

Duality

How do we get an upper bound to a maximization LP?

$$\begin{aligned} \max \quad & 13a + 23b \\ \text{s.t.} \quad & 5a + 15b \leq 480 \\ & 4a + 4b \leq 160 \\ & 35a + 20b \leq 1190 \\ & a, b \geq 0 \end{aligned}$$

Note that a lower bound is easy to derive. Every choice of $a, b \geq 0$ gives us a lower bound (e.g. $a = 12, b = 28$ gives us a lower bound of 800).

If you take a conic combination of the rows (multiply the i -th row with $y_i \geq 0$) such that $\sum_i y_i a_{ij} \geq c_j$ then $\sum_i y_i b_i$ will be an upper bound.

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Definition 30

Let $z = \max\{c^T x \mid Ax \leq b, x \geq 0\}$ be a linear program P (called the primal linear program).

The linear program D defined by

$$w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$$

is called the **dual problem**.

Lemma 31

The dual of the dual problem is the primal problem.

Proof:

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Weak Duality

Let $z = \max\{c^T x \mid Ax \leq b, x \geq 0\}$ and
 $w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$ be a primal dual pair.

x is primal feasible iff $x \in \{x \mid Ax \leq b, x \geq 0\}$

y is dual feasible, iff $y \in \{y \mid A^T y \geq c, y \geq 0\}$.

Theorem 32 (Weak Duality)

Let \hat{x} be primal feasible and let \hat{y} be dual feasible. Then

$$c^T \hat{x} \leq z \leq w \leq b^T \hat{y} .$$

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Weak Duality

$$A^T \hat{y} \geq c \Rightarrow \hat{x}^T A^T \hat{y} \geq \hat{x}^T c \quad (\hat{x} \geq 0)$$

$$A \hat{x} \leq b \Rightarrow y^T A \hat{x} \leq y^T b \quad (y \geq 0)$$

This gives

$$c^T \hat{x} \leq y^T A \hat{x} \leq b^T y .$$

Since, there exists primal feasible \hat{x} with $c^T \hat{x} = z$, and dual feasible \hat{y} with $b^T \hat{y} = w$ we get $z \leq w$.

If P is unbounded then D is infeasible.

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5.2 Simplex and Duality

The following linear programs form a primal dual pair:

$$z = \max\{c^T x \mid Ax = b, x \geq 0\}$$
$$w = \min\{b^T y \mid A^T y \geq c\}$$

This means for computing the dual of a standard form LP, we do not have non-negativity constraints for the dual variables.

Proof

Primal:

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Dual:

$$\min\{[b^T \ -b^T]y \mid [A^T \ -A^T]y \geq c, y \geq 0\}$$

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$$\begin{aligned} & \min\{[b^T \ -b^T]y \mid [A^T \ -A^T]y \geq c, y \geq 0\} \\ &= \min\left\{[b^T \ -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \mid [A^T \ -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \geq c, y^- \geq 0, y^+ \geq 0\right\} \end{aligned}$$

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Proof of Optimality Criterion for Simplex

Suppose that we have a basic feasible solution with **reduced cost**

$$\tilde{c} = c^T - c_B^T A_B^{-1} A \leq 0$$

This is equivalent to $A^T (A_B^{-1})^T c_B \geq c$

$y^* = (A_B^{-1})^T c_B$ is solution to the **dual** $\min\{b^T y \mid A^T y \geq c\}$.

Hence, the solution is optimal.

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5.3 Strong Duality

$$P = \max\{c^T x \mid Ax \leq b, x \geq 0\}$$

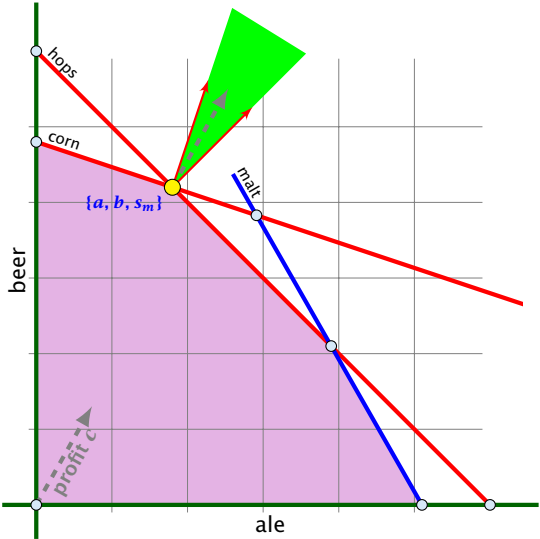
n_A : number of variables, m_A : number of constraints

