

Number of Simplex Iterations

Each iteration of Simplex can be implemented in polynomial time.

If we use lexicographic pivoting we know that Simplex requires at most $\binom{n}{m}$ iterations, because it will not visit a basis twice.

The input size is $L \cdot n \cdot m$, where n is the number of variables, m is the number of constraints, and L is the length of the binary representation of the largest coefficient in the matrix A .

If we really require $\binom{n}{m}$ iterations then Simplex is not a polynomial time algorithm.

Can we obtain a better analysis?

Number of Simplex Iterations

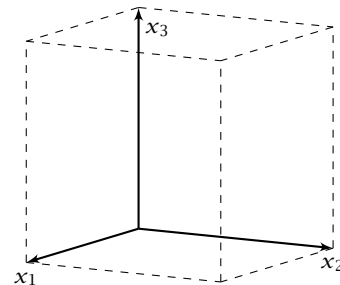
Observation

Simplex visits every **feasible** basis at most once.

However, also the number of feasible bases can be very large.

Example

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & 0 \leq x_1 \leq 1 \\ & 0 \leq x_2 \leq 1 \\ & \vdots \\ & 0 \leq x_n \leq 1 \end{aligned}$$

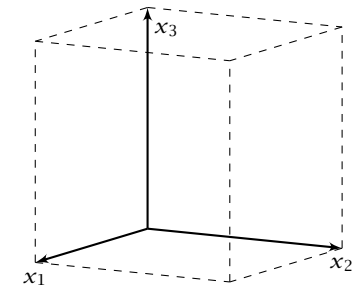


$2n$ constraint on n variables define an n -dimensional hypercube as feasible region.

The feasible region has 2^n vertices.

Example

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & 0 \leq x_1 \leq 1 \\ & 0 \leq x_2 \leq 1 \\ & \vdots \\ & 0 \leq x_n \leq 1 \end{aligned}$$



However, Simplex may still run quickly as it usually does not visit all feasible bases.

In the following we give an example of a feasible region for which there is a bad **Pivoting Rule**.

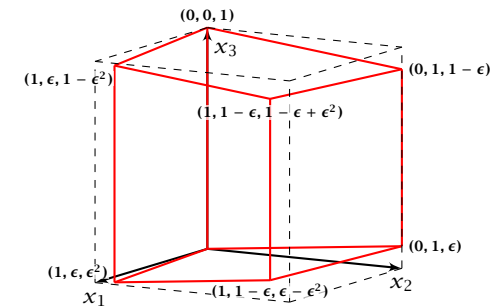
Pivoting Rule

A Pivoting Rule defines how to choose the entering and leaving variable for an iteration of Simplex.

In the non-degenerate case after choosing the entering variable the leaving variable is unique.

Klee Minty Cube

$$\begin{aligned} \max x_n \\ \text{s.t.} \quad & 0 \leq x_1 \leq 1 \\ & \epsilon x_1 \leq x_2 \leq 1 - \epsilon x_1 \\ & \epsilon x_2 \leq x_3 \leq 1 - \epsilon x_2 \\ & \vdots \\ & \epsilon x_{n-1} \leq x_n \leq 1 - \epsilon x_{n-1} \\ & x_i \geq 0 \end{aligned}$$



Observations

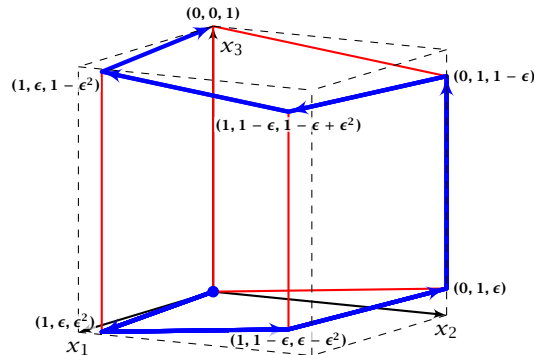
- ▶ We have $2n$ constraints, and $3n$ variables (after adding slack variables to every constraint).
- ▶ Every basis is defined by $2n$ variables, and n non-basic variables.
- ▶ There exist degenerate vertices.
- ▶ The degeneracies come from the non-negativity constraints, which are superfluous.
- ▶ In the following all variables x_i stay in the basis at all times.
- ▶ Then, we can uniquely specify a basis by choosing for each variable whether it should be equal to its lower bound, or equal to its upper bound (the slack variable corresponding to the non-tight constraint is part of the basis).
- ▶ We can also simply identify each basis/vertex with the corresponding hypercube vertex obtained by letting $\epsilon \rightarrow 0$.

