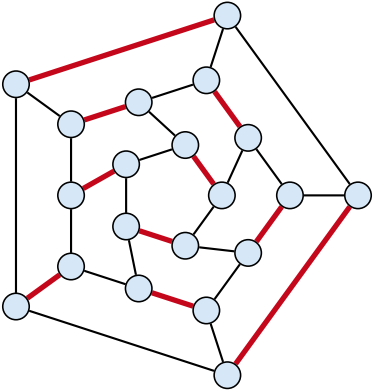
# Part V

# Matchings

# Matching

- Input: undirected graph $G = (V, E)$.
- $M \subseteq E$ is a matching if each node appears in at most one edge in $M$.
- Maximum Matching: find a matching of maximum cardinality



# 16 Bipartite Matching via Flows

**Which flow algorithm to use?**

- Generic augmenting path: $\mathcal{O}(m\,\mathrm{val}(f^*)) = \mathcal{O}(mn)$.
- Capacity scaling: $\mathcal{O}(m^2 \log C) = \mathcal{O}(m^2)$.
- Shortest augmenting path: $\mathcal{O}(mn^2)$.

For unit capacity simple graphs shortest augmenting path can be implemented in time $\mathcal{O}(m\sqrt{n})$.

# 17 Augmenting Paths for Matchings

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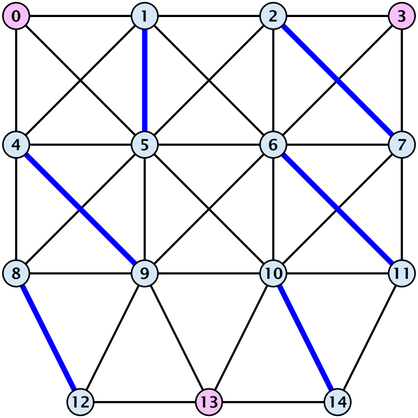
**Theorem 6**
*A matching $M$ is a maximum matching if and only if there is no augmenting path w. r. t. $M$.*

# Augmenting Paths in Action



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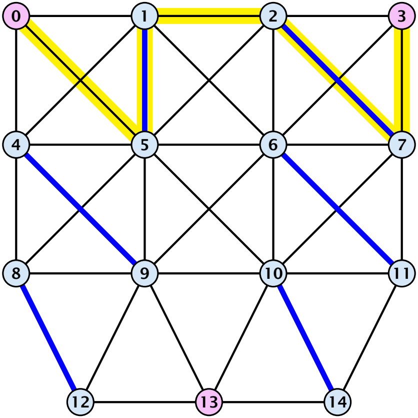

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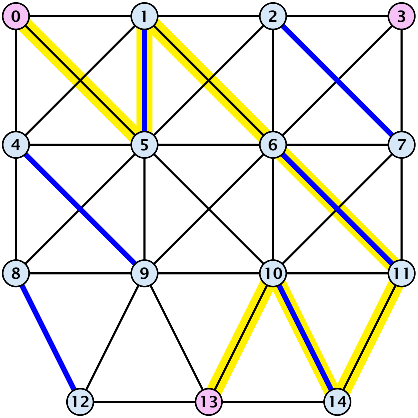

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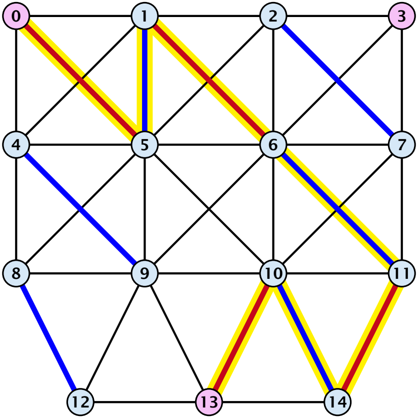

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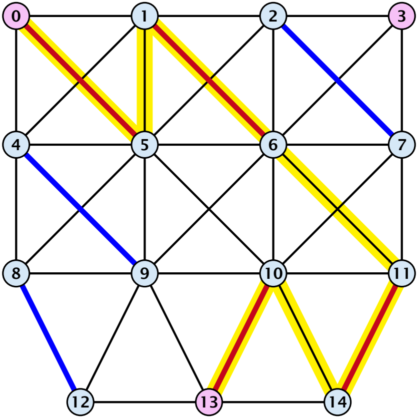

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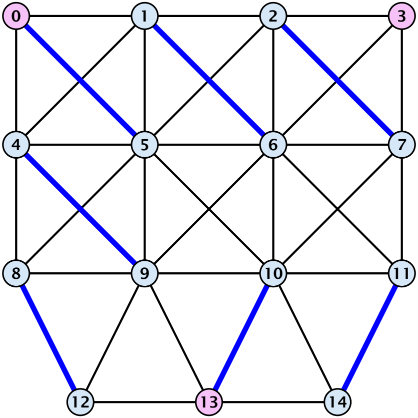

# 17 Augmenting Paths for Matchings

**Proof.**

- $\Rightarrow$ If $M$ is maximum there is no augmenting path $P$, because we could switch matching and non-matching edges along $P$. This gives matching $M' = M \oplus P$ with larger cardinality.

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  As $|M'| > |M|$ there is one connected component that is a path $P$ for which both endpoints are incident to edges from $M'$. $P$ is an alternating path.

# 17 Augmenting Paths for Matchings

**Algorithmic idea:**

As long as you find an augmenting path augment your matching using this path. When you arrive at a matching for which no augmenting path exists you have a maximum matching.

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**Theorem 7**

*Let $G$ be a graph, $M$ a matching in $G$, and let $u$ be a free vertex w.r.t. $M$. Further let $P$ denote an augmenting path w.r.t. $M$ and let $M' = M \oplus P$ denote the matching resulting from augmenting $M$ with $P$. If there was no augmenting path starting at $u$ in $M$ then there is no augmenting path starting at $u$ in $M'$.*

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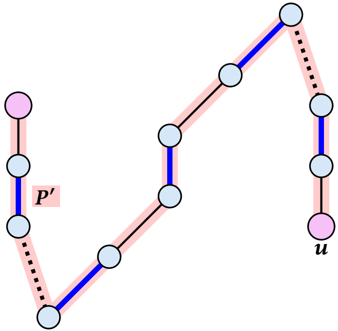
- Assume there is an augmenting path $P'$ w.r.t. $M'$ starting at $u$.



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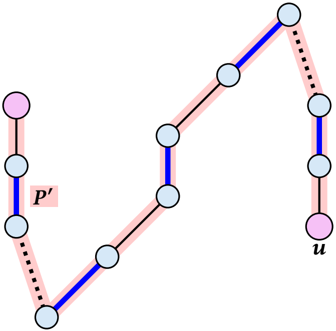
- Assume there is an augmenting path $P'$ w.r.t. $M'$ starting at $u$.
- If $P'$ and $P$ are node-disjoint, $P'$ is also augmenting path w.r.t. $M$ ($\lightning$).



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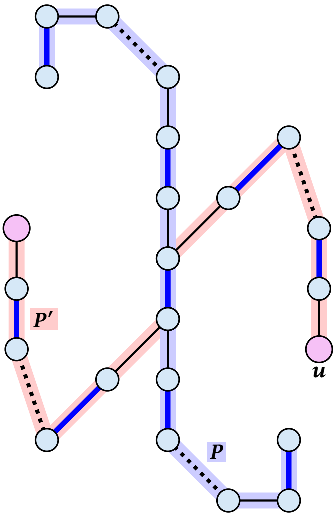

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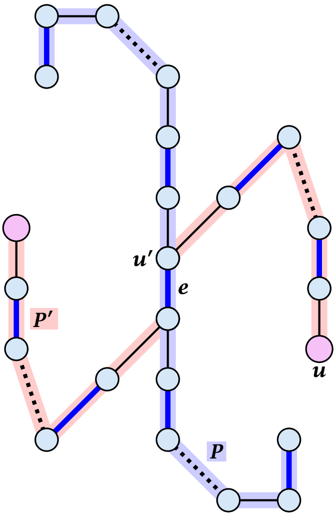

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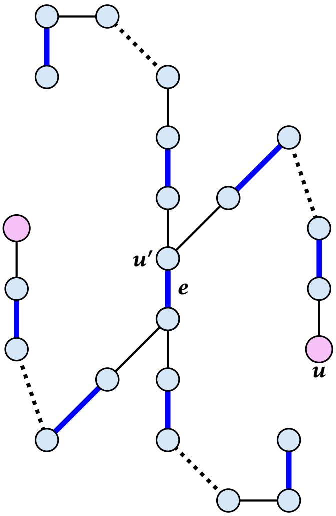

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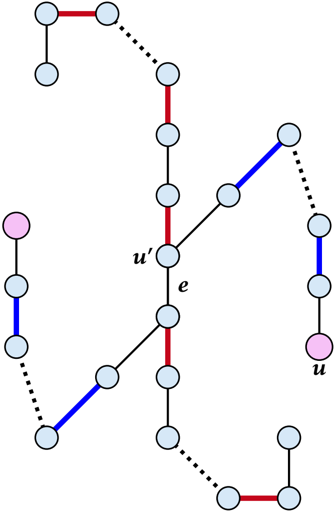

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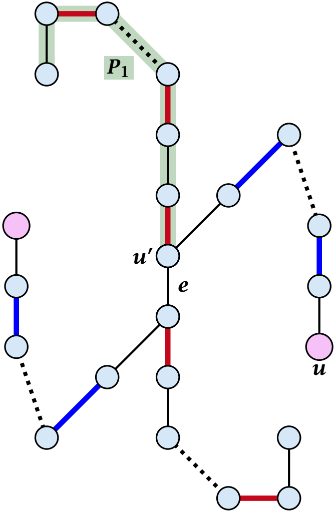

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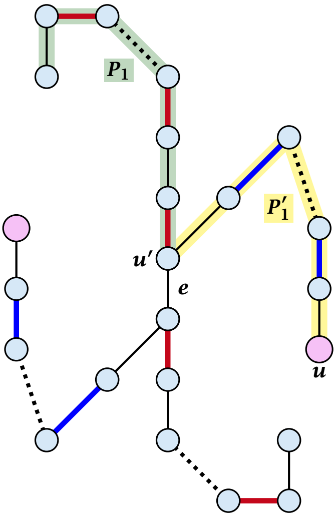

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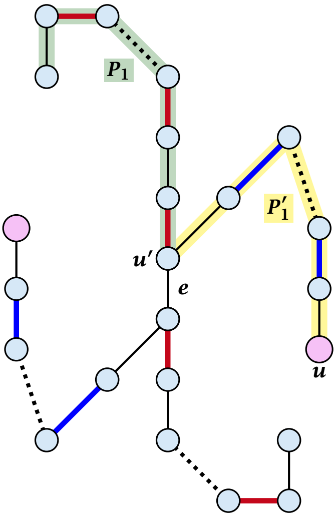
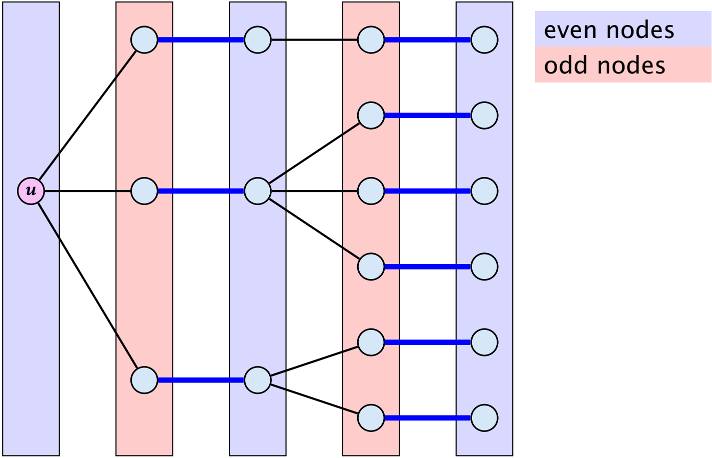
- $u'$ splits $P$ into two parts one of which does not contain $e$. Call this part $P_1$. Denote the sub-path of $P'$ from $u$ to $u'$ with $P_1'$.
- $P_1 \circ P_1'$ is augmenting path in $M$ ($\lightning$).



# How to find an augmenting path?

**Construct an alternating tree.**



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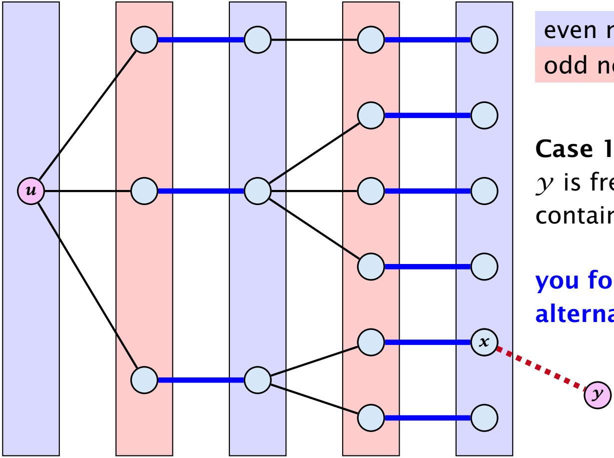


even nodes

odd nodes

**Case 1:**
$y$ is free vertex not contained in $T$

**you found alternating path**

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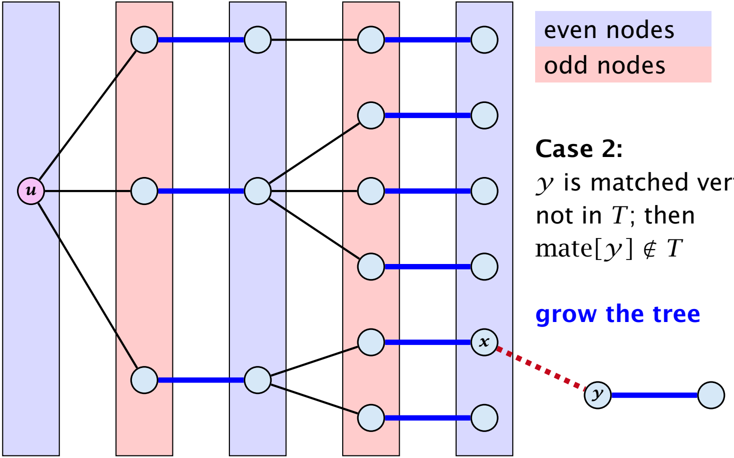