We can put the non-negativity constraints into A (which gives us unrestricted variables): $\bar{P} = \max\{c^T x \mid \bar{A}x \leq \bar{b}\}$

$$n_{\bar{A}} = n_A, m_{\bar{A}} = m_A + n_A$$

Dual $D = \min\{\bar{b}^T y \mid \bar{A}^T y = c, y \geq 0\}$.

5.3 Strong Duality



The profit vector c lies in the cone generated by the normals for the hops and the corn constraint (the tight constraints).

Strong Duality

Theorem 33 (Strong Duality)

Let P and D be a primal dual pair of linear programs, and let z^* and w^* denote the optimal solution to P and D , respectively.

Then

$$z^* = w^*$$

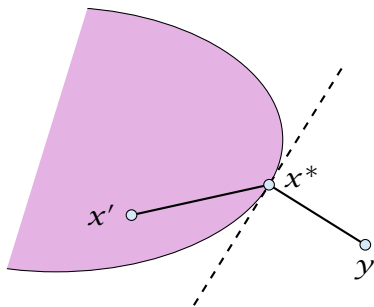
Lemma 34 (Weierstrass)

Let X be a compact set and let $f(x)$ be a continuous function on X . Then $\min\{f(x) : x \in X\}$ exists.

(without proof)

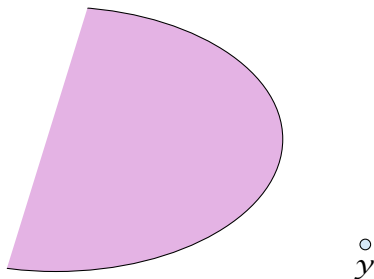
Lemma 35 (Projection Lemma)

Let $X \subseteq \mathbb{R}^m$ be a non-empty convex set, and let $y \notin X$. Then there exist $x^* \in X$ with minimum distance from y . Moreover for all $x \in X$ we have $(y - x^*)^T (x - x^*) \leq 0$.



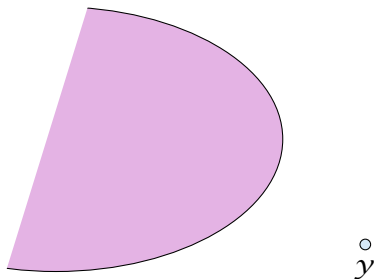
Proof of the Projection Lemma

- ▶ Define $f(x) = \|y - x\|$.
- ▶ We want to apply Weierstrass but X may not be bounded.
- ▶ $X \neq \emptyset$. Hence, there exists $x' \in X$.
- ▶ Define $X' = \{x \in X \mid \|y - x\| \leq \|y - x'\|\}$. This set is closed and bounded.
- ▶ Applying Weierstrass gives the existence.



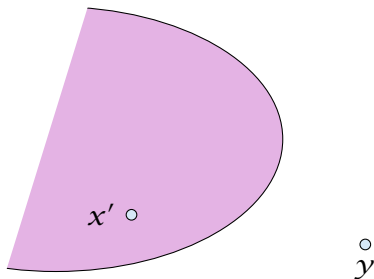
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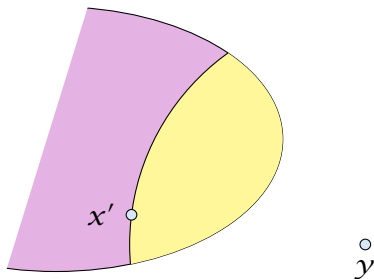
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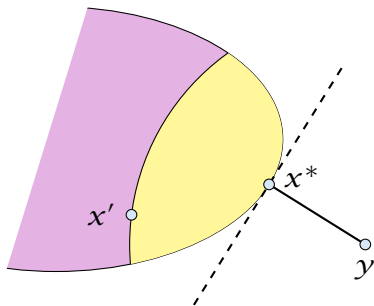
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By **convexity**: $x \in X$ then $x^* + \epsilon(x - x^*) \in X$ for all $0 \leq \epsilon \leq 1$.

