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Sometimes we also have

- ▶ **S . merge(S')**: $S := S \cup S'$; $S' := \emptyset$.

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- ▶ **S . delete(h)**: Deletes element specified through handle h .
- ▶ **S . decrease-key(h, k)**: Decreases the key of the element specified by handle h to k . Assumes that the key is at least k before the operation.

Dijkstra's Shortest Path Algorithm

Algorithm 1 Shortest-Path($G = (V, E, d), s \in V$)

```
1: Input: weighted graph  $G = (V, E, d)$ ; start vertex  $s$ ;  
2: Output: key-field of every node contains distance from  $s$ ;  
3:  $S.build()$ ; // build empty priority queue  
4: for all  $v \in V \setminus \{s\}$  do  
5:      $v.key \leftarrow \infty$ ;  
6:      $h_v \leftarrow S.insert(v)$ ;  
7:  $s.key \leftarrow 0$ ;  $S.insert(s)$ ;  
8: while  $S.is-empty() = false$  do  
9:      $v \leftarrow S.delete-min()$ ;  
10:    for all  $x \in V$  s.t.  $(v, x) \in E$  do  
11:        if  $x.key > v.key + d(v, x)$  then  
12:             $S.decrease-key(h_x, v.key + d(v, x))$ ;  
13:             $x.key \leftarrow v.key + d(v, x)$ ;
```

Prim's Minimum Spanning Tree Algorithm

Algorithm 2 Prim-MST($G = (V, E, d), s \in V$)

```
1: Input: weighted graph  $G = (V, E, d)$ ; start vertex  $s$ ;  
2: Output: pred-fields encode MST;  
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13:             $x.key \leftarrow d(v, x)$ ;  
14:             $x.pred \leftarrow v$ ;
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Analysis of Dijkstra and Prim

Both algorithms require:

- ▶ 1 build() operation
- ▶ $|V|$ insert() operations
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How good a running time can we obtain?

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<i>Operation</i>	<i>Binary Heap</i>	<i>BST</i>	<i>Binomial Heap</i>	<i>Fibonacci Heap*</i>
build	n	$n \log n$	$n \log n$	n
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n^{**}$	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
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Note that most applications use **build()** only to create an empty heap which then costs time 1.

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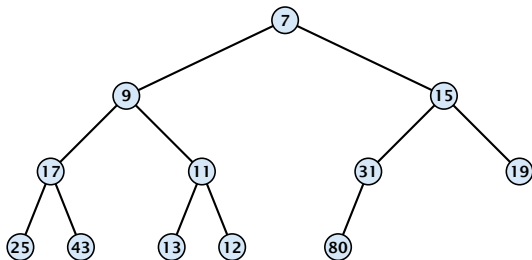
Fibonacci heaps only give an **amortized** guarantee.

8 Priority Queues

Using Binary Heaps, Prim and Dijkstra run in time $\mathcal{O}((|V| + |E|) \log |V|)$.

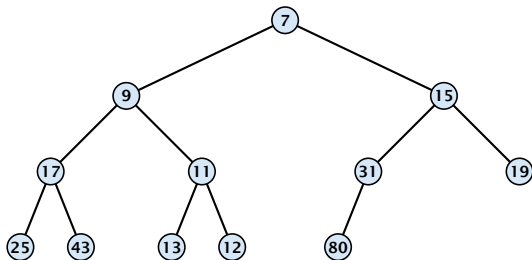
Using Fibonacci Heaps, Prim and Dijkstra run in time $\mathcal{O}(|V| \log |V| + |E|)$.

8.1 Binary Heaps



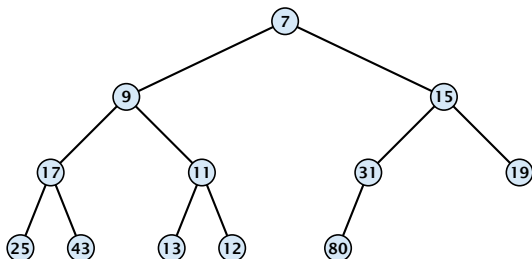
8.1 Binary Heaps

- ▶ Nearly complete binary tree; only the last level is not full, and this one is filled from left to right.



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- ▶ Nearly complete binary tree; only the last level is not full, and this one is filled from left to right.
- ▶ **Heap property:** A node's key is not larger than the key of one of its children.



Operations:

Binary Heaps

Operations:

- ▶ **minimum()**: return the root-element. Time $\mathcal{O}(1)$.

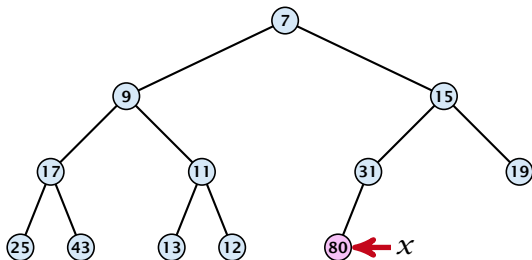
Binary Heaps

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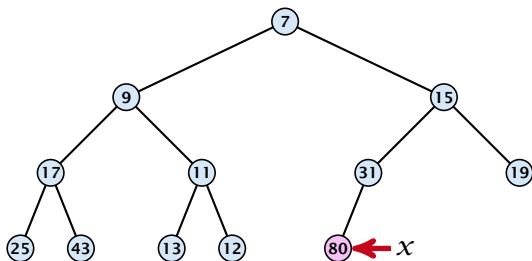
Maintain a pointer to the **last element** x .



8.1 Binary Heaps

Maintain a pointer to the **last element** x .

- ▶ We can compute the predecessor of x (last element when x is deleted) in time $\mathcal{O}(\log n)$.



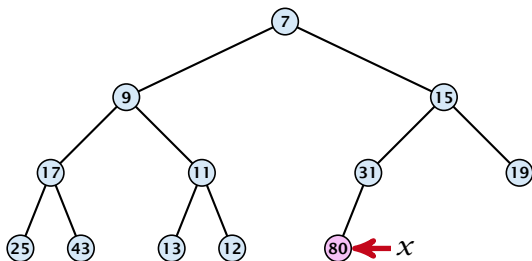
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go up until the last edge used was a right edge.

go left; go right until you reach a leaf



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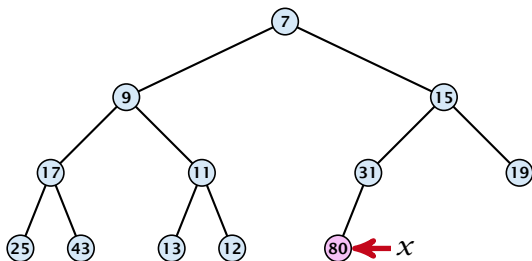
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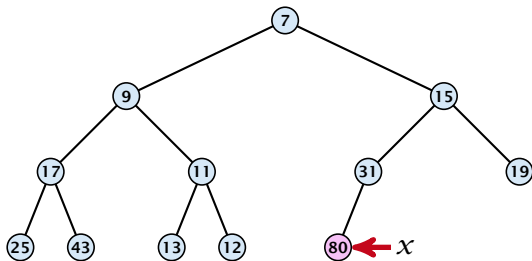
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if you hit the root on the way up, go to the rightmost element



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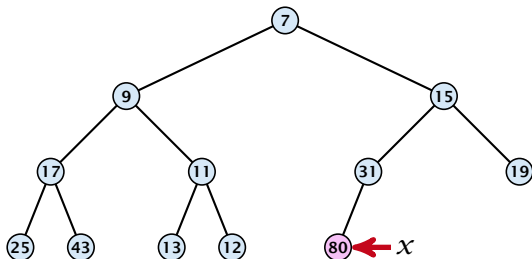
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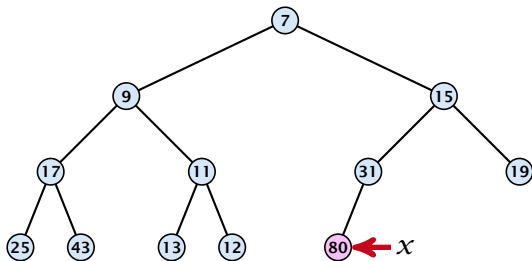
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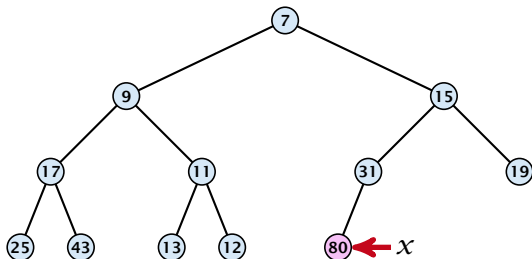
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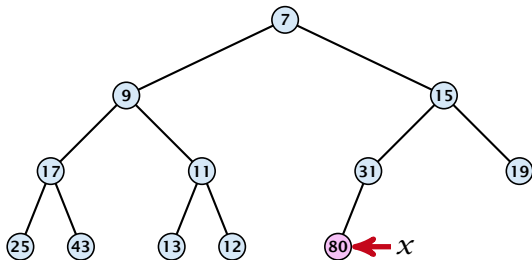
if you hit the root on the way up, go to the leftmost element;

insert a new element as a left child;



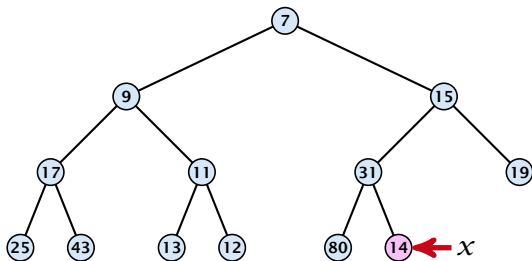
Insert

1. Insert element at successor of x .



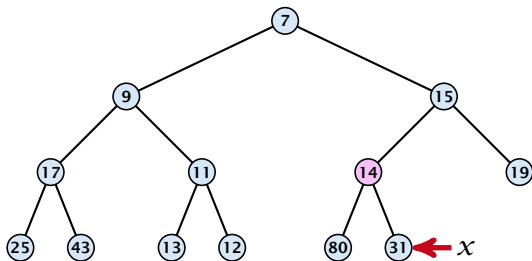
Insert

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2. Exchange with parent until heap property is fulfilled.



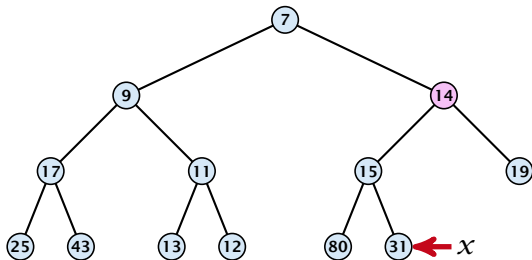
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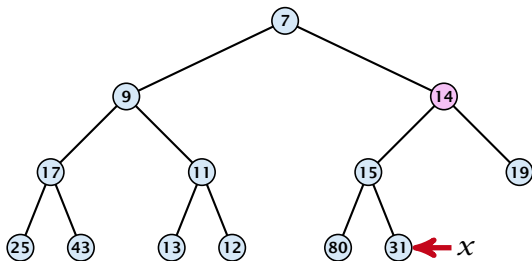
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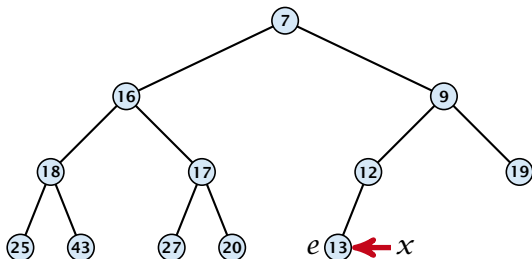
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Note that an exchange can either be done by moving the data or by changing pointers. The latter method leads to an addressable priority queue.

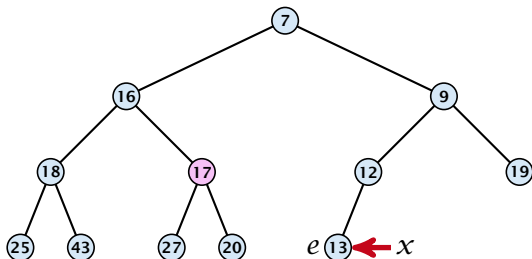
Delete

1. Exchange the element to be deleted with the element e pointed to by x .



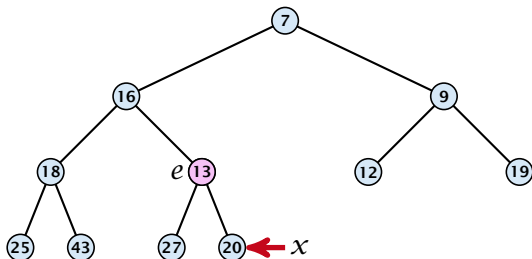
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2. Restore the heap-property for the element e .



