

Complexity

LP Feasibility Problem (LP feasibility A)

Given $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$. Does there exist $x \in \mathbb{R}^n$ with $Ax \leq b$, $x \geq 0$?

LP Feasibility Problem (LP feasibility B)

Given $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$. Find $x \in \mathbb{R}^n$ with $Ax \leq b$, $x \geq 0$!

LP Optimization A

Given $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$. What is the maximum value of $c^T x$ for a feasible point $x \in \mathbb{R}^n$?

LP Optimization B

Given $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$. Return feasible point $x \in \mathbb{R}^n$ with maximum value of $c^T x$?

Note that allowing A, b to contain rational numbers does not make a difference, as we can multiply every number by a suitable large constant so that everything becomes integral but the **feasible region** does not change.

The Bit Model

Input size

- ▶ The number of bits to represent a number $a \in \mathbb{Z}$ is

$$\lceil \log_2(|a|) \rceil + 1$$

- ▶ Let for an $m \times n$ matrix M , $L(M)$ denote the number of bits required to encode all the numbers in M .

$$\langle M \rangle := \sum_{i,j} \lceil \log_2(|m_{ij}|) \rceil + 1$$

- ▶ In the following we assume that input matrices are encoded in a standard way, where each number is encoded in binary and then suitable separators are added in order to separate distinct number from each other.
- ▶ Then the input length is $L = \Theta(\langle A \rangle + \langle b \rangle)$.

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- ▶ Then the input length is $L = \Theta(\langle A \rangle + \langle b \rangle)$.

- ▶ In the following we sometimes refer to $L := \langle A \rangle + \langle b \rangle$ as the input size (even though the real input size is something in $\Theta(\langle A \rangle + \langle b \rangle)$).
- ▶ Sometimes we may also refer to $L := \langle A \rangle + \langle b \rangle + n \log_2 n$ as the input size. Note that $n \log_2 n = \Theta(\langle A \rangle + \langle b \rangle)$.
- ▶ In order to show that LP-decision is in NP we show that if there is a solution x then there exists a small solution for which feasibility can be verified in polynomial time (polynomial in L).

Note that $m \log_2 m$ may be much larger than $\langle A \rangle + \langle b \rangle$.

Suppose that $\tilde{A}x = b; x \geq 0$ is feasible.

Then there exists a basic feasible solution. This means a set B of basic variables such that

$$x_B = \tilde{A}_B^{-1}b$$

and all other entries in x are 0.

In the following we show that this x has small encoding length and we give an explicit bound on this length. So far we have only been handwaving and have said that we can compute x via Gaussian elimination and it will be short...

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Size of a Basic Feasible Solution

Note that n in the theorem denotes the number of columns in A which may be much smaller than m .

- ▶ A : original input matrix
- ▶ \bar{A} : transformation of A into standard form
- ▶ \bar{A}_B : submatrix of \bar{A} corresponding to basis B

Lemma 47

Let $\bar{A}_B \in \mathbb{Z}^{m \times m}$ and $b \in \mathbb{Z}^m$. Define $L = \langle A \rangle + \langle b \rangle + n \log_2 n$.

Then a solution to $\bar{A}_B x_B = b$ has rational components x_j of the form $\frac{D_j}{D}$, where $|D_j| \leq 2^L$ and $|D| \leq 2^L$.

Proof:

Cramer's rule says that we can compute x_j as

$$x_j = \frac{\det(\bar{A}_B^j)}{\det(\bar{A}_B)}$$

where \bar{A}_B^j is the matrix obtained from \bar{A}_B by replacing the j -th column by the vector b .

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Bounding the Determinant

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Analogously for $\det(A_B^j)$.

When computing the determinant of $X = \bar{A}_B$ we first do expansions along columns that were introduced when transforming A into standard form, i.e., into \bar{A} .

Such a column contains a single 1 and the remaining entries of the column are 0. Therefore, these expansions do not increase the absolute value of the determinant. After we did expansions for all these columns we are left with a square sub-matrix of A of size at most $n \times n$.

Reducing LP-solving to LP decision.

Given an LP $\max\{c^T x \mid Ax \leq b; x \geq 0\}$ do a **binary search** for the optimum solution

(Add constraint $c^T x \geq M$). Then checking for feasibility shows whether optimum solution is larger or smaller than M).

If the LP is feasible then the binary search finishes in at most

$$\log_2 \left(\frac{2n2^{2L'}}{1/2^{L'}} \right) = \mathcal{O}(L') ,$$

as the range of the search is at most $-n2^{2L'}, \dots, n2^{2L'}$ and the distance between two adjacent values is at least $\frac{1}{\det(A)} \geq \frac{1}{2^{L'}}$.

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How do we detect whether the LP is unbounded?

Let $M_{\max} = n2^{2L'}$ be an upper bound on the objective value of a basic feasible solution.

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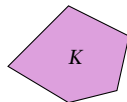
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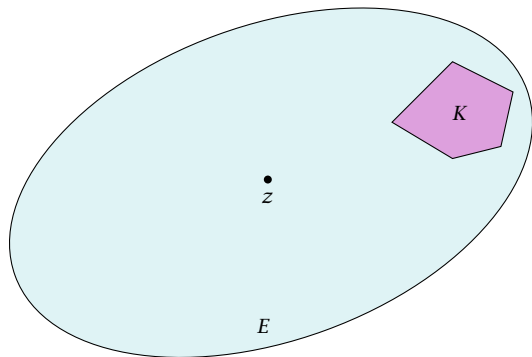
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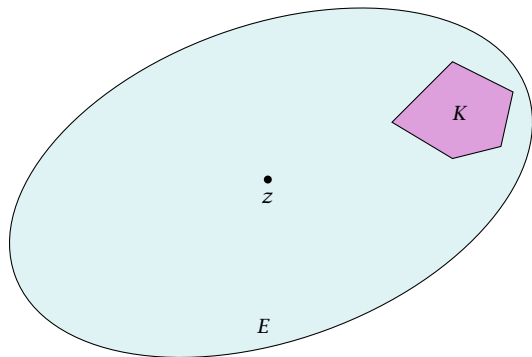
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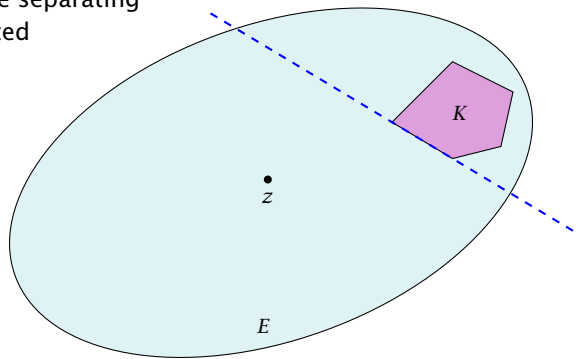
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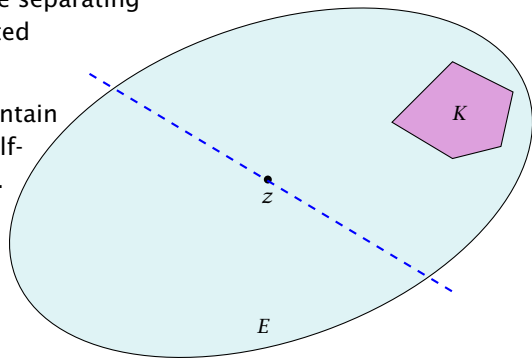
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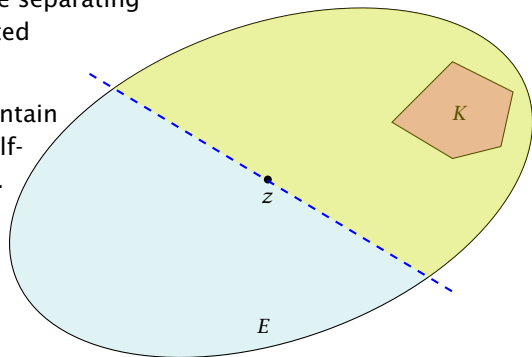
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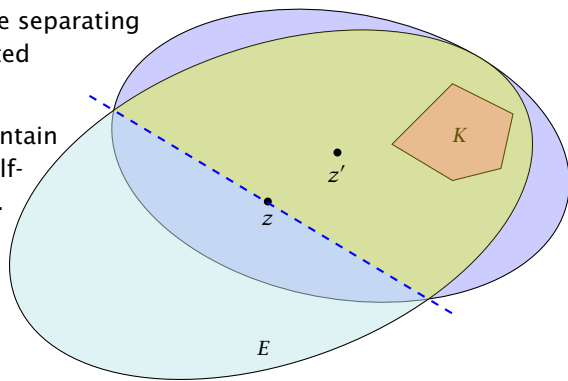
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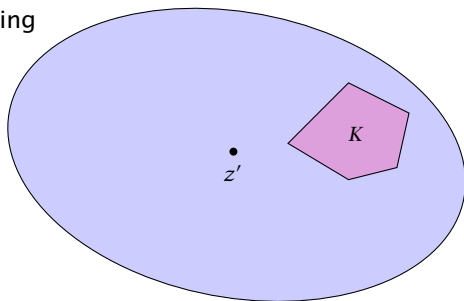
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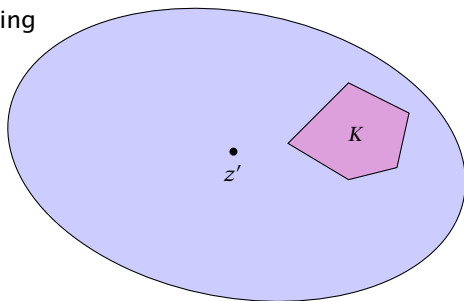
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- ▶ REPEAT



Issues/Questions:

- ▶ How do you choose the first Ellipsoid? What is its volume?
- ▶ How do you measure progress? By how much does the volume decrease in each iteration?
- ▶ When can you stop? What is the minimum volume of a non-empty polytop?

Definition 48

A mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $f(x) = Lx + t$, where L is an invertible matrix is called an **affine transformation**.

Definition 49

A ball in \mathbb{R}^n with center c and radius r is given by

$$\begin{aligned} B(c, r) &= \{x \mid (x - c)^T (x - c) \leq r^2\} \\ &= \{x \mid \sum_i (x - c)_i^2 / r^2 \leq 1\} \end{aligned}$$

$B(0, 1)$ is called the **unit ball**.

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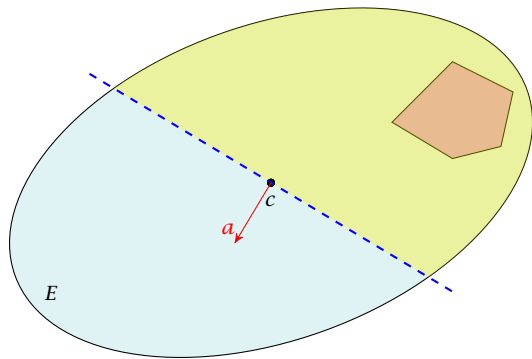
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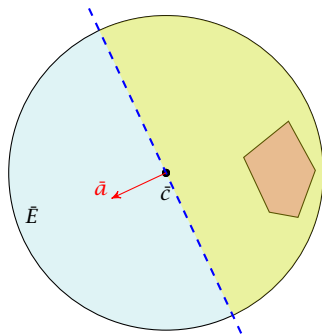
where $Q = LL^T$ is an invertible matrix.

How to Compute the New Ellipsoid



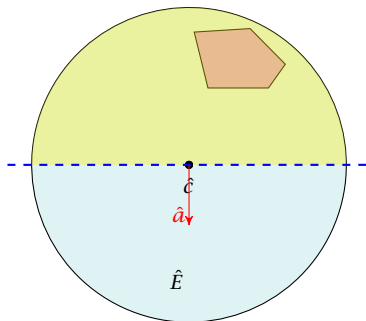
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- ▶ Use f^{-1} (recall that $f = Lx + t$ is the affine transformation of the unit ball) to rotate/distort the ellipsoid (back) into the unit ball.



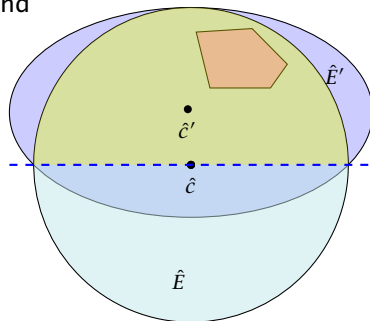
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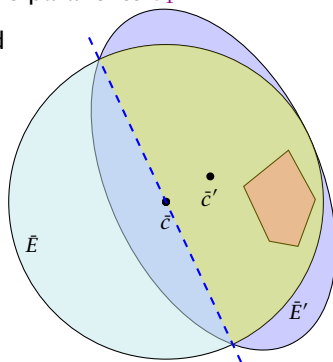
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- ▶ Compute the new center \hat{c}' and the new matrix \hat{Q}' for this simplified setting.



