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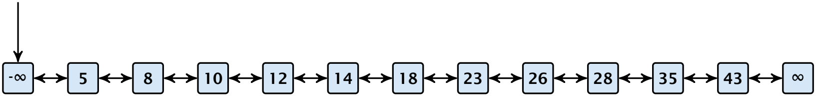

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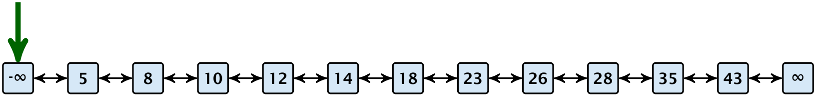

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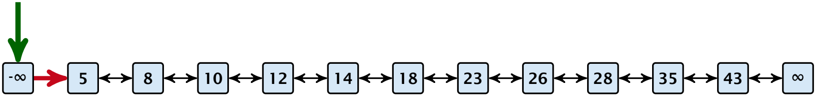

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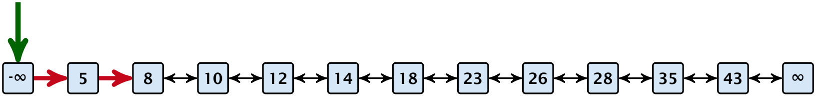

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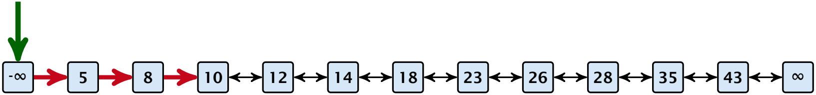

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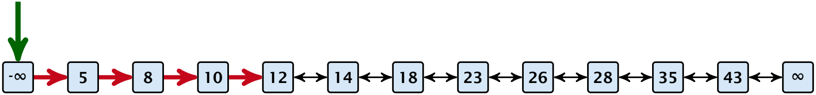

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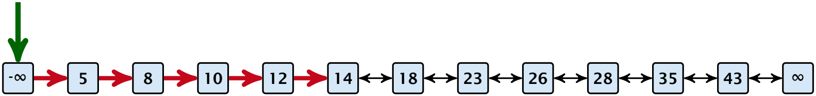

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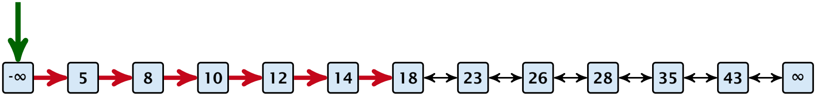

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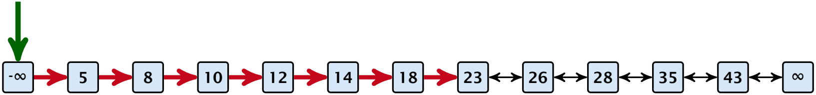

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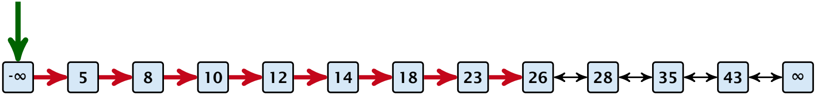

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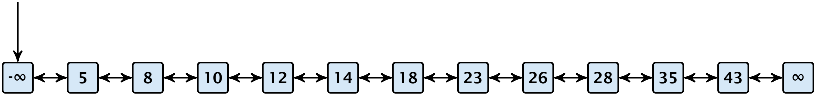

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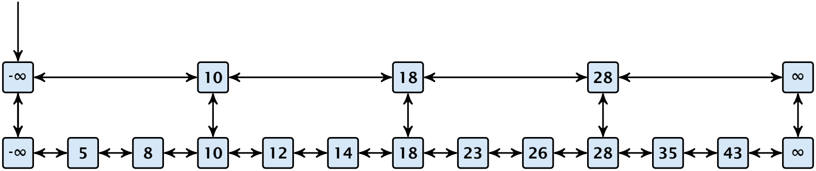

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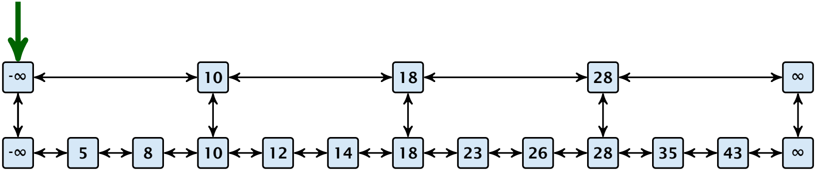

