

4 Simplex Algorithm

Enumerating all basic feasible solutions (BFS), in order to find the optimum is slow.

Simplex Algorithm [George Dantzig 1947]

Move from BFS to adjacent BFS, without decreasing objective function.

Two BFSs are called adjacent if the bases just differ in one variable.

4 Simplex Algorithm

Enumerating all basic feasible solutions (BFS), in order to find the optimum is slow.

Simplex Algorithm [George Dantzig 1947]

Move from BFS to **adjacent** BFS, without decreasing objective function.

Two BFSs are called **adjacent** if the bases just differ in one variable.

4 Simplex Algorithm

$$\begin{aligned} \max \quad & 13a + 23b \\ \text{s.t.} \quad & 5a + 15b + s_c = 480 \\ & 4a + 4b + s_h = 160 \\ & 35a + 20b + s_m = 1190 \\ & a, b, s_c, s_h, s_m \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & Z \\ & 13a + 23b - Z = 0 \\ & 5a + 15b + s_c = 480 \\ & 4a + 4b + s_h = 160 \\ & 35a + 20b + s_m = 1190 \\ & a, b, s_c, s_h, s_m \geq 0 \end{aligned}$$

$$\text{basis} = \{s_c, s_h, s_m\}$$

$$a = b = 0$$

$$Z = 0$$

$$s_c = 480$$

$$s_h = 160$$

$$s_m = 1190$$

4 Simplex Algorithm

$$\begin{aligned} \max \quad & 13a + 23b \\ \text{s.t.} \quad & 5a + 15b + s_c = 480 \\ & 4a + 4b + s_h = 160 \\ & 35a + 20b + s_m = 1190 \\ & a, b, s_c, s_h, s_m \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & Z \\ & 13a + 23b - Z = 0 \\ & 5a + 15b + s_c = 480 \\ & 4a + 4b + s_h = 160 \\ & 35a + 20b + s_m = 1190 \\ & a, b, s_c, s_h, s_m \geq 0 \end{aligned}$$

$$\begin{aligned} \text{basis} &= \{s_c, s_h, s_m\} \\ a &= b = 0 \\ Z &= 0 \\ s_c &= 480 \\ s_h &= 160 \\ s_m &= 1190 \end{aligned}$$

Pivoting Step

max Z

$$13a + 23b \quad \quad \quad - Z = 0$$

$$5a + 15b + s_c \quad \quad \quad = 480$$

$$4a + 4b \quad \quad + s_h \quad \quad \quad = 160$$

$$35a + 20b \quad \quad \quad + s_m \quad \quad \quad = 1190$$

$$a, \quad b, \quad s_c, \quad s_h, \quad s_m \quad \geq 0$$

basis = $\{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

Pivoting Step

max Z

$$13a + 23b \quad \quad \quad - Z = 0$$

$$5a + 15b + s_c \quad \quad \quad = 480$$

$$4a + 4b \quad \quad + s_h \quad \quad \quad = 160$$

$$35a + 20b \quad \quad \quad + s_m \quad \quad \quad = 1190$$

$$a, \quad b, \quad s_c, \quad s_h, \quad s_m \quad \geq 0$$

basis = $\{s_c, s_h, s_m\}$

$$a = b = 0$$

$$Z = 0$$

$$s_c = 480$$

$$s_h = 160$$

$$s_m = 1190$$

- ▶ choose variable to bring into the basis
- ▶ chosen variable should have positive coefficient in objective function
- ▶ apply min-ratio test to find out by how much the variable can be increased
- ▶ pivot on row found by min-ratio test
- ▶ the existing basis variable in this row leaves the basis

Pivoting Step

max Z

$$13a + 23b \quad \quad \quad - Z = 0$$

$$5a + 15b + s_c \quad \quad \quad = 480$$

$$4a + 4b \quad \quad + s_h \quad \quad \quad = 160$$

$$35a + 20b \quad \quad \quad + s_m \quad \quad \quad = 1190$$

$$a, \quad b, \quad s_c, \quad s_h, \quad s_m \quad \geq 0$$

basis = $\{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

- ▶ choose variable to bring into the basis
- ▶ chosen variable should have positive coefficient in objective function
- ▶ apply min-ratio test to find out by how much the variable can be increased
- ▶ pivot on row found by min-ratio test
- ▶ the existing basis variable in this row leaves the basis

Pivoting Step

max Z

$$13a + 23b \quad \quad \quad - Z = 0$$

$$5a + 15b + s_c \quad \quad \quad = 480$$

$$4a + 4b \quad \quad + s_h \quad \quad \quad = 160$$

$$35a + 20b \quad \quad \quad + s_m \quad \quad \quad = 1190$$

$$a, \quad b, \quad s_c, \quad s_h, \quad s_m \quad \geq 0$$

basis = $\{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

- ▶ choose variable to bring into the basis
- ▶ chosen variable should have positive coefficient in objective function
- ▶ apply **min-ratio** test to find out by how much the variable can be increased
 - ▶ pivot on row found by min-ratio test
 - ▶ the existing basis variable in this row leaves the basis

Pivoting Step

max Z

$$13a + 23b \quad \quad \quad - Z = 0$$

$$5a + 15b + s_c \quad \quad \quad = 480$$

$$4a + 4b \quad \quad + s_h \quad \quad \quad = 160$$

$$35a + 20b \quad \quad \quad + s_m \quad \quad \quad = 1190$$

$$a, \quad b, \quad s_c, \quad s_h, \quad s_m \quad \geq 0$$

basis = $\{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

- ▶ choose variable to bring into the basis
- ▶ chosen variable should have positive coefficient in objective function
- ▶ apply **min-ratio** test to find out by how much the variable can be increased
- ▶ pivot on row found by min-ratio test
- ▶ the existing basis variable in this row leaves the basis

Pivoting Step

max Z

$$13a + 23b \quad \quad \quad - Z = 0$$

$$5a + 15b + s_c \quad \quad \quad = 480$$

$$4a + 4b \quad \quad + s_h \quad \quad \quad = 160$$

$$35a + 20b \quad \quad \quad + s_m \quad \quad \quad = 1190$$

$$a, \quad b, \quad s_c, \quad s_h, \quad s_m \quad \geq 0$$

basis = $\{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

- ▶ choose variable to bring into the basis
- ▶ chosen variable should have positive coefficient in objective function
- ▶ apply **min-ratio** test to find out by how much the variable can be increased
- ▶ pivot on row found by min-ratio test
- ▶ the existing basis variable in this row leaves the basis

max Z

$$13a + 23b \qquad \qquad \qquad - Z = 0$$

$$5a + 15b + s_c \qquad \qquad \qquad = 480$$

$$4a + 4b \qquad \qquad + s_h \qquad \qquad = 160$$

$$35a + 20b \qquad \qquad \qquad + s_m \qquad \qquad = 1190$$

$$a, \quad b, \quad s_c, \quad s_h, \quad s_m \qquad \geq 0$$

basis = $\{s_c, s_h, s_m\}$

$$a = b = 0$$

$$Z = 0$$

$$s_c = 480$$

$$s_h = 160$$

$$s_m = 1190$$

max Z

$$13a + 23b \quad \quad \quad - Z = 0$$

$$5a + 15b + s_c \quad \quad \quad = 480$$

$$4a + 4b \quad \quad + s_h \quad \quad \quad = 160$$

$$35a + 20b \quad \quad \quad + s_m \quad \quad \quad = 1190$$

$$a, \quad b, \quad s_c, \quad s_h, \quad s_m \quad \quad \geq 0$$

basis = $\{s_c, s_h, s_m\}$

$$a = b = 0$$

$$Z = 0$$

$$s_c = 480$$

$$s_h = 160$$

$$s_m = 1190$$

- ▶ Choose variable with coefficient > 0 as entering variable.

max Z

$$13a + 23b \quad \quad \quad - Z = 0$$

$$5a + 15b + s_c \quad \quad \quad = 480$$

$$4a + 4b \quad \quad + s_h \quad \quad \quad = 160$$

$$35a + 20b \quad \quad \quad + s_m \quad \quad \quad = 1190$$

$$a, \quad b, \quad s_c, \quad s_h, \quad s_m \quad \geq 0$$

basis = $\{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

- ▶ Choose variable with coefficient > 0 as entering variable.
- ▶ If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.

max Z

$$13a + 23b - Z = 0$$

$$5a + 15b + s_c = 480$$

$$4a + 4b + s_h = 160$$

$$35a + 20b + s_m = 1190$$

$$a, b, s_c, s_h, s_m \geq 0$$

basis = $\{s_c, s_h, s_m\}$

$$a = b = 0$$

$$Z = 0$$

$$s_c = 480$$

$$s_h = 160$$

$$s_m = 1190$$

- ▶ Choose variable with coefficient > 0 as **entering variable**.
- ▶ If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.
- ▶ For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.