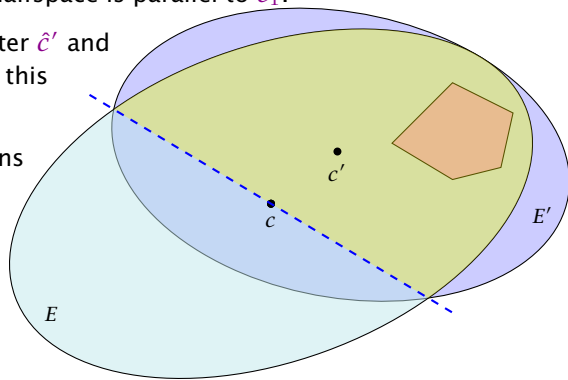
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- ▶ Use f^{-1} (recall that $f = Lx + t$ is the affine transformation of the unit ball) to rotate/distort the ellipsoid (back) into the unit ball.
- ▶ Use a rotation R^{-1} to rotate the unit ball such that the normal vector of the halfspace is parallel to e_1 .
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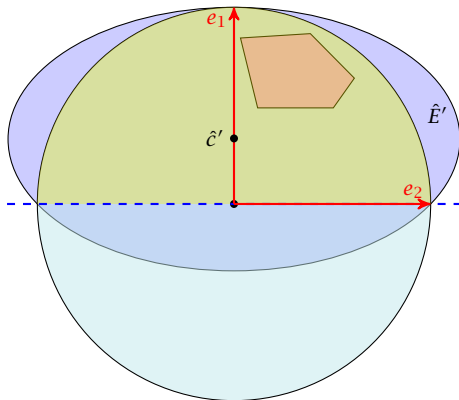


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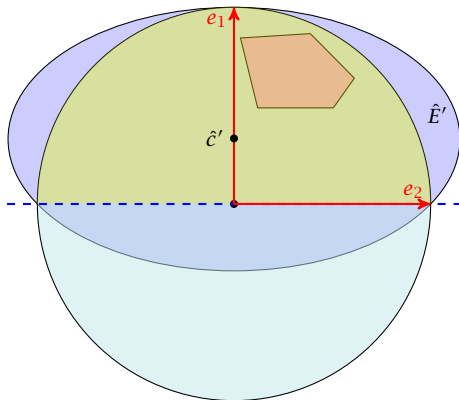


The Easy Case



- ▶ The new center lies on axis x_1 . Hence, $\hat{c}' = te_1$ for $t > 0$.
- ▶ The vectors e_1, e_2, \dots have to fulfill the ellipsoid constraint with equality. Hence $(e_i - \hat{c}')^T \hat{Q}'^{-1} (e_i - \hat{c}') = 1$.

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The Easy Case

- ▶ To obtain the matrix \hat{Q}'^{-1} for our ellipsoid \hat{E}' note that \hat{E}' is **axis-parallel**.
- ▶ Let a denote the radius along the x_1 -axis and let b denote the (common) radius for the other axes.
- ▶ The matrix

$$\hat{L}' = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b \end{pmatrix}$$

maps the unit ball (via function $\hat{f}'(x) = \hat{L}'x$) to an axis-parallel ellipsoid with radius a in direction x_1 and b in all other directions.

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The Easy Case

- ▶ As $\hat{Q}' = \hat{L}'\hat{L}'^t$ the matrix \hat{Q}'^{-1} is of the form

$$\hat{Q}'^{-1} = \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix}$$

The Easy Case

- ▶ $(e_1 - \hat{c}')^T \hat{Q}'^{-1} (e_1 - \hat{c}') = 1$ gives

$$\begin{pmatrix} 1-t \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T \cdot \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix} \cdot \begin{pmatrix} 1-t \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1$$

- ▶ This gives $(1-t)^2 = a^2$.

The Easy Case

- ▶ For $i \neq 1$ the equation $(e_i - \hat{c}')^T \hat{Q}'^{-1} (e_i - \hat{c}') = 1$ looks like (here $i = 2$)

$$\begin{pmatrix} -t \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T \cdot \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix} \cdot \begin{pmatrix} -t \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1$$

- ▶ This gives $\frac{t^2}{a^2} + \frac{1}{b^2} = 1$, and hence

$$\frac{1}{b^2} = 1 - \frac{t^2}{a^2}$$

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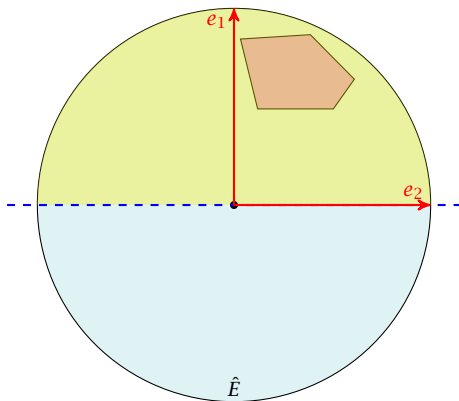
Summary

So far we have

$$a = 1 - t \quad \text{and} \quad b = \frac{1 - t}{\sqrt{1 - 2t}}$$

The Easy Case

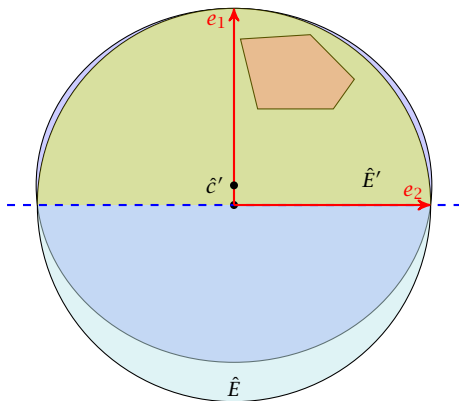
We still have many choices for t :



Choose t such that the volume of \hat{E}' is minimal!!!

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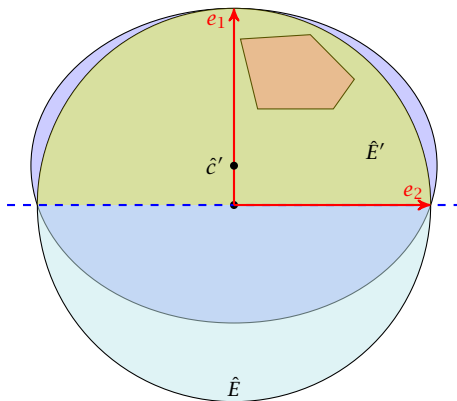
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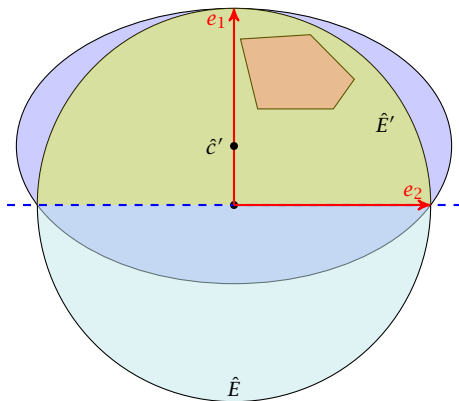
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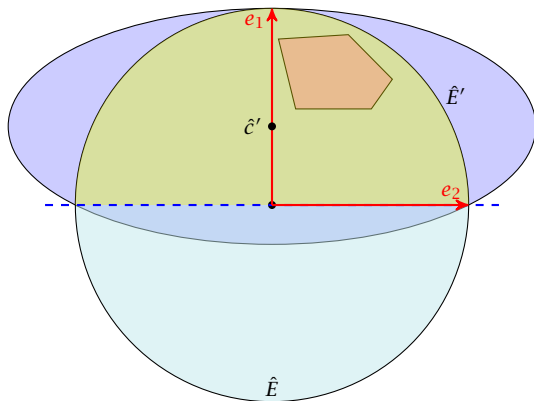
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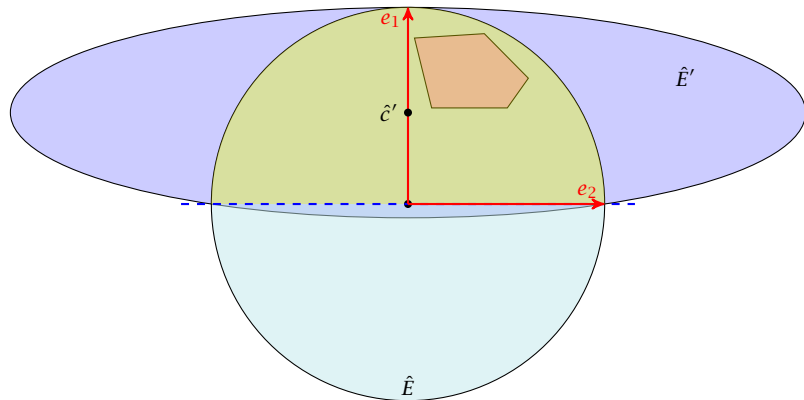
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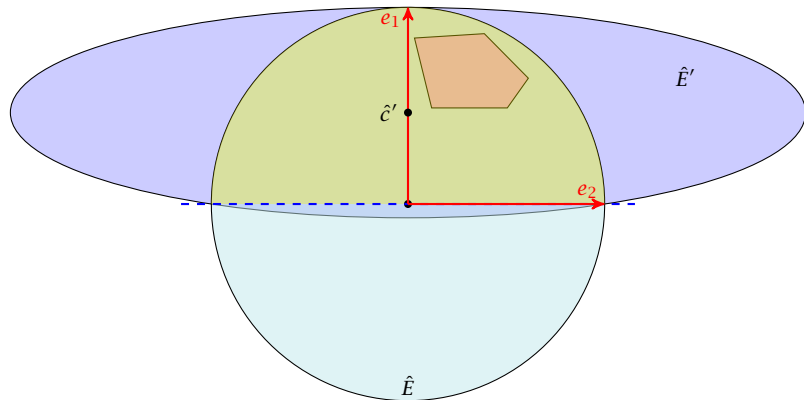
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Lemma 51

Let L be an affine transformation and $K \subseteq \mathbb{R}^n$. Then

$$\text{vol}(L(K)) = |\det(L)| \cdot \text{vol}(K) .$$

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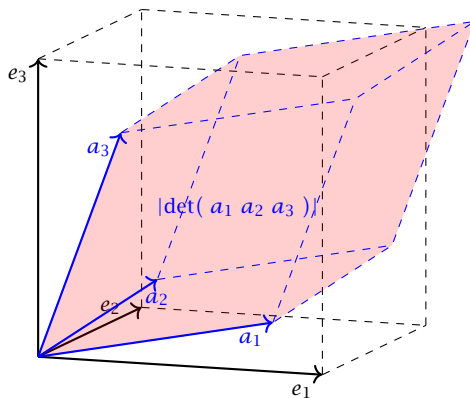
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n-dimensional volume



The Easy Case

- ▶ We want to choose t such that the volume of \hat{E}' is minimal.

$$\text{vol}(\hat{E}') = \text{vol}(B(0, 1)) \cdot |\det(\hat{L}')| ,$$

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- ▶ Note that a and b in the above equations depend on t , by the previous equations.

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We use the shortcut $\Phi := \text{vol}(B(0, 1))$.

The Easy Case

$$\frac{d \operatorname{vol}(\hat{E}')}{dt}$$

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$N = \text{denominator}$

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inner derivative

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numerator

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 &\quad \left. - (n-1) \cancel{(\sqrt{1-2t})^{n-2}} \cdot \frac{1}{2\sqrt{1-2t}} \cdot (-2) \cdot \cancel{(1-t)^n} \right) \\
 &= \frac{\Phi}{N^2} \cdot (\sqrt{1-2t})^{n-3} \cdot (1-t)^{n-1} \\
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The Easy Case

- ▶ We obtain the minimum for $t = \frac{1}{n+1}$.
- ▶ For this value we obtain

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$$b^2 = \frac{(1-t)^2}{1-2t} = \frac{\left(1 - \frac{1}{n+1}\right)^2}{1 - \frac{2}{n+1}}$$

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The Easy Case

- ▶ We obtain the minimum for $t = \frac{1}{n+1}$.
- ▶ For this value we obtain

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Let $\gamma_n = \frac{\text{vol}(\hat{E}^n)}{\text{vol}(B(0,1))} = ab^{n-1}$ be the ratio by which the volume changes:

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where we used $(1+x)^a \leq e^{ax}$ for $x \in \mathbb{R}$ and $a > 0$.

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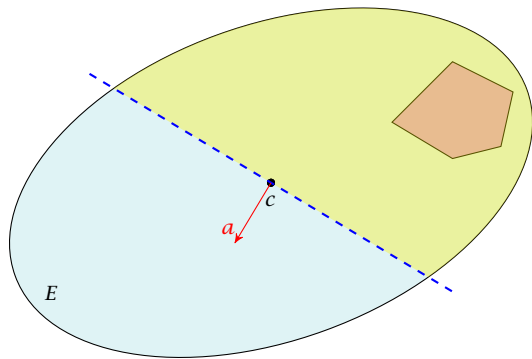
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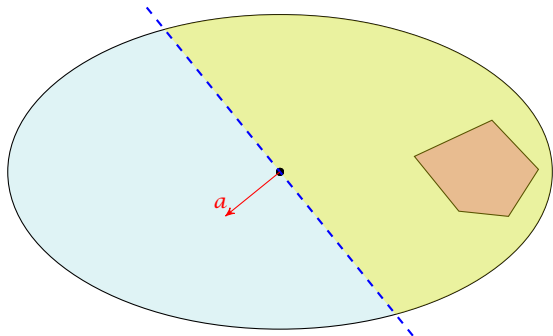
This gives $\gamma_n \leq e^{-\frac{1}{2(n+1)}}$.

How to Compute the New Ellipsoid



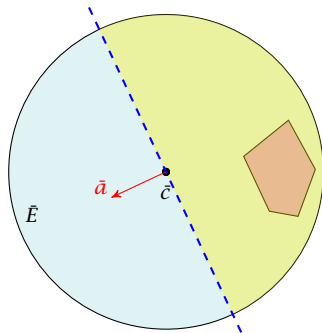
How to Compute the New Ellipsoid

- ▶ Use f^{-1} (recall that $f = Lx + t$ is the affine transformation of the unit ball) to translate/distort the ellipsoid (back) into the unit ball.