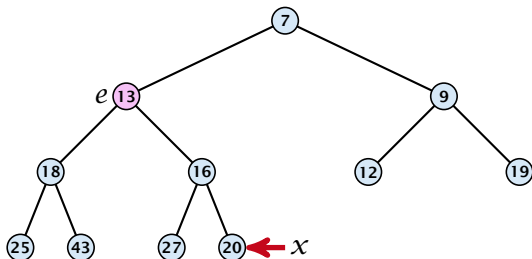
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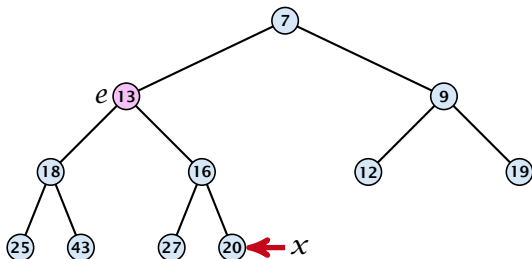
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At its new position e may either travel up or down in the tree (but not both directions).

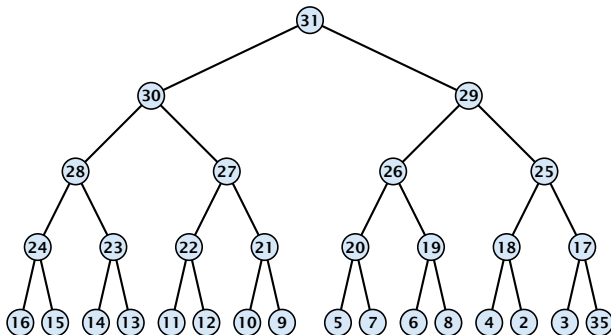
Binary Heaps

Operations:

- ▶ **minimum()**: return the root-element. Time $\mathcal{O}(1)$.
- ▶ **is-empty()**: check whether root-pointer is **null**. Time $\mathcal{O}(1)$.
- ▶ **insert(k)**: insert at successor of x and bubble up. Time $\mathcal{O}(\log n)$.
- ▶ **delete(h)**: swap with x and bubble up or sift-down. Time $\mathcal{O}(\log n)$.

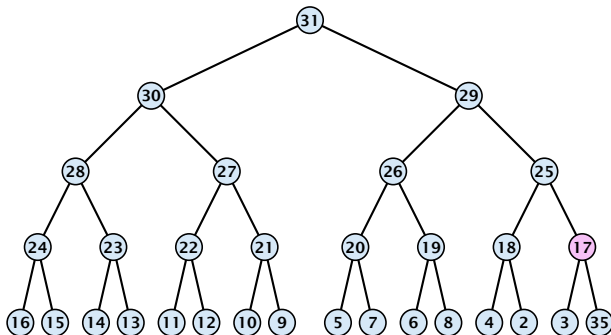
Build Heap

We can build a heap in linear time:



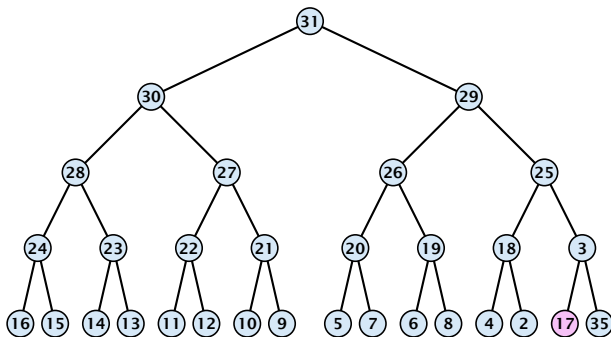
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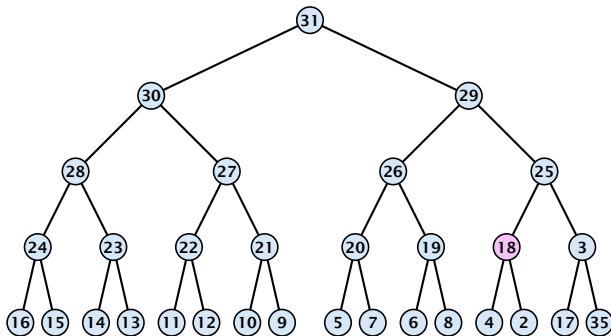
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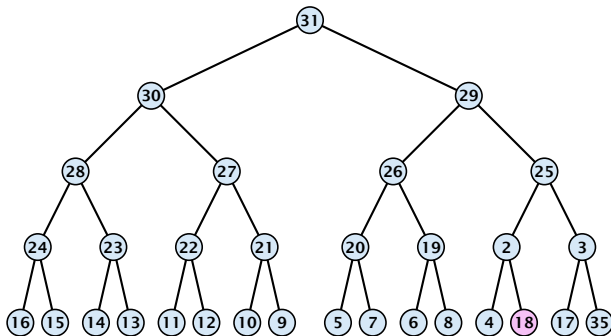
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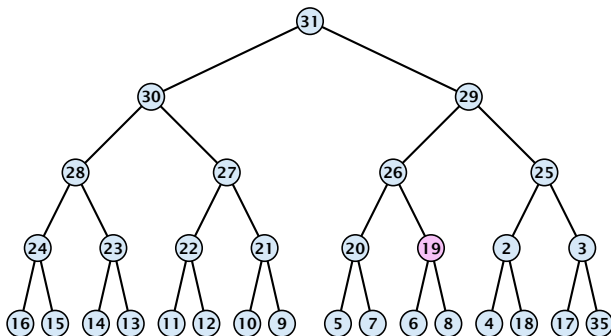
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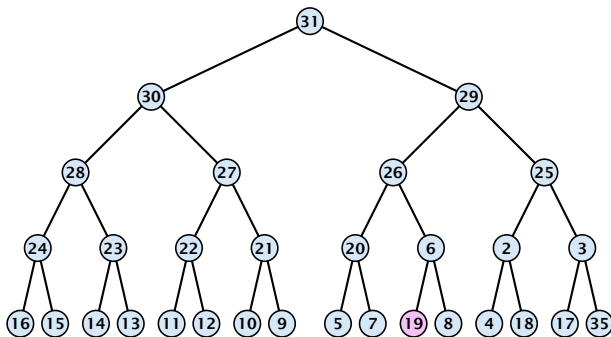
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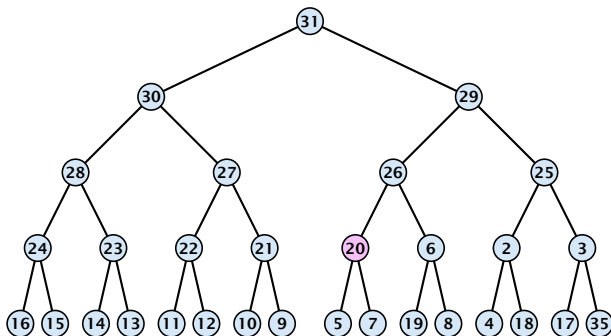
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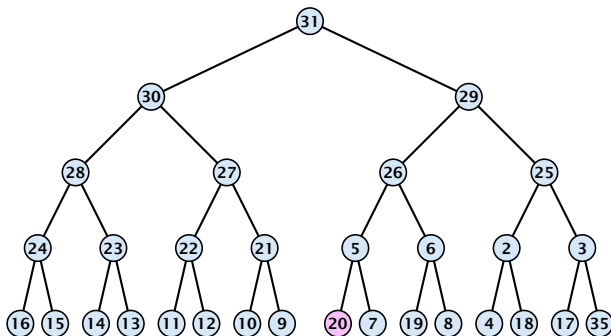
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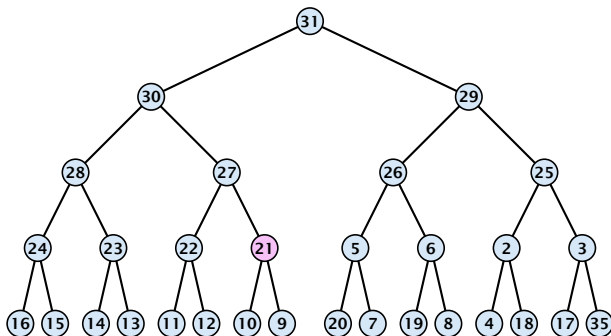
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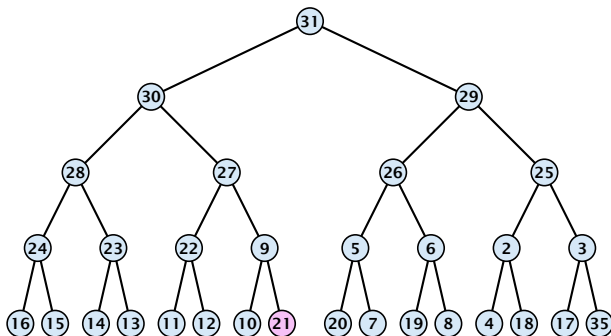
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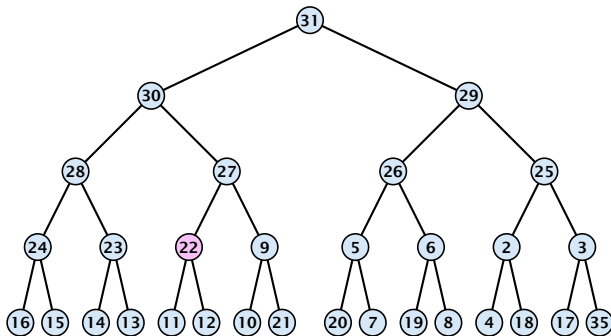
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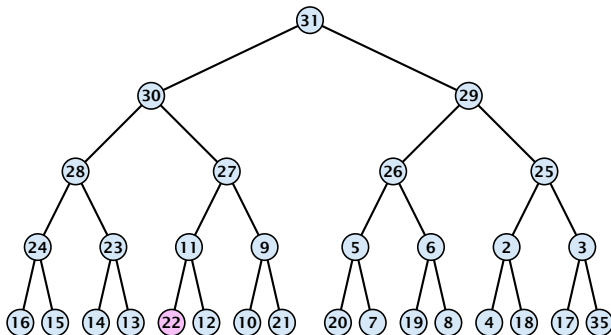
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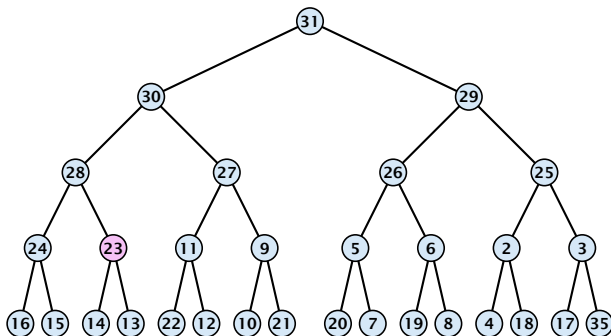
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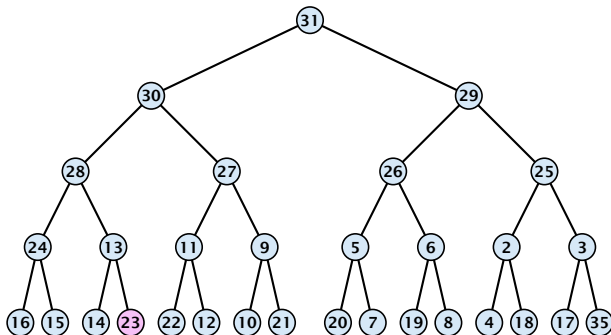
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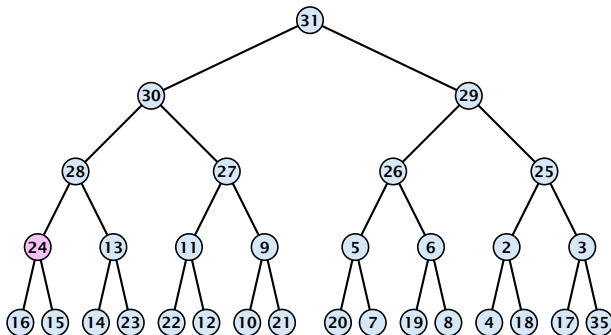