max Z

$$13a + 23b \quad \quad \quad - Z = 0$$

$$5a + 15b + s_c \quad \quad \quad = 480$$

$$4a + 4b \quad \quad + s_h \quad \quad \quad = 160$$

$$35a + 20b \quad \quad \quad + s_m \quad \quad \quad = 1190$$

$$a, \quad b, \quad s_c, \quad s_h, \quad s_m \quad \quad \geq 0$$

basis = $\{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

- ▶ Choose variable with coefficient > 0 as **entering variable**.
- ▶ If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.
- ▶ For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.
- ▶ Choosing $\theta = \min\{480/15, 160/4, 1190/20\}$ ensures that in the new solution one current basic variable becomes 0, and no variable goes negative.

max Z

$$13a + 23b \quad \quad \quad - Z = 0$$

$$5a + 15b + s_c \quad \quad \quad = 480$$

$$4a + 4b \quad \quad + s_h \quad \quad \quad = 160$$

$$35a + 20b \quad \quad \quad + s_m \quad \quad \quad = 1190$$

$$a, \quad b, \quad s_c, \quad s_h, \quad s_m \quad \quad \geq 0$$

basis = $\{s_c, s_h, s_m\}$

$$a = b = 0$$

$$Z = 0$$

$$s_c = 480$$

$$s_h = 160$$

$$s_m = 1190$$

- ▶ Choose variable with coefficient > 0 as **entering variable**.
- ▶ If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.
- ▶ For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.
- ▶ Choosing $\theta = \min\{480/15, 160/4, 1190/20\}$ ensures that in the new solution one current basic variable becomes 0, and no variable goes negative.
- ▶ The basic variable in the row that gives $\min\{480/15, 160/4, 1190/20\}$ becomes the **leaving variable**.

max Z

$$13a + 23b \qquad \qquad \qquad - Z = 0$$

$$5a + 15b + s_c \qquad \qquad \qquad = 480$$

$$4a + 4b \qquad \qquad + s_h \qquad \qquad = 160$$

$$35a + 20b \qquad \qquad \qquad + s_m \qquad \qquad = 1190$$

$$a, \quad b, \quad s_c, \quad s_h, \quad s_m \qquad \geq 0$$

basis = $\{s_c, s_h, s_m\}$

$$a = b = 0$$

$$Z = 0$$

$$s_c = 480$$

$$s_h = 160$$

$$s_m = 1190$$

max Z

$$13a + 23b \quad \quad \quad - Z = 0$$

$$5a + 15b + s_c \quad \quad \quad = 480$$

$$4a + 4b \quad \quad + s_h \quad \quad \quad = 160$$

$$35a + 20b \quad \quad \quad + s_m \quad \quad \quad = 1190$$

$$a, \quad b, \quad s_c, \quad s_h, \quad s_m \quad \quad \geq 0$$

basis = $\{s_c, s_h, s_m\}$

$$a = b = 0$$

$$Z = 0$$

$$s_c = 480$$

$$s_h = 160$$

$$s_m = 1190$$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

max Z

$$13a + 23b - Z = 0$$

$$5a + 15b + s_c = 480$$

$$4a + 4b + s_h = 160$$

$$35a + 20b + s_m = 1190$$

$$a, b, s_c, s_h, s_m \geq 0$$

basis = $\{s_c, s_h, s_m\}$

$$a = b = 0$$

$$Z = 0$$

$$s_c = 480$$

$$s_h = 160$$

$$s_m = 1190$$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

max Z

$$\frac{16}{3}a - \frac{23}{15}s_c - Z = -736$$

$$\frac{1}{3}a + b + \frac{1}{15}s_c = 32$$

$$\frac{8}{3}a - \frac{4}{15}s_c + s_h = 32$$

$$\frac{85}{3}a - \frac{4}{3}s_c + s_m = 550$$

$$a, b, s_c, s_h, s_m \geq 0$$

basis = $\{b, s_h, s_m\}$

$$a = s_c = 0$$

$$Z = 736$$

$$b = 32$$

$$s_h = 32$$

$$s_m = 550$$

max Z

$$\frac{16}{3}a - \frac{23}{15}s_c - Z = -736$$

$$\frac{1}{3}a + b + \frac{1}{15}s_c = 32$$

$$\frac{8}{3}a - \frac{4}{15}s_c + s_h = 32$$

$$\frac{85}{3}a - \frac{4}{3}s_c + s_m = 550$$

$$a, b, s_c, s_h, s_m \geq 0$$

basis = $\{b, s_h, s_m\}$

$$a = s_c = 0$$

$$Z = 736$$

$$b = 32$$

$$s_h = 32$$

$$s_m = 550$$

max Z

$$\frac{16}{3}a - \frac{23}{15}s_c - Z = -736$$

$$\frac{1}{3}a + b + \frac{1}{15}s_c = 32$$

$$\frac{8}{3}a - \frac{4}{15}s_c + s_h = 32$$

$$\frac{85}{3}a - \frac{4}{3}s_c + s_m = 550$$

$$a, b, s_c, s_h, s_m \geq 0$$

basis = $\{b, s_h, s_m\}$

$a = s_c = 0$

$Z = 736$

$b = 32$

$s_h = 32$

$s_m = 550$

Choose variable a to bring into basis.

max Z

$$\frac{16}{3}a - \frac{23}{15}s_c - Z = -736$$

$$\frac{1}{3}a + b + \frac{1}{15}s_c = 32$$

$$\frac{8}{3}a - \frac{4}{15}s_c + s_h = 32$$

$$\frac{85}{3}a - \frac{4}{3}s_c + s_m = 550$$

$$a, b, s_c, s_h, s_m \geq 0$$

basis = $\{b, s_h, s_m\}$

$a = s_c = 0$

$Z = 736$

$b = 32$

$s_h = 32$

$s_m = 550$

Choose variable a to bring into basis.

Computing $\min\{3 \cdot 32, 3 \cdot 32/8, 3 \cdot 550/85\}$ means pivot on line 2.

max Z

$$\frac{16}{3}a - \frac{23}{15}s_c - Z = -736$$

$$\frac{1}{3}a + b + \frac{1}{15}s_c = 32$$

$$\frac{8}{3}a - \frac{4}{15}s_c + s_h = 32$$

$$\frac{85}{3}a - \frac{4}{3}s_c + s_m = 550$$

$$a, b, s_c, s_h, s_m \geq 0$$

basis = $\{b, s_h, s_m\}$

$$a = s_c = 0$$

$$Z = 736$$

$$b = 32$$

$$s_h = 32$$

$$s_m = 550$$

Choose variable a to bring into basis.

Computing $\min\{3 \cdot 32, 3 \cdot 32/8, 3 \cdot 550/85\}$ means pivot on line 2.

Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

max Z

$$\frac{16}{3}a - \frac{23}{15}s_c - Z = -736$$

$$\frac{1}{3}a + b + \frac{1}{15}s_c = 32$$

$$\frac{8}{3}a - \frac{4}{15}s_c + s_h = 32$$

$$\frac{85}{3}a - \frac{4}{3}s_c + s_m = 550$$

$$a, b, s_c, s_h, s_m \geq 0$$

basis = $\{b, s_h, s_m\}$

$$a = s_c = 0$$

$$Z = 736$$

$$b = 32$$

$$s_h = 32$$

$$s_m = 550$$

Choose variable a to bring into basis.

Computing $\min\{3 \cdot 32, 3 \cdot 32/8, 3 \cdot 550/85\}$ means pivot on line 2.

Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

max Z

$$-s_c - 2s_h - Z = -800$$

$$b + \frac{1}{10}s_c - \frac{1}{8}s_h = 28$$

$$a - \frac{1}{10}s_c + \frac{3}{8}s_h = 12$$

$$\frac{3}{2}s_c - \frac{85}{8}s_h + s_m = 210$$

$$a, b, s_c, s_h, s_m \geq 0$$

basis = $\{a, b, s_m\}$

$$s_c = s_h = 0$$

$$Z = 800$$

$$b = 28$$

$$a = 12$$

$$s_m = 210$$

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- any feasible solution satisfies all constraints in the problem
- the current solution is optimal if and only if the objective function value is at most as large as the objective function value for any other feasible solution

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- ▶ any feasible solution satisfies all equations in the tableaux
- ▶ in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0$, $s_h \geq 0$
- ▶ hence optimum solution value is at most 800
- ▶ the current solution has value 800

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- ▶ any feasible solution satisfies all equations in the tableaux
- ▶ in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0, s_h \geq 0$
- ▶ hence optimum solution value is at most 800
- ▶ the current solution has value 800

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- ▶ any feasible solution satisfies all equations in the tableaux
- ▶ in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0$, $s_h \geq 0$
- ▶ hence optimum solution value is at most 800
- ▶ the current solution has value 800

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- ▶ any feasible solution satisfies all equations in the tableaux
- ▶ in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0, s_h \geq 0$
- ▶ hence optimum solution value is at most 800
- ▶ the current solution has value 800

Matrix View

Let our linear program be

$$\begin{aligned}c_B^T x_B + c_N^T x_N &= Z \\ A_B x_B + A_N x_N &= b \\ x_B, x_N &\geq 0\end{aligned}$$