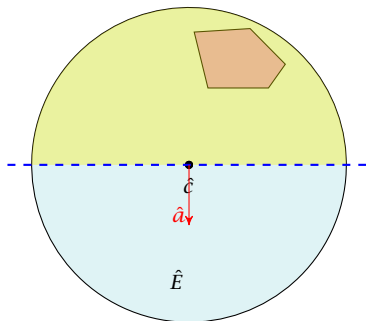
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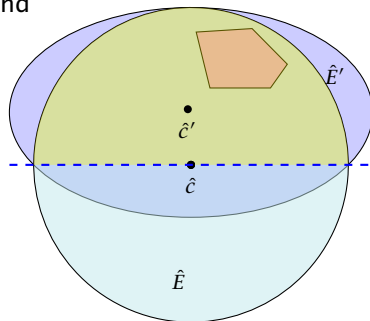
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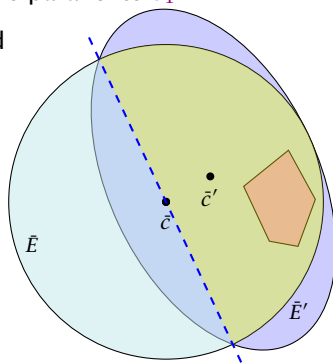
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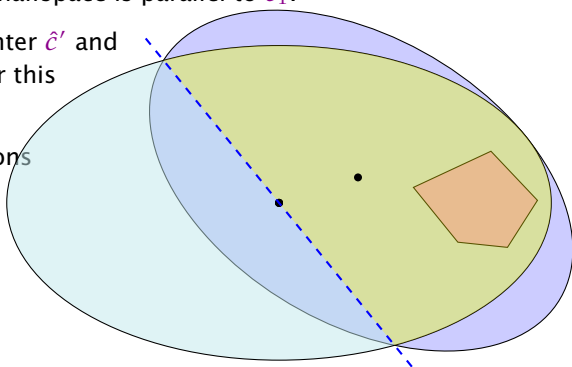
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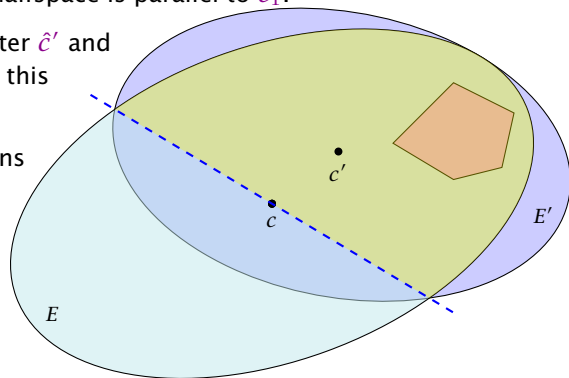
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Here it is important that mapping a set with affine function $f(x) = Lx + t$ changes the volume by factor $\det(L)$.

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This means $\bar{a} = L^T a$.

The center \bar{c} is of course at the origin.

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1 -direction. Hence,

$$R^{-1}\left(\frac{L^T a}{\|L^T a\|}\right) = -e_1 \quad \Rightarrow \quad -\frac{L^T a}{\|L^T a\|} = R \cdot e_1$$

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For computing the matrix Q' of the new ellipsoid we assume in the following that \hat{E}' , \bar{E}' and E' refer to the ellipsoids centered in the origin.

Recall that

$$\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}$$

This gives

$$\hat{Q}' = \frac{n^2}{n^2 - 1} \left(I - \frac{2}{n + 1} e_1 e_1^T \right)$$

Note that $e_1 e_1^T$ is a matrix M that has $M_{11} = 1$ and all other entries equal to 0.

because for $a^2 = n^2/(n+1)^2$ and $b^2 = n^2/n^2 - 1$

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$$\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}$$

This gives

$$\hat{Q}' = \frac{n^2}{n^2 - 1} \left(I - \frac{2}{n + 1} e_1 e_1^T \right)$$

Note that $e_1 e_1^T$ is a matrix M that has $M_{11} = 1$ and all other entries equal to 0.

because for $a^2 = n^2/(n+1)^2$ and $b^2 = n^2/n^2 - 1$

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9 The Ellipsoid Algorithm

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$$\begin{aligned}\bar{E}' &= R(\hat{E}') \\ &= \{R(x) \mid x^T \hat{Q}'^{-1} x \leq 1\}\end{aligned}$$

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Hence,

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Here we used the equation for Re_1 proved before, and the fact that $RR^T = I$, which holds for any rotation matrix. To see this observe that the length of a rotated vector x should not change, i.e.,

$$x^T I x = (Rx)^T (Rx) = x^T (R^T R) x$$

which means $x^T (I - R^T R) x = 0$ for every vector x . It is easy to see that this can only be fulfilled if $I - R^T R = 0$.

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Incomplete Algorithm

Algorithm 1 ellipsoid-algorithm

- 1: **input:** point $c \in \mathbb{R}^n$, convex set $K \subseteq \mathbb{R}^n$
- 2: **output:** point $x \in K$ or “ K is empty”
- 3: $Q \leftarrow ???$
- 4: **repeat**
- 5: **if** $c \in K$ **then return** c
- 6: **else**
- 7: choose a violated hyperplane a
- 8:
$$c \leftarrow c - \frac{1}{n+1} \frac{Qa}{\sqrt{a^T Q a}}$$
- 9:
$$Q \leftarrow \frac{n^2}{n^2 - 1} \left(Q - \frac{2}{n+1} \frac{Qaa^T Q}{a^T Q a} \right)$$
- 10: **endif**
- 11: **until** $???$
- 12: **return** “ K is empty”

Repeat: Size of basic solutions

Lemma 52

Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a bounded polyhedron. Let $L := 2\langle A \rangle + \langle b \rangle + 2n(1 + \log_2 n)$. Then every entry x_j in a basic solution fulfills $|x_j| = \frac{D_j}{D}$ with $D_j, D \leq 2^L$.

In the following we use $\delta := 2^L$.

Proof:

We can replace P by $P' := \{x \mid A'x \leq b; x \geq 0\}$ where $A' = \begin{bmatrix} A & -A \end{bmatrix}$. The lemma follows by applying Lemma 47, and observing that $\langle A' \rangle = 2\langle A \rangle$ and $n' = 2n$.

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How do we find the first ellipsoid?

For feasibility checking we can assume that the polytop P is bounded; it is sufficient to consider basic solutions.

Every entry x_i in a basic solution fulfills $|x_i| \leq \delta$.

Hence, P is contained in the cube $-\delta \leq x_i \leq \delta$.

A vector in this cube has at most distance $R := \sqrt{n}\delta$ from the origin.

Starting with the ball $E_0 := B(0, R)$ ensures that P is completely contained in the initial ellipsoid. This ellipsoid has volume at most $R^n \text{vol}(B(0, 1)) \leq (n\delta)^n \text{vol}(B(0, 1))$.

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When can we terminate?

Let $P := \{x \mid Ax \leq b\}$ with $A \in \mathbb{Z}$ and $b \in \mathbb{Z}$ be a bounded polytop.

Consider the following polyhedron

$$P_\lambda := \left\{ x \mid Ax \leq b + \frac{1}{\lambda} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\},$$

where $\lambda = \delta^2 + 1$.

Note that the volume of P_λ cannot be 0

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Making P full-dimensional

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P_λ is feasible if and only if P is feasible.

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Consider the polyhedrons

$$\bar{P} = \{x \mid [A \ -A \ I_m]x = b; x \geq 0\}$$

and

$$\bar{P}_\lambda = \left\{x \mid [A \ -A \ I_m]x = b + \frac{1}{\lambda} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}; x \geq 0\right\}.$$

P is feasible if and only if \bar{P} is feasible, and P_λ feasible if and only if \bar{P}_λ feasible.

\bar{P}_λ is bounded since P_λ and P are bounded.

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Let $\bar{A} = \begin{bmatrix} A & -A & I_m \end{bmatrix}$.

\bar{P}_λ feasible implies that there is a basic feasible solution represented by

$$x_B = \bar{A}_B^{-1}b + \frac{1}{\lambda}\bar{A}_B^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

(The other x -values are zero)

The only reason that this basic feasible solution is not feasible for \bar{P} is that one of the basic variables becomes negative.

Hence, there exists i with

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$$(\bar{A}_B^{-1}b)_i < 0 \quad \Rightarrow \quad (\bar{A}_B^{-1}b)_i \leq -\frac{1}{\det(\bar{A}_B)} \leq -1/\delta$$

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Lemma 54

If P_λ is feasible then it contains a ball of radius $r := 1/\delta^3$. This has a volume of at least $r^n \text{vol}(B(0, 1)) = \frac{1}{\delta^{3n}} \text{vol}(B(0, 1))$.

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Proof:

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Lemma 54

If P_λ is feasible then it contains a ball of radius $r := 1/\delta^3$. This has a volume of at least $r^n \text{vol}(B(0, 1)) = \frac{1}{\delta^{3n}} \text{vol}(B(0, 1))$.

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Hence, $x + \vec{\ell}$ is feasible for P_λ which proves the lemma.

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Algorithm 1 ellipsoid-algorithm

- 1: **input:** point $c \in \mathbb{R}^n$, convex set $K \subseteq \mathbb{R}^n$, radii R and r
- 2: with $K \subseteq B(c, R)$, and $B(x, r) \subseteq K$ for some x
- 3: **output:** point $x \in K$ or “ K is empty”
- 4: $Q \leftarrow \text{diag}(R^2, \dots, R^2)$ // i.e., $L = \text{diag}(R, \dots, R)$
- 5: **repeat**
- 6: **if** $c \in K$ **then return** c
- 7: **else**
- 8: choose a violated hyperplane a
- 9:
$$c \leftarrow c - \frac{1}{n+1} \frac{Qa}{\sqrt{a^T Q a}}$$
- 10:
$$Q \leftarrow \frac{n^2}{n^2 - 1} \left(Q - \frac{2}{n+1} \frac{Qaa^T Q}{a^T Q a} \right)$$
- 11: **endif**
- 12: **until** $\det(Q) \leq r^{2n}$ // i.e., $\det(L) \leq r^n$
- 13: **return** “ K is empty”

Separation Oracle

Let $K \subseteq \mathbb{R}^n$ be a convex set. A separation oracle for K is an algorithm A that gets as input a point $x \in \mathbb{R}^n$ and either

- ▶ certifies that $x \in K$,
- ▶ or finds a hyperplane separating x from K .

We will usually assume that A is a polynomial-time algorithm.

In order to find a point in K we need

- ▶ a guarantee that a ball of radius r is contained in K ,
- ▶ an initial ball B with radius R that contains K ,
- ▶ a separation oracle.

The Ellipsoid algorithm requires $\mathcal{O}(\text{poly}(n) \cdot \log(R/r))$ iterations. Each iteration is polytime for a polynomial-time Separation oracle.

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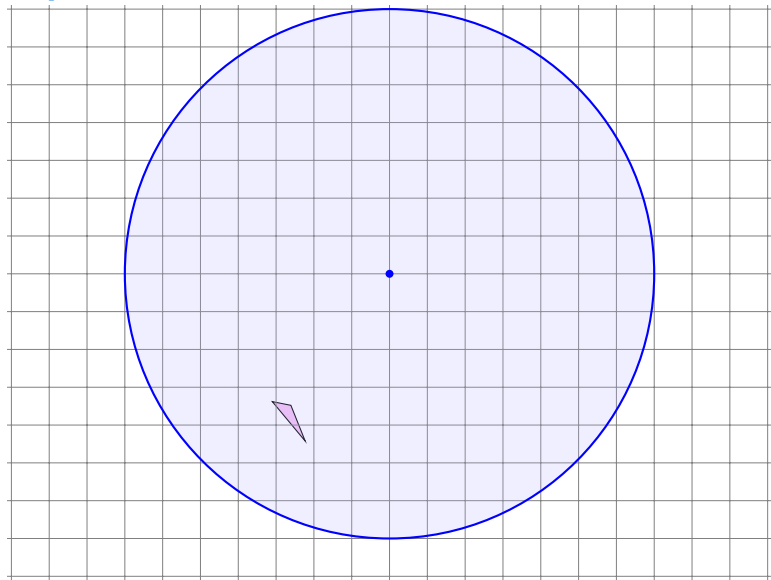
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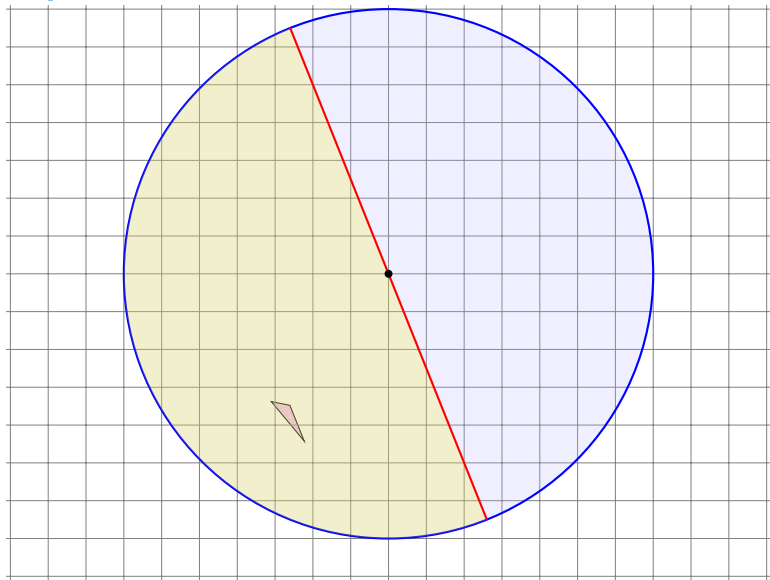
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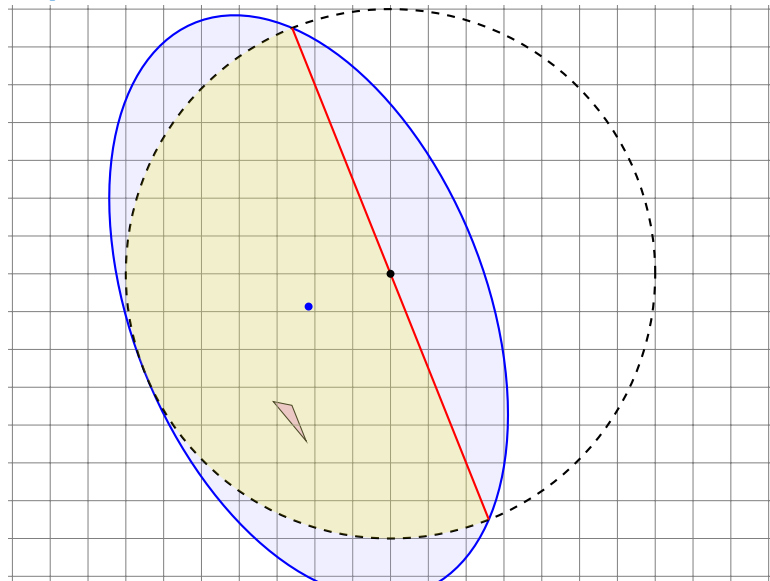
Example