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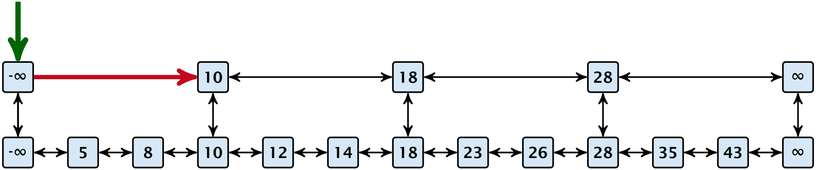

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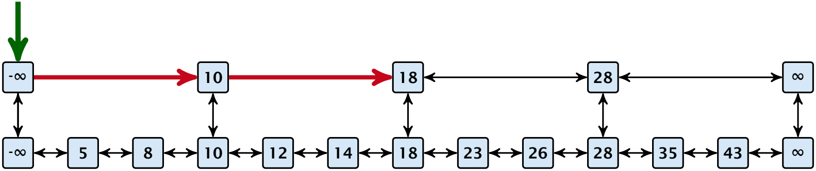

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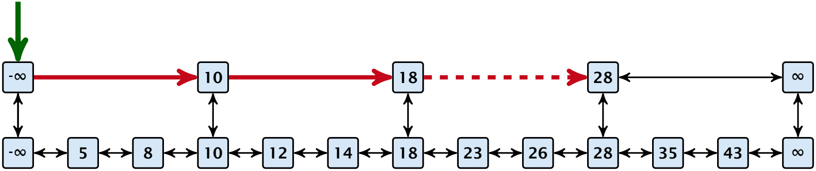

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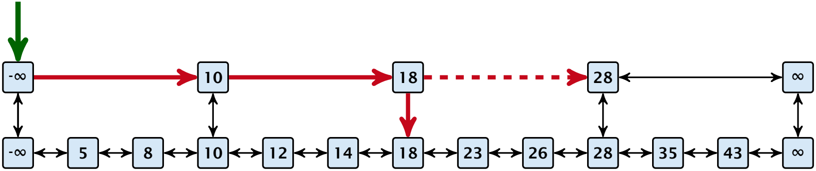

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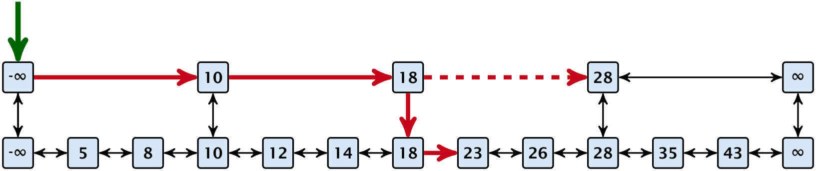

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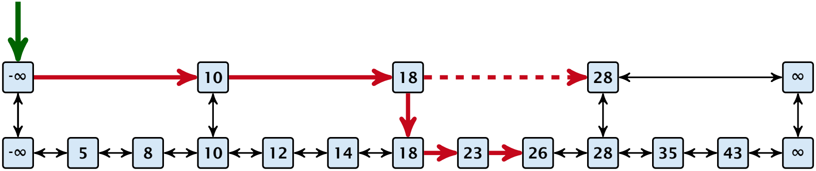

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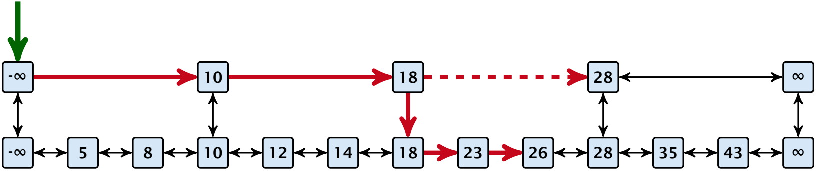

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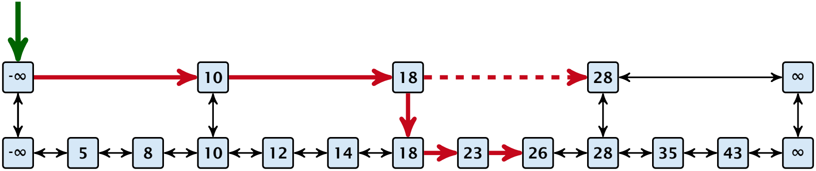

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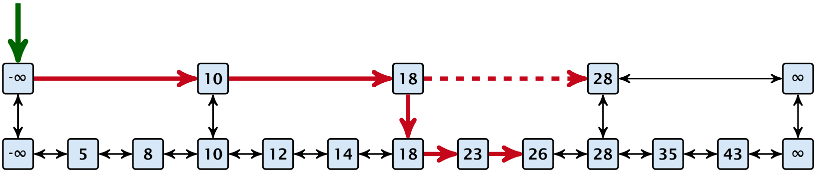

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Choose $|L_1| = \sqrt{n}$. Then search time $\Theta(\sqrt{n})$.