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$$\|y - x^*\|^2 \leq \|y - x^* - \epsilon(x - x^*)\|^2$$

Proof of the Projection Lemma (continued)

x^* is minimum. Hence $\|y - x^*\|^2 \leq \|y - x\|^2$ for all $x \in X$.

By **convexity**: $x \in X$ then $x^* + \epsilon(x - x^*) \in X$ for all $0 \leq \epsilon \leq 1$.

$$\begin{aligned}\|y - x^*\|^2 &\leq \|y - x^* - \epsilon(x - x^*)\|^2 \\ &= \|y - x^*\|^2 + \epsilon^2 \|x - x^*\|^2 - 2\epsilon(y - x^*)^T(x - x^*)\end{aligned}$$

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x^* is minimum. Hence $\|y - x^*\|^2 \leq \|y - x\|^2$ for all $x \in X$.

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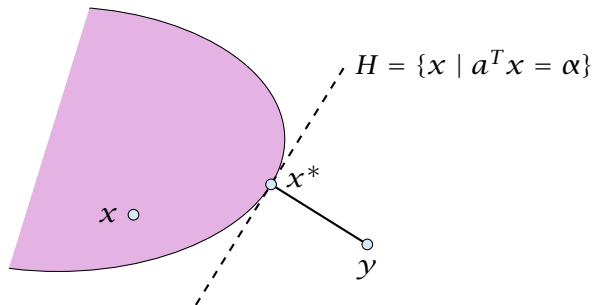
Letting $\epsilon \rightarrow 0$ gives the result.

Theorem 36 (Separating Hyperplane)

Let $X \subseteq \mathbb{R}^m$ be a non-empty closed convex set, and let $y \notin X$. Then there exists a *separating hyperplane* $\{x \in \mathbb{R}^m : a^T x = \alpha\}$ where $a \in \mathbb{R}^m$, $\alpha \in \mathbb{R}$ that *separates* y from X . ($a^T y < \alpha$; $a^T x \geq \alpha$ for all $x \in X$)

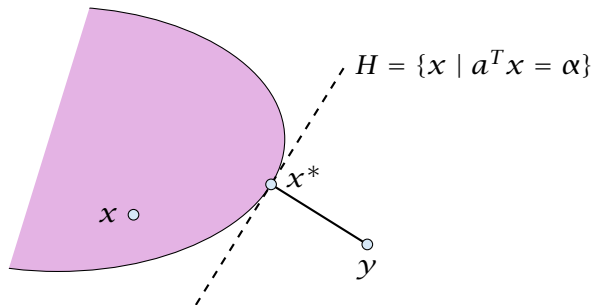
Proof of the Hyperplane Lemma

- ▶ Let $x^* \in X$ be closest point to y in X .
- ▶ By previous lemma $(y - x^*)^T(x - x^*) \leq 0$ for all $x \in X$.
- ▶ Choose $a = (x^* - y)$ and $\alpha = a^T x^*$.
- ▶ For $x \in X$: $a^T(x - x^*) \geq 0$, and, hence, $a^T x \geq \alpha$.
- ▶ Also, $a^T y = a^T(x^* - a) = \alpha - \|a\|^2 < \alpha$