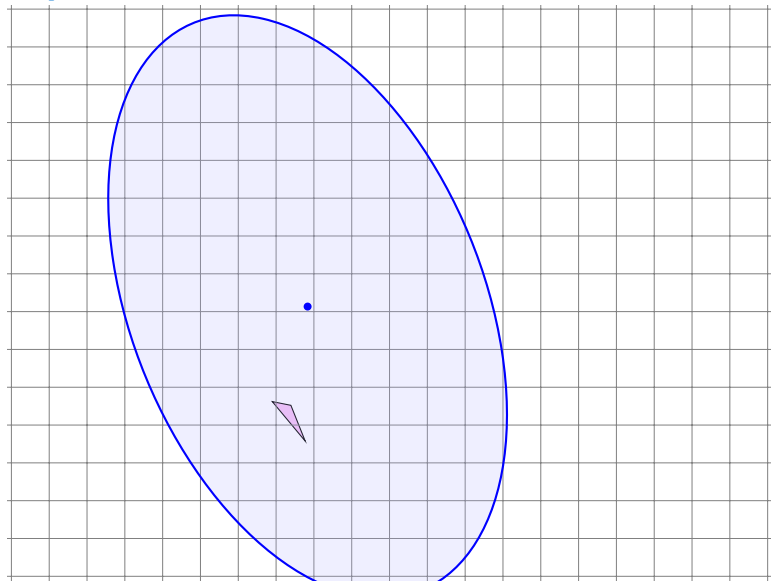
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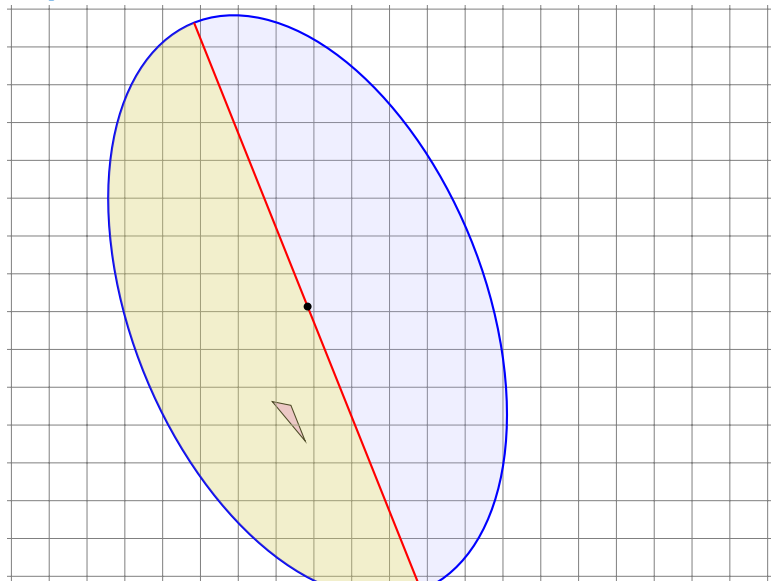
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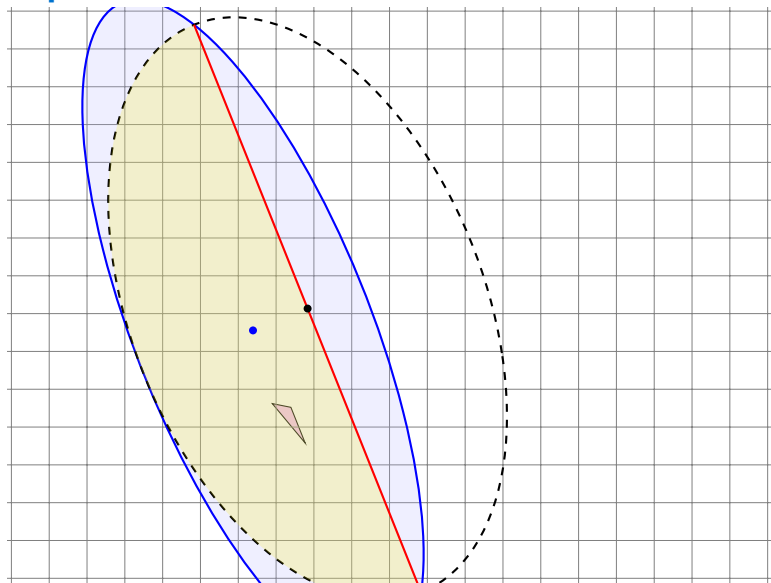
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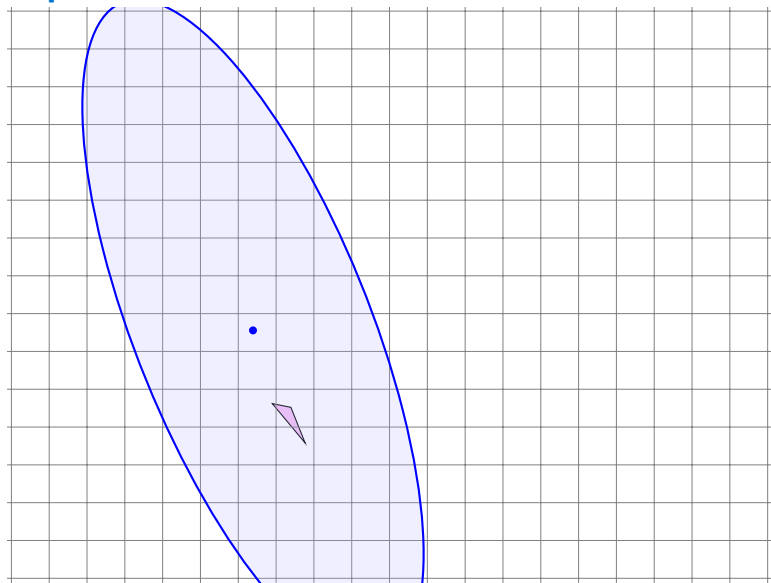
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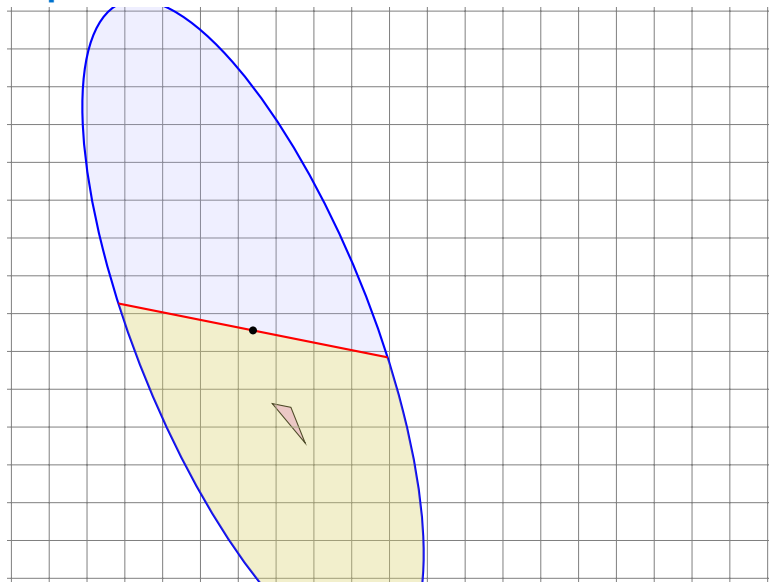
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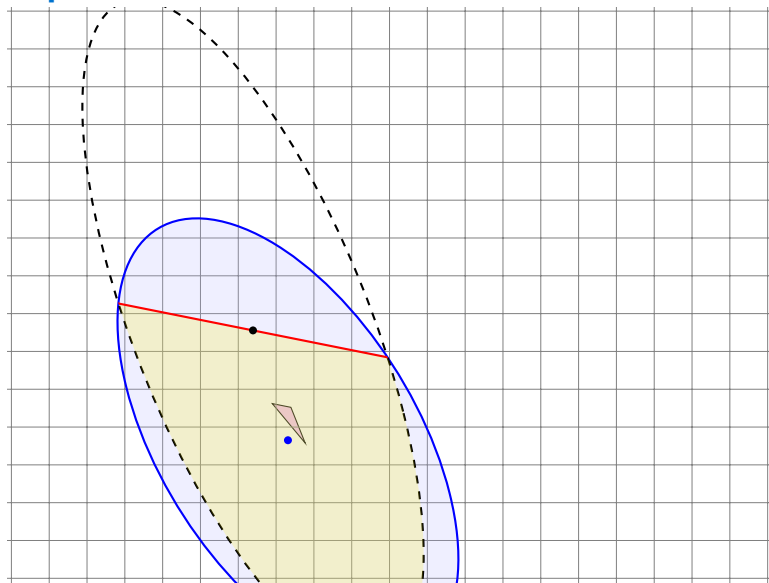
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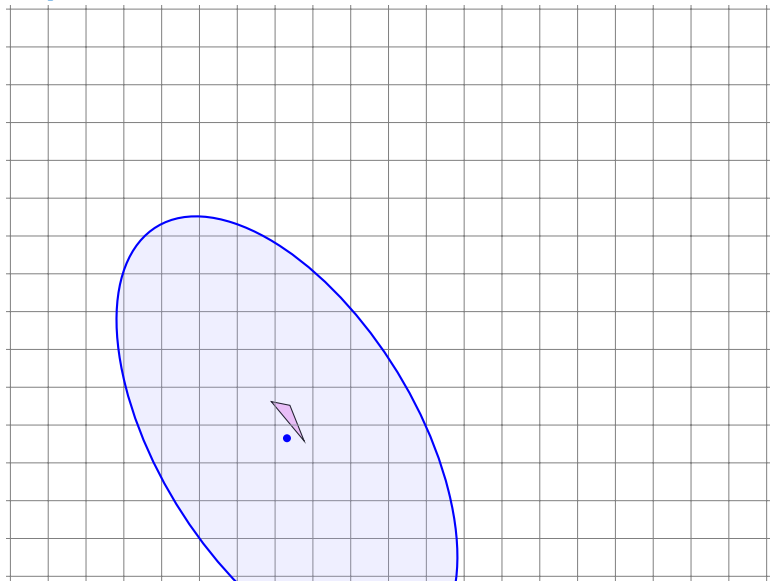
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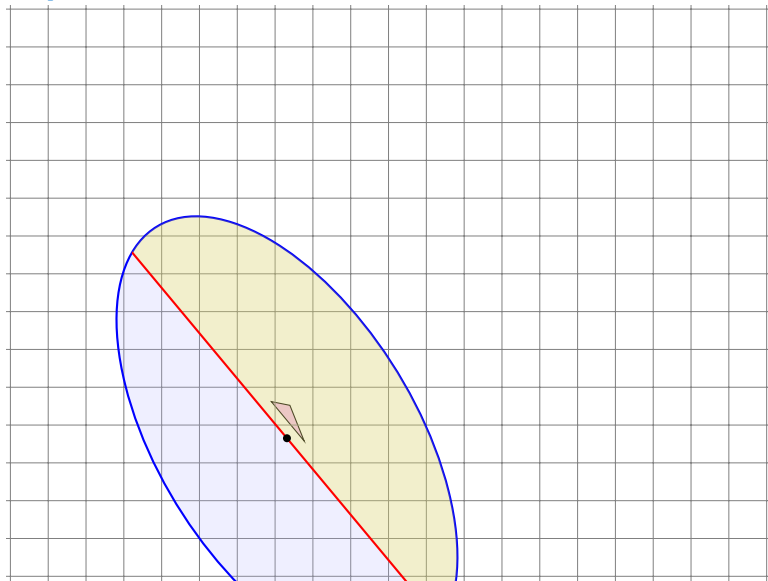
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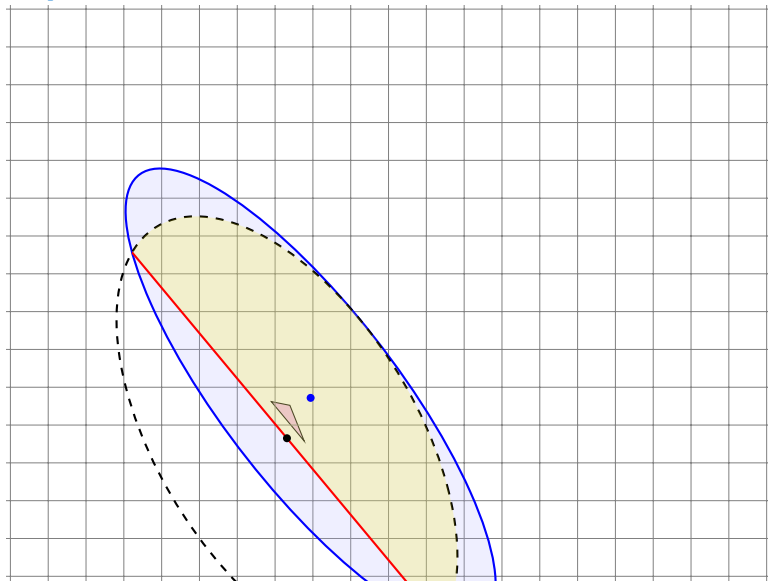
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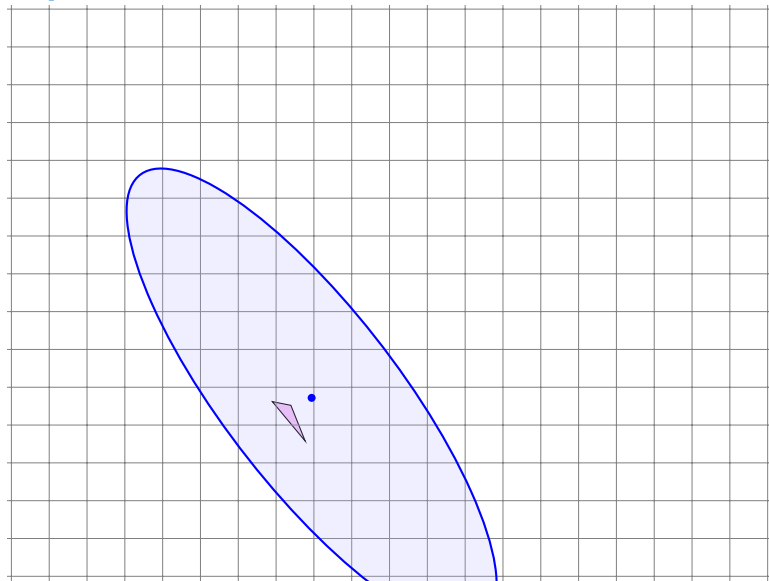