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Choosing $k = \Theta(\log n)$ gives a logarithmic running time.

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Use randomization instead!

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The time for both operations is dominated by the search time.

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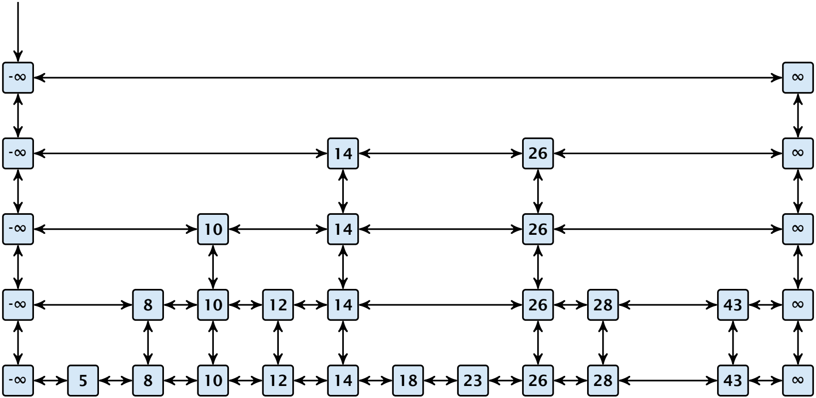

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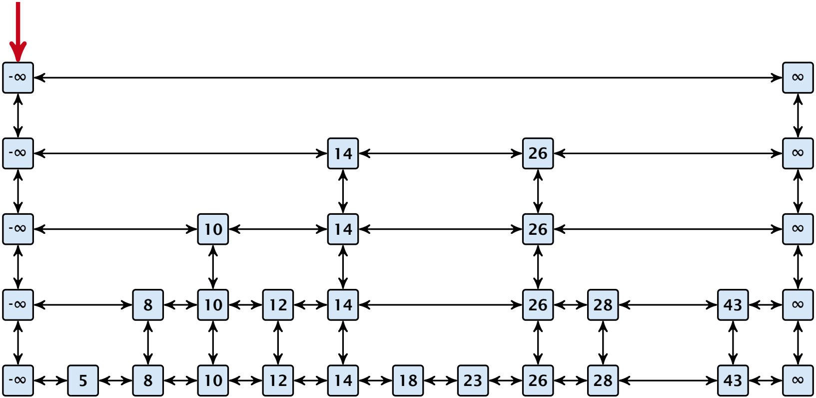

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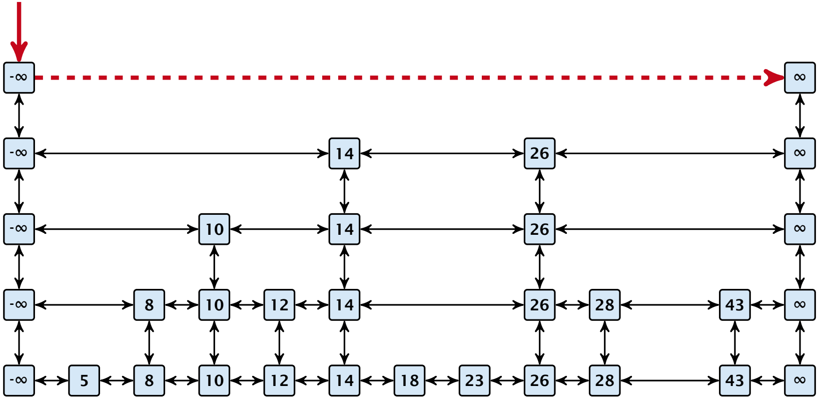

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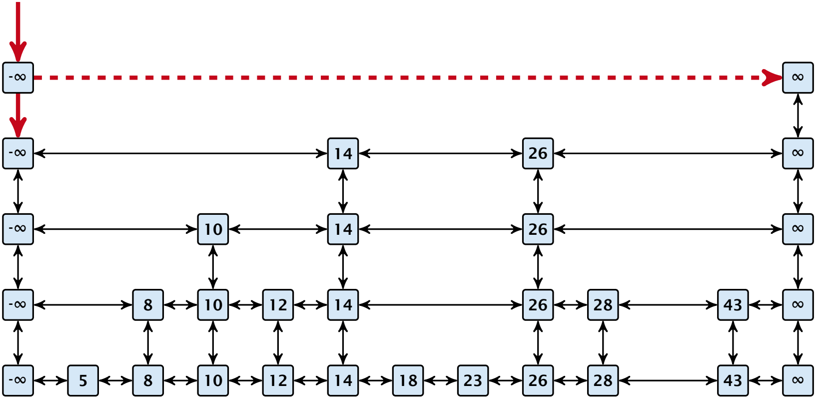