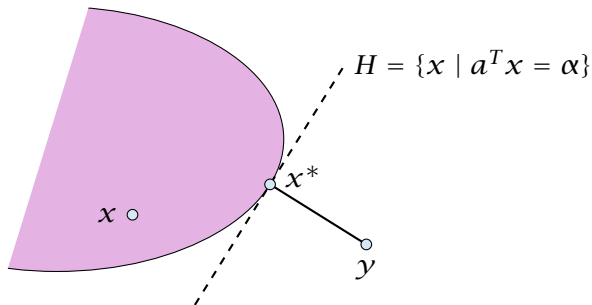
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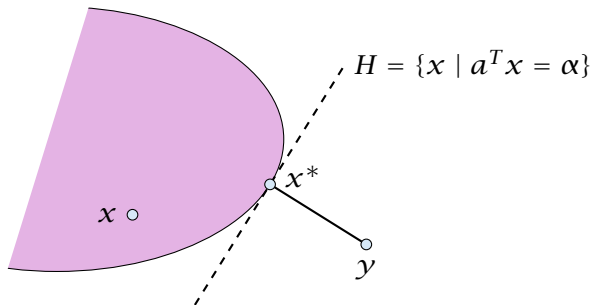
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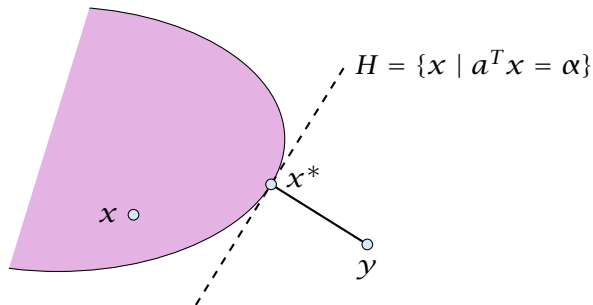
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Lemma 37 (Farkas Lemma)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Then *exactly one* of the following statements holds.

1. $\exists x \in \mathbb{R}^n$ with $Ax = b$, $x \geq 0$
2. $\exists y \in \mathbb{R}^m$ with $A^T y \geq 0$, $b^T y < 0$

Assume \hat{x} satisfies 1. and \hat{y} satisfies 2. Then

$$0 > \hat{y}^T b = \hat{y}^T A \hat{x} \geq 0$$

Hence, at most one of the statements can hold.

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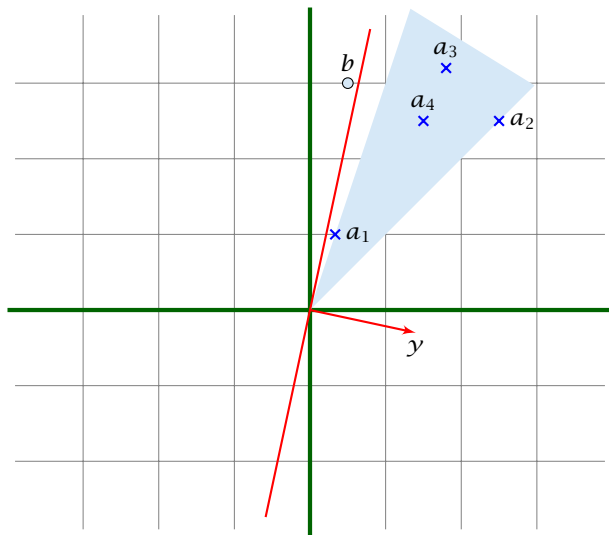
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Hence, at most one of the statements can hold.

Farkas Lemma



If b is not in the cone generated by the columns of A , there exists a hyperplane y that separates b from the cone.

Proof of Farkas Lemma

Now, assume that 1. does not hold.

Consider $S = \{Ax : x \geq 0\}$ so that S closed, convex, $b \notin S$.

We want to show that there is y with $A^T y \geq 0$, $b^T y < 0$.

Let y be a hyperplane that separates b from S . Hence, $y^T b < \alpha$ and $y^T s \geq \alpha$ for all $s \in S$.

$$0 \in S \Rightarrow \alpha \leq 0 \Rightarrow y^T b < 0$$

$y^T Ax \geq \alpha$ for all $x \geq 0$. Hence, $y^T A \geq 0$ as we can choose x arbitrarily large.

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Lemma 38 (Farkas Lemma; different version)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Then exactly one of the following statements holds.