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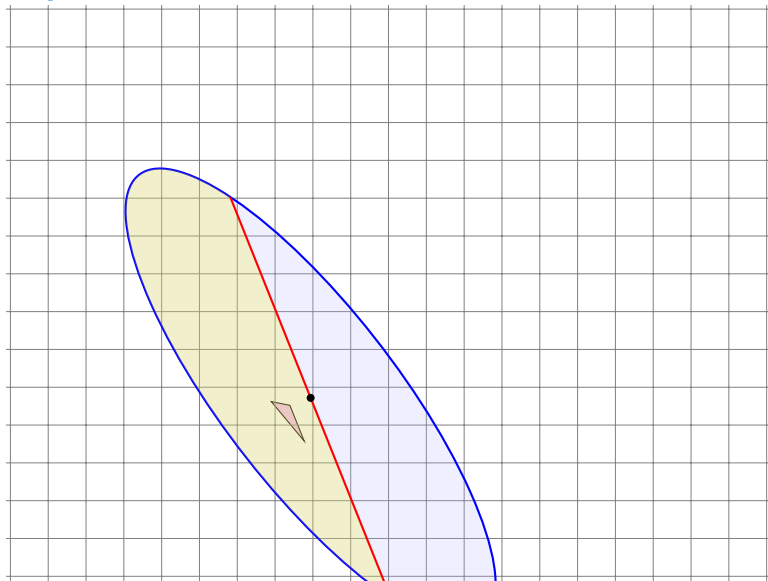
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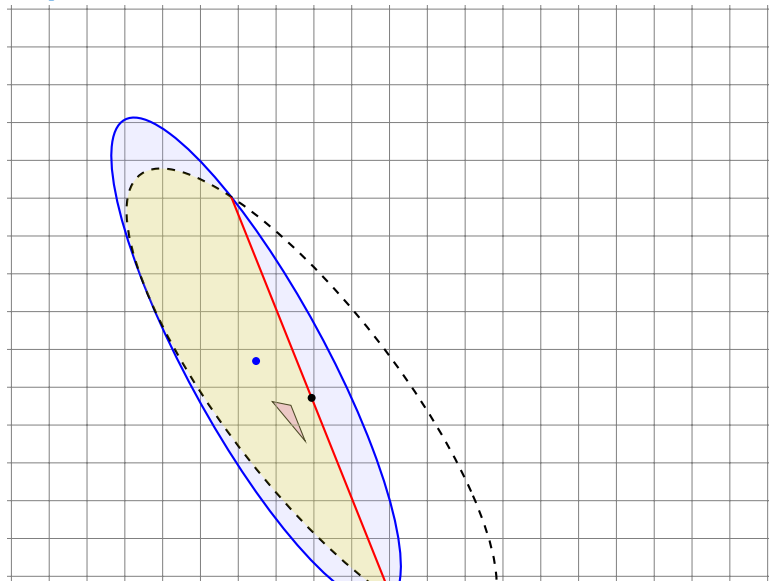
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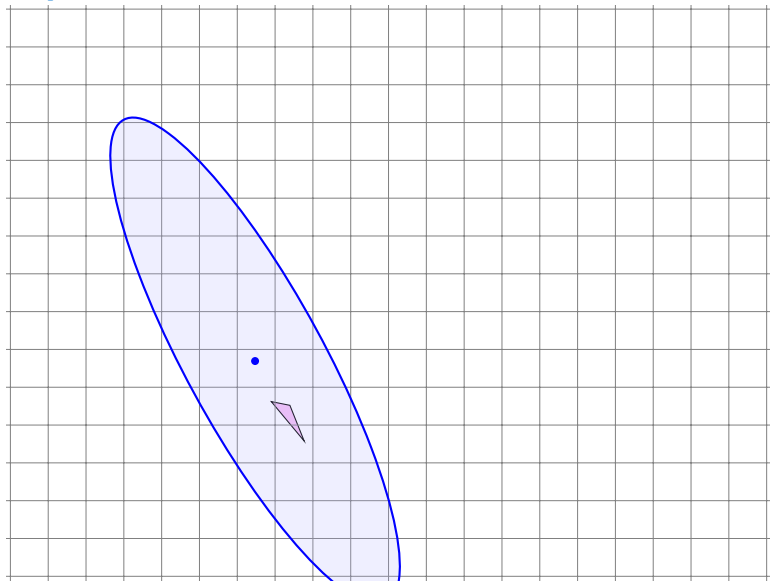
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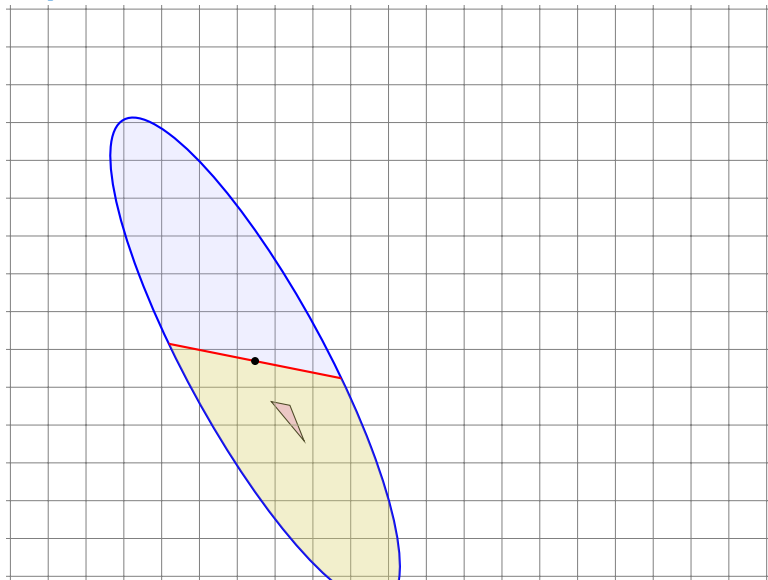
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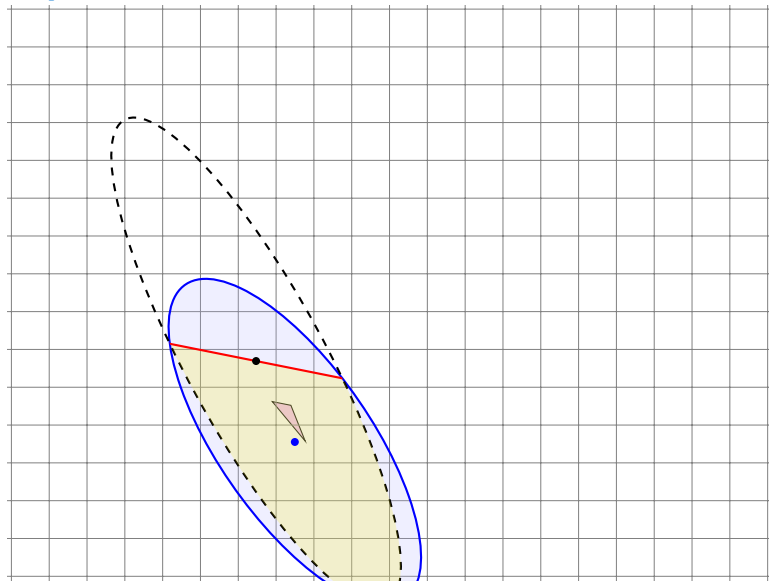
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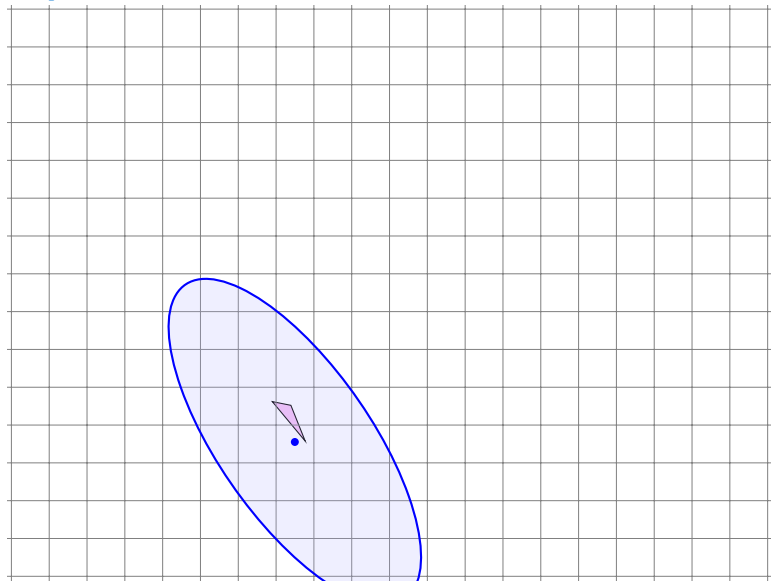
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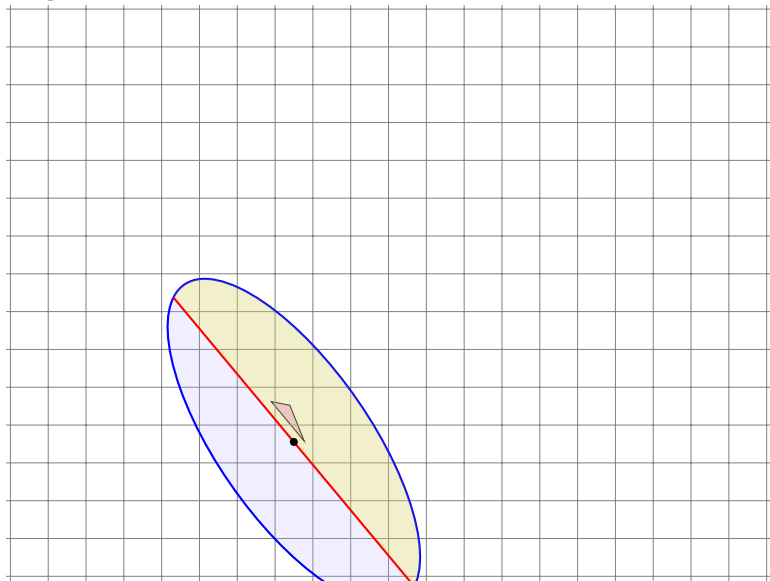
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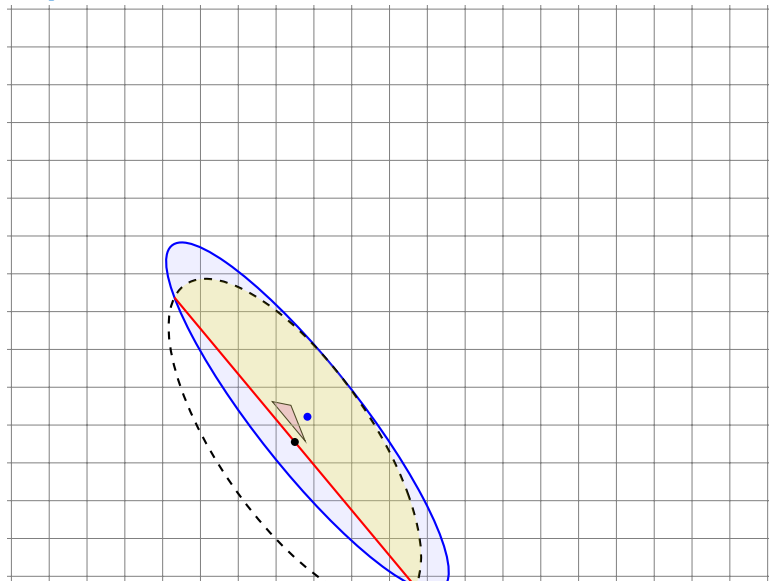
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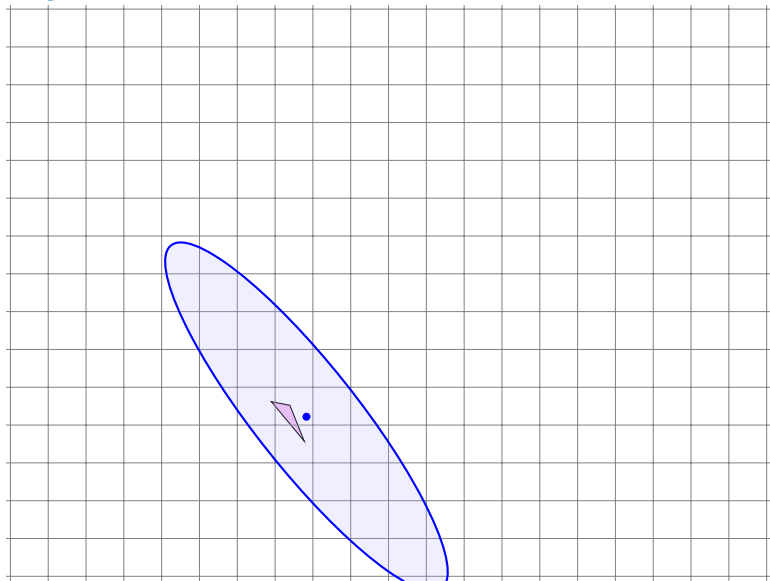
