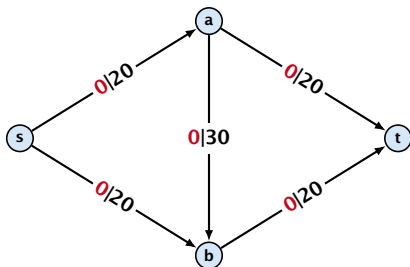


# 11 Augmenting Path Algorithms

## Greedy-algorithm:

- ▶ start with  $f(e) = 0$  everywhere
- ▶ find an  $s$ - $t$  path with  $f(e) < c(e)$  on every edge
- ▶ augment flow along the path
- ▶ repeat as long as possible

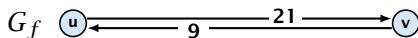
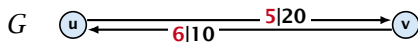


flow value: 00

# The Residual Graph

From the graph  $G = (V, E, c)$  and the current flow  $f$  we construct an auxiliary graph  $G_f = (V, E_f, c_f)$  (the residual graph):

- ▶ Suppose the original graph has edges  $e_1 = (u, v)$ , and  $e_2 = (v, u)$  between  $u$  and  $v$ .
- ▶  $G_f$  has edge  $e'_1$  with capacity  $\max\{0, c(e_1) - f(e_1) + f(e_2)\}$  and  $e'_2$  with with capacity  $\max\{0, c(e_2) - f(e_2) + f(e_1)\}$ .



# Augmenting Path Algorithm

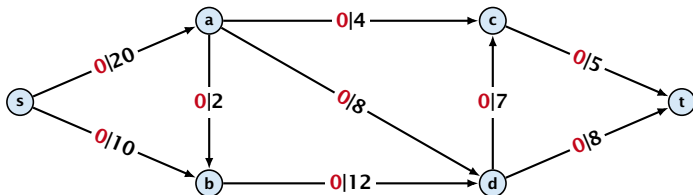
## Definition 4

An **augmenting path** with respect to flow  $f$ , is a path from  $s$  to  $t$  in the auxiliary graph  $G_f$  that contains only edges with non-zero capacity.

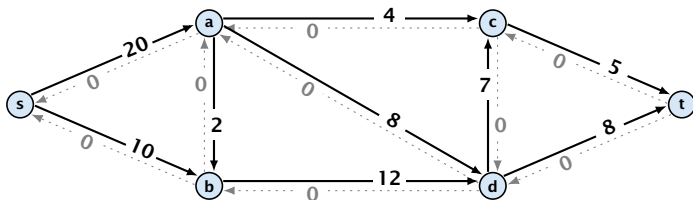
### Algorithm 1 FordFulkerson( $G = (V, E, c)$ )

- 1: Initialize  $f(e) \leftarrow 0$  for all edges.
- 2: **while**  $\exists$  augmenting path  $p$  in  $G_f$  **do**
- 3:     augment as much flow along  $p$  as possible.

# Augmenting Paths



flow value: 0



# Augmenting Path Algorithm

## Theorem 5

A flow  $f$  is a maximum flow **iff** there are no augmenting paths.

## Theorem 6

The value of a maximum flow is equal to the value of a minimum cut.

## Proof.

Let  $f$  be a flow. The following are equivalent:

1. There exists a cut  $A$  such that  $\text{val}(f) = \text{cap}(A, V \setminus A)$ .
2. Flow  $f$  is a maximum flow.
3. There is no augmenting path w.r.t.  $f$ .



# Augmenting Path Algorithm

1.  $\Rightarrow$  2.

This we already showed.

2.  $\Rightarrow$  3.

If there were an augmenting path, we could improve the flow.  
Contradiction.

3.  $\Rightarrow$  1.

- ▶ Let  $f$  be a flow with no augmenting paths.
- ▶ Let  $A$  be the set of vertices reachable from  $s$  in the residual graph along non-zero capacity edges.
- ▶ Since there is no augmenting path we have  $s \in A$  and  $t \notin A$ .