1. $\exists x \in \mathbb{R}^n$ with $Ax \leq b, x \geq 0$
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Rewrite the conditions:

1. $\exists x \in \mathbb{R}^n$ with $\begin{bmatrix} A & I \end{bmatrix} \cdot \begin{bmatrix} x \\ s \end{bmatrix} = b, x \geq 0, s \geq 0$
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Proof of Strong Duality

$$P: z = \max\{c^T x \mid Ax \leq b, x \geq 0\}$$

$$D: w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$$

Theorem 39 (Strong Duality)

Let P and D be a primal dual pair of linear programs, and let z and w denote the optimal solution to P and D , respectively (i.e., P and D are non-empty). Then

$$z = w .$$

Proof of Strong Duality

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$$\begin{array}{ll} \exists x \in \mathbb{R}^n & \\ \text{s.t.} & Ax \leq b \\ & -c^T x \leq -\alpha \\ & x \geq 0 \end{array}$$

Proof of Strong Duality

$z \leq w$: follows from weak duality

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We show $z < \alpha$ implies $w < \alpha$.

$$\exists x \in \mathbb{R}^n$$

$$\begin{aligned} \text{s.t.} \quad Ax &\leq b \\ -c^T x &\leq -\alpha \\ x &\geq 0 \end{aligned}$$

$$\exists y \in \mathbb{R}^m; v \in \mathbb{R}$$

$$\begin{aligned} \text{s.t.} \quad A^T y - cv &\geq 0 \\ b^T y - \alpha v &< 0 \\ y, v &\geq 0 \end{aligned}$$

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From the definition of α we know that the first system is infeasible; hence the second must be feasible.

Proof of Strong Duality

$$\begin{aligned} \exists \mathbf{y} \in \mathbb{R}^m; \nu \in \mathbb{R} \\ \text{s.t. } \quad A^T \mathbf{y} - c\nu &\geq 0 \\ \quad \quad b^T \mathbf{y} - \alpha\nu &< 0 \\ \quad \quad \mathbf{y}, \nu &\geq 0 \end{aligned}$$

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If the solution \mathbf{y}, \mathbf{v} has $\mathbf{v} = 0$ we have that

$$\begin{aligned} \exists \mathbf{y} \in \mathbb{R}^m \\ \text{s.t. } \quad A^T \mathbf{y} &\geq 0 \\ \quad \quad b^T \mathbf{y} &< 0 \\ \quad \quad \mathbf{y} &\geq 0 \end{aligned}$$

is feasible.

Proof of Strong Duality

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is feasible. By Farkas lemma this gives that LP P is infeasible.
Contradiction to the assumption of the lemma.

Proof of Strong Duality

Hence, there exists a solution y, v with $v > 0$.

We can rescale this solution (scaling both y and v) s.t. $v = 1$.

Then y is feasible for the dual but $b^T y < \alpha$. This means that $w < \alpha$.

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Fundamental Questions

Definition 40 (Linear Programming Problem (LP))

Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

Questions:

- ▶ Is LP in NP?
- ▶ Is LP in co-NP? **yes!**
- ▶ Is LP in P?

Proof:

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Proof:

- ▶ Given a primal maximization problem P and a parameter α . Suppose that $\alpha > \text{opt}(P)$.
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Complementary Slackness

Lemma 41

Assume a linear program $P = \max\{c^T x \mid Ax \leq b; x \geq 0\}$ has solution x^* and its dual $D = \min\{b^T y \mid A^T y \geq c; y \geq 0\}$ has solution y^* .

1. If $x_j^* > 0$ then the j -th constraint in D is tight.
2. If the j -th constraint in D is not tight then $x_j^* = 0$.
3. If $y_i^* > 0$ then the i -th constraint in P is tight.
4. If the i -th constraint in P is not tight then $y_i^* = 0$.

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2. If the j -th constraint in D is not tight then $x_j^* = 0$.
3. If $y_i^* > 0$ then the i -th constraint in P is tight.
4. If the i -th constraint in P is not tight then $y_i^* = 0$.

If we say that a variable x_j^* (y_i^*) has slack if $x_j^* > 0$ ($y_i^* > 0$), (i.e., the corresponding variable restriction is not tight) and a constraint has slack if it is not tight, then the above says that for a primal-dual solution pair it is not possible that a constraint **and** its corresponding (dual) variable has slack.