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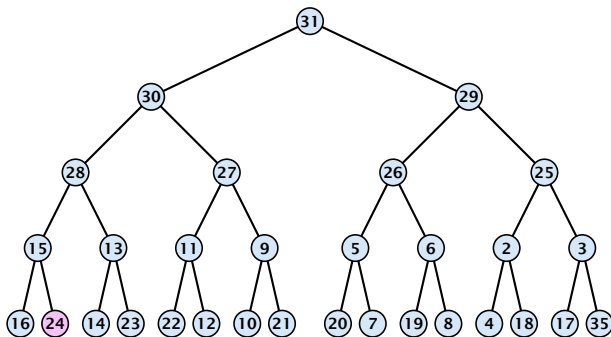
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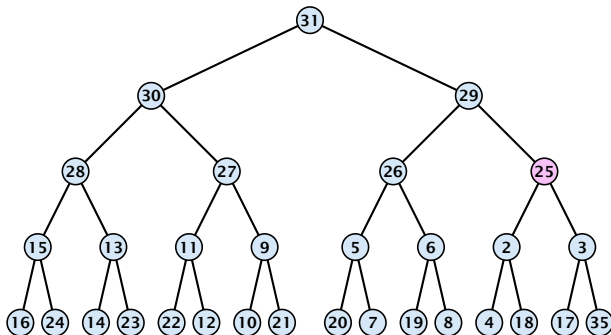
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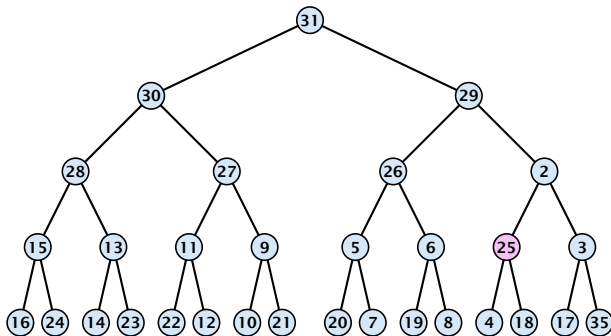
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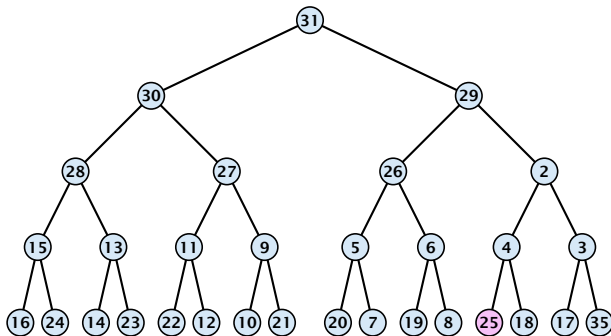
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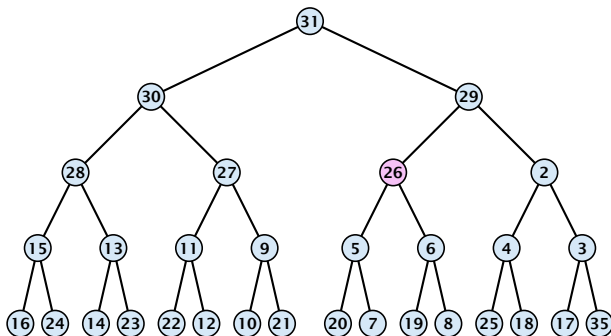
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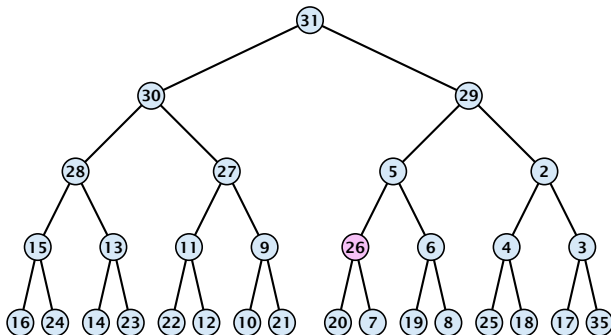
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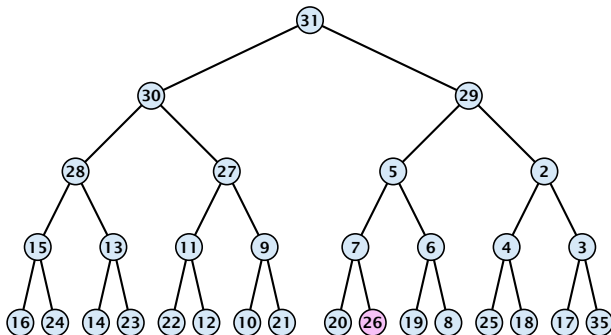
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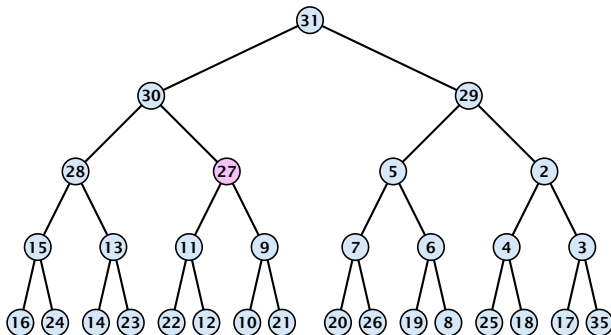
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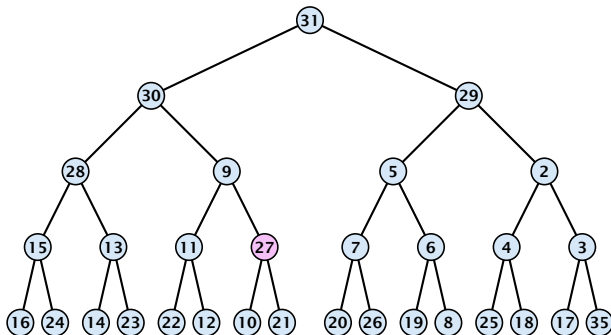
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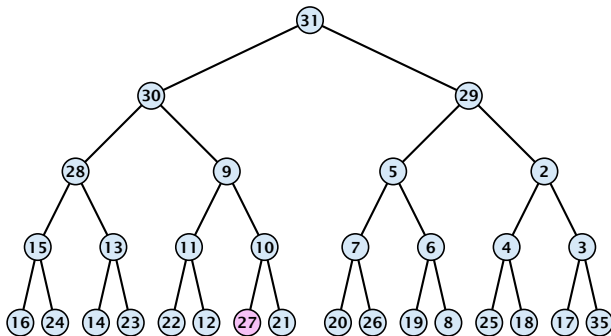
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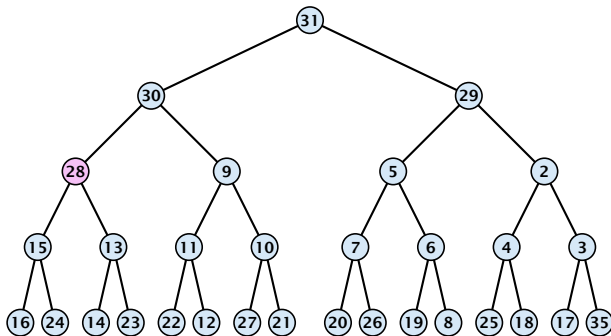
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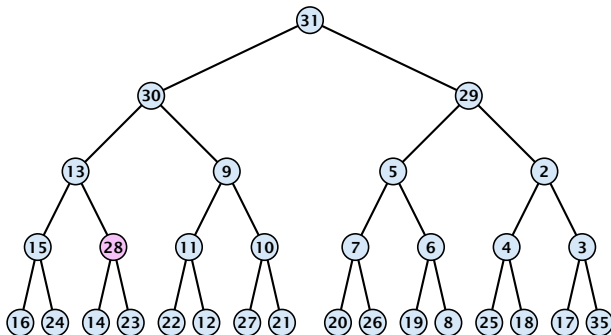
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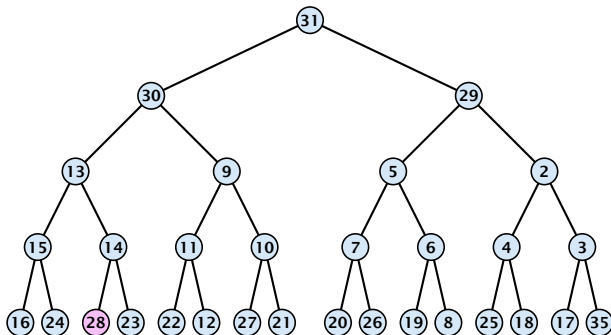
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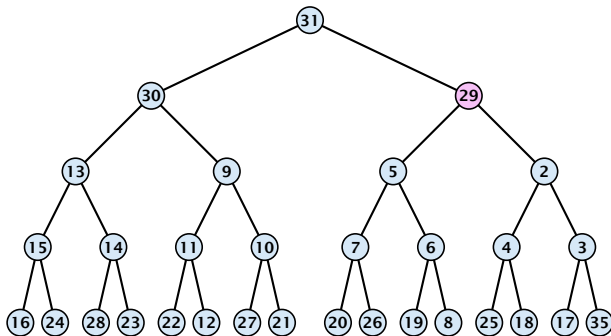
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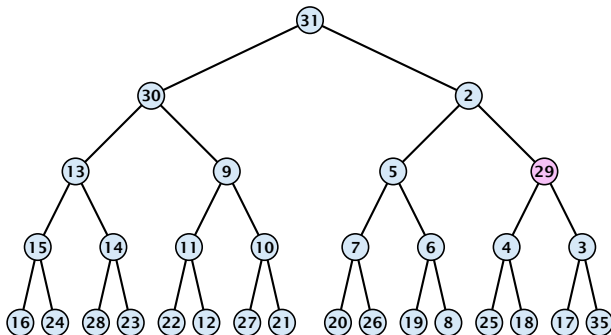
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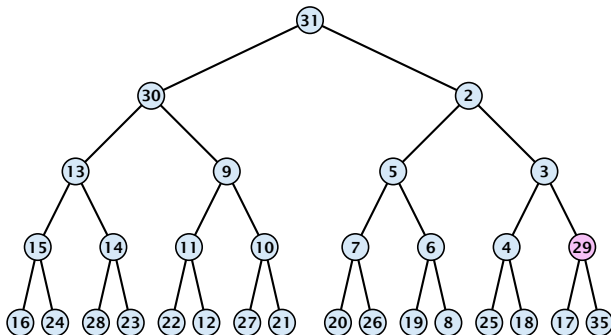
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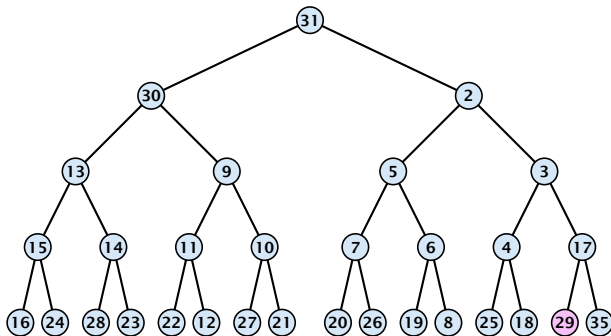
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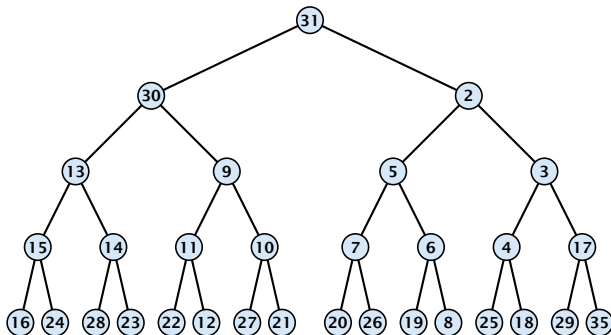
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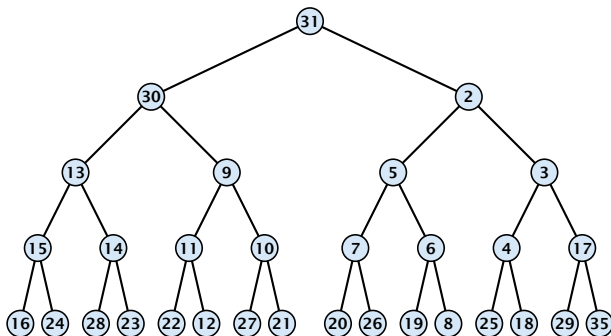
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Build Heap

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$$\sum_{\text{levels } \ell} 2^\ell \cdot (h - \ell) = \sum_i i 2^{h-i} = \mathcal{O}(2^h) = \mathcal{O}(n)$$

Binary Heaps

Operations:

- ▶ **minimum()**: Return the root-element. Time $\mathcal{O}(1)$.