Analysis

- ▶ In the following we specify a sequence of bases (identified by the corresponding hypercube node) along which the objective function strictly increases.
- ▶ The basis $(0, \dots, 0, 1)$ is the unique optimal basis.
- ▶ Our sequence S_n starts at $(0, \dots, 0)$ ends with $(0, \dots, 0, 1)$ and visits every node of the hypercube.
- ▶ An unfortunate Pivoting Rule may choose this sequence, and, hence, require an exponential number of iterations.

Klee Minty Cube

$$\begin{aligned} \max x_n \\ \text{s.t. } 0 \leq x_1 \leq 1 \\ \epsilon x_1 \leq x_2 \leq 1 - \epsilon x_1 \\ \epsilon x_2 \leq x_3 \leq 1 - \epsilon x_2 \end{aligned}$$



Analysis

The sequence S_n that visits every node of the hypercube is defined recursively

$$\begin{aligned} & (0, \dots, 0, 0, 0) \\ & \quad \downarrow S_{n-1} \\ & (0, \dots, 0, 1, 0) \\ & \quad \downarrow \\ & (0, \dots, 0, 1, 1) \\ & \quad \downarrow S_{n-1}^{\text{rev}} \\ & (0, \dots, 0, 0, 1) \end{aligned}$$

The non-recursive case is $S_1 = 0 \rightarrow 1$

Analysis

Lemma 45

The objective value x_n is increasing along path S_n .

Proof by induction:

$n = 1$: obvious, since $S_1 = 0 \rightarrow 1$, and $1 > 0$.

$n - 1 \rightarrow n$

- ▶ For the first part the value of $x_n = \epsilon x_{n-1}$.
- ▶ By induction hypothesis x_{n-1} is increasing along S_{n-1} , hence, also x_n .
- ▶ Going from $(0, \dots, 0, 1, 0)$ to $(0, \dots, 0, 1, 1)$ increases x_n for small enough ϵ .
- ▶ For the remaining path S_{n-1}^{rev} we have $x_n = 1 - \epsilon x_{n-1}$.
- ▶ By induction hypothesis x_{n-1} is increasing along S_{n-1} , hence $-\epsilon x_{n-1}$ is increasing along S_{n-1}^{rev} .

Remarks about Simplex

Observation

The simplex algorithm takes at most $\binom{n}{m}$ iterations. Each iteration can be implemented in time $\mathcal{O}(mn)$.

In practise it usually takes a linear number of iterations.

Remarks about Simplex

Theorem

For almost all known **deterministic** pivoting rules (rules for choosing entering and leaving variables) there exist lower bounds that require the algorithm to have exponential running time ($\Omega(2^{\Omega(n)})$) (e.g. Klee Minty 1972).

Remarks about Simplex

Theorem

For some standard **randomized** pivoting rules there exist subexponential lower bounds ($\Omega(2^{\Omega(n^\alpha)})$ for $\alpha > 0$) (Friedmann, Hansen, Zwick 2011).

Remarks about Simplex

Conjecture (Hirsch 1957)

The edge-vertex graph of an m -facet polytope in d -dimensional Euclidean space has diameter no more than $m - d$.

The conjecture has been proven wrong in 2010.

But the question whether the diameter is perhaps of the form $\mathcal{O}(\text{poly}(m, d))$ is open.