

# 7 Dictionary

## Dictionary:

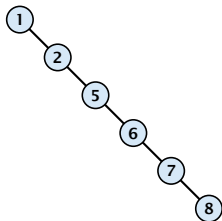
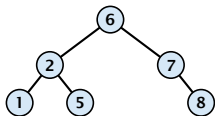
- ▶  **$S.$  insert( $x$ )**: Insert an element  $x$ .
- ▶  **$S.$  delete( $x$ )**: Delete the element pointed to by  $x$ .
- ▶  **$S.$  search( $k$ )**: Return a pointer to an element  $e$  with  $\text{key}[e] = k$  in  $S$  if it exists; otherwise return **null**.

## 7.1 Binary Search Trees

An (**internal**) **binary search tree** stores the elements in a binary tree. Each tree-node corresponds to an element. All elements in the left sub-tree of a node  $v$  have a smaller key-value than  $\text{key}[v]$  and elements in the right sub-tree have a larger-key value. We assume that all key-values are different.

(**External** Search Trees store objects only at leaf-vertices)

Examples:

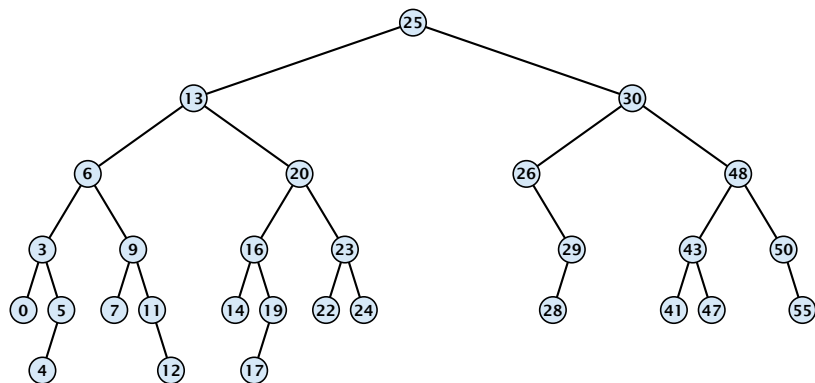


## 7.1 Binary Search Trees

We consider the following operations on binary search trees. Note that this is a super-set of the dictionary-operations.

- ▶  $T.\text{insert}(x)$
- ▶  $T.\text{delete}(x)$
- ▶  $T.\text{search}(k)$
- ▶  $T.\text{successor}(x)$
- ▶  $T.\text{predecessor}(x)$
- ▶  $T.\text{minimum}()$
- ▶  $T.\text{maximum}()$

# Binary Search Trees: Searching

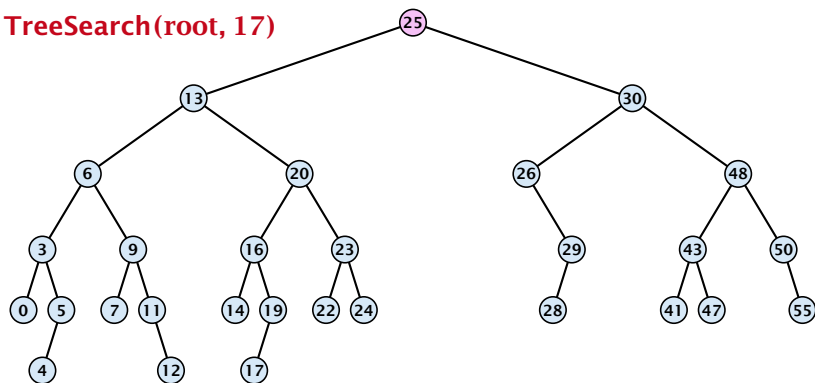


## Algorithm 1 TreeSearch( $x, k$ )

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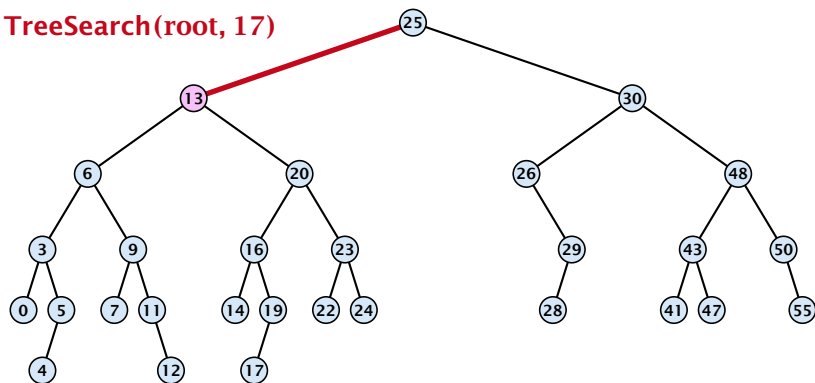


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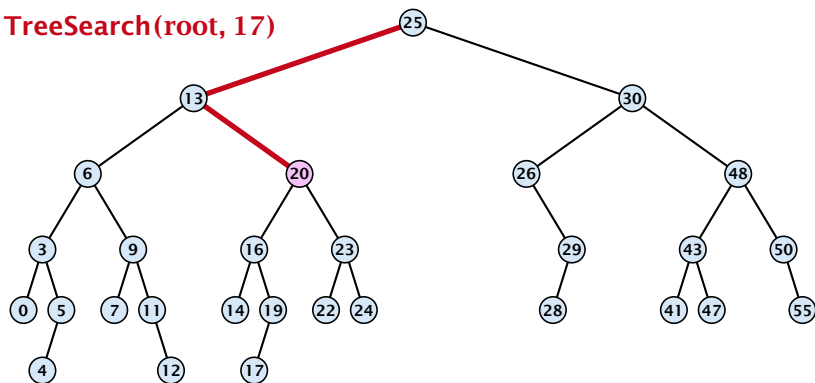


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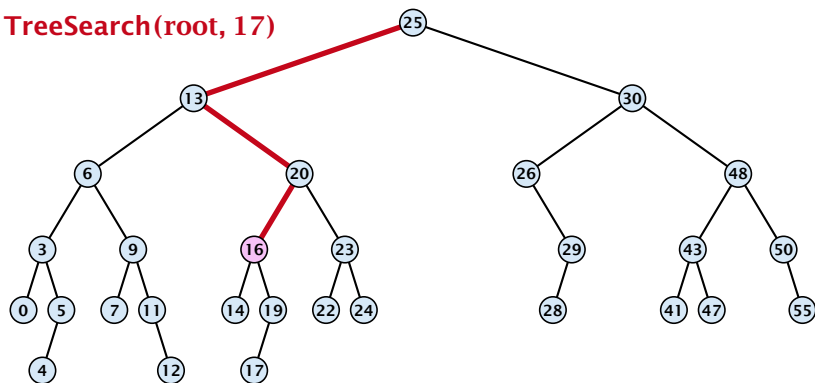


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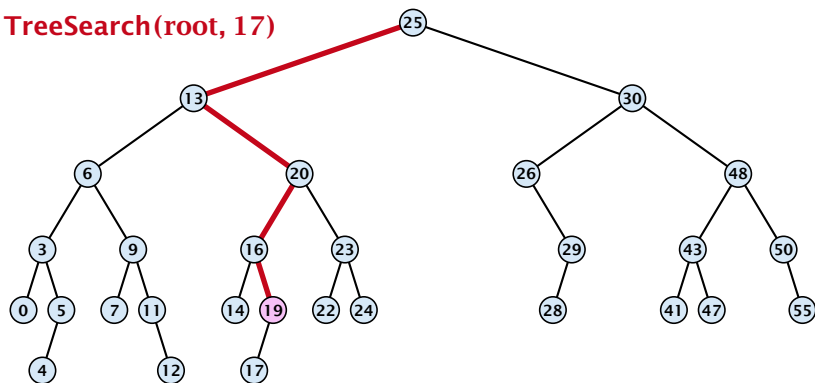
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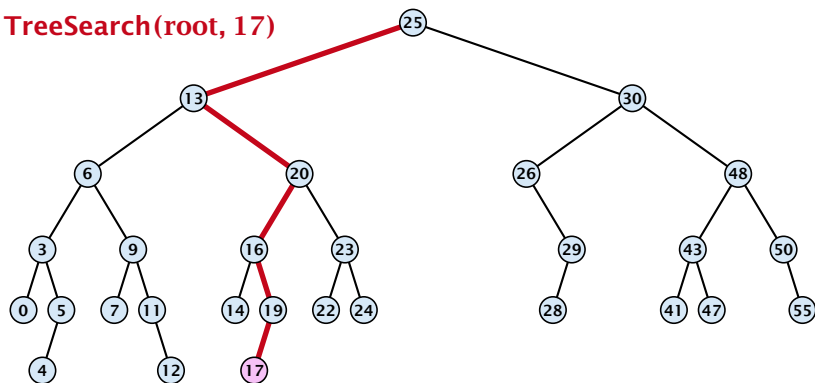


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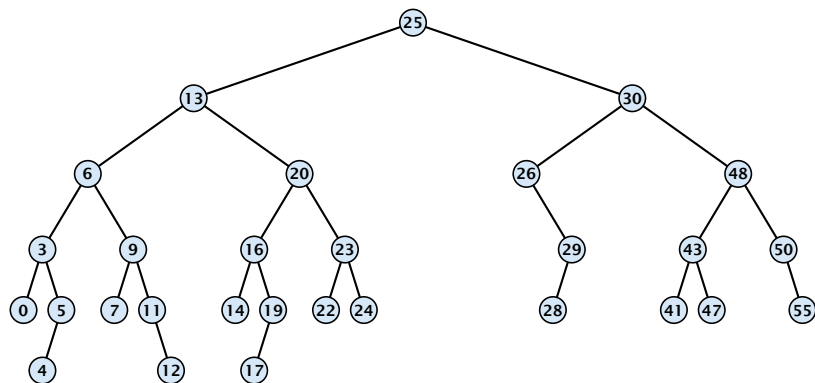
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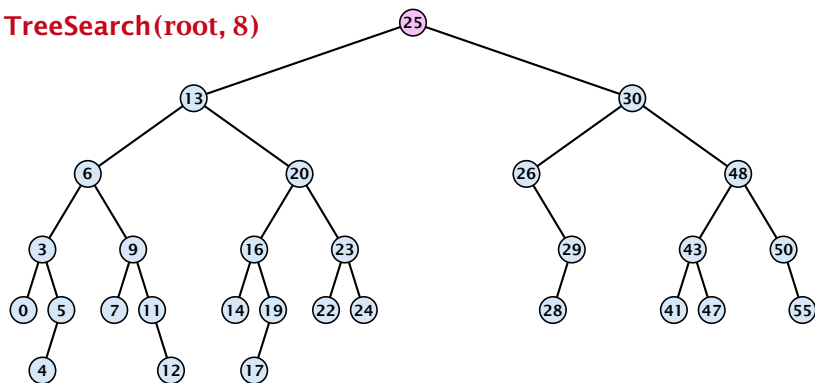


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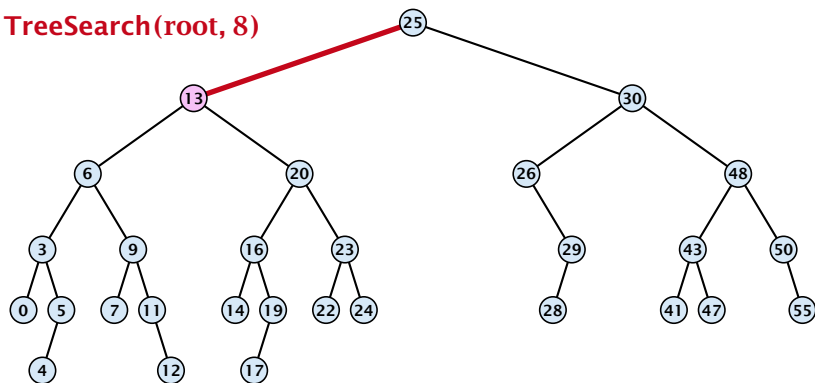


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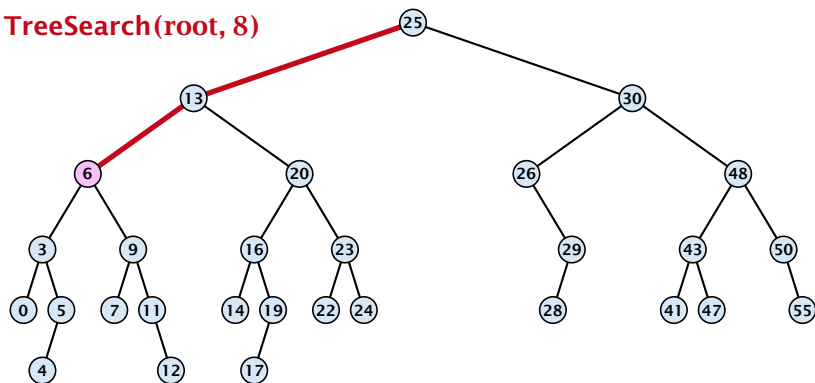


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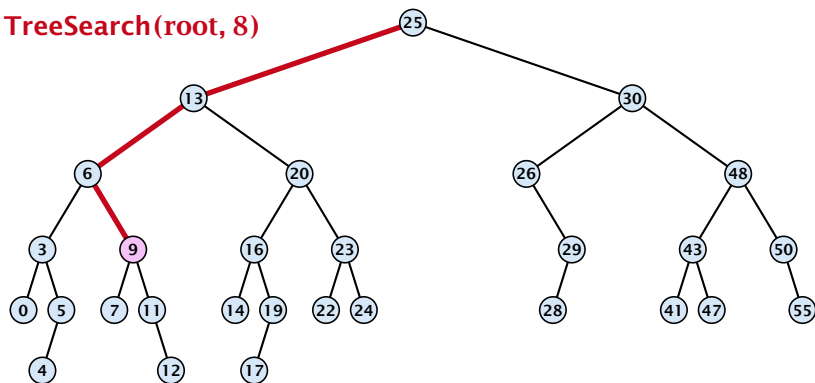


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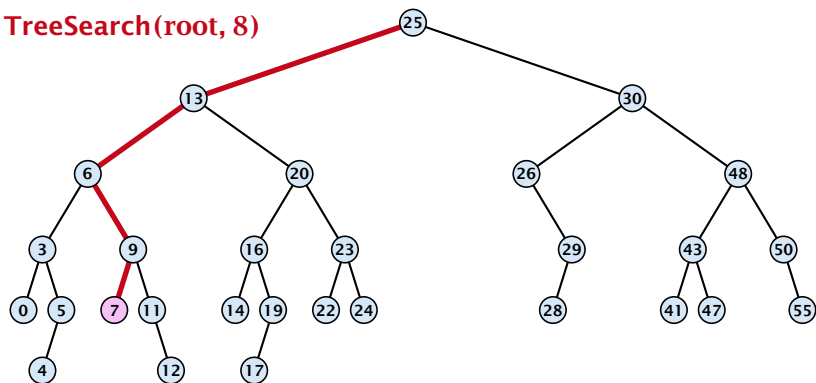


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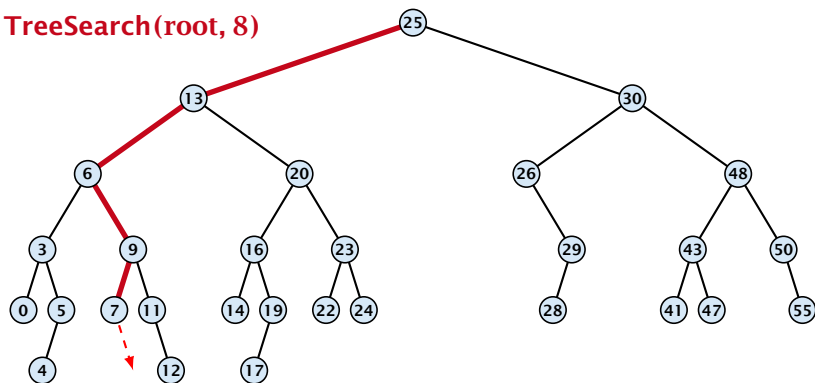
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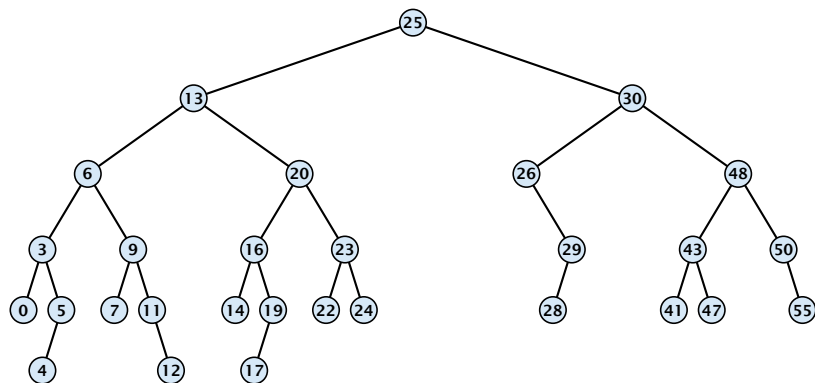
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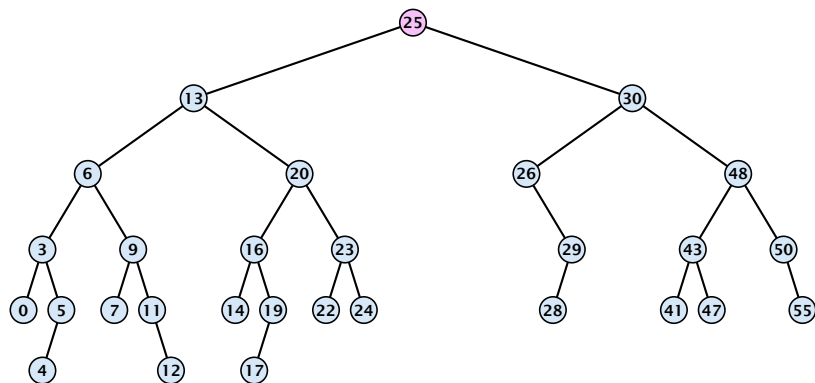
# Binary Search Trees: Minimum



## Algorithm 2 TreeMin( $x$ )

- 1: **if**  $x = \text{null}$  **or**  $\text{left}[x] = \text{null}$  **return**  $x$
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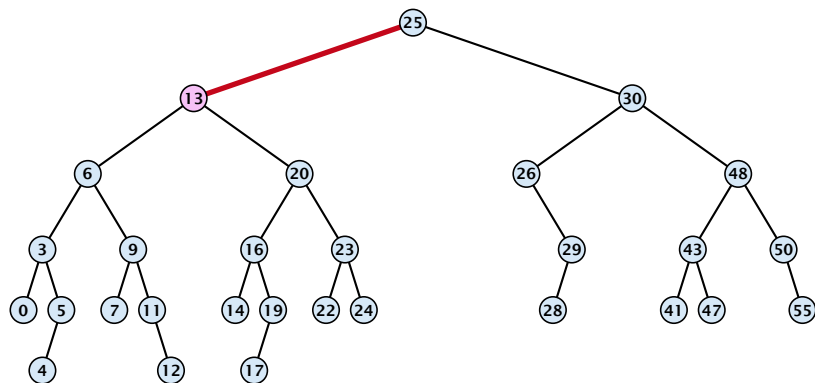
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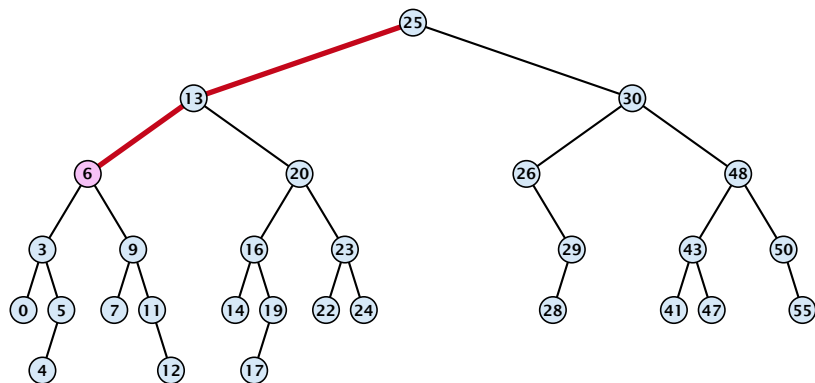
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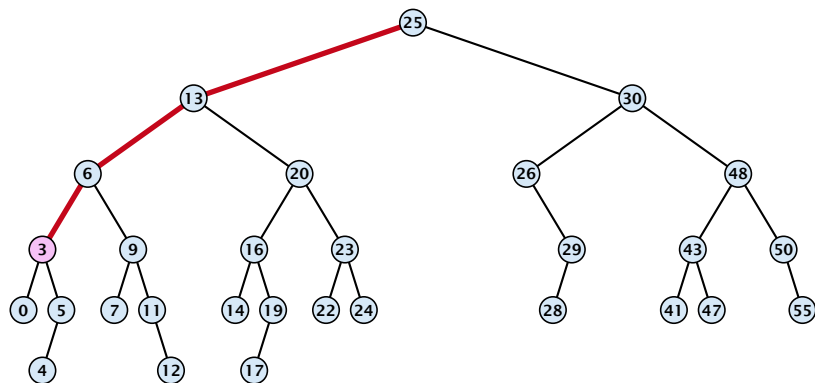
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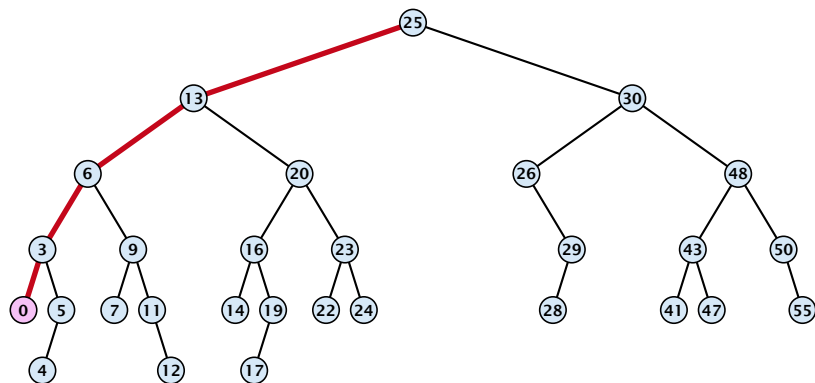
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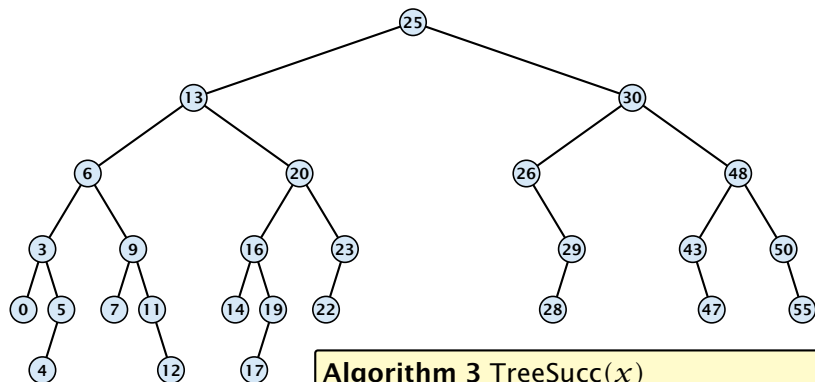
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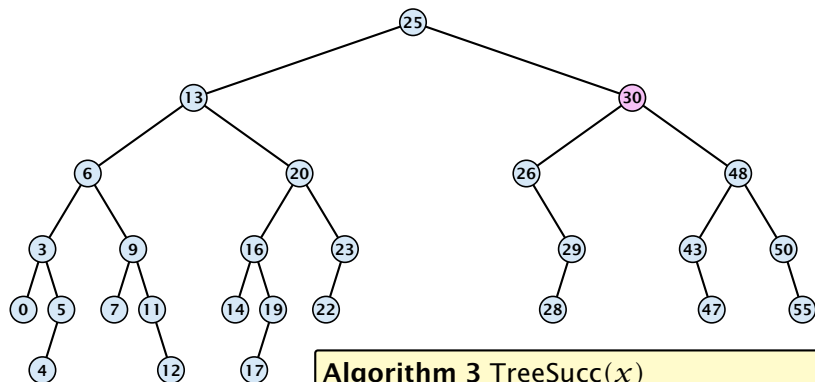


## Algorithm 3 TreeSucc( $x$ )

- 1: **if**  $\text{right}[x] \neq \text{null}$  **return**  $\text{TreeMin}(\text{right}[x])$
- 2:  $y \leftarrow \text{parent}[x]$
- 3: **while**  $y \neq \text{null}$  **and**  $x = \text{right}[y]$  **do**
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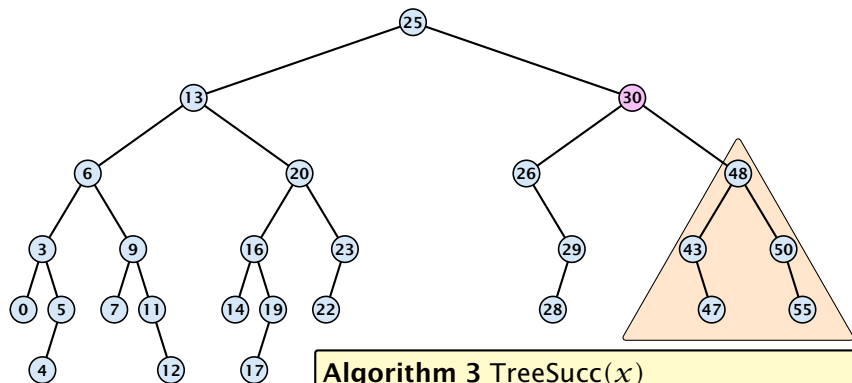
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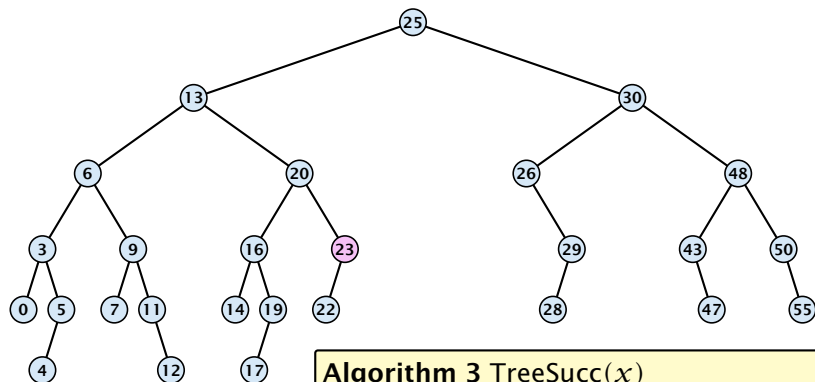
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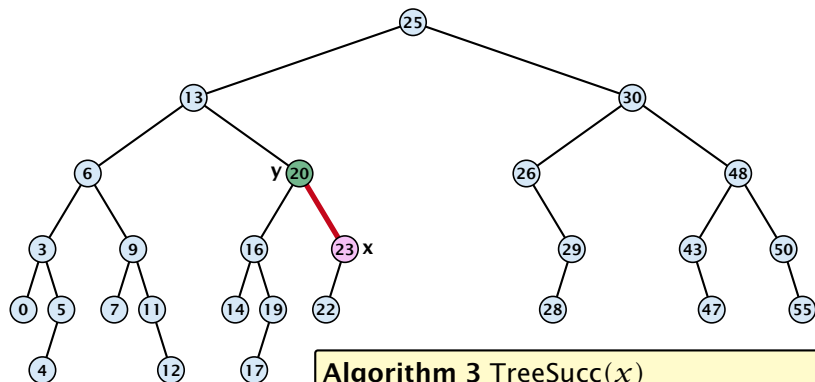
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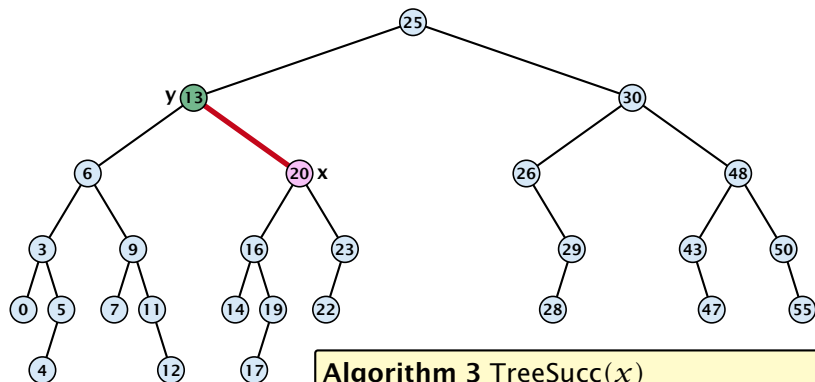
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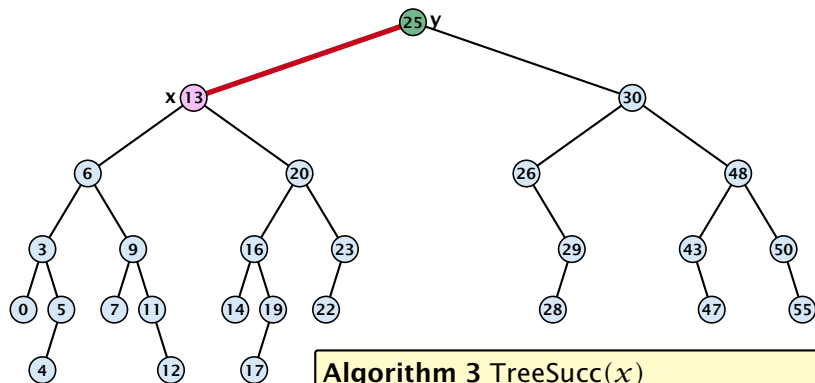
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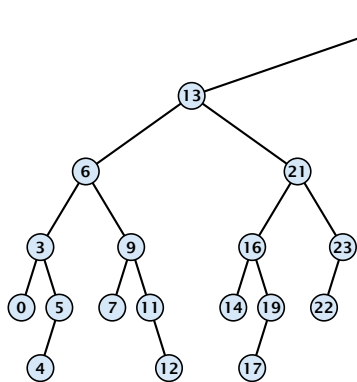
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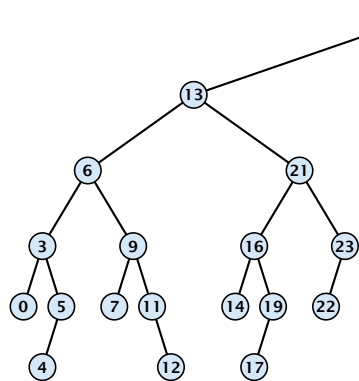


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3:   return;
4: if key[ $x$ ] > key[ $z$ ] then
5:   if left[ $x$ ] = null then
6:     left[ $x$ ]  $\leftarrow z$ ; parent[ $z$ ]  $\leftarrow x$ ;
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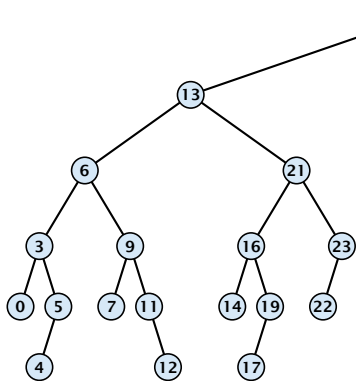
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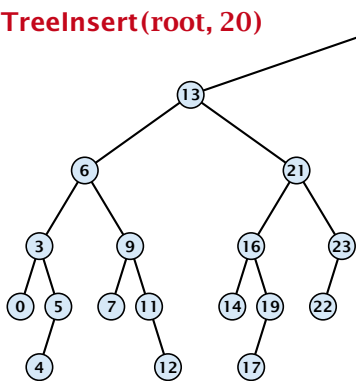
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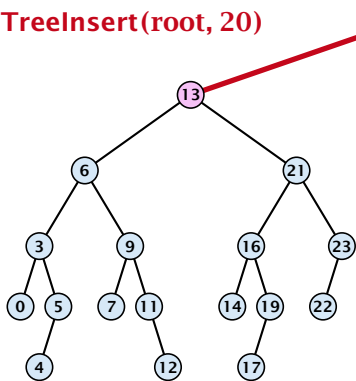
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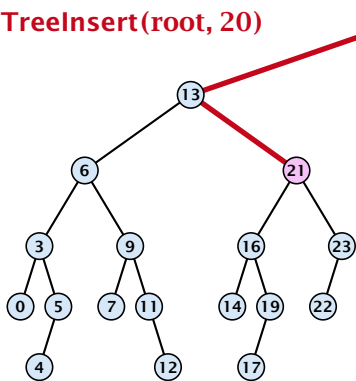
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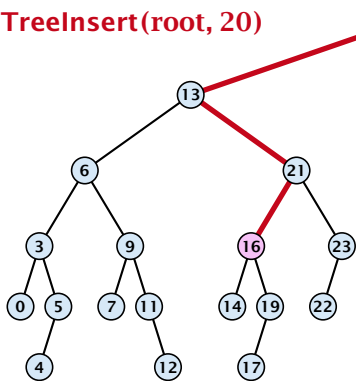
## Algorithm 4 TreeInsert( $x, z$ )

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1: if  $x = \text{null}$  then
2:   root[ $T$ ]  $\leftarrow z$ ; parent[ $z$ ]  $\leftarrow \text{null}$ ;
3:   return;
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7:   else TreeInsert(left[ $x$ ],  $z$ );
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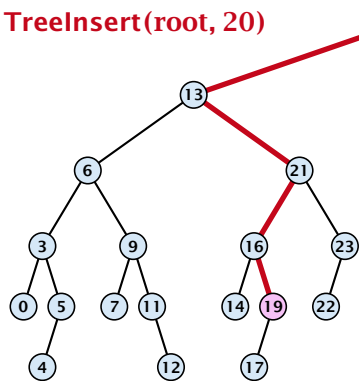
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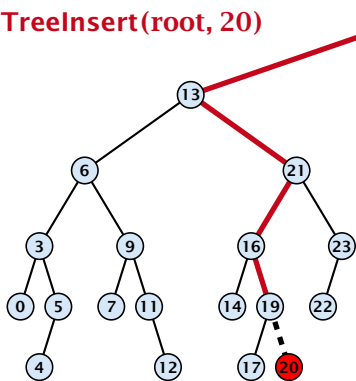
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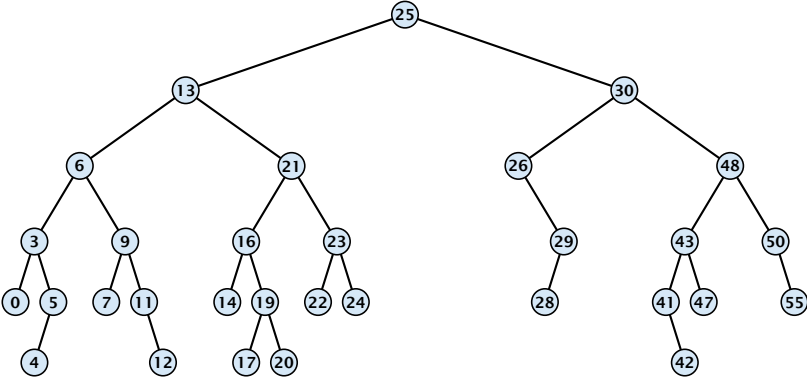


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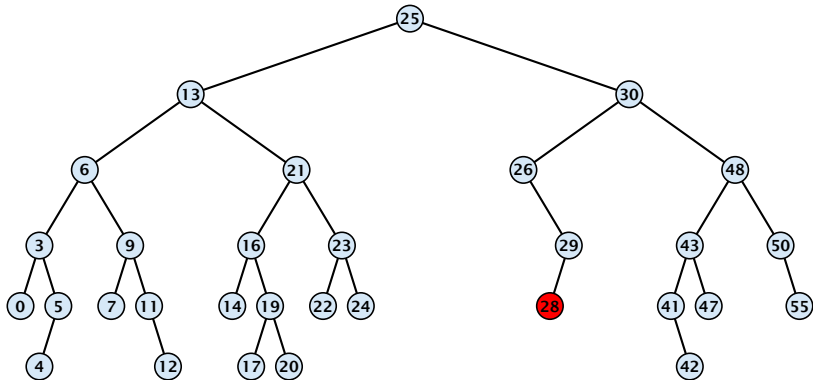
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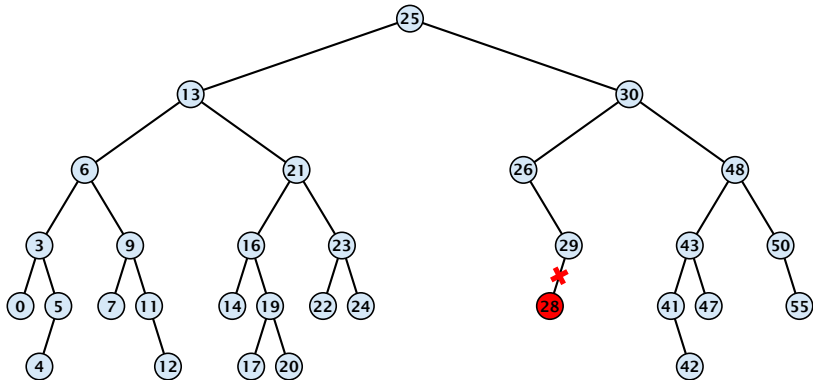


## Case 1:

Element does not have any children

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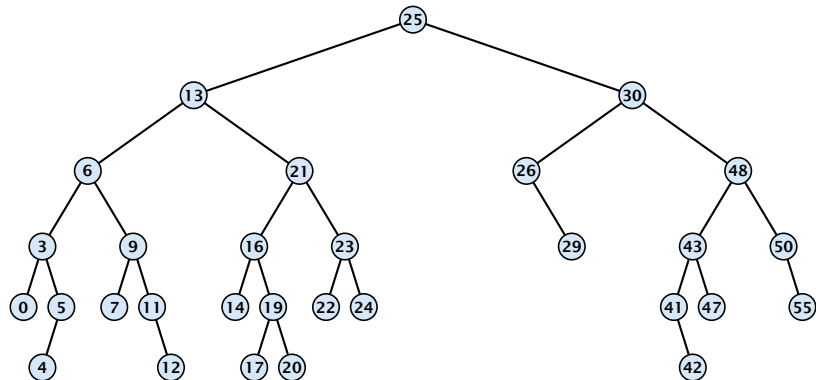