Proof: Complementary Slackness

Analogous to the proof of weak duality we obtain

$$c^T x^* \leq y^{*T} A x^* \leq b^T y^*$$

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$$\sum_j (y^T A - c^T)_j x_j^* = 0$$

From the constraint of the dual it follows that $y^T A \geq c^T$. Hence the left hand side is a sum over the product of non-negative numbers. Hence, if e.g. $(y^T A - c^T)_j > 0$ (the j -th constraint in the dual is not tight) then $x_j^* = 0$ (2.). The result for (1./3./4.) follows similarly.

Interpretation of Dual Variables

- ▶ Brewer: find mix of ale and beer that maximizes profits

$$\begin{aligned} \max \quad & 13a + 23b \\ \text{s.t.} \quad & 5a + 15b \leq 480 \\ & 4a + 4b \leq 160 \\ & 35a + 20b \leq 1190 \\ & a, b \geq 0 \end{aligned}$$

- ▶ Entrepreneur: buy resources from brewer at minimum cost
 C, H, M : unit price for corn, hops and malt.

$$\begin{aligned} \min \quad & 480C + 160H + 1190M \\ \text{s.t.} \quad & 5C + 4H + 35M \geq 13 \\ & 15C + 4H + 20M \geq 23 \\ & C, H, M \geq 0 \end{aligned}$$

Note that brewer won't sell (at least not all) if e.g.
 $5C + 4H + 35M < 13$ as then brewing ale would be advantageous.

Interpretation of Dual Variables

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Marginal Price:

- ▶ How much money is the brewer willing to pay for additional amount of Corn, Hops, or Malt?
- ▶ We are interested in the marginal price, i.e., what happens if we increase the amount of Corn, Hops, and Malt by ϵ_C , ϵ_H , and ϵ_M , respectively.

The profit increases to $\max\{c^T x \mid Ax \leq b + \epsilon; x \geq 0\}$. Because of strong duality this is equal to

$$\begin{array}{ll} \min & (b^T + \epsilon^T)y \\ \text{s.t.} & A^T y \geq c \\ & y \geq 0 \end{array}$$

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If ϵ is “small” enough then the optimum dual solution y^* might not change. Therefore the profit increases by $\sum_i \epsilon_i y_i^*$.

Therefore we can interpret the dual variables as **marginal prices**.

Note that with this interpretation, complementary slackness becomes obvious.

If the farmer has slack of some resource (i.e. only 1000 kg of land) he is not willing to pay anything for it (corresponding dual variable is zero).

If the dual variable for some resource is non-zero, then an increase of this resource increases the profit of the farmer. Hence, it makes no sense to have a surplus of this resource. Therefore, complementary slackness holds.