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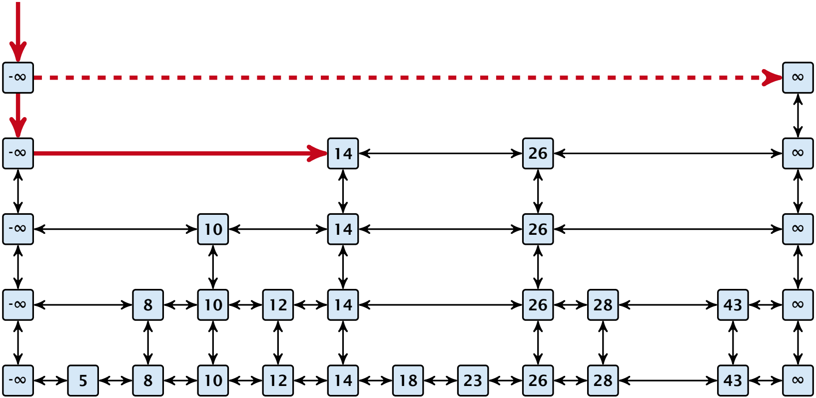

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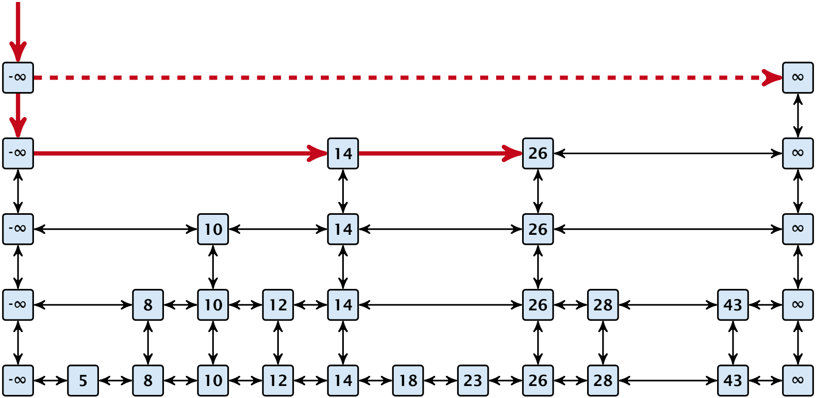

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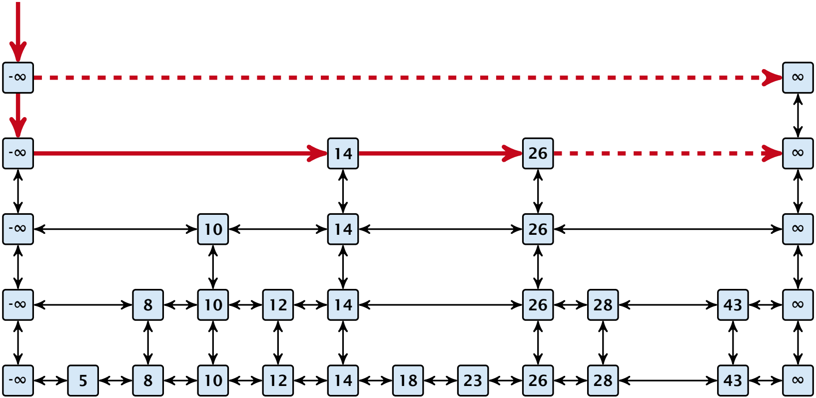