The simplex tableaux for basis B is

$$\begin{aligned}I x_B + (c_N^T - c_B^T A_B^{-1} A_N) x_N &= Z - c_B^T A_B^{-1} b \\ A_B^{-1} A_N x_N &= A_B^{-1} b \\ x_B, x_N &\geq 0\end{aligned}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

Matrix View

Let our linear program be

$$\begin{aligned}c_B^T x_B + c_N^T x_N &= Z \\ A_B x_B + A_N x_N &= b \\ x_B, x_N &\geq 0\end{aligned}$$

The simplex tableaux for basis B is

$$\begin{aligned}I x_B + (c_N^T - c_B^T A_B^{-1} A_N) x_N &= Z - c_B^T A_B^{-1} b \\ A_B^{-1} A_N x_N &= A_B^{-1} b \\ x_B, x_N &\geq 0\end{aligned}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

Matrix View

Let our linear program be

$$\begin{aligned}c_B^T x_B + c_N^T x_N &= Z \\ A_B x_B + A_N x_N &= b \\ x_B, x_N &\geq 0\end{aligned}$$

The simplex tableaux for basis B is

$$\begin{aligned}I x_B + (c_N^T - c_B^T A_B^{-1} A_N) x_N &= Z - c_B^T A_B^{-1} b \\ x_B, x_N &\geq 0\end{aligned}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

Matrix View

Let our linear program be

$$\begin{aligned}c_B^T x_B + c_N^T x_N &= Z \\ A_B x_B + A_N x_N &= b \\ x_B, x_N &\geq 0\end{aligned}$$

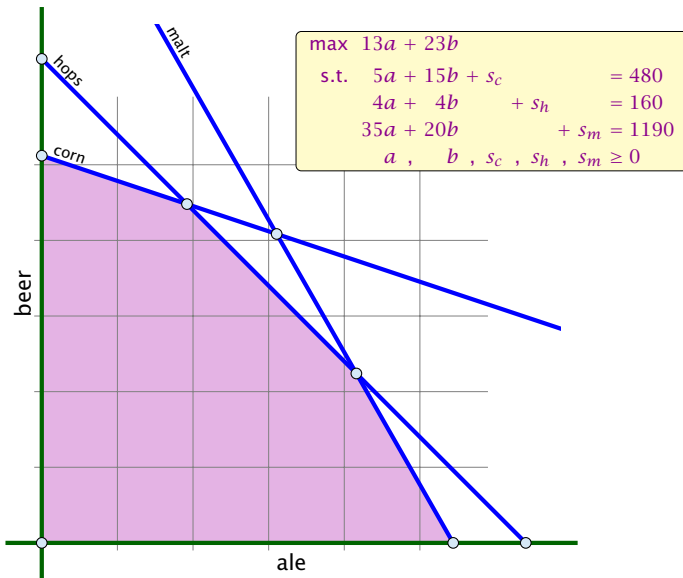
The simplex tableaux for basis B is

$$\begin{aligned}I x_B + (c_N^T - c_B^T A_B^{-1} A_N) x_N &= Z - c_B^T A_B^{-1} b \\ A_B^{-1} A_N x_N &= A_B^{-1} b \\ x_B, x_N &\geq 0\end{aligned}$$

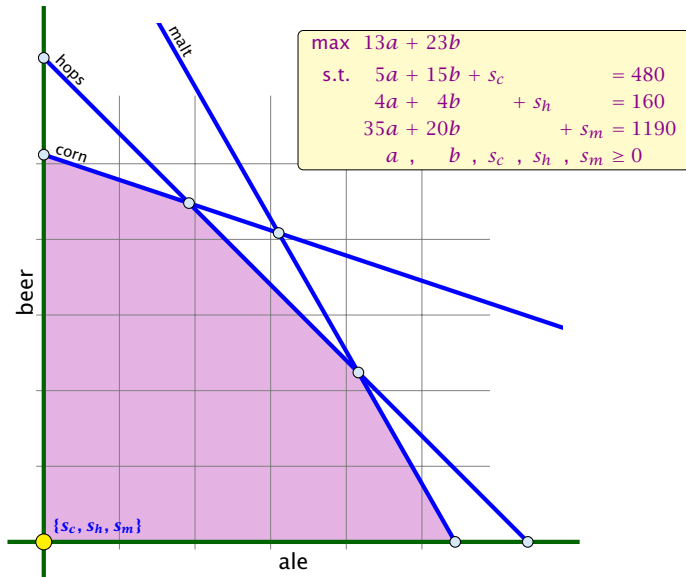
The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