- ▶ **is-empty()**: Check whether root-pointer is **null**. Time $\mathcal{O}(1)$.
- ▶ **insert(k)**: Insert at x and bubble up. Time $\mathcal{O}(\log n)$.
- ▶ **delete(h)**: Swap with x and bubble up or sift-down. Time $\mathcal{O}(\log n)$.
- ▶ **build(x_1, \dots, x_n)**: Insert elements arbitrarily; then do sift-down operations starting with the lowest layer in the tree. Time $\mathcal{O}(n)$.

Binary Heaps

Binary Heaps

The standard implementation of binary heaps is via arrays. Let $A[0, \dots, n - 1]$ be an array

- ▶ The parent of i -th element is at position $\lfloor \frac{i-1}{2} \rfloor$.
- ▶ The left child of i -th element is at position $2i + 1$.
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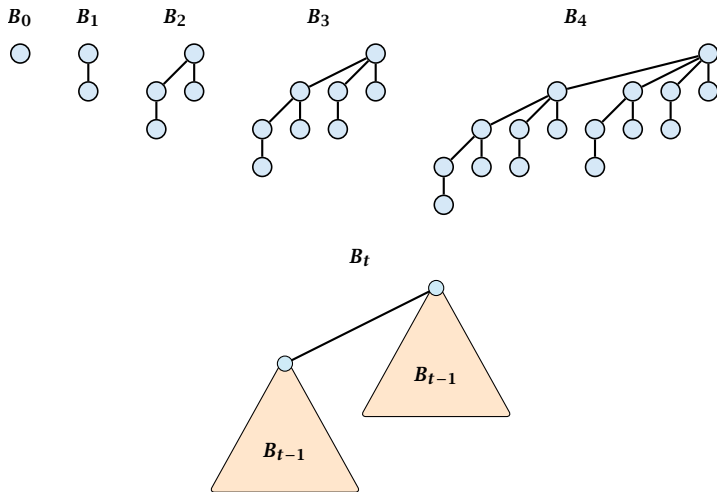
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The resulting binary heap is not addressable. The elements don't maintain their positions and therefore there are no stable handles.

8.2 Binomial Heaps

<i>Operation</i>	<i>Binary Heap</i>	<i>BST</i>	<i>Binomial Heap</i>	<i>Fibonacci Heap*</i>
build	n	$n \log n$	$n \log n$	n
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n^{**}$	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	n	$n \log n$	$\log n$	1

Binomial Trees



Binomial Trees

Properties of Binomial Trees

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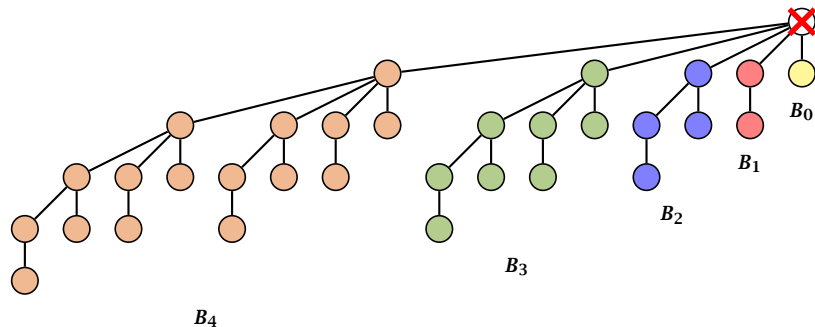
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Properties of Binomial Trees

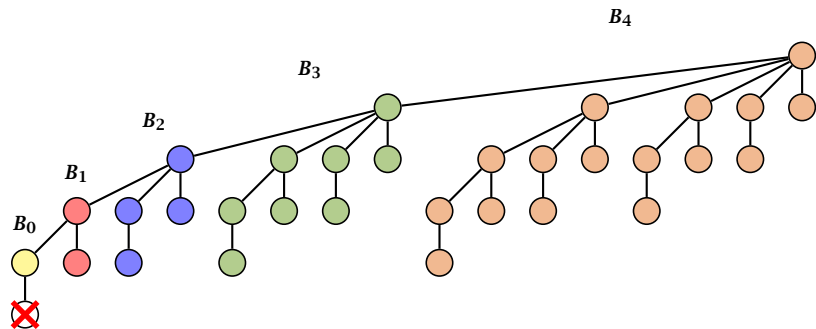
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- ▶ The root of B_k has degree k .
- ▶ B_k has $\binom{k}{\ell}$ nodes on level ℓ .
- ▶ Deleting the root of B_k gives trees B_0, B_1, \dots, B_{k-1} .

Binomial Trees



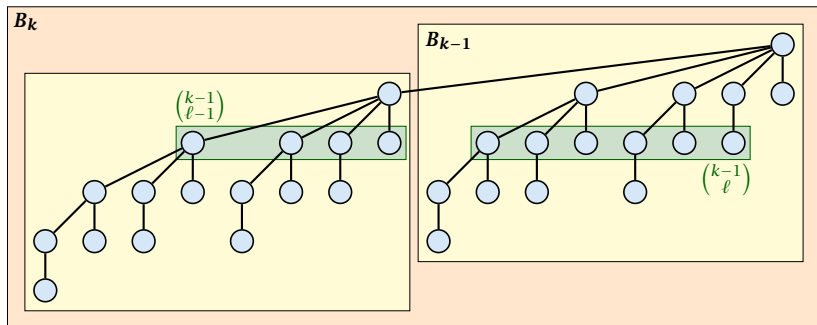
Deleting the root of B_5 leaves sub-trees B_4 , B_3 , B_2 , B_1 , and B_0 .

Binomial Trees



Deleting the leaf furthest from the root (in B_5) leaves a path that connects the roots of sub-trees B_4 , B_3 , B_2 , B_1 , and B_0 .

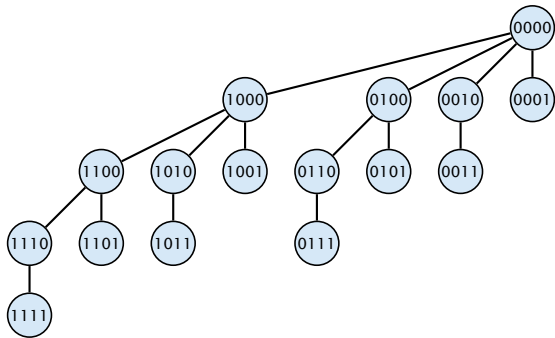
Binomial Trees



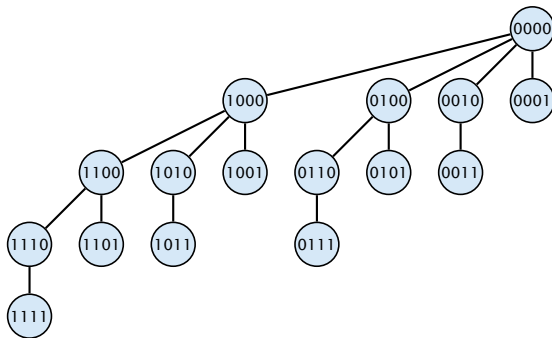
The number of nodes on level ℓ in tree B_k is therefore

$$\binom{k-1}{\ell-1} + \binom{k-1}{\ell} = \binom{k}{\ell}$$

Binomial Trees

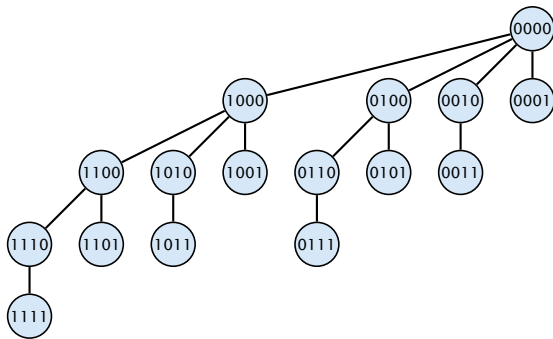


Binomial Trees



The binomial tree B_k is a sub-graph of the hypercube H_k .

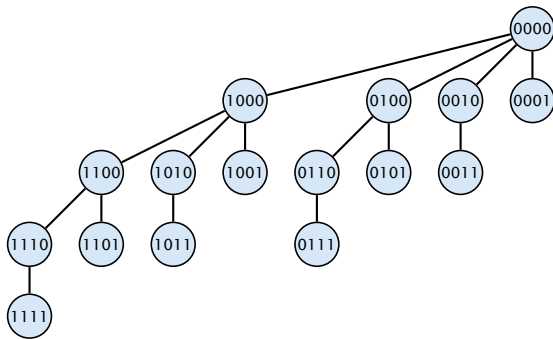
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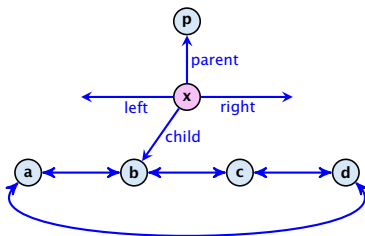
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The ℓ -th level contains nodes that have ℓ 1's in their label.

8.2 Binomial Heaps

How do we implement trees with non-constant degree?

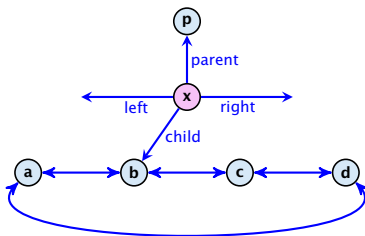
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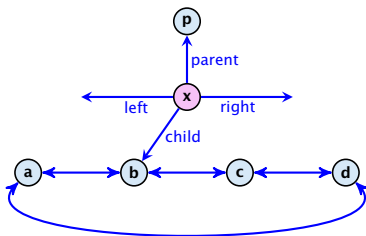
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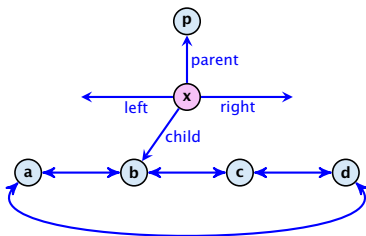
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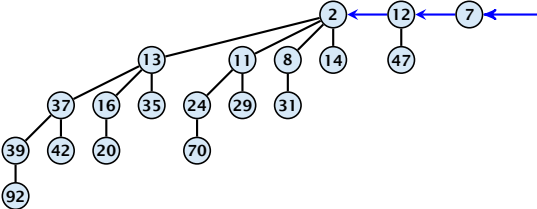
- ▶ The children of a node are arranged in a **circular linked list**.
- ▶ A child-pointer points to an arbitrary node within the list.
- ▶ A parent-pointer points to the parent node.
- ▶ Pointers $x.left$ and $x.right$ point to the left and right sibling of x (if x does not have siblings then $x.left = x.right = x$).



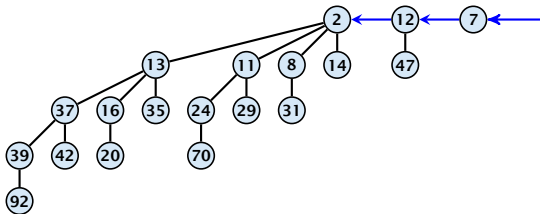
8.2 Binomial Heaps

- ▶ Given a pointer to a node x we can splice out the sub-tree rooted at x in constant time.
- ▶ We can add a child-tree T to a node x in constant time if we are given a pointer to x and a pointer to the root of T .

Binomial Heap

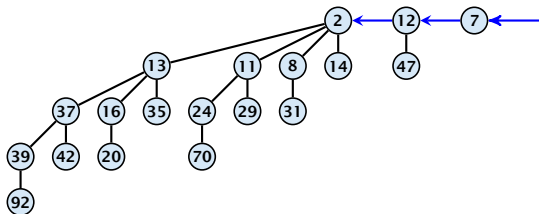


Binomial Heap



In a binomial heap the keys are arranged in a collection of binomial trees.

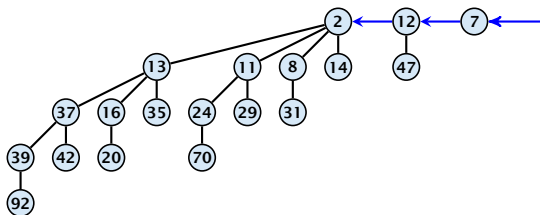
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There is at most one tree for every dimension/order. For example the above heap contains trees B_0 , B_1 , and B_4 .

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Let $B_{k_1}, B_{k_2}, B_{k_3}, k_i < k_{i+1}$ denote the binomial trees in the collection and recall that every tree may be contained at most once.

