1	1	1		
	0	1	1	0	0	1	0	0	0	

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		1	1	0	1	1	0	1	0	1	A
		1	0	0	0	1	0	0	1	1	B
		<hr/>									
		0	1	1	0	0	1	0	0	0	

Note: A light blue vertical box on the left contains a '1' at the bottom, indicating a carry-in.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{r} 110110101 \quad A \\ 1000010011 \quad B \\ \hline 1011001000 \end{array}$$

This gives that two n -bit integers can be added in time $\mathcal{O}(n)$.

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 10001 \\ \times 1011 \\ \hline \end{array}$$

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Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

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Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 10001 \times 1011 \\ \hline 10001 \\ 100010 \\ 0000000 \\ 00001000 \\ \hline 100011001 \end{array}$$

- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline \end{array}$$

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$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1 \end{array}$$

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Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

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- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

Time requirement:

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \\ \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1 \end{array}$$

- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

Time requirement:

- ▶ Computing intermediate results: $\mathcal{O}(nm)$.

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Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 10001 \times 1011 \\ \hline 10001 \\ 100010 \\ 0000000 \\ 10001000 \\ \hline 10111011 \end{array}$$

- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

Time requirement:

- ▶ Computing intermediate results: $\mathcal{O}(nm)$.
- ▶ Adding m numbers of length $\leq 2n$: $\mathcal{O}((m+n)m) = \mathcal{O}(nm)$.

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .

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Suppose that integers A and B are of length $n = 2^k$, for some k .

$$\boxed{b_{n-1} \quad \dots \quad b_0} \times \boxed{a_{n-1} \quad \dots \quad a_0}$$

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .

$$\boxed{b_{n-1} \quad \cdots \quad b_{\frac{n}{2}} \quad b_{\frac{n}{2}-1} \quad \cdots \quad b_0} \times \boxed{a_{n-1} \quad \cdots \quad a_{\frac{n}{2}} \quad a_{\frac{n}{2}-1} \quad \cdots \quad a_0}$$

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Suppose that integers A and B are of length $n = 2^k$, for some k .



Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .



Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 B_0$$

Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

```
1: if  $|A| = |B| = 1$  then  
2:   return  $a_0 \cdot b_0$   
3: split  $A$  into  $A_0$  and  $A_1$   
4: split  $B$  into  $B_0$  and  $B_1$   
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8: return  $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$ 
```


Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

```
1: if  $|A| = |B| = 1$  then  
2:   return  $a_0 \cdot b_0$   
3: split  $A$  into  $A_0$  and  $A_1$   
4: split  $B$  into  $B_0$  and  $B_1$   
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```

$\mathcal{O}(1)$

Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

1: **if** $|A| = |B| = 1$ **then**

$\mathcal{O}(1)$

2: **return** $a_0 \cdot b_0$

$\mathcal{O}(1)$

3: **split** A into A_0 and A_1

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We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) .$$

Example: Multiplying Two Integers

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- ▶ Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $T(n) = \Theta(n^{\log_b a})$
- ▶ Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
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In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$.

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We get a running time of $\mathcal{O}(n^2)$ for our algorithm.

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⇒ Not better than the “school method”.

Example: Multiplying Two Integers

We can use the following identity to compute Z_1 :

A more precise
(correct) analysis
would say that
computing Z_1
needs time
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We can use the following identity to compute Z_1 :

$$\begin{aligned}Z_1 &= A_1B_0 + A_0B_1 \\ &= (A_0 + A_1) \cdot (B_0 + B_1) - A_1B_1 - A_0B_0\end{aligned}$$

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1: if  $|A| = |B| = 1$  then
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Again we are in Case 1. We get a running time of $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$.

Example: Multiplying Two Integers

We get the following recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + \mathcal{O}(n) .$$

Master Theorem: Recurrence: $T[n] = aT\left(\frac{n}{b}\right) + f(n)$.

- ▶ Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $T(n) = \Theta(n^{\log_b a})$
- ▶ Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
- ▶ Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$ $T(n) = \Theta(f(n))$

Again we are in Case 1. We get a running time of $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$.

A huge improvement over the “school method”.

6.3 The Characteristic Polynomial

Consider the recurrence relation:

$$c_0T(n) + c_1T(n - 1) + c_2T(n - 2) + \cdots + c_kT(n - k) = f(n)$$

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Note that we ignore **boundary conditions** for the moment.

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- ▶ First determine all solutions that satisfy recurrence relation.
- ▶ Then pick the right one by analyzing boundary conditions.
- ▶ First consider the homogenous case.

The Homogenous Case

The solution space

$$S = \{ \mathcal{T} = T[1], T[2], T[3], \dots \mid \mathcal{T} \text{ fulfills recurrence relation} \}$$

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How do we find a non-trivial solution?

We guess that the solution is of the form λ^n , $\lambda \neq 0$, and see what happens. In order for this guess to fulfill the recurrence we need

$$c_0\lambda^n + c_1\lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \dots + c_k \cdot \lambda^{n-k} = 0$$

for all $n \geq k$.

The Homogenous Case

Dividing by λ^{n-k} gives that all these constraints are identical to

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Let $\lambda_1, \dots, \lambda_k$ be the k (complex) roots of $P[\lambda]$. Then, because of the vector space property

$$\alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \dots + \alpha_k\lambda_k^n$$

is a solution for arbitrary values α_i .

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Lemma 6

Assume that the characteristic polynomial has k *distinct* roots $\lambda_1, \dots, \lambda_k$. Then *all* solutions to the recurrence relation are of the form

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Proof.

There is one solution for every possible choice of boundary conditions for $T[1], \dots, T[k]$.

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Proof.

There is one solution for every possible choice of boundary conditions for $T[1], \dots, T[k]$.

We show that the above set of solutions contains one solution for every choice of boundary conditions.

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Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the α'_i 's such that these conditions are met:

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We show that the column vectors are linearly independent. Then the above equation has a solution.

Computing the Determinant

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} =$$

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$$\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2} \end{vmatrix}$$

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$$\prod_{i=2}^k (\lambda_i - \lambda_1) \cdot \begin{vmatrix} 1 & \lambda_2 & \cdots & \lambda_2^{k-3} & \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-3} & \lambda_k^{k-2} \end{vmatrix}$$

Computing the Determinant

Repeating the above steps gives:

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^k \lambda_i \cdot \prod_{i>\ell} (\lambda_i - \lambda_\ell)$$

Hence, if all λ_i 's are different, then the determinant is non-zero.

The Homogeneous Case

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Suppose we have a root λ_i with multiplicity (**Vielfachheit**) at least 2. Then not only is λ_i^n a solution to the recurrence but also $n\lambda_i^{n-1}$.

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To see this consider the polynomial