**Case 2:**
$y$ is matched vertex not in $T$; then $\text{mate}[y] \notin T$

**grow the tree**

# How to find an augmenting path?

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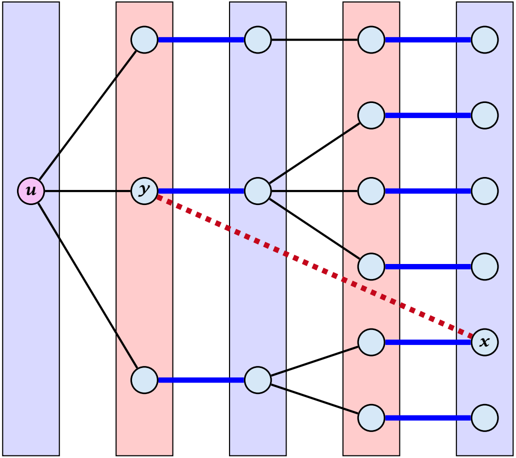


even nodes
odd nodes

**Case 3:**
$y$ is already contained in $T$ as an odd vertex

**ignore successor $y$**

# How to find an augmenting path?

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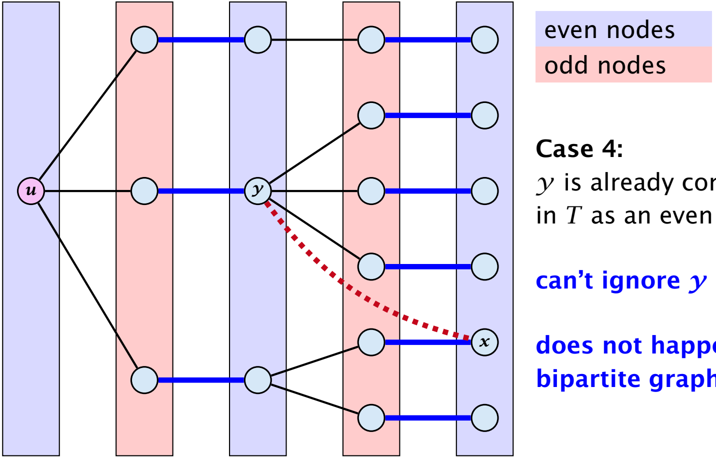


**Case 4:**
$y$ is already contained in $T$ as an even vertex

**can't ignore $y$**

**does not happen in bipartite graphs**

**Algorithm 49** BiMatch($G$, $match$)

```
for x ∈ V do mate[x] ← 0;
r ← 0; free ← n;
while free ≥ 1 and r < n do
    r ← r + 1
    if mate[r] = 0 then
        for i = 1 to n do parent[i′] ← 0
        Q ← ∅; Q.append(r); aug ← false;
        while aug = false and Q ≠ ∅ do
            x ← Q.dequeue();
            for y ∈ A_x do
                if mate[y] = 0 then
                    augm(mate, parent, y);
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graph $G = (S \cup S', E)$

$S = \{1, \ldots, n\}$

$S' = \{1', \ldots, n'\}$

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start with an empty matching

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*free*: number of unmatched nodes in $S$

$r$: root of current tree

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as long as there are unmatched nodes and we did not yet try to grow from all nodes we continue

**Algorithm 49** BiMatch$(G, match)$

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$r$ is the new node that we grow from.

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```

If $r$ is free start tree construction

**Algorithm 49** BiMatch$(G, match)$

**for** $x \in V$ **do** $mate[x] \leftarrow 0$;<br>
$r \leftarrow 0$; $free \leftarrow n$;<br>
**while** $free \geq 1$ **and** $r < n$ **do**<br>
    $r \leftarrow r + 1$<br>
    **if** $mate[r] = 0$ **then**<br>
        **for** $i = 1$ **to** $n$ **do** $parent[i'] \leftarrow 0$<br>
        $Q \leftarrow \emptyset$; $Q.\mathrm{append}(r)$; $aug \leftarrow \mathrm{false}$;<br>
        **while** $aug = \mathrm{false}$ **and** $Q \neq \emptyset$ **do**<br>
            $x \leftarrow Q.\mathrm{dequeue}()$;<br>
            **for** $y \in A_x$ **do**<br>
                **if** $mate[y] = 0$ **then**<br>
                    $\mathrm{augm}(mate, parent, y)$;<br>
                    $aug \leftarrow \mathrm{true}$;<br>
                    $free \leftarrow free - 1$;<br>
                **else**<br>
                    **if** $parent[y] = 0$ **then**<br>
                        $parent[y] \leftarrow x$;<br>
                        $Q.\mathrm{enqueue}(mate[y])$;

Initialize an empty tree. Note that only nodes $i'$ have parent pointers.

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$Q$ is a queue (BFS!!!).

*aug* is a Boolean that stores whether we already found an augmenting path.

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as long as we did not augment and there are still unexamined leaves continue...

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take next unexamined leaf

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if $x$ has unmatched neighbour we found an augmenting path (note that $y \neq r$ because we are in a bipartite graph)

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                        Q.enqueue(mate[y]);
```

do an augmentation...