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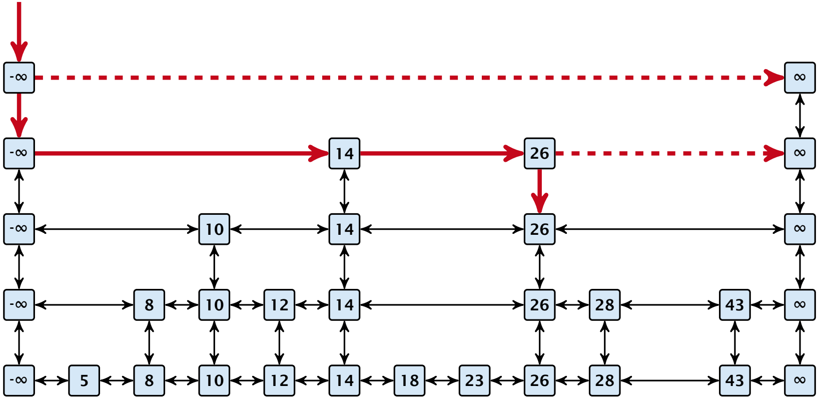

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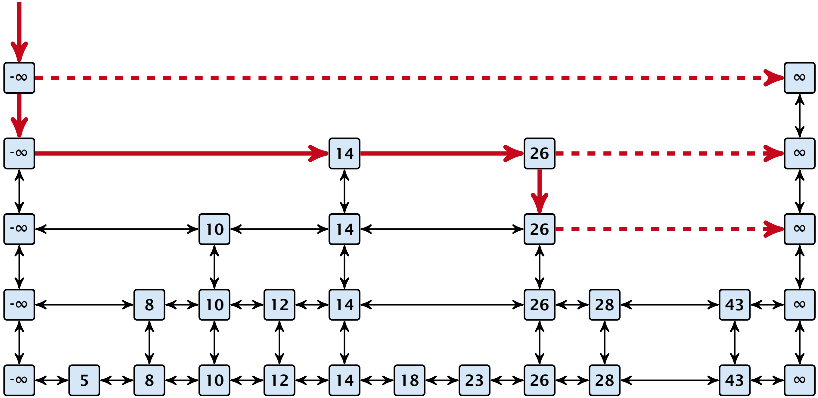

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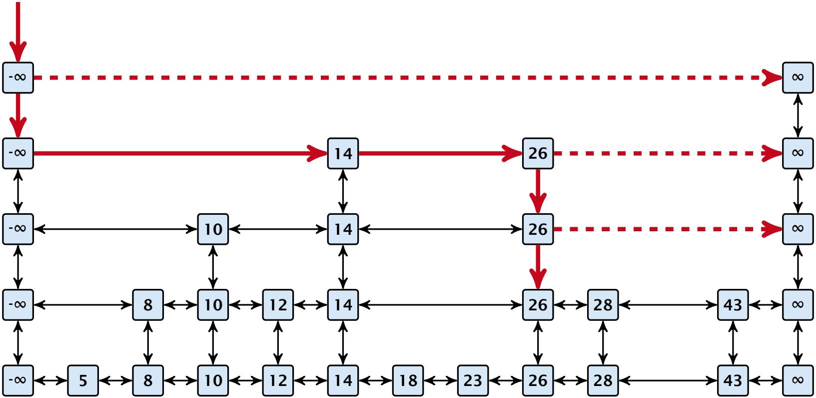

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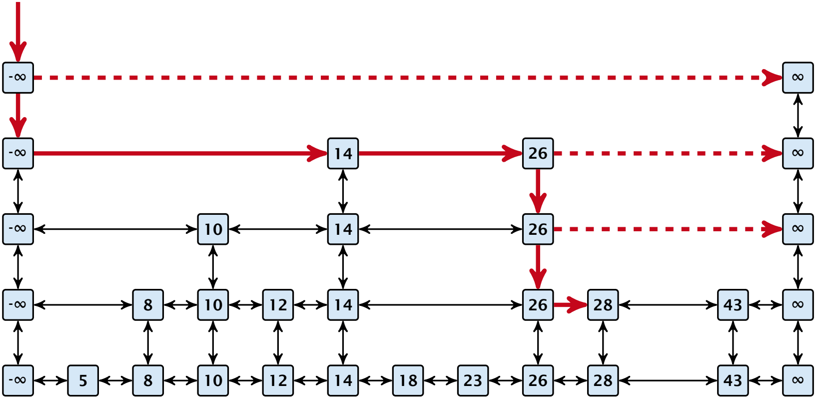

