# Brewery Problem

**Brewery brews ale and beer.**

- Production limited by supply of corn, hops and barley malt
- Recipes for ale and beer require different amounts of resources

| | ***Corn (kg)*** | ***Hops (kg)*** | ***Malt (kg)*** | ***Profit (€)*** |
|---|---|---|---|---|
| ale (barrel) | 5 | 4 | 35 | 13 |
| beer (barrel) | 15 | 4 | 20 | 23 |
| supply | 480 | 160 | 1190 | |

# Brewery Problem

| | ***Corn* (kg)** | ***Hops* (kg)** | ***Malt* (kg)** | ***Profit* (€)** |
|---|---|---|---|---|
| ale (barrel) | 5 | 4 | 35 | 13 |
| beer (barrel) | 15 | 4 | 20 | 23 |
| supply | 480 | 160 | 1190 | |

**How can brewer maximize profits?**

- only brew ale: $34$ barrels of ale $\Longrightarrow$ 442 €
- only brew beer: $32$ barrels of beer $\Longrightarrow$ 736 €
- $7.5$ barrels ale, $29.5$ barrels beer $\Longrightarrow$ 776 €
- $12$ barrels ale, $28$ barrels beer $\Longrightarrow$ 800 €

# Brewery Problem

## Linear Program

- Introduce variables $a$ and $b$ that define how much ale and beer to produce.
- Choose the variables in such a way that the objective function (profit) is maximized.
- Make sure that no constraints (due to limited supply) are violated.

$$
\begin{array}{lrcrl}
\max & 13a & + & 23b & \\
\text{s.t.} & 5a & + & 15b & \leq 480 \\
 & 4a & + & 4b & \leq 160 \\
 & 35a & + & 20b & \leq 1190 \\
 & & & a, b & \geq 0
\end{array}
$$

# Standard Form LPs

**LP in standard form:**

- input: numbers $a_{ij}$, $c_j$, $b_i$
- output: numbers $x_j$
- $n =$ #decision variables, $m =$ #constraints
- maximize linear objective function subject to linear (in)equalities

$$\begin{array}{llcll} \text{max} & \sum_{j=1}^{n} c_j x_j & & & \\ \text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j & = & b_i & 1 \le i \le m \\ & x_j & \ge & 0 & 1 \le j \le n \end{array}$$

$$\begin{array}{lrcl} \max & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \ge & 0 \end{array}$$

# Standard Form LPs

## Original LP

$$\begin{array}{lrcrl} \max & 13a & + & 23b & \\ \text{s.t.} & 5a & + & 15b & \leq 480 \\ & 4a & + & 4b & \leq 160 \\ & 35a & + & 20b & \leq 1190 \\ & & & a,b & \geq 0 \end{array}$$

## Standard Form

Add a slack variable to every constraint.

$$\begin{array}{lrcrcrcrcrl} \max & 13a & + & 23b & & & & & & & \\ \text{s.t.} & 5a & + & 15b & + & s_c & & & & & = 480 \\ & 4a & + & 4b & & & + & s_h & & & = 160 \\ & 35a & + & 20b & & & & & + & s_m & = 1190 \\ & a & , & b & , & s_c & , & s_h & , & s_m & \geq 0 \end{array}$$

# Standard Form LPs

There are different standard forms:

standard form

$$\begin{array}{llll} \max & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

$$\begin{array}{llll} \min & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

standard maximization form

$$\begin{array}{llll} \max & c^T x & & \\ \text{s.t.} & Ax & \leq & b \\ & x & \geq & 0 \end{array}$$

standard minimization form

$$\begin{array}{llll} \min & c^T x & & \\ \text{s.t.} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

# Standard Form LPs

It is easy to transform variants of LPs into (any) standard form:

- **less or equal to equality:**

$$a - 3b + 5c \le 12 \implies \begin{aligned} a - 3b + 5c + s &= 12 \\ s &\ge 0 \end{aligned}$$

- **greater or equal to equality:**

$$a - 3b + 5c \ge 12 \implies \begin{aligned} a - 3b + 5c - s &= 12 \\ s &\ge 0 \end{aligned}$$