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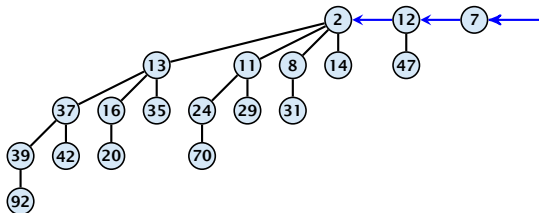
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Then $n = \sum_i 2^{k_i}$ must hold. But since the k_i are all distinct this means that the k_i define the non-zero bit-positions in the binary representation of n .

Binomial Heap

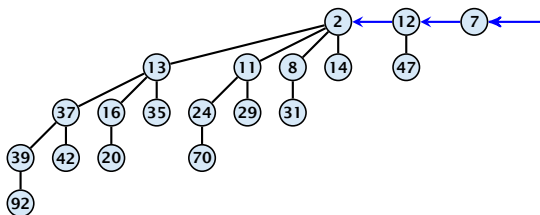
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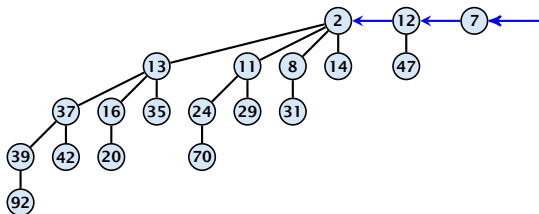
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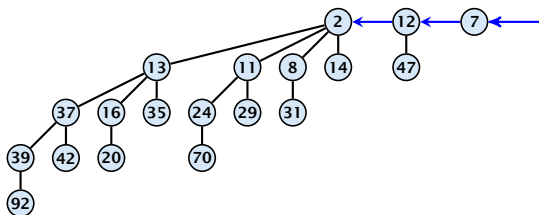
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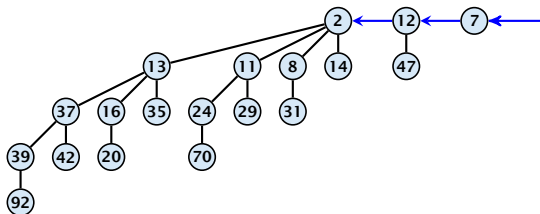
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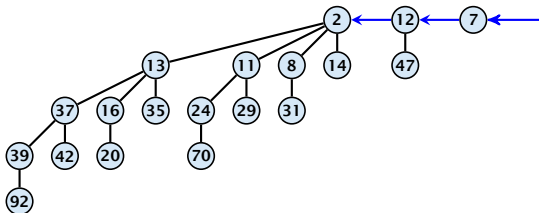
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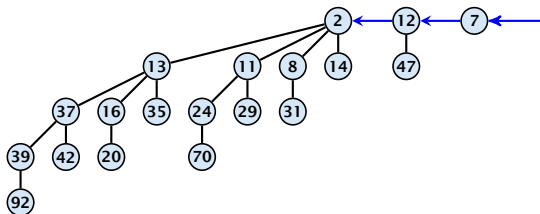
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- ▶ The trees are stored in a single-linked list; ordered by dimension/size.



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Otherwise, we cannot do this because the merged heap is not allowed to contain two trees of the same order.

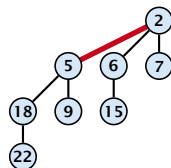
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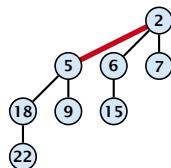
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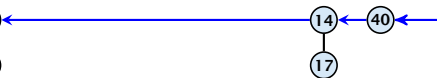
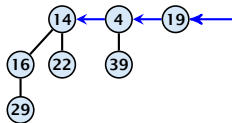
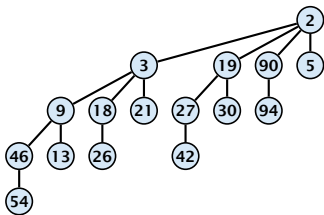
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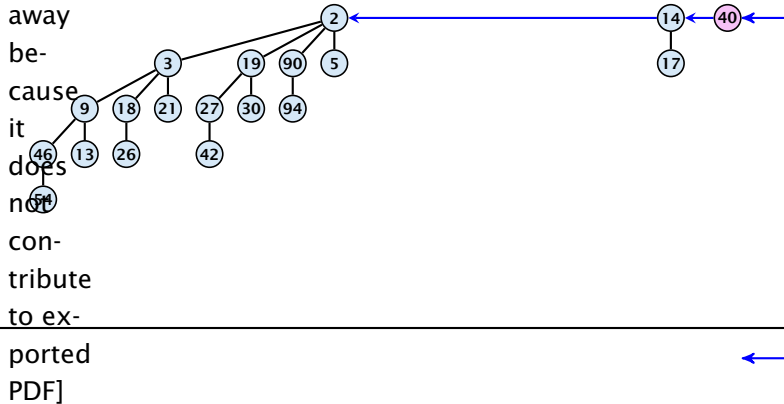
For more trees the technique is analogous to binary addition.





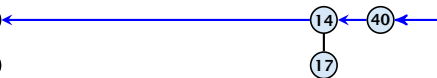
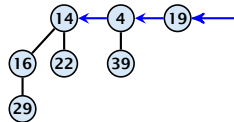
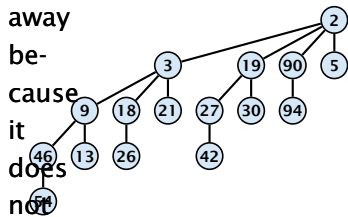
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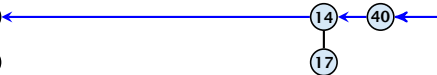
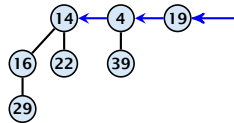
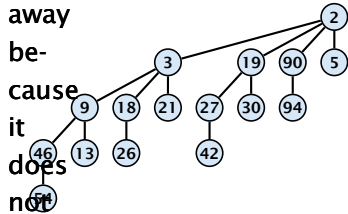
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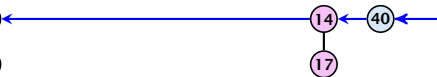
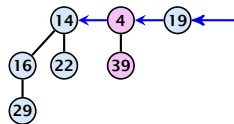
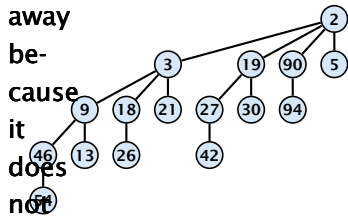
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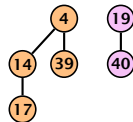
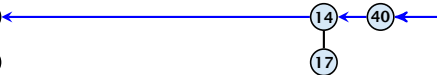
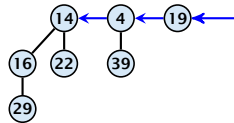
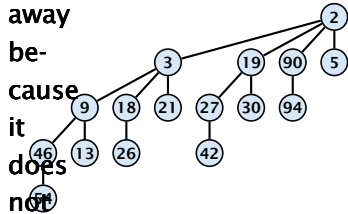
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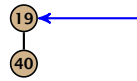
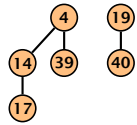
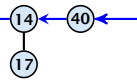
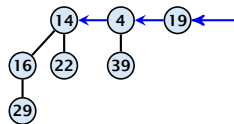
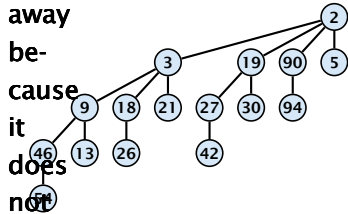
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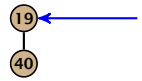
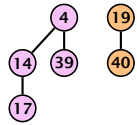
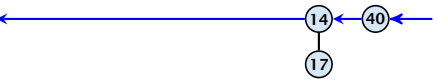
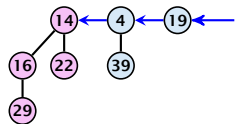
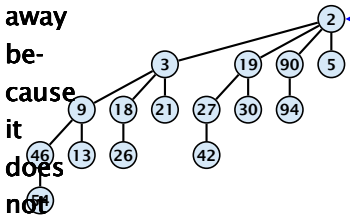
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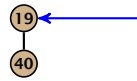
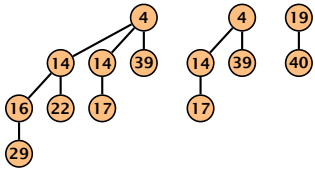
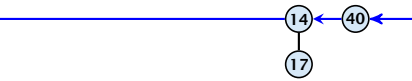
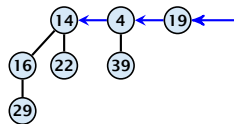
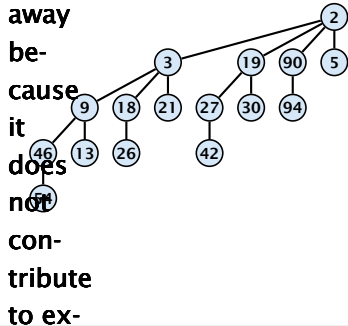
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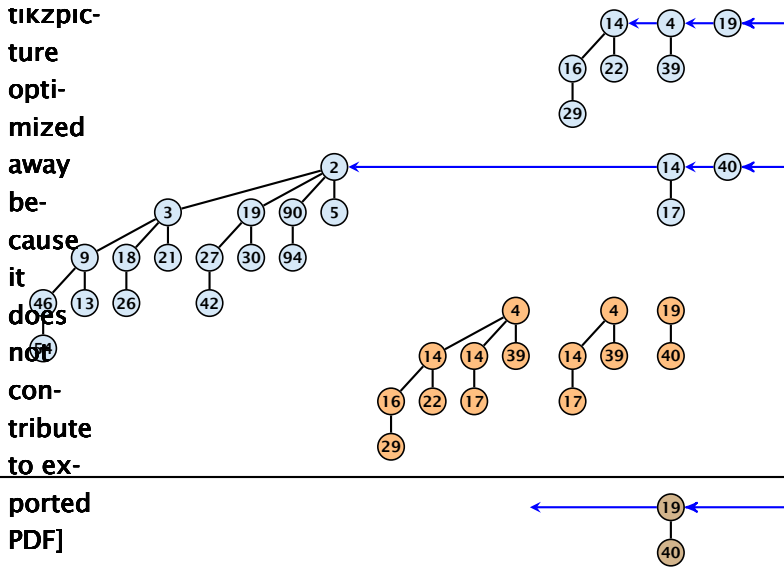
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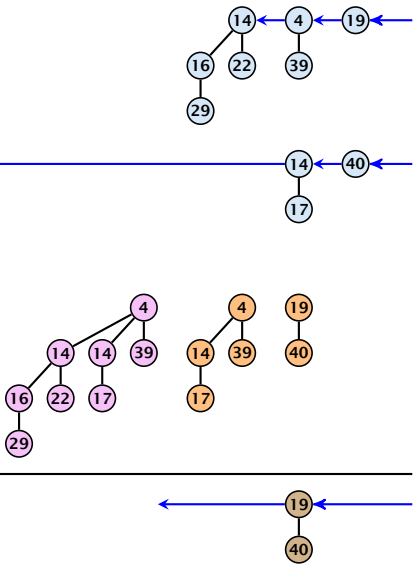
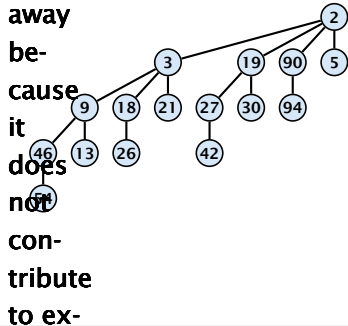
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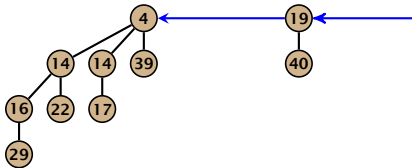
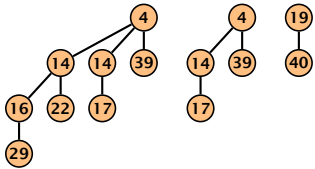
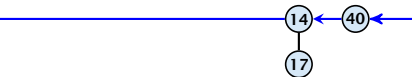
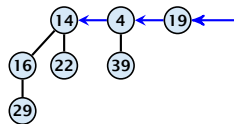
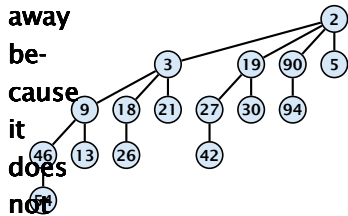
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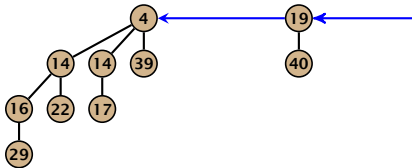
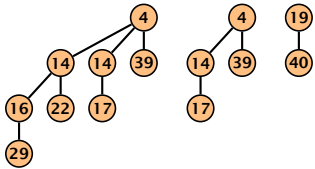
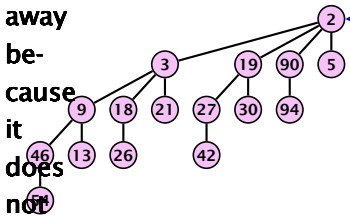
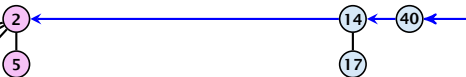
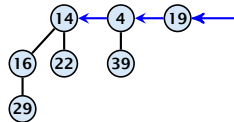
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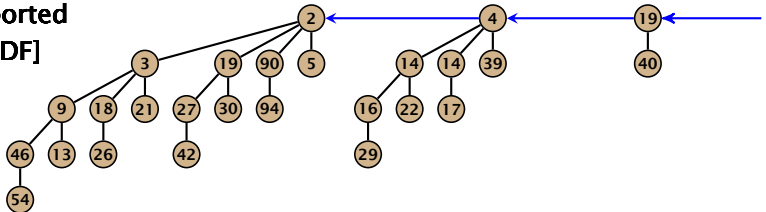
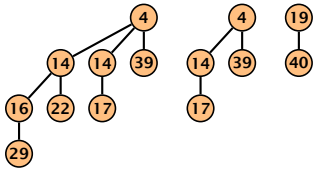
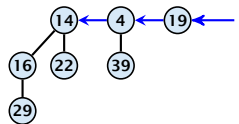
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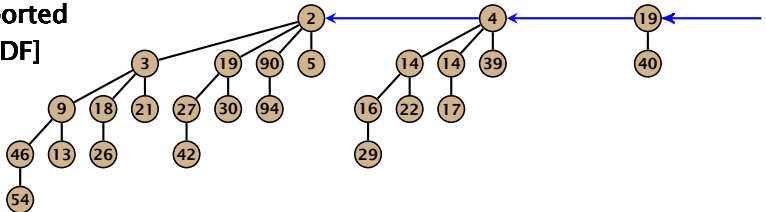
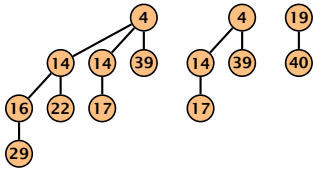
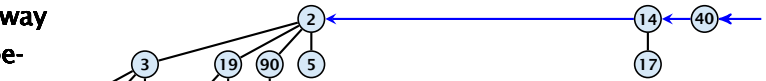
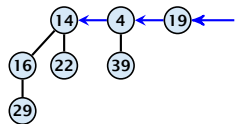
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8.2 Binomial Heaps

S_1 . merge(S_2):

- ▶ Analogous to binary addition.