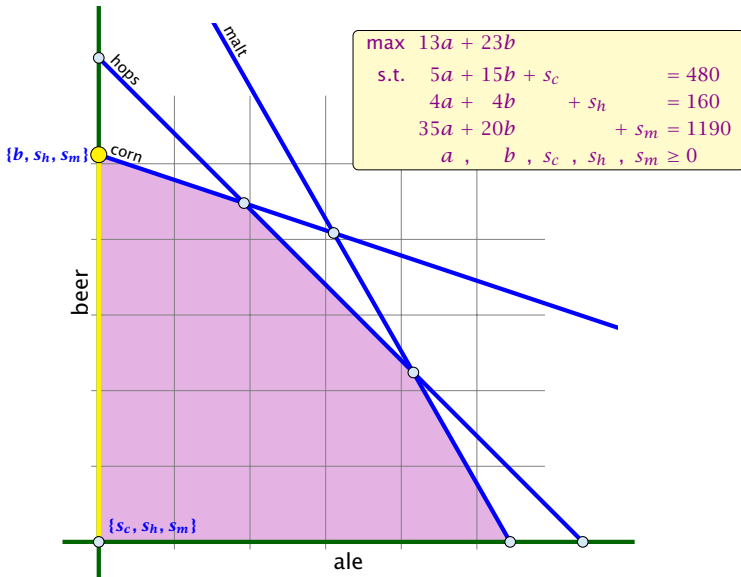
Geometric View of Pivoting



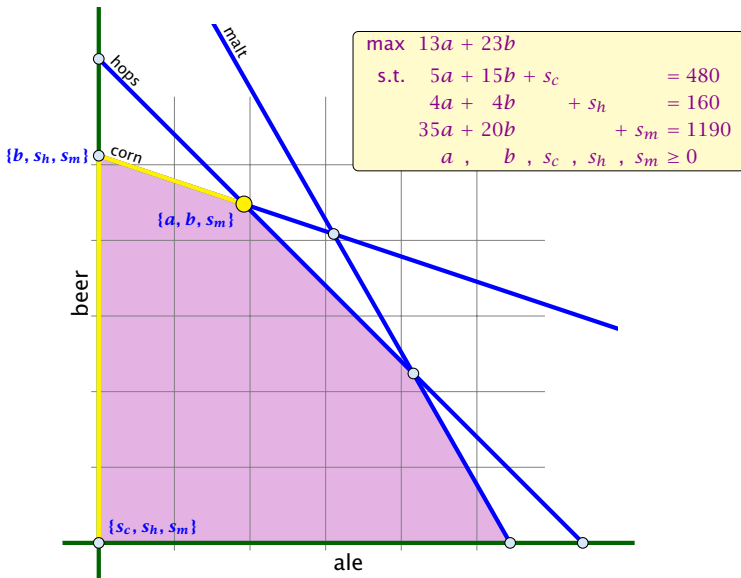
Geometric View of Pivoting



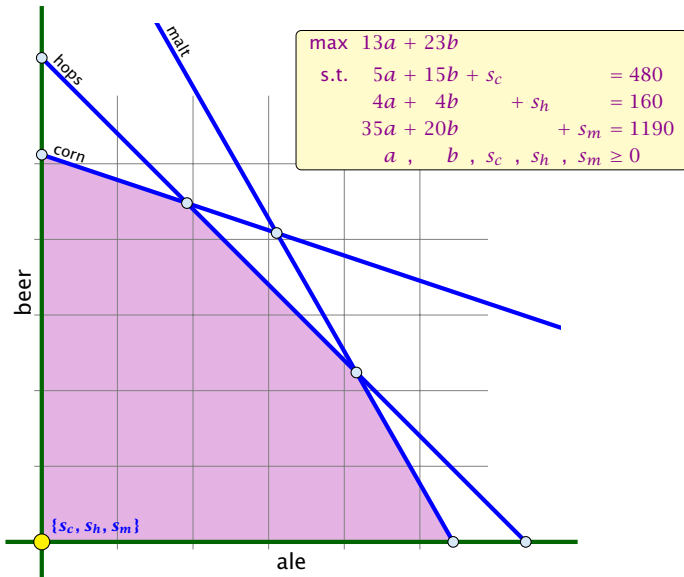
Geometric View of Pivoting



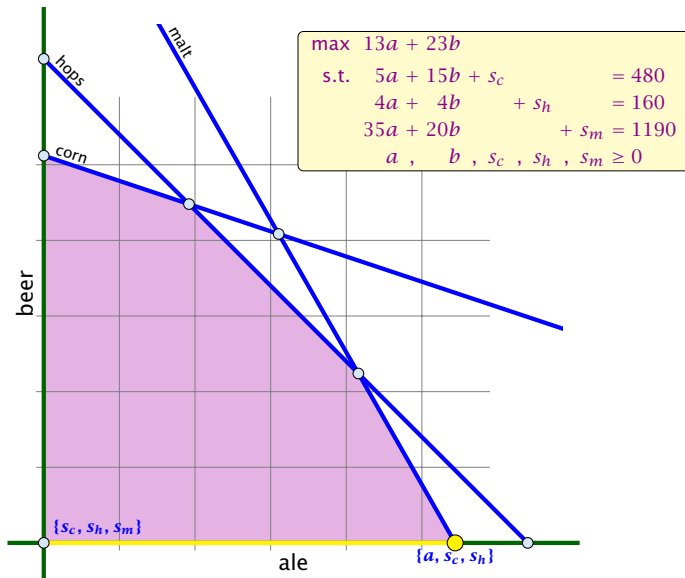
Geometric View of Pivoting



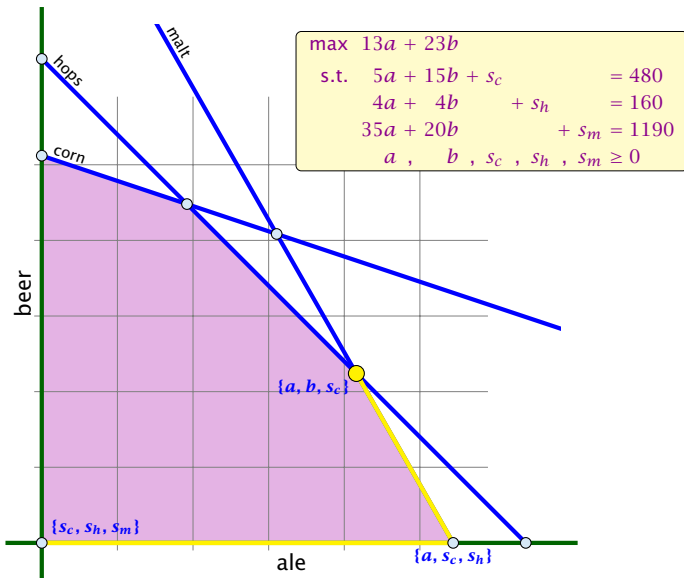
Geometric View of Pivoting



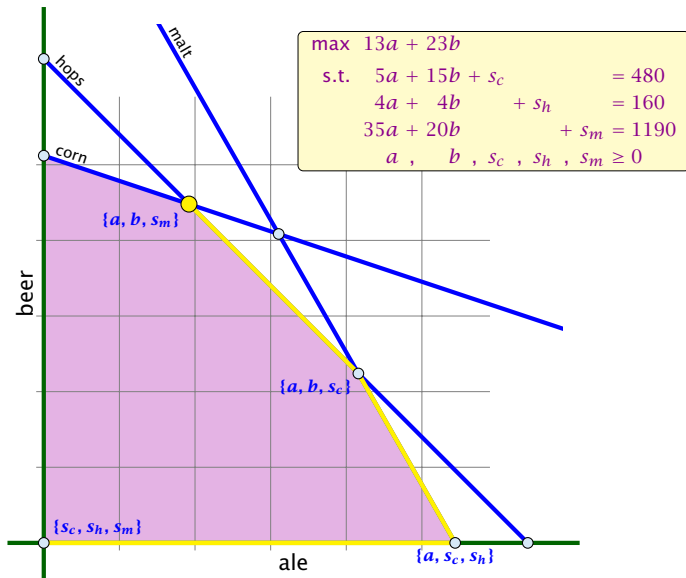
Geometric View of Pivoting



Geometric View of Pivoting



Geometric View of Pivoting



Algebraic Definition of Pivoting

- ▶ Given basis B with BFS x^* .
- ▶ Choose index $j \notin B$ in order to increase x_j^* from 0 to $\theta > 0$.
 - ▶ Other non-basis variables should stay at 0.
 - ▶ Basis variables change to maintain feasibility.
- ▶ Go from x^* to $x^* + \theta \cdot d$.

Requirements for d :

1. $d_j = 1$ (normalization)

2. $d_B = 0$ (non-basis variables)

3. d must be feasible, hence $d_B = -A_B^{-1} A_j$

4. Algorithm: $d = A_j - A_B A_B^{-1} A_j$, which gives $d_B = 0$

Algebraic Definition of Pivoting

- ▶ Given basis B with BFS x^* .
- ▶ Choose index $j \notin B$ in order to increase x_j^* from 0 to $\theta > 0$.
 - ▶ Other non-basis variables should stay at 0.
 - ▶ Basis variables change to maintain feasibility.
- ▶ Go from x^* to $x^* + \theta \cdot d$.

Requirements for d :

1. $d_j = 1$ (normalization)

2. $d_B = -a_{Bj}$ (maintain feasibility)

3. $d_{\text{non-basis}} = 0$ (only x_j increases)

4. $d_{\text{slack}} = 0$ (only x_j increases)

Algebraic Definition of Pivoting

- ▶ Given basis B with BFS x^* .
- ▶ Choose index $j \notin B$ in order to increase x_j^* from 0 to $\theta > 0$.
 - ▶ Other non-basis variables should stay at 0.
 - ▶ Basis variables change to maintain feasibility.
- ▶ Go from x^* to $x^* + \theta \cdot d$.

Requirements for d :

1. $d_j = 1$ (normalization)

2. $d_i = 0$ for all $i \in B$

3. d must satisfy all equality constraints

4. d must be a direction, i.e. $d \geq 0$ and $d \leq 0$ for all $i \in I$

Algebraic Definition of Pivoting

- ▶ Given basis B with BFS x^* .
- ▶ Choose index $j \notin B$ in order to increase x_j^* from 0 to $\theta > 0$.
 - ▶ Other non-basis variables should stay at 0.
 - ▶ Basis variables change to maintain feasibility.
- ▶ Go from x^* to $x^* + \theta \cdot d$.

Requirements for d :

• $d_j = 1$ (normalization)

• $d_i = 0$ for all $i \in B$ (no change in basis)

• $d_i = -a_{ij}$ for all $i \in B$ (maintain feasibility)

• $d_i = 0$ for all $i \notin B, i \neq j$ (no change in other non-basis variables)

Algebraic Definition of Pivoting

- ▶ Given basis B with BFS x^* .
- ▶ Choose index $j \notin B$ in order to increase x_j^* from 0 to $\theta > 0$.
 - ▶ Other non-basis variables should stay at 0.
 - ▶ Basis variables change to maintain feasibility.
- ▶ Go from x^* to $x^* + \theta \cdot d$.

Requirements for d :

1. $d_j = 1$ and $d_i = 0$ for $i \in B$.

2. $d_i = -a_{ij}$ for $i \in B$ and $i \neq j$.

3. $d_i = 0$ for $i \in B$ and $i = j$.

4. $d_i = a_{ij}$ for $i \notin B$ and $i \neq j$.

5. $d_i = 0$ for $i \notin B$ and $i = j$.

6. $d_i = 0$ for $i \in B$.

7. $d_i = 0$ for $i \notin B$.

Algebraic Definition of Pivoting

- ▶ Given basis B with BFS x^* .
- ▶ Choose index $j \notin B$ in order to increase x_j^* from 0 to $\theta > 0$.
 - ▶ Other non-basis variables should stay at 0.
 - ▶ Basis variables change to maintain feasibility.
- ▶ Go from x^* to $x^* + \theta \cdot d$.

Requirements for d :

- ▶ $d_j = 1$ (normalization)
- ▶ $d_\ell = 0, \ell \notin B, \ell \neq j$
- ▶ $A(x^* + \theta d) = b$ must hold. Hence $Ad = 0$.
- ▶ Altogether: $A_B d_B + A_{*j} = Ad = 0$, which gives $d_B = -A_B^{-1} A_{*j}$.

Algebraic Definition of Pivoting

- ▶ Given basis B with BFS x^* .
- ▶ Choose index $j \notin B$ in order to increase x_j^* from 0 to $\theta > 0$.
 - ▶ Other non-basis variables should stay at 0.
 - ▶ Basis variables change to maintain feasibility.
- ▶ Go from x^* to $x^* + \theta \cdot d$.

Requirements for d :

- ▶ $d_j = 1$ (normalization)
- ▶ $d_\ell = 0, \ell \notin B, \ell \neq j$
- ▶ $A(x^* + \theta d) = b$ must hold. Hence $Ad = 0$.
- ▶ Altogether: $A_B d_B + A_{*j} = Ad = 0$, which gives $d_B = -A_B^{-1} A_{*j}$.

Algebraic Definition of Pivoting

- ▶ Given basis B with BFS x^* .
- ▶ Choose index $j \notin B$ in order to increase x_j^* from 0 to $\theta > 0$.
 - ▶ Other non-basis variables should stay at 0.
 - ▶ Basis variables change to maintain feasibility.
- ▶ Go from x^* to $x^* + \theta \cdot d$.

