18 Cuts & Metrics

Shortest Path

$$\begin{array}{llrcl} \min & & \sum_e c(e) x_e & & \\ \text{s.t.} & \forall S \in \mathcal{S} & \sum_{e \in \delta(S)} x_e & \geq & 1 \\ & \forall e \in E & x_e & \in & \{0,1\} \end{array}$$

$\mathcal{S}$ is the set of subsets that separate s from t.

The Dual:

$$\begin{array}{llrcl} \max & & \sum_S y_S & & \\ \text{s.t.} & \forall e \in E & \sum_{S: e \in \delta(S)} y_S & \leq & c(e) \\ & \forall S \in \mathcal{S} & y_S & \geq & 0 \end{array}$$

The Separation Problem for the Shortest Path LP is the Minimum Cut Problem.

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Minimum Cut

$$\begin{array}{llrcl} \min & & \sum_e c(e)x_e & & \\ \text{s.t.} & \forall P \in \mathcal{P} & \sum_{e\in P} x_e & \geq & 1 \\ & \forall e \in E & x_e & \in & \{0,1\} \end{array}$$

$\mathcal{P}$ is the set of path that connect s and t.

The Dual:

$$\begin{array}{llrcl} \max & & \sum_P y_P & & \\ \text{s.t.} & \forall e \in E & \sum_{P:e\in P} y_P & \leq & c(e) \\ & \forall P \in \mathcal{P} & y_P & \geq & 0 \end{array}$$

The Separation Problem for the Minimum Cut LP is the Shortest Path Problem.

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Observations:

Suppose that ℓ_e-values are solution to Minimum Cut LP.

- We can view ℓ_e as defining the length of an edge.
- Define $d(u,v) = \min_{\text{path } P \text{ btw. } u \text{ and } v} \sum_{e\in P} \ell_e$ as the Shortest Path Metric induced by ℓ_e.
- We have $d(u,v) = \ell_e$ for every edge $e = (u,v)$, as otw. we could reduce ℓ_e without affecting the distance between s and t.

Remark for bean-counters:
d is not a metric on V but a semimetric as two nodes u and v could have distance zero.

How do we round the LP?

- Let $B(s,r)$ be the ball of radius r around s (w.r.t. metric d). Formally:

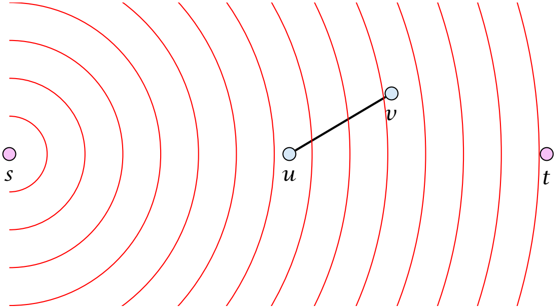
$$B = \{v \in V \mid d(s,v) \leq r\}$$

- For $0 \leq r < 1$, $B(s,r)$ is an s-t-cut.

Which value of r should we choose? choose randomly!!!

Formally:
choose r u.a.r. (uniformly at random) from interval $[0,1)$

What is the probability that an edge (u, v) is in the cut?



- asssume wlog. $d(s,u) \leq d(s,v)$

$$\Pr[e \text{ is cut}] = \Pr[r \in [d(s,u), d(s,v))] \leq \frac{d(s,v) - d(s,u)}{1 - 0}$$

$$\leq \ell_e$$

What is the expected size of a cut?

$$\begin{aligned}\mathrm{E}[\text{size of cut}] &= \mathrm{E}\left[\sum_e c(e)\Pr[e \text{ is cut}]\right] \\ &\le \sum_e c(e)\ell_e\end{aligned}$$

On the other hand:

$$\sum_e c(e)\ell_e \le \text{size of mincut}$$

as the ℓ_e are the solution to the Mincut LP *relaxation*.

Hence, our rounding gives an optimal solution.