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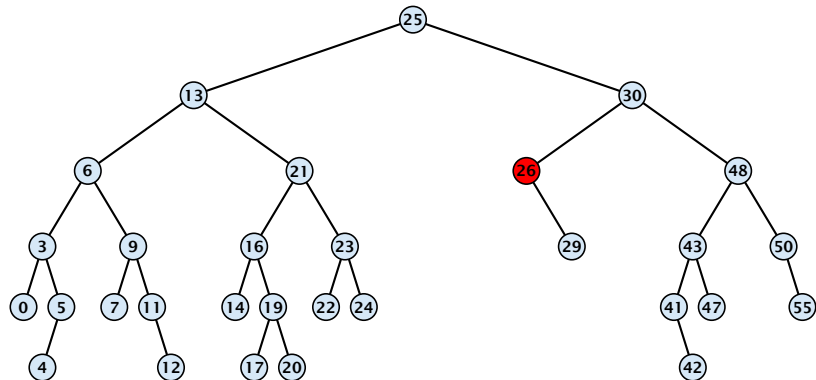


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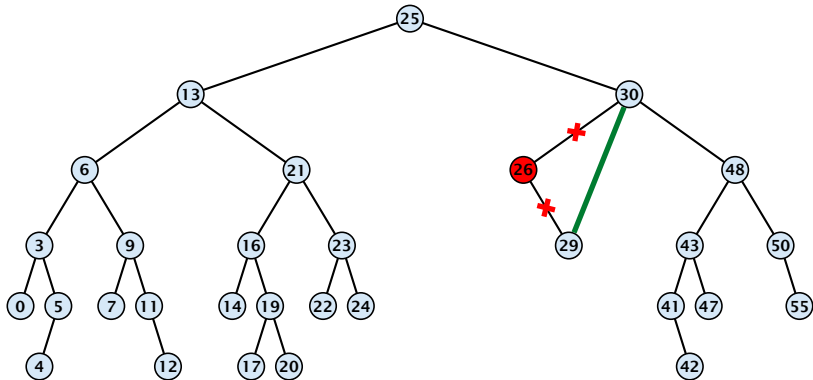


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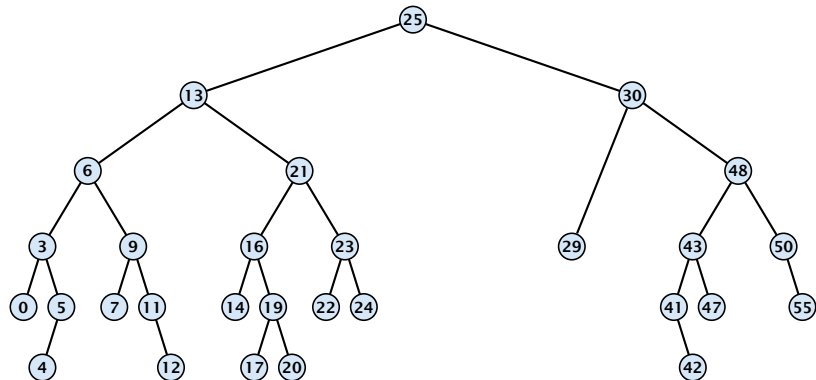


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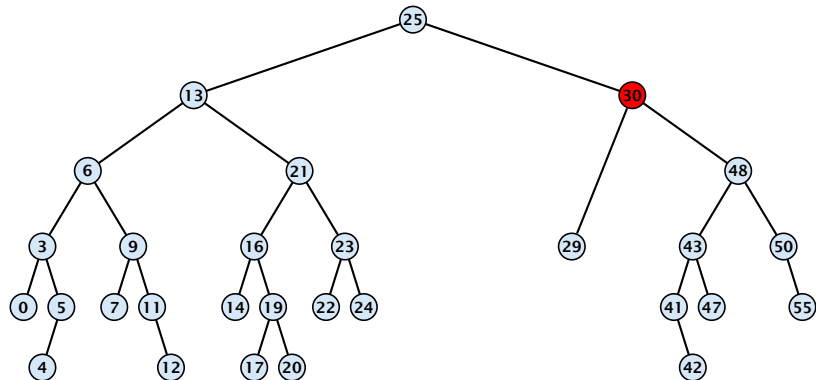


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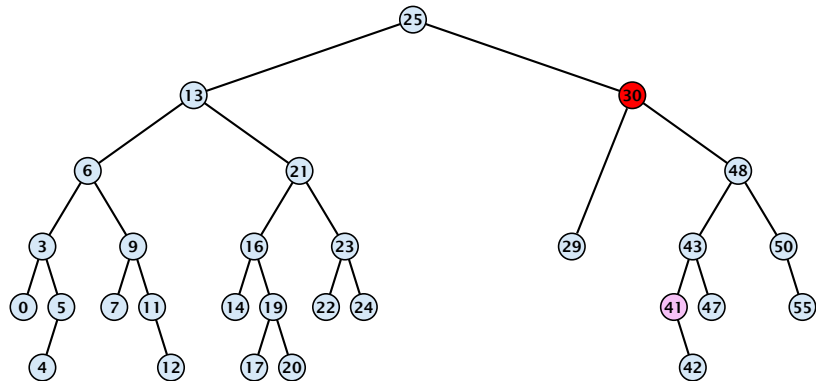


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Element has two children

- ▶ Find the successor of the element
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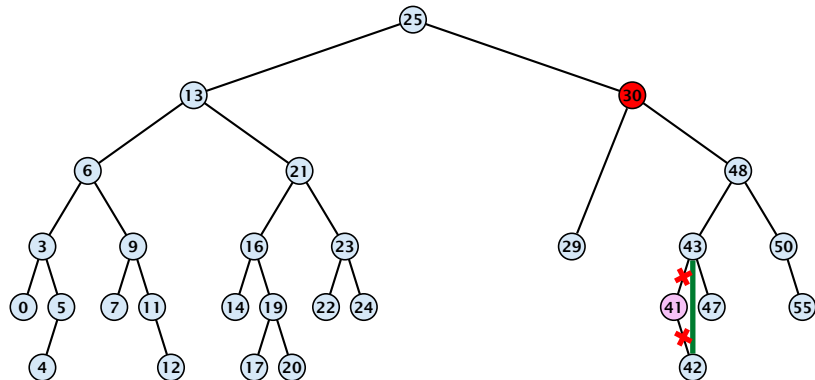
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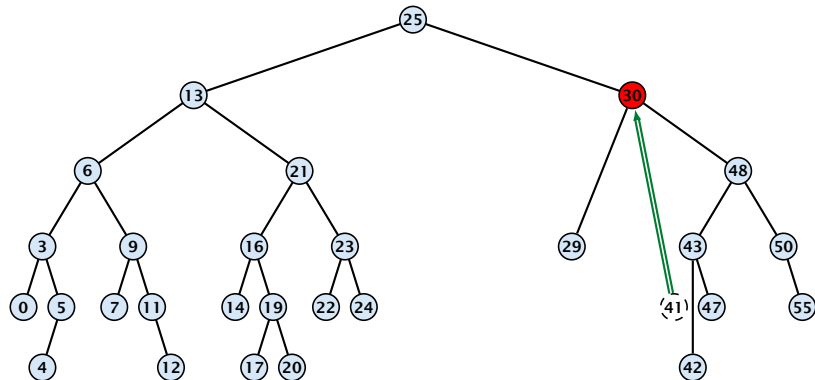


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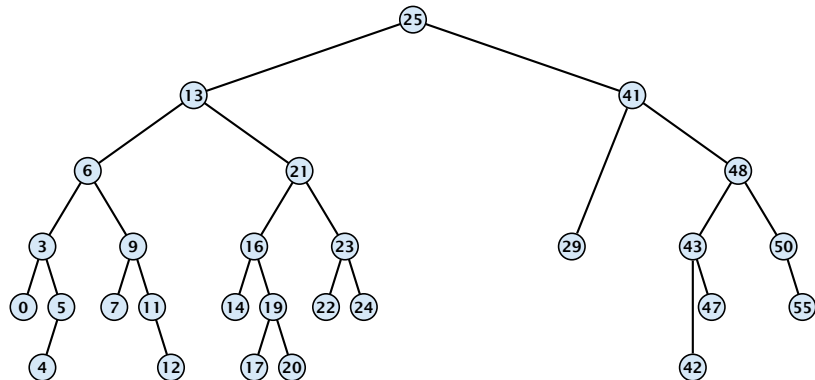


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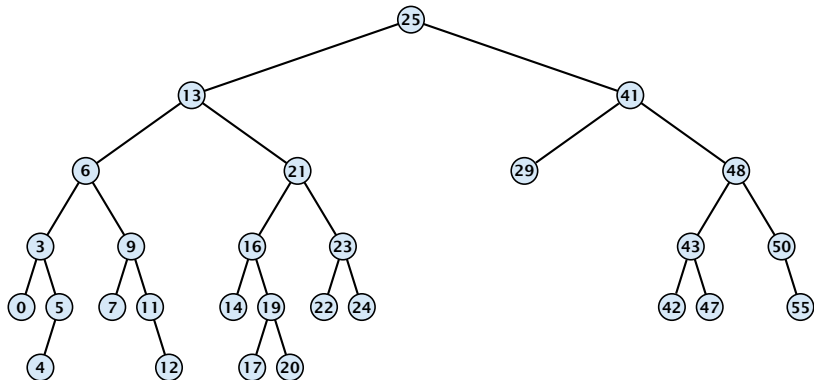


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# Binary Search Trees: Delete

## Algorithm 9 TreeDelete( $z$ )

```
1: if left[ $z$ ] = null or right[ $z$ ] = null
2:   then  $y \leftarrow z$  else  $y \leftarrow \text{TreeSucc}(z)$ ;   select  $y$  to splice out
3:   if left[ $y$ ]  $\neq$  null
4:     then  $x \leftarrow \text{left}[y]$  else  $x \leftarrow \text{right}[y]$ ;  $x$  is child of  $y$  (or null)
5:   if  $x \neq \text{null}$  then parent[ $x$ ]  $\leftarrow$  parent[ $y$ ];   parent[ $x$ ] is correct
6:   if parent[ $y$ ] = null then
7:     root[ $T$ ]  $\leftarrow x$ 
8:   else
9:     if  $y = \text{left}[\text{parent}[y]]$  then
10:      left[parent[ $y$ ]]  $\leftarrow x$ 
11:     else
12:      right[parent[ $y$ ]]  $\leftarrow x$ 
13:   if  $y \neq z$  then copy  $y$ -data to  $z$ 
```

} fix pointer to  $x$

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AVL-trees, Red-black trees, Scapegoat trees, 2-3 trees, B-trees, AA trees, Treaps

similar: SPLAY trees.

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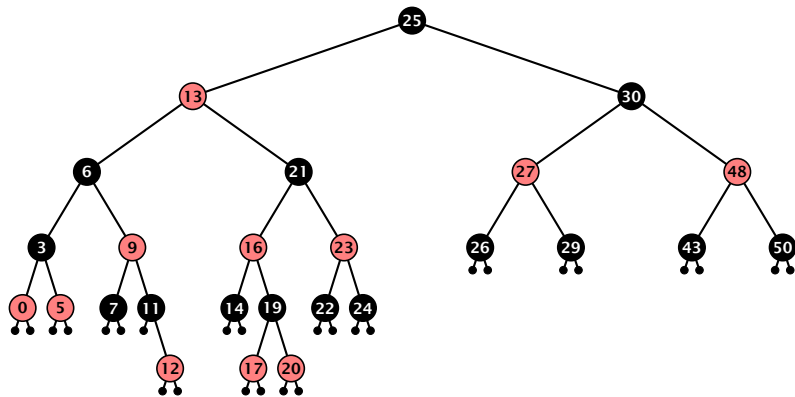
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# Red Black Trees: Example



## 7.2 Red Black Trees

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*A red-black tree with  $n$  internal nodes has height at most  $\mathcal{O}(\log n)$ .*

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We first show:

### Lemma 4

A sub-tree of black height  $\text{bh}(v)$  in a red black tree contains at least  $2^{\text{bh}(v)} - 1$  internal vertices.

## 7.2 Red Black Trees

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- ▶ The sub-tree rooted at  $v$  contains  $0 = 2^{\text{bh}(v)} - 1$  inner vertices.

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- ▶ Then  $T_v$  contains at least  $2(2^{\text{bh}(v)-1} - 1) + 1 \geq 2^{\text{bh}(v)} - 1$  vertices.



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Hence,  $h \leq 2 \log(n + 1) = \mathcal{O}(\log n)$ . □

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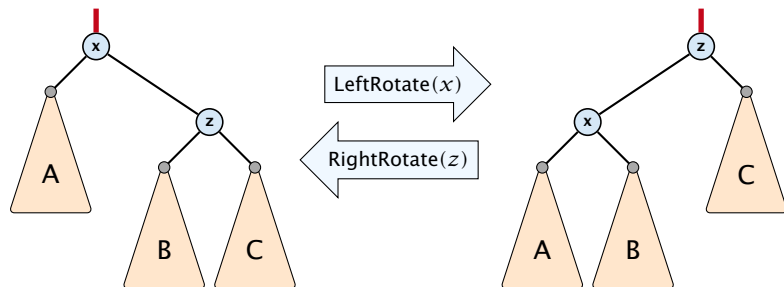
The **null**-pointers in a binary search tree are replaced by pointers to special null-vertices, that do not carry any object-data.

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We need to adapt the insert and delete operations so that the red black properties are maintained.

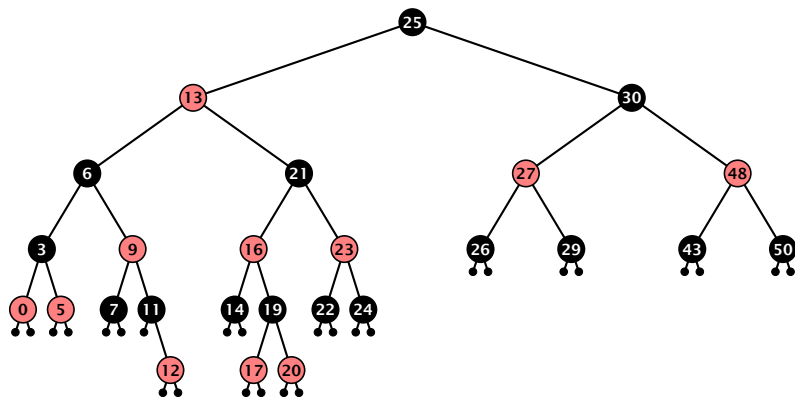
# Rotations

The properties will be maintained through rotations:





# Red Black Trees: Insert

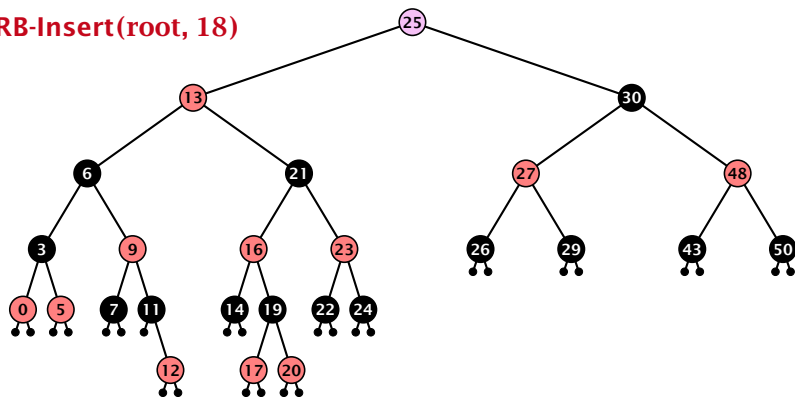


## Insert:

- ▶ first make a normal insert into a binary search tree
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# Red Black Trees: Insert

RB-Insert(root, 18)

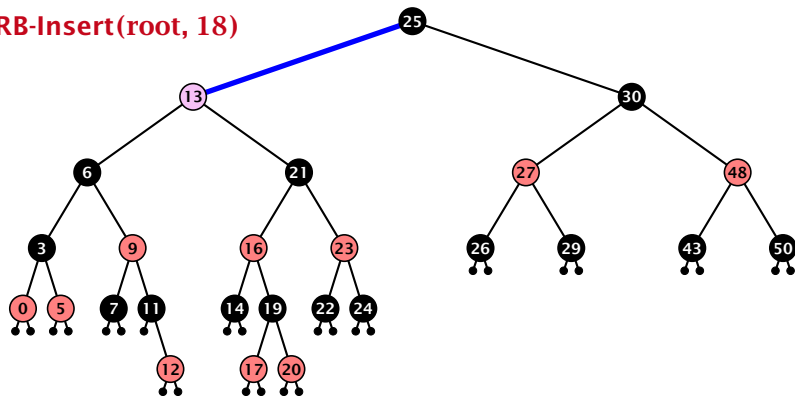


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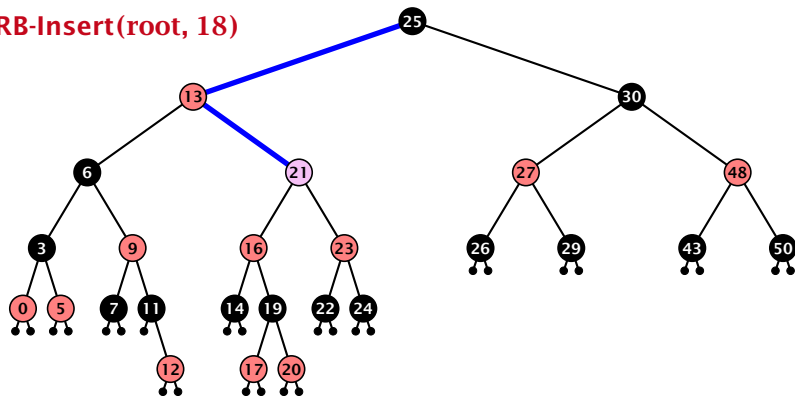


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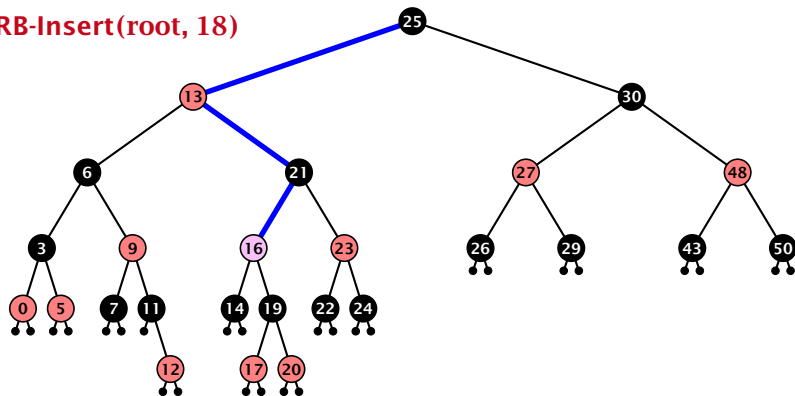


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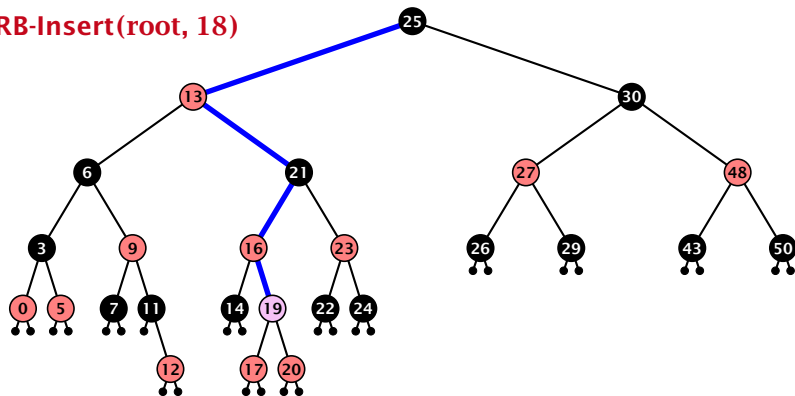


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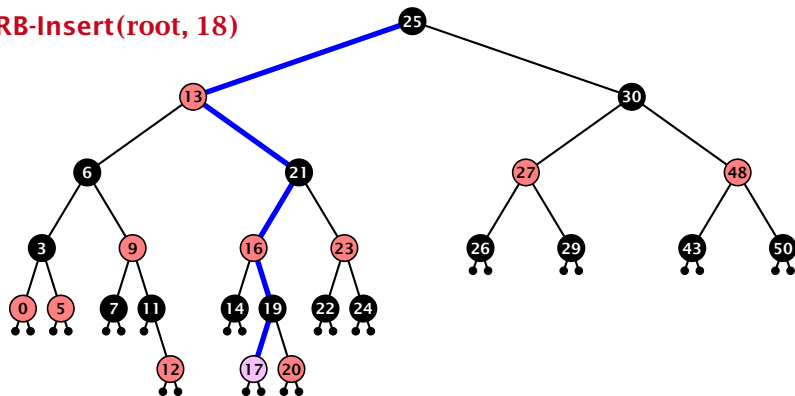


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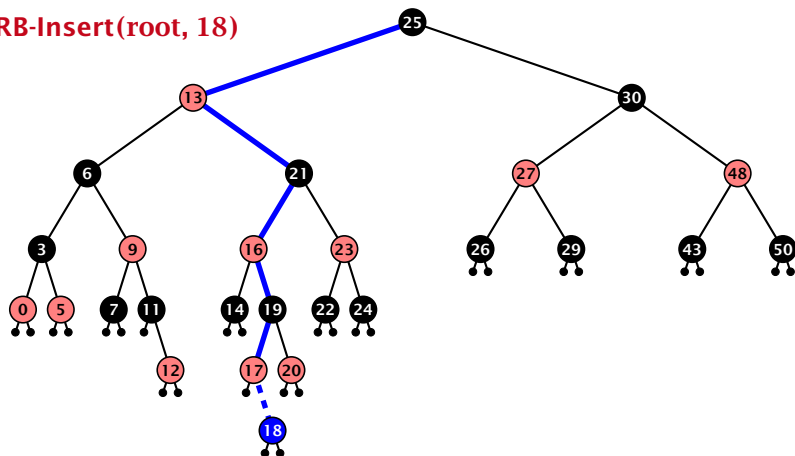


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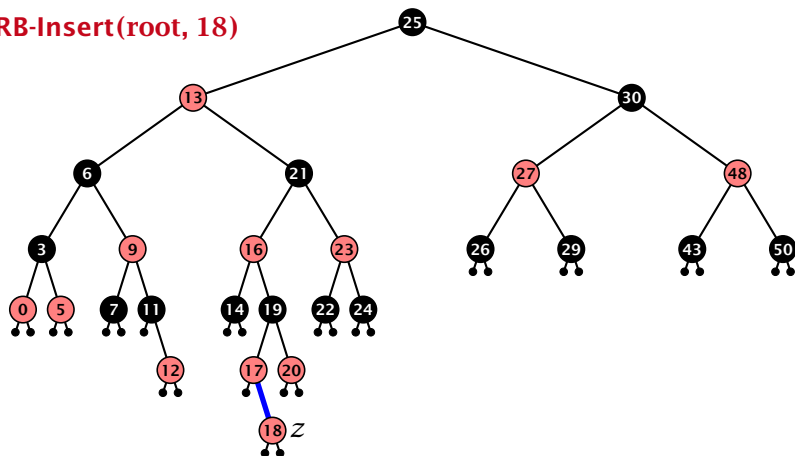
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  - ▶ either both of them are red (most important case)

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- ▶ the black-height property is fulfilled at every node
- ▶ the only violation of red-black properties occurs at  $z$  and  $\text{parent}[z]$ 
  - ▶ either both of them are red  
(most important case)
  - ▶ or the parent does not exist  
(violation since root must be black)

# Red Black Trees: Insert

## Invariant of the fix-up algorithm:

- ▶  $z$  is a red node
- ▶ the black-height property is fulfilled at every node
- ▶ the only violation of red-black properties occurs at  $z$  and  $\text{parent}[z]$ 
  - ▶ either both of them are red  
(most important case)
  - ▶ or the parent does not exist  
(violation since root must be black)

If  $z$  has a parent but no grand-parent we could simply color the parent/root black; however this case never happens.

# Red Black Trees: Insert

## Algorithm 10 InsertFix( $z$ )

```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
2:   if parent[ $z$ ] = left[gp[ $z$ ]] then
3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
7:     else
8:       if  $z$  = right[parent[ $z$ ]] then
9:          $z \leftarrow$  p[ $z$ ]; LeftRotate( $z$ );
10:      col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red;
11:      RightRotate(gp[ $z$ ]);
12:     else same as then-clause but right and left exchanged
13: col(root[ $T$ ])  $\leftarrow$  black;
```



# Red Black Trees: Insert

## Algorithm 10 InsertFix( $z$ )

```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
2:   if parent[ $z$ ] = left[gp[ $z$ ]] then  $z$  in left subtree of grandparent
3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
7:     else
8:       if  $z$  = right[parent[ $z$ ]] then
9:          $z \leftarrow$  p[ $z$ ]; LeftRotate( $z$ );
10:      col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red;
11:      RightRotate(gp[ $z$ ]);
12:     else same as then-clause but right and left exchanged
13: col(root[ $T$ ])  $\leftarrow$  black;
```

# Red Black Trees: Insert

## Algorithm 10 InsertFix( $z$ )

```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
2:   if parent[ $z$ ] = left[gp[ $z$ ]] then
3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then Case 1: uncle red
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
7:     else
8:       if  $z$  = right[parent[ $z$ ]] then
9:          $z \leftarrow$  p[ $z$ ]; LeftRotate( $z$ );
10:      col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red;
11:      RightRotate(gp[ $z$ ]);
12:     else same as then-clause but right and left exchanged
13: col(root[ $T$ ])  $\leftarrow$  black;
```

# Red Black Trees: Insert

## Algorithm 10 InsertFix( $z$ )

```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
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3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
7:   else Case 2: uncle black
8:     if  $z$  = right[parent[ $z$ ]] then
9:        $z \leftarrow$  p[ $z$ ]; LeftRotate( $z$ );
10:    col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red;
11:    RightRotate(gp[ $z$ ]);
12:   else same as then-clause but right and left exchanged
13: col(root[ $T$ ])  $\leftarrow$  black;
```

# Red Black Trees: Insert

## Algorithm 10 InsertFix( $z$ )

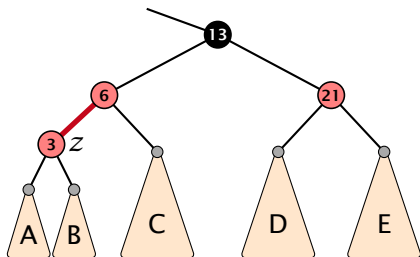
```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
2:   if parent[ $z$ ] = left[gp[ $z$ ]] then
3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
7:     else
8:       if  $z$  = right[parent[ $z$ ]] then 2a:  $z$  right child
9:          $z \leftarrow$  p[ $z$ ]; LeftRotate( $z$ );
10:        col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red;
11:        RightRotate(gp[ $z$ ]);
12:       else same as then-clause but right and left exchanged
13: col(root[ $T$ ])  $\leftarrow$  black;
```

# Red Black Trees: Insert

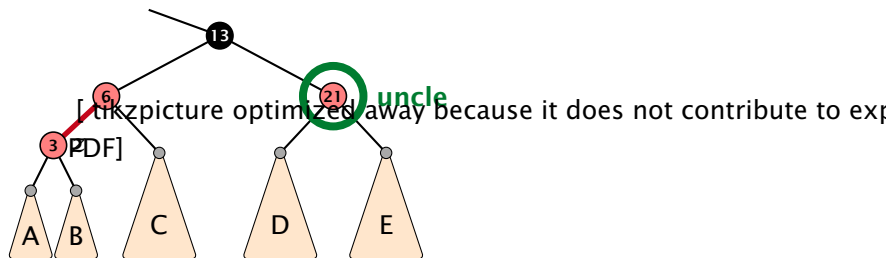
## Algorithm 10 InsertFix( $z$ )

```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
2:   if parent[ $z$ ] = left[gp[ $z$ ]] then
3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
7:     else
8:       if  $z$  = right[parent[ $z$ ]] then
9:          $z \leftarrow$  p[ $z$ ]; LeftRotate( $z$ );
10:      col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red; 2b:  $z$  left child
11:      RightRotate(gp[ $z$ ]);
12:     else same as then-clause but right and left exchanged
13: col(root[ $T$ ])  $\leftarrow$  black;
```

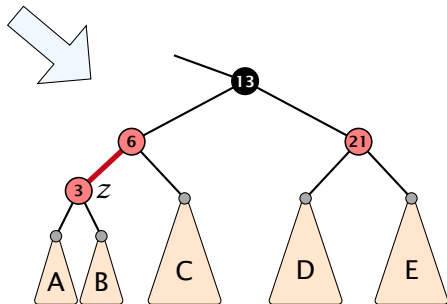
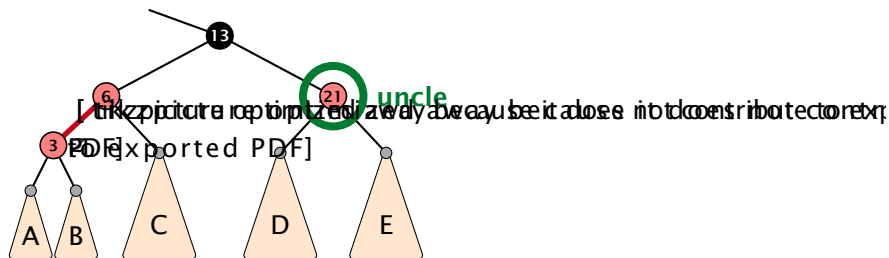
## Case 1: Red Uncle



## Case 1: Red Uncle

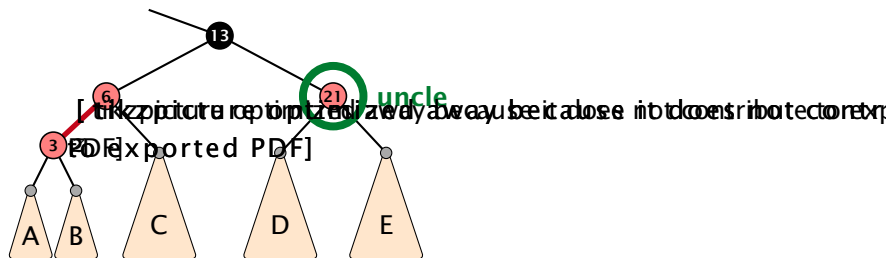


# Case 1: Red Uncle

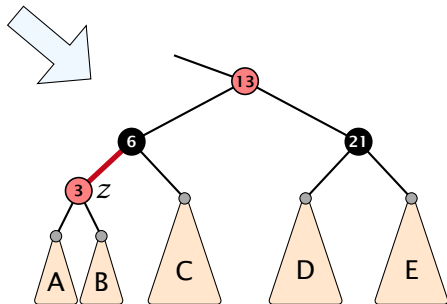




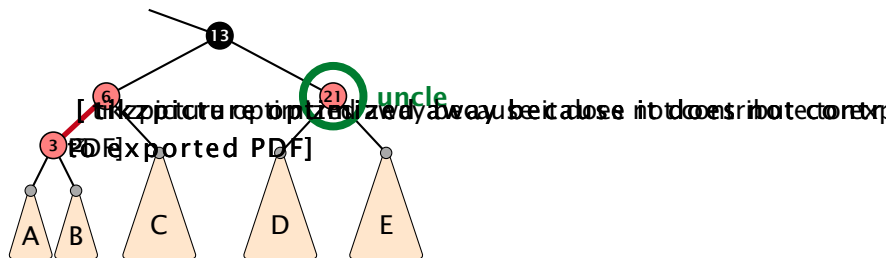
# Case 1: Red Uncle



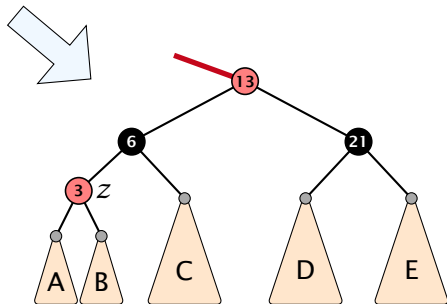
1. recolour



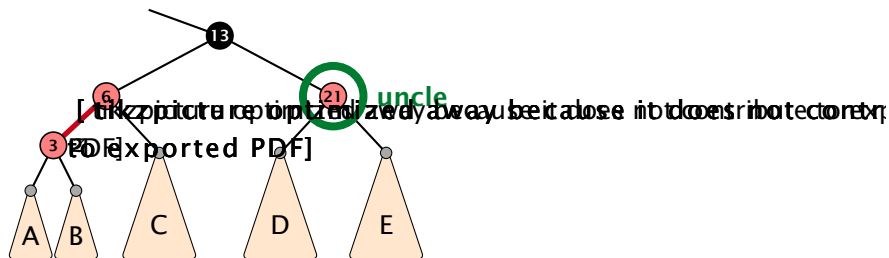
# Case 1: Red Uncle



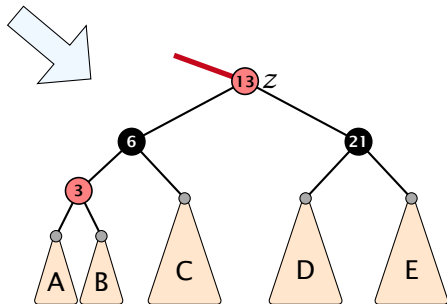
1. recolor



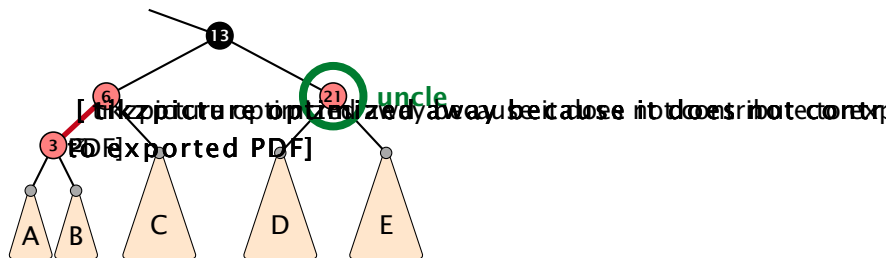
## Case 1: Red Uncle



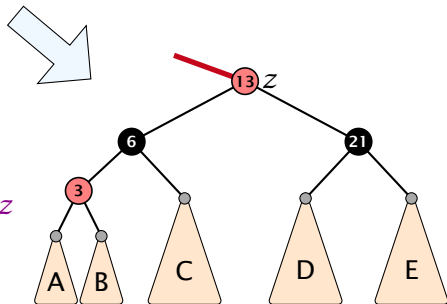
1. recolour
2. move  $z$  to grand-parent



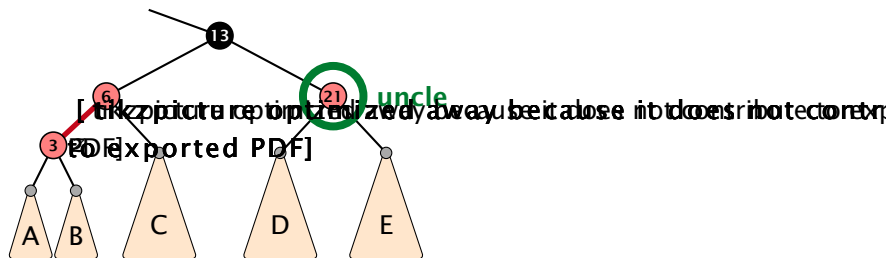
## Case 1: Red Uncle



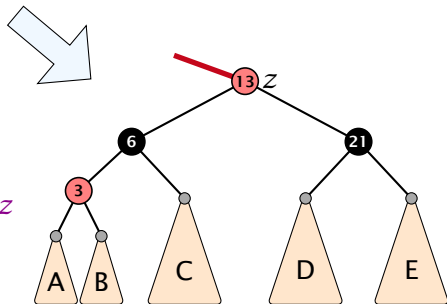
1. recolour
2. move  $z$  to grand-parent
3. invariant is fulfilled for new  $z$



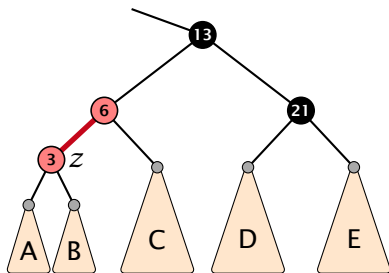
## Case 1: Red Uncle



1. recolour
2. move  $z$  to grand-parent
3. invariant is fulfilled for new  $z$
4. you made progress

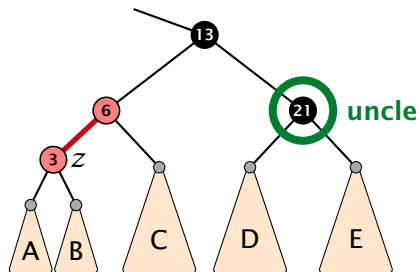


## Case 2b: Black uncle and z is left child



## Case 2b: Black uncle and z is left child

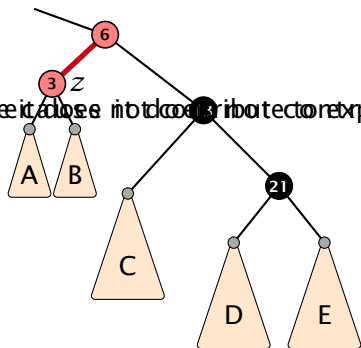
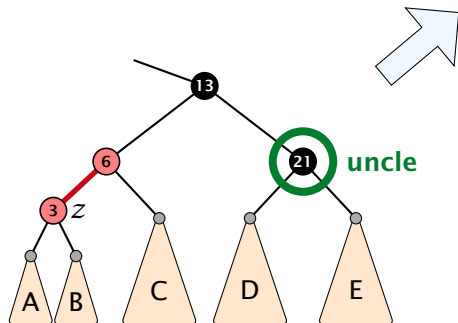
[ tikzpicture optimized away because it does not contribute to exp PDF]