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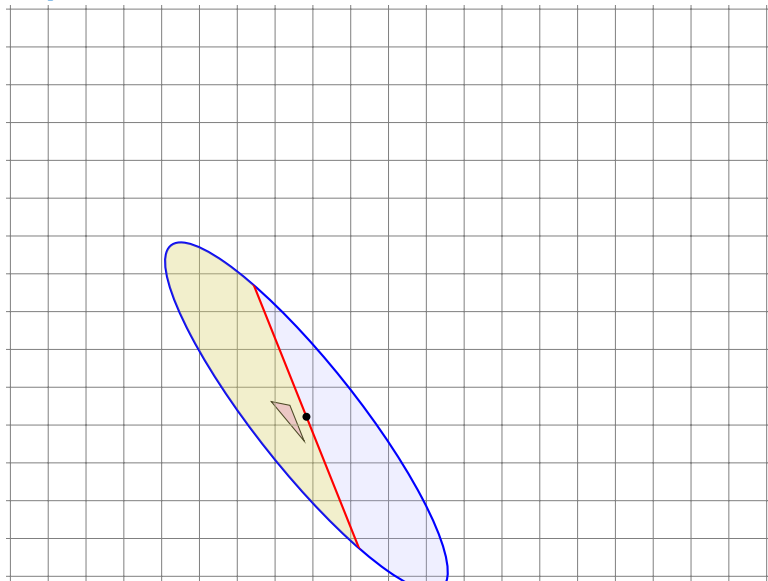
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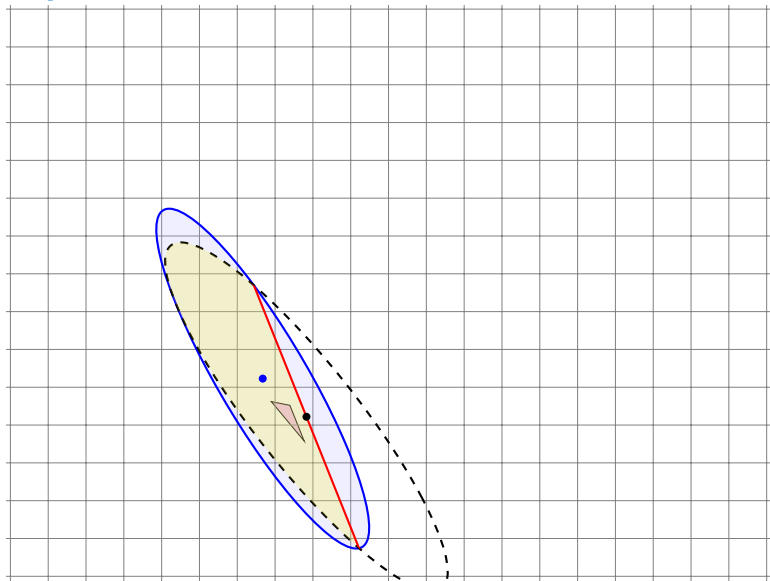
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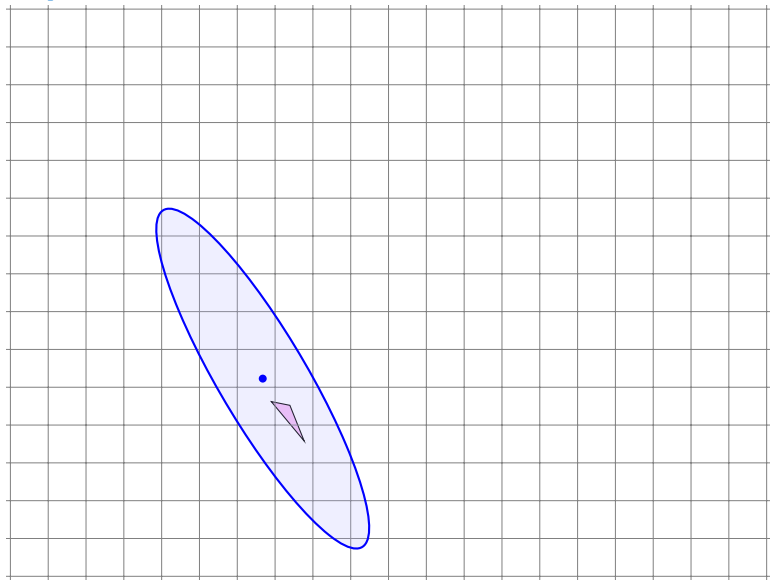
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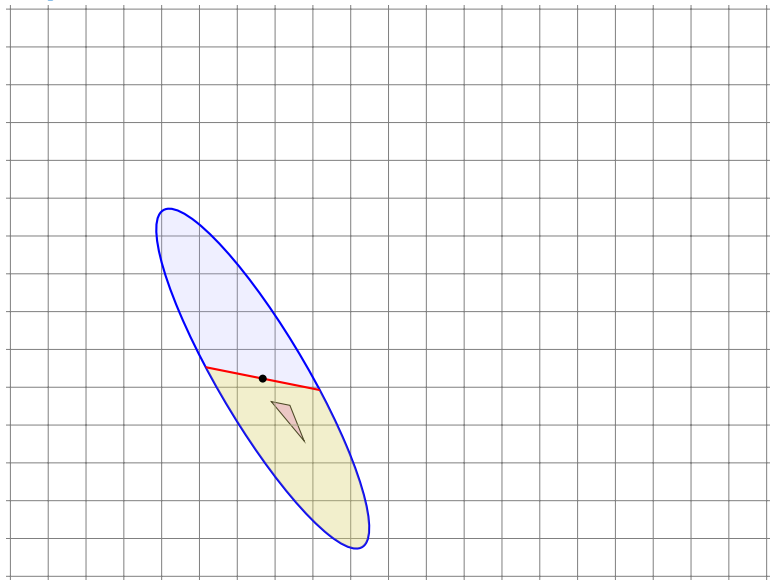
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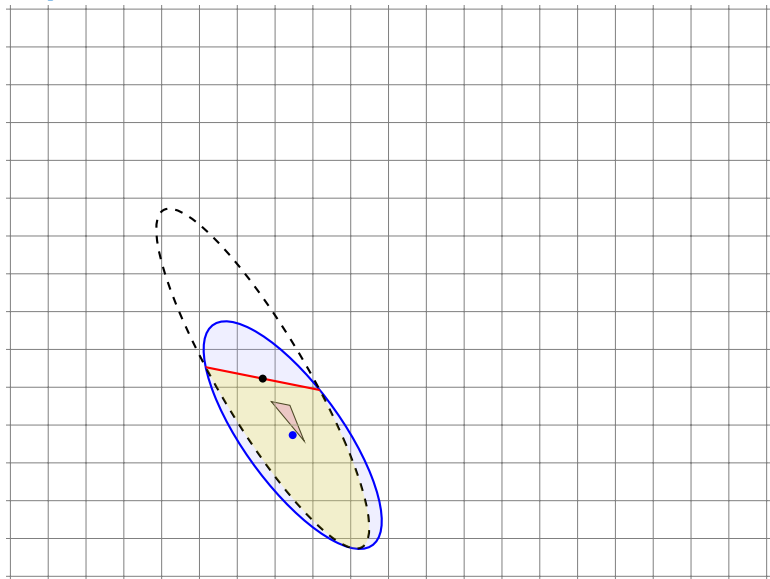
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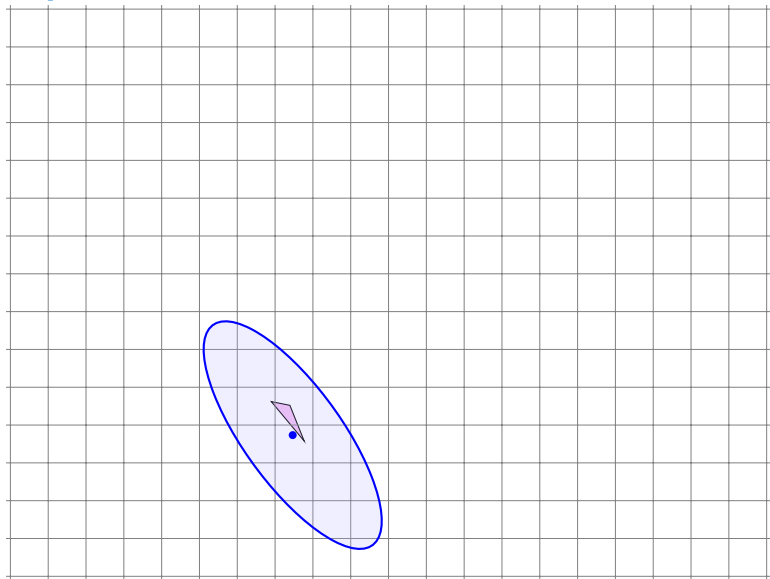
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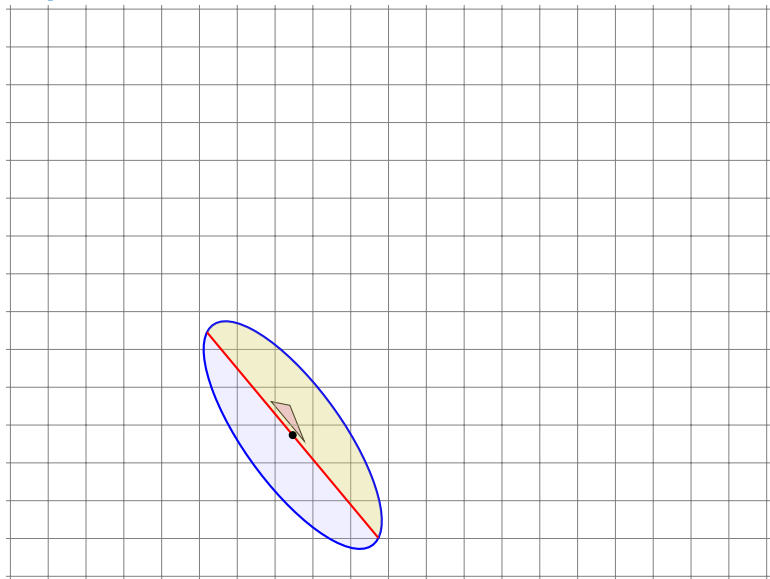
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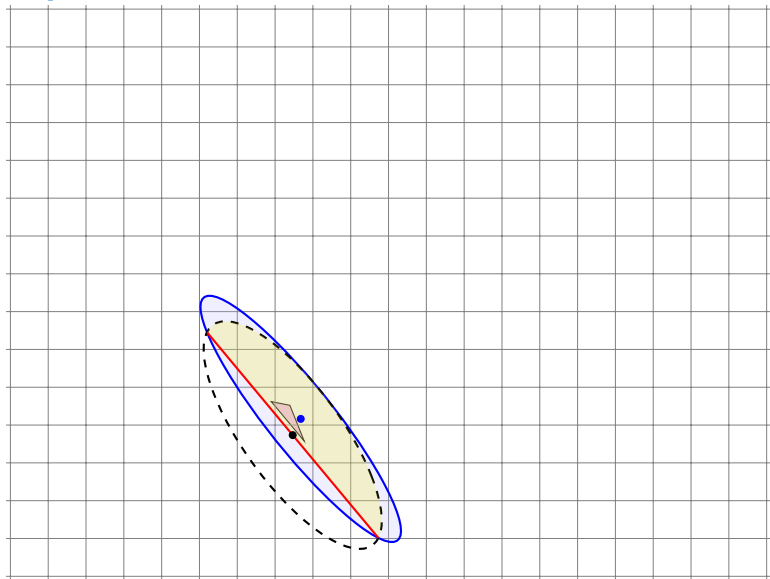
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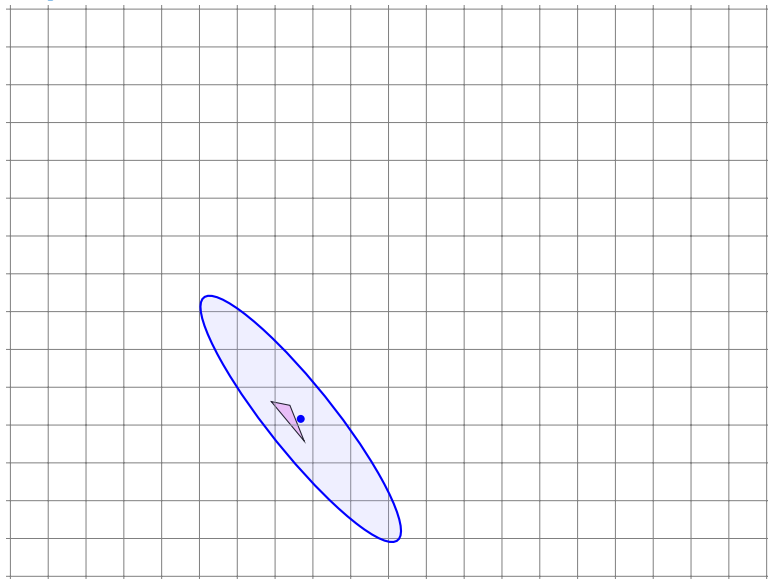
