

8 Seidels LP-algorithm

- ▶ Suppose we want to solve $\min\{c^T x \mid Ax \geq b; x \geq 0\}$, where $x \in \mathbb{R}^d$ and we have m constraints.
- ▶ In the worst-case Simplex runs in time roughly $\mathcal{O}(m(m+d)\binom{m+d}{m}) \approx (m+d)^m$. (slightly better bounds on the running time exist, but will not be discussed here).
- ▶ If d is much smaller than m one can do a lot better.
- ▶ In the following we develop an algorithm with running time $\mathcal{O}(d! \cdot m)$, i.e., **linear in m** .

8 Seidels LP-algorithm

Setting:

- ▶ We assume an LP of the form

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0 \end{array}$$

- ▶ We assume that the LP is **bounded**.

Ensuring Conditions

Given a **standard minimization LP**

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0 \end{array}$$

how can we obtain an LP of the required form?

- ▶ **Compute a lower bound on $c^T x$ for any basic feasible solution.**

Computing a Lower Bound

Let s denote the smallest common multiple of all denominators of entries in A, b .

Multiply entries in A, b by s to obtain integral entries. **This does not change the feasible region.**

Add slack variables to A ; denote the resulting matrix with \tilde{A} .

If B is an optimal basis then x_B with $\tilde{A}_B x_B = \tilde{b}$, gives an optimal assignment to the basis variables (non-basic variables are 0).

Theorem 46 (Cramers Rule)

Let M be a matrix with $\det(M) \neq 0$. Then the solution to the system $Mx = b$ is given by

$$x_i = \frac{\det(M_j)}{\det(M)},$$

where M_i is the matrix obtained from M by replacing the i -th column by the vector b .

Proof:

- Define

$$X_i = \begin{pmatrix} | & & | & | & | \\ e_1 & \cdots & e_{i-1} & x & e_{i+1} & \cdots & e_n \\ | & & | & | & | \end{pmatrix}$$

Note that expanding along the i -th column gives that $\det(X_i) = x_i$.

- Further, we have

$$MX_i = \begin{pmatrix} | & & | & | & | \\ Me_1 & \cdots & Me_{i-1} & Mx & Me_{i+1} & \cdots & Me_n \\ | & & | & | & | \end{pmatrix} = M_i$$

- Hence,

$$x_i = \det(X_i) = \frac{\det(M_i)}{\det(M)}$$

Bounding the Determinant

Let Z be the maximum absolute entry occurring in \bar{A} , \bar{b} or c . Let C denote the matrix obtained from \bar{A}_B by replacing the j -th column with vector \bar{b} (for some j).

Observe that

$$\begin{aligned} |\det(C)| &= \left| \sum_{\pi \in S_m} \text{sgn}(\pi) \prod_{1 \leq i \leq m} C_{i\pi(i)} \right| \\ &\leq \sum_{\pi \in S_m} \prod_{1 \leq i \leq m} |C_{i\pi(i)}| \\ &\leq m! \cdot Z^m \end{aligned}$$

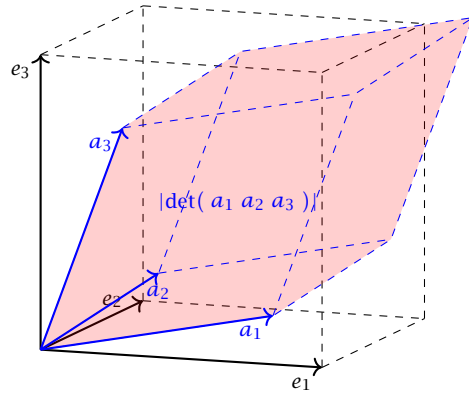
Here $\text{sgn}(\pi)$ denotes the sign of the permutation, which is 1 if the permutation can be generated by an even number of transpositions (exchanging two elements), and -1 if the number of transpositions is odd. The first identity is known as Leibniz formula.