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8.2 Binomial Heaps

All other operations can be reduced to `merge()`.

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- ▶ Execute `S.merge(S')`.
- ▶ Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

S. minimum():

- ▶ Find the minimum key-value among all roots.
- ▶ Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

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- ▶ Compute $S.\text{merge}(S')$.
- ▶ Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

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8.2 Binomial Heaps

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S. decrease-key(handle h):

- ▶ Decrease the key of the element pointed to by h .
- ▶ Bubble the element up in the tree until the heap property is fulfilled.
- ▶ Time: $\mathcal{O}(\log n)$ since the trees have height $\mathcal{O}(\log n)$.

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8.2 Binomial Heaps

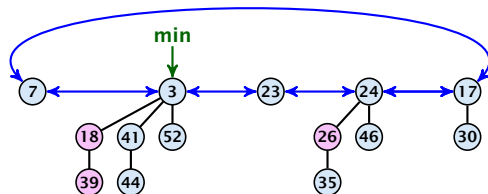
S . delete(handle h):

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- ▶ Execute S . delete-min().
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8.3 Fibonacci Heaps

Collection of trees that fulfill the heap property.

Structure is much more relaxed than binomial heaps.



8.3 Fibonacci Heaps

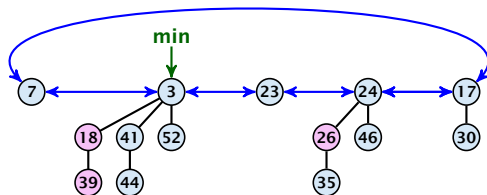
Additional implementation details:

- ▶ Every node x stores its degree in a field $x.degree$. Note that this can be updated in constant time when adding a child to x .
- ▶ Every node stores a boolean value $x.marked$ that specifies whether x is **marked** or not.

8.3 Fibonacci Heaps

The potential function:

- ▶ $t(S)$ denotes the number of trees in the heap.
- ▶ $m(S)$ denotes the number of marked nodes.
- ▶ We use the potential function $\Phi(S) = t(S) + 2m(S)$.



The potential is $\Phi(S) = 5 + 2 \cdot 3 = 11$.

8.3 Fibonacci Heaps

We assume that one unit of potential can pay for a constant amount of work, where the constant is chosen “big enough” (to take care of the constants that occur).

To make this more explicit we use c to denote the amount of work that a unit of potential can pay for.

8.3 Fibonacci Heaps

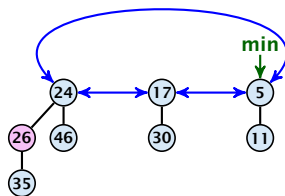
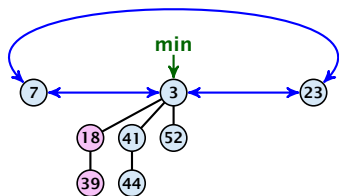
S. minimum()

- ▶ Access through the min-pointer.
- ▶ Actual cost $\mathcal{O}(1)$.
- ▶ No change in potential.
- ▶ Amortized cost $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

S . merge(S')

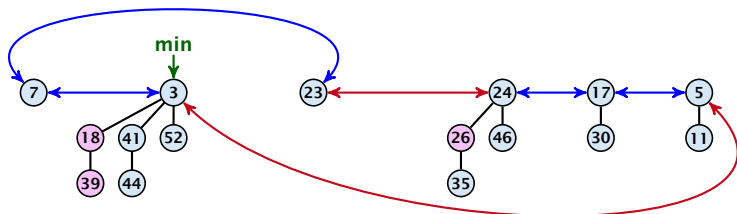
- ▶ Merge the root lists.
- ▶ Adjust the min-pointer



8.3 Fibonacci Heaps

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- ▶ Merge the root lists.
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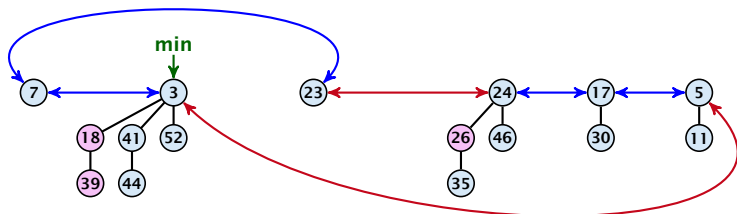
Running time:

- ▶ Actual cost $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

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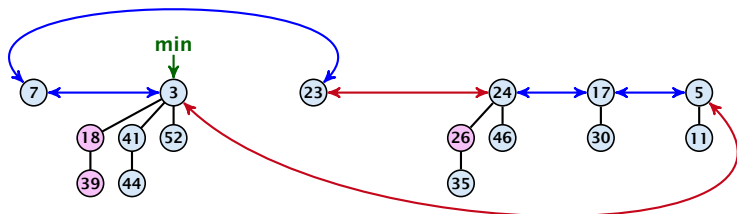
Running time:

- ▶ Actual cost $\mathcal{O}(1)$.
- ▶ No change in potential.

8.3 Fibonacci Heaps

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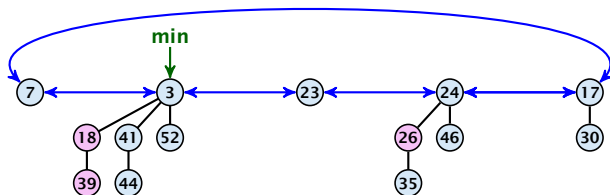
Running time:

- ▶ Actual cost $\mathcal{O}(1)$.
- ▶ No change in potential.
- ▶ Hence, amortized cost is $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

S. insert(x)

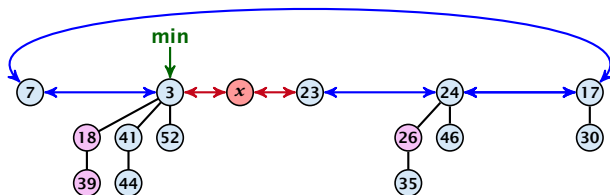
- ▶ Create a new tree containing x .
- ▶ Insert x into the root-list.
- ▶ Update min-pointer, if necessary.



8.3 Fibonacci Heaps

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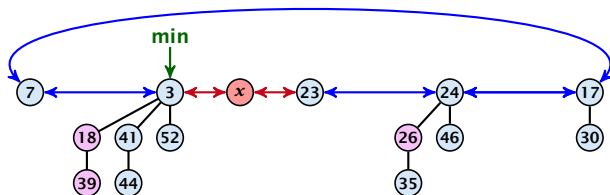
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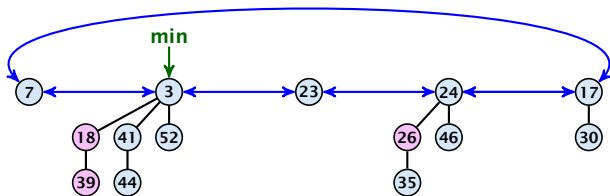


Running time:

- ▶ Actual cost $\mathcal{O}(1)$.
- ▶ Change in potential is $+1$.
- ▶ Amortized cost is $c + \mathcal{O}(1) = \mathcal{O}(1)$.

8.3 Fibonacci Heaps

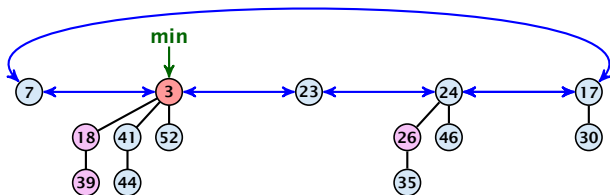
S. delete-min(x)



8.3 Fibonacci Heaps

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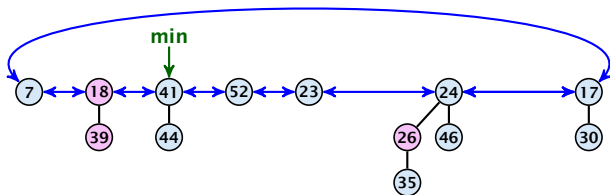
- ▶ Delete minimum; add child-trees to heap;
time: $D(\min) \cdot \mathcal{O}(1)$.



8.3 Fibonacci Heaps

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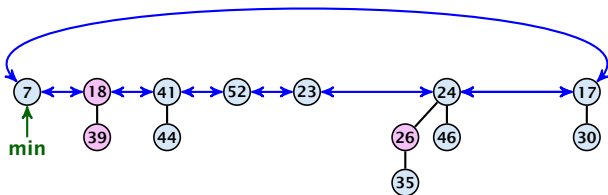
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- ▶ Update min-pointer; time: $(t + D(\min)) \cdot \mathcal{O}(1)$.



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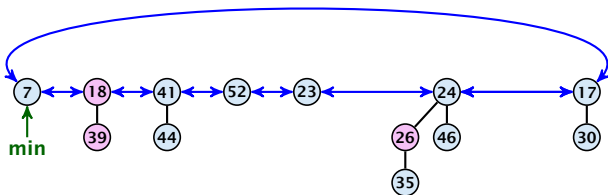
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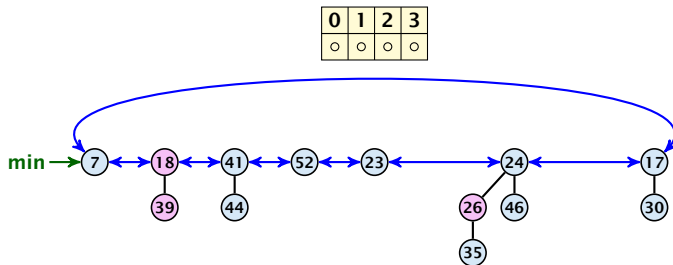
- ▶ Delete minimum; add child-trees to heap; time: $D(\min) \cdot \mathcal{O}(1)$.