# Augmenting Path Algorithm

$$\begin{aligned}\text{val}(f) &= \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e) \\ &= \sum_{e \in \text{out}(A)} c(e) \\ &= \text{cap}(A, V \setminus A)\end{aligned}$$

This finishes the proof.

Here the first equality uses the flow value lemma, and the second exploits the fact that the flow along incoming edges must be 0 as the residual graph does not have edges leaving  $A$ .

**Assumption:**

All capacities are integers between 1 and  $C$ .

**Invariant:**

Every flow value  $f(e)$  and every residual capacity  $c_f(e)$  remains integral throughout the algorithm.



## Lemma 7

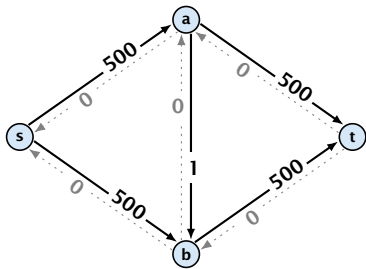
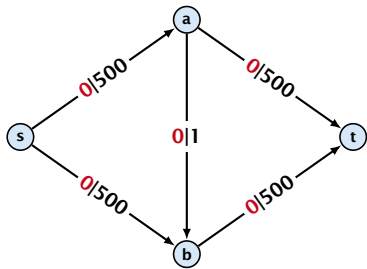
The algorithm terminates in at most  $\text{val}(f^*) \leq nC$  iterations, where  $f^*$  denotes the maximum flow. Each iteration can be implemented in time  $\mathcal{O}(m)$ . This gives a total running time of  $\mathcal{O}(nmC)$ .

## Theorem 8

If all capacities are integers, then there exists a maximum flow for which every flow value  $f(e)$  is integral.

# A Bad Input

**Problem:** The running time may not be polynomial



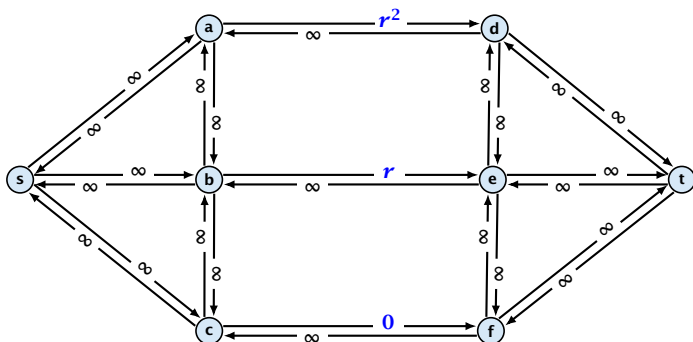
**flow value: 0**

**Question:**

Can we tweak the algorithm so that the running time is polynomial in the input length?

# A Pathological Input

Let  $r = \frac{1}{2}(\sqrt{5} - 1)$ . Then  $r^{n+2} = r^n - r^{n+1}$ .



flow value:  $r^0 r^3 r^4$

Running time may be infinite!!!

## How to choose augmenting paths?

- ▶ We need to find paths efficiently.
- ▶ We want to guarantee a small number of iterations.

### Several possibilities:

- ▶ Choose path with maximum bottleneck capacity.
- ▶ Choose path with sufficiently large bottleneck capacity.
- ▶ Choose the shortest augmenting path.

# Overview: Shortest Augmenting Paths

## Lemma 9

*The length of the shortest augmenting path never decreases.*

## Lemma 10

*After at most  $\mathcal{O}(m)$  augmentations, the length of the shortest augmenting path strictly increases.*

# Overview: Shortest Augmenting Paths

These two lemmas give the following theorem:

## Theorem 11

*The shortest augmenting path algorithm performs at most  $\mathcal{O}(mn)$  augmentations. This gives a running time of  $\mathcal{O}(m^2n)$ .*

## Proof.

- ▶ We can find the shortest augmenting paths in time  $\mathcal{O}(m)$  via BFS.
- ▶  $\mathcal{O}(m)$  augmentations for paths of exactly  $k < n$  edges.