$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \dots + c_k\lambda^{n-k}$$

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Since λ_i is a root we can write this as $Q[\lambda] \cdot (\lambda - \lambda_i)^2$. Calculating the derivative gives a polynomial that still has root λ_i .

This means

$$c_0 n \lambda_i^{n-1} + c_1 (n-1) \lambda_i^{n-2} + \dots + c_k (n-k) \lambda_i^{n-k-1} = 0$$

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Hence,

$$\underbrace{c_0 n \lambda_i^n}_{T[n]} + \underbrace{c_1 (n-1) \lambda_i^{n-1}}_{T[n-1]} + \dots + \underbrace{c_k (n-k) \lambda_i^{n-k}}_{T[n-k]} = 0$$

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Doing this again gives

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Doing this again gives

$$c_0 n^2 \lambda_i^n + c_1 (n-1)^2 \lambda_i^{n-1} + \dots + c_k (n-k)^2 \lambda_i^{n-k} = 0$$

We can continue $j-1$ times.

The Homogeneous Case

Suppose λ_i has multiplicity j . We know that

$$c_0 n \lambda_i^n + c_1 (n-1) \lambda_i^{n-1} + \dots + c_k (n-k) \lambda_i^{n-k} = 0$$

(after taking the derivative; multiplying with λ ; plugging in λ_i)

Doing this again gives

$$c_0 n^2 \lambda_i^n + c_1 (n-1)^2 \lambda_i^{n-1} + \dots + c_k (n-k)^2 \lambda_i^{n-k} = 0$$

We can continue $j-1$ times.

Hence, $n^\ell \lambda_i^n$ is a solution for $\ell \in 0, \dots, j-1$.

The Homogeneous Case

Lemma 7

Let $P[\lambda]$ denote the characteristic polynomial to the recurrence

$$c_0T[n] + c_1T[n-1] + \dots + c_kT[n-k] = 0$$

Let $\lambda_i, i = 1, \dots, m$ be the (complex) roots of $P[\lambda]$ with multiplicities ℓ_i . Then the general solution to the recurrence is given by

$$T[n] = \sum_{i=1}^m \sum_{j=0}^{\ell_i-1} \alpha_{ij} \cdot (n^j \lambda_i^n) .$$

The full proof is omitted. We have only shown that any choice of α_{ij} 's is a solution to the recurrence.

Example: Fibonacci Sequence

$$T[0] = 0$$

$$T[1] = 1$$

$$T[n] = T[n - 1] + T[n - 2] \text{ for } n \geq 2$$

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Finding the roots, gives

$$\lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2} (1 \pm \sqrt{5})$$

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Hence, the solution is of the form

$$\alpha \left(\frac{1 + \sqrt{5}}{2} \right)^n + \beta \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

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$$\alpha \left(\frac{1 + \sqrt{5}}{2} \right) + \beta \left(\frac{1 - \sqrt{5}}{2} \right) = 1 \implies \alpha - \beta = \frac{2}{\sqrt{5}}$$

Example: Fibonacci Sequence

Hence, the solution is

$$\frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right]$$

The Inhomogeneous Case

Consider the recurrence relation:

$$c_0T(n) + c_1T(n - 1) + c_2T(n - 2) + \cdots + c_kT(n - k) = f(n)$$

with $f(n) \neq 0$.

While we have a fairly general technique for solving **homogeneous**, linear recurrence relations the inhomogeneous case is different.

The Inhomogeneous Case

The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n) ,$$

where T_h is **any** solution to the homogeneous equation, and T_p is **one** particular solution to the inhomogeneous equation.

The Inhomogeneous Case

The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n) ,$$

where T_h is **any** solution to the homogeneous equation, and T_p is **one** particular solution to the inhomogeneous equation.

There is no general method to find a particular solution.

The Inhomogeneous Case

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$$T[n] = T[n - 1] + 1 \quad T[0] = 1$$

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$$T[n] = T[n - 1] + 1 \quad T[0] = 1$$

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$$T[n - 1] = T[n - 2] + 1 \quad (n \geq 2)$$

Subtracting the first from the second equation gives,

$$T[n] - T[n - 1] = T[n - 1] - T[n - 2] \quad (n \geq 2)$$

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$$T[n] - T[n - 1] = T[n - 1] - T[n - 2] \quad (n \geq 2)$$

or

$$T[n] = 2T[n - 1] - T[n - 2] \quad (n \geq 2)$$

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I get a completely determined recurrence if I add $T[0] = 1$ and $T[1] = 2$.

The Inhomogeneous Case

Example: Characteristic polynomial:

$$\lambda^2 - 2\lambda + 1 = 0$$

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$T[0] = 1$ gives $\alpha = 1$.

$T[1] = 2$ gives $1 + \beta = 2 \Rightarrow \beta = 1$.

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Shift:

$$T[n - 1] = 2T[n - 2] - T[n - 3] + 2(n - 1) - 1$$

$$T[n] = 2T[n - 1] - T[n - 2] + 2n - 1$$

Shift:

$$\begin{aligned} T[n - 1] &= 2T[n - 2] - T[n - 3] + 2(n - 1) - 1 \\ &= 2T[n - 2] - T[n - 3] + 2n - 3 \end{aligned}$$

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$$T[n] = 3T[n - 1] - 3T[n - 2] + T[n - 3] + 2$$

and so on...

6.4 Generating Functions

Definition 8 (Generating Function)

Let $(a_n)_{n \geq 0}$ be a sequence. The corresponding

- ▶ **generating function** (**Erzeugendenfunktion**) is

$$F(z) := \sum_{n \geq 0} a_n z^n ;$$

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- ▶ **exponential generating function** (**exponentielle Erzeugendenfunktion**) is

$$F(z) := \sum_{n \geq 0} \frac{a_n}{n!} z^n .$$

6.4 Generating Functions

Example 9

1. The generating function of the sequence $(1, 0, 0, \dots)$ is

$$F(z) = 1.$$

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Let $f = \sum_{n \geq 0} a_n z^n$ and $g = \sum_{n \geq 0} b_n z^n$.

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- ▶ **Multiplication:** $f \cdot g := \sum_{n \geq 0} c_n z^n$ with $c_n = \sum_{p=0}^n a_p b_{n-p}$.

There are no convergence issues here.

6.4 Generating Functions

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We view a power series as a function $f : \mathbb{C} \rightarrow \mathbb{C}$.