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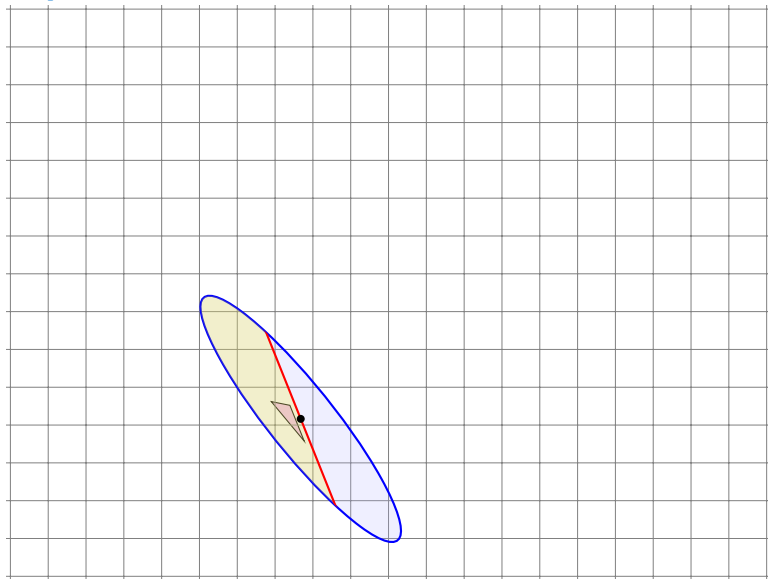
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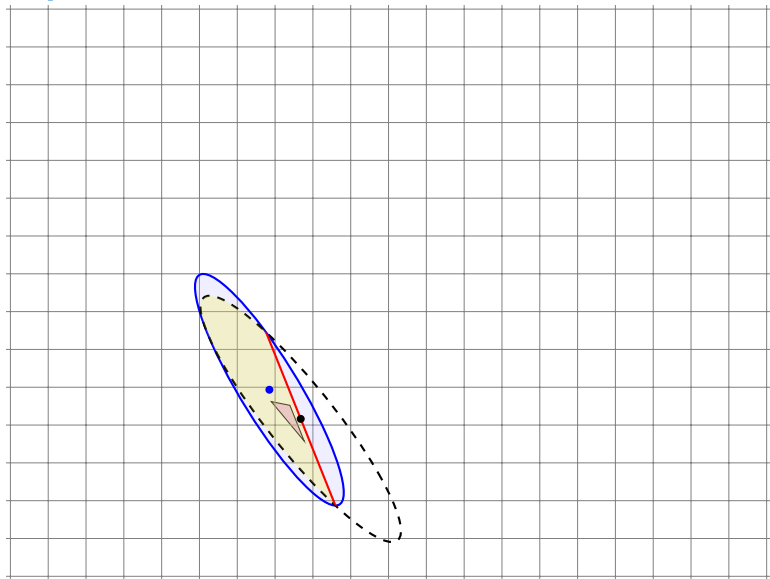
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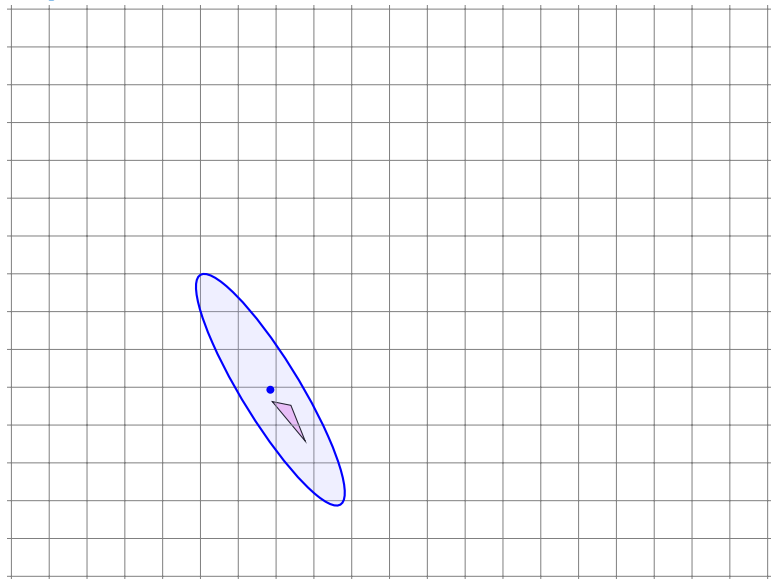
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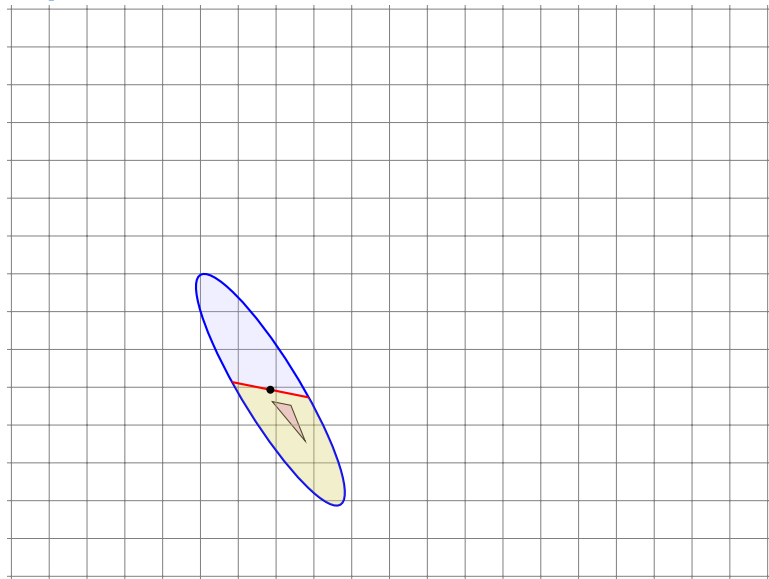
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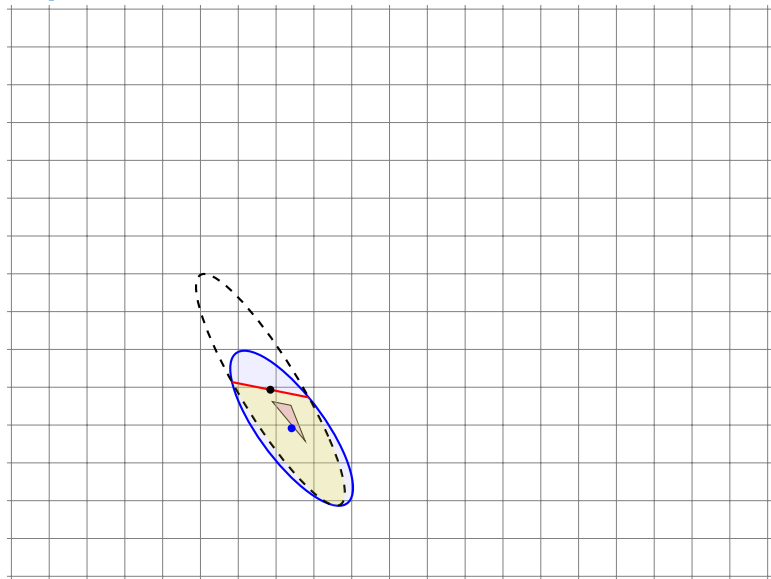
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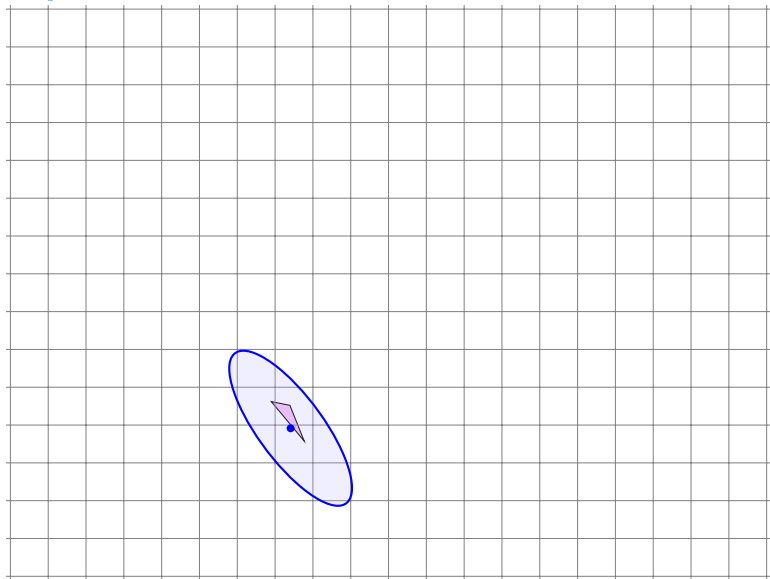
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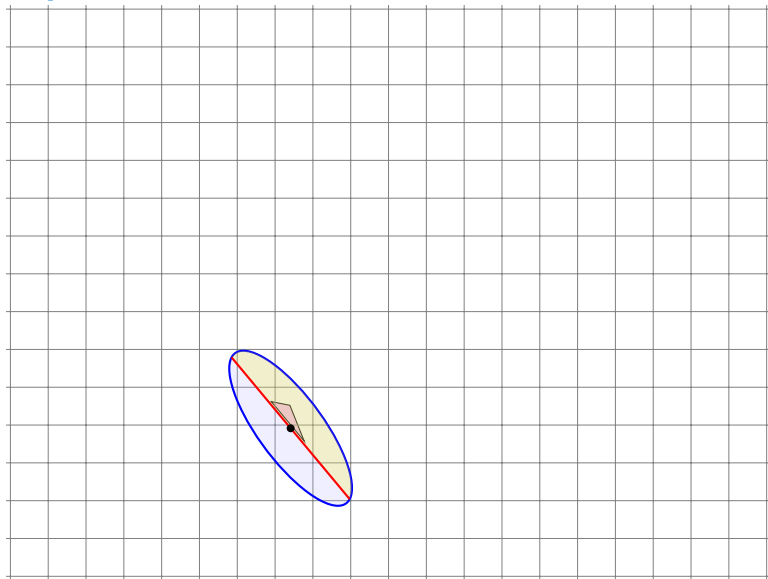
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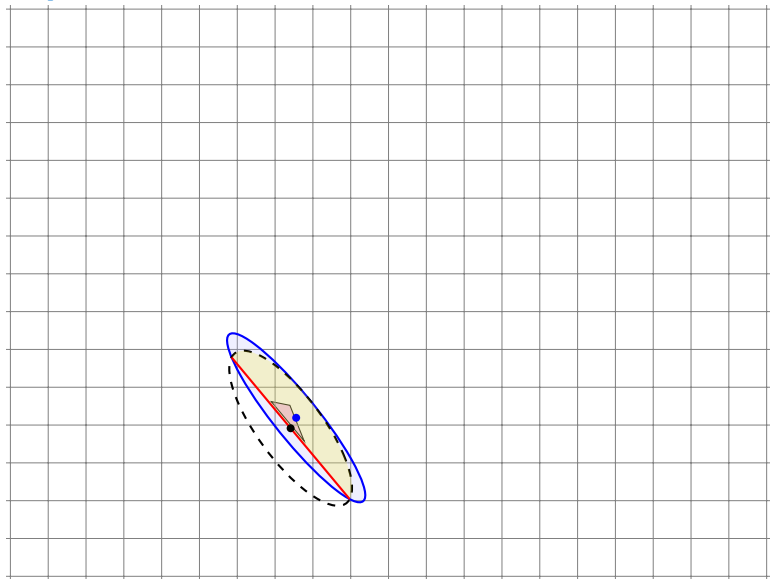
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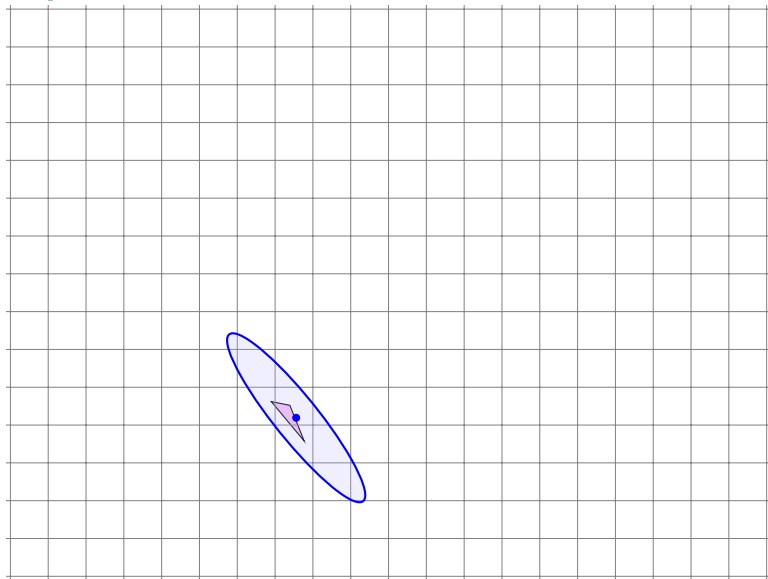