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- ▶ Consolidate root-list so that no roots have the same degree. Time $t \cdot \mathcal{O}(1)$ (see next slide).

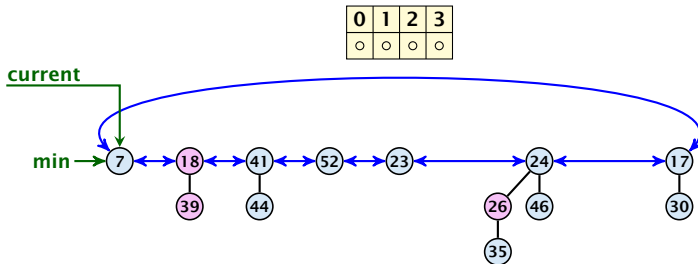
8.3 Fibonacci Heaps

Consolidate:



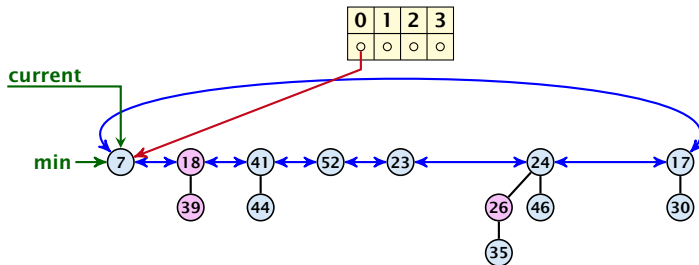
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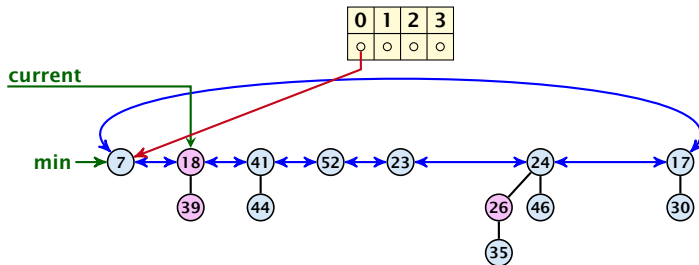
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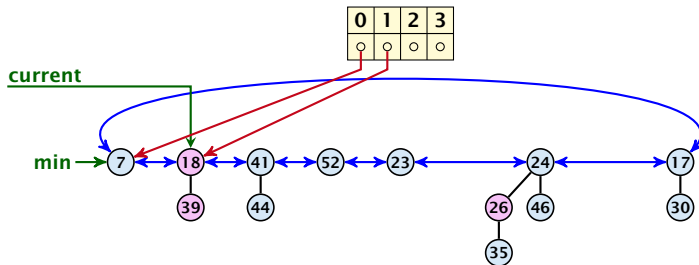
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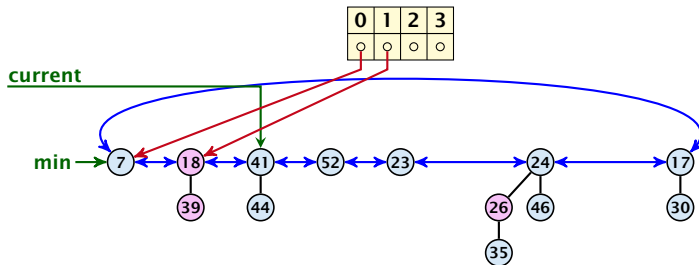
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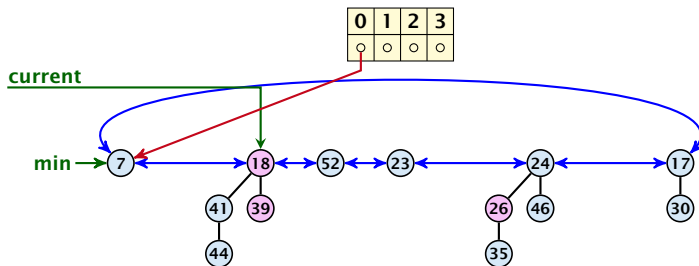
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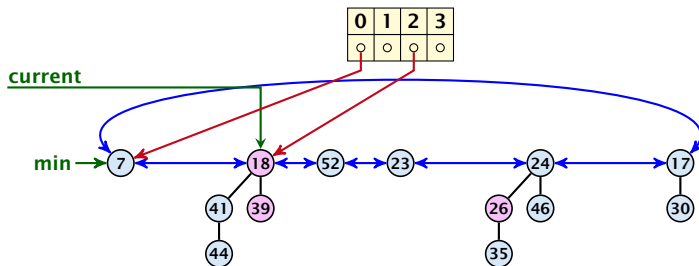
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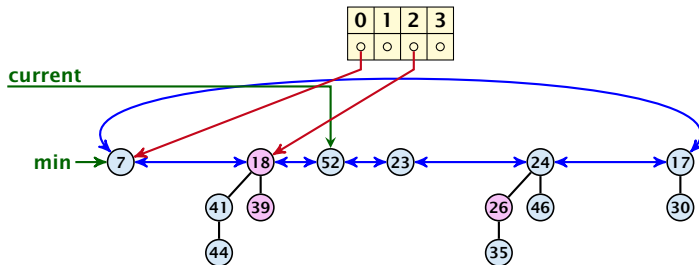
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Consolidate:



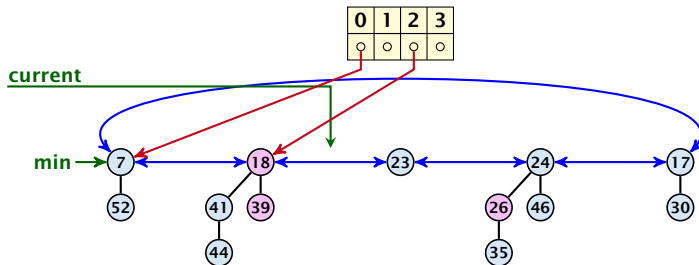
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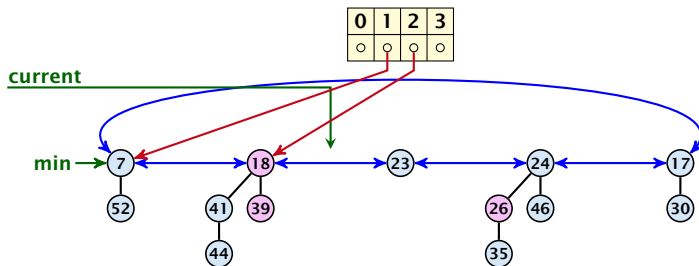
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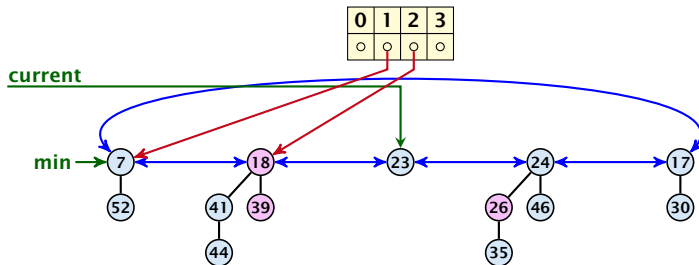
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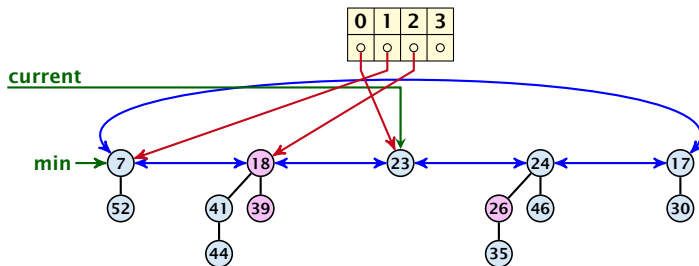
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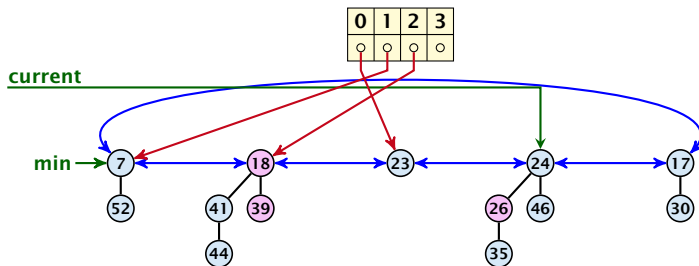
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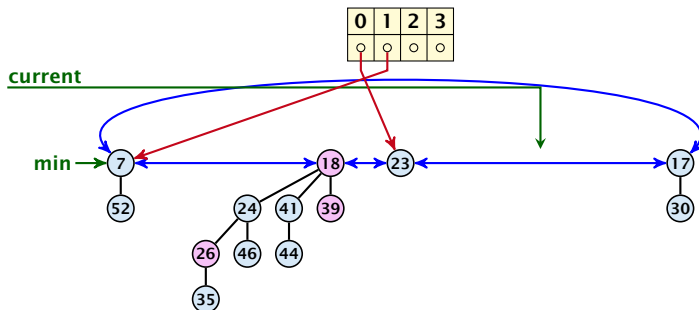
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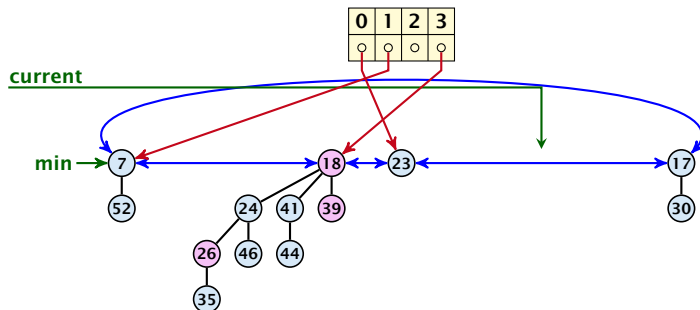
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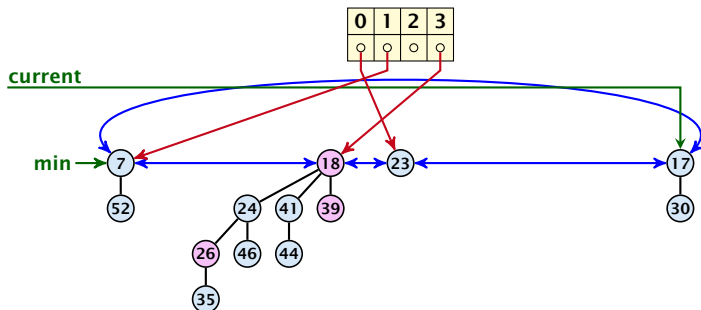
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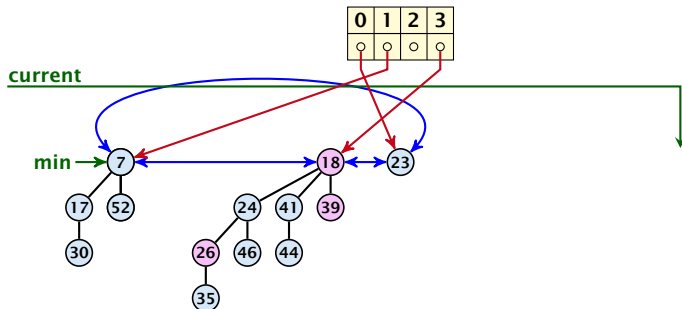
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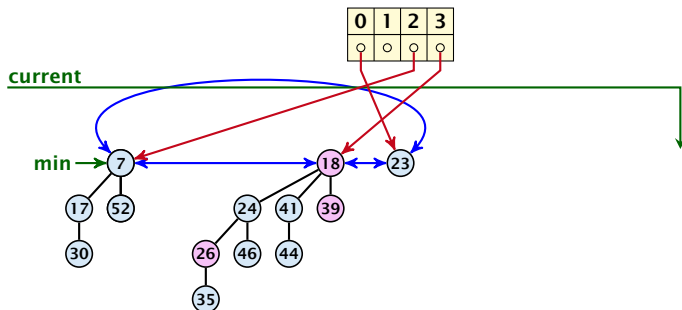
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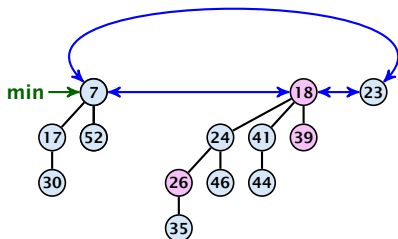
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for $c \geq c_1$.