**Algorithm 49** BiMatch($G$, $match$)

```
for x ∈ V do mate[x] ← 0;
r ← 0; free ← n;
while free ≥ 1 and r < n do
    r ← r + 1
    if mate[r] = 0 then
        for i = 1 to n do parent[i′] ← 0
        Q ← ∅; Q.append(r); aug ← false;
        while aug = false and Q ≠ ∅ do
            x ← Q.dequeue();
            for y ∈ A_x do
                if mate[y] = 0 then
                    augm(mate, parent, y);
                    aug ← true;
                    free ← free − 1;
                else
                    if parent[y] = 0 then
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```

setting $aug$ = true ensures that the tree construction will not continue

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reduce number of free nodes

**Algorithm 49** BiMatch($G$, *match*)

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if $y$ is not in the tree yet

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...put it into the tree

**Algorithm 49** BiMatch($G$, $match$)

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for x ∈ V do mate[x] ← 0;
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```

add its buddy to the set of unexamined leaves

# 18 Weighted Bipartite Matching

**Weighted Bipartite Matching/Assignment**

- Input: undirected, bipartite graph $G = L \cup R, E$.
- an edge $e = (\ell, r)$ has weight $w_e \geq 0$
- find a matching of maximum weight, where the weight of a matching is the sum of the weights of its edges

**Simplifying Assumptions (wlog [why?]):**

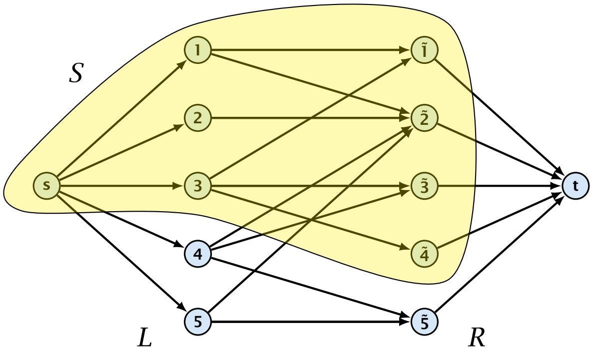
- assume that $|L| = |R| = n$
- assume that there is an edge between every pair of nodes $(\ell, r) \in V \times V$
- can assume goal is to construct maximum weight perfect matching

# Weighted Bipartite Matching

**Theorem 8 (Halls Theorem)**

*A bipartite graph $G = (L \cup R, E)$ has a perfect matching if and only if for all sets $S \subseteq L$, $|\Gamma(S)| \geq |S|$, where $\Gamma(S)$ denotes the set of nodes in $R$ that have a neighbour in $S$.*

# 18 Weighted Bipartite Matching



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  - Let $S$ denote a minimum cut and let $L_S \overset{\text{def}}{=} L \cap S$ and $R_S \overset{\text{def}}{=} R \cap S$ denote the portion of $S$ inside $L$ and $R$, respectively.

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  - The size of the cut is $|L| - |L_S| + |R_S|$.
  - Using the fact that $|\Gamma(L_S)| \geq L_S$ gives that this is at least $|L|$.

# Algorithm Outline

**Idea:**

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- Try to compute a perfect matching in the subgraph $H(\vec{x})$. If you are successful you found an optimal matching.

# Algorithm Outline

**Reason:**

- The weight of your matching $M^*$ is

$$\sum_{(u,v)\in M^*} w_{(u,v)} = \sum_{(u,v)\in M^*} (x_u + x_v) = \sum_v x_v \enspace .$$

- Any other perfect matching $M$ (in $G$, not necessarily in $H(\vec{x})$) has

$$\sum_{(u,v)\in M} w_{(u,v)} \le \sum_{(u,v)\in M} (x_u + x_v) = \sum_v x_v \enspace .$$

# Algorithm Outline

**What if you don't find a perfect matching?**

Then, Halls theorem guarantees you that there is a set $S \subseteq L$, with $|\Gamma(S)| < |S|$, where $\Gamma$ denotes the neighbourhood w.r.t. the subgraph $H(\vec{x})$.

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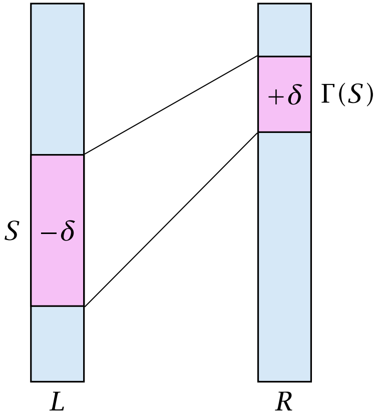
**Idea:** reweight such that:

- the total weight assigned to nodes decreases
- the weight function still dominates the edge-weights

If we can do this we have an algorithm that terminates with an optimal solution (we analyze the running time later).

# Changing Node Weights

Increase node-weights in $\Gamma(S)$ by $+\delta$, and decrease the node-weights in $S$ by $-\delta$.



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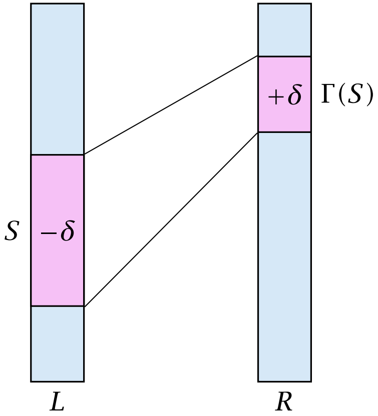

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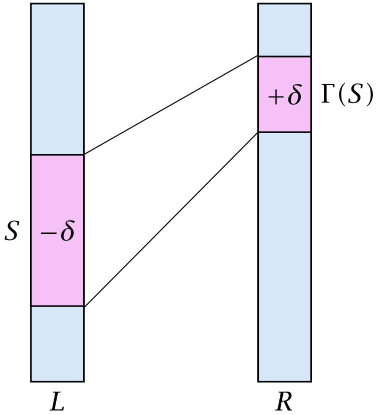

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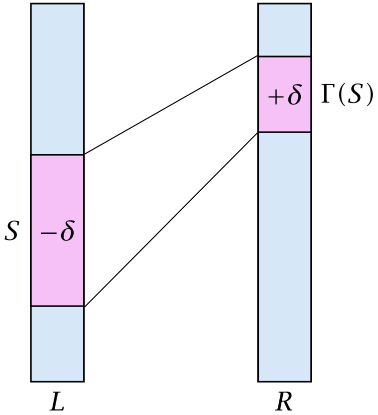
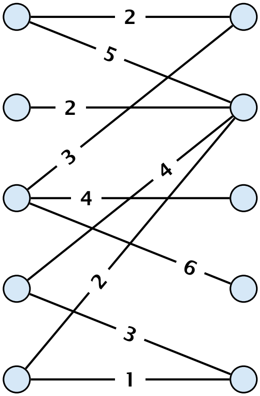
- Since, none of these edges is tight (otw. the edge would be contained in $H(\vec{x})$, and hence would go between $S$ and $\Gamma(S)$) we can do this decrement for small enough $\delta > 0$ until a new edge gets tight.



# Weighted Bipartite Matching

Edges not drawn have weight $0$.



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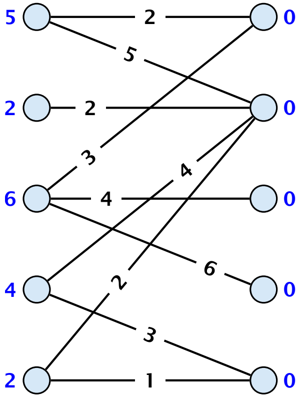

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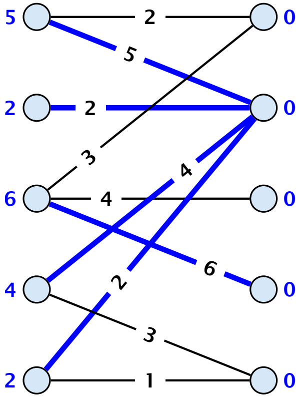

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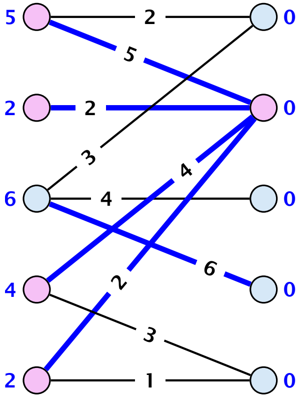

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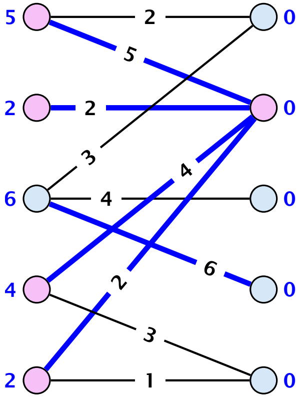

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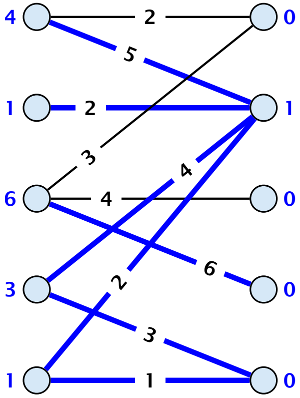

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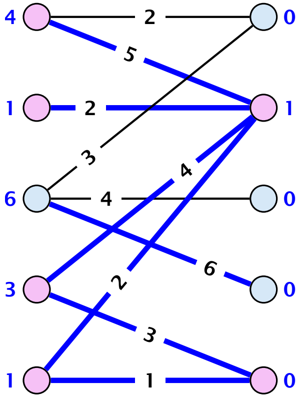

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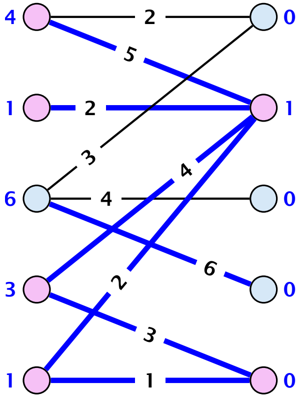