Minimum Multicut:
Given a graph $G = (V, E)$, together with source-target pairs s_i, t_i, $i = 1, \dots, k$, and a capacity function $c : E \to \mathbb{R}^+$ on the edges. Find a subset $F \subseteq E$ of the edges such that all s_i-t_i pairs lie in different components in $G = (V, E \setminus F)$.

$$\begin{array}{llrcl}
\min & & \sum_e c(e)\ell_e & & \\
\text{s.t.} & \forall P \in \mathcal{P}_i \text{ for some } i & \sum_{e \in P} \ell_e & \geq & 1 \\
 & \forall e \in E & \ell_e & \in & \{0, 1\}
\end{array}$$

Here $\mathcal{P}_i$ contains all path P between s_i and t_i.

Re-using the analysis for the single-commodity case is difficult.

$$\Pr[e \text{ is cut}] \le \textbf{?}$$

- If for some R the balls $B(s_i, R)$ are disjoint between different sources, we get a $1/R$ approximation.
- However, this cannot be guaranteed.

- Assume for simplicity that all edge-length ℓ_e are multiples of $\delta \ll 1$.
- Replace the graph G by a graph G', where an edge of length ℓ_e is replaced by ℓ_e/δ edges of length δ.
- Let $B(s_i, z)$ be the ball in G' that contains nodes v with distance $d(s_i, v) \leq z\delta$.

Algorithm 1 RegionGrowing(s_i, p)

```
z ← 0
repeat
    flip a coin (Pr[heads] = p)
    z ← z + 1
until heads
return B(s_i, z)
```

Algorithm 1 Multicut(G')

1: **while** $\exists s_i$-t_i pair in G' **do**
2: $C \leftarrow$ RegionGrowing(s_i, p)
3: $G' = G' \setminus C$ // cuts edges leaving C
4: **return** $B(s_i, z)$

- probability of cutting an edge is only $\boldsymbol{p}$
- a source either does not reach an edge during Region Growing; then it is not cut
- if it reaches the edge then it either cuts the edge or protects the edge from being cut by other sources
- if we choose $p = \delta$ the probability of cutting an edge is only its LP-value; our expected cost are at most OPT.

Problem:
We may not cut all source-target pairs.

A component that we remove may contain an s_i-t_i pair.

If we ensure that we cut before reaching radius $1/2$ we are in good shape.

- choose $p = 6\ln k \cdot \delta$
- we make $\frac{1}{2\delta}$ trials before reaching radius $1/2$.
- we say a Region Growing is not successful if it does not terminate before reaching radius $1/2$.

$$\Pr[\text{not successful}] \le (1-p)^{\frac{1}{2\delta}} = \left((1-p)^{1/p}\right)^{\frac{p}{2\delta}} \le e^{-\frac{p}{2\delta}} \le \frac{1}{k^3}$$

- Hence,

$$\Pr[\exists i \text{ that is not successful}] \le \frac{1}{k^2}$$

What is expected cost?

$$\mathrm{E}[\text{cutsize}] = \Pr[\text{success}] \cdot \mathrm{E}[\text{cutsize} \mid \text{success}] + \Pr[\text{no success}] \cdot \mathrm{E}[\text{cutsize} \mid \text{no success}]$$

$$\mathrm{E}[\text{cutsize} \mid \text{succ.}] = \frac{\mathrm{E}[\text{cutsize}] - \Pr[\text{no succ.}] \cdot \mathrm{E}[\text{cutsize} \mid \text{no succ.}]}{\Pr[\text{success}]} \le \frac{\mathrm{E}[\text{cutsize}]}{\Pr[\text{success}]} \le \frac{1}{1-\frac{1}{k^2}} 6 \ln k \cdot \mathrm{OPT} \le 8 \ln k \cdot \mathrm{OPT}$$

Note: success means all source-target pairs separated

We assume $k \ge 2$.

If we are not successful we simply perform a trivial k-approximation.

This only increases the expected cost by at most $\frac{1}{k^2} \cdot k\mathrm{OPT} \leq \mathrm{OPT}/k$.

Hence, our final cost is $\mathcal{O}(\ln k) \cdot \mathrm{OPT}$ in expectation.