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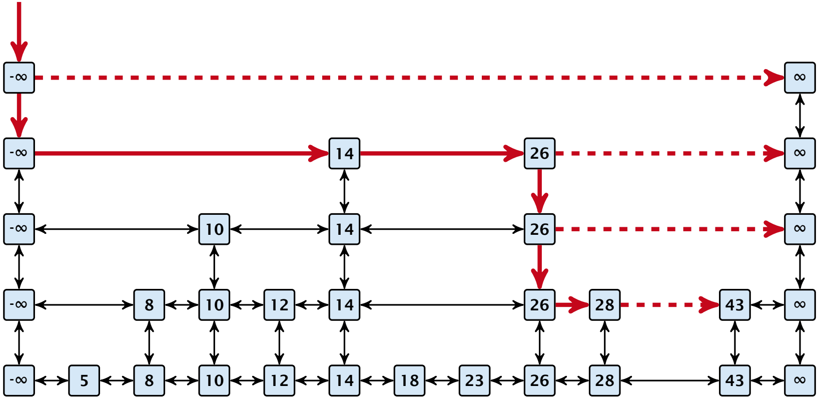

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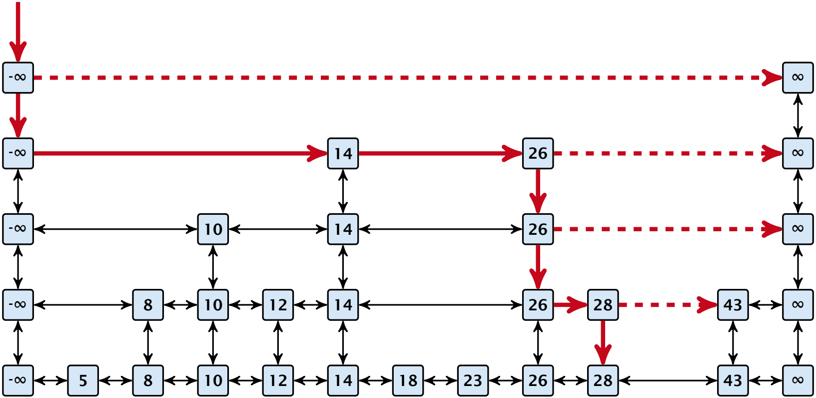

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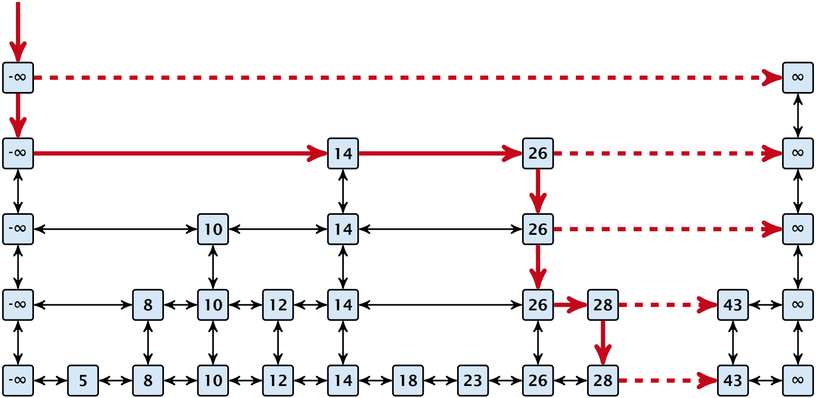

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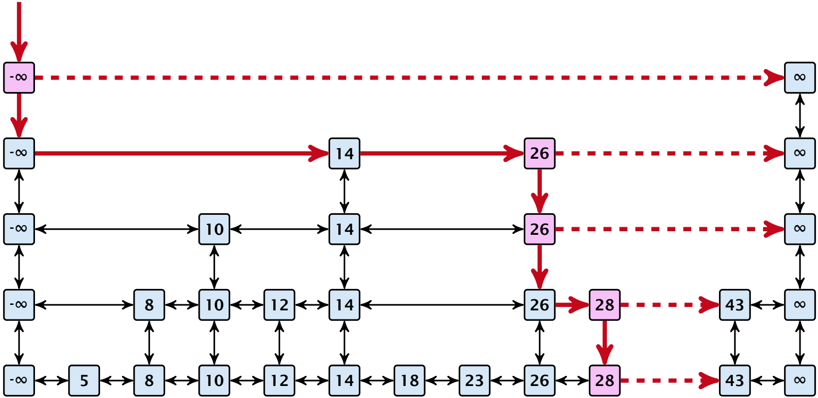