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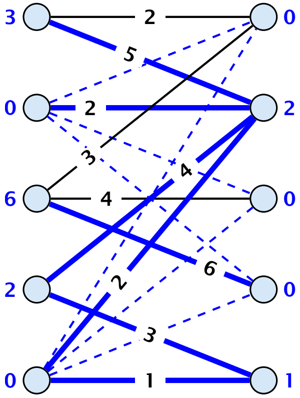

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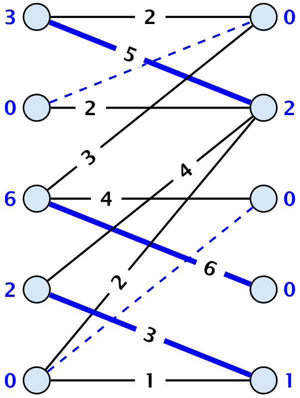

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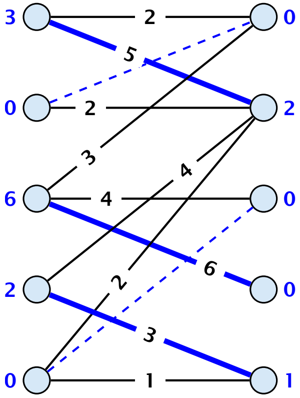

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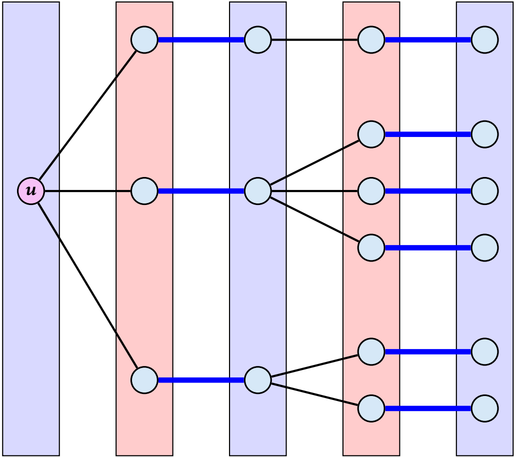
- This matching is still contained in the new graph, because all its edges either go between $\Gamma(S)$ and $S$ or between $L - S$ and $R - \Gamma(S)$.
- Hence, reweighting does not decrease the size of a maximum matching in the tight sub-graph.

# Analysis

- We will show that after at most $n$ reweighting steps the size of the maximum matching can be increased by finding an augmenting path.
- This gives a polynomial running time.

# How to find an augmenting path?

**Construct an alternating tree.**



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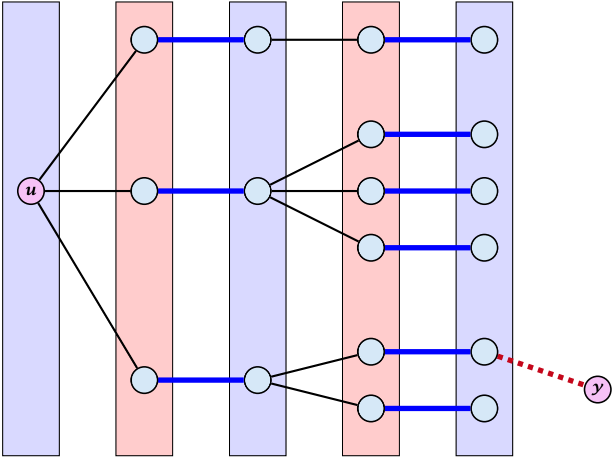

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- The set of even vertices is on the left and the set of odd vertices is on the right and contains all neighbours of even nodes.
- All odd vertices are matched to even vertices. Furthermore, the even vertices additionally contain the free vertex $u$. Hence, $|V_{\text{odd}}| = |\Gamma(V_{\text{even}})| < |V_{\text{even}}|$, and all odd vertices are saturated in the current matching.

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- A more careful implementation of the algorithm obtains a running time of $\mathcal{O}(n^3)$.

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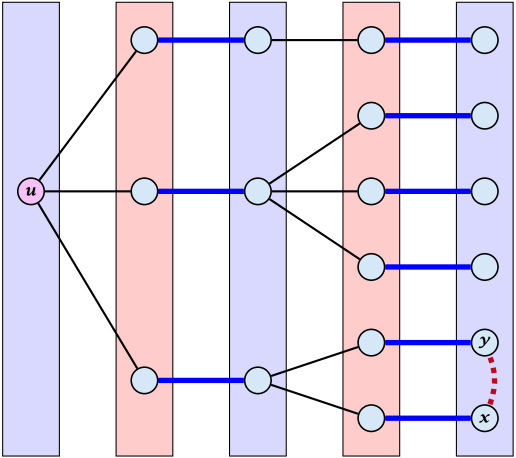


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$y$ is already contained in $T$ as an even vertex

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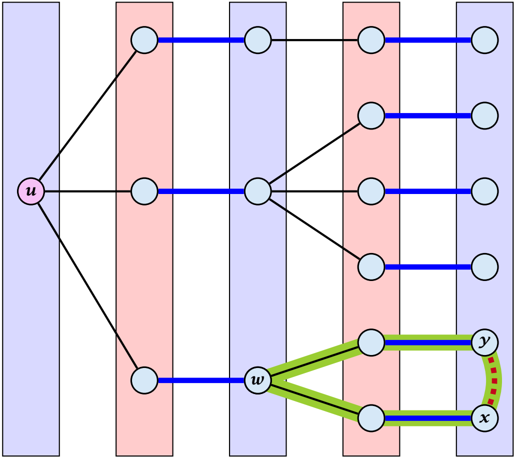


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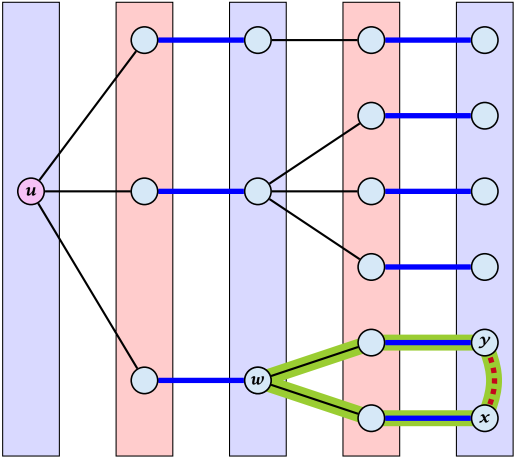


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The cycle $w \leftrightarrow y - x \leftrightarrow w$ is called a blossom. $w$ is called the base of the blossom (even node!!!). The path $u$-$w$ is called the stem of the blossom.

# Flowers and Blossoms

**Definition 9**
A flower in a graph $G = (V, E)$ w.r.t. a matching $M$ and a (free) root node $r$, is a subgraph with two components:

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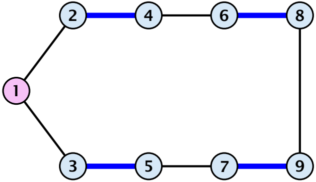
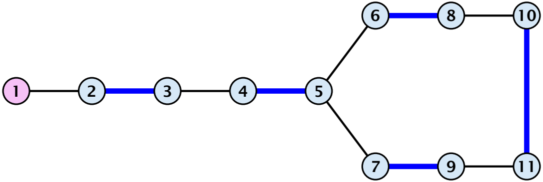
- A stem is an even length alternating path that starts at the root node $r$ and terminates at some node $w$. We permit the possibility that $r = w$ (empty stem).
- A blossom is an odd length alternating cycle that starts and terminates at the terminal node $w$ of a stem and has no other node in common with the stem. $w$ is called the base of the blossom.

# Flowers and Blossoms



# Flowers and Blossoms

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2. A blossom spans $2k + 1$ nodes and contains $k$ matched edges for some integer $k \geq 1$. The matched edges match all nodes of the blossom except the base.
3. The base of a blossom is an even node (if the stem is part of an alternating tree starting at $r$).

# Flowers and Blossoms

**Properties:**

4. Every node $x$ in the blossom (except its base) is reachable from the root (or from the base of the blossom) through two distinct alternating paths; one with even and one with odd length.

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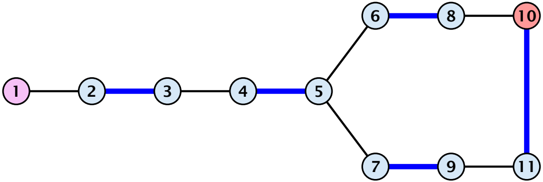
4. Every node $x$ in the blossom (except its base) is reachable from the root (or from the base of the blossom) through two distinct alternating paths; one with even and one with odd length.
5. The even alternating path to $x$ terminates with a matched edge and the odd path with an unmatched edge.

# Flowers and Blossoms



# Shrinking Blossoms

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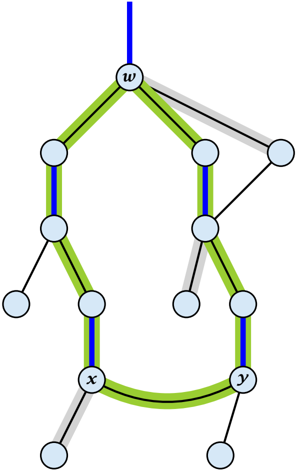
- Delete all vertices in $B$ (and its incident edges) from $G$.
- Add a new (pseudo-)vertex $b$. The new vertex $b$ is connected to all vertices in $V \setminus B$ that had at least one edge to a vertex from $B$.

# Shrinking Blossoms

- Edges of $T$ that connect a node $u$ not in $B$ to a node in $B$ become tree edges in $T'$ connecting $u$ to $b$.
- Matching edges (there is at most one) that connect a node $u$ not in $B$ to a node in $B$ become matching edges in $M'$.
- Nodes that are connected in $G$ to at least one node in $B$ become connected to $b$ in $G'$.



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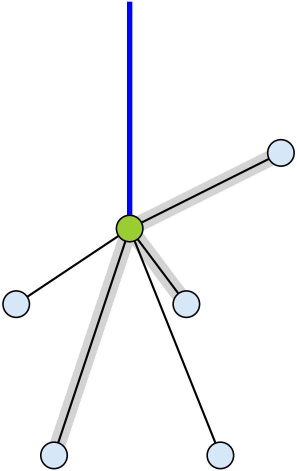
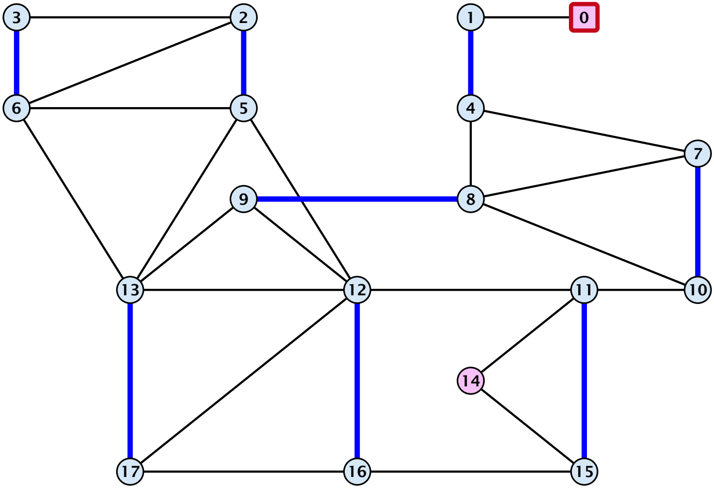


# Example: Blossom Algorithm



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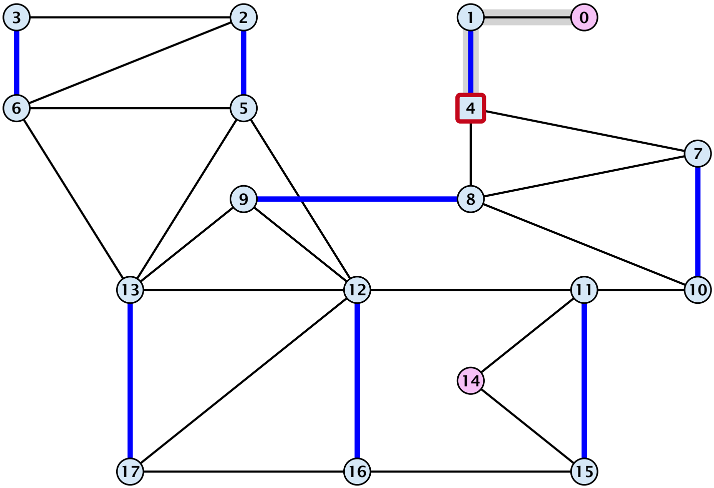

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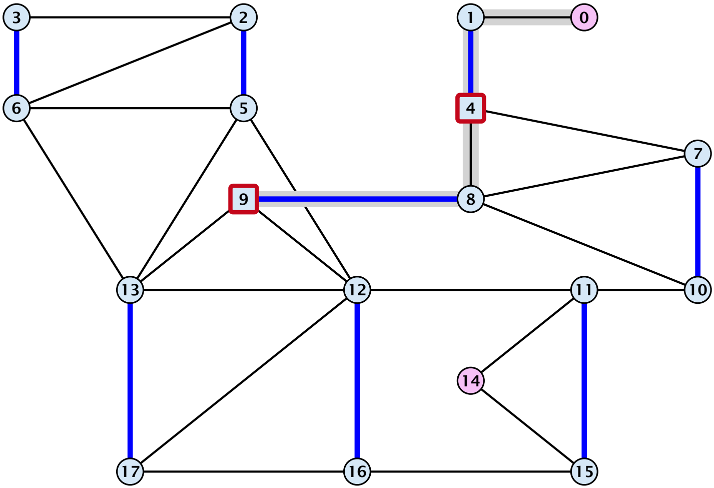

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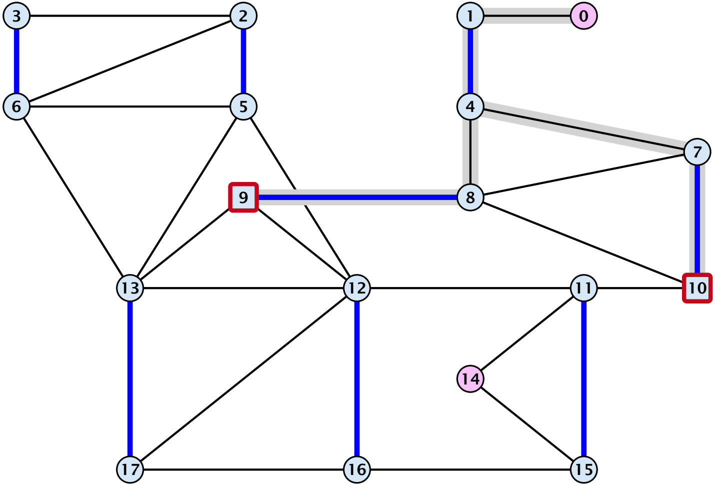

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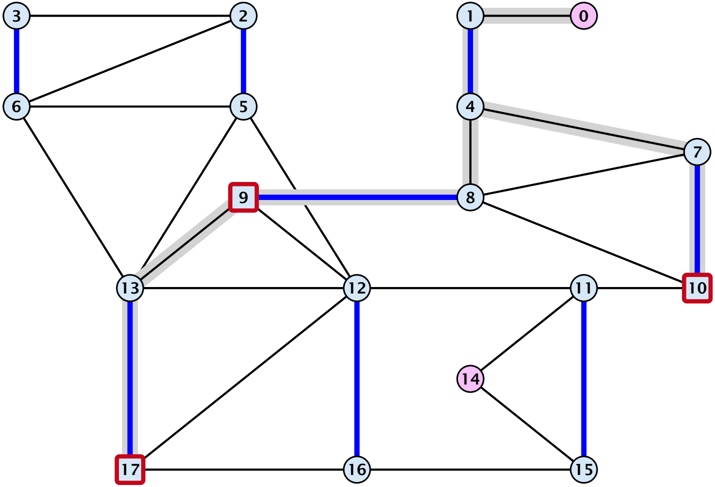

