

4 Simplex Algorithm

Enumerating all basic feasible solutions (BFS), in order to find the optimum is slow.

Simplex Algorithm [George Dantzig 1947]

Move from BFS to **adjacent** BFS, without decreasing objective function.

Two BFSs are called **adjacent** if the bases just differ in one variable.

4 Simplex Algorithm

$$\begin{aligned} \max \quad & 13a + 23b \\ \text{s.t.} \quad & 5a + 15b + s_c = 480 \\ & 4a + 4b + s_h = 160 \\ & 35a + 20b + s_m = 1190 \\ & a, b, s_c, s_h, s_m \geq 0 \end{aligned}$$

$$\begin{aligned} \max Z \quad & 13a + 23b - Z = 0 \\ & 5a + 15b + s_c = 480 \\ & 4a + 4b + s_h = 160 \\ & 35a + 20b + s_m = 1190 \\ & a, b, s_c, s_h, s_m \geq 0 \end{aligned}$$

$$\begin{aligned} \text{basis} &= \{s_c, s_h, s_m\} \\ a &= b = 0 \\ Z &= 0 \\ s_c &= 480 \\ s_h &= 160 \\ s_m &= 1190 \end{aligned}$$

Pivoting Step

$$\begin{aligned} \max Z \quad & 13a + 23b - Z = 0 \\ & 5a + 15b + s_c = 480 \\ & 4a + 4b + s_h = 160 \\ & 35a + 20b + s_m = 1190 \\ & a, b, s_c, s_h, s_m \geq 0 \end{aligned}$$

$$\begin{aligned} \text{basis} &= \{s_c, s_h, s_m\} \\ a &= b = 0 \\ Z &= 0 \\ s_c &= 480 \\ s_h &= 160 \\ s_m &= 1190 \end{aligned}$$

- ▶ choose variable to bring into the basis
- ▶ chosen variable should have positive coefficient in objective function
- ▶ apply **min-ratio** test to find out by how much the variable can be increased
- ▶ pivot on row found by min-ratio test
- ▶ the existing basis variable in this row leaves the basis

$$\begin{aligned} \max Z \quad & 13a + 23b - Z = 0 \\ & 5a + 15b + s_c = 480 \\ & 4a + 4b + s_h = 160 \\ & 35a + 20b + s_m = 1190 \\ & a, b, s_c, s_h, s_m \geq 0 \end{aligned}$$

$$\begin{aligned} \text{basis} &= \{s_c, s_h, s_m\} \\ a &= b = 0 \\ Z &= 0 \\ s_c &= 480 \\ s_h &= 160 \\ s_m &= 1190 \end{aligned}$$

- ▶ Choose variable with coefficient > 0 as **entering variable**.
- ▶ If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.
- ▶ For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.
- ▶ Choosing $\theta = \min\{480/15, 160/4, 1190/20\}$ ensures that in the new solution one current basic variable becomes 0, and no variable goes negative.
- ▶ The basic variable in the row that gives $\min\{480/15, 160/4, 1190/20\}$ becomes the **leaving variable**.

$$\begin{array}{rcl}
\max Z & & \\
13a + 23b & - Z = & 0 \\
5a + 15b + s_c & = & 480 \\
4a + 4b + s_h & = & 160 \\
35a + 20b + s_m & = & 1190 \\
a, b, s_c, s_h, s_m & \geq & 0
\end{array}$$

$$\begin{array}{l}
\text{basis} = \{s_c, s_h, s_m\} \\
a = b = 0 \\
Z = 0 \\
s_c = 480 \\
s_h = 160 \\
s_m = 1190
\end{array}$$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

$$\begin{array}{rcl}
\max Z & & \\
\frac{16}{3}a - \frac{23}{15}s_c & - Z = & -736 \\
\frac{1}{3}a + b + \frac{1}{15}s_c & = & 32 \\
\frac{8}{3}a - \frac{4}{15}s_c + s_h & = & 32 \\
\frac{85}{3}a - \frac{4}{3}s_c + s_m & = & 550 \\
a, b, s_c, s_h, s_m & \geq & 0
\end{array}$$

$$\begin{array}{l}
\text{basis} = \{b, s_h, s_m\} \\
a = s_c = 0 \\
Z = 736 \\
b = 32 \\
s_h = 32 \\
s_m = 550
\end{array}$$