## Case 2b: Black uncle and z is left child

### 1. rotate around grandparent

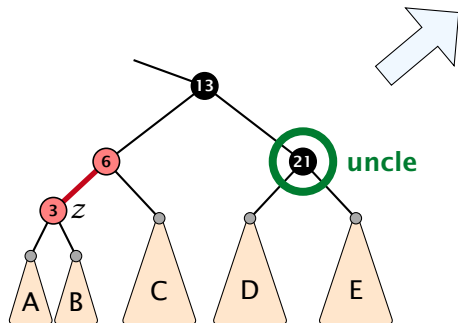
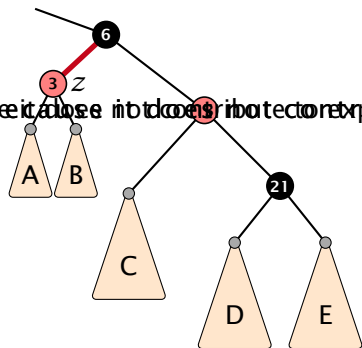
[tikzpicture optimized away because it does not close into closed form; PDF exported PDF]





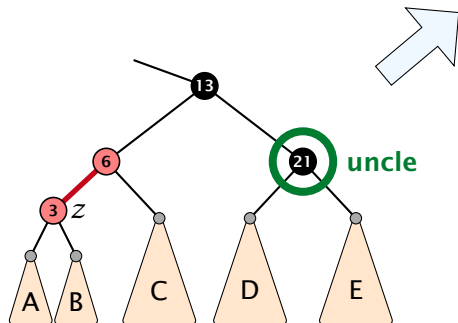
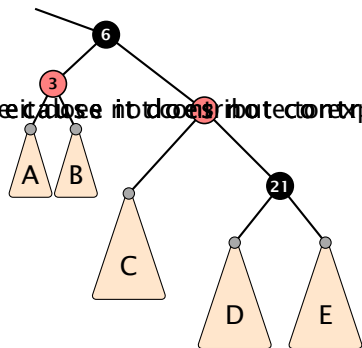
## Case 2b: Black uncle and z is left child

1. rotate around grandparent
2. [uncle is black] because it does not contain black height property holds

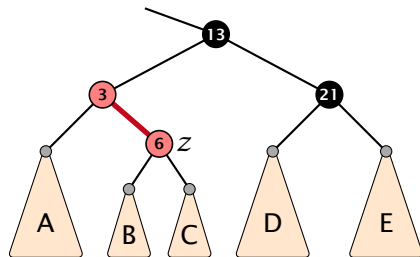


## Case 2b: Black uncle and z is left child

1. rotate around grandparent
2. ~~if you rotate around the grandparent, you will be close to doing a left-right rotation, but the black height property holds~~
3. you have a red black tree

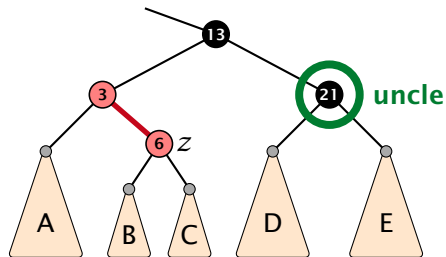


## Case 2a: Black uncle and z is right child



## Case 2a: Black uncle and z is right child

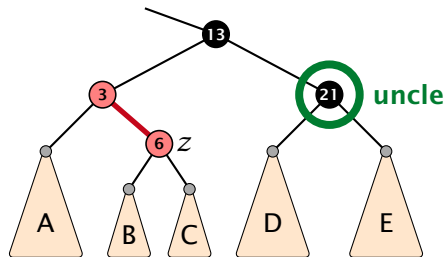
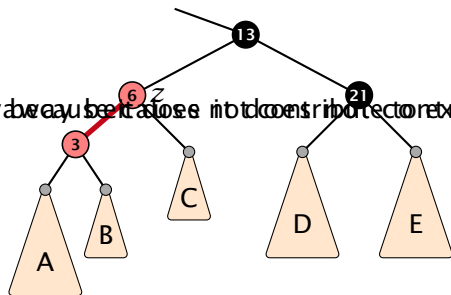
[ tikzpicture optimized away because it does not contribute to ex PDF]



## Case 2a: Black uncle and z is right child

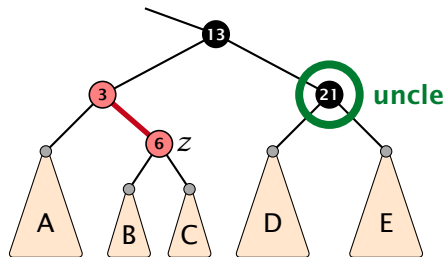
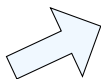
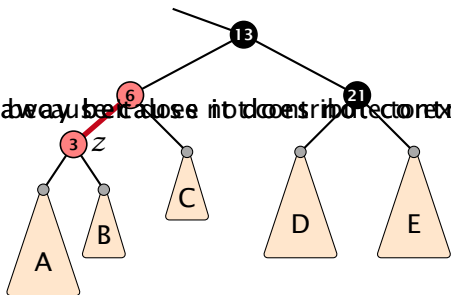
### 1. rotate around parent

[tikzpicture optimized away because it does not describe a PDF]  
[PDF exported PDF]



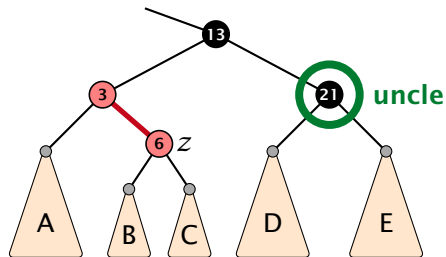
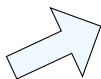
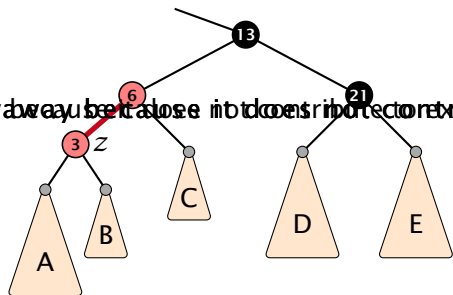
## Case 2a: Black uncle and z is right child

1. rotate around parent
2. [uncle is now uncle, z is now z, because it does not rotate] [PDF exported PDF]



## Case 2a: Black uncle and z is right child

1. rotate around parent
2. [initially, the uncles are always black, but close it does not matter]
3. [PDF exports PDF]



# Red Black Trees: Insert

## Running time:

- ▶ Only Case 1 may repeat; but only  $h/2$  many steps, where  $h$  is the height of the tree.



# Red Black Trees: Insert

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- ▶ Case 2a  $\rightarrow$  Case 2b  $\rightarrow$  red-black tree

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- ▶ Case 2b  $\rightarrow$  red-black tree

# Red Black Trees: Insert

## Running time:

- ▶ Only Case 1 may repeat; but only  $h/2$  many steps, where  $h$  is the height of the tree.
- ▶ Case 2a  $\rightarrow$  Case 2b  $\rightarrow$  red-black tree
- ▶ Case 2b  $\rightarrow$  red-black tree

Performing Case 1 at most  $\mathcal{O}(\log n)$  times and every other case at most once, we get a red-black tree. Hence  $\mathcal{O}(\log n)$  re-colorings and at most 2 rotations.

# Red Black Trees: Delete

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First do a standard delete.

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# Red Black Trees: Delete

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If the spliced out node  $x$  was red everything is fine.

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- ▶ Parent and child of  $x$  were red; two adjacent red vertices.
- ▶ If you delete the root, the root may now be red.

# Red Black Trees: Delete

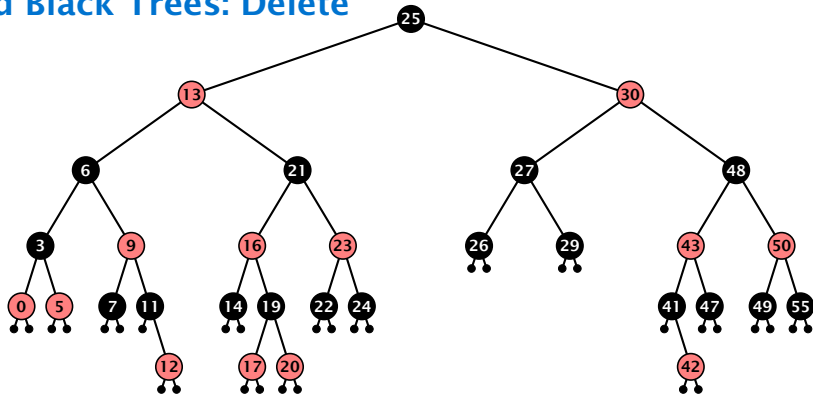
First do a standard delete.

If the spliced out node  $x$  was red everything is fine.

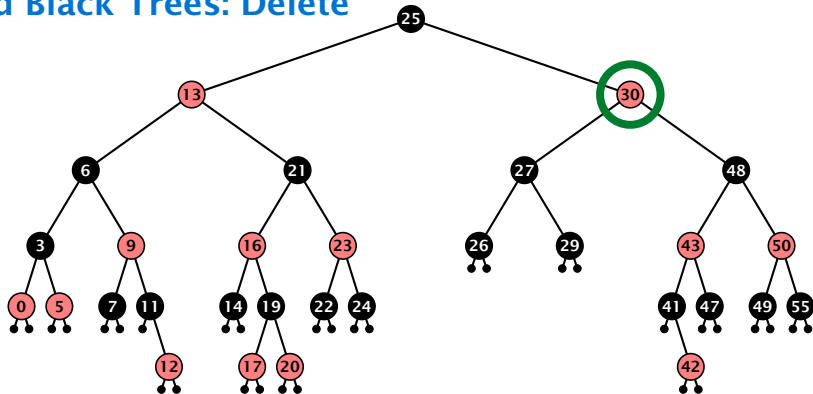
If it was black there may be the following problems.

- ▶ Parent and child of  $x$  were red; two adjacent red vertices.
- ▶ If you delete the root, the root may now be red.
- ▶ Every path from an ancestor of  $x$  to a descendant leaf of  $x$  changes the number of black nodes. Black height property might be violated.

## Red Black Trees: Delete



## Red Black Trees: Delete

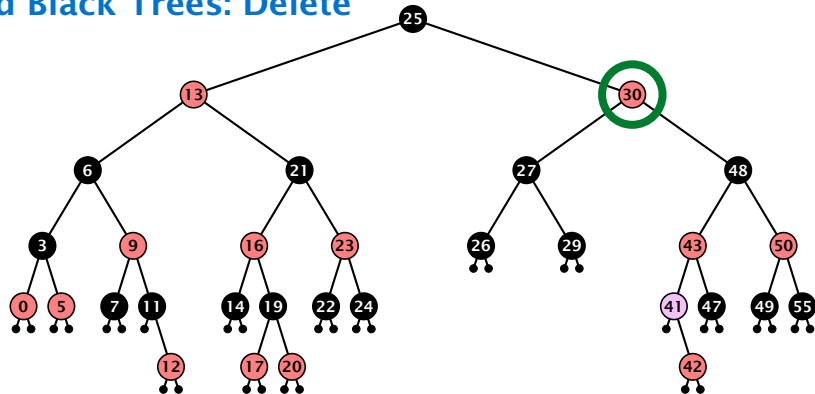


### Case 3:

Element has two children

- ▶ do normal delete
- ▶ when replacing content by content of successor, don't change color of node

## Red Black Trees: Delete

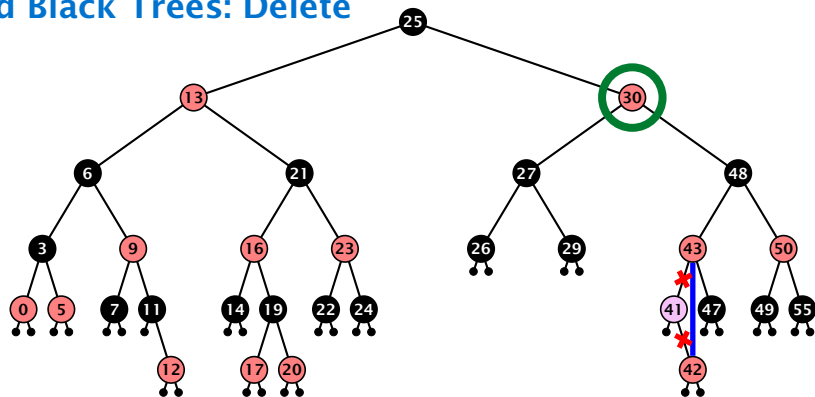


### Case 3:

Element has two children

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# Red Black Trees: Delete

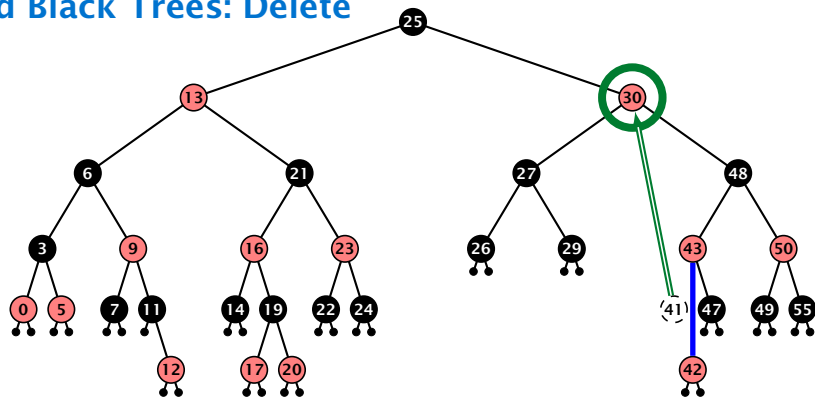


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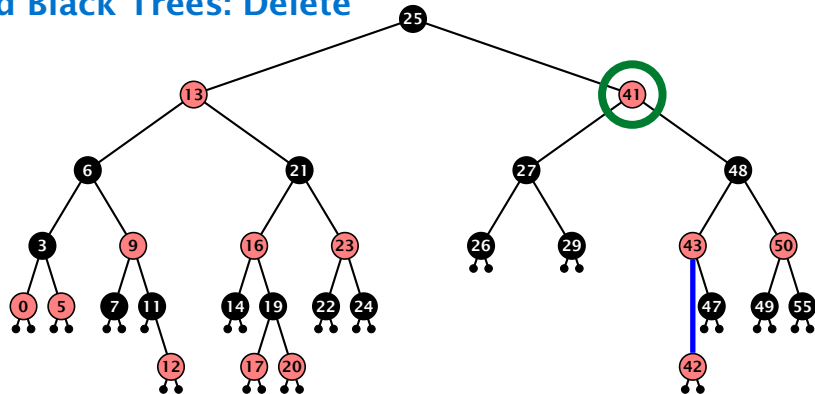


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Element has two children

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## Red Black Trees: Delete



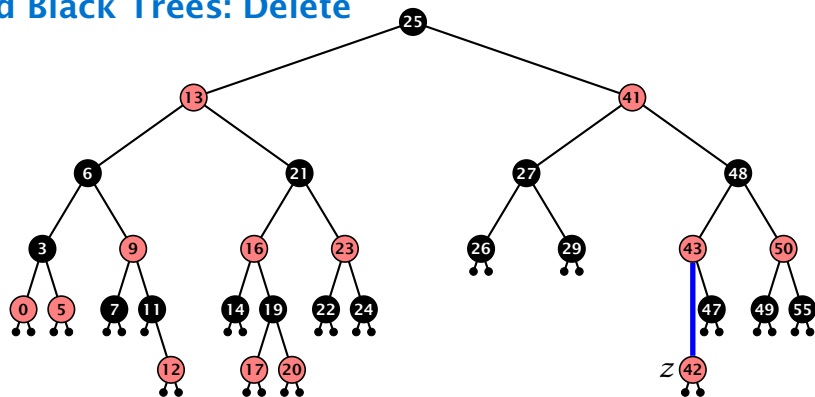
### Case 3:

Element has two children

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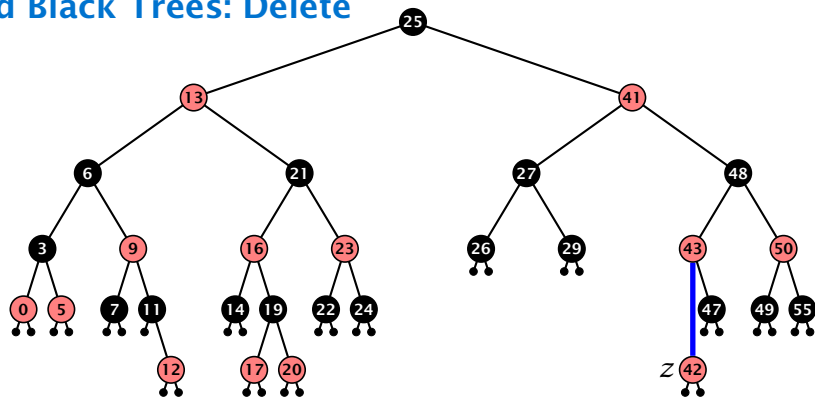
## Red Black Trees: Delete



Delete:

- ▶ deleting black node messes up black-height property

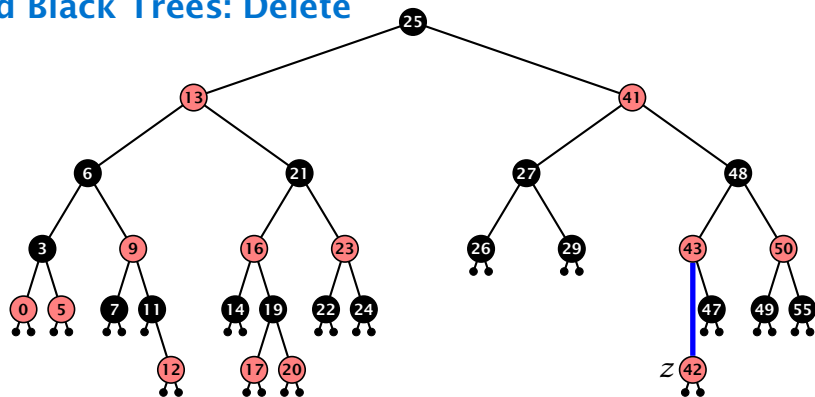
## Red Black Trees: Delete



### Delete:

- ▶ deleting black node messes up black-height property
- ▶ if  $z$  is red, we can simply color it black and everything is fine

## Red Black Trees: Delete



### Delete:

- ▶ deleting black node messes up black-height property
- ▶ if  $z$  is red, we can simply color it black and everything is fine
- ▶ the problem is if  $z$  is black (e.g. a dummy-leaf); we call a fix-up procedure to fix the problem.

# Red Black Trees: Delete

## Invariant of the fix-up algorithm

- ▶ the node  $z$  is black

# Red Black Trees: Delete

## Invariant of the fix-up algorithm

- ▶ the node  $z$  is black
- ▶ if we “assign” a fake black unit to the edge from  $z$  to its parent then the black-height property is fulfilled

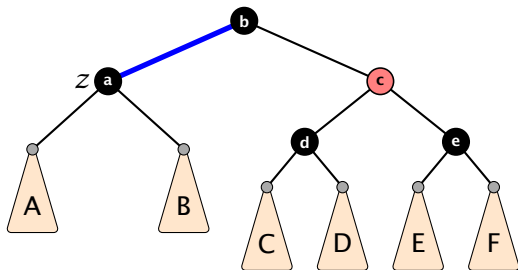
# Red Black Trees: Delete

## Invariant of the fix-up algorithm

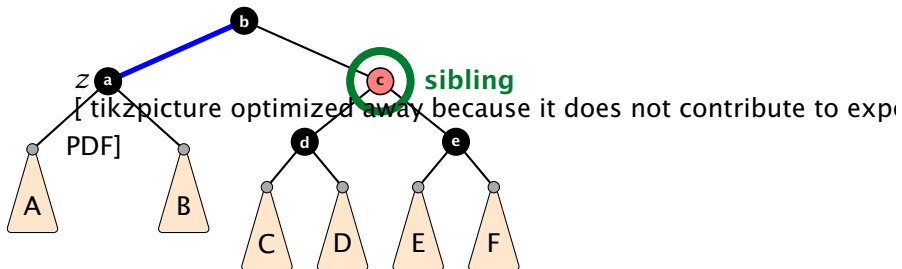
- ▶ the node  $z$  is black
- ▶ if we “assign” a fake black unit to the edge from  $z$  to its parent then the black-height property is fulfilled

**Goal:** make rotations in such a way that you at some point can remove the fake black unit from the edge.

## Case 1: Sibling of $z$ is red

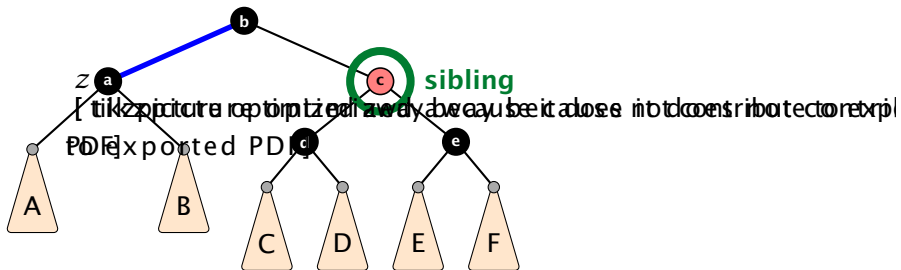


## Case 1: Sibling of z is red

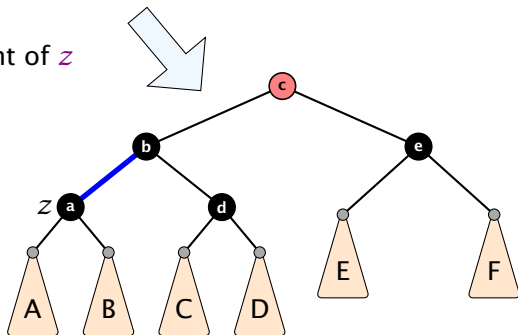




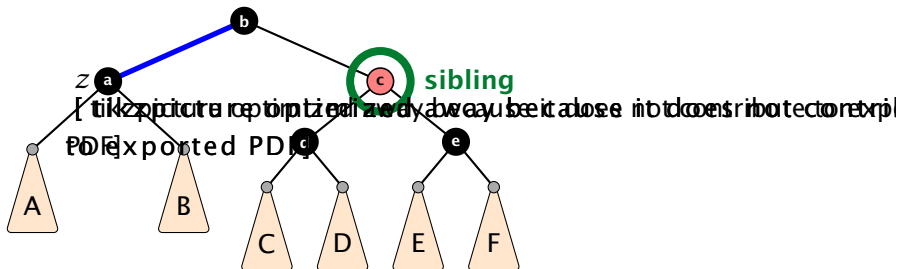
## Case 1: Sibling of $z$ is red



1. left-rotate around parent of  $z$

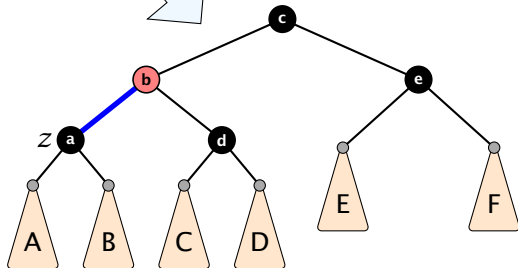


## Case 1: Sibling of $z$ is red

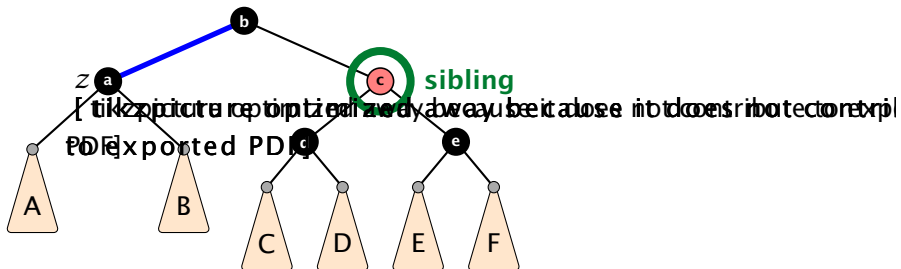


[tilt picture to optimize readability because it does not rotate properly]  
PDF exported PDF

1. left-rotate around parent of  $z$
2. recolor nodes  $b$  and  $c$

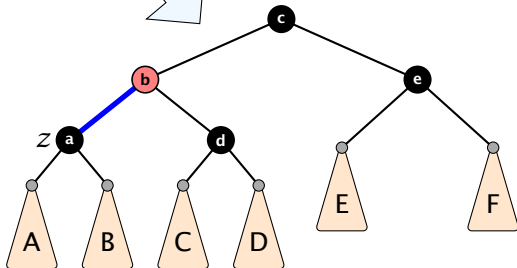


## Case 1: Sibling of $z$ is red

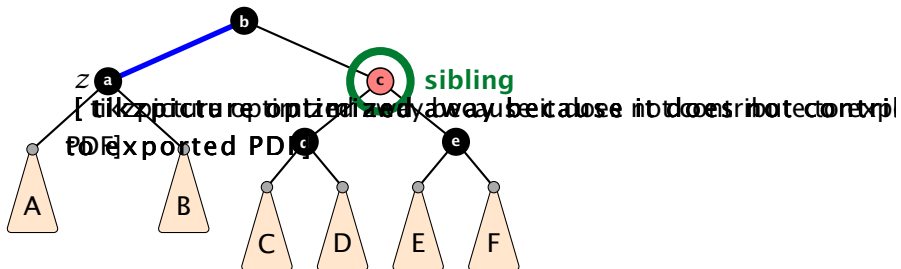


[tilt picture to optimize readability, because it does not control PD] exported PDF

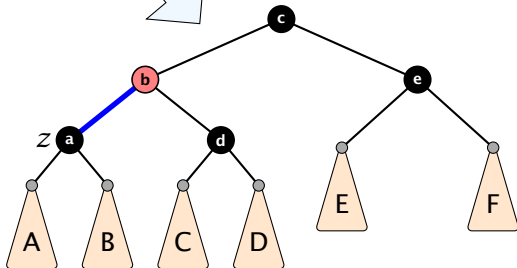
1. left-rotate around parent of  $z$
2. recolor nodes  $b$  and  $c$
3. the new sibling is black (and parent of  $z$  is red)



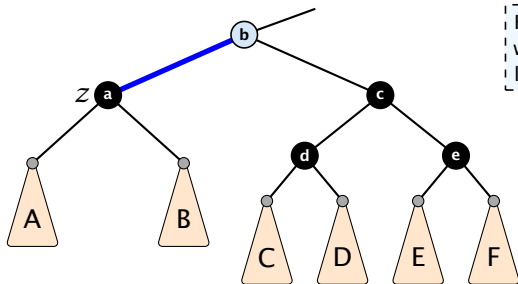
## Case 1: Sibling of $z$ is red



1. left-rotate around parent of  $z$
2. recolor nodes  $b$  and  $c$
3. the new sibling is black (and parent of  $z$  is red)
4. Case 2 (special), or Case 3, or Case 4

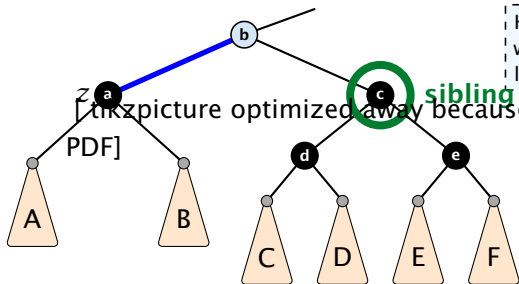


## Case 2: Sibling is black with two black children



Here b is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

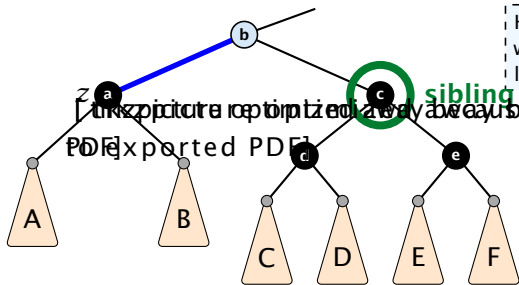
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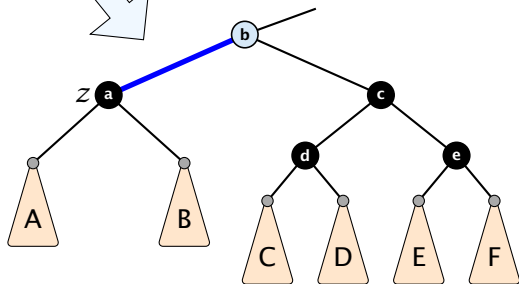
[tikzpicture optimized away because it does not contribute to exp PDF]

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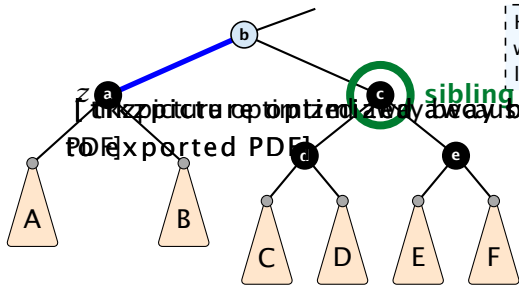


Here **b** is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

[The picture is optimized away because it does not describe a tree structure]



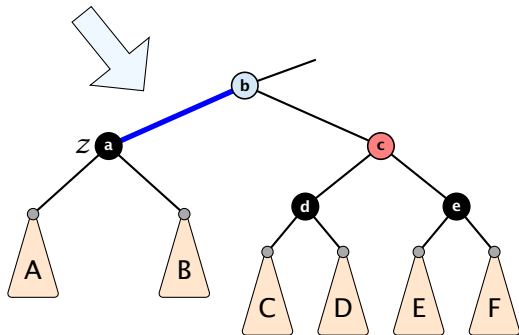
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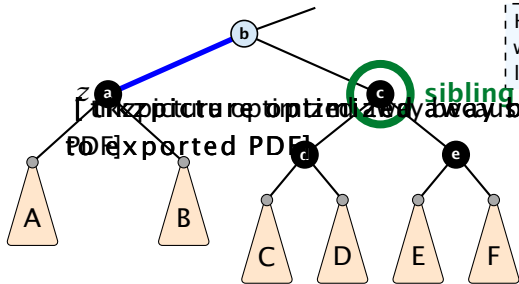
[The picture is optimized only because it does not describe a PDF exported PDF]

1. re-color node **c**



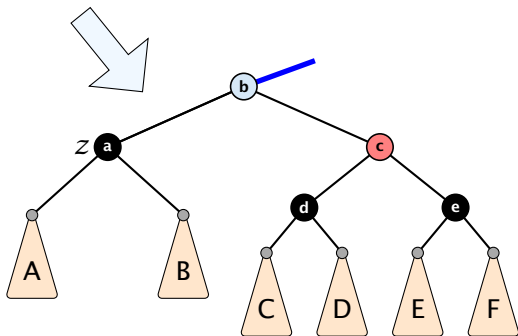


## Case 2: Sibling is black with two black children

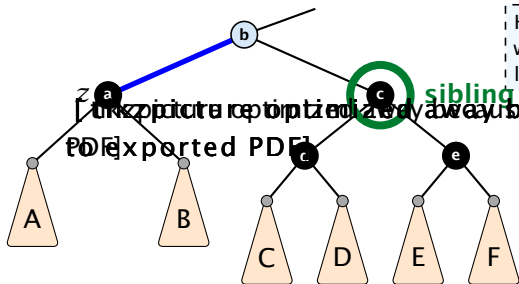


Here **b** is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

1. re-color node **c**
2. move fake black unit upwards

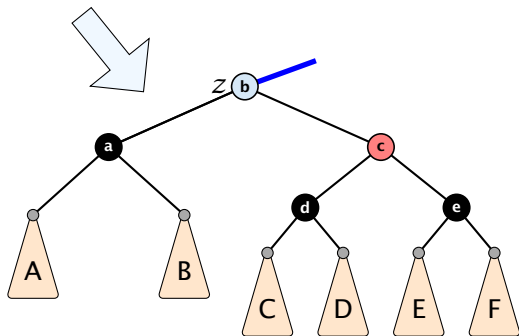


## Case 2: Sibling is black with two black children

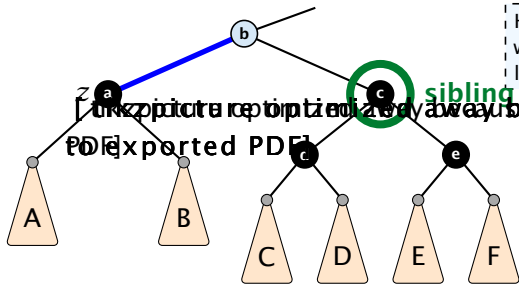


Here  $b$  is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

1. re-color node  $c$
2. move fake black unit upwards
3. move  $z$  upwards



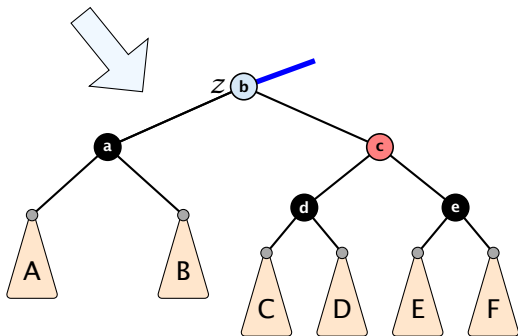
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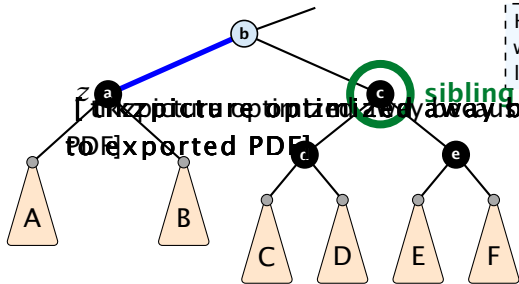
Here **b** is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

[The picture is optimized by always being close in order to control the exported PDF]

1. re-color node **c**
2. move fake black unit upwards
3. move **z** upwards
4. we made progress

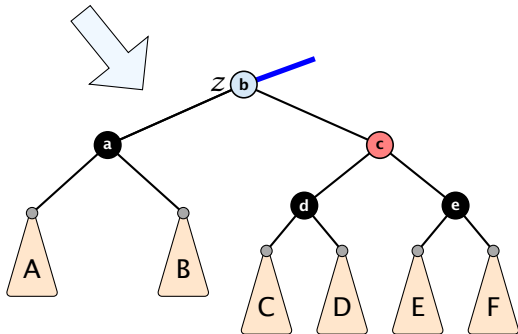


## Case 2: Sibling is black with two black children

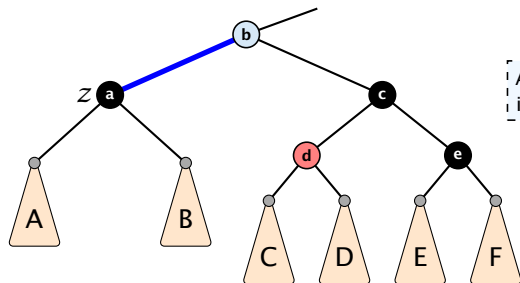


Here  $b$  is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

1. re-color node  $c$
2. move fake black unit upwards
3. move  $z$  upwards
4. we made progress
5. if  $b$  is red we color it black and are done



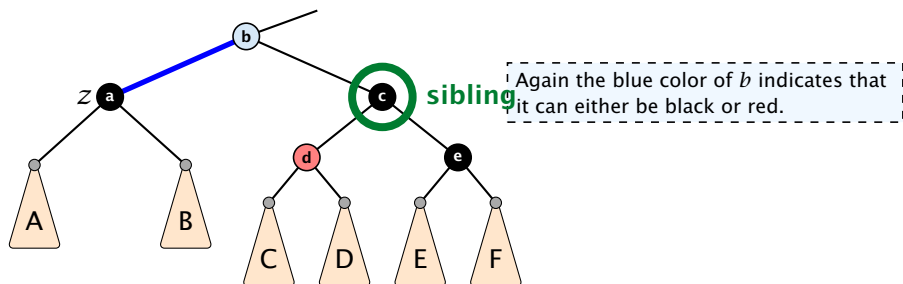
## Case 3: Sibling black with one black child to the right



Again the blue color of  $b$  indicates that it can either be black or red.

## Case 3: Sibling black with one black child to the right

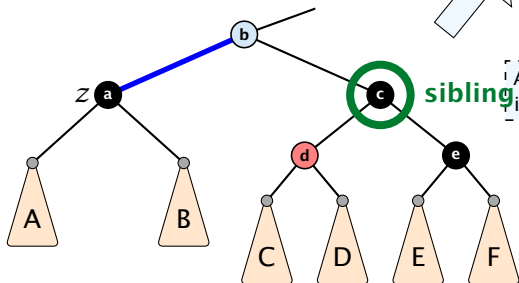
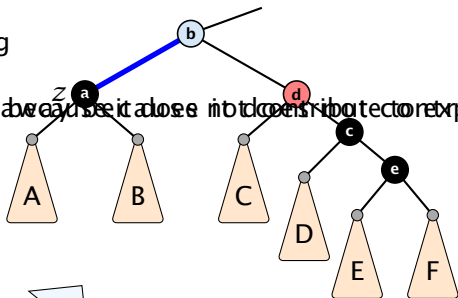
[ tikzpicture optimized away because it does not contribute to exp PDF]



# Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling

[tikzpicture optimized away because it does not describe a vector, PDF exported PDF]

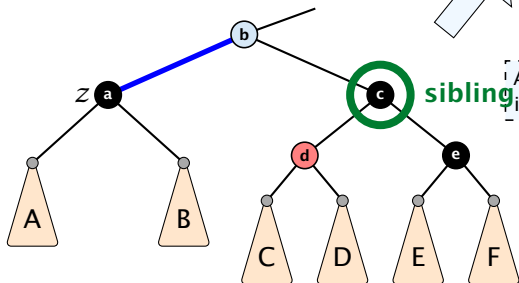
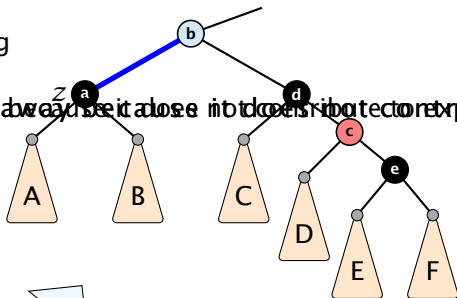


Again the blue color of **b** indicates that it can either be black or red.

# Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling
2. recolor  $c$  and  $d$ .

[tikzpicture optimized away because it does not describe a tree, PDF exported PDF]



Again the blue color of  $b$  indicates that it can either be black or red.