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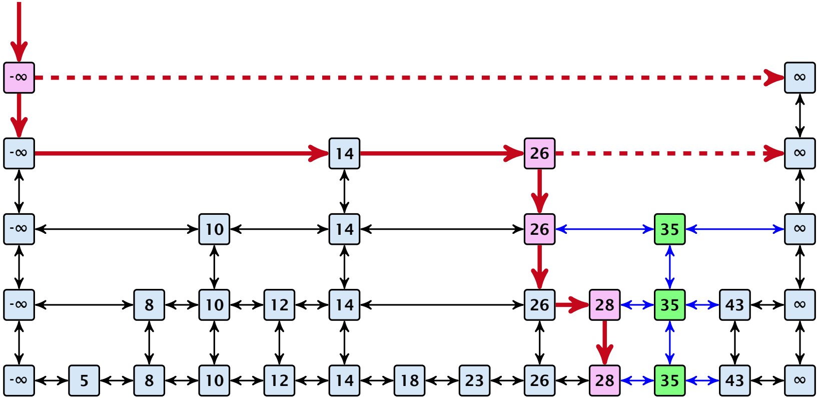

High Probability

Definition 1 (High Probability)

We say a **randomized** algorithm has running time $\mathcal{O}(\log n)$ with high probability if for any constant α the running time is at most $\mathcal{O}(\log n)$ with probability at least $1 - \frac{1}{n^\alpha}$.

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Here the $\mathcal{O}$-notation hides a constant that may depend on α.

High Probability

Suppose there are polynomially many events $E_1, E_2, \ldots, E_\ell$, $\ell = n^c$ each holding with high probability (e.g. E_i may be the event that the i-th search in a skip list takes time at most $\mathcal{O}(\log n)$).

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Then the probability that all E_i hold is at least

$$\Pr[E_1 \wedge \cdots \wedge E_\ell]$$

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This means $\Pr[E_1 \wedge \cdots \wedge E_\ell]$ holds with high probability.

7.5 Skip Lists

Lemma 2

A search (and, hence, also insert and delete) in a skip list with n elements takes time $\mathcal{O}(\log n)$ with high probability (w. h. p.).

7.5 Skip Lists

Backward analysis:

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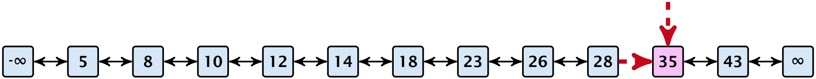

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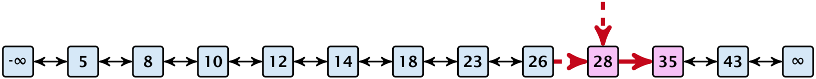

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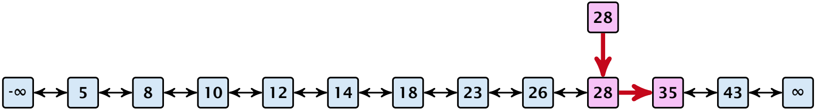

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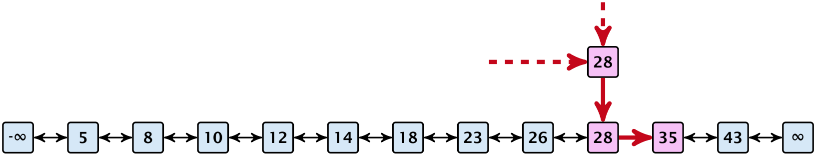

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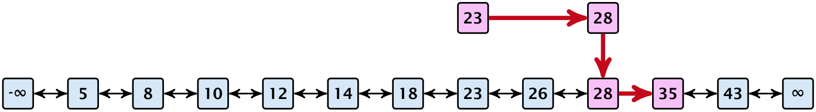

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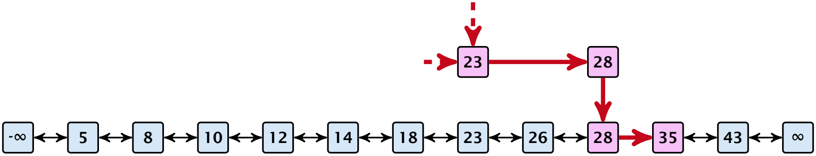

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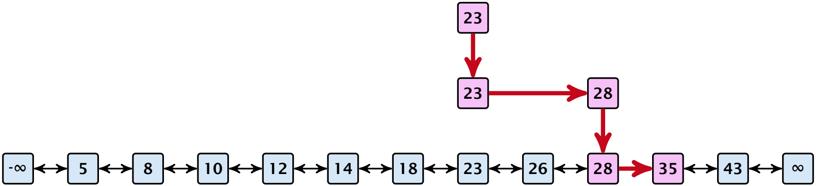

