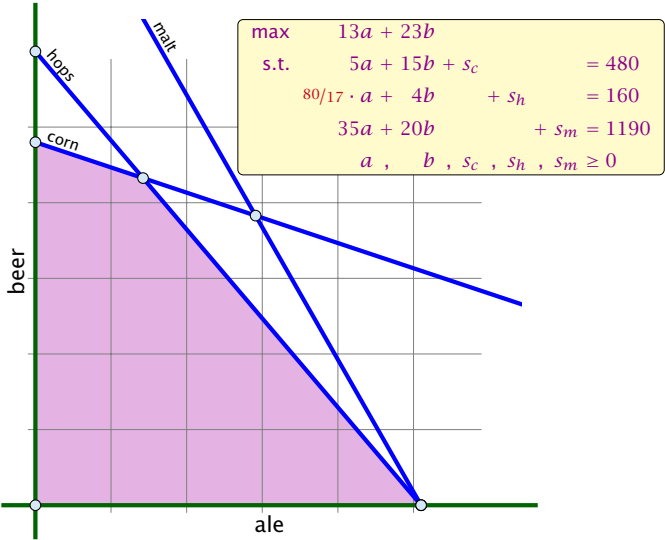


Degeneracy Revisited

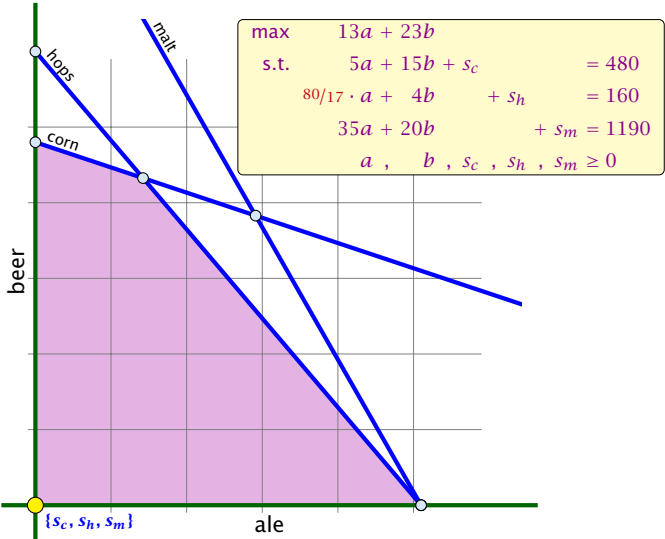
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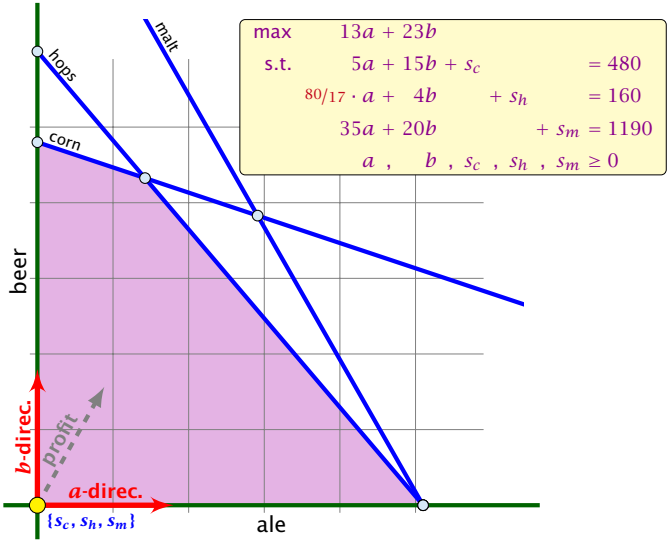
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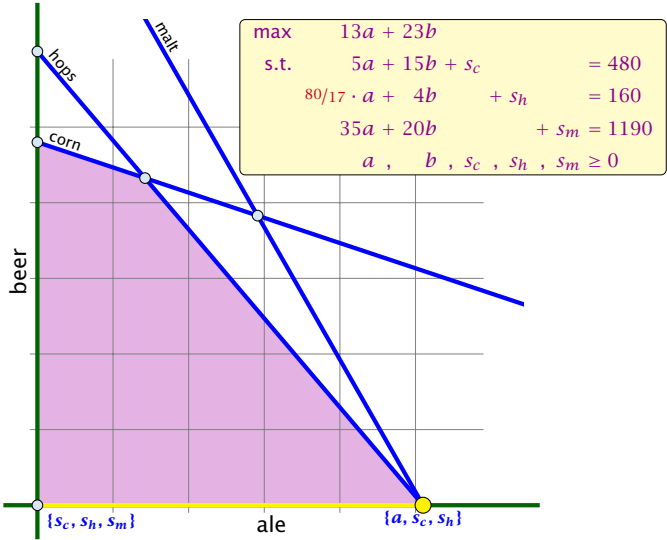
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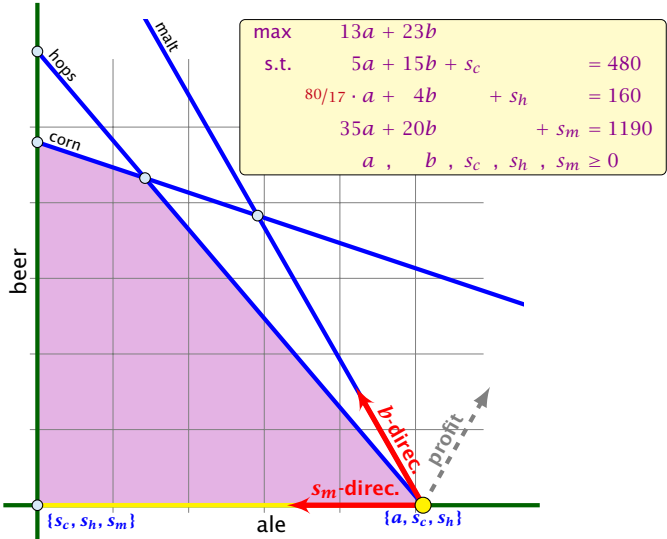