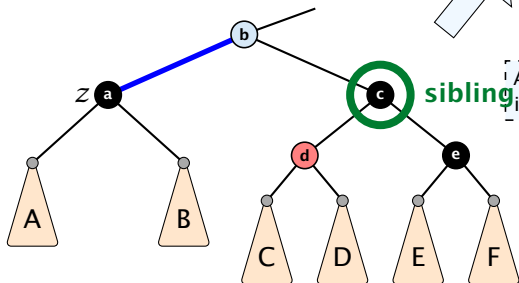
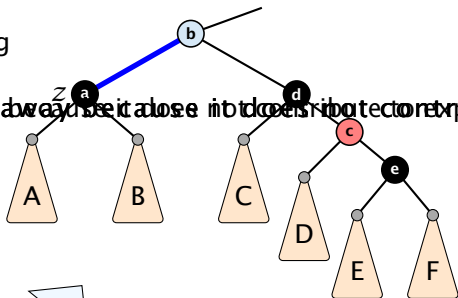


## Case 3: Sibling black with one black child to the right

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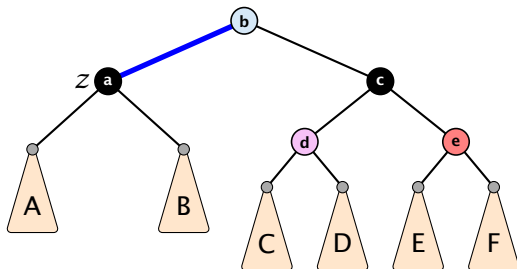
2. recolor  $c$  and  $d$ .

3. new sibling is black with red right child (Case 4)



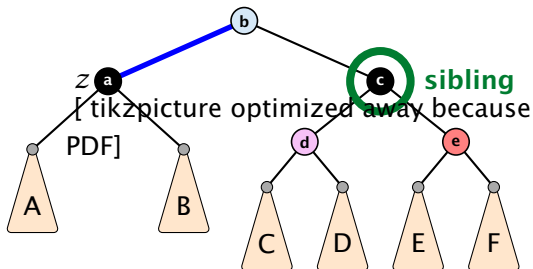
Again the blue color of  $b$  indicates that it can either be black or red.

## Case 4: Sibling is black with red right child



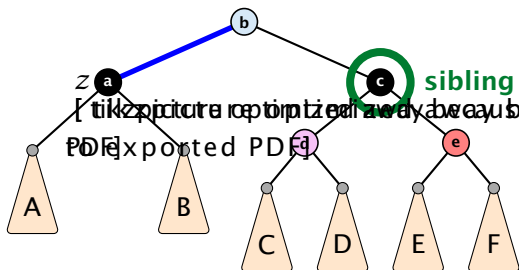
- Here b and d are either red or black but have possibly different colors.
- We recolor c by giving it the color of b.

## Case 4: Sibling is black with red right child



- Here b and d are either red or black but have possibly different colors.
- Does not contribute to exp color of b.

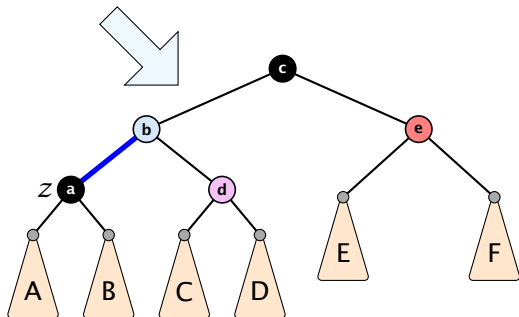
## Case 4: Sibling is black with red right child



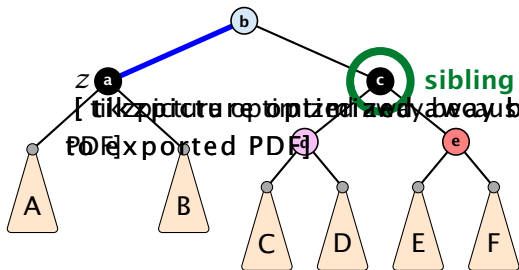
- Here  $b$  and  $d$  are either red or black but have possibly different colors.

[til picture optimized away because it does not contribute to the PDF] exported PDF

1. left-rotate around  $b$

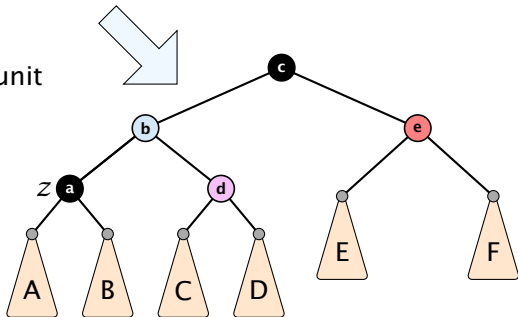


## Case 4: Sibling is black with red right child

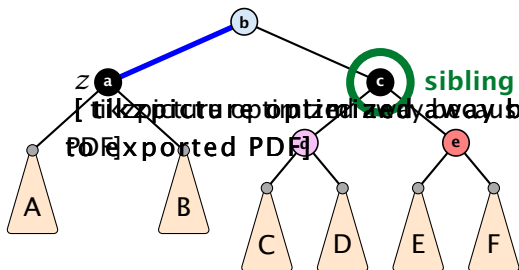


- Here **b** and **d** are either red or black but have possibly different colors.

- left-rotate around **b**
- remove the fake black unit

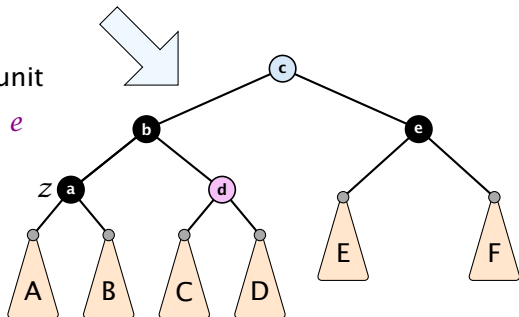


## Case 4: Sibling is black with red right child

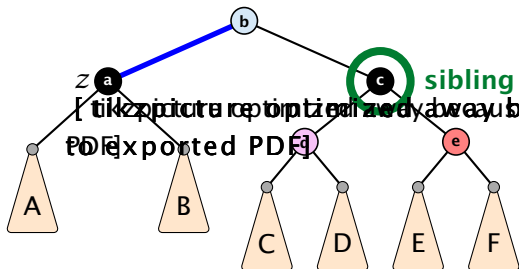


• Here  $b$  and  $d$  are either red or black but have possibly different colors.

1. left-rotate around  $b$
2. remove the fake black unit
3. recolor nodes  $b$ ,  $c$ , and  $e$

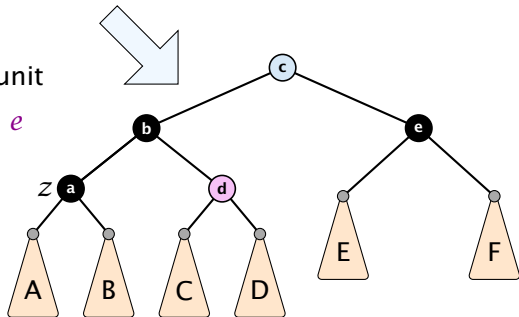


## Case 4: Sibling is black with red right child



- Here  $b$  and  $d$  are either red or black but have possibly different colors.

1. left-rotate around  $b$
2. remove the fake black unit
3. recolor nodes  $b$ ,  $c$ , and  $e$
4. you have a valid red black tree



## Running time:

- ▶ only Case 2 can repeat; but only  $h$  many steps, where  $h$  is the height of the tree



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Case 1 → Case 3 → Case 4 → red black tree  
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- ▶ Case 3 → Case 4 → red black tree
- ▶ Case 4 → red black tree

Performing Case 2 at most  $\mathcal{O}(\log n)$  times and every other step at most once, we get a red black tree. Hence,  $\mathcal{O}(\log n)$  re-colorings and at most 3 rotations.

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- only amortized guarantee
- read-operations change the tree

# Splay Trees

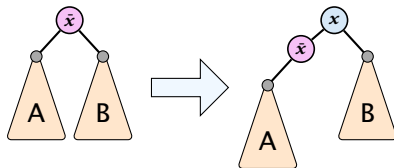
## **find( $x$ )**

- ▶ search for  $x$  according to a search tree
- ▶ let  $\tilde{x}$  be last element on search-path
- ▶ splay( $\tilde{x}$ )

# Splay Trees

## insert( $x$ )

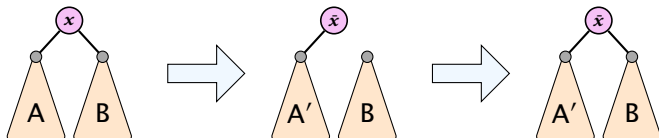
- ▶ search for  $x$ ;  $\bar{x}$  is last visited element during search (successor or predecessor of  $x$ )
- ▶ splay( $\bar{x}$ ) moves  $\bar{x}$  to the root
- ▶ insert  $x$  as new root



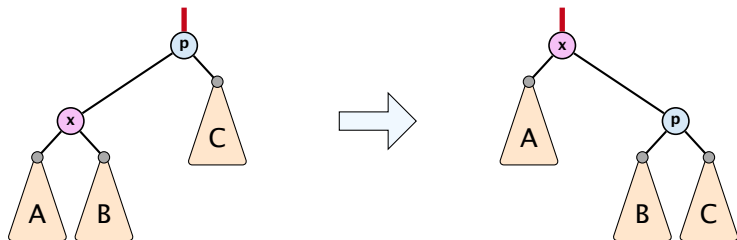
# Splay Trees

## delete( $x$ )

- ▶ search for  $x$ ; splay( $x$ ); remove  $x$
- ▶ search largest element  $\bar{x}$  in  $A$
- ▶ splay( $\bar{x}$ ) (on subtree  $A$ )
- ▶ connect root of  $B$  as right child of  $\bar{x}$



# Move to Root

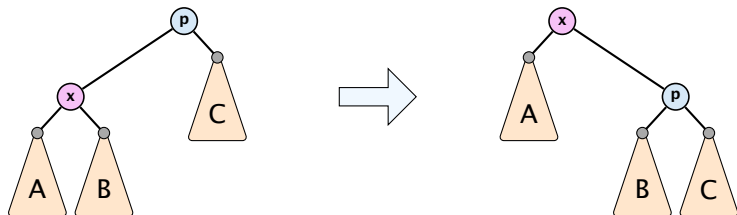


## How to bring element to root?

- ▶ one (bad) option: `moveToRoot( $x$ )`
- ▶ iteratively do rotation around parent of  $x$  until  $x$  is root
- ▶ if  $x$  is left child do right rotation otw. left rotation



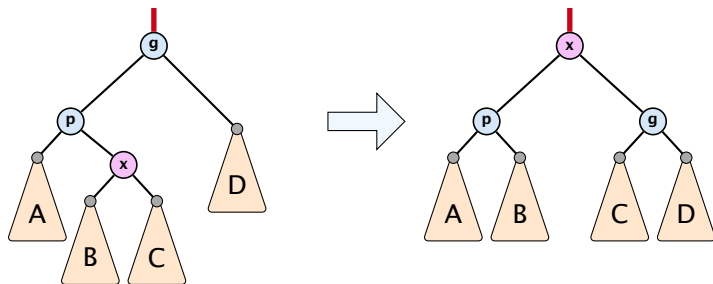
## Splay: Zig Case



**better option splay( $x$ ):**

- ▶ zig case: if  $x$  is child of root do left rotation or right rotation around parent

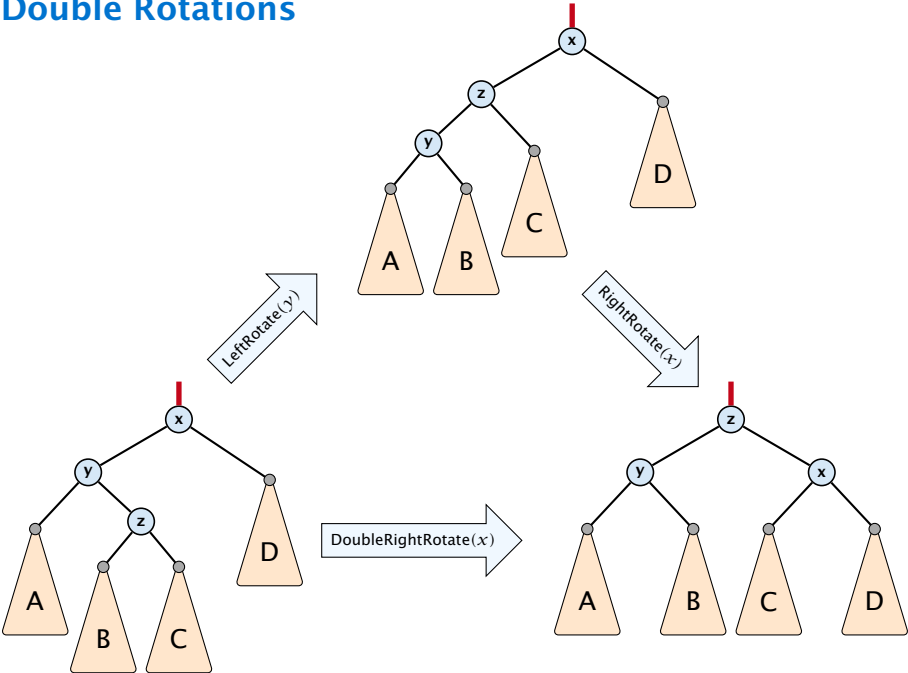
## Splay: Zigzag Case



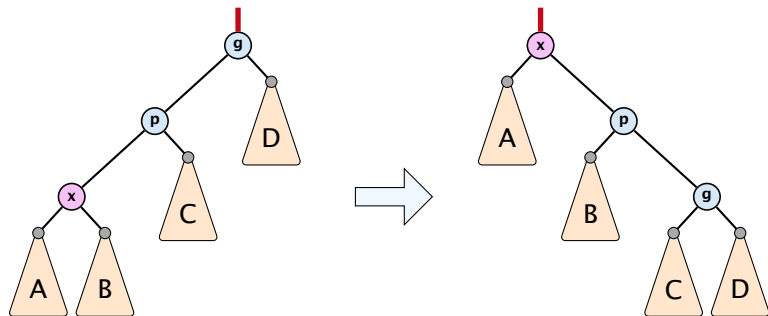
### better option $\text{splay}(x)$ :

- ▶ zigzag case: if  $x$  is right child and parent of  $x$  is left child (or  $x$  left child parent of  $x$  right child)
- ▶ do double right rotation around grand-parent (resp. double left rotation)

# Double Rotations



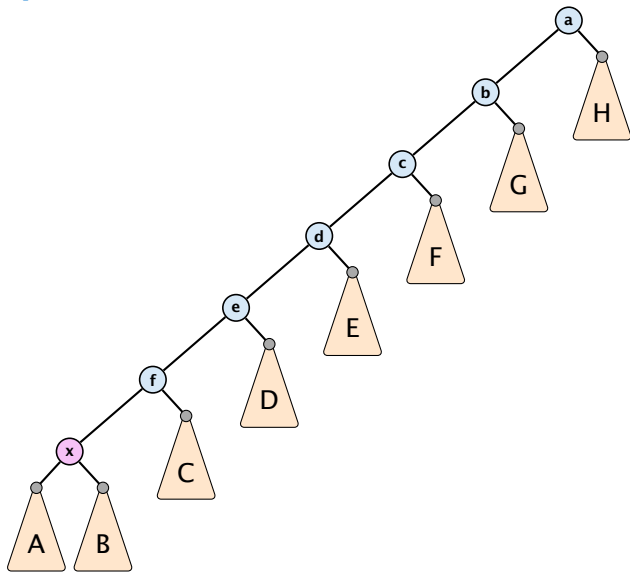
## Splay: Zigzig Case



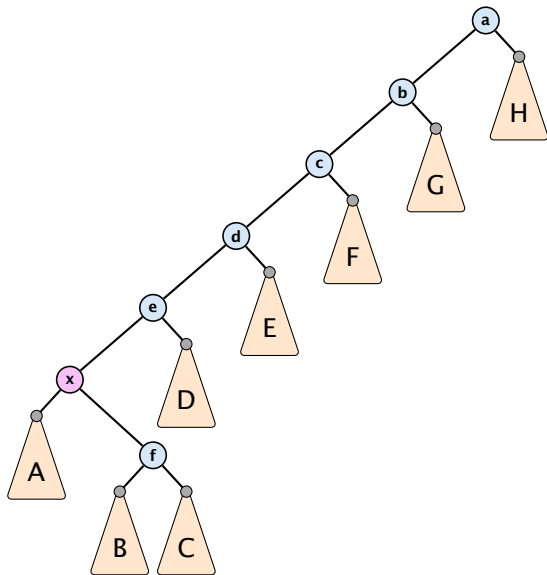
### better option $\text{splay}(x)$ :

- ▶ zigzig case: if  $x$  is left child and parent of  $x$  is left child (or  $x$  right child, parent of  $x$  right child)
- ▶ do right rotation around grand-parent followed by right rotation around parent (resp. left rotations)

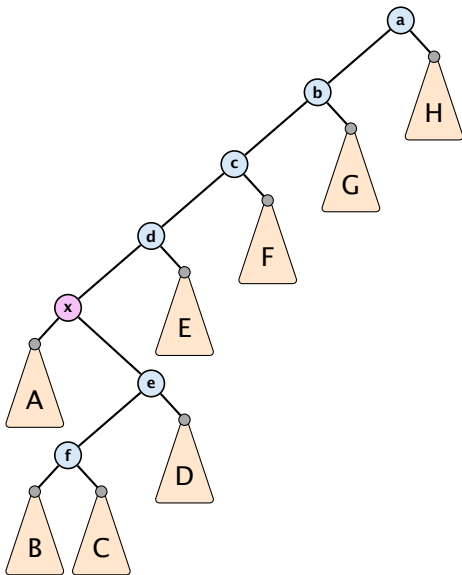
# Splay vs. Move to Root



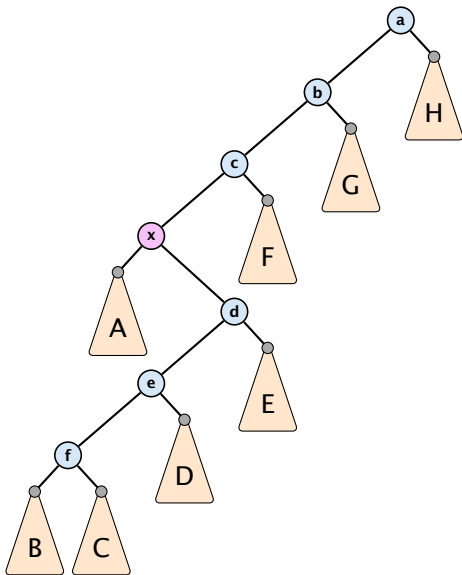
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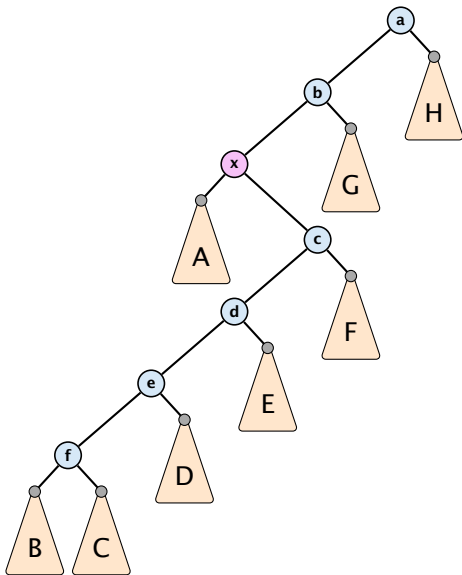


# Splay vs. Move to Root

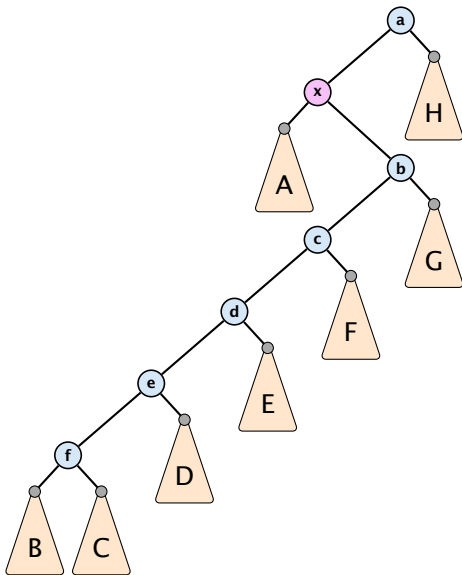




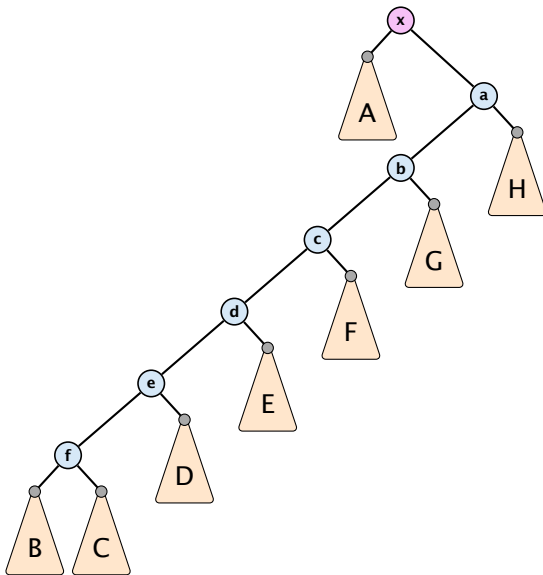
# Splay vs. Move to Root



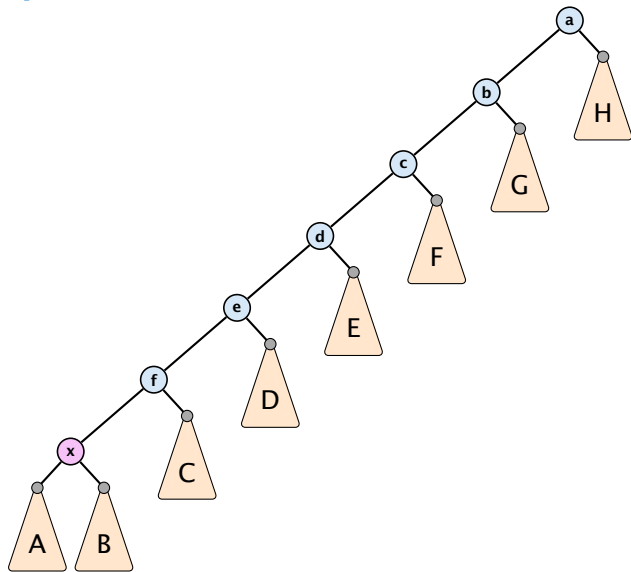
# Splay vs. Move to Root



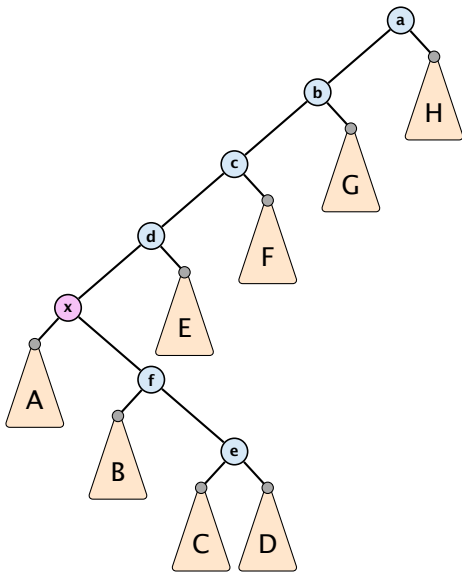
# Splay vs. Move to Root



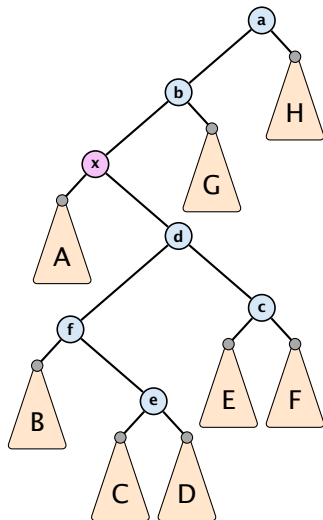
# Splay vs. Move to Root



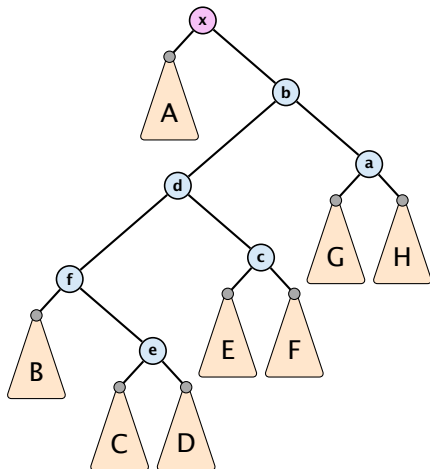
# Splay vs. Move to Root



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# Splay vs. Move to Root



# Static Optimality

Suppose we have a sequence of  $m$  find-operations.  $\text{find}(x)$  appears  $h_x$  times in this sequence.

The cost of a **static** search tree  $T$  is:

$$\text{cost}(T) = m + \sum_x h_x \text{depth}_T(x)$$

The total cost for processing the sequence on a splay-tree is  $\mathcal{O}(\text{cost}(T_{\min}))$ , where  $T_{\min}$  is an **optimal static search tree**.



# Dynamic Optimality

Let  $S$  be a sequence with  $m$  find-operations.

Let  $A$  be a data-structure based on a search tree:

- ▶ the cost for accessing element  $x$  is  $1 + \text{depth}(x)$ ;
- ▶ after accessing  $x$  the tree may be re-arranged through rotations;

## Conjecture:

A splay tree that only contains elements from  $S$  has cost  $\mathcal{O}(\text{cost}(A, S))$ , for processing  $S$ .

## Lemma 5

*Splay Trees have an **amortized** running time of  $\mathcal{O}(\log n)$  for all operations.*

# Amortized Analysis

## Definition 6

A data structure with operations  $\text{op}_1(), \dots, \text{op}_k()$  has amortized running times  $t_1, \dots, t_k$  for these operations if the following holds.

Suppose you are given a sequence of operations (**starting with an empty data-structure**) that operate on at most  $n$  elements, and let  $k_i$  denote the number of occurrences of  $\text{op}_i()$  within this sequence. Then the actual running time must be at most  $\sum_i k_i \cdot t_i(n)$ .

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$$\sum_{i=1}^k c_i \leq \sum_{i=1}^k c_i + \Phi(D_k) - \Phi(D_0) = \sum_{i=1}^k \hat{c}_i$$

This means the amortized costs can be used to derive a bound on the total cost.

# Example: Stack

## Stack

- ▶  $S.$  push()
- ▶  $S.$  pop()
- ▶  $S.$  multipop( $k$ ): removes  $k$  items from the stack. If the stack currently contains less than  $k$  items it empties the stack.
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## Actual cost:

- ▶  $S.$  push(): cost 1.
- ▶  $S.$  pop(): cost 1.
- ▶  $S.$  multipop( $k$ ): cost  $\min\{\text{size}, k\} = k$ .

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- ▶  **$S$ . multipop( $k$ ):** cost

$$\hat{C}_{\text{mp}} = C_{\text{mp}} + \Delta\Phi = \min\{\text{size}, k\} - \min\{\text{size}, k\} \leq 0 .$$



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### **Incrementing a binary counter:**

Consider a computational model where each bit-operation costs one time-unit.

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### Actual cost:

- ▶ Changing bit from 0 to 1: cost 1.
- ▶ Changing bit from 1 to 0: cost 1.
- ▶ **Increment**: cost is  $k + 1$ , where  $k$  is the number of consecutive ones in the least significant bit-positions (e.g, 001101 has  $k = 1$ ).

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- ▶ **Increment:** Let  $k$  denotes the number of consecutive ones in the least significant bit-positions. An increment involves  $k$   $(1 \rightarrow 0)$ -operations, and one  $(0 \rightarrow 1)$ -operation.

Hence, the amortized cost is  $k\hat{C}_{1 \rightarrow 0} + \hat{C}_{0 \rightarrow 1} \leq 2$ .



# Splay Trees

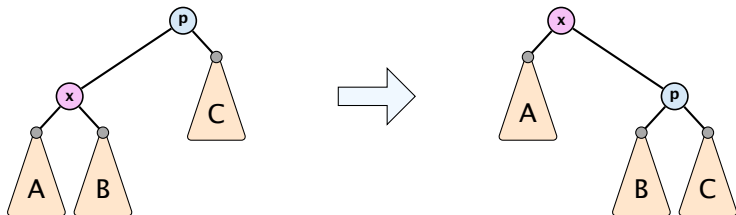
## potential function for splay trees:

- ▶ size  $s(x) = |T_x|$
- ▶ rank  $r(x) = \log_2(s(x))$
- ▶  $\Phi(T) = \sum_{v \in T} r(v)$

amortized cost = real cost + potential change

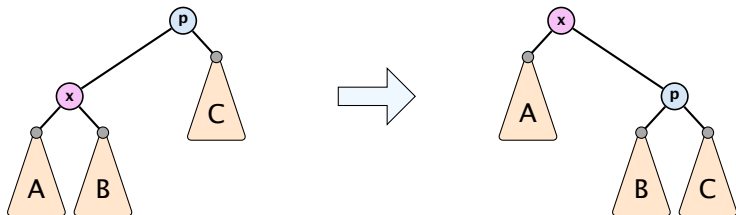
The cost is essentially the cost of the splay-operation, which is 1 plus the number of rotations.

## Splay: Zig Case



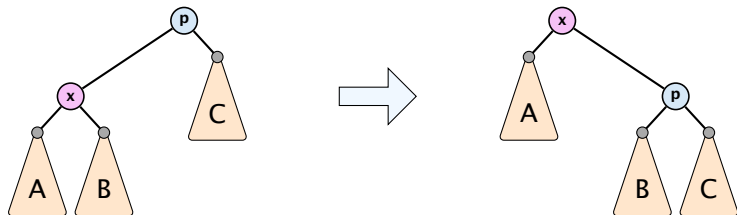
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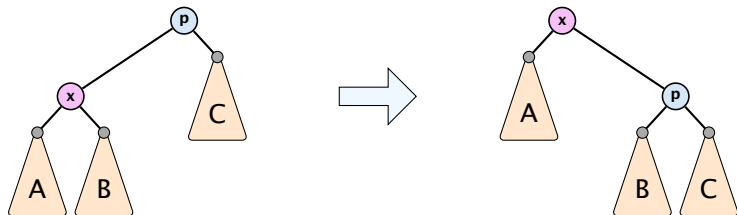
$$\Delta\Phi = r'(x) + r'(p) - r(x) - r(p)$$

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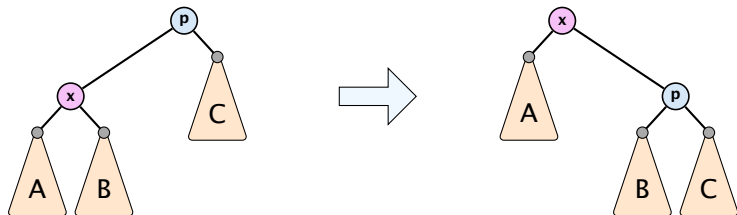
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p) \\ &= r'(p) - r(x)\end{aligned}$$

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$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p) \\ &= r'(p) - r(x) \\ &\leq r'(x) - r(x)\end{aligned}$$

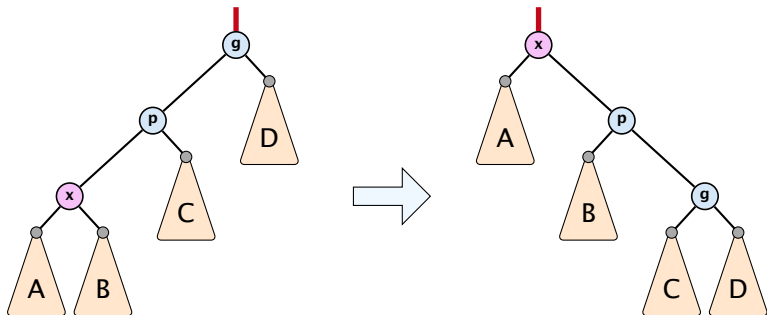
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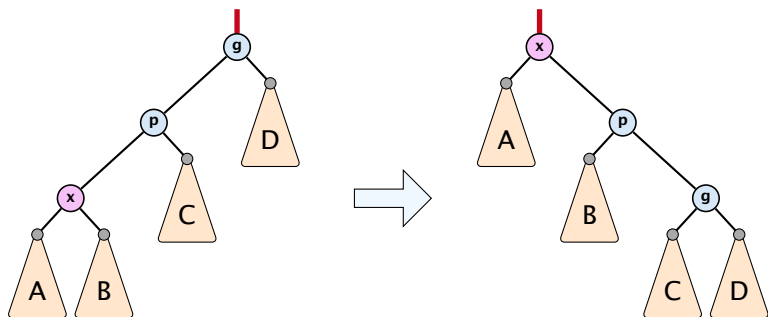
$$\text{cost}_{\text{zig}} \leq 1 + 3(r'(x) - r(x))$$

## Splay: Zigzig Case



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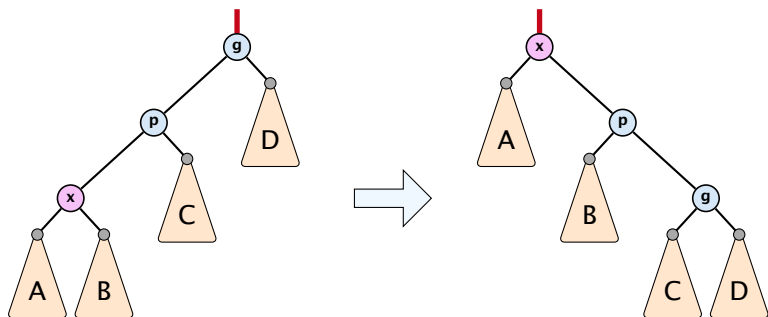
## Splay: Zigzig Case



$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$

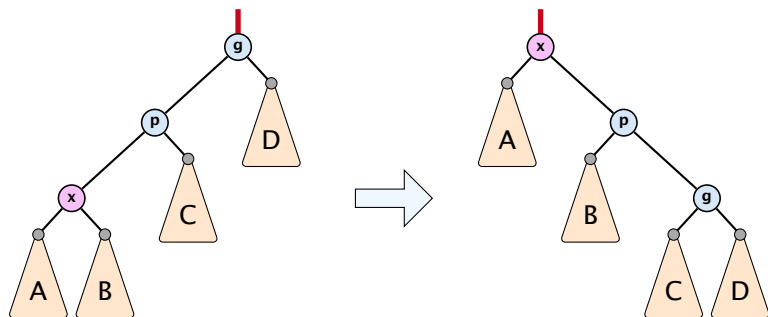


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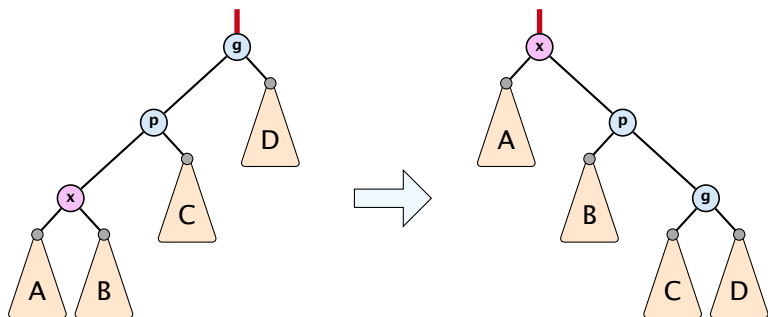
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p)\end{aligned}$$

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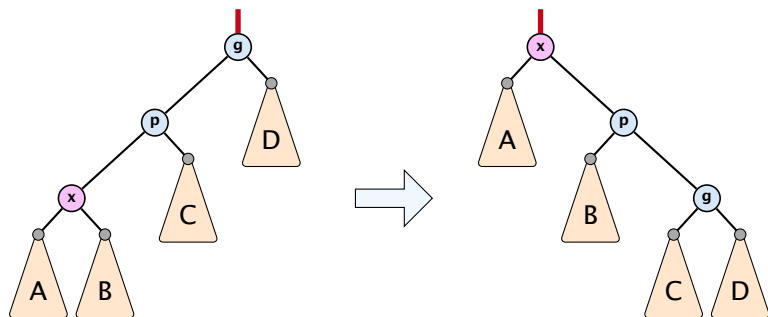
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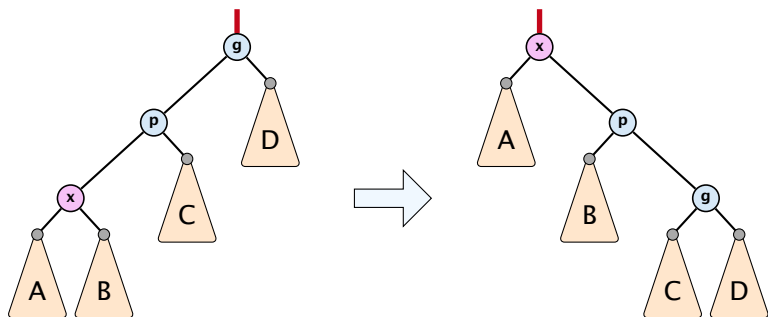
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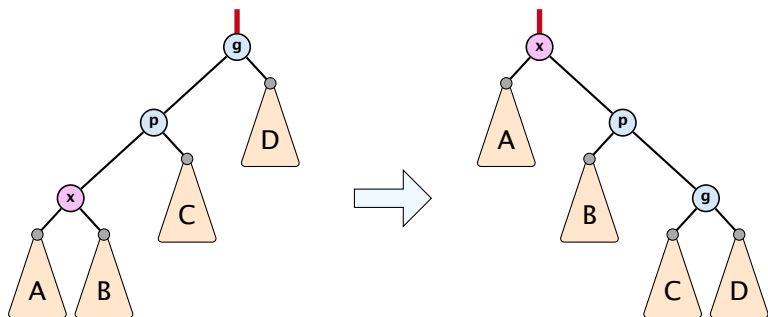
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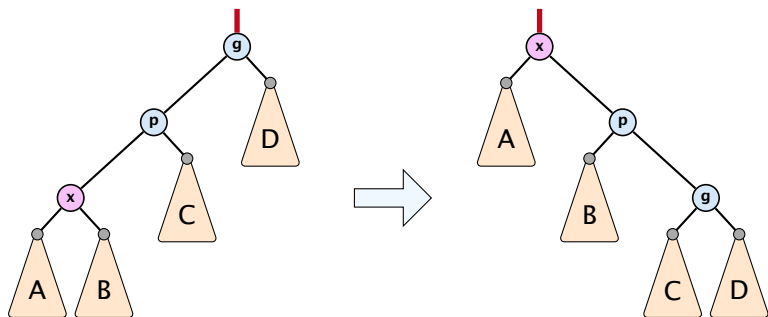
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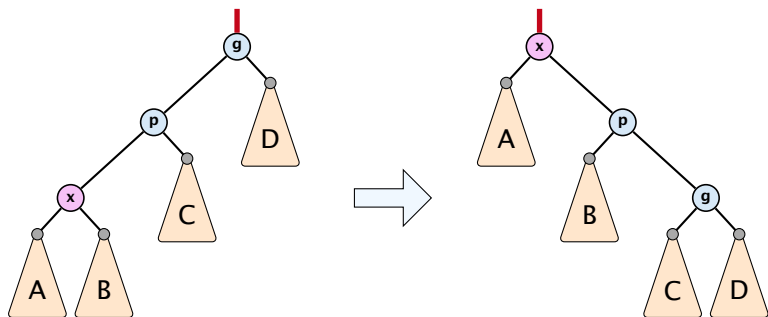
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## Splay: Zigzig Case



$$\frac{1}{2}(r(x) + r'(g) - 2r'(x))$$

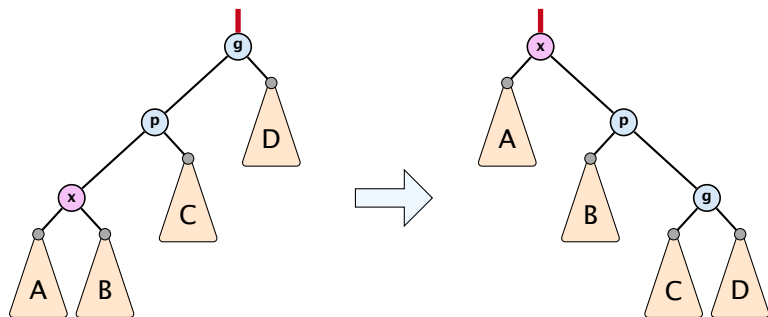
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$$\begin{aligned} & \frac{1}{2} (r(x) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} (\log(s(x)) + \log(s'(g)) - 2\log(s'(x))) \end{aligned}$$