- **min to max:**

$$\min a - 3b + 5c \implies \max -a + 3b - 5c$$

# Standard Form LPs

It is easy to transform variants of LPs into (any) standard form:

- **equality to less or equal:**

$$a - 3b + 5c = 12 \implies \begin{array}{r} a - 3b + 5c \le 12 \\ -a + 3b - 5c \le -12 \end{array}$$

- **equality to greater or equal:**

$$a - 3b + 5c = 12 \implies \begin{array}{r} a - 3b + 5c \ge 12 \\ -a + 3b - 5c \ge -12 \end{array}$$

- **unrestricted to nonnegative:**

$$x \text{ unrestricted} \implies x = x^+ - x^-,\, x^+ \ge 0,\, x^- \ge 0$$

# Standard Form LPs

**Observations:**

- a linear program does not contain $x^2$, $\cos(x)$, etc.
- transformations between standard forms can be done efficiently and only change the size of the LP by a small constant factor
- for the standard minimization or maximization LPs we could include the nonnegativity constraints into the set of ordinary constraints; this is of course not possible for the standard form

# Fundamental Questions

**Definition 1 (Linear Programming Problem (LP))**

Let $A \in \mathbb{Q}^{m\times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

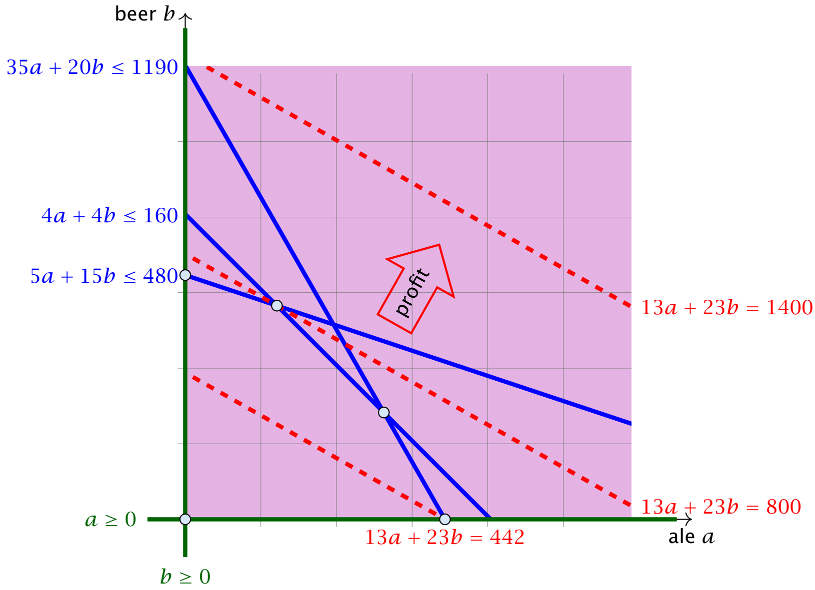
**Questions**:

- Is LP in NP?
- Is LP in co-NP?
- Is LP in P?

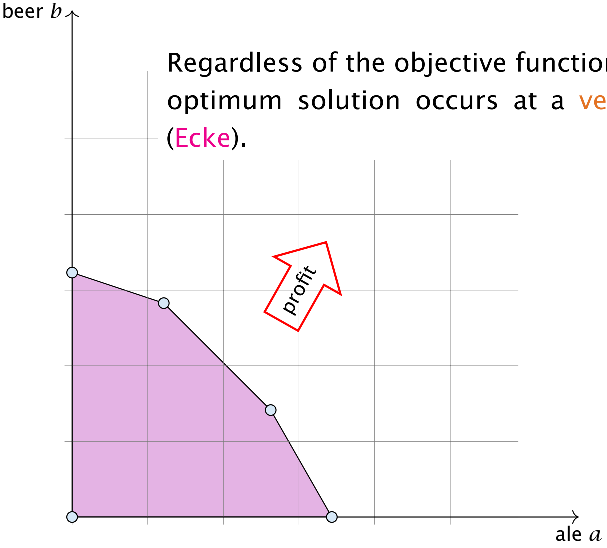
**Input size**:

- $n$ number of variables, $m$ constraints, $L$ number of bits to encode the input

# Geometry of Linear Programming



# Geometry of Linear Programming




Regardless of the objective function an optimum solution occurs at a vertex (Ecke).