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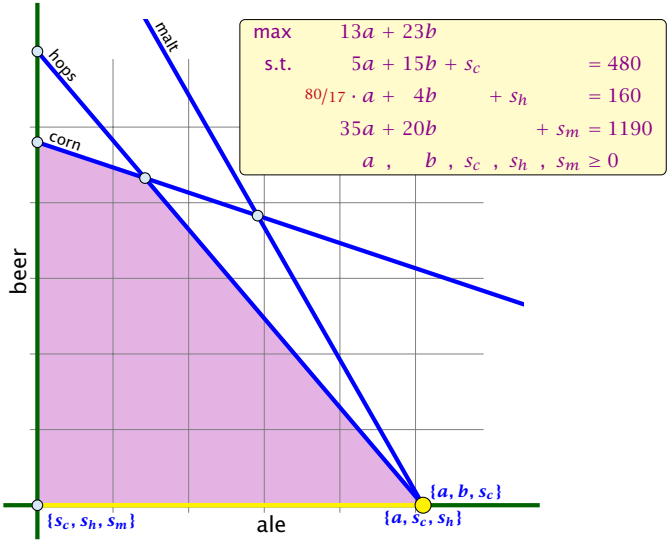
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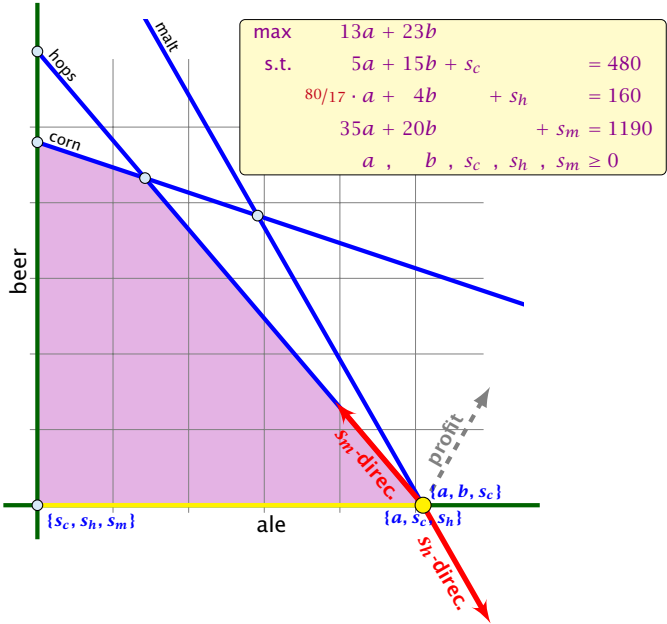
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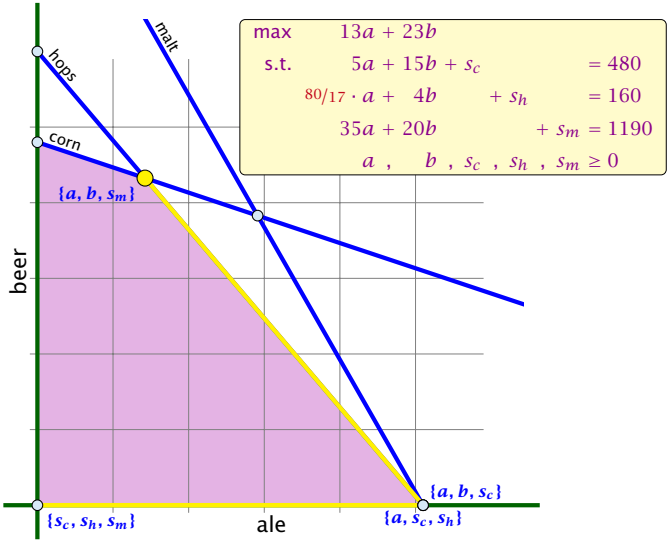
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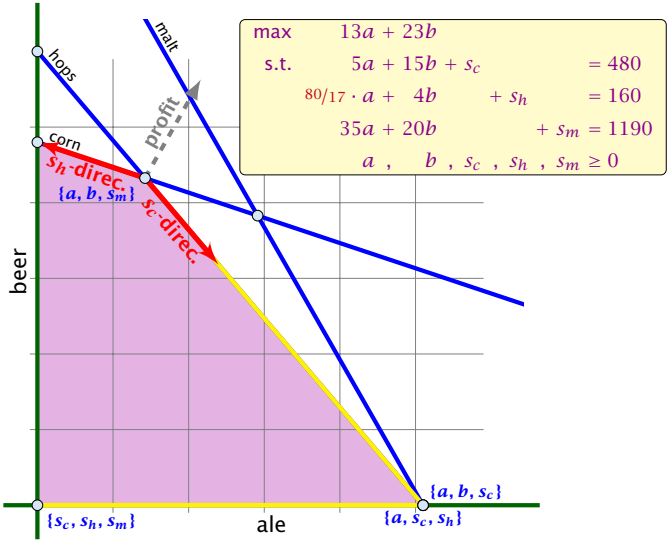
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Idea:

Given feasible LP := $\max\{c^T x, Ax = b; x \geq 0\}$. Change it into LP' := $\max\{c^T x, Ax = b', x \geq 0\}$ such that

LP' is feasible

if a set of basic variables corresponds to an optimal solution of LP

if a set of basic variables corresponds to an infeasible basis then

LP is unfeasible (but should not be a linearly independent set)

if a set of basic variables is not optimal

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- II. If a set B of basis variables corresponds to an infeasible basis (i.e. $A_B^{-1}b \not\geq 0$) then B corresponds to an infeasible basis in LP' (note that columns in A_B are linearly independent).
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Perturbation

Let B be index set of **some** basis with basic solution

$$x_B^* = A_B^{-1}b \geq 0, x_N^* = 0 \quad (\text{i.e. } B \text{ is feasible})$$

Fix

$$b' := b + A_B \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix} \quad \text{for } \varepsilon > 0 .$$

This is the perturbation that we are using.

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The new LP is feasible because the set B of basis variables provides a feasible basis:

$$A_B^{-1} \left(b + A_B \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix} \right) = x_B^* + \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix} \geq 0 .$$

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Hence, \tilde{B} is not feasible.

Property III

Let \tilde{B} be a basis. It has an associated solution

$$x_{\tilde{B}}^* = A_{\tilde{B}}^{-1}b + A_{\tilde{B}}^{-1}A_B \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix}$$

in the perturbed instance.

We can view each component of the vector as a polynomial with variable ε of degree at most m .

$A_{\tilde{B}}^{-1}A_B$ has rank m . Therefore no polynomial is 0.

A polynomial of degree at most m has at most m roots (Nullstellen).

Hence, $\varepsilon > 0$ small enough gives that no component of the above vector is 0. Hence, no degeneracies.

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- ▶ If it terminates because it finds a variable x_j with $\tilde{c}_j > 0$ for which the j -th basis direction d , fulfills $d \geq 0$ we know that LP' is unbounded. The basis direction **does not depend on b .** Hence, we also know that LP is unbounded.

Lexicographic Pivoting

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Simulate behaviour of LP' without explicitly doing a perturbation.

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We choose the entering variable arbitrarily as before ($\tilde{c}_e > 0$, of course).

If we do not have a choice for the leaving variable then LP' and LP do the same (i.e., choose the same variable).

Otherwise we have to be careful.

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In the following we assume that $b \geq 0$. This can be obtained by replacing the initial system $(A \mid b)$ by $(A_B^{-1}A \mid A_B^{-1}b)$ where B is the index set of a feasible basis (found e.g. by the first phase of the Two-phase algorithm).

Then the perturbed instance is

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Matrix View

Let our linear program be

$$\begin{aligned}c_B^T x_B + c_N^T x_N &= Z \\ A_B x_B + A_N x_N &= b \\ x_B, x_N &\geq 0\end{aligned}$$

The simplex tableaux for basis B is

$$\begin{aligned}I x_B + (c_N^T - c_B^T A_B^{-1} A_N) x_N &= Z - c_B^T A_B^{-1} b \\ A_B^{-1} A_N x_N &= A_B^{-1} b \\ x_B, x_N &\geq 0\end{aligned}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

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LP chooses an arbitrary leaving variable that has $\hat{A}_{\ell e} > 0$ and minimizes

$$\theta_{\ell} = \frac{\hat{b}_{\ell}}{\hat{A}_{\ell e}} = \frac{(A_B^{-1}b)_{\ell}}{(A_B^{-1}A_{*e})_{\ell}}.$$

ℓ is the index of a leaving variable within B . This means if e.g. $B = \{1, 3, 7, 14\}$ and leaving variable is 3 then $\ell = 2$.

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Definition 44

$u \leq_{\text{lex}} v$ if and only if the first component in which u and v differ fulfills $u_i \leq v_i$.

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This means you can choose the variable/row ℓ for which the vector

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is lexicographically minimal.

Of course only including rows with $(A_B^{-1}A_{*e})_\ell > 0$.

This technique guarantees that your pivoting is the same as in the perturbed case. This guarantees that cycling does not occur.

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