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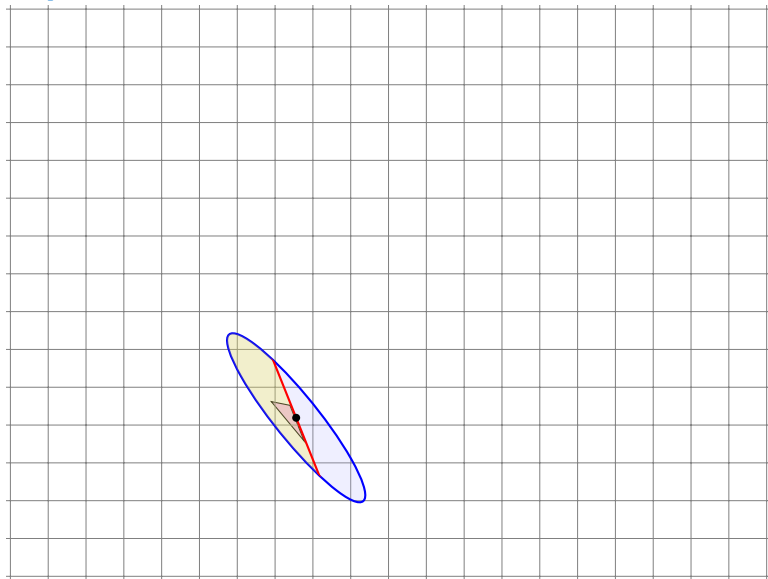
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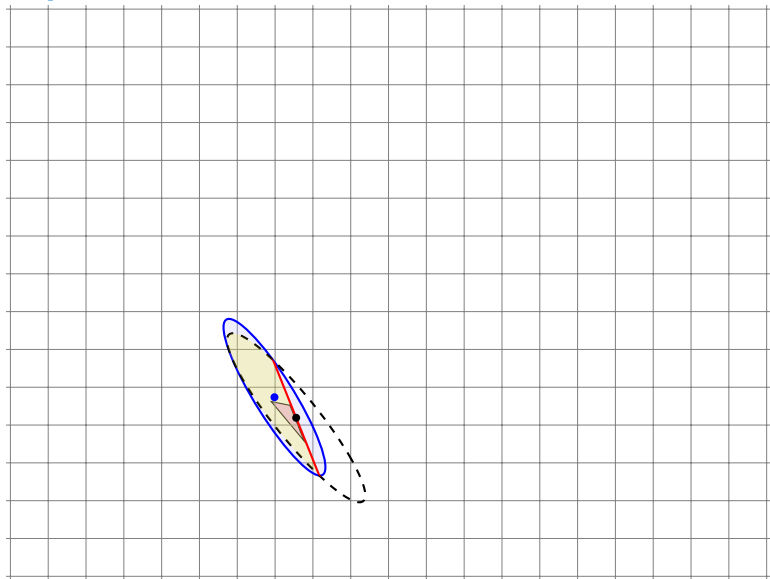
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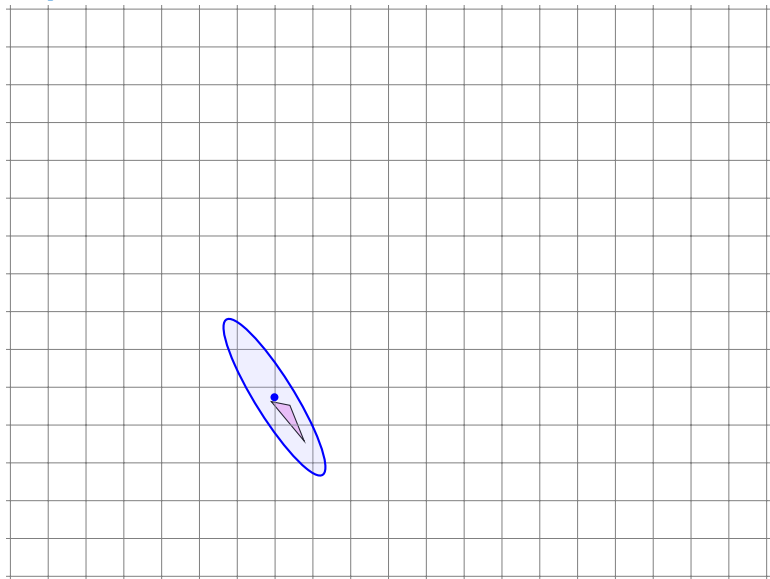
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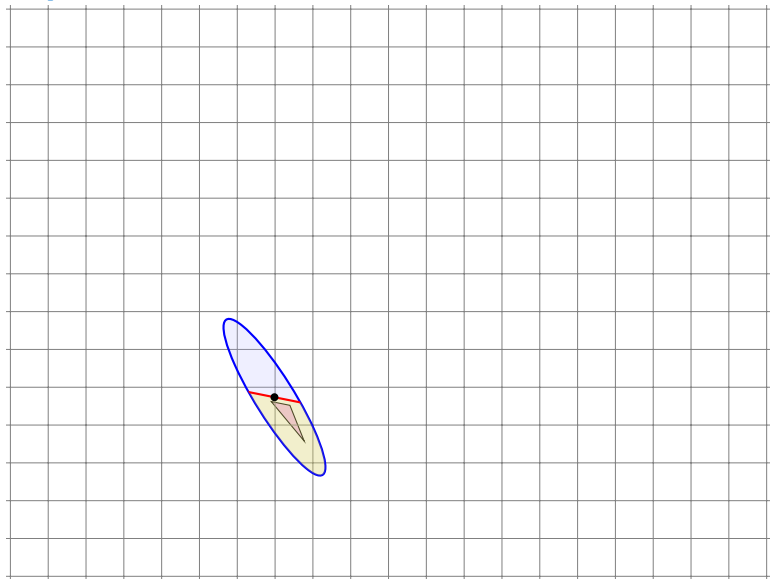
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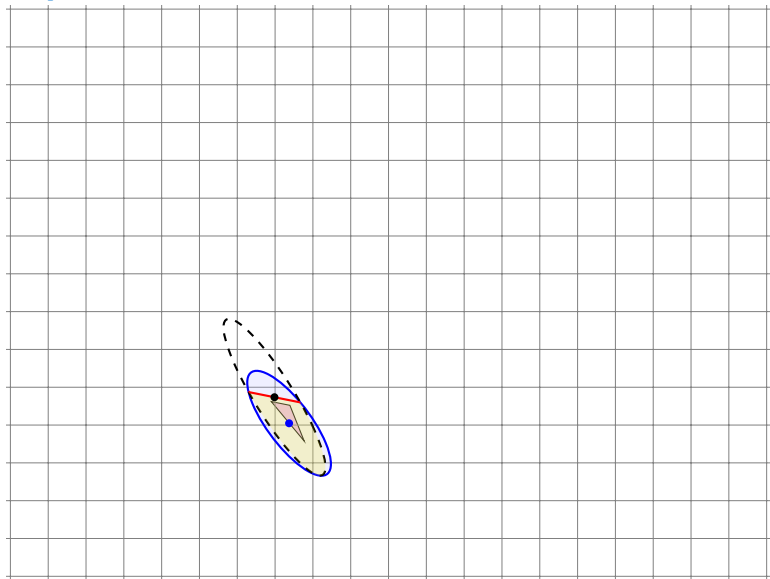
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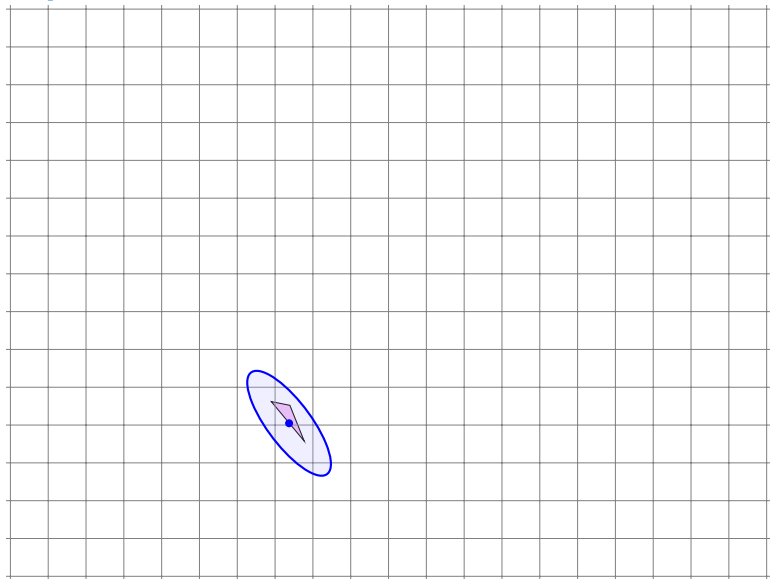
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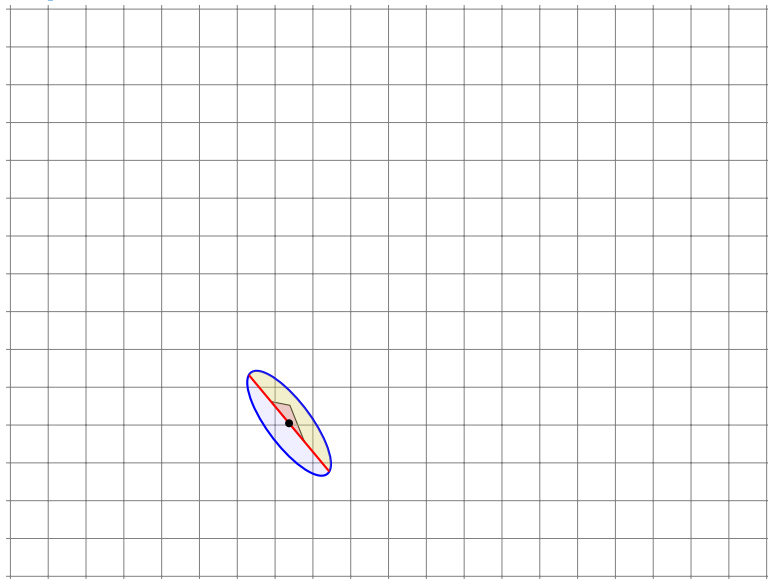
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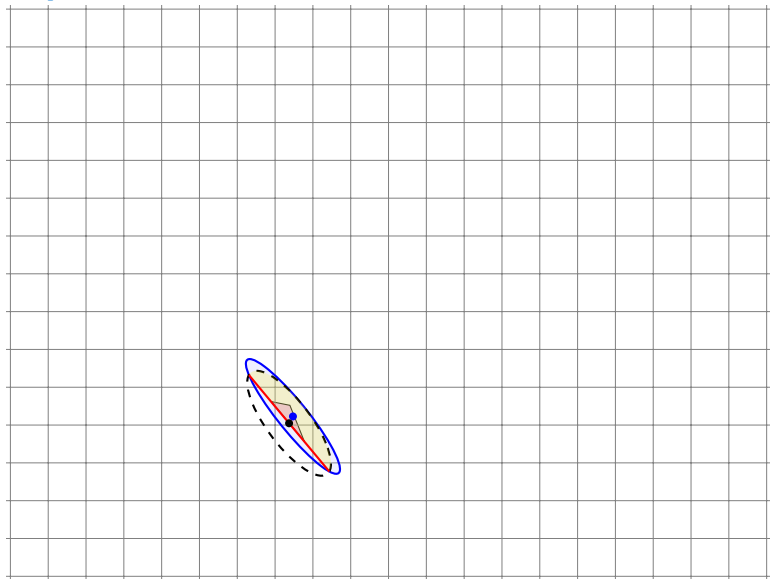
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