# Definitions

Let for a Linear Program in standard form
$P = \{x \mid Ax = b, x \geq 0\}$.

- $P$ is called the feasible region (Lösungsraum) of the LP.
- A point $x \in P$ is called a feasible point (gültige Lösung).
- If $P \neq \emptyset$ then the LP is called feasible (erfüllbar). Otherwise, it is called infeasible (unerfüllbar).
- An LP is bounded (beschränkt) if it is feasible and
  - $c^T x < \infty$ for all $x \in P$ (for maximization problems)
  - $c^T x > -\infty$ for all $x \in P$ (for minimization problems)

**Definition 2**

Given vectors/points $x_1, \dots, x_k \in \mathbb{R}^n$, $\sum \lambda_i x_i$ is called

- linear combination if $\lambda_i \in \mathbb{R}$.
- affine combination if $\lambda_i \in \mathbb{R}$ and $\sum_i \lambda_i = 1$.
- convex combination if $\lambda_i \in \mathbb{R}$ and $\sum_i \lambda_i = 1$ and $\lambda_i \geq 0$.
- conic combination if $\lambda_i \in \mathbb{R}$ and $\lambda_i \geq 0$.

Note that a combination involves only finitely many vectors.

## Definition 3

A set $X \subseteq \mathbb{R}^n$ is called

- a linear subspace if it is closed under linear combinations.
- an affine subspace if it is closed under affine combinations.
- convex if it is closed under convex combinations.
- a convex cone if it is closed under conic combinations.

Note that an affine subspace is **not** a vector space

**Definition 4**
Given a set $X \subseteq \mathbb{R}^n$.

- $\text{span}(X)$ is the set of all linear combinations of $X$ (linear hull, span)
- $\text{aff}(X)$ is the set of all affine combinations of $X$ (affine hull)
- $\text{conv}(X)$ is the set of all convex combinations of $X$ (convex hull)
- $\text{cone}(X)$ is the set of all conic combinations of $X$ (conic hull)

**Definition 5**
A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if for $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$ we have

$$f(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda)f(y)$$

**Lemma 6**
*If $P \subseteq \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}$ convex then also*

$$Q = \{x \in P \mid f(x) \le t\}$$

# Dimensions

**Definition 7**
The dimension $\dim(A)$ of an affine subspace $A \subseteq \mathbb{R}^n$ is the dimension of the vector space $\{x - a \mid x \in A\}$, where $a \in A$.

**Definition 8**
The dimension $\dim(X)$ of a convex set $X \subseteq \mathbb{R}^n$ is the dimension of its affine hull $\mathrm{aff}(X)$.

**Definition 9**

A set $H \subseteq \mathbb{R}^n$ is a hyperplane if $H = \{x \mid a^T x = b\}$, for $a \neq 0$.

**Definition 10**

A set $H' \subseteq \mathbb{R}^n$ is a (closed) halfspace if $H = \{x \mid a^T x \leq b\}$, for $a \neq 0$.

# Definitions

**Definition 11**
A polytop is a set $P \subseteq \mathbb{R}^n$ that is the convex hull of a finite set of points, i.e., $P = \text{conv}(X)$ where $|X| = c$.