$$\begin{array}{rcl}
\max Z & & \\
\frac{16}{3}a - \frac{23}{15}s_c & - Z = & -736 \\
\frac{1}{3}a + b + \frac{1}{15}s_c & = & 32 \\
\frac{8}{3}a - \frac{4}{15}s_c + s_h & = & 32 \\
\frac{85}{3}a - \frac{4}{3}s_c + s_m & = & 550 \\
a, b, s_c, s_h, s_m & \geq & 0
\end{array}$$

$$\begin{array}{l}
\text{basis} = \{b, s_h, s_m\} \\
a = s_c = 0 \\
Z = 736 \\
b = 32 \\
s_h = 32 \\
s_m = 550
\end{array}$$

Choose variable a to bring into basis.

Computing $\min\{3 \cdot 32, 3 \cdot 32/8, 3 \cdot 550/85\}$ means pivot on line 2.

Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$\begin{array}{rcl}
\max Z & & \\
- s_c - 2s_h & - Z = & -800 \\
b + \frac{1}{10}s_c - \frac{1}{8}s_h & = & 28 \\
a - \frac{1}{10}s_c + \frac{3}{8}s_h & = & 12 \\
\frac{3}{2}s_c - \frac{85}{8}s_h + s_m & = & 210 \\
a, b, s_c, s_h, s_m & \geq & 0
\end{array}$$

$$\begin{array}{l}
\text{basis} = \{a, b, s_m\} \\
s_c = s_h = 0 \\
Z = 800 \\
b = 28 \\
a = 12 \\
s_m = 210
\end{array}$$

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- ▶ any feasible solution satisfies all equations in the tableaux
- ▶ in particular: $Z = 800 - s_c - 2s_h, s_c \geq 0, s_h \geq 0$
- ▶ hence optimum solution value is at most 800
- ▶ the current solution has value 800

Matrix View

Let our linear program be

$$\begin{array}{l}
c_B^T x_B + c_N^T x_N = Z \\
A_B x_B + A_N x_N = b \\
x_B, x_N \geq 0
\end{array}$$

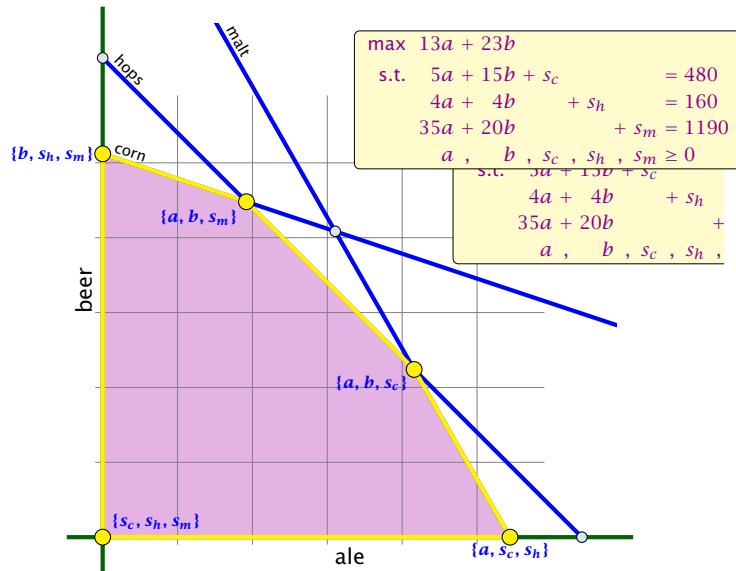
The simplex tableaux for basis B is

$$\begin{array}{l}
(c_N^T - c_B^T A_B^{-1} A_N) x_N = Z - c_B^T A_B^{-1} b \\
I x_B + A_B^{-1} A_N x_N = A_B^{-1} b \\
x_B, x_N \geq 0
\end{array}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

Geometric View of Pivoting



Algebraic Definition of Pivoting

- ▶ Given basis B with BFS x^* .
- ▶ Choose index $j \notin B$ in order to increase x_j^* from 0 to $\theta > 0$.
 - ▶ Other non-basis variables should stay at 0.
 - ▶ Basis variables change to maintain feasibility.
- ▶ Go from x^* to $x^* + \theta \cdot d$.