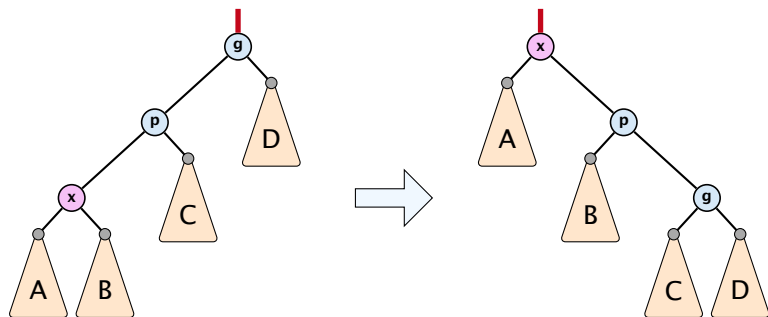


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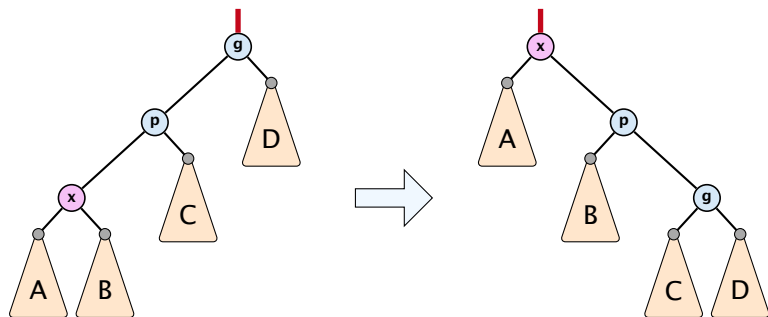
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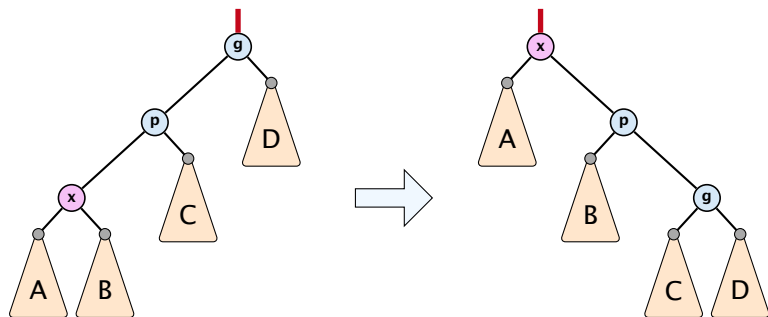
$$\begin{aligned} & \frac{1}{2} (r(x) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} \left( \log(s(x)) + \log(s'(g)) - 2 \log(s'(x)) \right) \\ &= \frac{1}{2} \log \left( \frac{s(x)}{s'(x)} \right) + \frac{1}{2} \log \left( \frac{s'(g)}{s'(x)} \right) \\ &\leq \log \left( \frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)} \right) \end{aligned}$$

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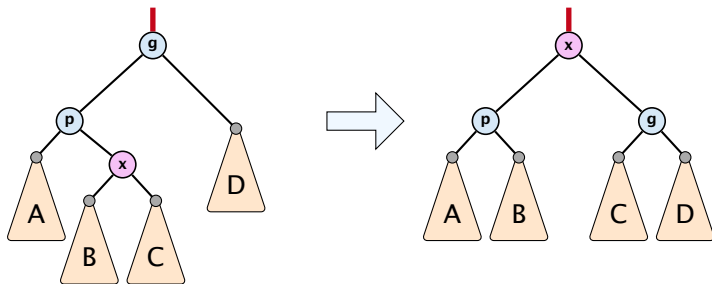
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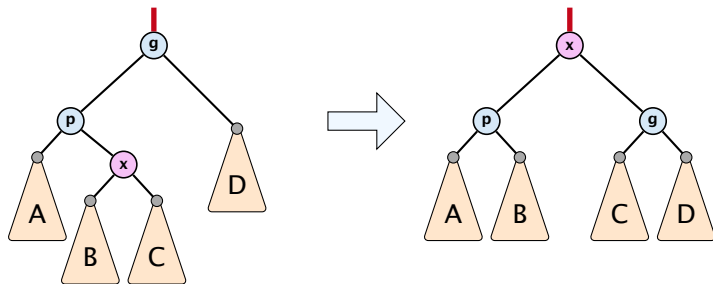
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## Splay: Zigzag Case



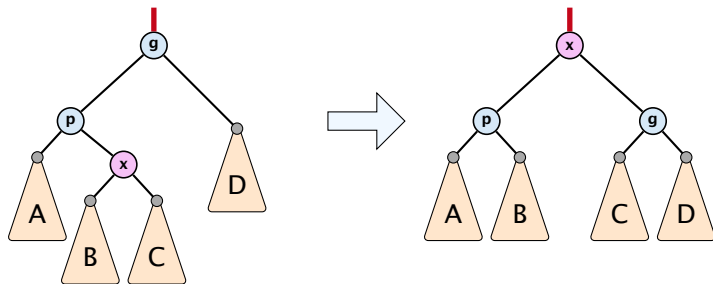
$\Delta\Phi =$

## Splay: Zigzag Case



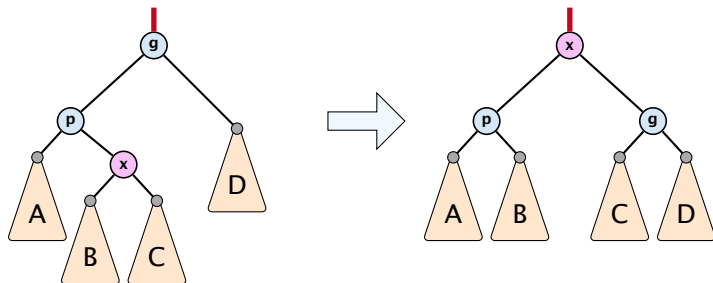
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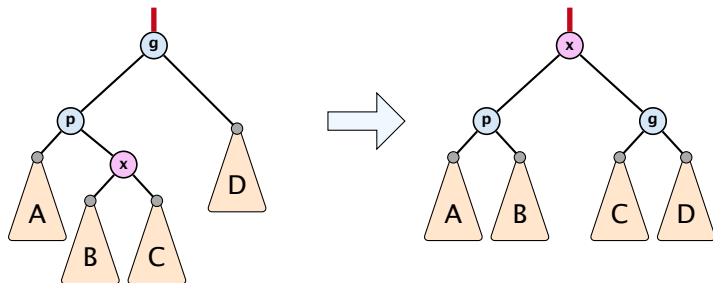
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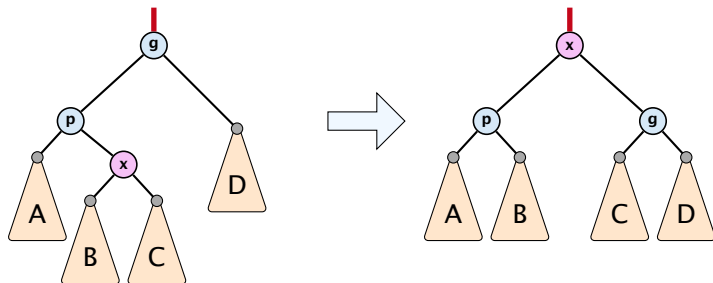


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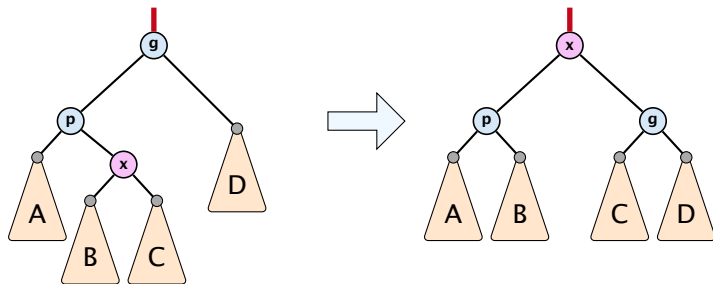
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## Splay: Zigzag Case



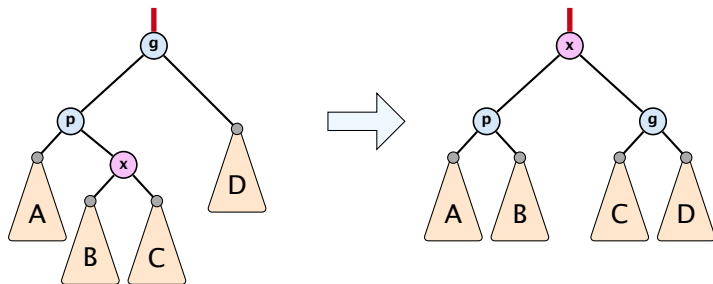
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## Splay: Zigzag Case



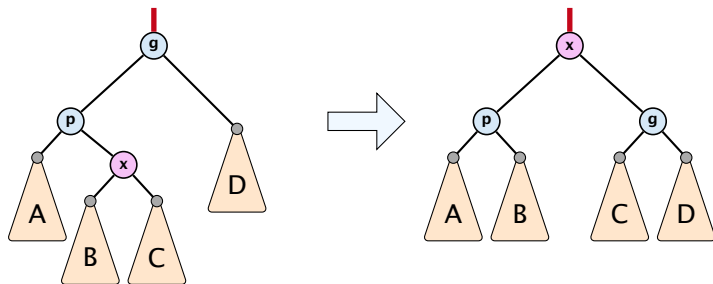
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## Splay: Zigzag Case



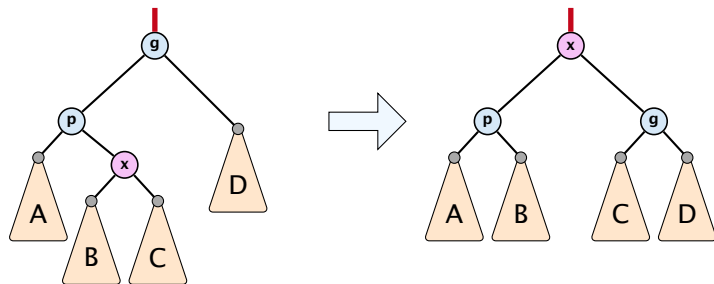
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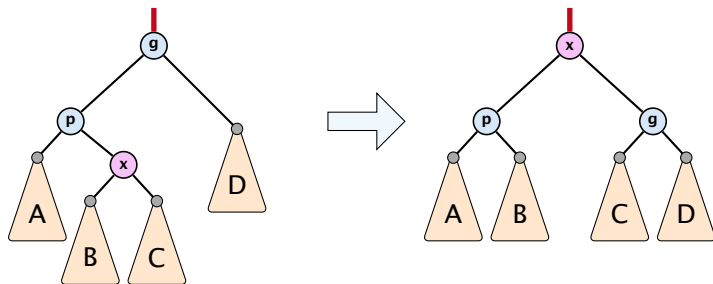
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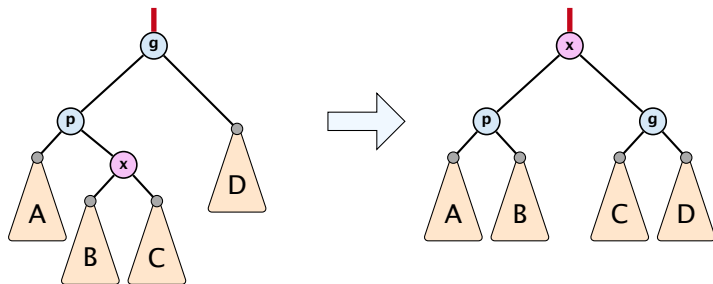
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## Splay: Zigzag Case



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Amortized cost of the whole splay operation:

$$\begin{aligned} &\leq 1 + 1 + \sum_{\text{steps } t} 3(r_t(x) - r_{t-1}(x)) \\ &= 2 + 3(r(\text{root}) - r_0(x)) \\ &\leq \mathcal{O}(\log n) \end{aligned}$$

## 7.4 Augmenting Data Structures

Suppose you want to develop a data structure with:

- ▶ **Insert( $x$ )**: insert element  $x$ .
- ▶ **Search( $k$ )**: search for element with key  $k$ .
- ▶ **Delete( $x$ )**: delete element referenced by pointer  $x$ .
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**Augment an existing data-structure instead of developing a new one.**

## 7.4 Augmenting Data Structures

### How to augment a data-structure

1. choose an underlying data-structure

- Of course, the above steps heavily depend on each other. For example it makes no sense to choose additional information to be stored (Step 2), and later realize that either the information cannot be maintained efficiently (Step 3) or is not sufficient to support the new operations (Step 4).
- However, the above outline is a good way to describe/document a new data-structure.

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### How to augment a data-structure

1. choose an underlying data-structure
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## 7.4 Augmenting Data Structures

### How to augment a data-structure

1. choose an underlying data-structure
2. determine additional information to be stored in the underlying structure
3. verify/show how the additional information can be maintained for the basic modifying operations on the underlying structure.

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## 7.4 Augmenting Data Structures

**Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time  $\mathcal{O}(\log n)$ .**

1. We choose a red-black tree as the underlying data-structure.



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3. We need to be able to update the size-field in each node without asymptotically affecting the running time of insert, delete, and search. We come back to this step later...

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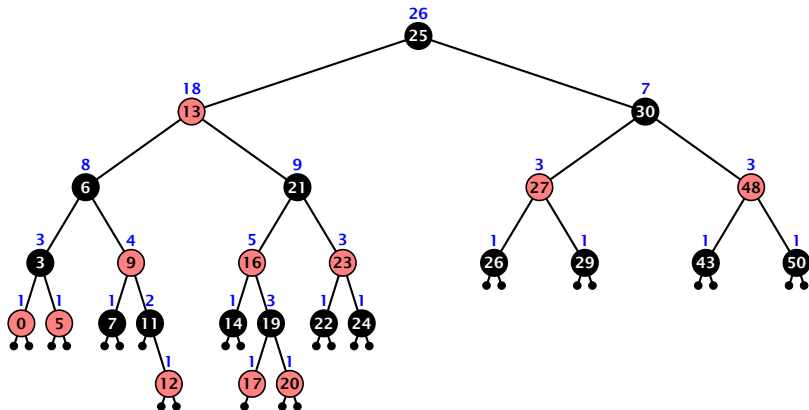
4. How does find-by-rank work?

Find-by-rank( $k$ ) := Select( $\text{root}, k$ ) with

**Algorithm 1** Select( $x, i$ )

```
1: if  $x = \text{null}$  then return error
2: if  $\text{left}[x] \neq \text{null}$  then  $r \leftarrow \text{left}[x].\text{size} + 1$  else  $r \leftarrow 1$ 
3: if  $i = r$  then return  $x$ 
4: if  $i < r$  then
5:     return Select( $\text{left}[x], i$ )
6: else
7:     return Select( $\text{right}[x], i - r$ )
```

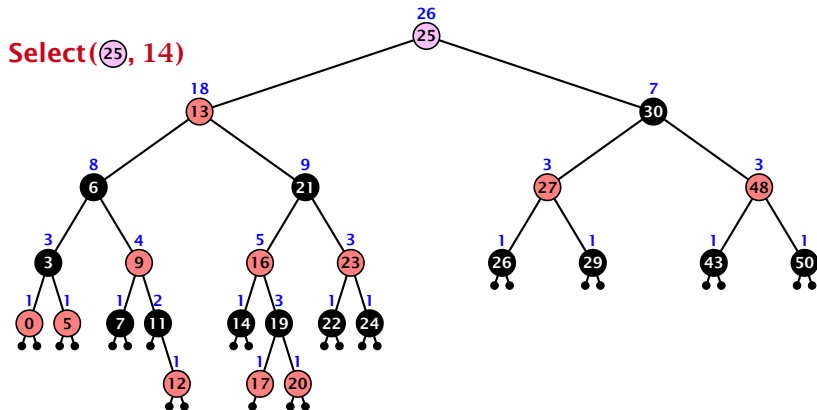
## Select( $x, i$ )



### Find-by-rank:

- ▶ decide whether you have to proceed into the left or right sub-tree
- ▶ adjust the rank that you are searching for if you go right

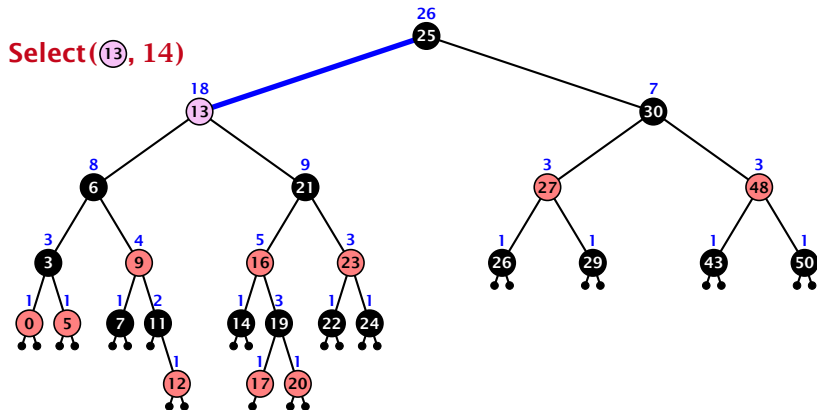
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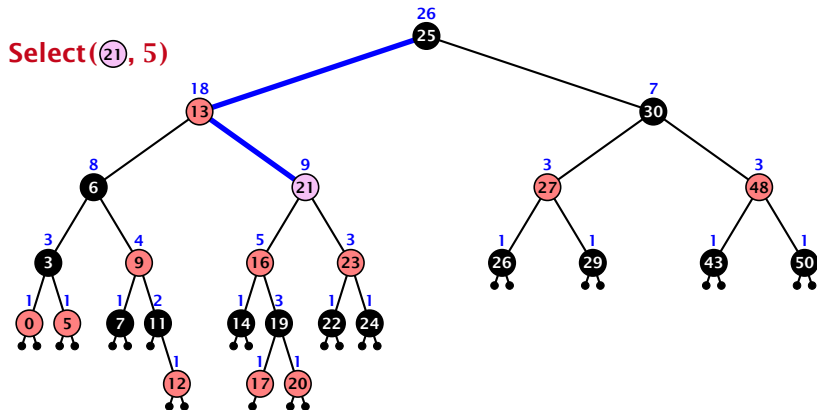
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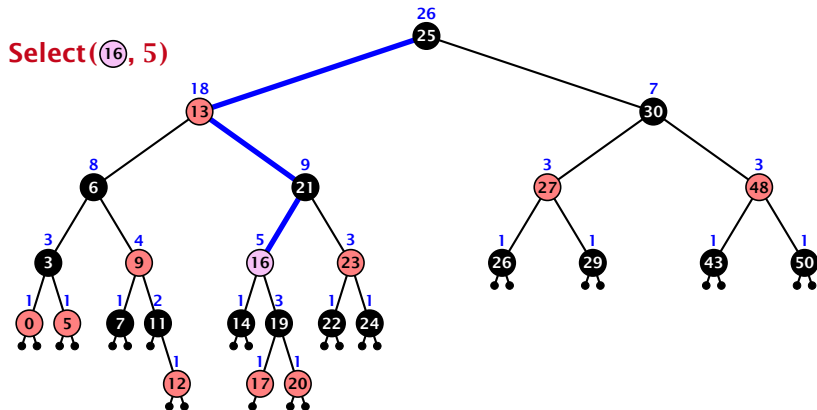
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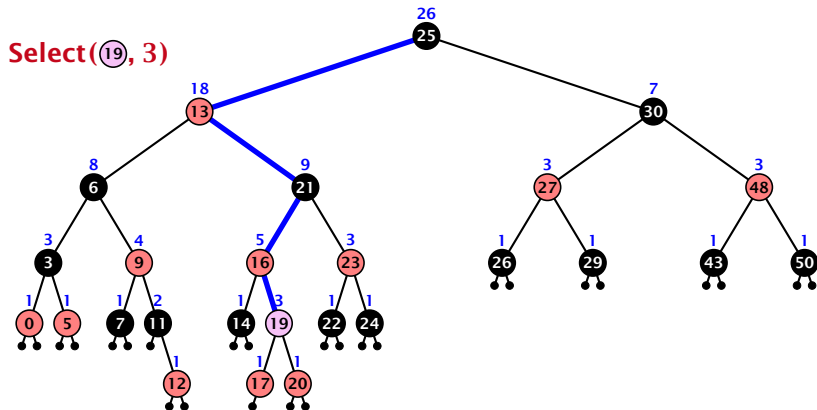


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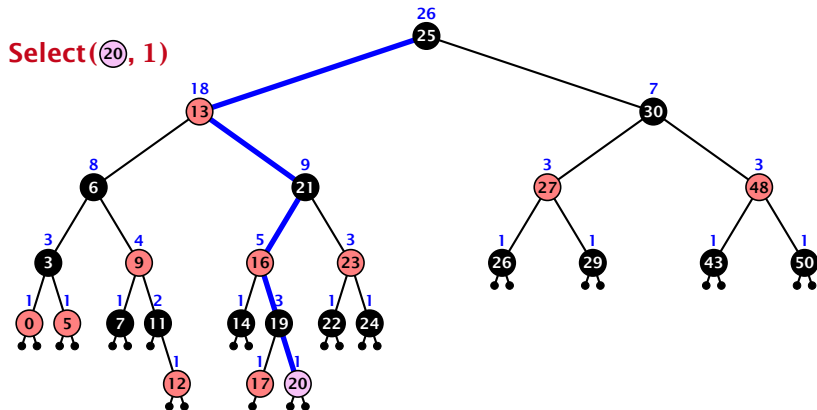
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**Insert( $x$ ):** When going down the search path increase the size field for each visited node. **Maintain the size field during rotations.**

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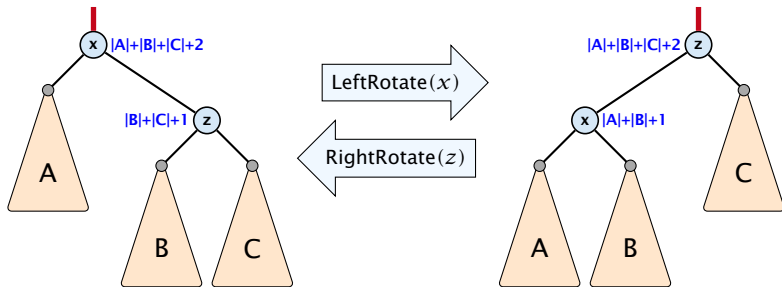
**Search( $k$ ):** Nothing to do.

**Insert( $x$ ):** When going down the search path increase the size field for each visited node. **Maintain the size field during rotations.**

**Delete( $x$ ):** Directly after splicing out a node traverse the path from the spliced out node upwards, and decrease the size counter on every node on this path. **Maintain the size field during rotations.**

# Rotations

The only operation during the fix-up procedure that alters the tree and requires an update of the size-field:



The nodes  $x$  and  $z$  are the only nodes changing their size-fields.

The new size-fields can be computed **locally** from the size-fields of the children.

## 7.5 Skip Lists

**Why do we not use a list for implementing the ADT Dynamic Set?**



## 7.5 Skip Lists

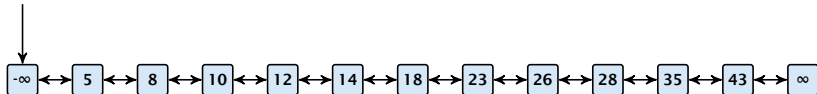
### Why do we not use a list for implementing the ADT Dynamic Set?

- ▶ time for search  $\Theta(n)$
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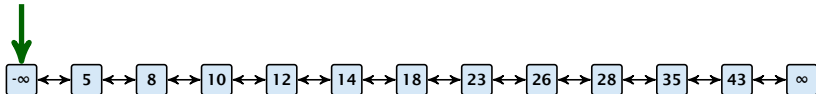
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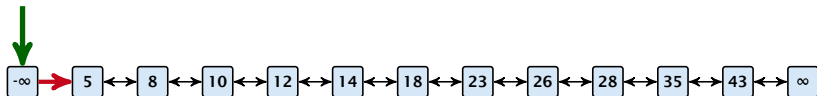
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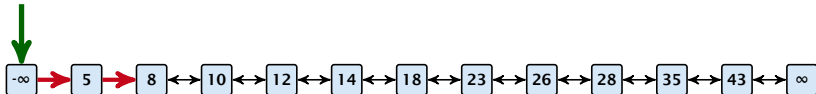
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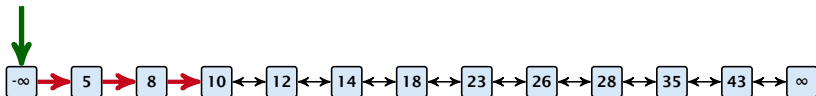
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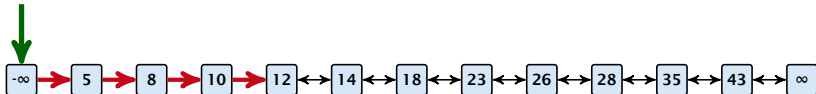
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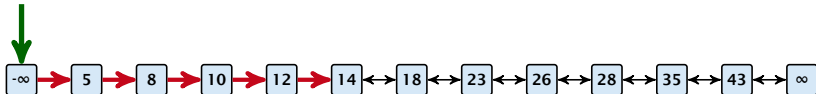
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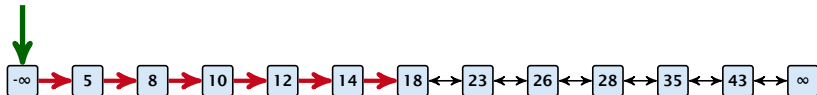




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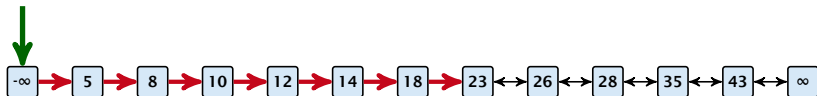
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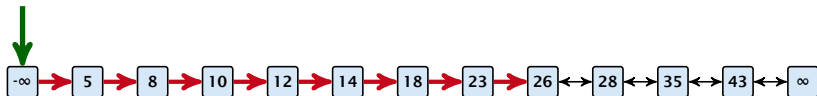
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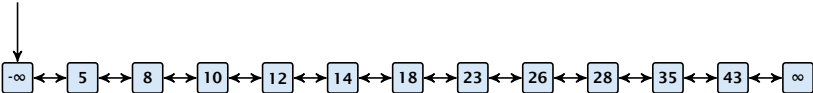
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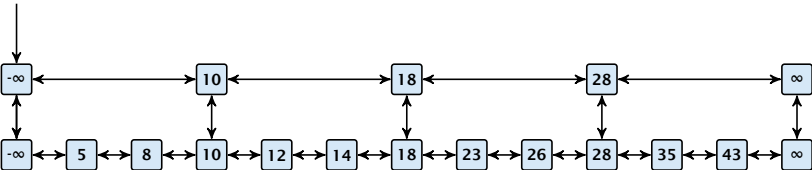
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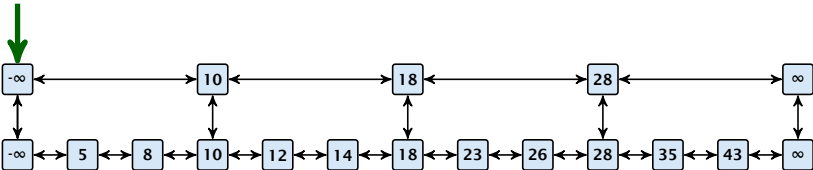
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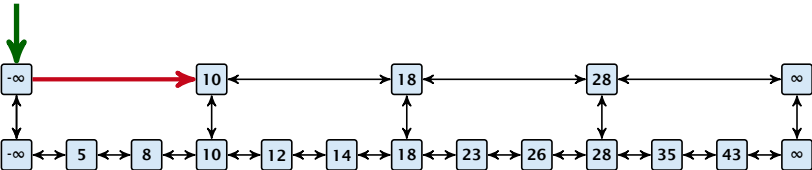




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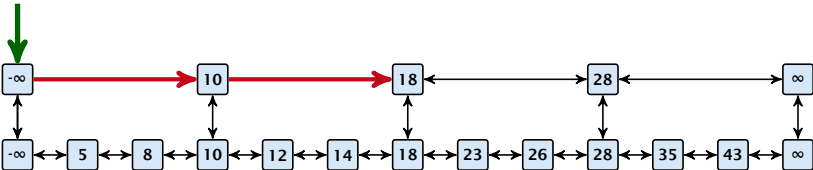
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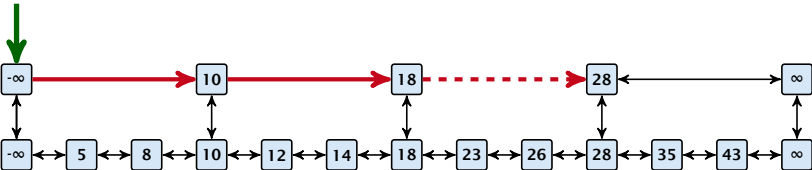
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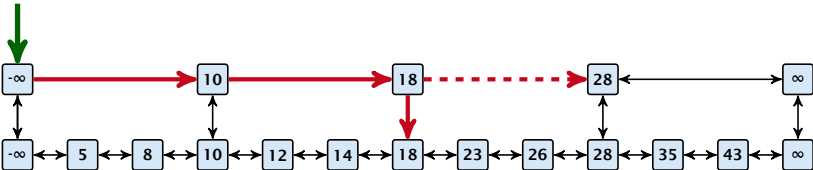
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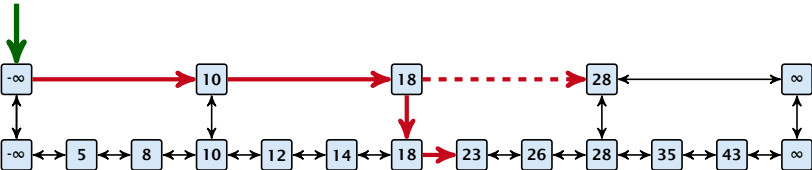
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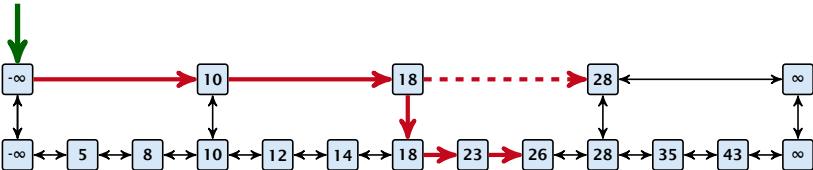
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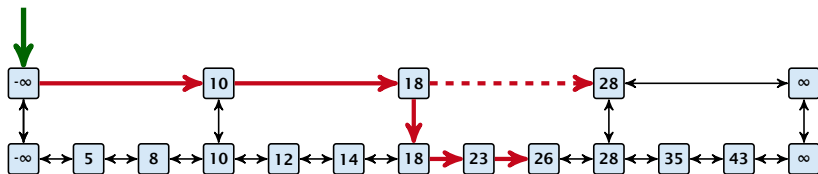
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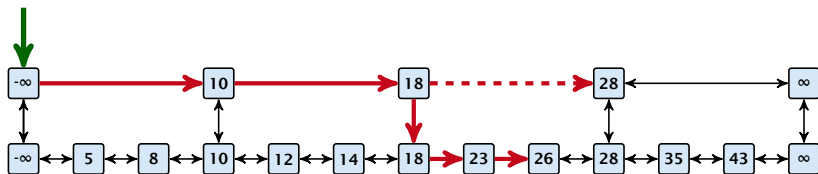


Let  $|L_1|$  denote the number of elements in the “express lane”, and  $|L_0| = n$  the number of all elements (ignoring dummy elements).

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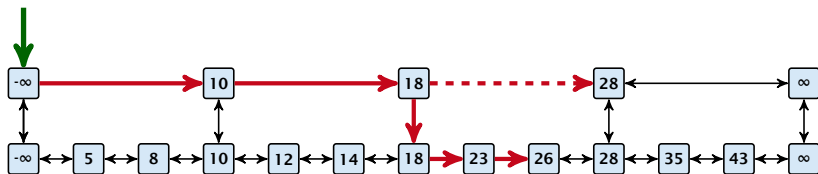
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Choose  $|L_1| = \sqrt{n}$ . Then search time  $\Theta(\sqrt{n})$ .

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Add more express lanes. Lane  $L_i$  contains roughly every  $\frac{L_{i-1}}{L_i}$ -th item from list  $L_{i-1}$ .

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- ▶ ...
- ▶ At most  $|L_k| + \sum_{i=1}^k \frac{L_{i-1}}{L_i} + 3(k + 1)$  steps.



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Choose ratios between list-lengths evenly, i.e.,  $\frac{|L_{i-1}|}{|L_i|} = r$ , and, hence,  $L_k \approx r^{-k}n$ .

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Choosing  $k = \Theta(\log n)$  gives a logarithmic running time.

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**How to do insert and delete?**



## 7.5 Skip Lists

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- ▶ If we want that in  $L_i$  we always skip over roughly the same number of elements in  $L_{i-1}$  an insert or delete may require a lot of re-organisation.

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- ▶ If we want that in  $L_i$  we always skip over roughly the same number of elements in  $L_{i-1}$  an insert or delete may require a lot of re-organisation.