# Definitions

**Definition 12**
A polyhedron is a set $P \subseteq \mathbb{R}^n$ that can be represented as the intersection of finitely many half-spaces $\{H(a_1, b_1), \ldots, H(a_m, b_m)\}$, where

$$H(a_i, b_i) = \{x \in \mathbb{R}^n \mid a_i x \leq b_i\} \ .$$

**Definition 13**
A polyhedron $P$ is bounded if there exists $B$ s.t. $\|x\|_2 \leq B$ for all $x \in P$.

# Definitions

**Theorem 14**
*$P$ is a bounded polyhedron iff $P$ is a polytop.*

## Definition 15

Let $P \subseteq \mathbb{R}^n$, $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. The hyperplane

$$H(a,b) = \{x \in \mathbb{R}^n \mid a^T x = b\}$$

is a supporting hyperplane of $P$ if $\max\{a^T x \mid x \in P\} = b$.

## Definition 16

Let $P \subseteq \mathbb{R}^n$. $F$ is a face of $P$ if $F = P$ or $F = P \cap H$ for some supporting hyperplane $H$.

## Definition 17

Let $P \subseteq \mathbb{R}^n$.

- a face $v$ is a vertex of $P$ if $\{v\}$ is a face of $P$.
- a face $e$ is an edge of $P$ if $e$ is a face and $\dim(e) = 1$.
- a face $F$ is a facet of $P$ if $F$ is a face and $\dim(F) = \dim(P) - 1$.

**Equivalent definition for vertex:**

**Definition 18**

Given polyhedron $P$. A point $x \in P$ is a vertex if $\exists c \in \mathbb{R}^n$ such that $c^T y < c^T x$, for all $y \in P$, $y \neq x$.

**Definition 19**

Given polyhedron $P$. A point $x \in P$ is an extreme point if $\nexists a, b \neq x$, $a, b \in P$, with $\lambda a + (1 - \lambda) b = x$ for $\lambda \in [0, 1]$.

**Lemma 20**

*A vertex is also an extreme point.*

**Observation**

The feasible region of an LP is a Polyhedron.

# Convex Sets

### Theorem 21

*If there exists an optimal solution to an LP (in standard form) then there exists an optimum solution that is an extreme point.*

**Proof**

- suppose $x$ is optimal solution that is not extreme point
- there exists direction $d \neq 0$ such that $x \pm d \in P$
- $Ad = 0$ because $A(x \pm d) = b$
- Wlog. assume $c^T d \geq 0$ (by taking either $d$ or $-d$)
- Consider $x + \lambda d$, $\lambda > 0$

# Convex Sets

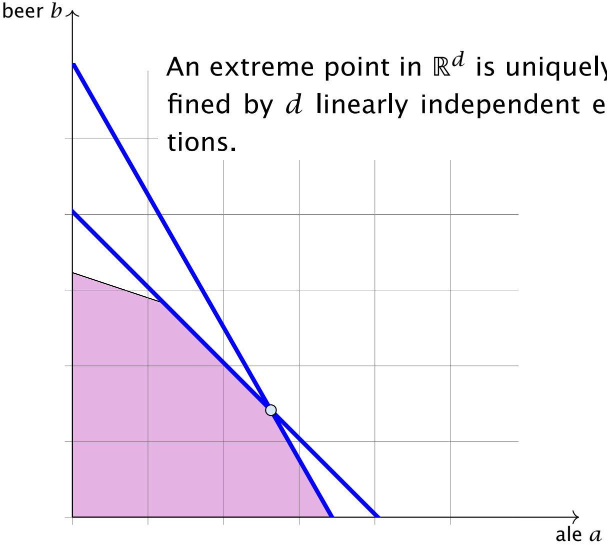
**Case 1.** [$\exists j$ s.t. $d_j < 0$]

- increase $\lambda$ to $\lambda'$ until first component of $x + \lambda d$ hits 0
- $x + \lambda' d$ is feasible. Since $A(x + \lambda' d) = b$ and $x + \lambda' d \geq 0$
- $x + \lambda' d$ has one more zero-component ($d_k = 0$ for $x_k = 0$ as $x \pm d \in P$)
- $c^T x' = c^T(x + \lambda' d) = c^T x + \lambda' c^T d \geq c^T x$

**Case 2.** [$d_j \geq 0$ for all $j$ and $c^T d > 0$]

- $x + \lambda d$ is feasible for all $\lambda \geq 0$ since $A(x + \lambda d) = b$ and $x + \lambda d \geq x \geq 0$
- as $\lambda \to \infty$, $c^T(x + \lambda d) \to \infty$ as $c^T d > 0$

# Algebraic View




An extreme point in $\mathbb{R}^d$ is uniquely defined by $d$ linearly independent equations.