# Shortest Augmenting Paths

Define the level  $\ell(v)$  of a node as the length of the shortest  $s$ - $v$  path in  $G_f$  (along non-zero edges).

Let  $L_G$  denote the **subgraph** of the residual graph  $G_f$  that contains only those edges  $(u, v)$  with  $\ell(v) = \ell(u) + 1$ .

A path  $P$  is a shortest  $s$ - $u$  path in  $G_f$  **iff** it is an  $s$ - $u$  path in  $L_G$ .



In the following we assume that the residual graph  $G_f$  does not contain zero capacity edges.

This means, we construct it in the usual sense and then delete edges of zero capacity.



# Shortest Augmenting Path

## First Lemma:

The length of the shortest augmenting path never decreases.

After an augmentation  $G_f$  changes as follows:

- ▶ Bottleneck edges on the chosen path are deleted.
- ▶ Back edges are added to all edges that don't have back edges so far.

These changes cannot decrease the distance between  $s$  and  $t$ .



## Shortest Augmenting Path

**Second Lemma:** After at most  $m$  augmentations the length of the shortest augmenting path strictly increases.

Let  $M$  denote the set of edges in graph  $L_G$  at the beginning of a round when the distance between  $s$  and  $t$  is  $k$ .

An  $s$ - $t$  path in  $G_f$  that uses edges not in  $M$  has length larger than  $k$ , even when using edges added to  $G_f$  during the round.

In each augmentation an edge is deleted from  $M$ .

  
edge of  $G_f$

  
edge in  $M$

Note that an edge cannot enter  $M$  again during the round as this would require an augmentation along a non-shortest path.

# Shortest Augmenting Paths

## Theorem 12

*The shortest augmenting path algorithm performs at most  $\mathcal{O}(mn)$  augmentations. Each augmentation can be performed in time  $\mathcal{O}(m)$ .*

## Theorem 13 (without proof)

*There exist networks with  $m = \Theta(n^2)$  that require  $\Omega(mn)$  augmentations, when we restrict ourselves to only augment along shortest augmenting paths.*

### Note:

There always exists a set of  $m$  augmentations that gives a maximum flow (why?).

# Shortest Augmenting Paths

When sticking to shortest augmenting paths we cannot improve (asymptotically) on the number of augmentations.

However, we can improve the running time to  $\mathcal{O}(mn^2)$  by improving the running time for finding an augmenting path (currently we assume  $\mathcal{O}(m)$  per augmentation for this).

# Shortest Augmenting Paths

We maintain a subset  $M$  of the edges of  $G_f$  with the guarantee that a shortest  $s$ - $t$  path using only edges from  $M$  is a shortest augmenting path.

With each augmentation some edges are deleted from  $M$ .

When  $M$  does not contain an  $s$ - $t$  path anymore the distance between  $s$  and  $t$  strictly increases.

Note that  $M$  is not the set of edges of the level graph but a subset of level-graph edges.

Suppose that the initial distance between  $s$  and  $t$  in  $G_f$  is  $k$ .

$M$  is initialized as the level graph  $L_G$ .

Perform a **DFS search** to find a path from  $s$  to  $t$  using edges from  $M$ .

Either you find  $t$  after at most  $n$  steps, or you end at a node  $v$  that does not have any outgoing edges.

You can delete incoming edges of  $v$  from  $M$ .

## Analysis

Let a phase of the algorithm be defined by the time between two augmentations during which the distance between  $s$  and  $t$  strictly increases.

Initializing  $M$  for the phase takes time  $\mathcal{O}(m)$ .