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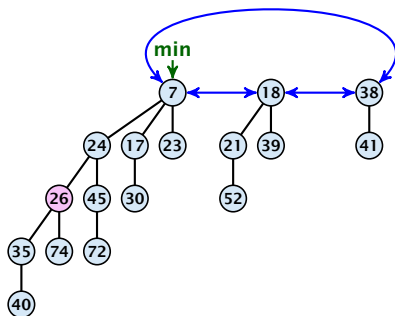
If the input trees of the consolidation procedure are binomial trees (for example only singleton vertices) then the output will be a set of distinct binomial trees, and, hence, the Fibonacci heap will be (more or less) a Binomial heap right after the consolidation.

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If we do not have delete or decrease-key operations then
 $D_n \leq \log n$.

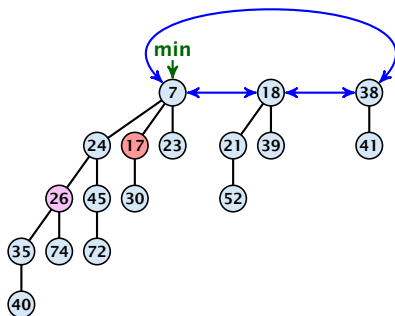
Fibonacci Heaps: decrease-key(handle h, v)



Case 1: decrease-key does not violate heap-property

- ▶ Just decrease the key-value of element referenced by h . Nothing else to do.

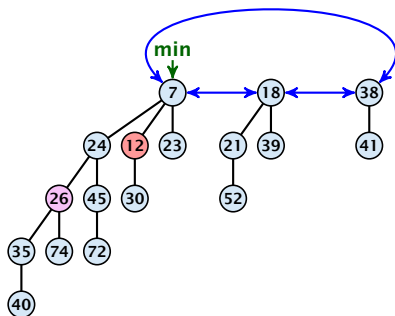
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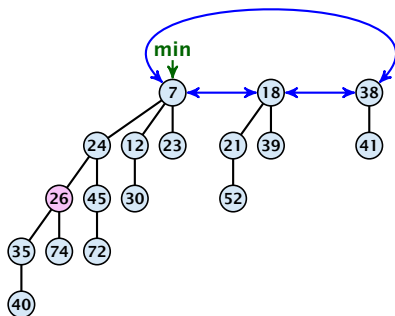
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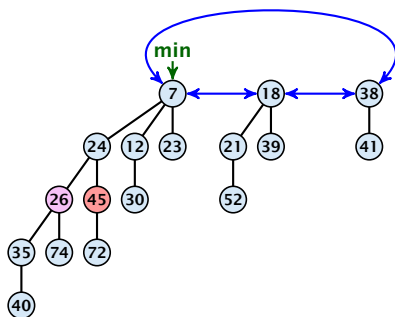
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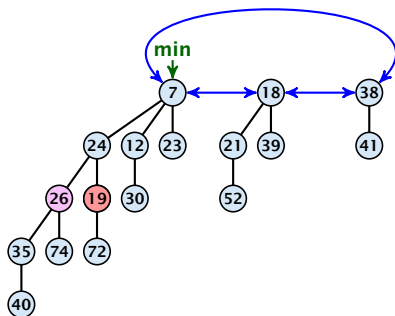
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Case 2: heap-property is violated, but parent is not marked

- ▶ Decrease key-value of element x reference by h .
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- ▶ Adjust min-pointers, if necessary.
- ▶ Mark the (previous) parent of x (unless it's a root).

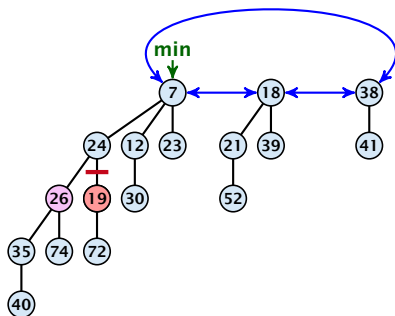
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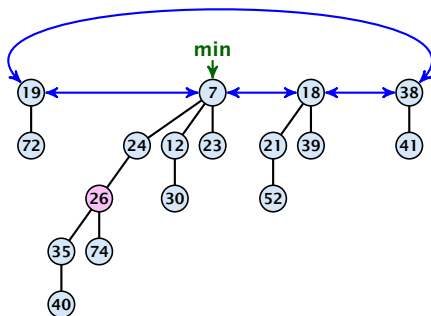
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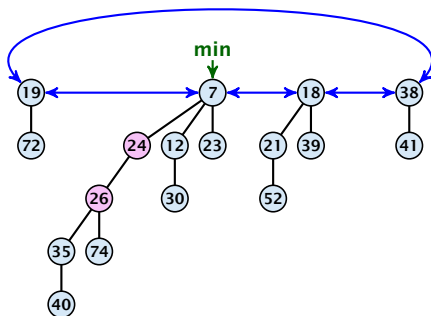
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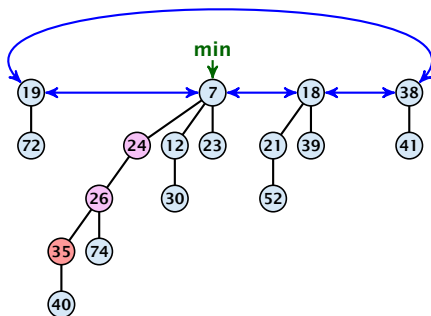
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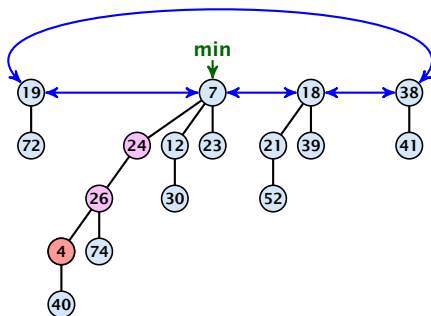
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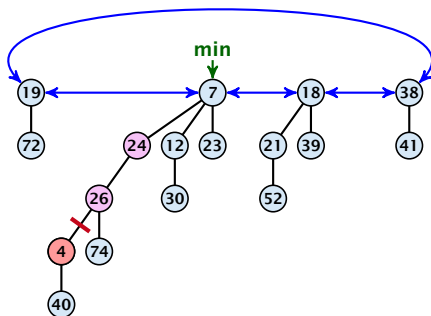
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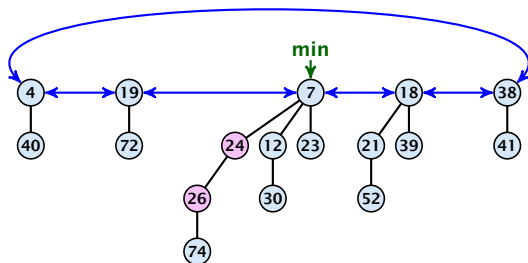
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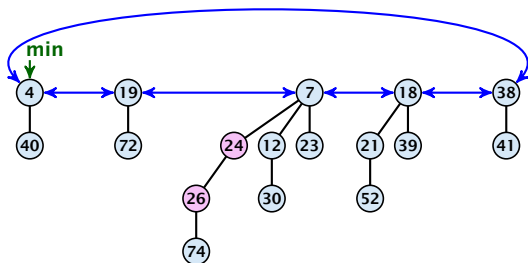
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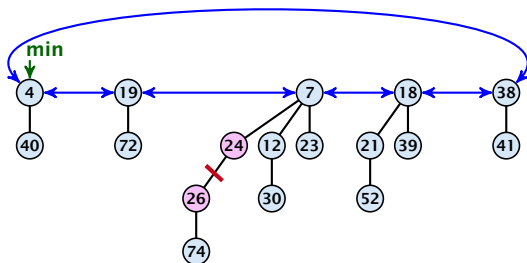
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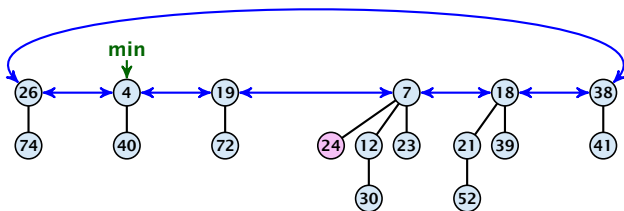
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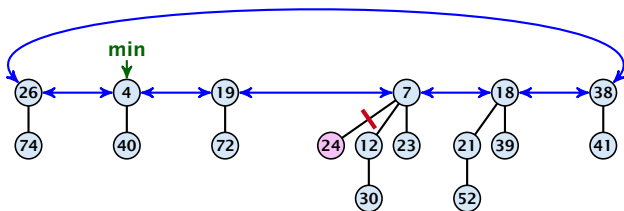
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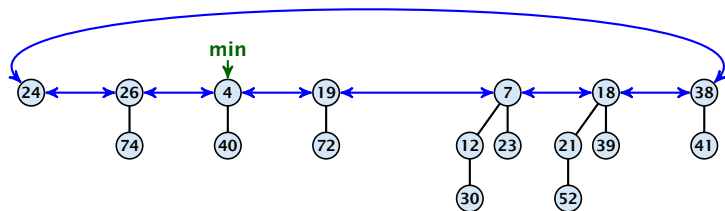
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- ▶ Cut the parent edge of x , and make x into a root.
- ▶ Adjust min-pointers, if necessary.
- ▶ Execute the following:

```
 $p \leftarrow \text{parent}[x];$   
while ( $p$  is marked)  
     $pp \leftarrow \text{parent}[p];$   
    cut of  $p$ ; make it into a root; unmark it;  
     $p \leftarrow pp;$   
if  $p$  is unmarked and not a root mark it;
```

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

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- ▶ $t' = t + \ell$, as every cut creates one new root.

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- ▶ $t' = t + \ell$, as every cut creates one new root.
- ▶ $m' \leq m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.

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$$c_2(\ell + 1) + c(4 - \ell) \leq (c_2 - c)\ell + 4c + c_2$$

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- ▶ Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2 .

Amortized cost:

- ▶ $t' = t + \ell$, as every cut creates one new root.
- ▶ $m' \leq m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.
- ▶ $\Delta\Phi \leq \ell + 2(-\ell + 2) = 4 - \ell$
- ▶ Amortized cost is at most
$$c_2(\ell + 1) + c(4 - \ell) \leq (c_2 - c)\ell + 4c + c_2 = \mathcal{O}(1),$$
if $c \geq c_2$.

Delete node

H. delete(x):

- ▶ decrease value of x to $-\infty$.
- ▶ delete-min.

Amortized cost: $\mathcal{O}(D_n)$

- ▶ $\mathcal{O}(1)$ for decrease-key.
- ▶ $\mathcal{O}(D_n)$ for delete-min.

8.3 Fibonacci Heaps

Lemma 1

Let x be a node with degree k and let y_1, \dots, y_k denote the children of x in the order that they were linked to x . Then

$$\text{degree}(y_i) \geq \begin{cases} 0 & \text{if } i = 1 \\ i - 2 & \text{if } i > 1 \end{cases}$$

8.3 Fibonacci Heaps

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- ▶ Therefore, $\text{degree}(y_i) \geq i - 2$.

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Let x be a degree k node of size s_k and let y_1, \dots, y_k be its children.

$$s_k = 2 + \sum_{i=2}^k \text{size}(y_i)$$

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8.3 Fibonacci Heaps

Definition 2

Consider the following non-standard Fibonacci type sequence:

$$F_k = \begin{cases} 1 & \text{if } k = 0 \\ 2 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

Facts:

1. $F_k \geq \phi^k$.
2. For $k \geq 2$: $F_k = 2 + \sum_{i=0}^{k-2} F_i$.

The above facts can be easily proved by induction. From this it follows that $s_k \geq F_k \geq \phi^k$, which gives that the maximum degree in a Fibonacci heap is logarithmic.

$$k=0: \quad 1 = F_0 \geq \Phi^0 = 1$$

$$k=1: \quad 2 = F_1 \geq \Phi^1 \approx 1.61$$

$$k-2, k-1 \rightarrow k: \quad F_k = F_{k-1} + F_{k-2} \geq \Phi^{k-1} + \Phi^{k-2} = \Phi^{k-2} \underbrace{(\Phi + 1)}_{\Phi^2} = \Phi^k$$

$$k=2: \quad 3 = F_2 = 2 + 1 = 2 + F_0$$

$$k-1 \rightarrow k: \quad F_k = F_{k-1} + F_{k-2} = 2 + \sum_{i=0}^{k-3} F_i + F_{k-2} = 2 + \sum_{i=0}^{k-2} F_i$$