Requirements for d :

- ▶ $d_j = 1$ (normalization)
- ▶ $d_\ell = 0, \ell \notin B, \ell \neq j$
- ▶ $A(x^* + \theta d) = b$ must hold. Hence $Ad = 0$.
- ▶ Altogether: $A_B d_B + A_{*j} = Ad = 0$, which gives
 $d_B = -A_B^{-1} A_{*j}$.

Algebraic Definition of Pivoting

- ▶ Given basis B with BFS x^* .
- ▶ Choose index $j \notin B$ in order to increase x_j^* from 0 to $\theta > 0$.
 - ▶ Other non-basis variables should stay at 0.
 - ▶ Basis variables change to maintain feasibility.
- ▶ Go from x^* to $x^* + \theta \cdot d$.

Requirements for d :

- ▶ $d_j = 1$ (normalization)
- ▶ $d_\ell = 0, \ell \notin B, \ell \neq j$
- ▶ $A(x^* + \theta d) = b$ must hold. Hence $Ad = 0$.
- ▶ Altogether: $A_B d_B + A_{*j} = Ad = 0$, which gives $d_B = -A_B^{-1} A_{*j}$.

Algebraic Definition of Pivoting

Definition 26 (j -th basis direction)

Let B be a basis, and let $j \notin B$. The vector d with $d_j = 1$ and $d_\ell = 0, \ell \notin B, \ell \neq j$ and $d_B = -A_B^{-1}A_{*j}$ is called the j -th basis direction for B .

Going from x^* to $x^* + \theta \cdot d$ the objective function changes by

$$\theta \cdot c^T d = \theta(c_j - c_B^T A_B^{-1} A_{*j})$$

Algebraic Definition of Pivoting

Definition 26 (j -th basis direction)

Let B be a basis, and let $j \notin B$. The vector d with $d_j = 1$ and $d_\ell = 0, \ell \notin B, \ell \neq j$ and $d_B = -A_B^{-1}A_{*j}$ is called the j -th basis direction for B .

Going from x^* to $x^* + \theta \cdot d$ the objective function changes by

$$\theta \cdot c^T d = \theta(c_j - c_B^T A_B^{-1} A_{*j})$$

Algebraic Definition of Pivoting

Definition 27 (Reduced Cost)

For a basis B the value

$$\tilde{c}_j = c_j - c_B^T A_B^{-1} A_{*j}$$

is called the **reduced cost** for variable x_j .

Note that this is defined for every j . If $j \in B$ then the above term is 0.

Algebraic Definition of Pivoting

Let our linear program be

$$\begin{aligned}c_B^T x_B + c_N^T x_N &= Z \\ A_B x_B + A_N x_N &= b \\ x_B, x_N &\geq 0\end{aligned}$$

The simplex tableaux for basis B is

$$\begin{aligned}I x_B + (c_N^T - c_B^T A_B^{-1} A_N) x_N &= Z - c_B^T A_B^{-1} b \\ A_B^{-1} A_N x_N &= A_B^{-1} b \\ x_B, x_N &\geq 0\end{aligned}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

Algebraic Definition of Pivoting

Let our linear program be

$$\begin{aligned}c_B^T x_B + c_N^T x_N &= Z \\ A_B x_B + A_N x_N &= b \\ x_B, x_N &\geq 0\end{aligned}$$

The simplex tableaux for basis B is

$$\begin{aligned}I x_B + (c_N^T - c_B^T A_B^{-1} A_N) x_N &= Z - c_B^T A_B^{-1} b \\ x_B, x_N &\geq 0\end{aligned}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

Algebraic Definition of Pivoting

Let our linear program be

$$\begin{aligned}c_B^T x_B + c_N^T x_N &= Z \\A_B x_B + A_N x_N &= b \\x_B, x_N &\geq 0\end{aligned}$$

The simplex tableaux for basis B is

$$\begin{aligned}I x_B + (c_N^T - c_B^T A_B^{-1} A_N) x_N &= Z - c_B^T A_B^{-1} b \\x_B, x_N &\geq 0\end{aligned}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

Algebraic Definition of Pivoting

Let our linear program be

$$\begin{aligned}c_B^T x_B + c_N^T x_N &= Z \\ A_B x_B + A_N x_N &= b \\ x_B, x_N &\geq 0\end{aligned}$$

The simplex tableaux for basis B is

$$\begin{aligned}I x_B + (c_N^T - c_B^T A_B^{-1} A_N) x_N &= Z - c_B^T A_B^{-1} b \\ A_B^{-1} A_N x_N &= A_B^{-1} b \\ x_B, x_N &\geq 0\end{aligned}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

4 Simplex Algorithm

Questions:

• The ratio test suggests if the min ratio test tells us give us a value θ , how much can we safely increase the entering variable?

• How do we find the initial basic feasible solution?

• What always a basis? Is \mathbb{R}^n a basis?

• When we can terminate because we know that the solution is

• How can we be sure that we reach such a basis?

4 Simplex Algorithm

Questions:

- ▶ What happens if the min ratio test fails to give us a value θ by which we can safely increase the entering variable?
- ▶ How do we find the initial basic feasible solution?
- ▶ Is there always a basis B such that

$$(c_N^T - c_B^T A_B^{-1} A_N) \leq 0 ?$$

Then we can terminate because we know that the solution is optimal.

- ▶ If yes how do we make sure that we reach such a basis?

4 Simplex Algorithm

Questions:

- ▶ What happens if the min ratio test fails to give us a value θ by which we can safely increase the entering variable?
- ▶ How do we find the initial basic feasible solution?
- ▶ Is there always a basis B such that

$$(c_N^T - c_B^T A_B^{-1} A_N) \leq 0 ?$$

Then we can terminate because we know that the solution is optimal.

- ▶ If yes how do we make sure that we reach such a basis?

4 Simplex Algorithm

Questions:

- ▶ What happens if the min ratio test fails to give us a value θ by which we can safely increase the entering variable?
- ▶ How do we find the initial basic feasible solution?
- ▶ Is there always a basis B such that

$$(c_N^T - c_B^T A_B^{-1} A_N) \leq 0 \quad ?$$

Then we can terminate because we know that the solution is optimal.

- ▶ If yes how do we make sure that we reach such a basis?

4 Simplex Algorithm

Questions:

- ▶ What happens if the min ratio test fails to give us a value θ by which we can safely increase the entering variable?
- ▶ How do we find the initial basic feasible solution?
- ▶ Is there always a basis B such that

$$(c_N^T - c_B^T A_B^{-1} A_N) \leq 0 \quad ?$$

Then we can terminate because we know that the solution is optimal.

- ▶ If yes how do we make sure that we reach such a basis?

Min Ratio Test

The min ratio test computes a value $\theta \geq 0$ such that after setting the entering variable to θ the leaving variable becomes 0 and all other variables stay non-negative.

For this, one computes b_i/A_{ie} for all constraints i and calculates the minimum positive value.

What does it mean that the ratio b_i/A_{ie} (and hence A_{ie}) is negative for a constraint?

This means that the corresponding basic variable will increase if we increase b . Hence, there is no danger of this basic variable becoming negative

What happens if all b_i/A_{ie} are negative? Then we do not have a leaving variable. Then the LP is unbounded!

Min Ratio Test

The min ratio test computes a value $\theta \geq 0$ such that after setting the entering variable to θ the leaving variable becomes 0 and all other variables stay non-negative.

For this, one computes b_i/A_{ie} for all constraints i and calculates the minimum positive value.

What does it mean that the ratio b_i/A_{ie} (and hence A_{ie}) is negative for a constraint?

This means that the corresponding basic variable will increase if we increase b . Hence, there is no danger of this basic variable becoming negative

What happens if all b_i/A_{ie} are negative? Then we do not have a leaving variable. Then the LP is unbounded!

Min Ratio Test

The min ratio test computes a value $\theta \geq 0$ such that after setting the entering variable to θ the leaving variable becomes 0 and all other variables stay non-negative.

For this, one computes b_i/A_{ie} for all constraints i and calculates the minimum positive value.

What does it mean that the ratio b_i/A_{ie} (and hence A_{ie}) is negative for a constraint?

This means that the corresponding basic variable will increase if we increase b . Hence, there is no danger of this basic variable becoming negative

What happens if all b_i/A_{ie} are negative? Then we do not have a leaving variable. Then the LP is unbounded!

Min Ratio Test

The min ratio test computes a value $\theta \geq 0$ such that after setting the entering variable to θ the leaving variable becomes 0 and all other variables stay non-negative.

For this, one computes b_i/A_{ie} for all constraints i and calculates the minimum positive value.

What does it mean that the ratio b_i/A_{ie} (and hence A_{ie}) is negative for a constraint?

This means that the corresponding basic variable will increase if we increase b . Hence, there is no danger of this basic variable becoming negative

What happens if all b_i/A_{ie} are negative? Then we do not have a leaving variable. Then the LP is unbounded!