Requirements for d :

- ▶ $d_j = 1$ (normalization)
- ▶ $d_\ell = 0, \ell \notin B, \ell \neq j$
- ▶ $A(x^* + \theta d) = b$ must hold. Hence $Ad = 0$.
- ▶ Altogether: $A_B d_B + A_{*j} = Ad = 0$, which gives $d_B = -A_B^{-1} A_{*j}$.

Algebraic Definition of Pivoting

Definition 26 (j -th basis direction)

Let B be a basis, and let $j \notin B$. The vector d with $d_j = 1$ and $d_\ell = 0, \ell \notin B, \ell \neq j$ and $d_B = -A_B^{-1} A_{*j}$ is called the j -th basis direction for B .

Going from x^* to $x^* + \theta \cdot d$ the objective function changes by

$$\theta \cdot c^T d = \theta(c_j - c_B^T A_B^{-1} A_{*j})$$

Algebraic Definition of Pivoting

Definition 27 (Reduced Cost)

For a basis B the value

$$\tilde{c}_j = c_j - c_B^T A_B^{-1} A_{*j}$$

is called the **reduced cost** for variable x_j .

Note that this is defined for every j . If $j \in B$ then the above term is 0.

Algebraic Definition of Pivoting

Let our linear program be

$$\begin{aligned}c_B^T x_B + c_N^T x_N &= Z \\A_B x_B + A_N x_N &= b \\x_B, x_N &\geq 0\end{aligned}$$

The simplex tableaux for basis B is

$$\begin{aligned}(c_N^T - c_B^T A_B^{-1} A_N) x_N &= Z - c_B^T A_B^{-1} b \\I x_B + A_B^{-1} A_N x_N &= A_B^{-1} b \\x_B, x_N &\geq 0\end{aligned}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

4 Simplex Algorithm

Questions:

- ▶ What happens if the min ratio test fails to give us a value θ by which we can safely increase the entering variable?
- ▶ How do we find the initial basic feasible solution?
- ▶ Is there always a basis B such that

$$(c_N^T - c_B^T A_B^{-1} A_N) \leq 0 ?$$

Then we can terminate because we know that the solution is optimal.

- ▶ If yes how do we make sure that we reach such a basis?

Min Ratio Test

The min ratio test computes a value $\theta \geq 0$ such that after setting the entering variable to θ the leaving variable becomes 0 and all other variables stay non-negative.

For this, one computes b_i/A_{ie} for all constraints i and calculates the minimum positive value.

What does it mean that the ratio b_i/A_{ie} (and hence A_{ie}) is negative for a constraint?

This means that the corresponding basic variable will increase if we increase b . Hence, there is no danger of this basic variable becoming negative

What happens if **all** b_i/A_{ie} are negative? Then we do not have a leaving variable. **Then the LP is unbounded!**

Termination

The objective function does not decrease during one iteration of the simplex-algorithm.

Does it always increase?

Termination

The objective function may not increase!

Because a variable x_ℓ with $\ell \in B$ is already 0.

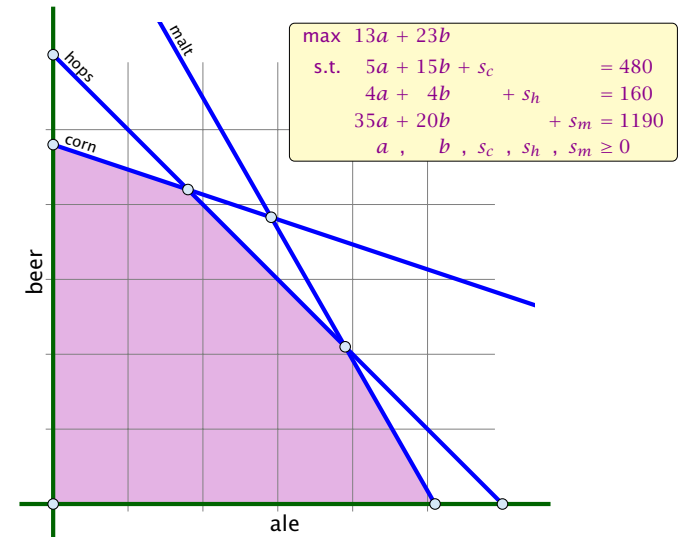
The set of inequalities is **degenerate** (also the basis is degenerate).