Bounding the Determinant

Alternatively, Hadamards inequality gives

$$\begin{aligned} |\det(C)| &\leq \prod_{i=1}^m \|C_{*i}\| \leq \prod_{i=1}^m (\sqrt{m}Z) \\ &\leq m^{m/2} Z^m \end{aligned}$$

Hadamards Inequality



Hadamards inequality says that the volume of the red parallelepiped (Spat) is smaller than the volume in the black cube (if $\|e_1\| = \|a_1\|$, $\|e_2\| = \|a_2\|$, $\|e_3\| = \|a_3\|$).

Ensuring Conditions

Given a **standard minimization LP**

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0 \end{array}$$

how can we obtain an LP of the required form?

- **Compute a lower bound on $c^T x$ for any basic feasible solution.** Add the constraint $c^T x \geq -dZ(m! \cdot Z^m) - 1$. **Note that this constraint is superfluous unless the LP is unbounded.**

Ensuring Conditions

Compute an optimum basis for the new LP.

- If the cost is $c^T x = -(dZ)(m! \cdot Z^m) - 1$ we know that the original LP is unbounded.
- Otw. we have an optimum basis.

In the following we use \mathcal{H} to denote the set of all constraints apart from the constraint $c^T x \geq -dZ(m! \cdot Z^m) - 1$.

We give a routine $\text{SeidelLP}(\mathcal{H}, d)$ that is given a set \mathcal{H} of **explicit, non-degenerate** constraints over d variables, and minimizes $c^T x$ over all feasible points.

In addition it obeys the implicit constraint $c^T x \geq -(dZ)(m! \cdot Z^m) - 1$.

Algorithm 1 SeidelLP(\mathcal{H}, d)

```
1: if  $d = 1$  then solve 1-dimensional problem and return;
2: if  $\mathcal{H} = \emptyset$  then return  $x$  on implicit constraint hyperplane
3: choose random constraint  $h \in \mathcal{H}$ 
4:  $\hat{\mathcal{H}} \leftarrow \mathcal{H} \setminus \{h\}$ 
5:  $\hat{x}^* \leftarrow \text{SeidelLP}(\hat{\mathcal{H}}, d)$ 
6: if  $\hat{x}^* = \text{infeasible}$  then return infeasible
7: if  $\hat{x}^*$  fulfills  $h$  then return  $\hat{x}^*$ 
8: // optimal solution fulfills  $h$  with equality, i.e.,  $a_h^T x = b_h$ 
9: solve  $a_h^T x = b_h$  for some variable  $x_\ell$ ;
10: eliminate  $x_\ell$  in constraints from  $\hat{\mathcal{H}}$  and in implicit constr.;
11:  $\hat{x}^* \leftarrow \text{SeidelLP}(\hat{\mathcal{H}}, d - 1)$ 
12: if  $\hat{x}^* = \text{infeasible}$  then
13:     return infeasible
14: else
15:     add the value of  $x_\ell$  to  $\hat{x}^*$  and return the solution
```

8 Seidels LP-algorithm

Note that for the case $d = 1$, the asymptotic bound $\mathcal{O}(\max\{m, 1\})$ is valid also for the case $m = 0$.

- ▶ If $d = 1$ we can solve the 1-dimensional problem in time $\mathcal{O}(\max\{m, 1\})$.
- ▶ If $d > 1$ and $m = 0$ we take time $\mathcal{O}(d)$ to return d -dimensional vector x .
- ▶ The first recursive call takes time $T(m - 1, d)$ for the call plus $\mathcal{O}(d)$ for checking whether the solution fulfills h .
- ▶ If we are unlucky and \hat{x}^* does not fulfill h we need time $\mathcal{O}(d(m + 1)) = \mathcal{O}(dm)$ to eliminate x_ℓ . Then we make a recursive call that takes time $T(m - 1, d - 1)$.
- ▶ The probability of being unlucky is at most d/m as there are at most d constraints whose removal will decrease the objective function

8 Seidels LP-algorithm

This gives the recurrence

$$T(m, d) = \begin{cases} \mathcal{O}(\max\{1, m\}) & \text{if } d = 1 \\ \mathcal{O}(d) & \text{if } d > 1 \text{ and } m = 0 \\ \mathcal{O}(d) + T(m - 1, d) + \\ \frac{d}{m}(\mathcal{O}(dm) + T(m - 1, d - 1)) & \text{otw.} \end{cases}$$

Note that $T(m, d)$ denotes the **expected running time**.