**Use randomization instead!**

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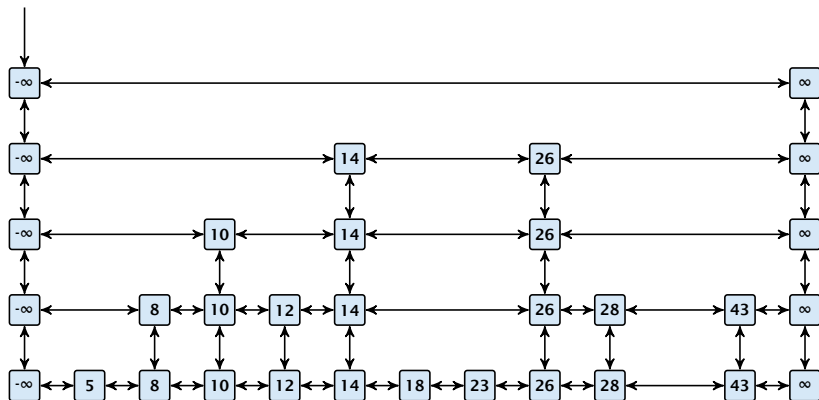
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**The time for both operations is dominated by the search time.**

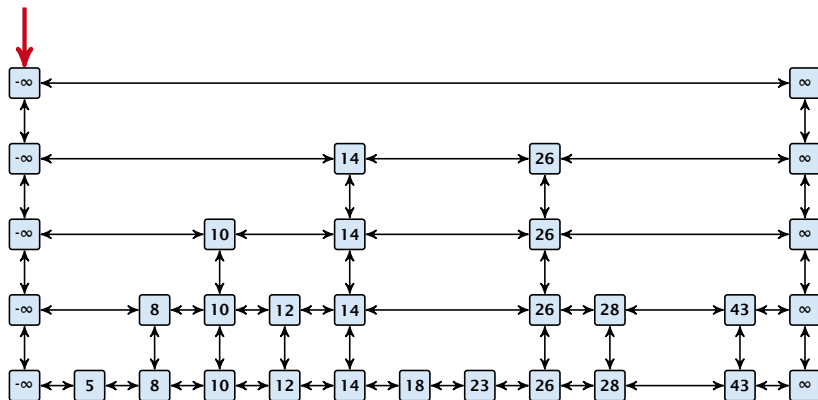
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Insert (35):



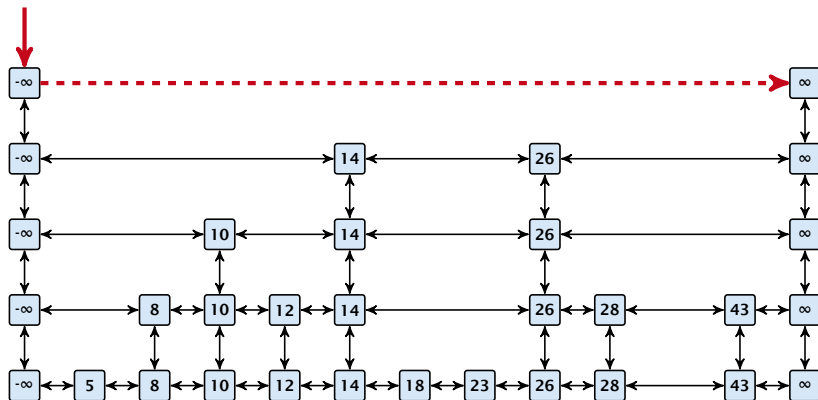
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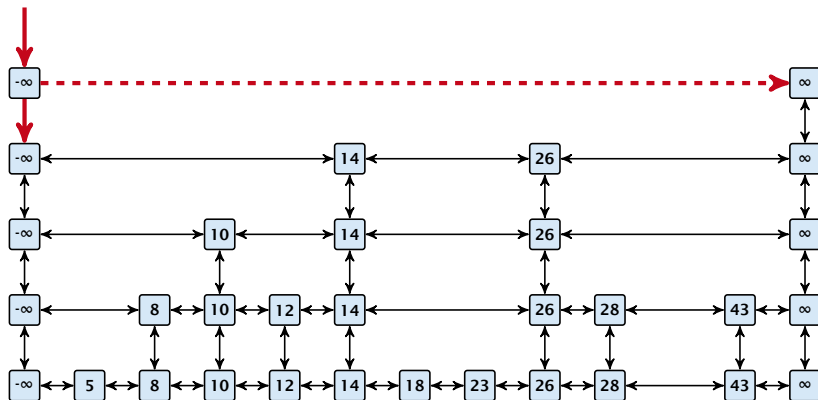
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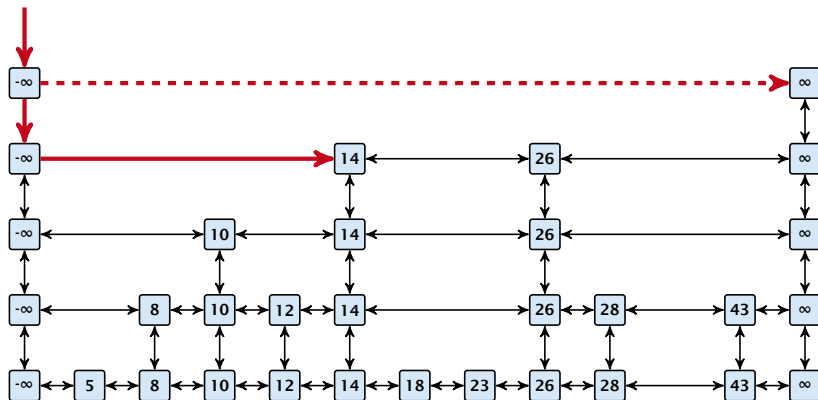
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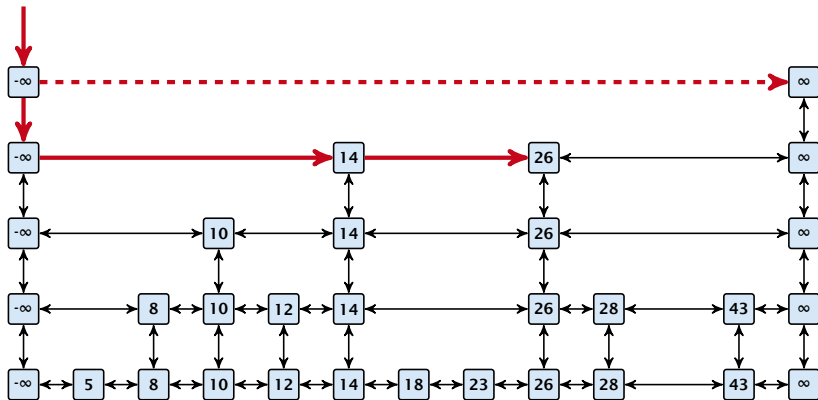
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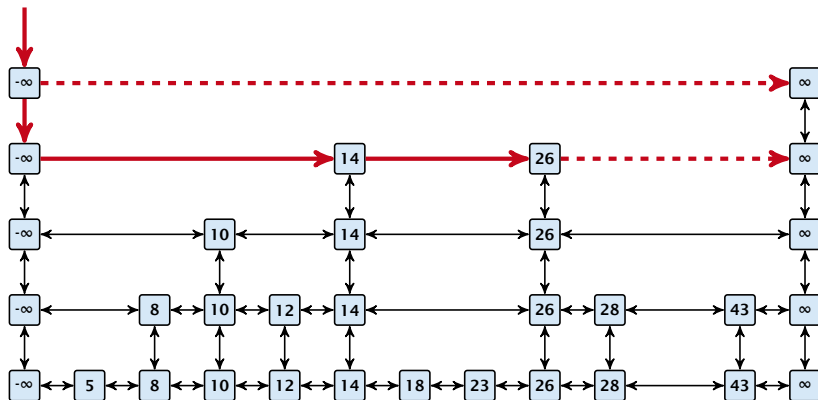
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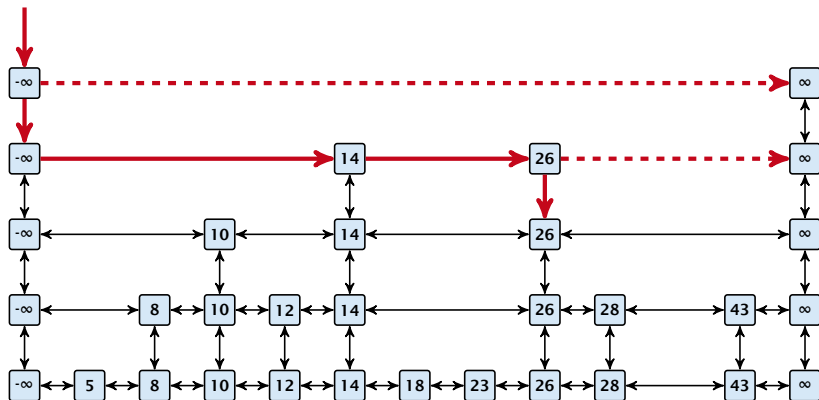
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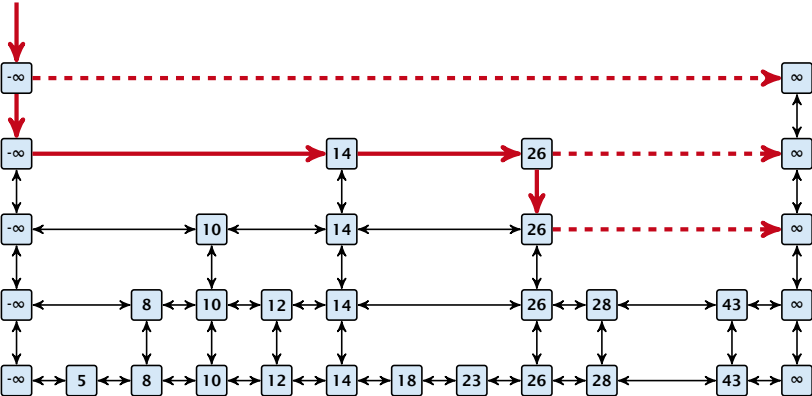
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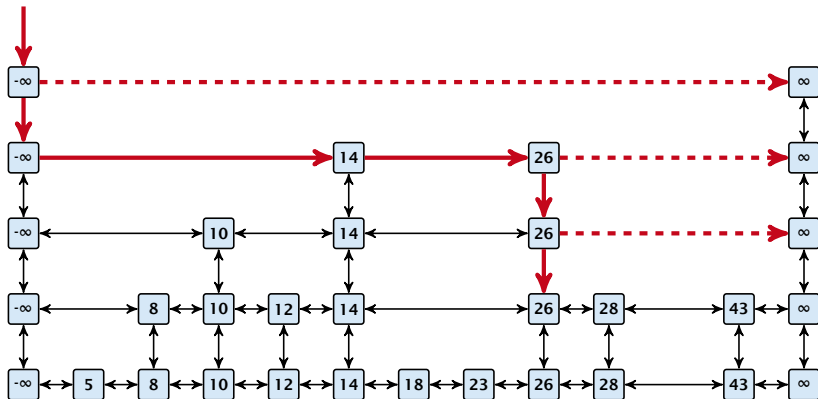
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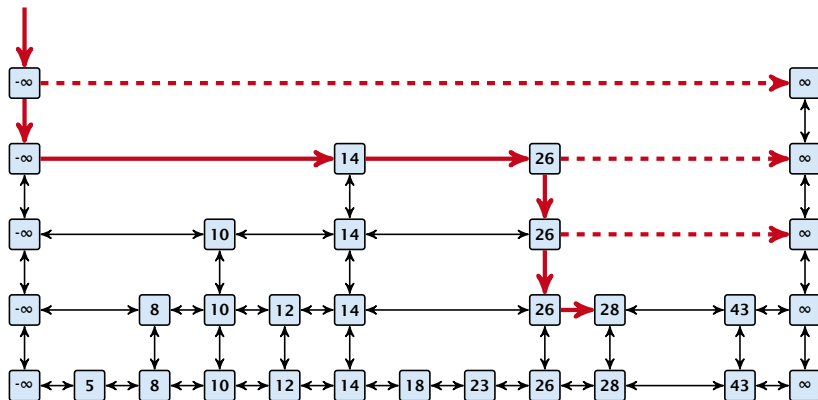
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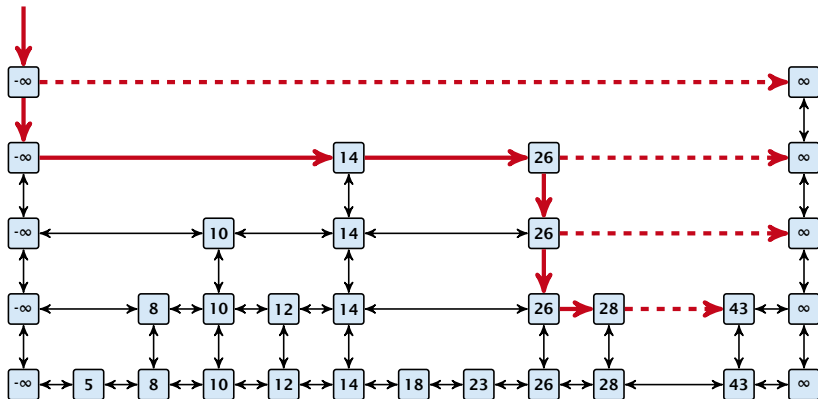
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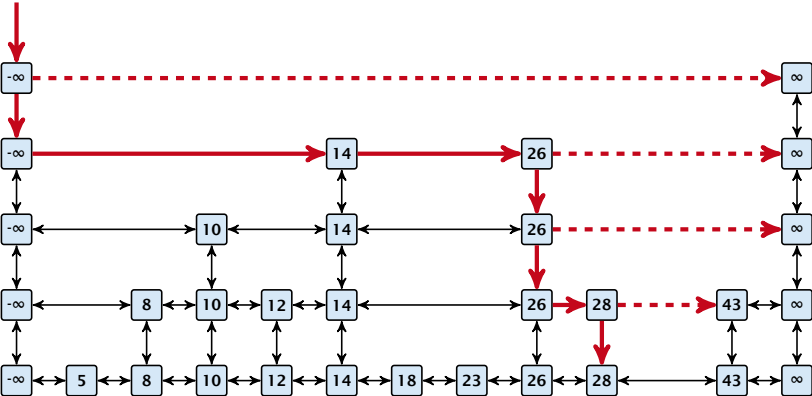
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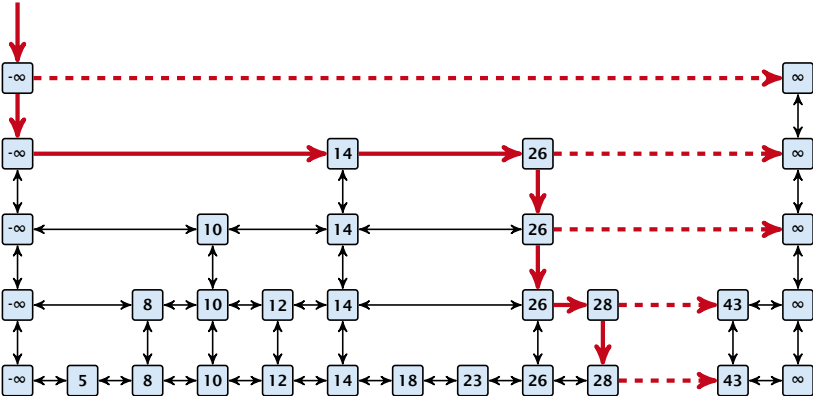
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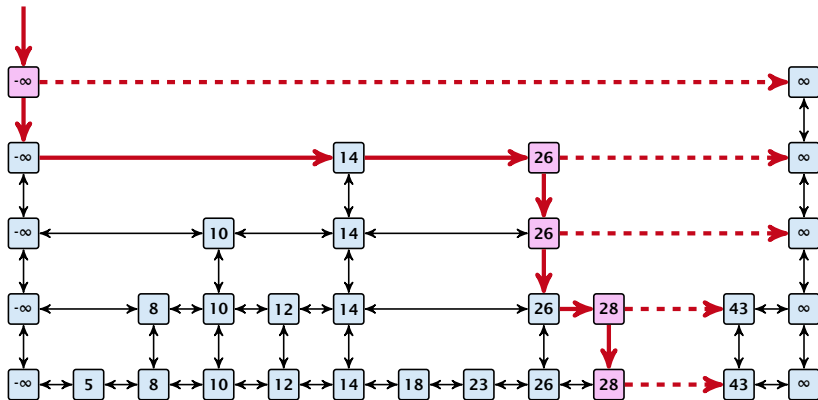
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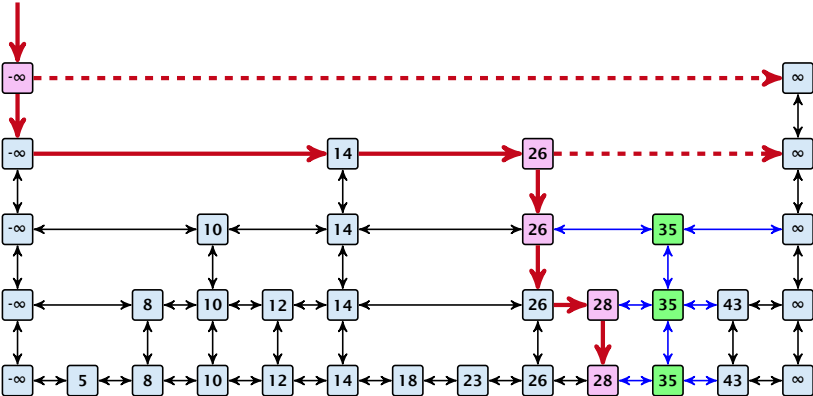
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# High Probability

## Definition 7 (High Probability)

We say a **randomized** algorithm has running time  $\mathcal{O}(\log n)$  with **high probability** if for any constant  $\alpha$  the running time is at most  $\mathcal{O}(\log n)$  with probability at least  $1 - \frac{1}{n^\alpha}$ .

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Here the  $\mathcal{O}$ -notation hides a constant that may depend on  $\alpha$ .

# High Probability

Suppose there are **polynomially** many events  $E_1, E_2, \dots, E_\ell$ ,  $\ell = n^c$  each holding with high probability (e.g.  $E_i$  may be the event that the  $i$ -th search in a skip list takes time at most  $\mathcal{O}(\log n)$ ).

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This means  $\Pr[E_1 \wedge \dots \wedge E_\ell]$  holds with high probability.

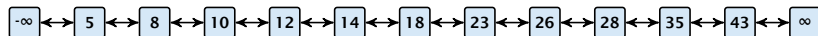
## 7.5 Skip Lists

### Lemma 8

*A search (and, hence, also insert and delete) in a skip list with  $n$  elements takes time  $\mathcal{O}(\log n)$  with high probability (w. h. p.).*

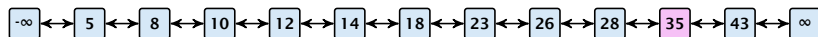
## 7.5 Skip Lists

Backward analysis:



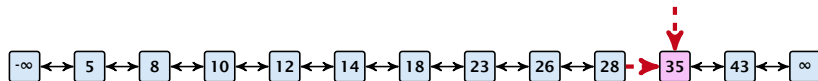
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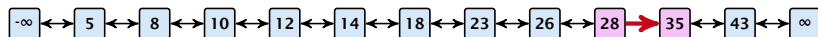
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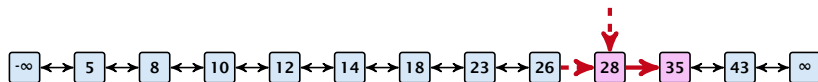
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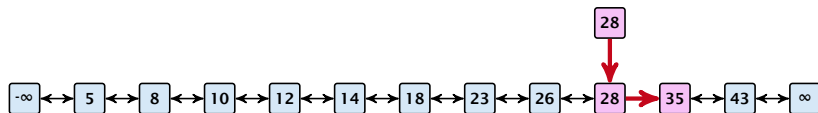
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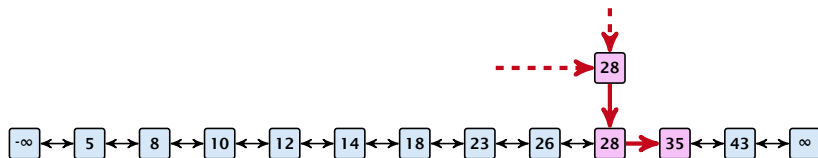
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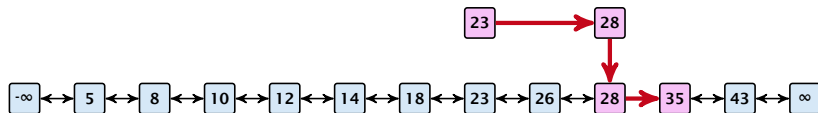
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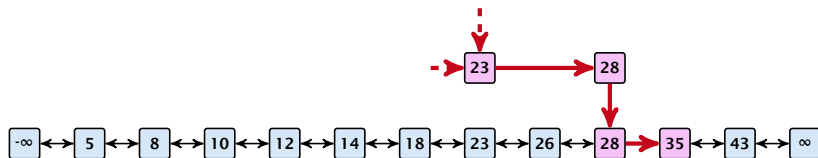
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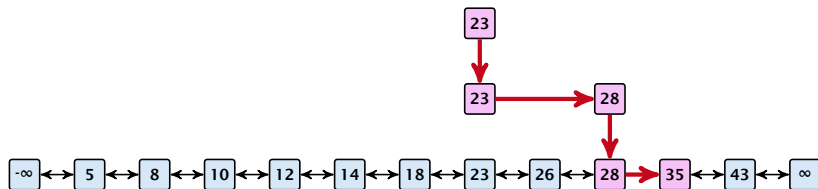
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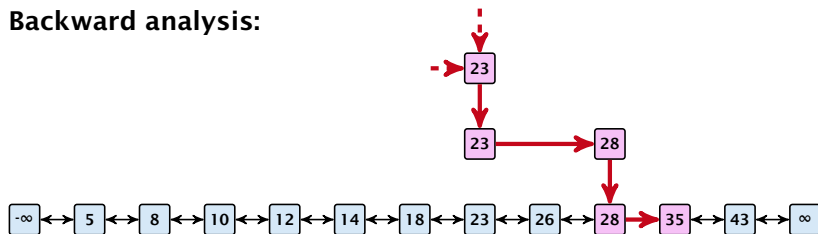
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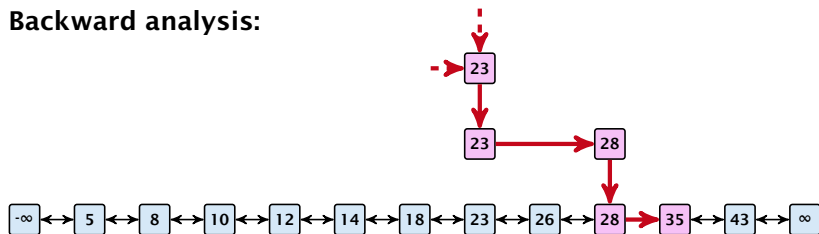
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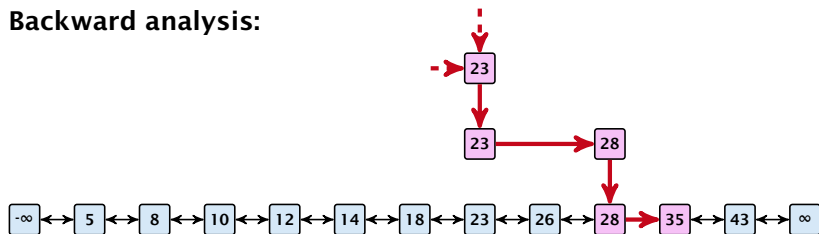
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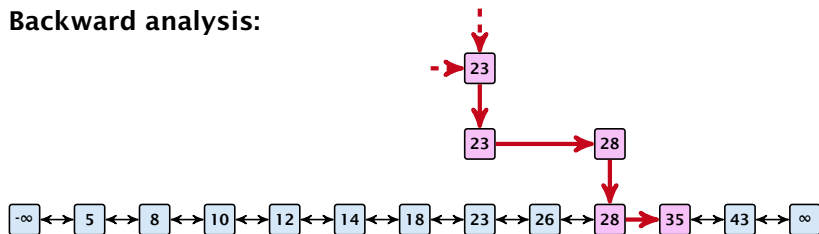
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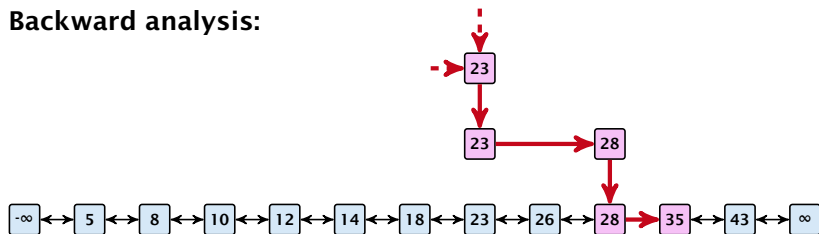
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From this it follows that w.h.p. there are no long paths.

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### Estimation for Binomial Coefficients

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$$= \left(\frac{n}{k}\right)^k \cdot \frac{k^k}{k!} \leq \left(\frac{n}{k}\right)^k \cdot \sum_{i \geq 0} \frac{k^i}{i!}$$

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### Estimation for Binomial Coefficients

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot \dots \cdot (n-k+1)}{k \cdot \dots \cdot 1} \geq \left(\frac{n}{k}\right)^k$$

$$\begin{aligned} \binom{n}{k} &= \frac{n \cdot \dots \cdot (n-k+1)}{k!} \leq \frac{n^k}{k!} = \frac{n^k \cdot k^k}{k^k \cdot k!} \\ &= \left(\frac{n}{k}\right)^k \cdot \frac{k^k}{k!} \leq \left(\frac{n}{k}\right)^k \cdot \sum_{i \geq 0} \frac{k^i}{i!} = \left(\frac{en}{k}\right)^k \end{aligned}$$

# 7.5 Skip Lists

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Let  $E_{z,k}$  denote the event that a search path is of length  $z$  (number of edges) but does not visit a list above  $L_k$ .



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In particular, this means that during the construction in the backward analysis we see at most  $k$  heads (i.e., coin flips that tell you to go up) in  $z$  trials.

## 7.5 Skip Lists

$$\Pr[E_{z,k}]$$

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$$\Pr[E_{z,k}] \leq \Pr[\text{at most } k \text{ heads in } z \text{ trials}]$$

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## 7.5 Skip Lists

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Let  $A_{k+1}$  denote the event that the list  $L_{k+1}$  is non-empty. Then

$$\Pr[A_{k+1}] \leq n2^{-(k+1)} \leq n^{-(\gamma-1)} .$$

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For the search to take at least  $z = 7\alpha\gamma \log n$  steps either the event  $E_{z,k}$  or the event  $A_{k+1}$  must hold.

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Hence,

$$\Pr[\text{search requires } z \text{ steps}]$$

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For the search to take at least  $z = 7\alpha\gamma \log n$  steps either the event  $E_{z,k}$  or the event  $A_{k+1}$  must hold.

Hence,

$$\Pr[\text{search requires } z \text{ steps}] \leq \Pr[E_{z,k}] + \Pr[A_{k+1}]$$

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For the search to take at least  $z = 7\alpha\gamma \log n$  steps either the event  $E_{z,k}$  or the event  $A_{k+1}$  must hold.

Hence,

$$\begin{aligned} \Pr[\text{search requires } z \text{ steps}] &\leq \Pr[E_{z,k}] + \Pr[A_{k+1}] \\ &\leq n^{-\alpha} + n^{-(\gamma-1)} \end{aligned}$$

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For the search to take at least  $z = 7\alpha\gamma \log n$  steps either the event  $E_{z,k}$  or the event  $A_{k+1}$  must hold.

Hence,

$$\begin{aligned} \Pr[\text{search requires } z \text{ steps}] &\leq \Pr[E_{z,k}] + \Pr[A_{k+1}] \\ &\leq n^{-\alpha} + n^{-(\gamma-1)} \end{aligned}$$

This means, the search requires at most  $z$  steps, w. h. p.



## 7.6 van Emde Boas Trees

### Dynamic Set Data Structure $S$ :

- ▶  $S.insert(x)$
- ▶  $S.delete(x)$
- ▶  $S.search(x)$
- ▶  $S.min()$
- ▶  $S.max()$
- ▶  $S.succ(x)$
- ▶  $S.pred(x)$

## 7.6 van Emde Boas Trees

For this chapter we ignore the problem of storing satellite data:

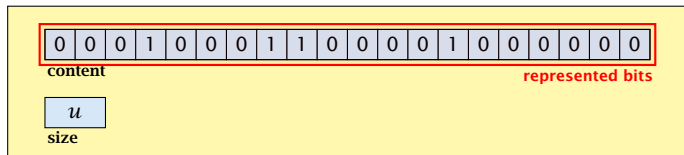
- ▶  **$S$ . insert( $x$ ):** Inserts  $x$  into  $S$ .
- ▶  **$S$ . delete( $x$ ):** Deletes  $x$  from  $S$ . Usually assumes that  $x \in S$ .
- ▶  **$S$ . member( $x$ ):** Returns 1 if  $x \in S$  and 0 otherwise.
- ▶  **$S$ . min():** Returns the value of the minimum element in  $S$ .
- ▶  **$S$ . max():** Returns the value of the maximum element in  $S$ .
- ▶  **$S$ . succ( $x$ ):** Returns successor of  $x$  in  $S$ . Returns **null** if  $x$  is maximum or larger than any element in  $S$ . Note that  $x$  needs not to be in  $S$ .
- ▶  **$S$ . pred( $x$ ):** Returns the predecessor of  $x$  in  $S$ . Returns **null** if  $x$  is minimum or smaller than any element in  $S$ . Note that  $x$  needs not to be in  $S$ .

## 7.6 van Emde Boas Trees

Can we improve the existing algorithms when the keys are from a restricted set?

In the following we assume that the keys are from  $\{0, 1, \dots, u - 1\}$ , where  $u$  denotes the size of the universe.

# Implementation 1: Array



one array of  $u$  bits

Use an array that encodes the indicator function of the dynamic set.

# Implementation 1: Array

**Algorithm 1** `array.insert( $x$ )`

1: `content[ $x$ ] ← 1;`

**Algorithm 2** `array.delete( $x$ )`

1: `content[ $x$ ] ← 0;`

**Algorithm 3** `array.member( $x$ )`

1: **return** `content[ $x$ ];`

- ▶ Note that we assume that  $x$  is valid, i.e., it falls within the array boundaries.
- ▶ Obviously(?) the running time is constant.

## Implementation 1: Array

### Algorithm 4 `array.max()`

```
1: for ( $i = \text{size} - 1; i \geq 0; i--$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
3: return null;
```

# Implementation 1: Array

## Algorithm 4 `array.max()`

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1: for ( $i = \text{size} - 1; i \geq 0; i--$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
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## Algorithm 5 `array.min()`

```
1: for ( $i = 0; i < \text{size}; i++$ ) do  
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## Implementation 1: Array

### Algorithm 4 `array.max()`

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### Algorithm 5 `array.min()`

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1: for ( $i = 0; i < \text{size}; i++$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
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```

- ▶ Running time is  $\mathcal{O}(u)$  in the worst case.



## Implementation 1: Array

### Algorithm 6 `array.succ(x)`

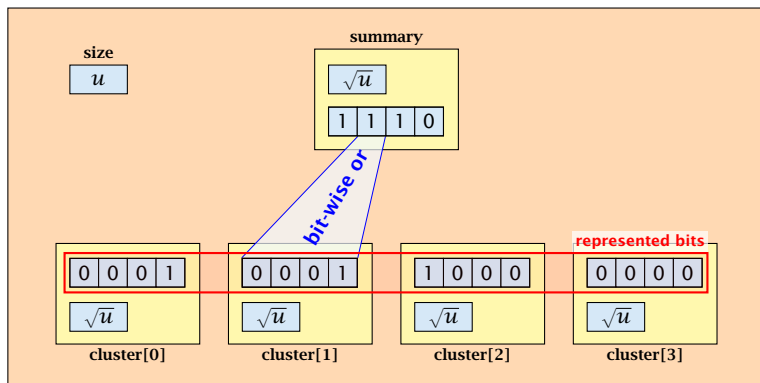
```
1: for ( $i = x + 1$ ;  $i < \text{size}$ ;  $i++$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
3: return null;
```

### Algorithm 7 `array.pred(x)`

```
1: for ( $i = x - 1$ ;  $i \geq 0$ ;  $i--$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
3: return null;
```

- ▶ Running time is  $\mathcal{O}(u)$  in the worst case.

## Implementation 2: Summary Array



- ▶  $\sqrt{u}$  cluster-arrays of  $\sqrt{u}$  bits.
- ▶ One summary-array of  $\sqrt{u}$  bits. The  $i$ -th bit in the summary array stores the bit-wise or of the bits in the  $i$ -th cluster.

# Implementation 2: Summary Array

## Implementation 2: Summary Array

The bit for a key  $x$  is contained in cluster number  $\left\lfloor \frac{x}{\sqrt{u}} \right\rfloor$ .

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Within the cluster-array the bit is at position  $x \bmod \sqrt{u}$ .

## Implementation 2: Summary Array

The bit for a key  $x$  is contained in cluster number  $\lfloor \frac{x}{\sqrt{u}} \rfloor$ .

Within the cluster-array the bit is at position  $x \bmod \sqrt{u}$ .

For simplicity we assume that  $u = 2^{2k}$  for some  $k \geq 1$ . Then we can compute the cluster-number for an entry  $x$  as  $\text{high}(x)$  (the upper half of the dual representation of  $x$ ) and the position of  $x$  within its cluster as  $\text{low}(x)$  (the lower half of the dual representation).

## Implementation 2: Summary Array

**Algorithm 8**  $\text{member}(x)$

1: **return**  $\text{cluster}[\text{high}(x)].\text{member}(\text{low}(x));$

## Implementation 2: Summary Array

### Algorithm 8 $\text{member}(x)$

1: **return**  $\text{cluster}[\text{high}(x)].\text{member}(\text{low}(x));$

### Algorithm 9 $\text{insert}(x)$

1:  $\text{cluster}[\text{high}(x)].\text{insert}(\text{low}(x));$

2:  $\text{summary}.\text{insert}(\text{high}(x));$



## Implementation 2: Summary Array

### Algorithm 8 $\text{member}(x)$

```
1: return cluster[high(x)].member(low(x));
```

### Algorithm 9 $\text{insert}(x)$

```
1: cluster[high(x)].insert(low(x));  
2: summary.insert(high(x));
```

- ▶ The running times are constant, because the corresponding array-functions have constant running times.

## Implementation 2: Summary Array

### Algorithm 10 delete( $x$ )

- 1: cluster[high( $x$ )].delete(low( $x$ ));
- 2: **if** cluster[high( $x$ )].min() = null **then**
- 3:     summary.delete(high( $x$ ));

## Implementation 2: Summary Array

### Algorithm 10 delete( $x$ )

```
1: cluster[high( $x$ )].delete(low( $x$ ));  
2: if cluster[high( $x$ )].min() = null then  
3:     summary.delete(high( $x$ ));
```

- ▶ The running time is dominated by the cost of a minimum computation on an array of size  $\sqrt{u}$ . Hence,  $\mathcal{O}(\sqrt{u})$ .

## Implementation 2: Summary Array

### Algorithm 11 $\text{max}()$

- 1:  $\text{maxcluster} \leftarrow \text{summary}.\text{max}();$
- 2: **if**  $\text{maxcluster} = \text{null}$  **return**  $\text{null};$
- 3:  $\text{offs} \leftarrow \text{cluster}[\text{maxcluster}].\text{max}()$
- 4: **return**  $\text{maxcluster} \circ \text{offs};$

## Implementation 2: Summary Array

### Algorithm 11 $\text{max}()$

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1:  $\text{maxcluster} \leftarrow \text{summary.max}();$   
2: if  $\text{maxcluster} = \text{null}$  return  $\text{null}$ ;  
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4: return  $\text{maxcluster} \circ \text{offs};$ 
```

### Algorithm 12 $\text{min}()$

```
1:  $\text{mincluster} \leftarrow \text{summary.min}();$   
2: if  $\text{mincluster} = \text{null}$  return  $\text{null}$ ;  
3:  $\text{offs} \leftarrow \text{cluster}[\text{mincluster}].\text{min}();$   
4: return  $\text{mincluster} \circ \text{offs};$ 
```

## Implementation 2: Summary Array

### Algorithm 11 $\text{max}()$

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1:  $\text{maxcluster} \leftarrow \text{summary.max}();$   
2: if  $\text{maxcluster} = \text{null}$  return  $\text{null}$ ;  
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4: return  $\text{maxcluster} \circ \text{offs};$ 
```

### Algorithm 12 $\text{min}()$

```
1:  $\text{mincluster} \leftarrow \text{summary.min}();$   
2: if  $\text{mincluster} = \text{null}$  return  $\text{null}$ ;  
3:  $\text{offs} \leftarrow \text{cluster}[\text{mincluster}].\text{min}();$   
4: return  $\text{mincluster} \circ \text{offs};$ 
```

The operator  $\circ$  stands for the concatenation of two bitstrings.

This means if  $x = 0111_2$  and  $y = 0001_2$  then  $x \circ y = 01110001_2$ .

- ▶ Running time is roughly  $2\sqrt{u} = \mathcal{O}(\sqrt{u})$  in the worst case.

## Implementation 2: Summary Array

### Algorithm 13 $\text{succ}(x)$

```
1:  $m \leftarrow \text{cluster}[\text{high}(x)].\text{succ}(\text{low}(x))$ 
2: if  $m \neq \text{null}$  then return  $\text{high}(x) \circ m$ ;
3:  $\text{succcluster} \leftarrow \text{summary}.\text{succ}(\text{high}(x))$ ;
4: if  $\text{succcluster} \neq \text{null}$  then
5:      $\text{offs} \leftarrow \text{cluster}[\text{succcluster}].\text{min}()$ ;
6:     return  $\text{succcluster} \circ \text{offs}$ ;
7: return  $\text{null}$ ;
```

## Implementation 2: Summary Array

### Algorithm 13 $\text{succ}(x)$

```
1:  $m \leftarrow \text{cluster}[\text{high}(x)].\text{succ}(\text{low}(x))$ 
2: if  $m \neq \text{null}$  then return  $\text{high}(x) \circ m$ ;
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4: if  $\text{succcluster} \neq \text{null}$  then
5:      $\text{offs} \leftarrow \text{cluster}[\text{succcluster}].\text{min}()$ ;
6:     return  $\text{succcluster} \circ \text{offs}$ ;
7: return  $\text{null}$ ;
```

- ▶ Running time is roughly  $3\sqrt{u} = \mathcal{O}(\sqrt{u})$  in the worst case.



## Implementation 2: Summary Array

### Algorithm 14 $\text{pred}(x)$

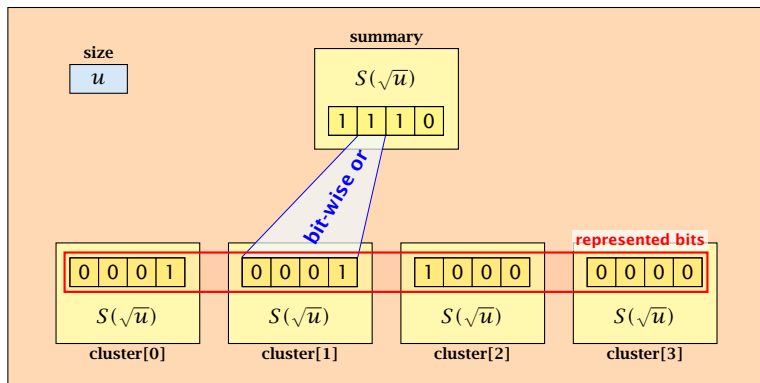
```
1:  $m \leftarrow \text{cluster}[\text{high}(x)].\text{pred}(\text{low}(x))$ 
2: if  $m \neq \text{null}$  then return  $\text{high}(x) \circ m$ ;
3:  $\text{predcluster} \leftarrow \text{summary}.\text{pred}(\text{high}(x))$ ;
4: if  $\text{predcluster} \neq \text{null}$  then
5:      $\text{offs} \leftarrow \text{cluster}[\text{predcluster}].\text{max}()$ ;
6:     return  $\text{predcluster} \circ \text{offs}$ ;
7: return  $\text{null}$ ;
```

- ▶ Running time is roughly  $3\sqrt{u} = \mathcal{O}(\sqrt{u})$  in the worst case.

## Implementation 3: Recursion

Instead of using sub-arrays, we build a recursive data-structure.

$S(u)$  is a dynamic set data-structure representing  $u$  bits:



## Implementation 3: Recursion

We assume that  $u = 2^{2^k}$  for some  $k$ .

The data-structure  $S(2)$  is defined as an array of 2-bits (end of the recursion).

# Implementation 3: Recursion

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The code from Implementation 2 can be used **unchanged**. We only need to redo the analysis of the running time.

## Implementation 3: Recursion

The code from Implementation 2 can be used **unchanged**. We only need to redo the analysis of the running time.

Note that in the code we do not need to specifically address the non-recursive case. This is achieved by the fact that an  $S(4)$  will contain  $S(2)$ 's as sub-datastructures, which are **arrays**. Hence, a call like `cluster[1].min()` from within the data-structure  $S(4)$  is **not** a recursive call as it will call the function `array.min()`.

## Implementation 3: Recursion

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Note that in the code we do not need to specifically address the non-recursive case. This is achieved by the fact that an  $S(4)$  will contain  $S(2)$ 's as sub-datastructures, which are **arrays**. Hence, a call like `cluster[1].min()` from within the data-structure  $S(4)$  is **not** a recursive call as it will call the function `array.min()`.

This means that the non-recursive case is been dealt with while initializing the data-structure.

## Implementation 3: Recursion

**Algorithm 15**  $\text{member}(x)$

1: **return**  $\text{cluster}[\text{high}(x)].\text{member}(\text{low}(x));$

- ▶  $T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1.$



## Implementation 3: Recursion

### Algorithm 16 insert( $x$ )