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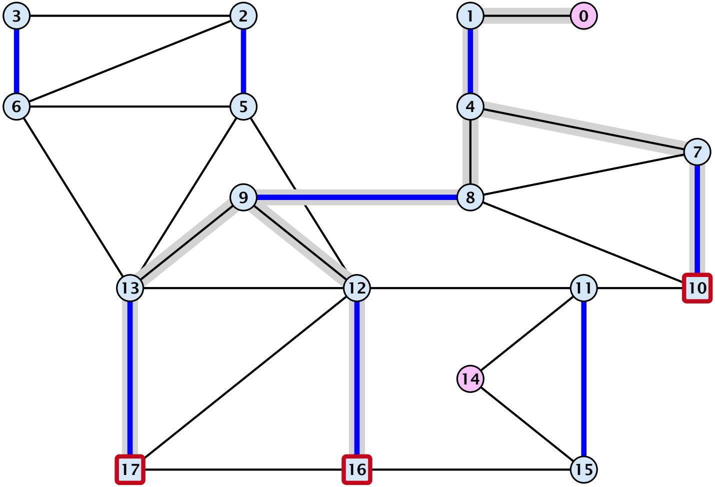

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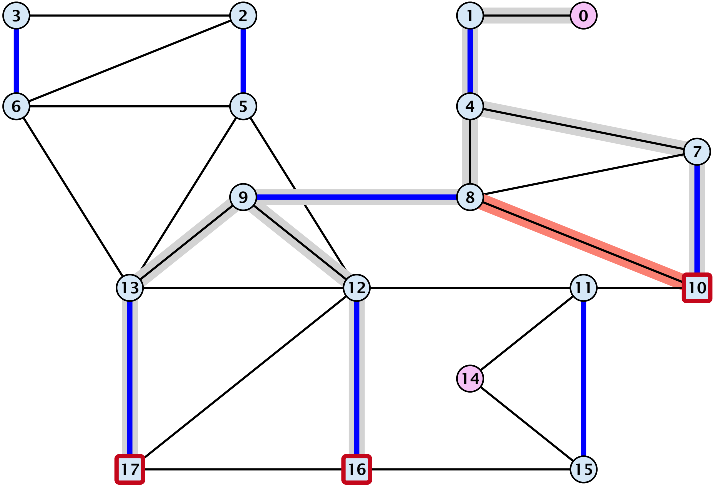

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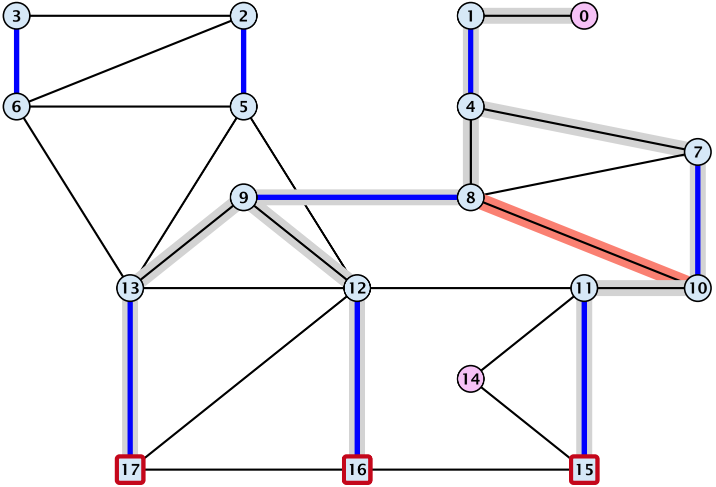

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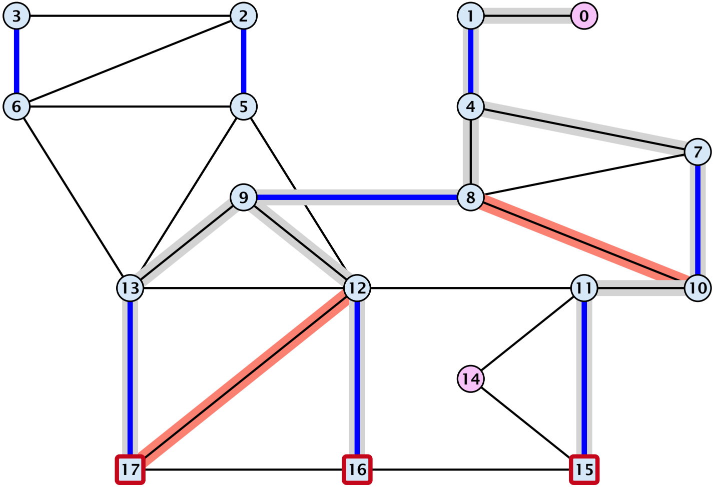

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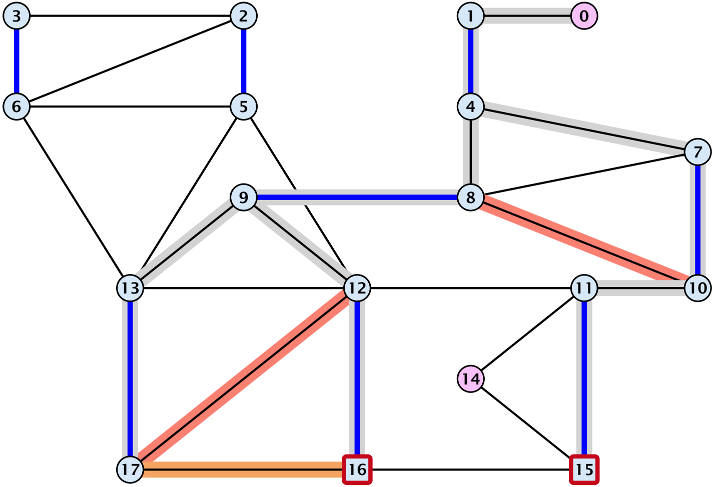

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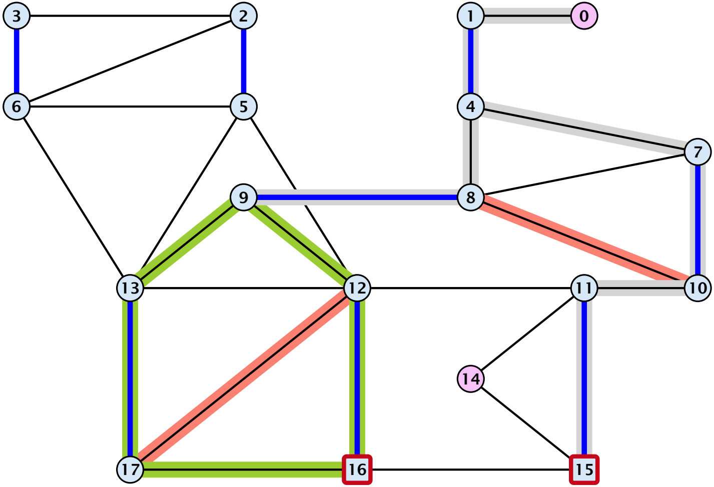

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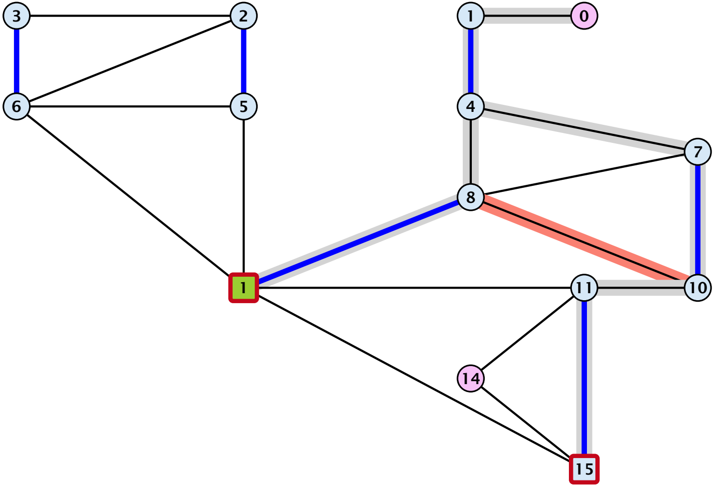

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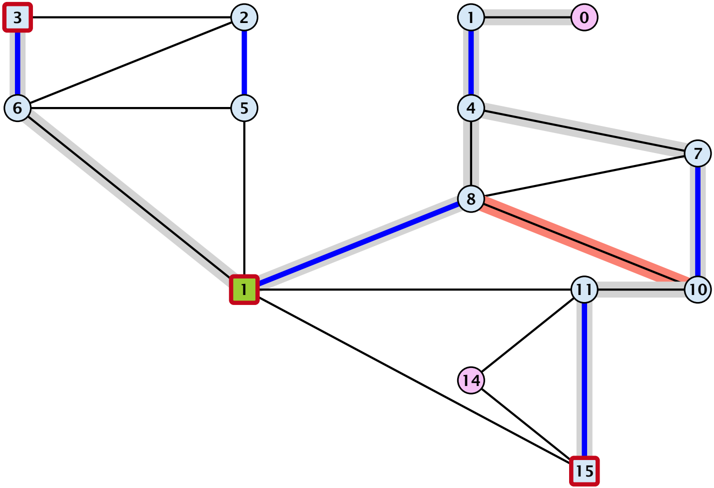

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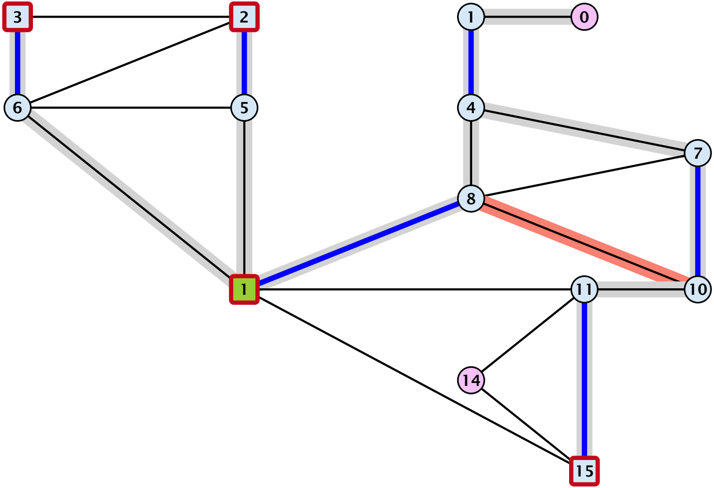

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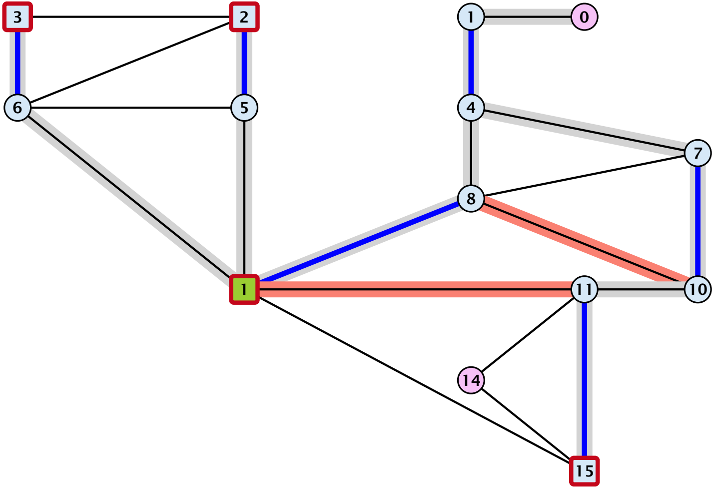

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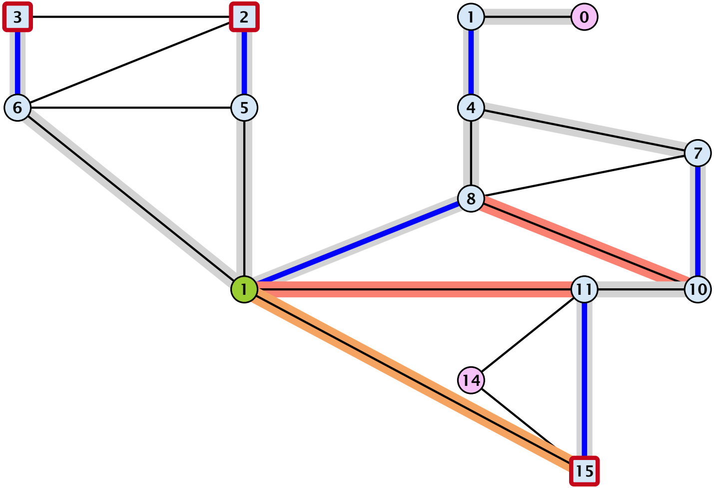

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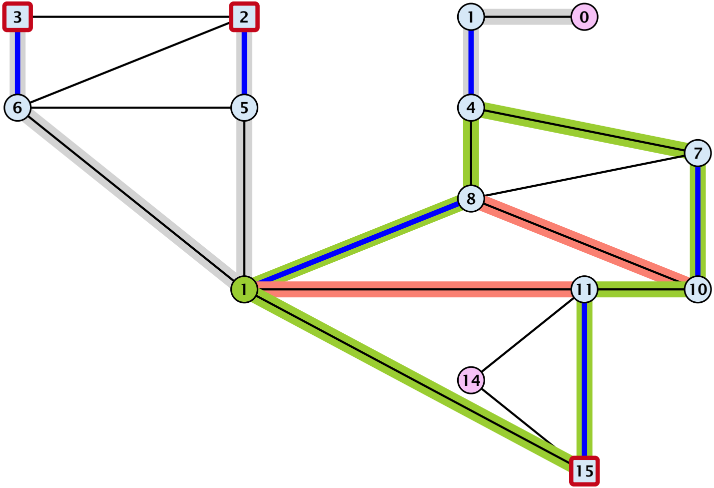

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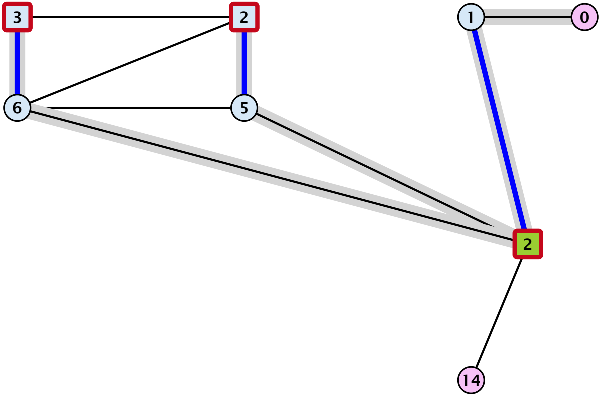

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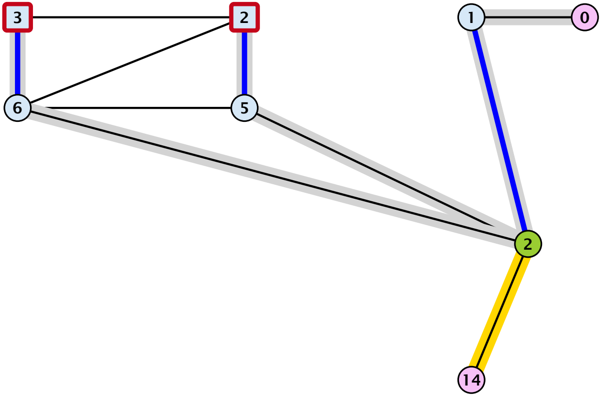

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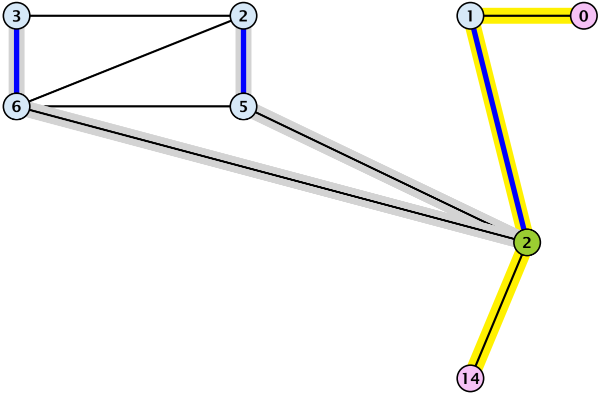

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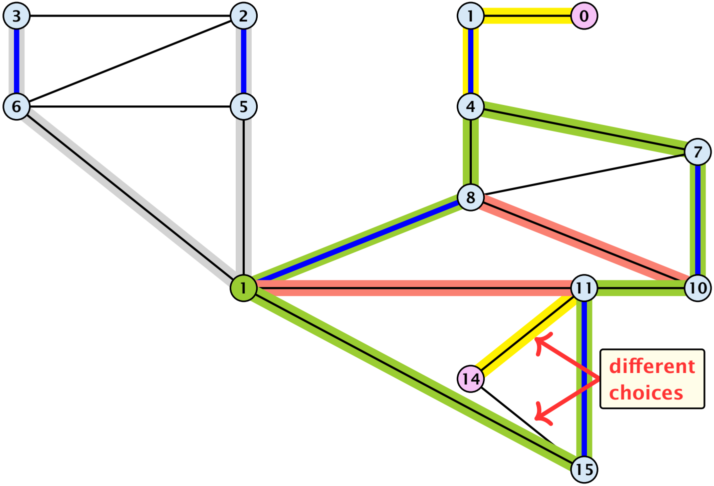

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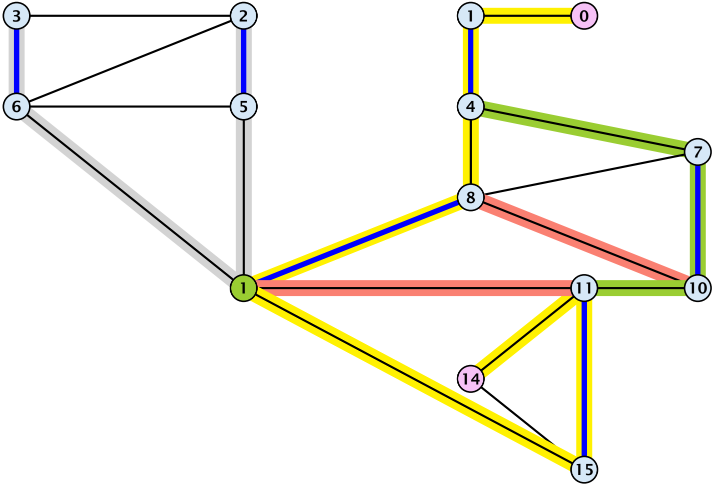

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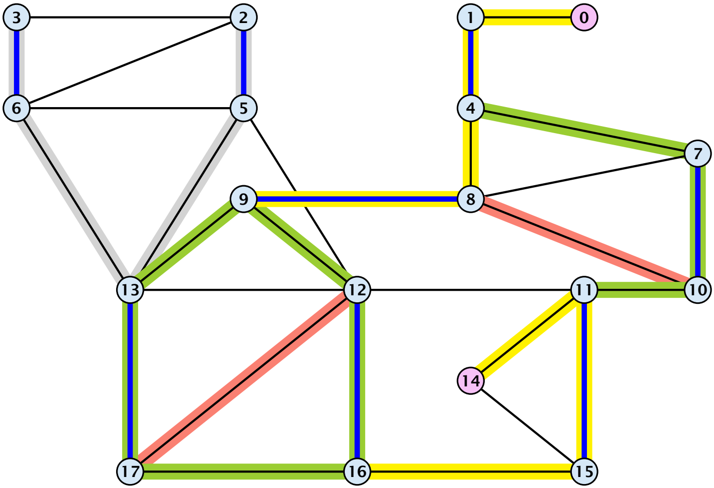

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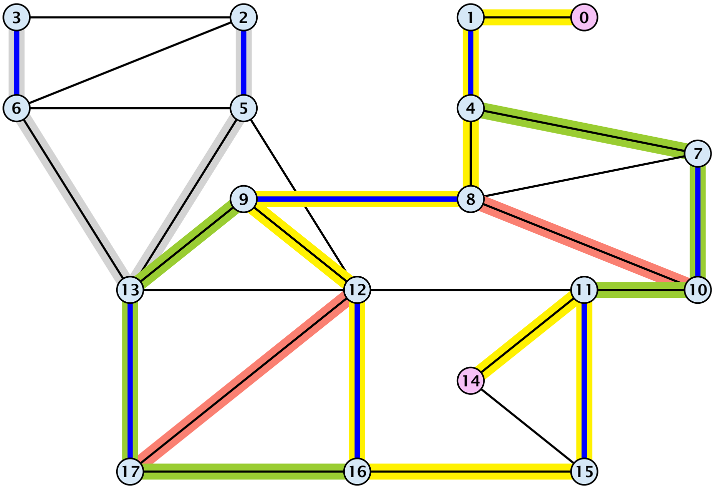

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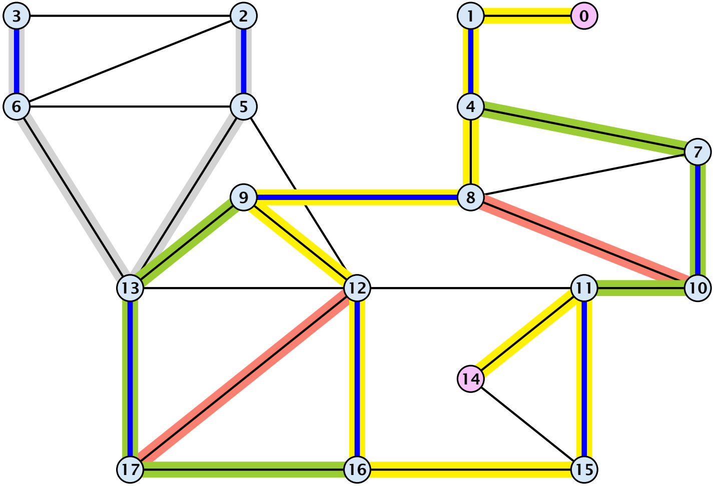

# Correctness

Assume that in $G$ we have a flower w.r.t. matching $M$. Let $r$ be the root, $B$ the blossom, and $w$ the base. Let graph $G' = G/B$ with pseudonode $b$. Let $M'$ be the matching in the contracted graph.

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**Lemma 10**

*If $G'$ contains an augmenting path $P'$ starting at $r$ (or the pseudo-node containing $r$) w.r.t. the matching $M'$ then $G$ contains an augmenting path starting at $r$ w.r.t. matching $M$.*

# Correctness

**Proof.**

If $P'$ does not contain $b$ it is also an augmenting path in $G$.

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**Case 1: non-empty stem**

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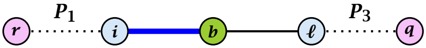

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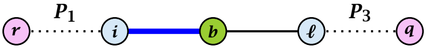


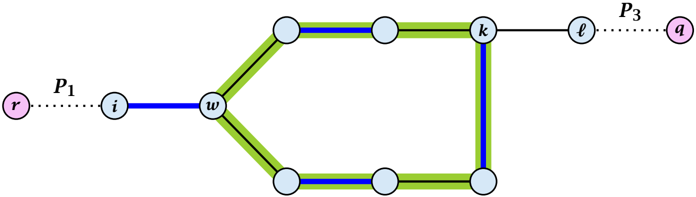

# Correctness

- After the expansion $\ell$ must be incident to some node in the blossom. Let this node be $k$.
- If $k \neq w$ there is an alternating path $P_2$ from $w$ to $k$ that ends in a matching edge.
- $P_1 \circ (i, w) \circ P_2 \circ (k, \ell) \circ P_3$ is an alternating path.
- If $k = w$ then $P_1 \circ (i, w) \circ (w, \ell) \circ P_3$ is an alternating path.

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**Proof.**

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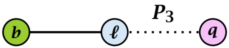

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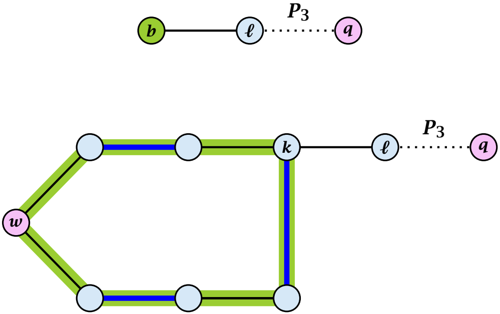

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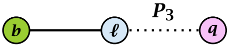


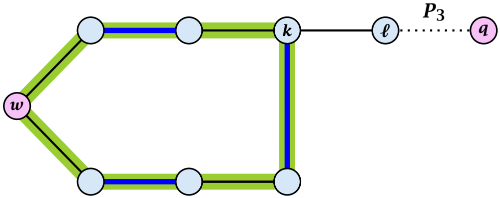


- The path $r \circ P_2 \circ (k, \ell) \circ P_3$ is an alternating path.

# Correctness

**Lemma 11**
*If $G$ contains an augmenting path $P$ from $r$ to $q$ w.r.t. matching $M$ then $G'$ contains an augmenting path from $r$ (or the pseudo-node containing $r$) to $q$ w.r.t. $M'$.*

# Correctness

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Let $i$ be the last node on the path $P$ that is part of the blossom.

$P$ is of the form $P_1 \circ (i,j) \circ P_2$, for some node $j$ and $(i,j)$ is unmatched.

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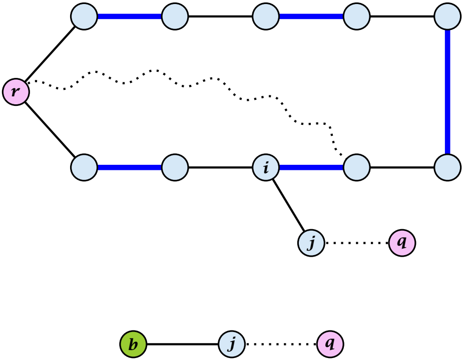
$(b, j) \circ P_2$ is an augmenting path in the contracted network.

# Correctness

## Illustration for Case 1:



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This path must go between $w$ and $q$ as these are the only unmatched vertices w.r.t. $M_+$.

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**Algorithm 50** search($r$, *found*)

set $\bar{A}(i) \leftarrow A(i)$ for all nodes $i$
*found* $\leftarrow$ false
unlabel all nodes;
give an even label to $r$ and initialize *list* $\leftarrow \{r\}$
**while** *list* $\neq \emptyset$ **do**
    delete a node $i$ from *list*
    examine($i$, *found*)
    **if** *found* $=$ true **then return**

Search for an augmenting path starting at $r$.

**Algorithm 50** search($r$, *found*)

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    if found = true then return
```