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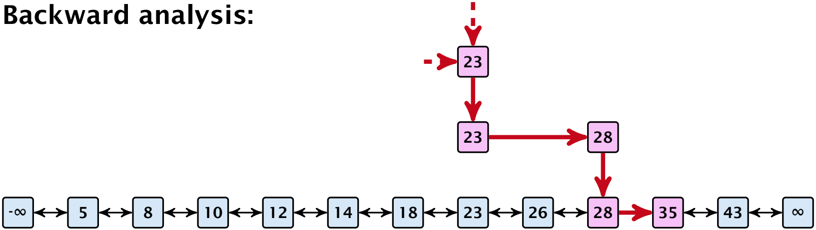

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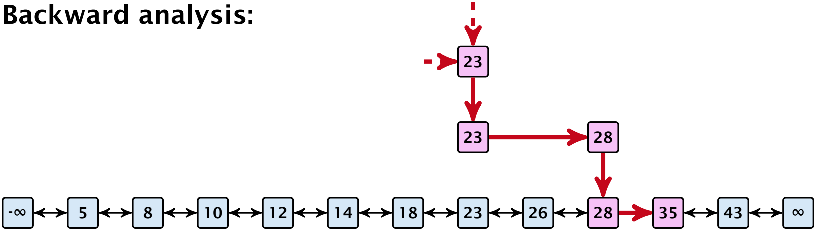

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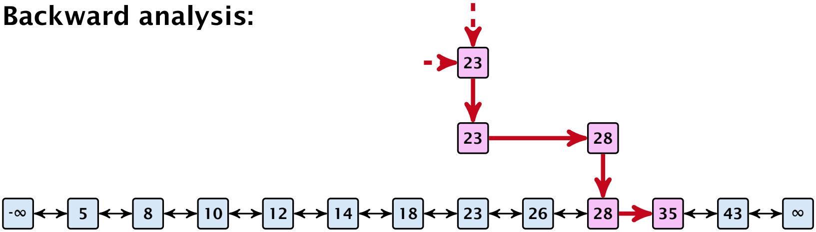

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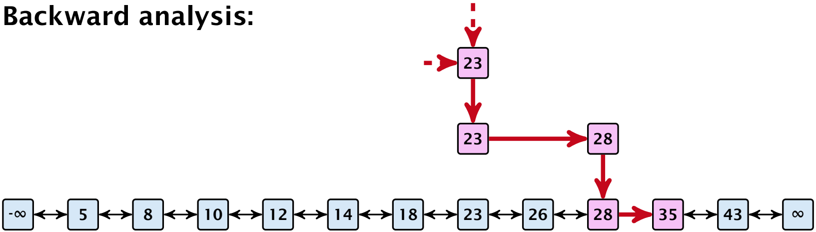

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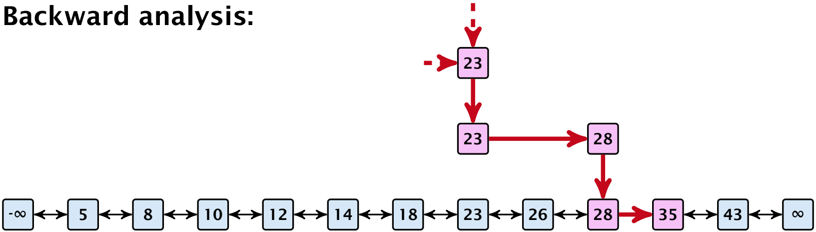

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We show that w.h.p:

- A “long” search path must also go very high.
- There are no elements in high lists.

From this it follows that w.h.p. there are no long paths.

7.5 Skip Lists

Estimation for Binomial Coefficients

$$\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \left(\frac{en}{k}\right)^k$$

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In particular, this means that during the construction in the backward analysis we see at most k heads (i.e., coin flips that tell you to go up) in z trials.

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7.5 Skip Lists

So far we fixed $k = \gamma \log n$, $\gamma \geq 1$, and $z = 7\alpha\gamma \log n$, $\alpha \geq 1$.

This means that a search path of length $\Omega(\log n)$ visits a list on a level $\Omega(\log n)$, w.h.p.

Let A_{k+1} denote the event that the list L_{k+1} is non-empty. Then

$$\Pr[A_{k+1}] \leq n2^{-(k+1)} \leq n^{-(\gamma-1)} \ .$$

For the search to take at least $z = 7\alpha\gamma \log n$ steps either the event $E_{z,k}$ or the event A_{k+1} must hold.
Hence,

$$\begin{aligned}\Pr[\text{search requires } z \text{ steps}] &\leq \Pr[E_{z,k}] + \Pr[A_{k+1}] \\ &\leq n^{-\alpha} + n^{-(\gamma-1)}\end{aligned}$$

This means, the search requires at most z steps, w. h. p.