```
1: cluster[high( $x$ )].insert(low( $x$ ));  
2: summary.insert(high( $x$ ));
```

►  $T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$

## Implementation 3: Recursion

### Algorithm 17 delete( $x$ )

```
1: cluster[high( $x$ )].delete(low( $x$ ));  
2: if cluster[high( $x$ )].min() = null then  
3:     summary.delete(high( $x$ ));
```

►  $T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + T_{\text{min}}(\sqrt{u}) + 1.$

## Implementation 3: Recursion

### Algorithm 18 $\text{min}()$

```
1: mincluster  $\leftarrow$  summary.min();  
2: if mincluster = null return null;  
3: offs  $\leftarrow$  cluster[mincluster].min();  
4: return mincluster  $\circ$  offs;
```

►  $T_{\min}(u) = 2T_{\min}(\sqrt{u}) + 1.$

## Implementation 3: Recursion

### Algorithm 19 $\text{succ}(x)$

```
1:  $m \leftarrow \text{cluster}[\text{high}(x)].\text{succ}(\text{low}(x))$ 
2: if  $m \neq \text{null}$  then return  $\text{high}(x) \circ m$ ;
3:  $\text{succcluster} \leftarrow \text{summary}.\text{succ}(\text{high}(x))$ ;
4: if  $\text{succcluster} \neq \text{null}$  then
5:      $\text{offs} \leftarrow \text{cluster}[\text{succcluster}].\text{min}()$ ;
6:     return  $\text{succcluster} \circ \text{offs}$ ;
7: return  $\text{null}$ ;
```

►  $T_{\text{succ}}(u) = 2T_{\text{succ}}(\sqrt{u}) + T_{\text{min}}(\sqrt{u}) + 1.$

## Implementation 3: Recursion

$$T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + \mathbf{1}:$$

## Implementation 3: Recursion

$$T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + \mathbf{1}:$$

Set  $\ell := \log u$  and  $X(\ell) := T_{\text{mem}}(2^\ell)$ .

## Implementation 3: Recursion

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Set  $\ell := \log u$  and  $X(\ell) := T_{\text{mem}}(2^\ell)$ . Then

$$X(\ell)$$



## Implementation 3: Recursion

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Set  $\ell := \log u$  and  $X(\ell) := T_{\text{mem}}(2^\ell)$ . Then

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Set  $\ell := \log u$  and  $X(\ell) := T_{\text{mem}}(2^\ell)$ . Then

$$X(\ell) = T_{\text{mem}}(2^\ell) = T_{\text{mem}}(u)$$

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$$T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + 1:$$

Set  $\ell := \log u$  and  $X(\ell) := T_{\text{mem}}(2^\ell)$ . Then

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$$\begin{aligned} X(\ell) = T_{\text{mem}}(2^\ell) &= T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + 1 \\ &= T_{\text{mem}}(2^{\frac{\ell}{2}}) + 1 \end{aligned}$$

## Implementation 3: Recursion

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Set  $\ell := \log u$  and  $X(\ell) := T_{\text{mem}}(2^\ell)$ . Then

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## Implementation 3: Recursion

$$T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + 1:$$

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Using Master theorem gives  $X(\ell) = \mathcal{O}(\log \ell)$ , and hence  $T_{\text{mem}}(u) = \mathcal{O}(\log \log u)$ .

## Implementation 3: Recursion

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## Implementation 3: Recursion

$$T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$$

Set  $\ell := \log u$  and  $X(\ell) := T_{\text{ins}}(2^\ell)$ . Then

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Using Master theorem gives  $X(\ell) = \mathcal{O}(\ell)$ , and hence  $T_{\text{ins}}(u) = \mathcal{O}(\log u)$ .



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Using Master theorem gives  $X(\ell) = \mathcal{O}(\ell)$ , and hence  $T_{\text{ins}}(\mathbf{u}) = \mathcal{O}(\log u)$ .

The same holds for  $T_{\text{max}}(\mathbf{u})$  and  $T_{\text{min}}(\mathbf{u})$ .

## Implementation 3: Recursion

$$T_{\text{del}}(\mathbf{u}) = 2T_{\text{del}}(\sqrt{\mathbf{u}}) + T_{\text{min}}(\sqrt{\mathbf{u}}) + 1 \leq 2T_{\text{del}}(\sqrt{\mathbf{u}}) + \mathbf{c} \log(\mathbf{u}).$$

## Implementation 3: Recursion

$$T_{\text{del}}(\mathbf{u}) = 2T_{\text{del}}(\sqrt{\mathbf{u}}) + T_{\text{min}}(\sqrt{\mathbf{u}}) + 1 \leq 2T_{\text{del}}(\sqrt{\mathbf{u}}) + c \log(\mathbf{u}).$$

Set  $\ell := \log u$  and  $X(\ell) := T_{\text{del}}(2^\ell)$ .

## Implementation 3: Recursion

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Set  $\ell := \log u$  and  $X(\ell) := T_{\text{del}}(2^\ell)$ . Then

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Set  $\ell := \log u$  and  $X(\ell) := T_{\text{del}}(2^\ell)$ . Then

$$X(\ell) = T_{\text{del}}(2^\ell) = T_{\text{del}}(u)$$

## Implementation 3: Recursion

$$T_{\text{del}}(\mathbf{u}) = 2T_{\text{del}}(\sqrt{\mathbf{u}}) + T_{\text{min}}(\sqrt{\mathbf{u}}) + 1 \leq 2T_{\text{del}}(\sqrt{\mathbf{u}}) + c \log(\mathbf{u}).$$

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## Implementation 3: Recursion

$$T_{\text{del}}(\mathbf{u}) = 2T_{\text{del}}(\sqrt{\mathbf{u}}) + T_{\text{min}}(\sqrt{\mathbf{u}}) + 1 \leq 2T_{\text{del}}(\sqrt{\mathbf{u}}) + c \log(\mathbf{u}).$$

Set  $\ell := \log u$  and  $X(\ell) := T_{\text{del}}(2^\ell)$ . Then

$$\begin{aligned} X(\ell) &= T_{\text{del}}(2^\ell) = T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + c \log u \\ &= 2T_{\text{del}}(2^{\frac{\ell}{2}}) + c\ell \end{aligned}$$

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$$T_{\text{del}}(\mathbf{u}) = 2T_{\text{del}}(\sqrt{\mathbf{u}}) + T_{\text{min}}(\sqrt{\mathbf{u}}) + 1 \leq 2T_{\text{del}}(\sqrt{\mathbf{u}}) + c \log(\mathbf{u}).$$

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Using Master theorem gives  $X(\ell) = \Theta(\ell \log \ell)$ , and hence  $T_{\text{del}}(u) = \mathcal{O}(\log u \log \log u)$ .

## Implementation 3: Recursion

$$T_{\text{del}}(\mathbf{u}) = 2T_{\text{del}}(\sqrt{\mathbf{u}}) + T_{\text{min}}(\sqrt{\mathbf{u}}) + 1 \leq 2T_{\text{del}}(\sqrt{\mathbf{u}}) + c \log(\mathbf{u}).$$

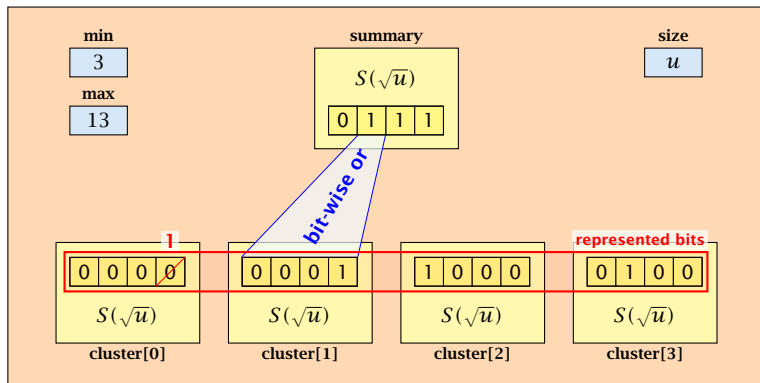
Set  $\ell := \log u$  and  $X(\ell) := T_{\text{del}}(2^\ell)$ . Then

$$\begin{aligned} X(\ell) &= T_{\text{del}}(2^\ell) = T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + c \log u \\ &= 2T_{\text{del}}(2^{\frac{\ell}{2}}) + c\ell = 2X\left(\frac{\ell}{2}\right) + c\ell . \end{aligned}$$

Using Master theorem gives  $X(\ell) = \Theta(\ell \log \ell)$ , and hence  $T_{\text{del}}(u) = \mathcal{O}(\log u \log \log u)$ .

The same holds for  $T_{\text{pred}}(u)$  and  $T_{\text{succ}}(u)$ .

# Implementation 4: van Emde Boas Trees



- ▶ The bit referenced by **min** is **not** set within sub-datastructures.
- ▶ The bit referenced by **max** is set within sub-datastructures (if  $\text{max} \neq \text{min}$ ).

# Implementation 4: van Emde Boas Trees

**Advantages of having max/min pointers:**

## Implementation 4: van Emde Boas Trees

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# Implementation 4: van Emde Boas Trees

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- ▶ Recursive calls for **min** and **max** are constant time.
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- ▶ **min = max  $\neq$  null** means that the data-structure contains exactly one element.

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## Implementation 4: van Emde Boas Trees

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- ▶ Recursive calls for **min** and **max** are constant time.
- ▶ **min = null** means that the data-structure is empty.
- ▶ **min = max  $\neq$  null** means that the data-structure contains exactly one element.
- ▶ We can insert into an empty datastructure in constant time by only setting **min = max =  $x$** .
- ▶ We can delete from a data-structure that just contains one element in constant time by setting **min = max = null**.

## Implementation 4: van Emde Boas Trees

**Algorithm 20** max()

1: **return** max;

**Algorithm 21** min()

1: **return** min;

- ▶ Constant time.

## Implementation 4: van Emde Boas Trees

### Algorithm 22 `member(x)`

```
1: if  $x = \min$  then return 1; // TRUE  
2: return cluster[high(x)].member(low(x));
```

- ▶  $T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1 \Rightarrow T(u) = \mathcal{O}(\log \log u)$ .

## Implementation 4: van Emde Boas Trees

### Algorithm 23 $\text{succ}(x)$

```
1: if  $\text{min} \neq \text{null} \wedge x < \text{min}$  then return  $\text{min}$ ;  
2:  $\text{maxincluster} \leftarrow \text{cluster}[\text{high}(x)].\text{max}()$ ;  
3: if  $\text{maxincluster} \neq \text{null} \wedge \text{low}(x) < \text{maxincluster}$  then  
4:    $\text{offs} \leftarrow \text{cluster}[\text{high}(x)].\text{succ}(\text{low}(x))$ ;  
5:   return  $\text{high}(x) \circ \text{offs}$ ;  
6: else  
7:    $\text{succcluster} \leftarrow \text{summary}.\text{succ}(\text{high}(x))$ ;  
8:   if  $\text{succcluster} = \text{null}$  then return  $\text{null}$ ;  
9:    $\text{offs} \leftarrow \text{cluster}[\text{succcluster}].\text{min}()$ ;  
10:  return  $\text{succcluster} \circ \text{offs}$ ;
```

►  $T_{\text{succ}}(u) = T_{\text{succ}}(\sqrt{u}) + 1 \implies T_{\text{succ}}(u) = \mathcal{O}(\log \log u)$ .

## Implementation 4: van Emde Boas Trees

### Algorithm 35 insert( $x$ )

```
1: if min = null then
2:     min =  $x$ ; max =  $x$ ;
3: else
4:     if  $x < \text{min}$  then exchange  $x$  and min;
5:     if  $x > \text{max}$  then max =  $x$ ;
6:     if cluster[high( $x$ )].min = null; then
7:         summary.insert(high( $x$ ));
8:         cluster[high( $x$ )].insert(low( $x$ ));
9:     else
10:        cluster[high( $x$ )].insert(low( $x$ ));
```

►  $T_{\text{ins}}(u) = T_{\text{ins}}(\sqrt{u}) + 1 \Rightarrow T_{\text{ins}}(u) = \mathcal{O}(\log \log u)$ .

## Implementation 4: van Emde Boas Trees

Note that the recursive call in Line 8 takes constant time as the if-condition in Line 6 ensures that we are inserting in an empty sub-tree.

The only non-constant recursive calls are the call in Line 7 and in Line 10. These are mutually exclusive, i.e., only one of these calls will actually occur.

From this we get that  $T_{\text{ins}}(u) = T_{\text{ins}}(\sqrt{u}) + 1$ .



## Implementation 4: van Emde Boas Trees

- ▶ Assumes that  $x$  is contained in the structure.

**Algorithm 36** delete( $x$ )

```
1: if min = max then
2:     min = max = null;
3: else
4:     if  $x$  = min then
5:         firstcluster  $\leftarrow$  summary.min();
6:         offs  $\leftarrow$  cluster[firstcluster].min();
7:          $x \leftarrow$  firstcluster  $\circ$  offs;
8:         min  $\leftarrow$   $x$ ;
9:         cluster[high( $x$ )].delete(low( $x$ ));
```

continued...

## Implementation 4: van Emde Boas Trees

- Assumes that  $x$  is contained in the structure.

**Algorithm 36** delete( $x$ )

```
1: if min = max then
2:     min = max = null;
3: else
4:     if  $x = \text{min}$  then find new minimum
5:          $\text{firstcluster} \leftarrow \text{summary.min}()$ ;
6:          $\text{offs} \leftarrow \text{cluster}[\text{firstcluster}].\text{min}()$ ;
7:          $x \leftarrow \text{firstcluster} \circ \text{offs}$ ;
8:         min  $\leftarrow x$ ;
9:         cluster[high( $x$ )].delete(low( $x$ ));
continued...
```

## Implementation 4: van Emde Boas Trees

- ▶ **Assumes that  $x$  is contained in the structure.**

**Algorithm 36** delete( $x$ )

```
1: if min = max then  
2:     min = max = null;  
3: else  
4:     if  $x$  = min then  
5:         firstcluster  $\leftarrow$  summary.min();  
6:         offs  $\leftarrow$  cluster[firstcluster].min();  
7:          $x \leftarrow$  firstcluster  $\circ$  offs;  
8:         min  $\leftarrow$   $x$ ;  
9:     cluster[high( $x$ )].delete(low( $x$ )); delete
```

continued...

## Implementation 4: van Emde Boas Trees

### Algorithm 36 delete( $x$ )

...continued

```
10:   if cluster[high( $x$ )].min() = null then
11:       summary.delete(high( $x$ ));
12:   if  $x$  = max then
13:       summax  $\leftarrow$  summary.max();
14:       if summax = null then max  $\leftarrow$  min;
15:       else
16:           offs  $\leftarrow$  cluster[summax].max();
17:           max  $\leftarrow$  summax  $\circ$  offs
18:   else
19:       if  $x$  = max then
20:           offs  $\leftarrow$  cluster[high( $x$ )].max();
21:           max  $\leftarrow$  high( $x$ )  $\circ$  offs;
```

## Implementation 4: van Emde Boas Trees

### Algorithm 36 delete( $x$ )

...continued

fix maximum

```
10:   if cluster[high( $x$ )].min() = null then
11:       summary.delete(high( $x$ ));
12:       if  $x$  = max then
13:           summax  $\leftarrow$  summary.max();
14:           if summax = null then max  $\leftarrow$  min;
15:           else
16:               offs  $\leftarrow$  cluster[summax].max();
17:               max  $\leftarrow$  summax  $\circ$  offs
18:       else
19:           if  $x$  = max then
20:               offs  $\leftarrow$  cluster[high( $x$ )].max();
21:               max  $\leftarrow$  high( $x$ )  $\circ$  offs;
```

## Implementation 4: van Emde Boas Trees

Note that only one of the possible recursive calls in Line 9 and Line 11 in the deletion-algorithm may take non-constant time.

To see this observe that the call in Line 11 only occurs if the cluster where  $x$  was deleted is now empty. But this means that the call in Line 9 deleted the last element in  $\text{cluster}[\text{high}(x)]$ . Such a call only takes constant time.

Hence, we get a recurrence of the form

$$T_{\text{del}}(u) = T_{\text{del}}(\sqrt{u}) + c .$$

This gives  $T_{\text{del}}(u) = \mathcal{O}(\log \log u)$ .

## 7.6 van Emde Boas Trees

### Space requirements:

- ▶ The space requirement fulfills the recurrence

$$S(u) = (\sqrt{u} + 1)S(\sqrt{u}) + \mathcal{O}(\sqrt{u}) .$$

- ▶ Note that we cannot solve this recurrence by the Master theorem as the branching factor is not constant.
- ▶ One can show by induction that the space requirement is  $S(u) = \mathcal{O}(u)$ . Exercise.

- ▶ Let the “real” recurrence relation be

$$S(k^2) = (k + 1)S(k) + c_1 \cdot k; S(4) = c_2$$

- ▶ Replacing  $S(k)$  by  $R(k) := S(k)/c_2$  gives the recurrence

$$R(k^2) = (k + 1)R(k) + ck; R(4) = 1$$

where  $c = c_1/c_2 < 1$ .

- ▶ Now, we show  $R(k) \leq k - 2$  for squares  $k \geq 4$ .
  - ▶ Obviously, this holds for  $k = 4$ .
  - ▶ For  $k = \ell^2 > 4$  with  $\ell$  integral we have

$$\begin{aligned} R(k) &= (1 + \ell)R(\ell) + c\ell \\ &\leq (1 + \ell)(\ell - 2) + \ell \leq k - 2 \end{aligned}$$

- ▶ This shows that  $R(k)$  and, hence,  $S(k)$  grows linearly.



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**Hashing** tries to **directly** compute the memory location from the given key. The goal is to have constant search time.

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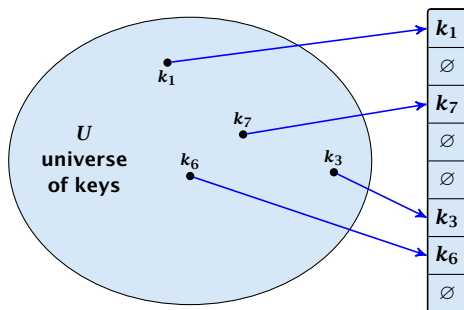
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- ▶ Small storage requirement.
- ▶ Good distribution of elements over the whole table.

# Direct Addressing

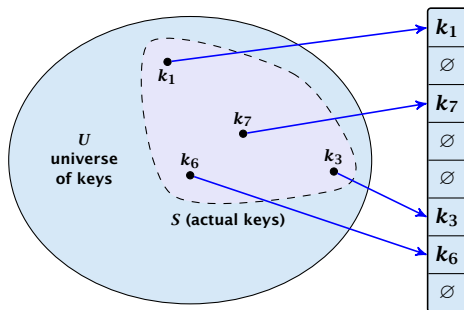
Ideally the hash function maps **all** keys to different memory locations.



This special case is known as **Direct Addressing**. It is usually very unrealistic as the universe of keys typically is quite large, and in particular larger than the available memory.

# Perfect Hashing

Suppose that we **know** the set  $S$  of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



Such a hash function  $h$  is called a **perfect hash function** for set  $S$ .

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Usually the universe  $U$  is much larger than the table-size  $n$ .

Hence, there may be two elements  $k_1, k_2$  from the set  $S$  that map to the same memory location (i.e.,  $h(k_1) = h(k_2)$ ). This is called a **collision**.

# Collisions

Typically, collisions do not appear once the size of the set  $S$  of actual keys gets close to  $n$ , but already when  $|S| \geq \omega(\sqrt{n})$ .

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## Lemma 9

*The probability of having a collision when hashing  $m$  elements into a table of size  $n$  under uniform hashing is at least*

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## **Uniform hashing:**

Choose a hash function uniformly at random from all functions  $f : U \rightarrow [0, \dots, n-1]$ .

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Let  $A_{m,n}$  denote the event that inserting  $m$  keys into a table of size  $n$  does **not** generate a collision. Then

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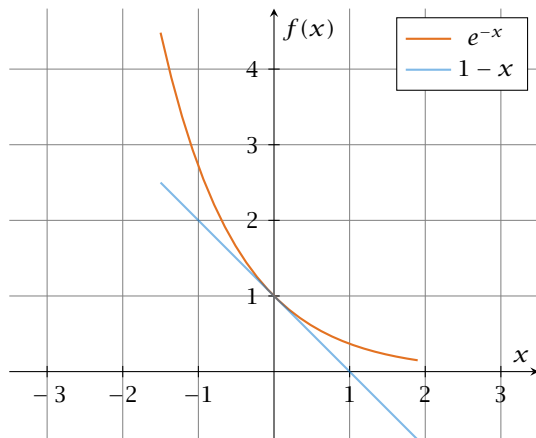
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Here the first equality follows since the  $\ell$ -th element that is hashed has a probability of  $\frac{n-\ell+1}{n}$  to not generate a collision under the condition that the previous elements did not induce collisions. □

# Collisions



The inequality  $1 - x \leq e^{-x}$  is derived by stopping the Taylor-expansion of  $e^{-x}$  after the second term.

# Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

- ▶ **open addressing**, aka. closed hashing
- ▶ **hashing with chaining**, aka. closed addressing, open hashing.

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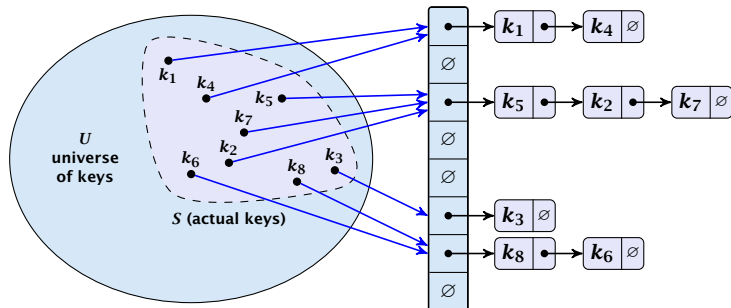
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- ▶ **hashing with chaining**, aka. closed addressing, open hashing.

There are applications e.g. computer chess where you do not resolve collisions at all.

# Hashing with Chaining

Arrange elements that map to the same position in a linear list.

- ▶ Access: compute  $h(x)$  and search list for  $\text{key}[x]$ .
- ▶ Insert: insert at the front of the list.





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We assume **uniform hashing** for the following analysis.

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$$A^- = 1 + \alpha .$$



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Hence, the expected cost for a successful search is  $A^+ \leq 1 + \frac{\alpha}{2}$ .

# Hashing with Chaining

## Disadvantages:

- ▶ pointers increase memory requirements
- ▶ pointers may lead to bad cache efficiency

## Advantages:

- ▶ no à priori limit on the number of elements
- ▶ deletion can be implemented efficiently
- ▶ by using balanced trees instead of linked list one can also obtain worst-case guarantees.



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**Insert( $x$ ):** Search until you find an empty slot; insert your element there. If your search reaches  $h(k, n - 1)$ , and this slot is non-empty then your table is full.

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Choices for  $h(k, j)$ :

- ▶ **Linear probing:**

$$h(k, i) = h(k) + i \bmod n$$

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$$h(k, i) = h(k) + c_1i + c_2i^2 \pmod n.$$

- ▶ **Double hashing:**

$$h(k, i) = h_1(k) + ih_2(k) \pmod n.$$



# Open Addressing

Choices for  $h(k, j)$ :

- ▶ **Linear probing:**

$$h(k, i) = h(k) + i \pmod n$$

(sometimes:  $h(k, i) = h(k) + ci \pmod n$ ).

- ▶ **Quadratic probing:**

$$h(k, i) = h(k) + c_1i + c_2i^2 \pmod n.$$

- ▶ **Double hashing:**

$$h(k, i) = h_1(k) + ih_2(k) \pmod n.$$

For quadratic probing and double hashing one has to ensure that the search covers all positions in the table (i.e., for double hashing  $h_2(k)$  must be relatively prime to  $n$  (**teilerfremd**); for quadratic probing  $c_1$  and  $c_2$  have to be chosen carefully).

# Linear Probing

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## Lemma 10

Let  $L$  be the method of linear probing for resolving collisions:

$$L^+ \approx \frac{1}{2} \left( 1 + \frac{1}{1 - \alpha} \right)$$

$$L^- \approx \frac{1}{2} \left( 1 + \frac{1}{(1 - \alpha)^2} \right)$$

# Quadratic Probing

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## Lemma 11

Let  $Q$  be the method of quadratic probing for resolving collisions:

$$Q^+ \approx 1 + \ln\left(\frac{1}{1-\alpha}\right) - \frac{\alpha}{2}$$

$$Q^- \approx \frac{1}{1-\alpha} + \ln\left(\frac{1}{1-\alpha}\right) - \alpha$$

# Double Hashing

- ▶ Any probe into the hash-table usually creates a cache-miss.

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## Lemma 12

Let  $D$  be the method of double hashing for resolving collisions:

$$D^+ \approx \frac{1}{\alpha} \ln \left( \frac{1}{1 - \alpha} \right)$$

$$D^- \approx \frac{1}{1 - \alpha}$$

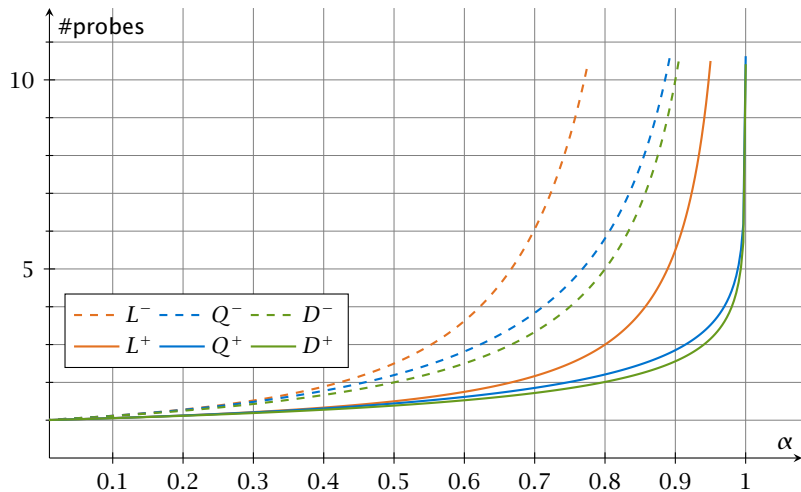


# Open Addressing

Some values:

$\alpha$	<i>Linear Probing</i>		<i>Quadratic Probing</i>		<i>Double Hashing</i>	
	$L^+$	$L^-$	$Q^+$	$Q^-$	$D^+$	$D^-$
0.5	1.5	2.5	1.44	2.19	1.39	2
0.9	5.5	50.5	2.85	11.40	2.55	10
0.95	10.5	200.5	3.52	22.05	3.15	20

# Open Addressing



# Analysis of Idealized Open Address Hashing

We analyze the time for a search in a very idealized Open Addressing scheme.

- ▶ The probe sequence  $h(k, 0), h(k, 1), h(k, 2), \dots$  is equally likely to be any permutation of  $\langle 0, 1, \dots, n - 1 \rangle$ .

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$$\Pr[A_1 \cap A_2 \cap \cdots \cap A_{i-1}]$$

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$$\Pr[X \geq i] = \frac{m}{n} \cdot \frac{m-1}{n-1} \cdot \frac{m-2}{n-2} \cdot \dots \cdot \frac{m-i+2}{n-i+2}$$

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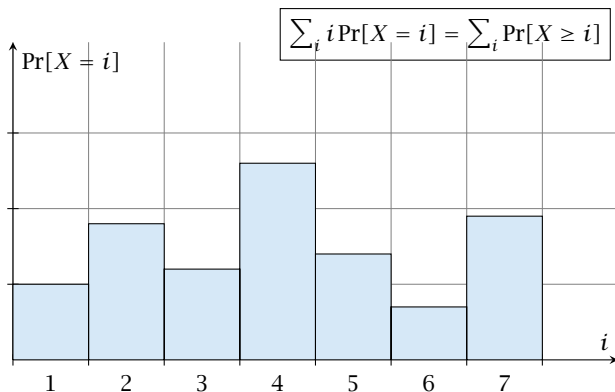


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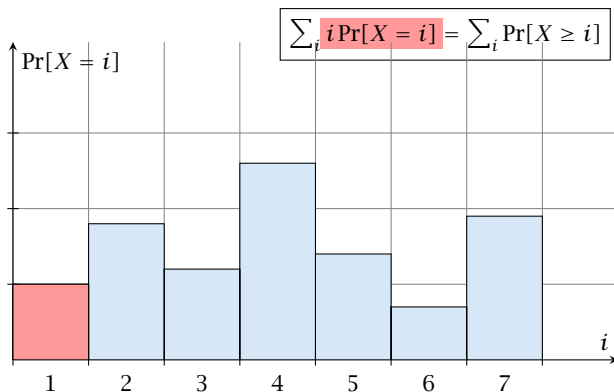
$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \dots$$

# Analysis of Idealized Open Address Hashing



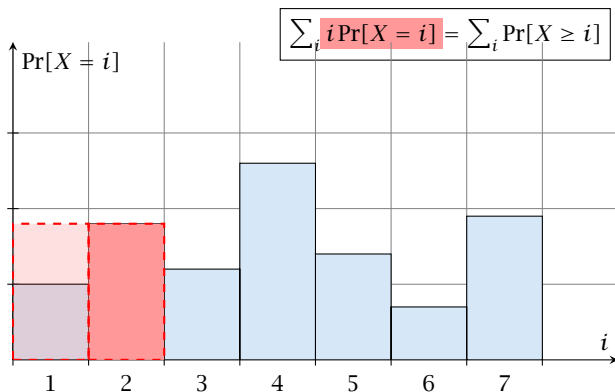
# Analysis of Idealized Open Address Hashing

$i = 1$



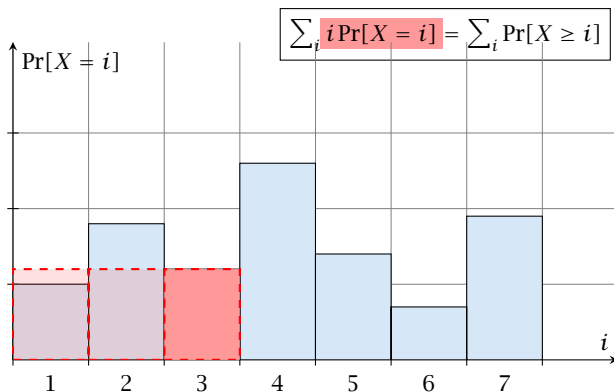
# Analysis of Idealized Open Address Hashing

$i = 2$



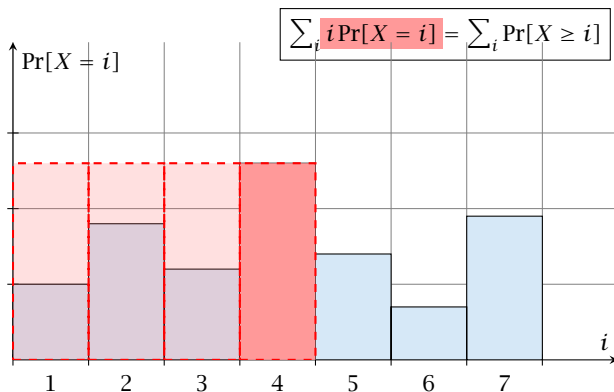
# Analysis of Idealized Open Address Hashing

$i = 3$



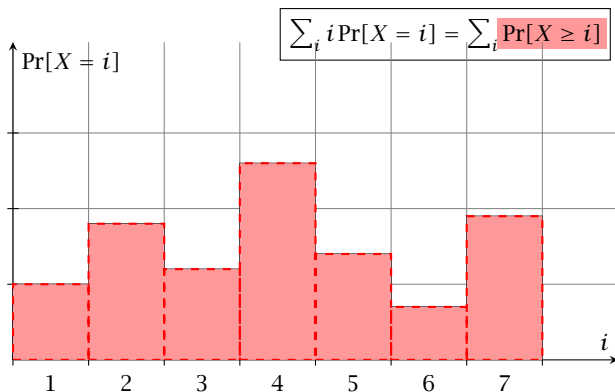
# Analysis of Idealized Open Address Hashing

$i = 4$



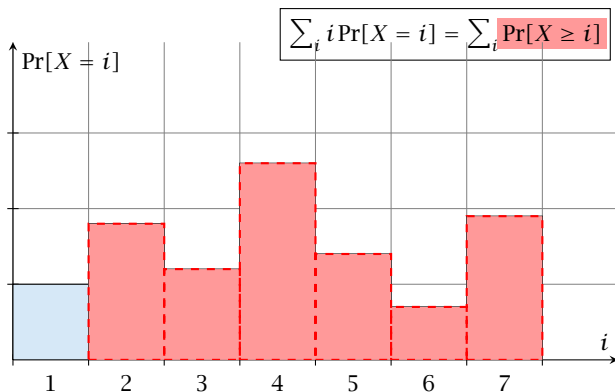
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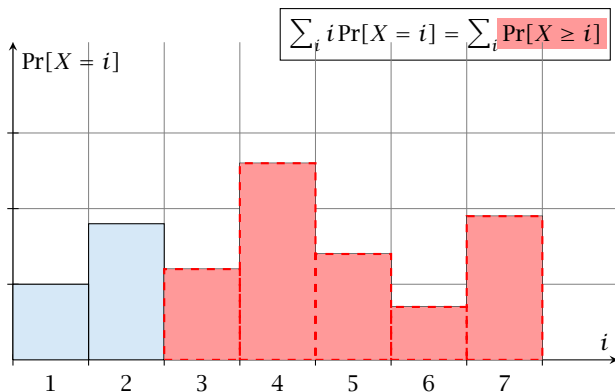
$i = 2$





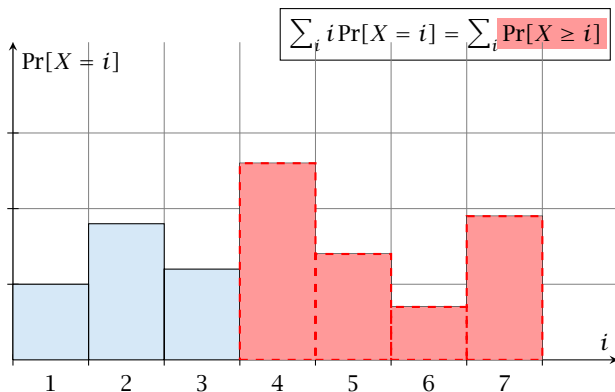
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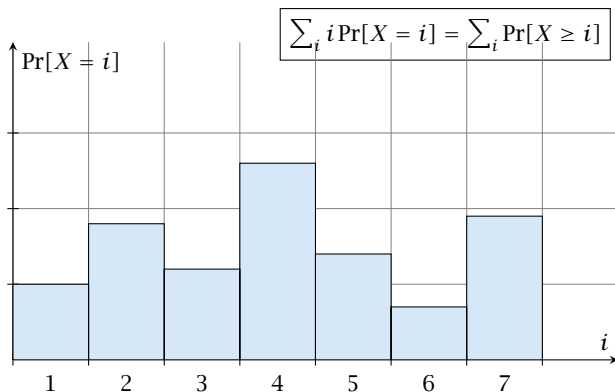


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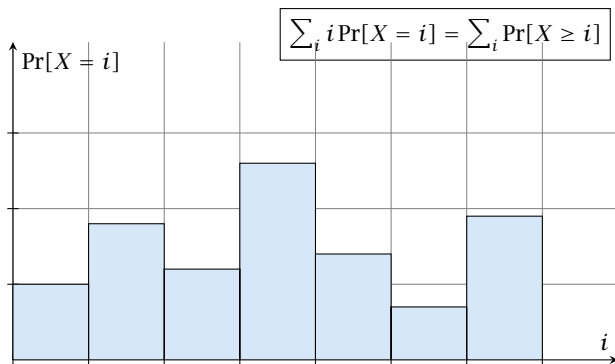
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# Analysis of Idealized Open Address Hashing



# Analysis of Idealized Open Address Hashing



The  $j$ -th rectangle appears in both sums  $j$  times. ( $j$  times in the first due to multiplication with  $j$ ; and  $j$  times in the second for summands  $i = 1, 2, \dots, j$ )

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$$\frac{1}{m} \sum_{i=0}^{m-1} \frac{n}{n-i} = \frac{n}{m} \sum_{i=0}^{m-1} \frac{1}{n-i} = \frac{1}{\alpha} \sum_{k=n-m+1}^n \frac{1}{k}$$

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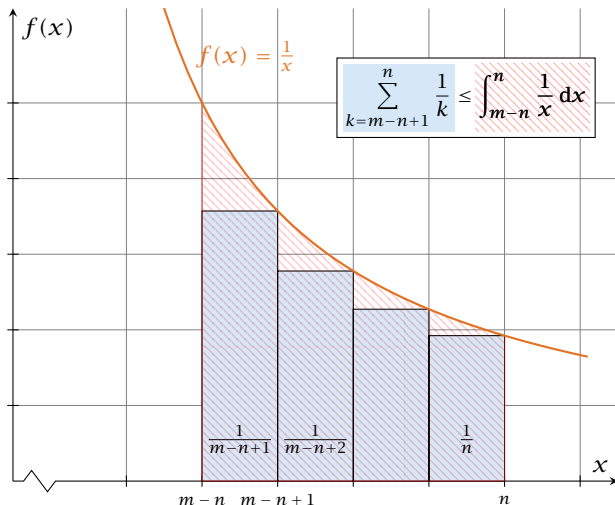
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# Analysis of Idealized Open Address Hashing



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- ▶ For hashing with chaining this is not a problem. Simply search for the key, and delete the item in the corresponding list.
- ▶ For open addressing this is difficult.



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  - ▶ During a search a **deleted**-marker must not be used to terminate the probe sequence.
- ▶ The table could fill up with **deleted**-markers leading to bad performance.
- ▶ If a table contains many deleted-markers (linear fraction of the keys) one can rehash the whole table and amortize the cost for this rehash against the cost for the deletions.

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- ▶ Upon a deletion elements that are further down in the probe-sequence may be moved to guarantee that they are still found during a search.



## Deletions for Linear Probing

### Algorithm 37 delete( $p$ )

```
1:  $T[p] \leftarrow \text{null}$ 
2:  $p \leftarrow \text{succ}(p)$ 
3: while  $T[p] \neq \text{null}$  do
4:    $y \leftarrow T[p]$ 
5:    $T[p] \leftarrow \text{null}$ 
6:    $p \leftarrow \text{succ}(p)$ 
7:   insert( $y$ )
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$p$  is the index into the table-cell that contains the object to be deleted.