**Notation**

Suppose $B \subseteq \{1 \dots n\}$ is a set of column-indices. Define $A_B$ as the subset of columns of $A$ indexed by $B$.

**Theorem 22**

*Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. Then $x$ is extreme point **iff** $A_B$ has linearly independent columns.*

## Theorem 22

*Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. Then $x$ is extreme point **iff** $A_B$ has linearly independent columns.*

**Proof ($\Leftarrow$)**

- assume $x$ is not extreme point
- there exists direction $d$ s.t. $x \pm d \in P$
- $Ad = 0$ because $A(x \pm d) = b$
- define $B' = \{j \mid d_j \neq 0\}$
- $A_{B'}$ has linearly dependent columns as $Ad = 0$
- $d_j = 0$ for all $j$ with $x_j = 0$ as $x \pm d \geq 0$
- Hence, $B' \subseteq B$, $A_{B'}$ is sub-matrix of $A_B$

## Theorem 22

*Let* $P = \{x \mid Ax = b, x \geq 0\}$*. For* $x \in P$*, define* $B = \{j \mid x_j > 0\}$*. Then* $x$ *is extreme point* ***iff*** $A_B$ *has linearly independent columns.*

**Proof (⇒)**

- assume $A_B$ has linearly dependent columns
- there exists $d \neq 0$ such that $A_B d = 0$
- extend $d$ to $\mathbb{R}^n$ by adding $0$-components
- now, $Ad = 0$ and $d_j = 0$ whenever $x_j = 0$
- for sufficiently small $\lambda$ we have $x \pm \lambda d \in P$
- hence, $x$ is not extreme point

## Theorem 23

*Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. If $A_B$ has linearly independent columns then $x$ is a vertex of $P$.*

- define $c_j = \begin{cases} 0 & j \in B \\ -1 & j \notin B \end{cases}$
- then $c^T x = 0$ and $c^T y \leq 0$ for $y \in P$
- assume $c^T y = 0$; then $y_j = 0$ for all $j \notin B$
- $b = Ay = A_B y_B = Ax = A_B x_B$ gives that $A_B(x_B - y_B) = 0$;
- this means that $x_B = y_B$ since $A_B$ has linearly independent columns
- we get $y = x$
- hence, $x$ is a vertex of $P$

**Observation**

For an LP we can assume wlog. that the matrix $A$ has full row-rank. This means $\text{rank}(A) = m$.

- assume that $\text{rank}(A) < m$
- assume wlog. that the first row $A_1$ lies in the span of the other rows $A_2, \ldots, A_m$; this means

$$A_1 = \sum_{i=2}^{m} \lambda_i \cdot A_i, \text{ for suitable } \lambda_i$$

**C1** if now $b_1 = \sum_{i=2}^{m} \lambda_i \cdot b_i$ then for all $x$ with $A_i x = b_i$ we also have $A_1 x = b_1$; hence the first constraint is superfluous

**C2** if $b_1 \neq \sum_{i=2}^{m} \lambda_i \cdot b_i$ then the LP is infeasible, since for all $x$ that fulfill constraints $A_2, \ldots, A_m$ we have

$$A_1 x = \sum_{i=2}^{m} \lambda_i \cdot A_i x = \sum_{i=2}^{m} \lambda_i \cdot b_i \neq b_1$$

From now on we will always assume that the constraint matrix of a standard form LP has full row rank.

## Theorem 24

*Given* $P = \{x \mid Ax = b, x \geq 0\}$*.* $x$ *is extreme point iff there exists* $B \subseteq \{1, \ldots, n\}$ *with* $|B| = m$ *and*

- $A_B$ is non-singular
- $x_B = A_B^{-1} b \geq 0$
- $x_N = 0$

where $N = \{1, \ldots, n\} \setminus B$.