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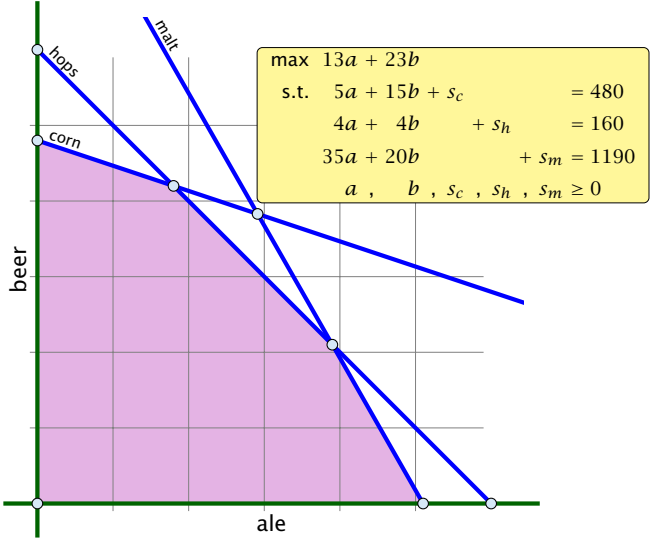
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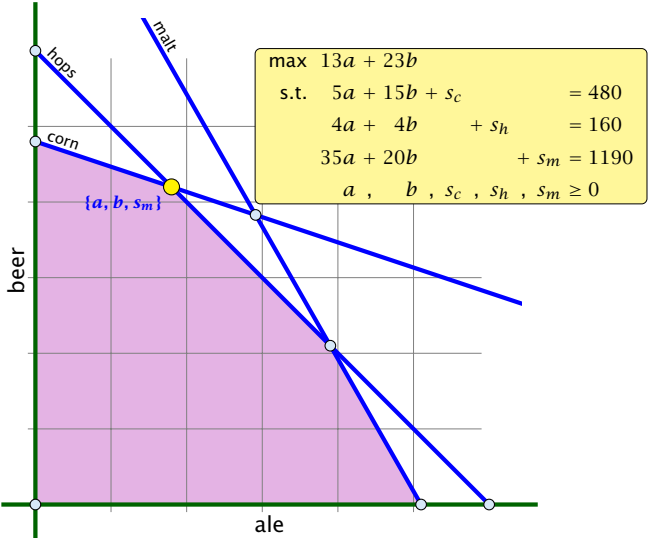
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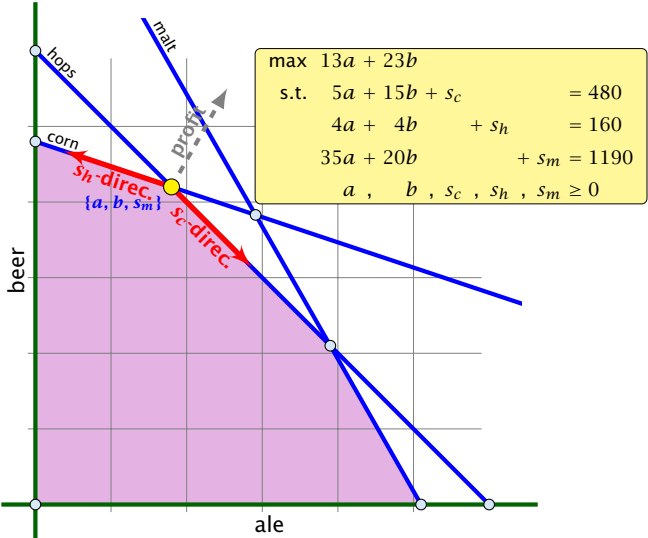
Example