The total cost for searching for augmenting paths during a phase is at most  $\mathcal{O}(mn)$ , since every search (successful (i.e., reaching  $t$ ) or unsuccessful) decreases the number of edges in  $M$  and takes time  $\mathcal{O}(n)$ .

The total cost for performing an augmentation **during** a phase is only  $\mathcal{O}(n)$ . For every edge in the augmenting path one has to update the residual graph  $G_f$  and has to check whether the edge is still in  $M$  for the next search.

There are at most  $n$  phases. Hence, total cost is  $\mathcal{O}(mn^2)$ .

## How to choose augmenting paths?

- ▶ We need to find paths efficiently.
- ▶ We want to guarantee a small number of iterations.

### Several possibilities:

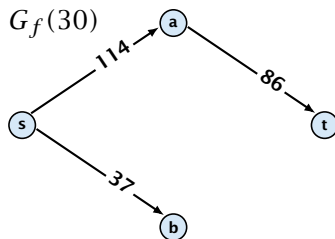
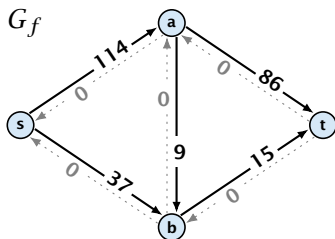
- ▶ Choose path with maximum bottleneck capacity.
- ▶ Choose path with sufficiently large bottleneck capacity.
- ▶ Choose the shortest augmenting path.



# Capacity Scaling

## Intuition:

- ▶ Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.
- ▶ Don't worry about finding the exact bottleneck.
- ▶ Maintain scaling parameter  $\Delta$ .
- ▶  $G_f(\Delta)$  is a sub-graph of the residual graph  $G_f$  that contains only edges with capacity at least  $\Delta$ .



# Capacity Scaling

## Algorithm 1 maxflow( $G, s, t, c$ )

```
1: foreach  $e \in E$  do  $f_e \leftarrow 0$ ;  
2:  $\Delta \leftarrow 2^{\lceil \log_2 C \rceil}$   
3: while  $\Delta \geq 1$  do  
4:    $G_f(\Delta) \leftarrow \Delta$ -residual graph  
5:   while there is augmenting path  $P$  in  $G_f(\Delta)$  do  
6:      $f \leftarrow \text{augment}(f, c, P)$   
7:      $\text{update}(G_f(\Delta))$   
8:    $\Delta \leftarrow \Delta/2$   
9: return  $f$ 
```

# Capacity Scaling

## Assumption:

All capacities are integers between 1 and  $C$ .

## Invariant:

All flows and capacities are/remain integral throughout the algorithm.

## Correctness:

The algorithm computes a maxflow:

- ▶ because of integrality we have  $G_f(1) = G_f$
- ▶ therefore after the last phase there are no augmenting paths anymore
- ▶ this means we have a maximum flow.

# Capacity Scaling

## Lemma 14

There are  $\lceil \log C \rceil + 1$  iterations over  $\Delta$ .

**Proof:** obvious.

## Lemma 15

Let  $f$  be the flow at the end of a  $\Delta$ -phase. Then the maximum flow is smaller than  $\text{val}(f) + m\Delta$ .

**Proof:** less obvious, but simple:

- ▶ There must exist an  $s$ - $t$  cut in  $G_f(\Delta)$  of zero capacity.
- ▶ In  $G_f$  this cut can have capacity at most  $m\Delta$ .
- ▶ This gives me an upper bound on the flow that I can still add.

# Capacity Scaling

## Lemma 16

*There are at most  $2m$  augmentations per scaling-phase.*

**Proof:**

- ▶ Let  $f$  be the flow at the end of the previous phase.
- ▶  $\text{val}(f^*) \leq \text{val}(f) + 2m\Delta$
- ▶ Each augmentation increases flow by  $\Delta$ .

## Theorem 17

*We need  $\mathcal{O}(m \log C)$  augmentations. The algorithm can be implemented in time  $\mathcal{O}(m^2 \log C)$ .*