$A(i)$ contains neighbours of node $i$.

We create a copy $\bar{A}(i)$ so that we later can shrink blossoms.

**Algorithm 50** search($r$, *found*)

set $\bar{A}(i) \leftarrow A(i)$ for all nodes $i$
*found* $\leftarrow$ false
unlabel all nodes;
give an even label to $r$ and initialize *list* $\leftarrow \{r\}$
**while** *list* $\neq \emptyset$ **do**
    delete a node $i$ from *list*
    examine($i$, *found*)
    **if** *found* $=$ true **then return**

*found* is just a Boolean that allows to abort the search process...

**Algorithm 50** search($r$, *found*)

1: set $\bar{A}(i) \leftarrow A(i)$ for all nodes $i$
2: $\mathit{found} \leftarrow$ false
3: unlabel all nodes;
4: give an even label to $r$ and initialize $\mathit{list} \leftarrow \{r\}$
5: **while** $\mathit{list} \neq \emptyset$ **do**
6:     delete a node $i$ from *list*
7:     examine($i$, *found*)
8:     **if** $\mathit{found} =$ true **then return**

In the beginning no node is in the tree.

**Algorithm 50** search($r$, *found*)

set $\bar{A}(i) \leftarrow A(i)$ for all nodes $i$
*found* $\leftarrow$ false
unlabel all nodes;
give an even label to $r$ and initialize *list* $\leftarrow \{r\}$
**while** *list* $\neq \emptyset$ **do**
    delete a node $i$ from *list*
    examine($i$, *found*)
    **if** *found* $=$ true **then return**

Put the root in the tree.

*list* could also be a set or a stack.

**Algorithm 50** search($r$, *found*)

set $\bar{A}(i) \leftarrow A(i)$ for all nodes $i$
*found* $\leftarrow$ false
unlabel all nodes;
give an even label to $r$ and initialize *list* $\leftarrow \{r\}$
**while** *list* $\neq \emptyset$ **do**
  delete a node $i$ from *list*
  examine($i$, *found*)
  **if** *found* $=$ true **then return**

As long as there are nodes with unexamined neighbours...

**Algorithm 50** search($r$, *found*)

```
set Ā(i) ← A(i) for all nodes i
found ← false
unlabel all nodes;
give an even label to r and initialize list ← {r}
while list ≠ ∅ do
    delete a node i from list
    examine(i, found)
    if found = true then return
```

...examine the next one

**Algorithm 50** search($r$, *found*)

```
set Ā(i) ← A(i) for all nodes i
found ← false
unlabel all nodes;
give an even label to r and initialize list ← {r}
while list ≠ ∅ do
    delete a node i from list
    examine(i, found)
    if found = true then return
```

If you found augmenting path
abort and start from next root.

**Algorithm 51** examine($i$, $found$)

**for all** $j \in \bar{A}(i)$ **do**
    **if** $j$ is even **then** contract($i, j$) and **return**
    **if** $j$ is unmatched **then**
        $q \leftarrow j$;
        $\text{pred}(q) \leftarrow i$;
        $found \leftarrow \text{true}$;
        **return**
    **if** $j$ is matched and unlabeled **then**
        $\text{pred}(j) \leftarrow i$;
        $\text{pred}(\text{mate}(j)) \leftarrow j$;
        add $\text{mate}(j)$ to $list$

Examine the neighbours of a node $i$

**Algorithm 51** examine($i$, $found$)

```
for all j ∈ Ā(i) do
    if j is even then contract(i, j) and return
    if j is unmatched then
        q ← j;
        pred(q) ← i;
        found ← true;
        return
    if j is matched and unlabeled then
        pred(j) ← i;
        pred(mate(j)) ← j;
        add mate(j) to list
```

For all neighbours $j$ do...

**Algorithm 51** examine$(i, \mathit{found})$

```
for all j ∈ Ā(i) do
    if j is even then contract(i, j) and return
    if j is unmatched then
        q ← j;
        pred(q) ← i;
        found ← true;
        return
    if j is matched and unlabeled then
        pred(j) ← i;
        pred(mate(j)) ← j;
        add mate(j) to list
```

You have found a blossom...

**Algorithm 51** examine($i$, $found$)

**for all** $j \in \bar{A}(i)$ **do**
    **if** $j$ is even **then** contract($i, j$) and **return**
    **if** $j$ is unmatched **then**
        $q \leftarrow j$;
        $\text{pred}(q) \leftarrow i$;
        $found \leftarrow \text{true}$;
        **return**
    **if** $j$ is matched and unlabeled **then**
        $\text{pred}(j) \leftarrow i$;
        $\text{pred}(\text{mate}(j)) \leftarrow j$;
        add mate($j$) to $list$

You have found a free node which gives you an augmenting path.

**Algorithm 51** examine($i$, *found*)

```
for all j ∈ Ā(i) do
    if j is even then contract(i, j) and return
    if j is unmatched then
        q ← j;
        pred(q) ← i;
        found ← true;
        return
    if j is matched and unlabeled then
        pred(j) ← i;
        pred(mate(j)) ← j;
        add mate(j) to list
```

If you find a matched node that is not in the tree you grow...

**Algorithm 51** examine($i$, *found*)

```
for all j ∈ Ā(i) do
    if j is even then contract(i, j) and return
    if j is unmatched then
        q ← j;
        pred(q) ← i;
        found ← true;
        return
    if j is matched and unlabeled then
        pred(j) ← i;
        pred(mate(j)) ← j;
        add mate(j) to list
```

*mate*($j$) is a new node from which you can grow further.

**Algorithm 52** contract$(i, j)$

1: trace pred-indices of $i$ and $j$ to identify a blossom $B$
2: create new node $b$ and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label $b$ even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in $B$
6: delete nodes in $B$ from the graph

Contract blossom identified by nodes $i$ and $j$

**Algorithm 52** contract$(i, j)$

trace pred-indices of $i$ and $j$ to identify a blossom $B$
create new node $b$ and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
label $b$ even and add to *list*
update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
form a circular double linked list of nodes in $B$
delete nodes in $B$ from the graph

Get all nodes of the blossom.

Time: $\mathcal{O}(m)$

**Algorithm 52** contract$(i, j)$

1: trace pred-indices of $i$ and $j$ to identify a blossom $B$
2: create new node $b$ and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label $b$ even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in $B$
6: delete nodes in $B$ from the graph

Identify all neighbours of $b$.

Time: $\mathcal{O}(m)$ (how?)

**Algorithm 52** contract$(i, j)$

1: trace pred-indices of $i$ and $j$ to identify a blossom $B$
2: create new node $b$ and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label $b$ even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in $B$
6: delete nodes in $B$ from the graph

$b$ will be an even node, and it has unexamined neighbours.

**Algorithm 52** contract$(i, j)$

1: trace pred-indices of $i$ and $j$ to identify a blossom $B$
2: create new node $b$ and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label $b$ even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in $B$
6: delete nodes in $B$ from the graph

Every node that was adjacent to a node in $B$ is now adjacent to $b$

**Algorithm 52** contract$(i, j)$

1: trace pred-indices of $i$ and $j$ to identify a blossom $B$
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5: form a circular double linked list of nodes in $B$
6: delete nodes in $B$ from the graph

Only for making a blossom expansion easier.

**Algorithm 52** contract$(i, j)$

1: trace pred-indices of $i$ and $j$ to identify a blossom $B$
2: create new node $b$ and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label $b$ even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in $B$
6: delete nodes in $B$ from the graph

Only delete links from nodes not in $B$ to $B$.

When expanding the blossom again we can recreate these links in time $\mathcal{O}(m)$.

# Analysis

- A contraction operation can be performed in time $\mathcal{O}(m)$. Note, that any graph created will have at most $m$ edges.

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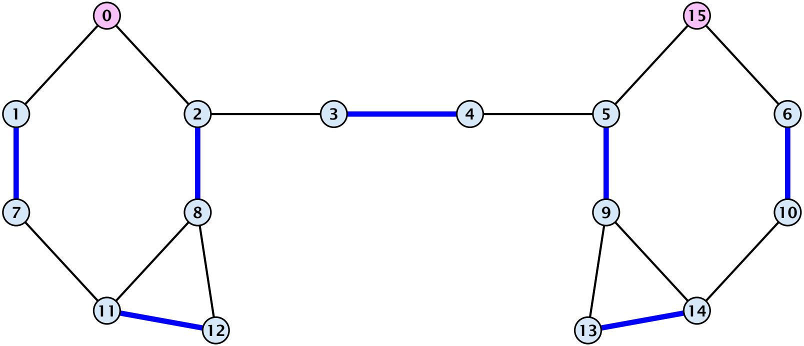
- There are at most $n$ contractions as each contraction reduces the number of vertices.
- The expansion can trivially be done in the same time as needed for all contractions.
- An augmentation requires time $\mathcal{O}(n)$. There are at most $n$ of them.
- In total the running time is at most

$$n \cdot (\mathcal{O}(mn) + \mathcal{O}(n)) = \mathcal{O}(mn^2) \ .$$

# Example: Blossom Algorithm



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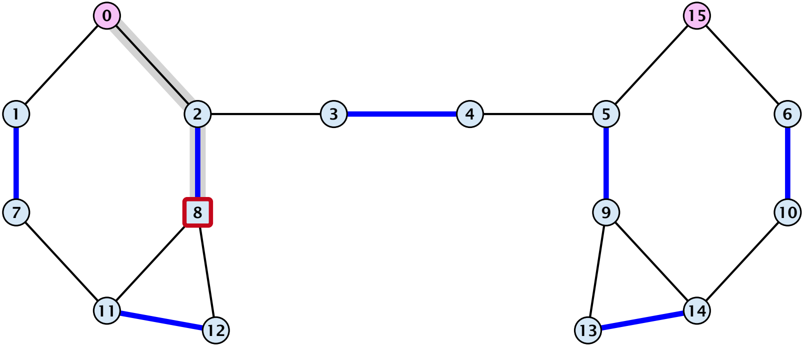

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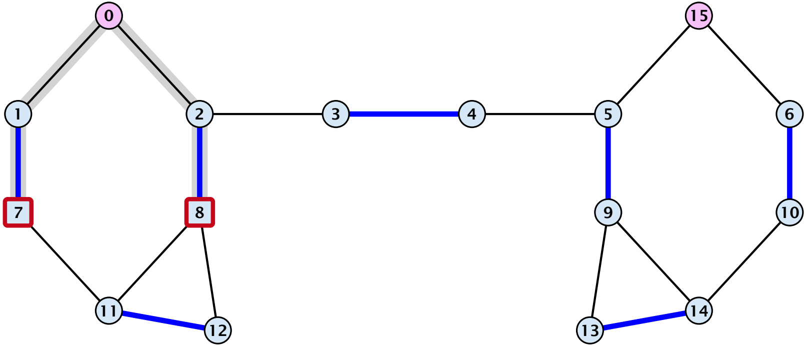

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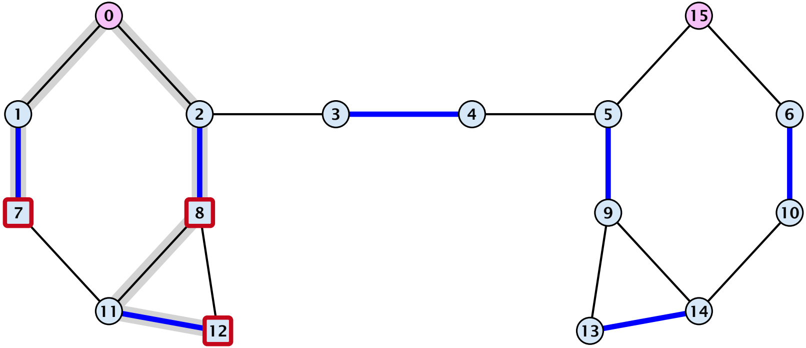

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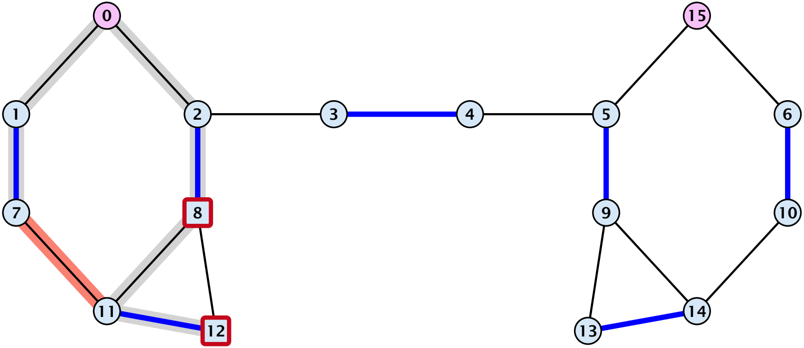

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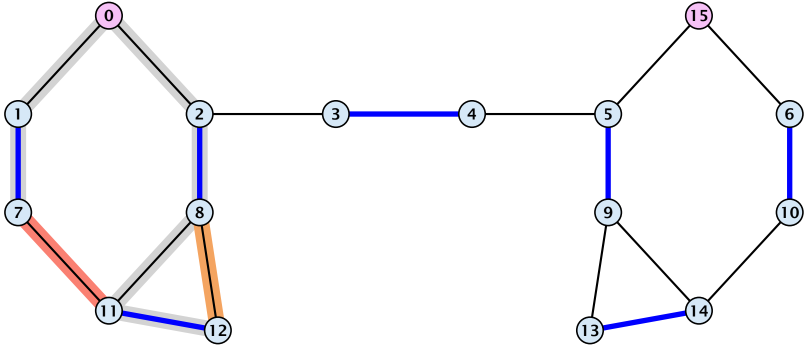