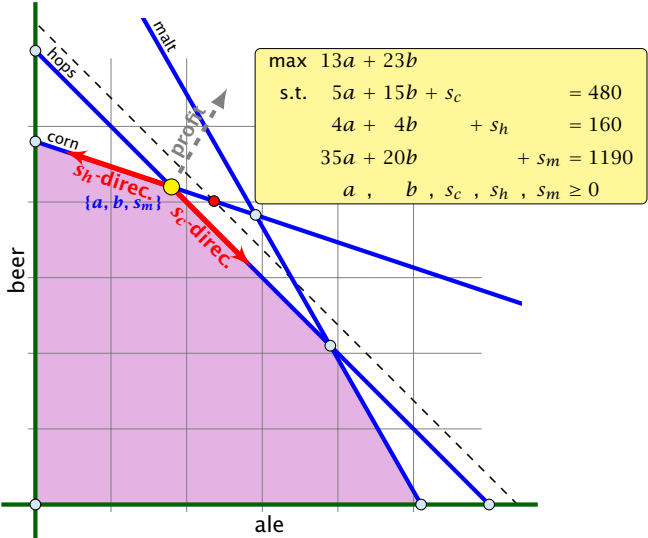
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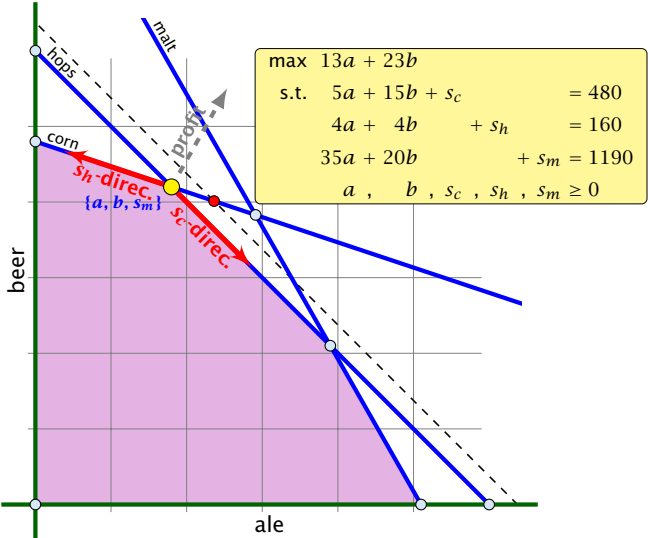
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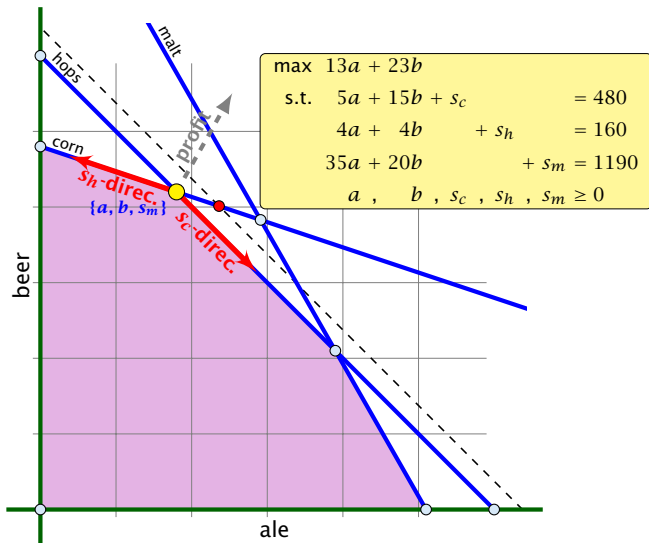
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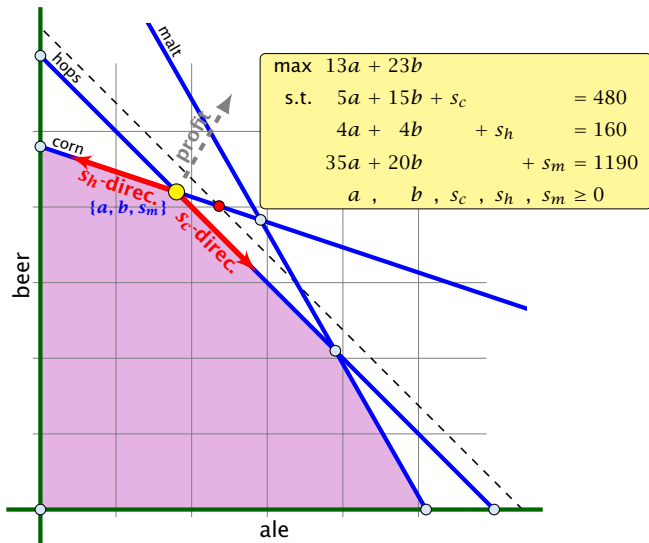
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Of course, the previous argument about the increase in the primal objective only holds for the non-degenerate case.

If the optimum basis is degenerate then increasing the supply of one resource may not allow the objective value to increase.

Flows

Definition 42

An (s, t) -flow in a (complete) directed graph $G = (V, V \times V, c)$ is a function $f : V \times V \rightarrow \mathbb{R}_0^+$ that satisfies

1. For each edge (x, y)

$$0 \leq f_{xy} \leq c_{xy} .$$

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Maximum Flow Problem:

Find an (s, t) -flow with maximum value.

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We can interpret the ℓ_{xy} value as assigning a length to every edge.

The value p_x for a variable, then can be seen as the distance of x to t (where the distance from s to t is required to be 1 since $p_s = 1$).

The constraint $p_x \leq \ell_{xy} + p_y$ then simply follows from triangle inequality ($d(x, t) \leq d(x, y) + d(y, t) \Rightarrow d(x, t) \leq \ell_{xy} + d(y, t)$).

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One can show that there is an optimum LP-solution for the dual problem that gives an integral assignment of variables.

This means $p_x = 1$ or $p_x = 0$ for our case. This gives rise to a cut in the graph with vertices having value 1 on one side and the other vertices on the other side. The objective function then evaluates the capacity of this cut.

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