Min Ratio Test

The min ratio test computes a value $\theta \geq 0$ such that after setting the entering variable to θ the leaving variable becomes 0 and all other variables stay non-negative.

For this, one computes b_i/A_{ie} for all constraints i and calculates the minimum positive value.

What does it mean that the ratio b_i/A_{ie} (and hence A_{ie}) is negative for a constraint?

This means that the corresponding basic variable will increase if we increase b . Hence, there is no danger of this basic variable becoming negative

What happens if **all** b_i/A_{ie} are negative? Then we do not have a leaving variable. Then the LP is unbounded!

Min Ratio Test

The min ratio test computes a value $\theta \geq 0$ such that after setting the entering variable to θ the leaving variable becomes 0 and all other variables stay non-negative.

For this, one computes b_i/A_{ie} for all constraints i and calculates the minimum positive value.

What does it mean that the ratio b_i/A_{ie} (and hence A_{ie}) is negative for a constraint?

This means that the corresponding basic variable will increase if we increase b . Hence, there is no danger of this basic variable becoming negative

What happens if **all** b_i/A_{ie} are negative? Then we do not have a leaving variable. **Then the LP is unbounded!**

Termination

The objective function does not decrease during one iteration of the simplex-algorithm.

Does it always increase?

Termination

The objective function does not decrease during one iteration of the simplex-algorithm.

Does it always increase?

Termination

The objective function does not decrease during one iteration of the simplex-algorithm.

Does it always increase?

Termination

The objective function may not increase!

Because a variable x_ℓ with $\ell \in B$ is already 0.

The set of inequalities is **degenerate** (also the basis is degenerate).

Definition 28 (Degeneracy)

A BFS x^* is called **degenerate** if the set $J = \{j \mid x_j^* > 0\}$ fulfills $|J| < m$.

It is possible that the algorithm **cycles**, i.e., it cycles through a sequence of different bases without ever terminating. Happens, very rarely in practise.

Termination

The objective function may not increase!

Because a variable x_ℓ with $\ell \in B$ is already 0.

The set of inequalities is **degenerate** (also the basis is degenerate).

Definition 28 (Degeneracy)

A BFS x^* is called **degenerate** if the set $J = \{j \mid x_j^* > 0\}$ fulfills $|J| < m$.

It is possible that the algorithm **cycles**, i.e., it cycles through a sequence of different bases without ever terminating. Happens, very rarely in practise.

Termination

The objective function may not increase!

Because a variable x_ℓ with $\ell \in B$ is already 0.

The set of inequalities is **degenerate** (also the basis is degenerate).

Definition 28 (Degeneracy)

A BFS x^* is called **degenerate** if the set $J = \{j \mid x_j^* > 0\}$ fulfills $|J| < m$.

It is possible that the algorithm **cycles**, i.e., it cycles through a sequence of different bases without ever terminating. Happens, very rarely in practise.

Termination

The objective function may not increase!

Because a variable x_ℓ with $\ell \in B$ is already 0.

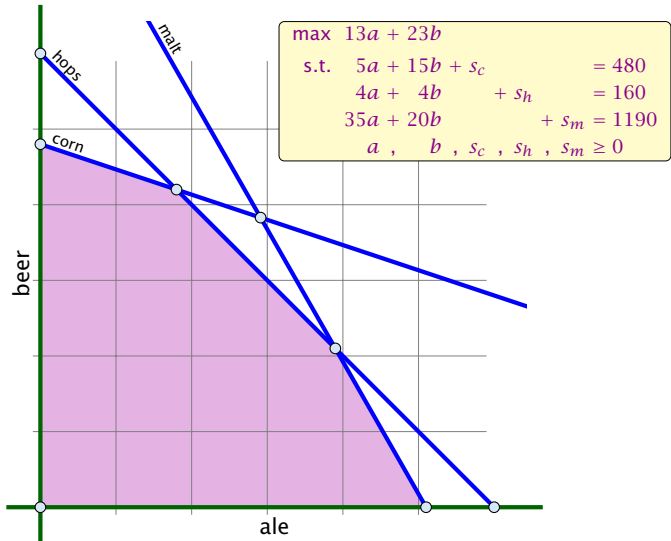
The set of inequalities is **degenerate** (also the basis is degenerate).

Definition 28 (Degeneracy)

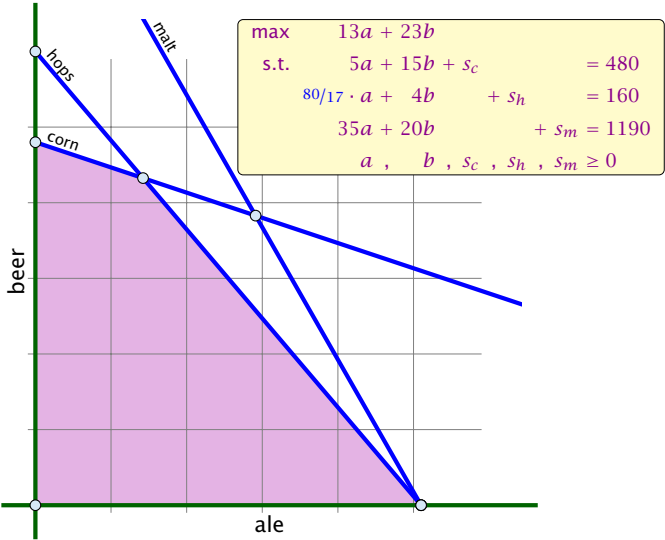
A BFS x^* is called **degenerate** if the set $J = \{j \mid x_j^* > 0\}$ fulfills $|J| < m$.

It is possible that the algorithm **cycles**, i.e., it cycles through a sequence of different bases without ever terminating. Happens, very rarely in practise.