8 Seidels LP-algorithm

Let C be the largest constant in the \mathcal{O} -notations.

$$T(m, d) = \begin{cases} C \max\{1, m\} & \text{if } d = 1 \\ Cd & \text{if } d > 1 \text{ and } m = 0 \\ Cd + T(m - 1, d) + \\ \frac{d}{m}(Cdm + T(m - 1, d - 1)) & \text{otw.} \end{cases}$$

Note that $T(m, d)$ denotes the **expected running time**.

8 Seidels LP-algorithm

Let C be the largest constant in the \mathcal{O} -notations.

We show $T(m, d) \leq Cf(d) \max\{1, m\}$.

$d = 1$:

$$T(m, 1) \leq C \max\{1, m\} \leq Cf(1) \max\{1, m\} \text{ for } f(1) \geq 1$$

$d > 1; m = 0$:

$$T(0, d) \leq \mathcal{O}(d) \leq Cd \leq Cf(d) \max\{1, m\} \text{ for } f(d) \geq d$$

$d > 1; m = 1$:

$$\begin{aligned} T(1, d) &= \mathcal{O}(d) + T(0, d) + d(\mathcal{O}(d) + T(0, d-1)) \\ &\leq Cd + Cd + Cd^2 + dCf(d-1) \\ &\leq Cf(d) \max\{1, m\} \text{ for } f(d) \geq 3d^2 + df(d-1) \end{aligned}$$

8 Seidels LP-algorithm

$d > 1; m > 1$:

(by induction hypothesis statm. true for $d' < d, m' \geq 0$;

and for $d' = d, m' < m$)

$$\begin{aligned} T(m, d) &= \mathcal{O}(d) + T(m-1, d) + \frac{d}{m} (\mathcal{O}(dm) + T(m-1, d-1)) \\ &\leq Cd + Cf(d)(m-1) + Cd^2 + \frac{d}{m} Cf(d-1)(m-1) \\ &\leq 2Cd^2 + Cf(d)(m-1) + dCf(d-1) \\ &\leq Cf(d)m \end{aligned}$$

if $f(d) \geq df(d-1) + 2d^2$.

8 Seidels LP-algorithm

- Define $f(1) = 3 \cdot 1^2$ and $f(d) = df(d-1) + 3d^2$ for $d > 1$.

Then

$$\begin{aligned} f(d) &= 3d^2 + df(d-1) \\ &= 3d^2 + d[3(d-1)^2 + (d-1)f(d-2)] \\ &= 3d^2 + d[3(d-1)^2 + (d-1)[3(d-2)^2 + (d-2)f(d-3)]] \\ &= 3d^2 + 3d(d-1)^2 + 3d(d-1)(d-2)^2 + \dots \\ &\quad + 3d(d-1)(d-2) \cdot \dots \cdot 4 \cdot 3 \cdot 2 \cdot 1^2 \\ &= 3d! \left(\frac{d^2}{d!} + \frac{(d-1)^2}{(d-1)!} + \frac{(d-2)^2}{(d-2)!} + \dots \right) \\ &= \mathcal{O}(d!) \end{aligned}$$

since $\sum_{i \geq 1} \frac{i^2}{i!}$ is a constant.

$$\sum_{i \geq 1} \frac{i^2}{i!} = \sum_{i \geq 0} \frac{i+1}{i!} = e + \sum_{i \geq 1} \frac{i}{i!} = 2e$$