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Pointers into the hash-table become invalid.

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However, the assumption of uniform hashing that  $h$  is chosen randomly from all functions  $f : U \rightarrow [0, \dots, n - 1]$  is clearly unrealistic as there are  $n^{|U|}$  such functions. Even writing down such a function would take  $|U| \log n$  bits.

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Universal hashing tries to define a set  $\mathcal{H}$  of functions that is much smaller but still leads to good average case behaviour when selecting a hash-function uniformly at random from  $\mathcal{H}$ .

# Universal Hashing

## Definition 13

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called **universal** if for all  $u_1, u_2 \in U$  with  $u_1 \neq u_2$

$$\Pr[h(u_1) = h(u_2)] \leq \frac{1}{n} ,$$

where the probability is w. r. t. the choice of a random hash-function from set  $\mathcal{H}$ .



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Note that this means that the probability of a collision between two arbitrary elements is at most  $\frac{1}{n}$ .

# Universal Hashing

## Definition 14

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called **2-independent** (pairwise independent) if the following two conditions hold

- ▶ For any key  $u \in U$ , and  $t \in \{0, \dots, n-1\}$   $\Pr[h(u) = t] = \frac{1}{n}$ ,  
i.e., a key is distributed uniformly within the hash-table.
- ▶ For all  $u_1, u_2 \in U$  with  $u_1 \neq u_2$ , and for any two hash-positions  $t_1, t_2$ :

$$\Pr[h(u_1) = t_1 \wedge h(u_2) = t_2] \leq \frac{1}{n^2} .$$

# Universal Hashing

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This requirement clearly implies a universal hash-function.

## Definition 15

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called  **$k$ -independent** if for any choice of  $\ell \leq k$  distinct keys  $u_1, \dots, u_\ell \in U$ , and for any set of  $\ell$  not necessarily distinct hash-positions  $t_1, \dots, t_\ell$ :

$$\Pr[h(u_1) = t_1 \wedge \dots \wedge h(u_\ell) = t_\ell] \leq \frac{1}{n^\ell} ,$$

where the probability is w. r. t. the choice of a random hash-function from set  $\mathcal{H}$ .

# Universal Hashing

## Definition 16

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called  $(\mu, k)$ -independent if for any choice of  $\ell \leq k$  distinct keys  $u_1, \dots, u_\ell \in U$ , and for any set of  $\ell$  not necessarily distinct hash-positions  $t_1, \dots, t_\ell$ :

$$\Pr[h(u_1) = t_1 \wedge \dots \wedge h(u_\ell) = t_\ell] \leq \frac{\mu}{n^\ell},$$

where the probability is w. r. t. the choice of a random hash-function from set  $\mathcal{H}$ .

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$$h_{a,b}(x) := (ax + b \bmod p) \bmod n$$



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## Lemma 17

*The class*

$$\mathcal{H} = \{h_{a,b} \mid a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$$

*is a universal class of hash-functions from  $U$  to  $\{0, \dots, n-1\}$ .*

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where we use that  $\mathbb{Z}_p$  is a field (**Körper**) and, hence, has no zero divisors (**nullteilerfrei**).

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$$a \equiv (t_x - t_y)(x - y)^{-1} \pmod{p}$$

$$b \equiv t_y - ay \pmod{p}$$

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From the range  $0, \dots, p - 1$  the values  $t_x, t_x + n, t_x + 2n, \dots$  map to  $t_x$  after the modulo-operation. These are at most  $\lceil p/n \rceil$  values.

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This happens with probability at most  $\frac{1}{n}$ .



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It is also possible to show that  $\mathcal{H}$  is an (almost) pairwise independent class of hash-functions.

$$\Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[ \begin{array}{l} t_x \bmod n = h_1 \\ t_y \bmod n = h_2 \end{array} \right]$$

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Note that the middle is the probability that  $h(x) = h_1$  and  $h(y) = h_2$ . The total number of choices for  $(t_x, t_y)$  is  $p(p-1)$ . The number of choices for  $t_x$  ( $t_y$ ) such that  $t_x \bmod n = h_1$  ( $t_y \bmod n = h_2$ ) lies between  $\lfloor \frac{p}{n} \rfloor$  and  $\lceil \frac{p}{n} \rceil$ .

# Universal Hashing

## Definition 18

Let  $d \in \mathbb{N}$ ;  $q \geq (d + 1)n$  be a prime; and let  $\bar{a} \in \{0, \dots, q - 1\}^{d+1}$ . Define for  $x \in \{0, \dots, q - 1\}$

$$h_{\bar{a}}(x) := \left( \sum_{i=0}^d a_i x^i \bmod q \right) \bmod n .$$

Let  $\mathcal{H}_n^d := \{h_{\bar{a}} \mid \bar{a} \in \{0, \dots, q - 1\}^{d+1}\}$ . The class  $\mathcal{H}_n^d$  is  $(e, d + 1)$ -independent.

Note that in the previous case we had  $d = 1$  and chose  $a_d \neq 0$ .

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For the coefficients  $\bar{a} \in \{0, \dots, q-1\}^{d+1}$  let  $f_{\bar{a}}$  denote the polynomial

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The polynomial is defined by  $d+1$  distinct points.



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Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

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We first fix the values for inputs  $x_1, \dots, x_\ell$ .

We have

$$|B_1| \cdot \dots \cdot |B_\ell|$$

possibilities to do this (so that  $h_{\bar{a}}(x_i) = t_i$ ).

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Now, we choose  $d - \ell + 1$  other inputs and choose their value arbitrarily. We have  $q^{d-\ell+1}$  possibilities to do this.



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Therefore we have

$$|B_1| \cdot \dots \cdot |B_\ell| \cdot q^{d-\ell+1} \leq \left\lceil \frac{q}{n} \right\rceil^\ell \cdot q^{d-\ell+1}$$

possibilities to choose  $\bar{a}$  such that  $h_{\bar{a}} \in A_\ell$ .

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Therefore the probability of choosing  $h_{\bar{a}}$  from  $A_\ell$  is only

$$\frac{\lceil \frac{q}{n} \rceil^\ell \cdot q^{d-\ell+1}}{q^{d+1}}$$

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$$\frac{\left[\frac{q}{n}\right]^\ell \cdot q^{d-\ell+1}}{q^{d+1}} \leq \frac{\left(\frac{q+n}{n}\right)^\ell}{q^\ell} \leq \left(\frac{q+n}{q}\right)^\ell \cdot \frac{1}{n^\ell}$$

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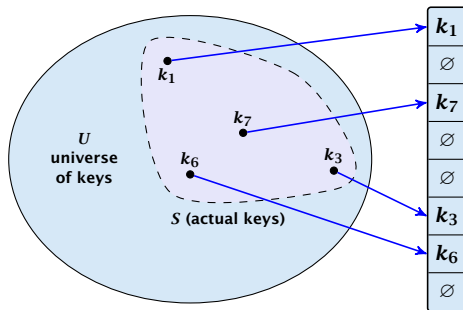
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This shows that the  $\mathcal{H}$  is  $(e, d+1)$ -universal.

The last step followed from  $q \geq (d+1)n$ , and  $\ell \leq d+1$ .

# Perfect Hashing

Suppose that we **know** the set  $S$  of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.





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The probability of having **1** or more collisions can be at most  $\frac{1}{2}$  as otherwise the expectation would be larger than  $\frac{1}{2}$ .

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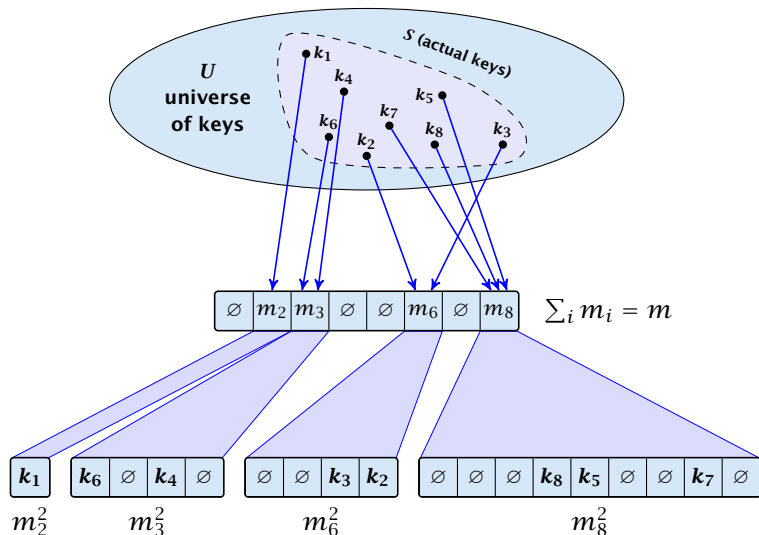
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Let  $m_j$  denote the number of items that are hashed to the  $j$ -th bucket. For each bucket we choose a second hash-function that maps the elements of the bucket into a table of size  $m_j^2$ . The second function can be chosen such that all elements are mapped to different locations.

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The first expectation is simply the expected number of collisions, for the first level. Since we use universal hashing we have

$$= 2 \binom{m}{2} \frac{1}{m} + m = 2m - 1 .$$

# Perfect Hashing

We need only  $\mathcal{O}(m)$  time to construct a hash-function  $h$  with  $\sum_j m_j^2 = \mathcal{O}(4m)$ , because with probability at least  $1/2$  a random function from a universal family will have this property.

Then we construct a hash-table  $h_j$  for every bucket. This takes expected time  $\mathcal{O}(m_j)$  for every bucket. A random function  $h_j$  is collision-free with probability at least  $1/2$ . We need  $\mathcal{O}(m_j)$  to test this.

We only need that the hash-functions are chosen from a universal family!!!

# Cuckoo Hashing

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## Goal:

Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

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- ▶ A search clearly takes constant time if the above constraint is met.



# Cuckoo Hashing

Insert:



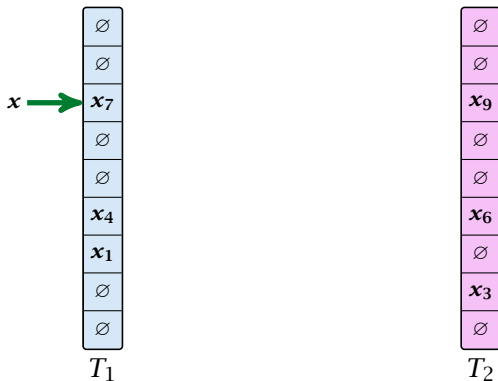
$T_1$



$T_2$

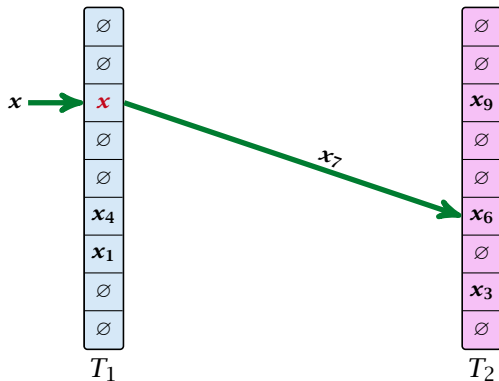
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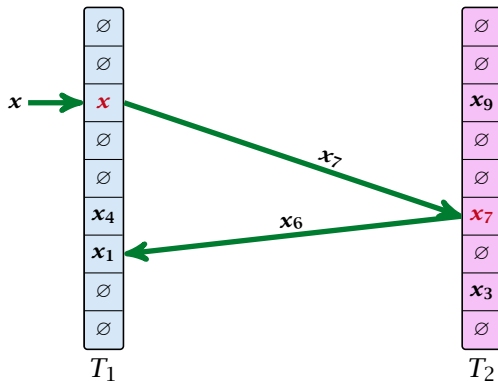
# Cuckoo Hashing

Insert:



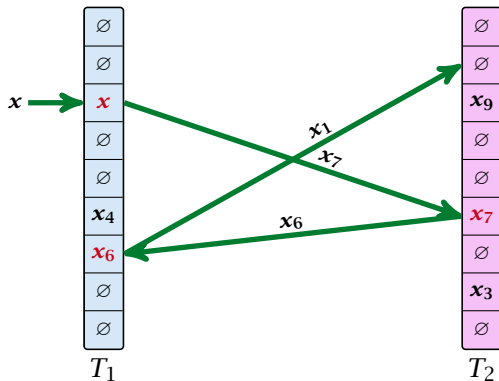
# Cuckoo Hashing

Insert:



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Insert:



## Algorithm 38 Cuckoo-Insert( $x$ )

```
1: if  $T_1[h_1(x)] = x \vee T_2[h_2(x)] = x$  then return  
2: steps  $\leftarrow$  1  
3: while steps  $\leq$  maxsteps do  
4:     exchange  $x$  and  $T_1[h_1(x)]$   
5:     if  $x = \text{null}$  then return  
6:     exchange  $x$  and  $T_2[h_2(x)]$   
7:     if  $x = \text{null}$  then return  
8:     steps  $\leftarrow$  steps + 1  
9: rehash() // change hash-functions; rehash everything  
10: Cuckoo-Insert( $x$ )
```

# Cuckoo Hashing

- ▶ We call one iteration through the while-loop a **step** of the algorithm.

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- ▶ We call one iteration through the while-loop a **step** of the algorithm.
- ▶ We call a sequence of iterations through the while-loop without the termination condition becoming true a **phase** of the algorithm.
- ▶ We say a phase is **successful** if it is not terminated by the **maxstep**-condition, but the while loop is left because  $x = \text{null}$ .

# Cuckoo Hashing

**What is the expected time for an insert-operation?**

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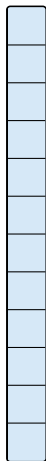
We first analyze the probability that we end-up in an infinite loop (that is then terminated after **maxsteps** steps).

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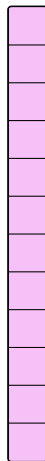
We first analyze the probability that we end-up in an infinite loop (that is then terminated after **maxsteps** steps).

Formally what is the probability to enter an infinite loop that touches  $s$  different keys?

# Cuckoo Hashing: Insert

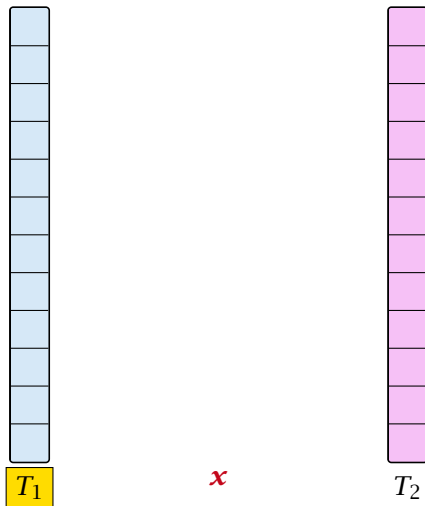


$T_1$

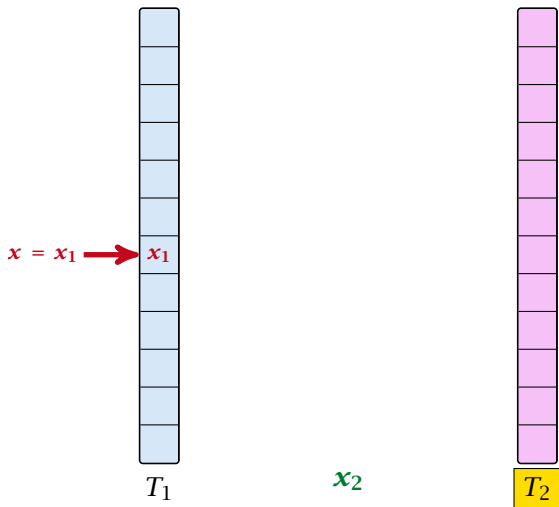


$T_2$

# Cuckoo Hashing: Insert

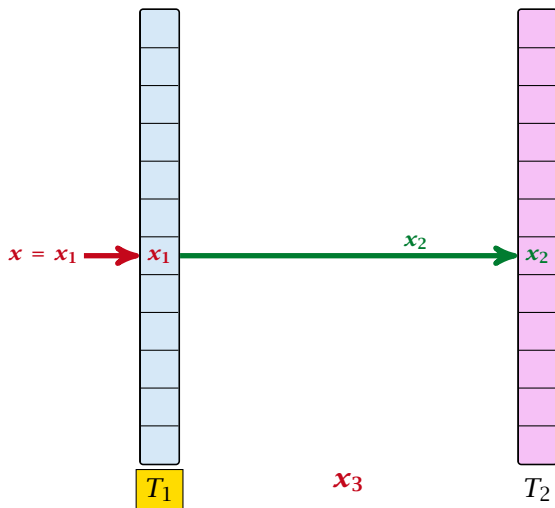


# Cuckoo Hashing: Insert

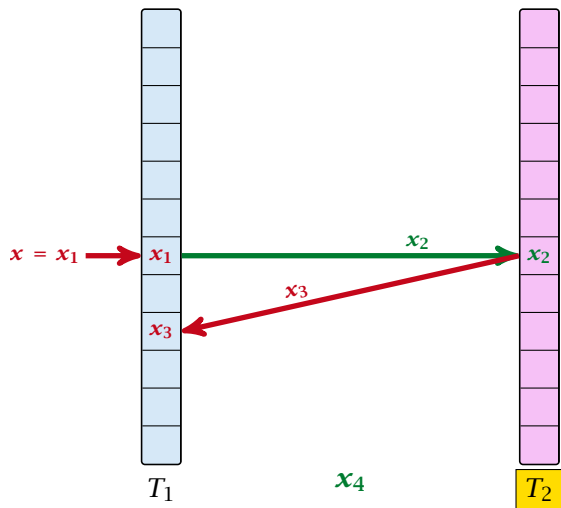




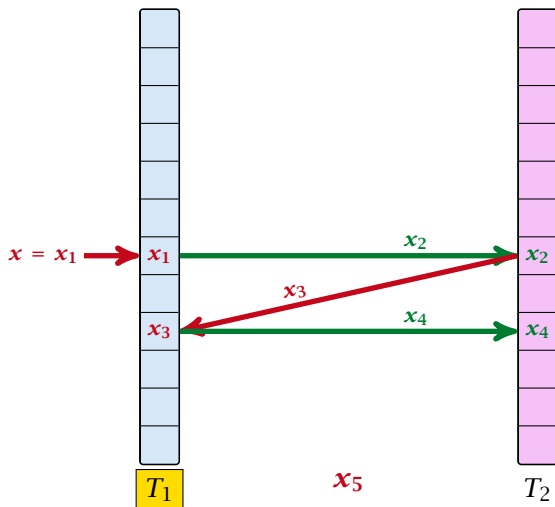
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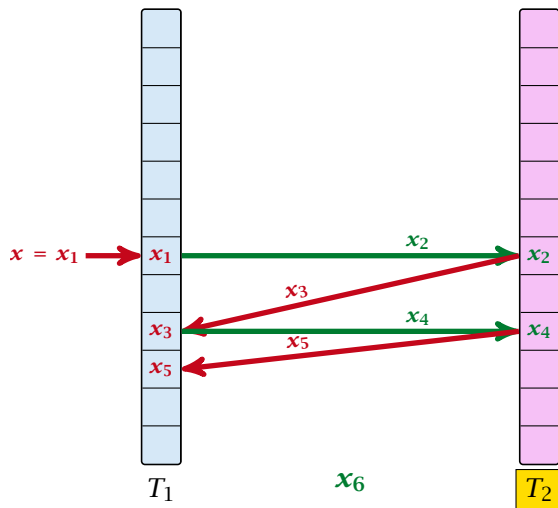
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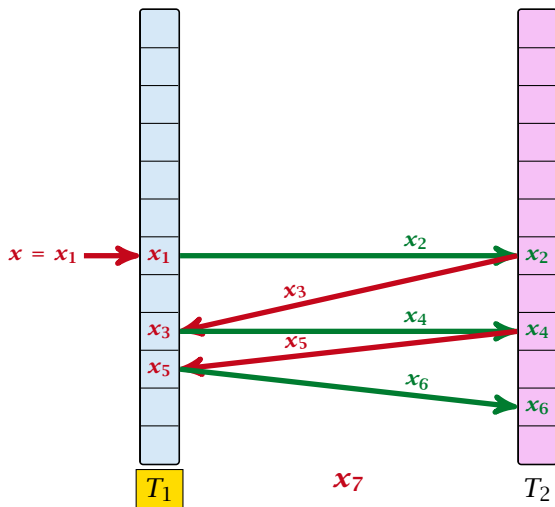
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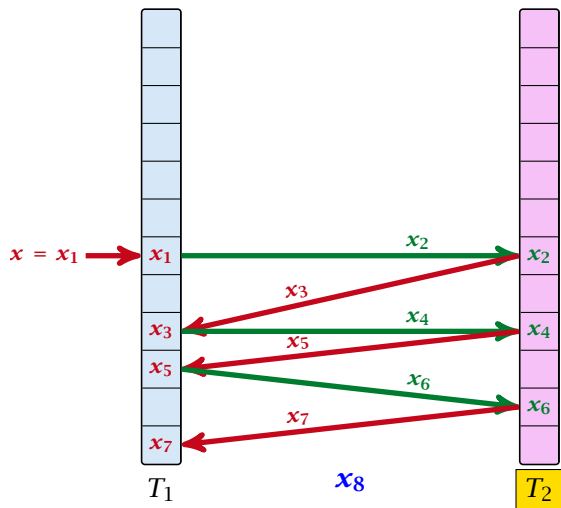
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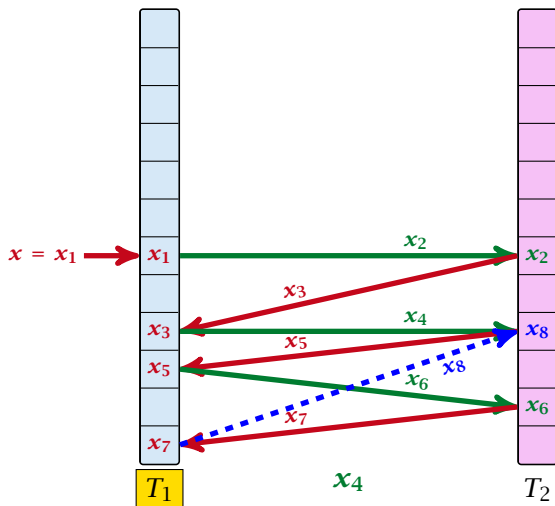
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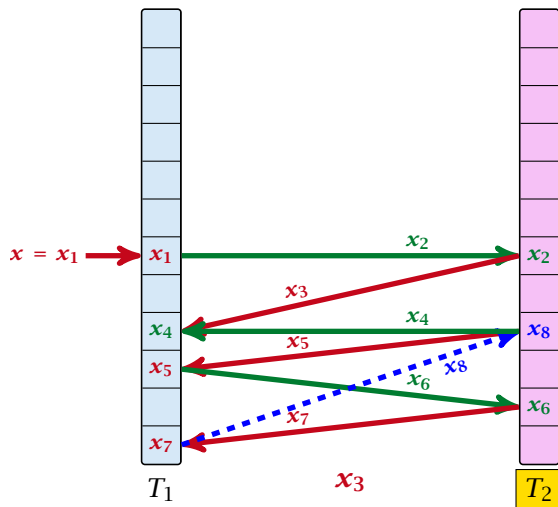
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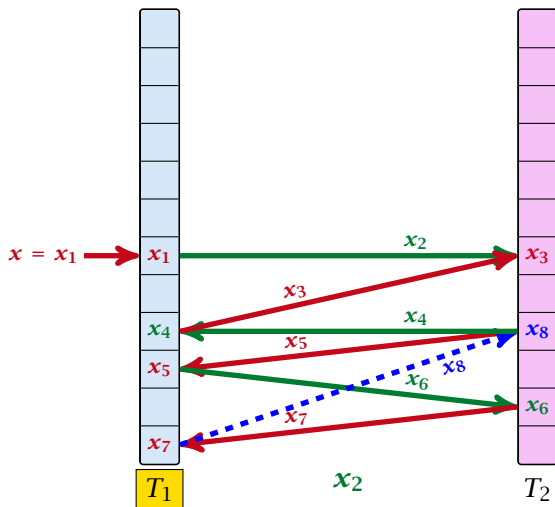


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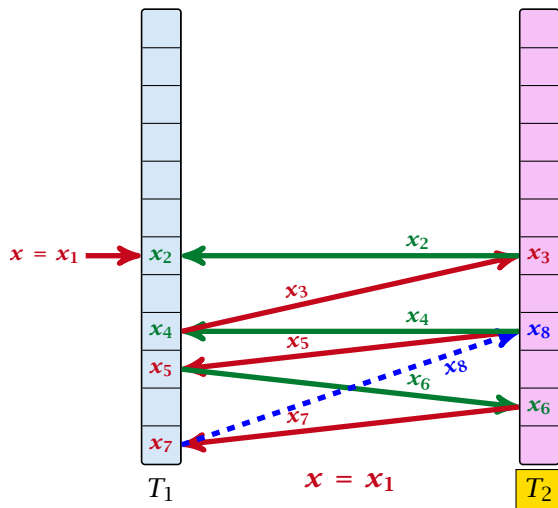




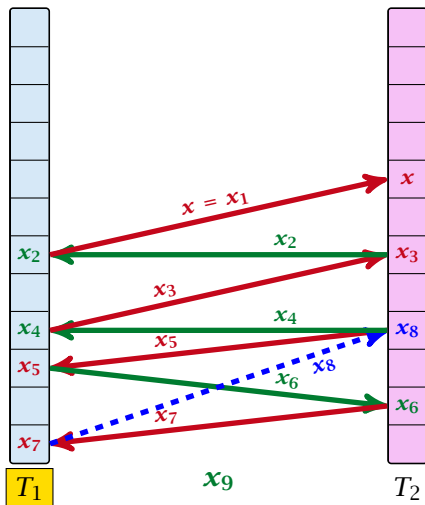
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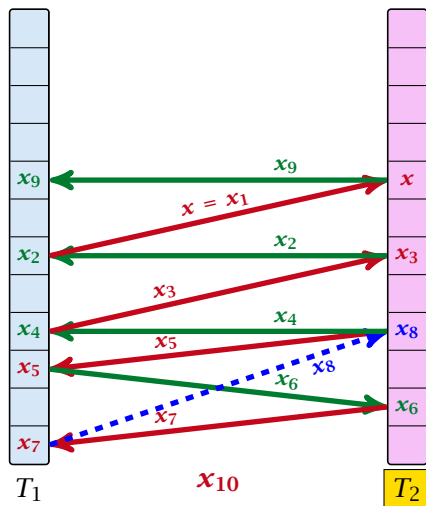
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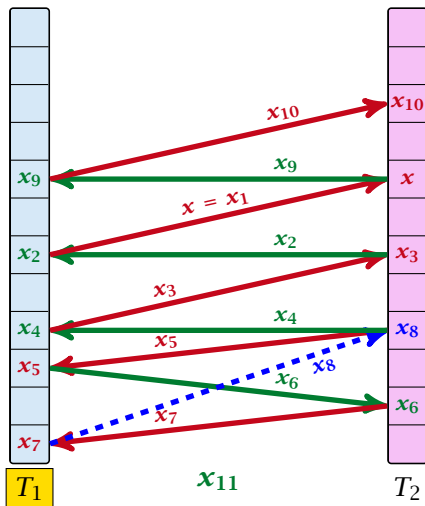
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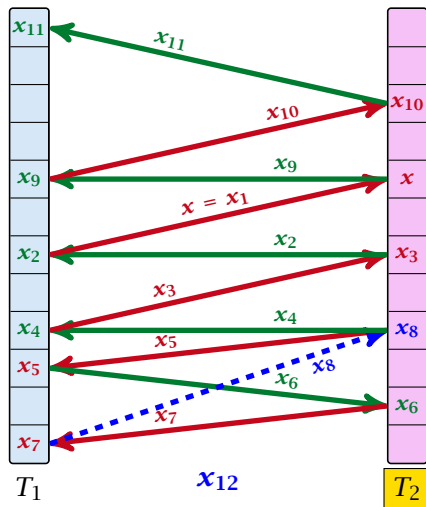
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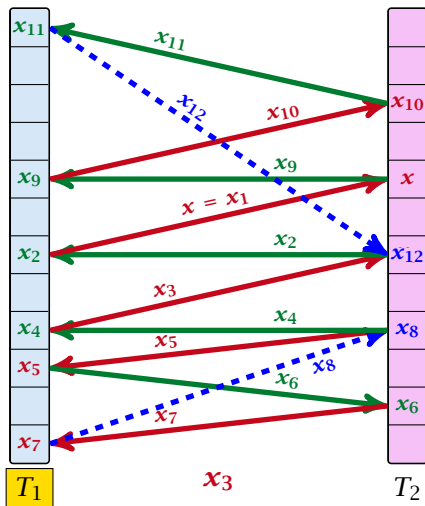
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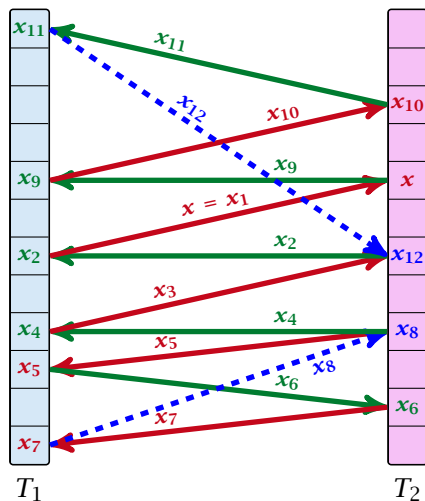
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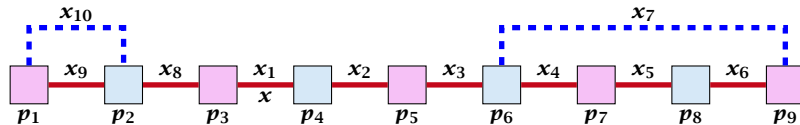


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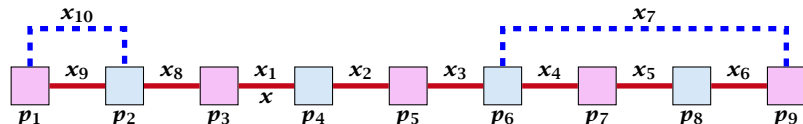


# Cuckoo Hashing



A cycle-structure of size  $s$  is defined by

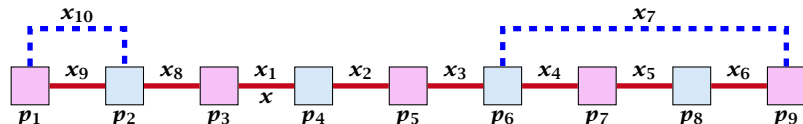
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A cycle-structure of size  $s$  is defined by

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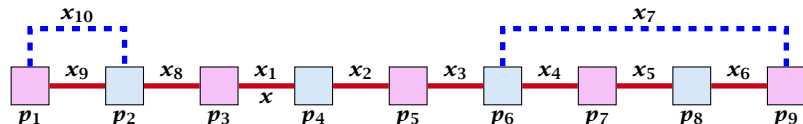
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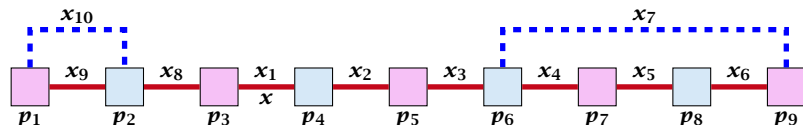
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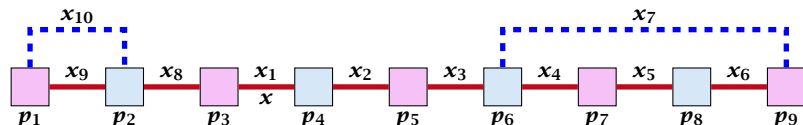
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- ▶ One link represents key  $x$ ; this is where the counting starts.

# Cuckoo Hashing

A cycle-structure is **active** if for every key  $x_\ell$  (linking a cell  $p_i$  from  $T_1$  and a cell  $p_j$  from  $T_2$ ) we have

$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

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## Observation:

If during a phase the insert-procedure runs into a cycle there must exist an active cycle structure of size  $s \geq 3$ .



# Cuckoo Hashing

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These events are independent.

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What is the probability that **there exists** an active cycle structure of size  $s$ ?

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The number of cycle-structures of size  $s$  is at most

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- ▶ There are  $n^{s-1}$  possibilities to choose the cells.

# Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$

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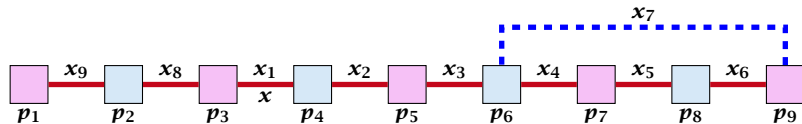
Hence,

$$\Pr[\text{cycle}] = \mathcal{O}\left(\frac{1}{m^2}\right).$$

# Cuckoo Hashing

Now, we analyze the probability that a phase is not successful without running into a closed cycle.

# Cuckoo Hashing



Sequence of visited keys:

$x = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_3, x_2, x_1 = x, x_8, x_9, \dots$

# Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with  $x$  in the order that they are visited during the phase.

# Cuckoo Hashing

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## Lemma 19

*If the sequence is of length  $p$  then there exists a sub-sequence of at least  $\frac{p+2}{3}$  keys starting with  $x$  of *distinct* keys.*

# Cuckoo Hashing

## Proof.

Let  $i$  be the number of keys (including  $x$ ) that we see before the first repeated key. Let  $j$  denote the total number of distinct keys.

The sequence is of the form:

$$x = x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i \rightarrow x_r \rightarrow x_{r-1} \rightarrow \dots \rightarrow x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$$

As  $r \leq i - 1$  the length  $p$  of the sequence is

$$p = i + r + (j - i) \leq i + j - 1 .$$

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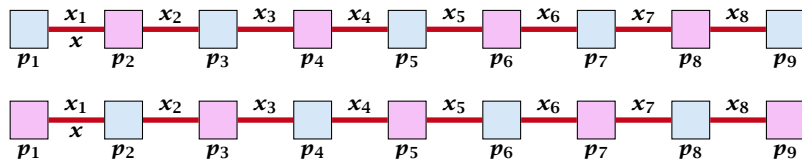
As  $r \leq i - 1$  the length  $p$  of the sequence is

$$p = i + r + (j - i) \leq i + j - 1 .$$

Either sub-sequence  $x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i$  or sub-sequence  $x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$  has at least  $\frac{p+2}{3}$  elements. □

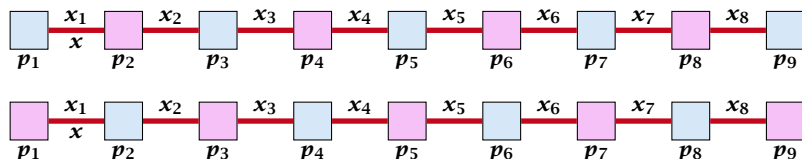


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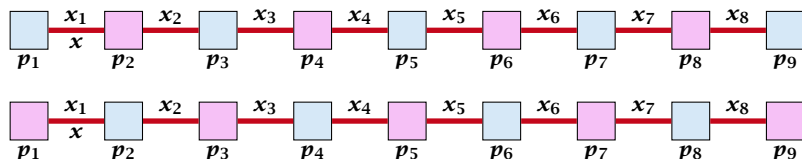
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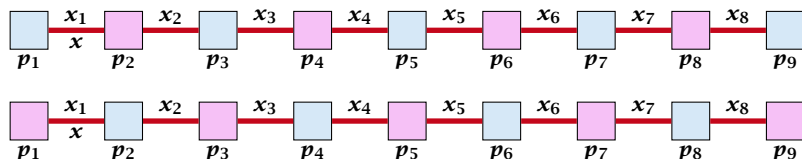
# Cuckoo Hashing



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A path-structure is **active** if for every key  $x_\ell$  (linking a cell  $p_i$  from  $T_1$  and a cell  $p_j$  from  $T_2$ ) we have

$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

## Observation:

If a phase takes at least  $t$  steps without running into a cycle there must exist an active path-structure of size  $(2t + 2)/3$ .

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This gives  $\text{maxsteps} = \Theta(\log m)$ .

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So far we estimated

$$\Pr[\text{cycle}] \leq \mathcal{O}\left(\frac{1}{m^2}\right)$$

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for a suitable constant  $c > 0$ .

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This means the expected cost for a successful phase is constant (even after accounting for the cost of the incomplete step that finishes the phase).



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Therefore the expected cost for re-hashes is  $\mathcal{O}(m) \cdot \mathcal{O}(p) = \mathcal{O}(1)$ .

## Formal Proof

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The 0-th (re)hash is the initial configuration when doing the insert.

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Therefore, it is sufficient to have  $(\mu, \Theta(\log m))$ -independent hash-functions.

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- ▶ Therefore we can amortize the rehash cost after a change in table-size against the cost for insertions and deletions.

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Note that the above lemma only holds if the fill-factor (number of keys/total number of hash-table slots) is at most  $\frac{1}{2(1+\epsilon)}$ .

The  $1/(2(1+\epsilon))$  fill-factor comes from the fact that the total hash-table is of size  $2n$  (because we have two tables of size  $n$ ); moreover  $m \leq (1+\epsilon)n$ .