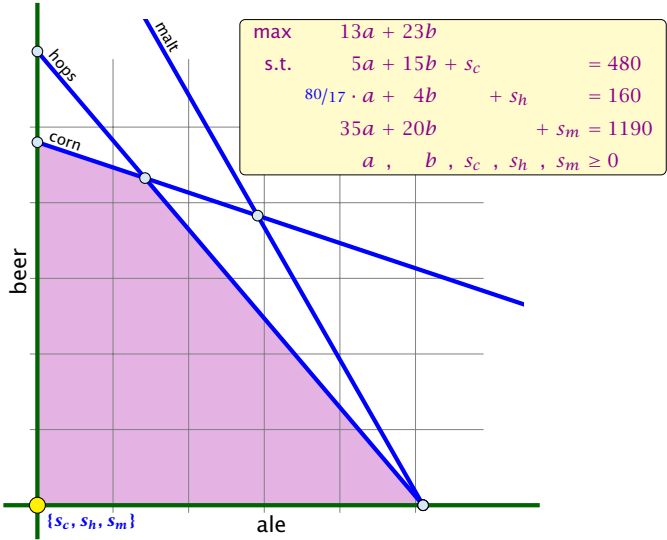
Non Degenerate Example



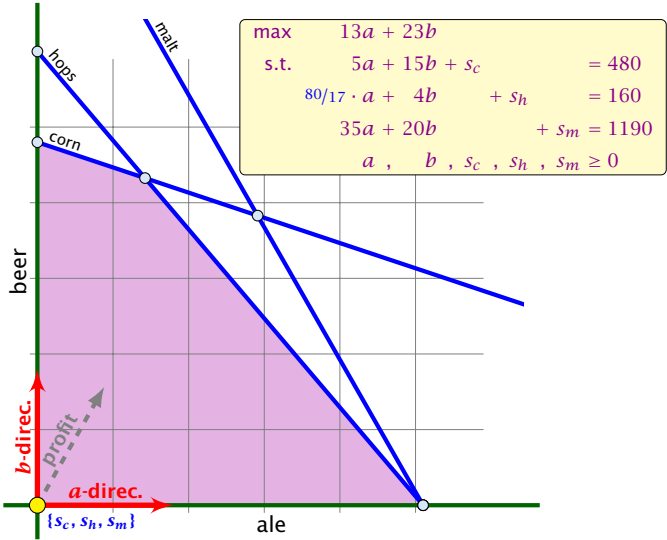
Degenerate Example



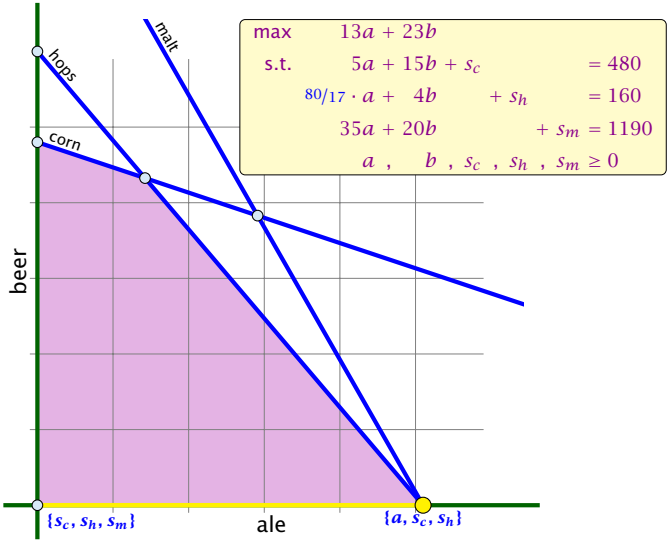
Degenerate Example



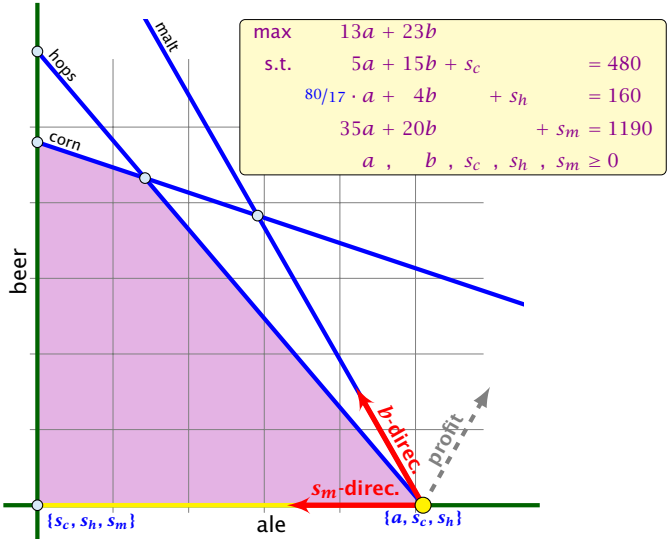
Degenerate Example



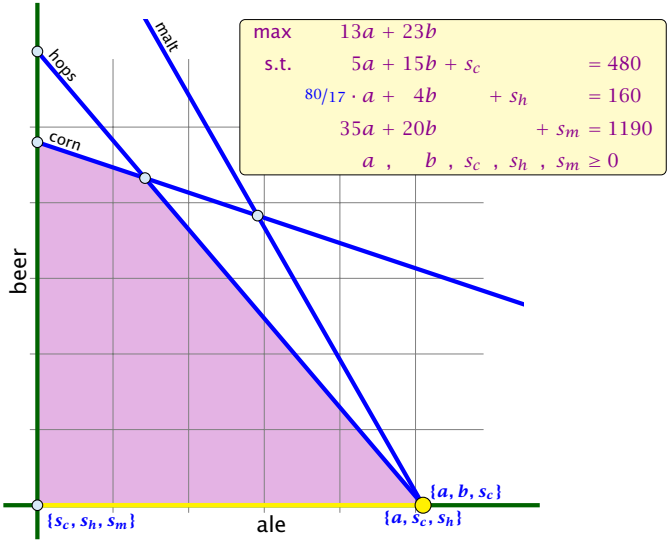
Degenerate Example



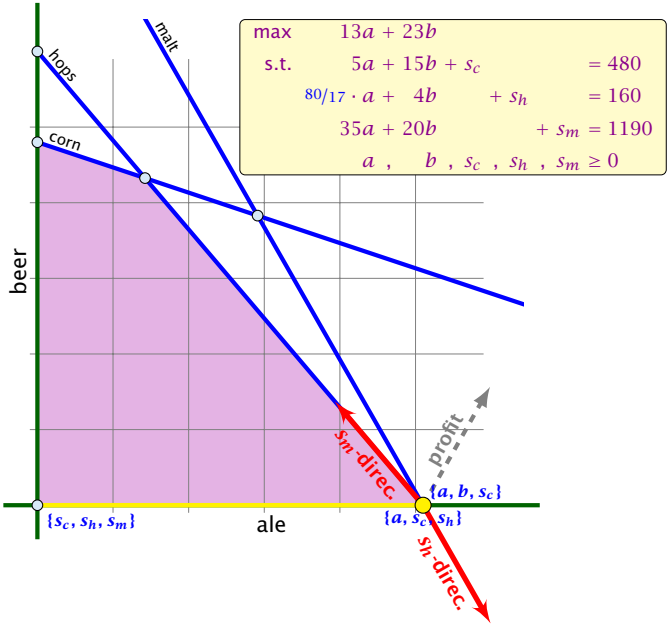
Degenerate Example



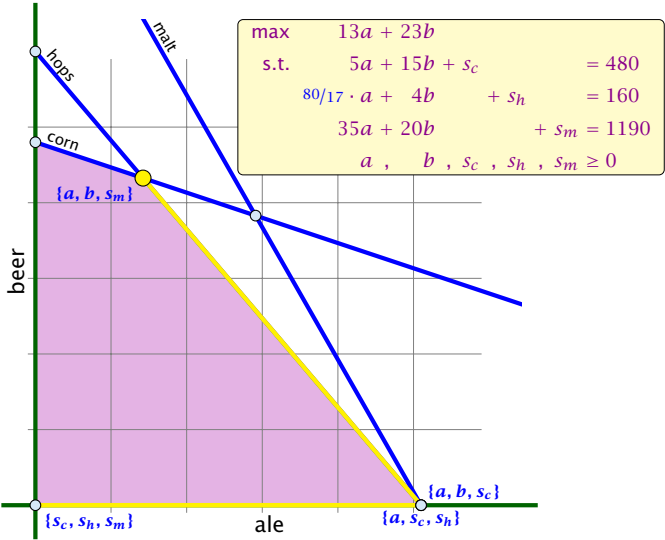
Degenerate Example



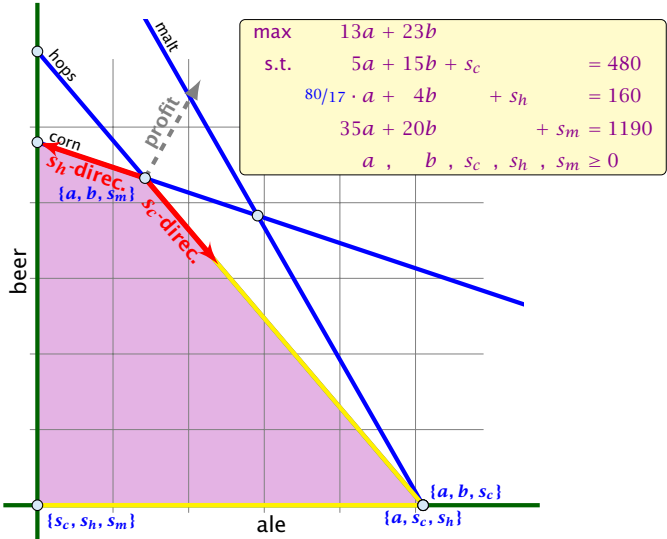
Degenerate Example



Degenerate Example



Degenerate Example



Summary: How to choose pivot-elements

- ▶ We can choose a column e as an entering variable if $\tilde{c}_e > 0$ (\tilde{c}_e is reduced cost for x_e).
- ▶ The standard choice is the column that maximizes \tilde{c}_e .
- ▶ If $A_{ie} \leq 0$ for all $i \in \{1, \dots, m\}$ then the maximum is not bounded.
- ▶ Otw. choose a leaving variable ℓ such that $b_\ell / A_{\ell e}$ is minimal among all variables i with $A_{ie} > 0$.
- ▶ If several variables have minimum $b_\ell / A_{\ell e}$ you reach a **degenerate** basis.
- ▶ Depending on the choice of ℓ it may happen that the algorithm runs into a cycle where it does not escape from a degenerate vertex.

Summary: How to choose pivot-elements

- ▶ We can choose a column e as an entering variable if $\tilde{c}_e > 0$ (\tilde{c}_e is reduced cost for x_e).
- ▶ The standard choice is the column that maximizes \tilde{c}_e .
- ▶ If $A_{ie} \leq 0$ for all $i \in \{1, \dots, m\}$ then the maximum is not bounded.
- ▶ Otw. choose a leaving variable ℓ such that $b_\ell / A_{\ell e}$ is minimal among all variables i with $A_{ie} > 0$.
- ▶ If several variables have minimum $b_\ell / A_{\ell e}$ you reach a **degenerate** basis.
- ▶ Depending on the choice of ℓ it may happen that the algorithm runs into a cycle where it does not escape from a degenerate vertex.

Summary: How to choose pivot-elements

- ▶ We can choose a column e as an entering variable if $\tilde{c}_e > 0$ (\tilde{c}_e is reduced cost for x_e).
- ▶ The standard choice is the column that maximizes \tilde{c}_e .
- ▶ If $A_{ie} \leq 0$ for all $i \in \{1, \dots, m\}$ then the maximum is not bounded.
- ▶ Otw. choose a leaving variable ℓ such that $b_\ell / A_{\ell e}$ is minimal among all variables i with $A_{ie} > 0$.
- ▶ If several variables have minimum $b_\ell / A_{\ell e}$ you reach a **degenerate** basis.
- ▶ Depending on the choice of ℓ it may happen that the algorithm runs into a cycle where it does not escape from a degenerate vertex.