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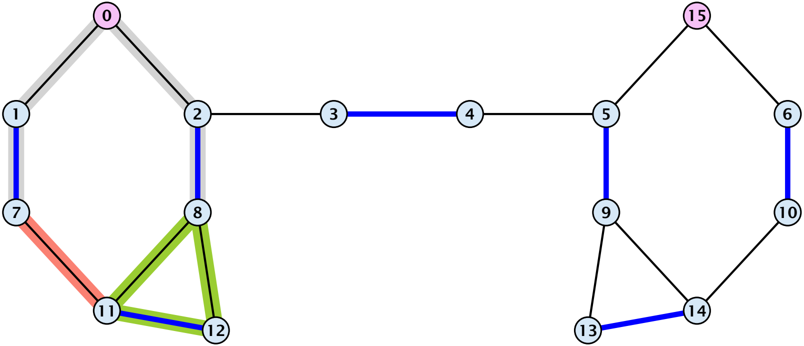

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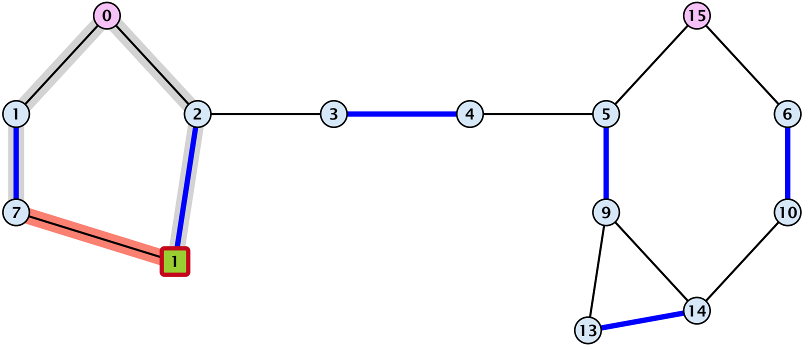

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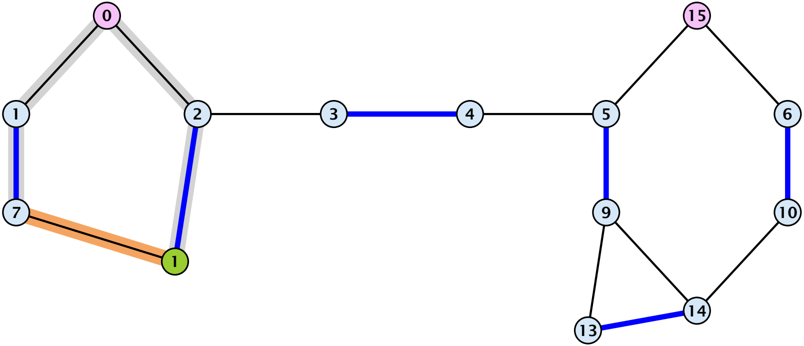

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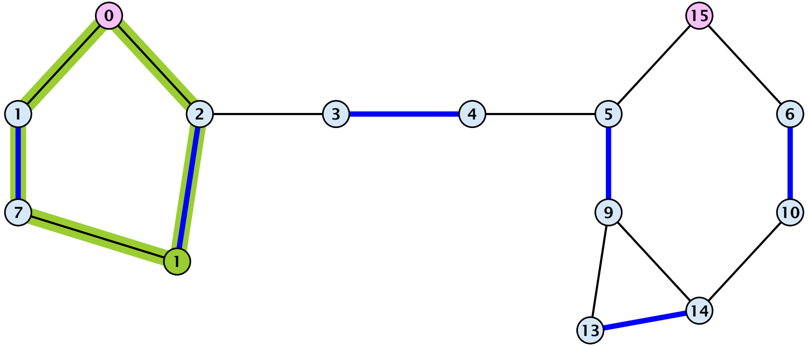

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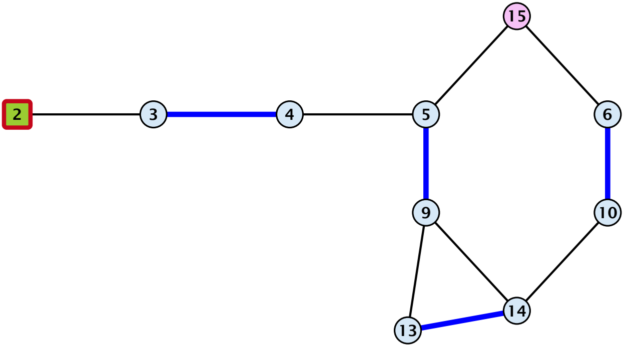

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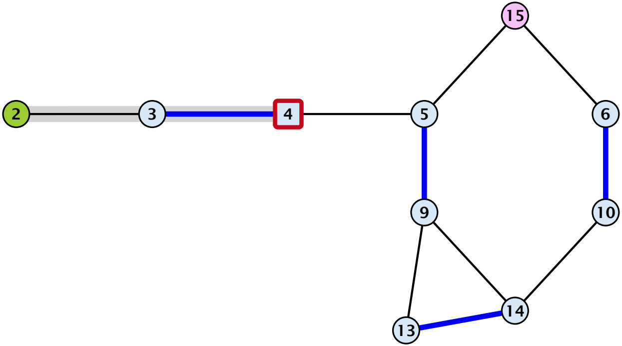

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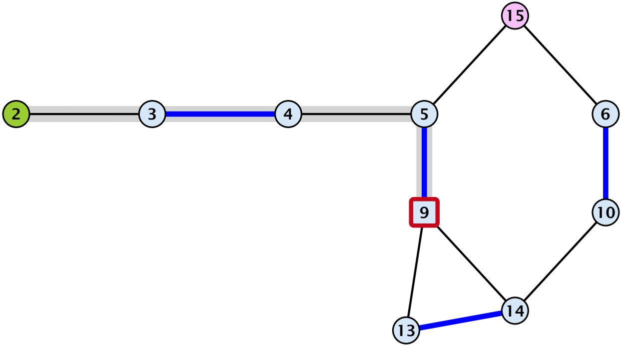

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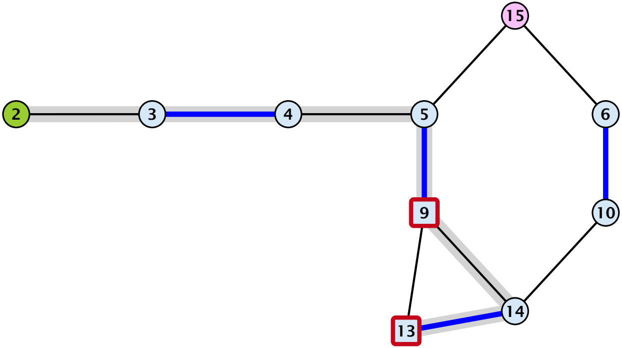

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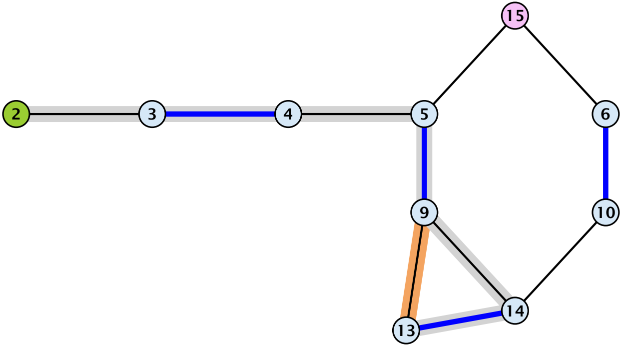

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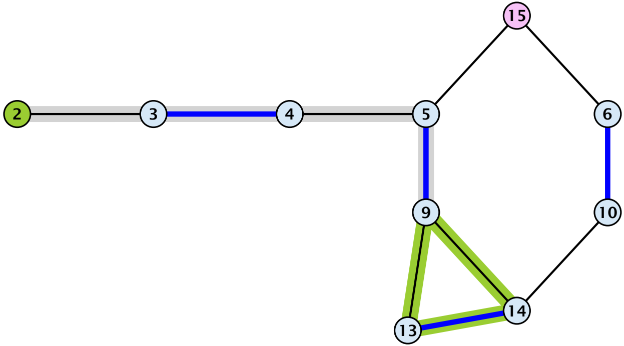

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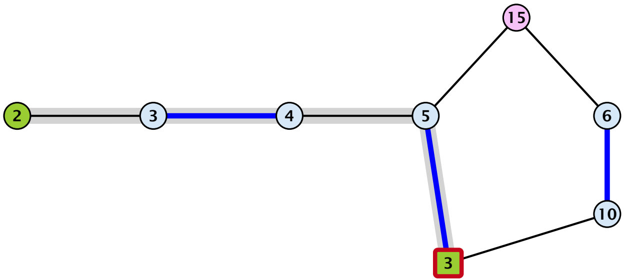

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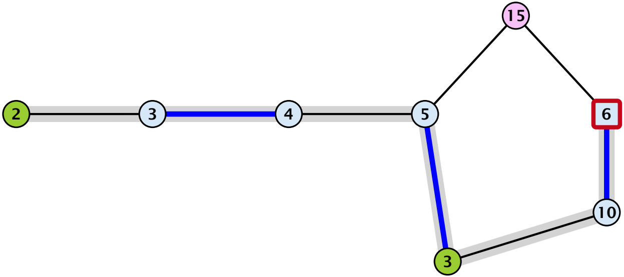

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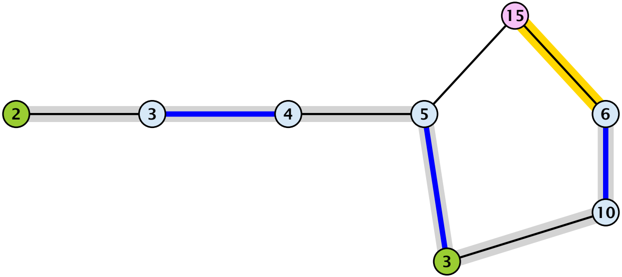

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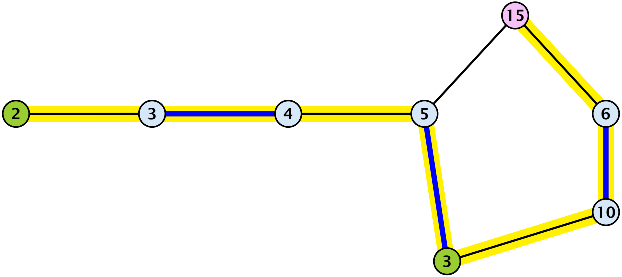

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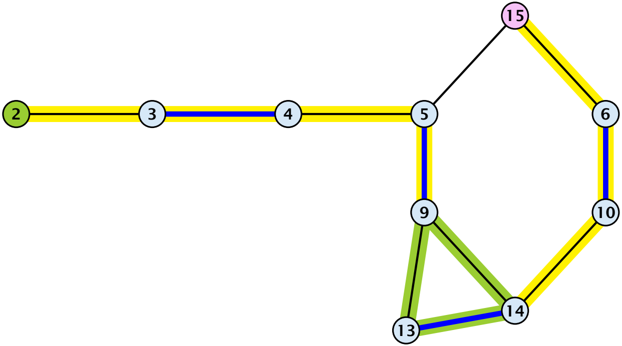

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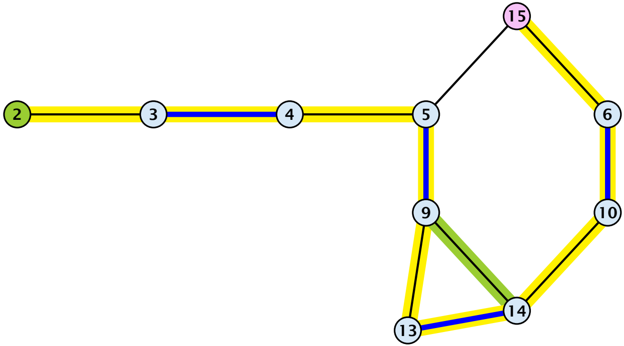

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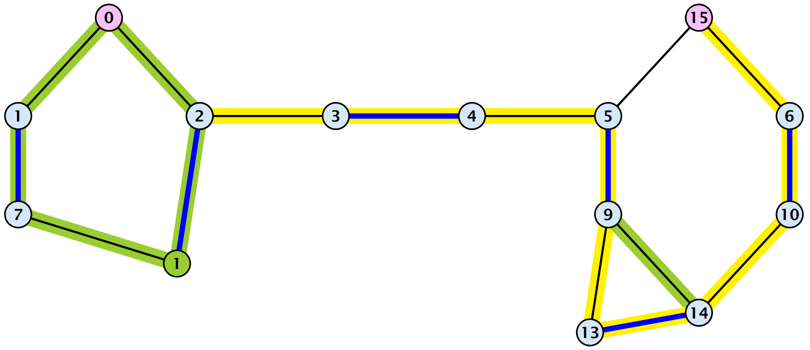

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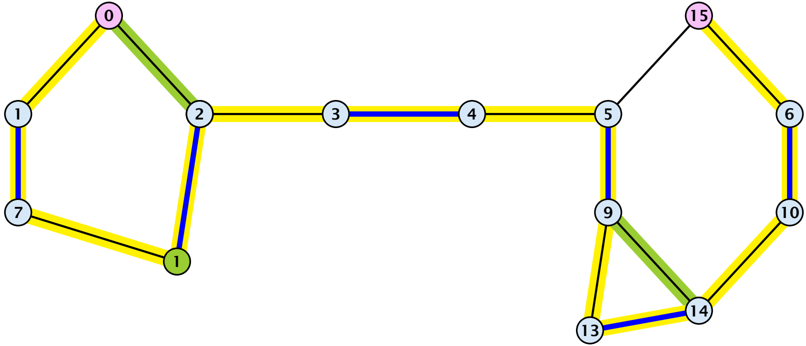

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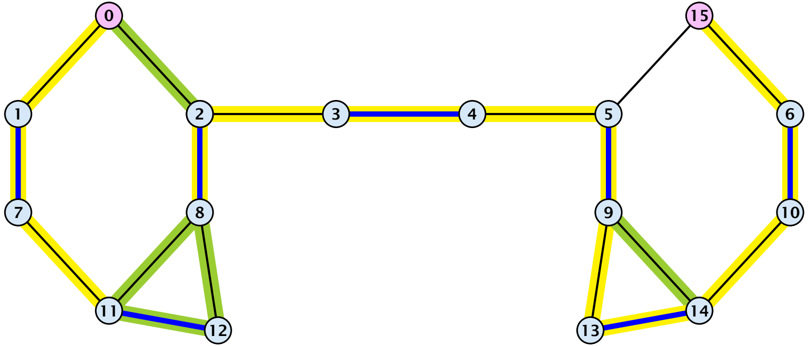