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Then, it is important to think about convergence/convergence radius etc.

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It means that the power series $1 - z$ and the power series $\sum_{n \geq 0} z^n$ are invers, i.e.,

$$(1 - z) \cdot \left(\sum_{n \geq 0} z^n \right) = 1 .$$

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This is well-defined.

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Suppose we are given the generating function

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Hence, the generating function of the sequence $a_n = n + 1$ is $1/(1-z)^2$.

Formally the derivative of a formal power series $\sum_{n \geq 0} a_n z^n$ is defined as $\sum_{n \geq 0} n a_n z^{n-1}$.

The known rules for differentiation work for this definition. In particular, e.g. the derivative of $\frac{1}{1-z}$ is $\frac{1}{(1-z)^2}$.

Note that this requires a proof if we consider power series as algebraic objects. However, we did not prove this in the lecture.

6.4 Generating Functions

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Hence, the generating function of the sequence

$$a_n = (n+1)(n+2) \text{ is } \frac{2}{(1-z)^3} .$$

6.4 Generating Functions

Computing the k -th derivative of $\sum z^n$.

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$$\begin{aligned}\sum_{n \geq k} n(n-1) \cdot \dots \cdot (n-k+1)z^{n-k} &= \sum_{n \geq 0} (n+k) \cdot \dots \cdot (n+1)z^n \\ &= \frac{k!}{(1-z)^{k+1}} \cdot\end{aligned}$$

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Hence:

$$\sum_{n \geq 0} \binom{n+k}{k} z^n = \frac{1}{(1-z)^{k+1}} \cdot$$

The generating function of the sequence $a_n = \binom{n+k}{k}$ is $\frac{1}{(1-z)^{k+1}}$.

6.4 Generating Functions

$$\sum_{n \geq 0} n z^n = \sum_{n \geq 0} (n+1) z^n - \sum_{n \geq 0} z^n$$

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$$\begin{aligned}\sum_{n \geq 0} n z^n &= \sum_{n \geq 0} (n+1) z^n - \sum_{n \geq 0} z^n \\ &= \frac{1}{(1-z)^2} - \frac{1}{1-z}\end{aligned}$$

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The generating function of the sequence $a_n = n$ is $\frac{z}{(1-z)^2}$.

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Hence,

$$\sum_{n \geq 0} a^n z^n = \frac{1}{1-az}$$

The generating function of the sequence $f_n = a^n$ is $\frac{1}{1-az}$.

Example: $a_n = a_{n-1} + 1, a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

$$A(z)$$

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Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

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$$\begin{aligned} A(z) &= \sum_{n \geq 0} a_n z^n \\ &= a_0 + \sum_{n \geq 1} (a_{n-1} + 1) z^n \\ &= 1 + z \sum_{n \geq 1} a_{n-1} z^{n-1} + \sum_{n \geq 1} z^n \end{aligned}$$

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Hence, $a_n = n + 1$.

Some Generating Functions

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$\frac{1}{n!}$	e^z

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$\sum_{i=0}^n f_i$	$\frac{F(z)}{1-z}$

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$c^n f_n$	$F(cz)$

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1. Set $A(z) = \sum_{n \geq 0} a_n z^n$.

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6. The coefficients of the resulting power series are the a_n .

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

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$$\sum_{n \geq 0} a_n z^n = A(z) = \frac{1}{1 - 2z} = \sum_{n \geq 0} 2^n z^n$$

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We use partial fraction decomposition:

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$$\frac{z^2 - z + 1}{(1 - 3z)(1 - z)^2} \stackrel{!}{=} \frac{A}{1 - 3z} + \frac{B}{1 - z} + \frac{C}{(1 - z)^2}$$

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This gives

$$z^2 - z + 1 = A(1 - z)^2 + B(1 - 3z)(1 - z) + C(1 - 3z)$$

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$$\begin{aligned} z^2 - z + 1 &= A(1 - z)^2 + B(1 - 3z)(1 - z) + C(1 - 3z) \\ &= A(1 - 2z + z^2) + B(1 - 4z + 3z^2) + C(1 - 3z) \end{aligned}$$

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This gives

$$\begin{aligned} z^2 - z + 1 &= A(1 - z)^2 + B(1 - 3z)(1 - z) + C(1 - 3z) \\ &= A(1 - 2z + z^2) + B(1 - 4z + 3z^2) + C(1 - 3z) \\ &= (A + 3B)z^2 + (-2A - 4B - 3C)z + (A + B + C) \end{aligned}$$

Example: $a_n = 3a_{n-1} + n, a_0 = 1$

5. Write $f(z)$ as a formal power series:

This leads to the following conditions:

$$A + B + C = 1$$

$$2A + 4B + 3C = 1$$

$$A + 3B = 1$$

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5. Write $f(z)$ as a formal power series:

This leads to the following conditions:

$$A + B + C = 1$$

$$2A + 4B + 3C = 1$$

$$A + 3B = 1$$

which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

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Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

5. Write $f(z)$ as a formal power series:

$$A(z) = \frac{7}{4} \cdot \frac{1}{1-3z} - \frac{1}{4} \cdot \frac{1}{1-z} - \frac{1}{2} \cdot \frac{1}{(1-z)^2}$$

Example: $a_n = 3a_{n-1} + n, a_0 = 1$

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Example: $a_n = 3a_{n-1} + n, a_0 = 1$

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Example: $a_n = 3a_{n-1} + n, a_0 = 1$

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6. This means $a_n = \frac{7}{4}3^n - \frac{1}{2}n - \frac{3}{4}$.

6.5 Transformation of the Recurrence

Example 10

$$f_0 = 1$$

$$f_1 = 2$$

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Let $n = 2^k$:

$$g_k = 3^{k+1} - 2^{k+1}, \text{ hence}$$

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