Summary: How to choose pivot-elements

- ▶ We can choose a column e as an entering variable if $\tilde{c}_e > 0$ (\tilde{c}_e is reduced cost for x_e).
- ▶ The standard choice is the column that maximizes \tilde{c}_e .
- ▶ If $A_{ie} \leq 0$ for all $i \in \{1, \dots, m\}$ then the maximum is not bounded.
- ▶ Otw. choose a leaving variable ℓ such that $b_\ell / A_{\ell e}$ is minimal among all variables i with $A_{ie} > 0$.
 - ▶ If several variables have minimum $b_\ell / A_{\ell e}$ you reach a **degenerate** basis.
 - ▶ Depending on the choice of ℓ it may happen that the algorithm runs into a cycle where it does not escape from a degenerate vertex.

Summary: How to choose pivot-elements

- ▶ We can choose a column e as an entering variable if $\tilde{c}_e > 0$ (\tilde{c}_e is reduced cost for x_e).
- ▶ The standard choice is the column that maximizes \tilde{c}_e .
- ▶ If $A_{ie} \leq 0$ for all $i \in \{1, \dots, m\}$ then the maximum is not bounded.
- ▶ Otw. choose a leaving variable ℓ such that $b_\ell / A_{\ell e}$ is minimal among all variables i with $A_{ie} > 0$.
- ▶ If several variables have minimum $b_\ell / A_{\ell e}$ you reach a **degenerate** basis.
- ▶ Depending on the choice of ℓ it may happen that the algorithm runs into a cycle where it does not escape from a degenerate vertex.

Summary: How to choose pivot-elements

- ▶ We can choose a column e as an entering variable if $\tilde{c}_e > 0$ (\tilde{c}_e is reduced cost for x_e).
- ▶ The standard choice is the column that maximizes \tilde{c}_e .
- ▶ If $A_{ie} \leq 0$ for all $i \in \{1, \dots, m\}$ then the maximum is not bounded.
- ▶ Otw. choose a leaving variable ℓ such that $b_\ell / A_{\ell e}$ is minimal among all variables i with $A_{ie} > 0$.
- ▶ If several variables have minimum $b_\ell / A_{\ell e}$ you reach a **degenerate** basis.
- ▶ Depending on the choice of ℓ it may happen that the algorithm runs into a cycle where it does not escape from a degenerate vertex.

What do we have so far?

Suppose we are given an initial feasible solution to an LP. If the LP is non-degenerate then Simplex will terminate.

Note that we either terminate because the min-ratio test fails and we can conclude that the LP is **unbounded**, or we terminate because the vector of reduced cost is non-positive. In the latter case we have an **optimum solution**.

How do we come up with an initial solution?

- ▶ $Ax \leq b, x \geq 0$, and $b \geq 0$.
- ▶ The standard slack form for this problem is $Ax + Is = b, x \geq 0, s \geq 0$, where s denotes the vector of slack variables.
- ▶ Then $s = b, x = 0$ is a basic feasible solution (how?).
- ▶ We directly can start the simplex algorithm.

How do we find an initial basic feasible solution for an arbitrary problem?

How do we come up with an initial solution?

- ▶ $Ax \leq b, x \geq 0$, and $b \geq 0$.
- ▶ The standard slack form for this problem is $Ax + Is = b, x \geq 0, s \geq 0$, where s denotes the vector of slack variables.
 - ▶ Then $s = b, x = 0$ is a basic feasible solution (how?).
 - ▶ We directly can start the simplex algorithm.

How do we find an initial basic feasible solution for an arbitrary problem?

How do we come up with an initial solution?

- ▶ $Ax \leq b, x \geq 0$, and $b \geq 0$.
- ▶ The standard slack form for this problem is $Ax + Is = b, x \geq 0, s \geq 0$, where s denotes the vector of slack variables.
- ▶ Then $s = b, x = 0$ is a basic feasible solution (how?).
- ▶ We directly can start the simplex algorithm.

How do we find an initial basic feasible solution for an arbitrary problem?

How do we come up with an initial solution?

- ▶ $Ax \leq b, x \geq 0$, and $b \geq 0$.
- ▶ The standard slack form for this problem is $Ax + Is = b, x \geq 0, s \geq 0$, where s denotes the vector of slack variables.
- ▶ Then $s = b, x = 0$ is a basic feasible solution (how?).
- ▶ We directly can start the simplex algorithm.

How do we find an initial basic feasible solution for an arbitrary problem?

How do we come up with an initial solution?

- ▶ $Ax \leq b, x \geq 0$, and $b \geq 0$.
- ▶ The standard slack form for this problem is $Ax + Is = b, x \geq 0, s \geq 0$, where s denotes the vector of slack variables.
- ▶ Then $s = b, x = 0$ is a basic feasible solution (how?).
- ▶ We directly can start the simplex algorithm.

How do we find an initial basic feasible solution for an arbitrary problem?

Two phase algorithm

Suppose we want to maximize $c^T x$ s.t. $Ax = b, x \geq 0$.

Multiply all rows with $b_i < 0$ by -1 .

maximize $0^T x$ s.t. $Ax = b, x \geq 0$ (Phase I problem)

Simplex, if b is non-negative is initial feasible.

If b is not, then the original problem is infeasible.

Once you have $x \geq 0$ with $Ax = b$.

From this you can get basic feasible solution.

Now you can start the Simplex for the original problem.

Two phase algorithm

Suppose we want to maximize $c^T x$ s.t. $Ax = b, x \geq 0$.

1. Multiply all rows with $b_i < 0$ by -1 .
2. maximize $-\sum_i v_i$ s.t. $Ax + Iv = b, x \geq 0, v \geq 0$ using Simplex. $x = 0, v = b$ is initial feasible.
3. If $\sum_i v_i > 0$ then the original problem is infeasible.
4. Otw. you have $x \geq 0$ with $Ax = b$.
5. From this you can get basic feasible solution.
6. Now you can start the Simplex for the original problem.

Two phase algorithm

Suppose we want to maximize $c^T x$ s.t. $Ax = b, x \geq 0$.

1. Multiply all rows with $b_i < 0$ by -1 .
2. maximize $-\sum_i v_i$ s.t. $Ax + Iv = b, x \geq 0, v \geq 0$ using Simplex. $x = 0, v = b$ is initial feasible.
3. If $\sum_i v_i > 0$ then the original problem is infeasible.
4. Otw. you have $x \geq 0$ with $Ax = b$.
5. From this you can get basic feasible solution.
6. Now you can start the Simplex for the original problem.

Two phase algorithm

Suppose we want to maximize $c^T x$ s.t. $Ax = b, x \geq 0$.

1. Multiply all rows with $b_i < 0$ by -1 .
2. maximize $-\sum_i v_i$ s.t. $Ax + Iv = b, x \geq 0, v \geq 0$ using Simplex. $x = 0, v = b$ is initial feasible.
3. If $\sum_i v_i > 0$ then the original problem is **infeasible**.
4. Otw. you have $x \geq 0$ with $Ax = b$.
5. From this you can get basic feasible solution.
6. Now you can start the Simplex for the original problem.

Two phase algorithm

Suppose we want to maximize $c^T x$ s.t. $Ax = b, x \geq 0$.

1. Multiply all rows with $b_i < 0$ by -1 .
2. maximize $-\sum_i v_i$ s.t. $Ax + Iv = b, x \geq 0, v \geq 0$ using Simplex. $x = 0, v = b$ is initial feasible.
3. If $\sum_i v_i > 0$ then the original problem is **infeasible**.
4. Otw. you have $x \geq 0$ with $Ax = b$.
5. From this you can get basic feasible solution.
6. Now you can start the Simplex for the original problem.

Two phase algorithm

Suppose we want to maximize $c^T x$ s.t. $Ax = b, x \geq 0$.

1. Multiply all rows with $b_i < 0$ by -1 .
2. maximize $-\sum_i v_i$ s.t. $Ax + Iv = b, x \geq 0, v \geq 0$ using Simplex. $x = 0, v = b$ is initial feasible.
3. If $\sum_i v_i > 0$ then the original problem is **infeasible**.
4. Otw. you have $x \geq 0$ with $Ax = b$.
5. From this you can get basic feasible solution.
6. Now you can start the Simplex for the original problem.

Two phase algorithm

Suppose we want to maximize $c^T x$ s.t. $Ax = b, x \geq 0$.

1. Multiply all rows with $b_i < 0$ by -1 .
2. maximize $-\sum_i v_i$ s.t. $Ax + Iv = b, x \geq 0, v \geq 0$ using Simplex. $x = 0, v = b$ is initial feasible.
3. If $\sum_i v_i > 0$ then the original problem is **infeasible**.
4. Otw. you have $x \geq 0$ with $Ax = b$.
5. From this you can get basic feasible solution.
6. Now you can start the Simplex for the original problem.

Lemma 29

Let B be a basis and x^* a BFS corresponding to basis B . $\tilde{c} \leq 0$ implies that x^* is an optimum solution to the LP.