Definition 28 (Degeneracy)

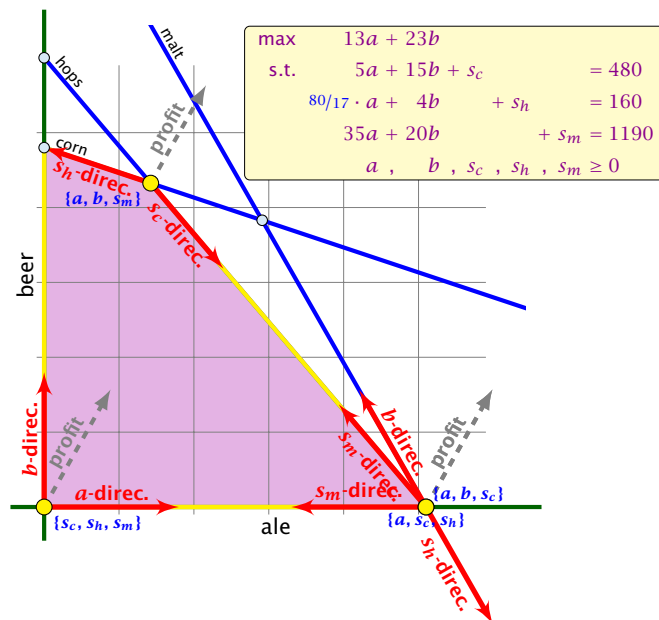
A BFS x^* is called **degenerate** if the set $J = \{j \mid x_j^* > 0\}$ fulfills $|J| < m$.

It is possible that the algorithm **cycles**, i.e., it cycles through a sequence of different bases without ever terminating. Happens, very rarely in practise.

Non Degenerate Example



Degenerate Example



Summary: How to choose pivot-elements

- ▶ We can choose a column e as an entering variable if $\tilde{c}_e > 0$ (\tilde{c}_e is reduced cost for x_e).
- ▶ The standard choice is the column that maximizes \tilde{c}_e .
- ▶ If $A_{ie} \leq 0$ for all $i \in \{1, \dots, m\}$ then the maximum is not bounded.
- ▶ Otw. choose a leaving variable ℓ such that $b_\ell / A_{\ell e}$ is minimal among all variables i with $A_{ie} > 0$.
- ▶ If several variables have minimum $b_\ell / A_{\ell e}$ you reach a **degenerate** basis.
- ▶ Depending on the choice of ℓ it may happen that the algorithm runs into a cycle where it does not escape from a degenerate vertex.

Termination

What do we have so far?

Suppose we are given an initial feasible solution to an LP. If the LP is non-degenerate then Simplex will terminate.

Note that we either terminate because the min-ratio test fails and we can conclude that the LP is **unbounded**, or we terminate because the vector of reduced cost is non-positive. In the latter case we have an **optimum solution**.

How do we come up with an initial solution?

- ▶ $Ax \leq b, x \geq 0$, and $b \geq 0$.
- ▶ The standard slack form for this problem is $Ax + Is = b, x \geq 0, s \geq 0$, where s denotes the vector of slack variables.
- ▶ Then $s = b, x = 0$ is a basic feasible solution (how?).
- ▶ We directly can start the simplex algorithm.

How do we find an initial basic feasible solution for an arbitrary problem?

Two phase algorithm

Suppose we want to maximize $c^T x$ s.t. $Ax = b, x \geq 0$.

1. Multiply all rows with $b_i < 0$ by -1 .
2. maximize $-\sum_i v_i$ s.t. $Ax + Iv = b, x \geq 0, v \geq 0$ using Simplex. $x = 0, v = b$ is initial feasible.
3. If $\sum_i v_i > 0$ then the original problem is **infeasible**.
4. Otw. you have $x \geq 0$ with $Ax = b$.
5. From this you can get basic feasible solution.
6. Now you can start the Simplex for the original problem.

Optimality

Lemma 29

Let B be a basis and x^* a BFS corresponding to basis B . $\tilde{c} \leq 0$ implies that x^* is an optimum solution to the LP.