**Proof**

Take $B = \{j \mid x_j > 0\}$ and augment with linearly independent columns until $|B| = m$; always possible since $\operatorname{rank}(A) = m$.

# Basic Feasible Solutions

$x \in \mathbb{R}^n$ is called basic solution (Basislösung) if $Ax = b$ and $\text{rank}(A_J) = |J|$ where $J = \{j \mid x_j \neq 0\}$;

$x$ is a basic **feasible** solution (gültige Basislösung) if in addition $x \geq 0$.

A basis (Basis) is an index set $B \subseteq \{1, \dots, n\}$ with $\text{rank}(A_B) = m$ and $|B| = m$.

$x \in \mathbb{R}^n$ with $A_B x_B = b$ and $x_j = 0$ for all $j \notin B$ is the basic solution associated to basis B (die zu $B$ assoziierte Basislösung)

# Basic Feasible Solutions

A BFS fulfills the $m$ equality constraints.

In addition, at least $n - m$ of the $x_i$'s are zero. The corresponding non-negativity constraint is fulfilled with equality.

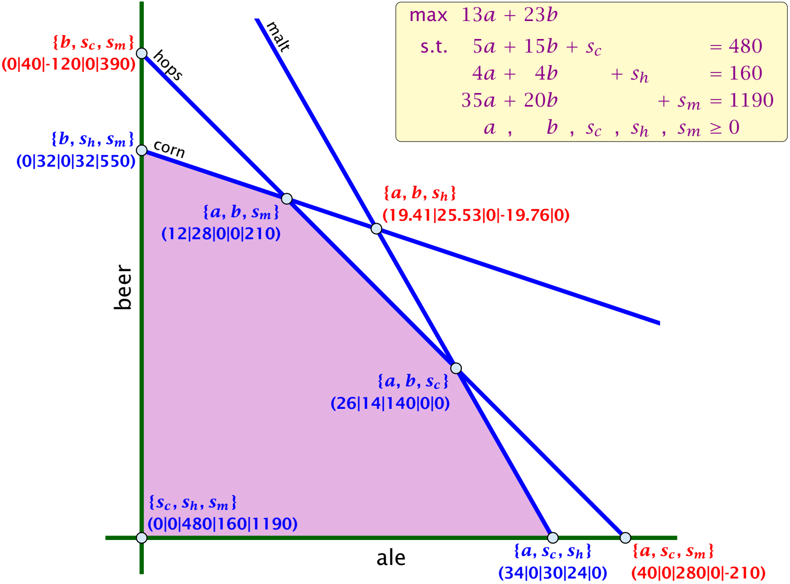
**Fact:**
In a BFS at least $n$ constraints are fulfilled with equality.

# Basic Feasible Solutions

**Definition 25**
For a general LP ($\max\{c^T x \mid Ax \leq b\}$) with $n$ variables a point $x$ is a basic feasible solution if $x$ is feasible and there exist $n$ (linearly independent) constraints that are tight.

# Algebraic View

$$
\begin{array}{llllllll}
\max & 13a + 23b & & & & & \\
\text{s.t.} & 5a + 15b & + s_c & & & = & 480 \\
& 4a + 4b & & + s_h & & = & 160 \\
& 35a + 20b & & & + s_m & = & 1190 \\
& a, \; b, & s_c, & s_h, & s_m & \geq & 0
\end{array}
$$



# Fundamental Questions

**Linear Programming Problem (LP)**
Let $A \in \mathbb{Q}^{m\times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

**Questions**:

- Is LP in NP? yes!
- Is LP in co-NP?
- Is LP in P?

**Proof**:

- Given a basis $B$ we can compute the associated basis solution by calculating $A_B^{-1}b$ in polynomial time; then we can also compute the profit.

**Observation**

We can compute an optimal solution to a linear program in time $\mathcal{O}\left(\binom{n}{m} \cdot \text{poly}(n, m)\right)$.

- there are only $\binom{n}{m}$ different bases.
- compute the profit of each of them and take the maximum

What happens if LP is unbounded?