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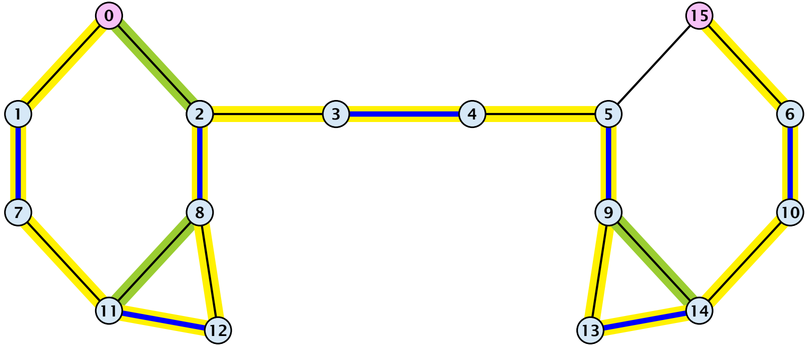

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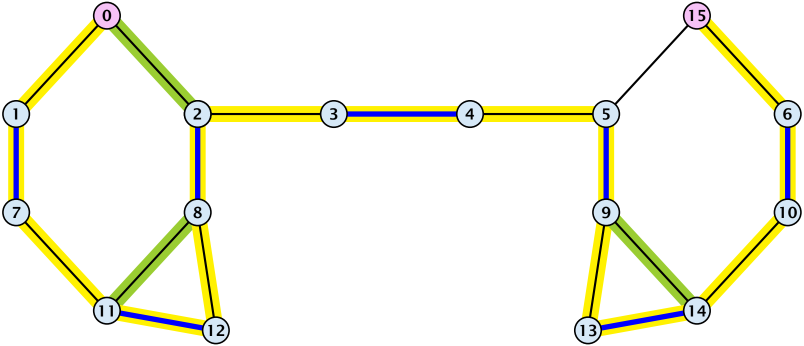

# A Fast Matching Algorithm

**Algorithm 53** Bimatch-Hopcroft-Karp($G$)

$M \leftarrow \emptyset$
**repeat**
    let $\mathcal{P} = \{P_1, \dots, P_k\}$ be maximal set of
    vertex-disjoint, shortest augmenting path w.r.t. $M$.
    $M \leftarrow M \oplus (P_1 \cup \cdots \cup P_k)$
**until** $\mathcal{P} = \emptyset$
**return** $M$

We call one iteration of the repeat-loop a phase of the algorithm.

# Analysis Hopcroft-Karp

**Lemma 12**

*Given a matching $M$ and a matching $M^*$ with $|M^*| - |M| \geq 0$. There exist $|M^*| - |M|$ vertex-disjoint augmenting path w.r.t. $M$.*

# Analysis Hopcroft-Karp

### Lemma 12

*Given a matching $M$ and a matching $M^*$ with $|M^*| - |M| \geq 0$. There exist $|M^*| - |M|$ vertex-disjoint augmenting path w.r.t. $M$.*

**Proof:**

- Similar to the proof that a matching is optimal iff it does not contain an augmenting path.

# Analysis Hopcroft-Karp

### Lemma 12

*Given a matching* $M$ *and a matching* $M^*$ *with* $|M^*| - |M| \geq 0$*.*
*There exist* $|M^*| - |M|$ *vertex-disjoint augmenting path w.r.t.* $M$*.*

**Proof:**

- Similar to the proof that a matching is optimal iff it does not contain an augmenting path.
- Consider the graph $G = (V, M \oplus M^*)$, and mark edges in this graph blue if they are in $M$ and red if they are in $M^*$.

# Analysis Hopcroft-Karp

### Lemma 12

*Given a matching* $M$ *and a matching* $M^*$ *with* $|M^*| - |M| \geq 0$*. There exist* $|M^*| - |M|$ *vertex-disjoint augmenting path w.r.t.* $M$*.*

**Proof:**

- Similar to the proof that a matching is optimal iff it does not contain an augmenting path.
- Consider the graph $G = (V, M \oplus M^*)$, and mark edges in this graph blue if they are in $M$ and red if they are in $M^*$.
- The connected components of $G$ are cycles and paths.

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- The graph contains $k \stackrel{\text{def}}{=} |M^*| - |M|$ more red edges than blue edges.

# Analysis Hopcroft-Karp

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*Given a matching* $M$ *and a matching* $M^*$ *with* $|M^*| - |M| \geq 0$*. There exist* $|M^*| - |M|$ *vertex-disjoint augmenting path w.r.t.* $M$*.*

**Proof:**

- Similar to the proof that a matching is optimal iff it does not contain an augmenting path.
- Consider the graph $G = (V, M \oplus M^*)$, and mark edges in this graph blue if they are in $M$ and red if they are in $M^*$.
- The connected components of $G$ are cycles and paths.
- The graph contains $k \stackrel{\text{def}}{=} |M^*| - |M|$ more red edges than blue edges.
- Hence, there are at least $k$ components that form a path starting and ending with a red edge. These are augmenting paths w.r.t. $M$.

# Analysis Hopcroft-Karp

- Let $P_1, \dots, P_k$ be a maximal collection of vertex-disjoint, shortest augmenting paths w.r.t. $M$ (let $\ell = |P_i|$).

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- Let $P_1, \dots, P_k$ be a maximal collection of vertex-disjoint, shortest augmenting paths w.r.t. $M$ (let $\ell = |P_i|$).
- $M' \stackrel{\text{def}}{=} M \oplus (P_1 \cup \cdots \cup P_k) = M \oplus P_1 \oplus \cdots \oplus P_k$.

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- Let $P$ be an augmenting path in $M'$.

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- $M' \stackrel{\text{def}}{=} M \oplus (P_1 \cup \dots \cup P_k) = M \oplus P_1 \oplus \dots \oplus P_k$.
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**Lemma 13**

*The set* $A \stackrel{\text{def}}{=} M \oplus (M' \oplus P) = (P_1 \cup \dots \cup P_k) \oplus P$ *contains at least* $(k+1)\ell$ *edges.*

# Analysis Hopcroft-Karp

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- Each of these paths is of length at least $\ell$.

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- Hence, $|A| \le k\ell + |P| - 1$.
- The lower bound on $|A|$ gives $(k+1)\ell \le |A| \le k\ell + |P| - 1$, and hence $|P| \ge \ell + 1$.

# Analysis Hopcroft-Karp

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**Proof.**
The symmetric difference between $M$ and $M^*$ contains $|M^*| - |M|$ vertex-disjoint augmenting paths. Each of these paths contains at least $\ell + 1$ vertices. Hence, there can be at most $\frac{|V|}{\ell+1}$ of them.

# Analysis Hopcroft-Karp

**Lemma 15**
*The Hopcroft-Karp algorithm requires at most* $2\sqrt{|V|}$ *phases.*

# Analysis Hopcroft-Karp

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*The Hopcroft-Karp algorithm requires at most* $2\sqrt{|V|}$ *phases.*

**Proof.**

- After iteration $\lfloor\sqrt{|V|}\rfloor$ the length of a shortest augmenting path must be at least $\lfloor\sqrt{|V|}\rfloor + 1 \geq \sqrt{|V|}$.
- Hence, there can be at most $|V|/(\sqrt{|V|} + 1) \leq \sqrt{|V|}$ additional augmentations.

# Analysis Hopcroft-Karp

**Lemma 16**
*One phase of the Hopcroft-Karp algorithm can be implemented in time $\mathcal{O}(m)$.*

construct a "level graph" $G'$:

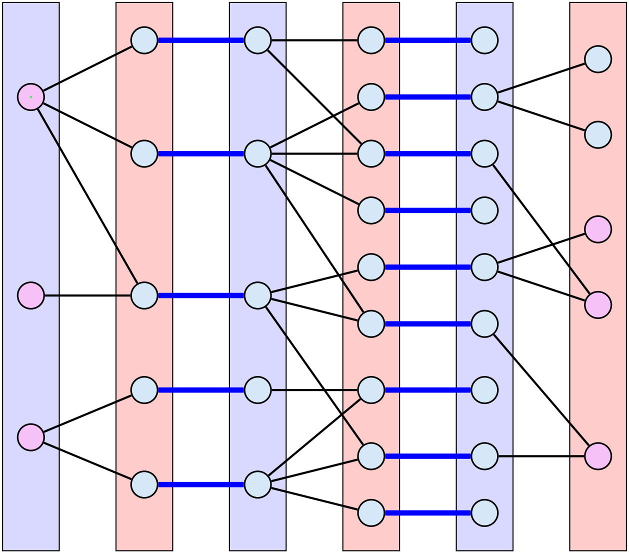
- construct Level 0 that includes all free vertices on left side $L$
- construct Level 1 containing all neighbors of Level 0
- construct Level 2 containing matching neighbors of Level 1
- construct Level 3 containing all neighbors of Level 2
- ...
- stop when a level (apart from Level 0) contains a free vertex

can be done in time $\mathcal{O}(m)$ by a modified BFS

# Analysis Hopcroft-Karp

- a shortest augmenting path must go from Level 0 to the last layer constructed
- it can only use edges between layers
- construct a maxim**al** set of vertex disjoint augmenting path connecting the layers
- for this, go forward until you either reach a free vertex or you reach a “dead end” $v$
- if you reach a free vertex delete the augmenting path and all incident edges from the graph
- if you reach a dead end backtrack and delete $v$ together with its incident edges

# Analysis Hopcroft-Karp



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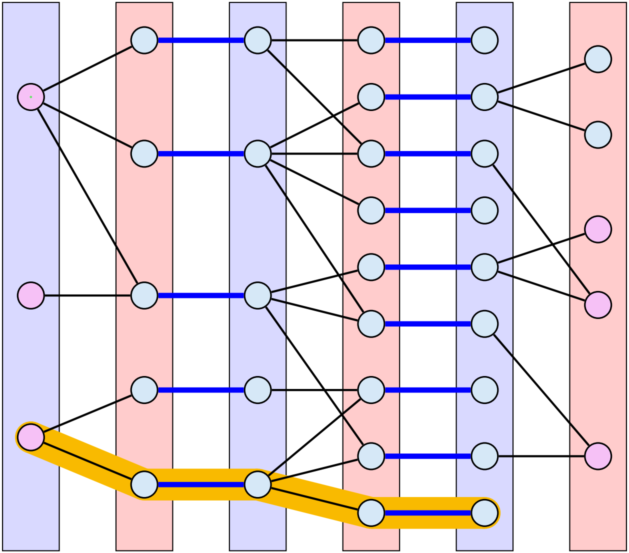

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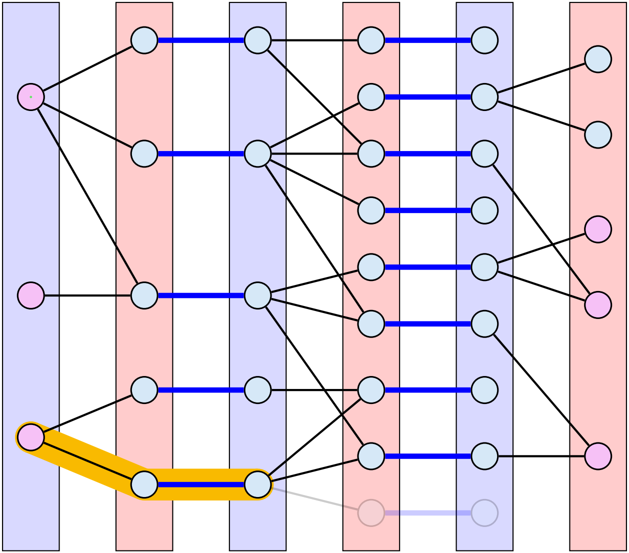

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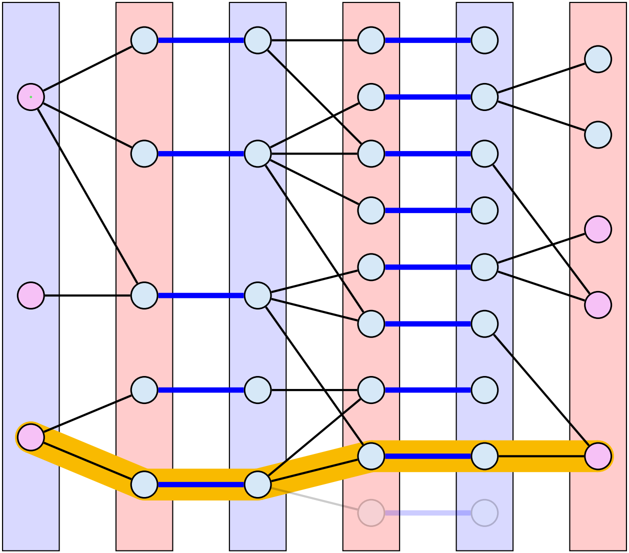

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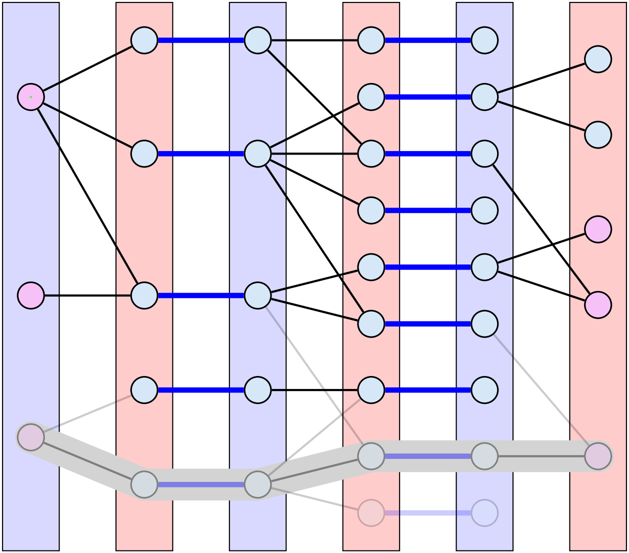

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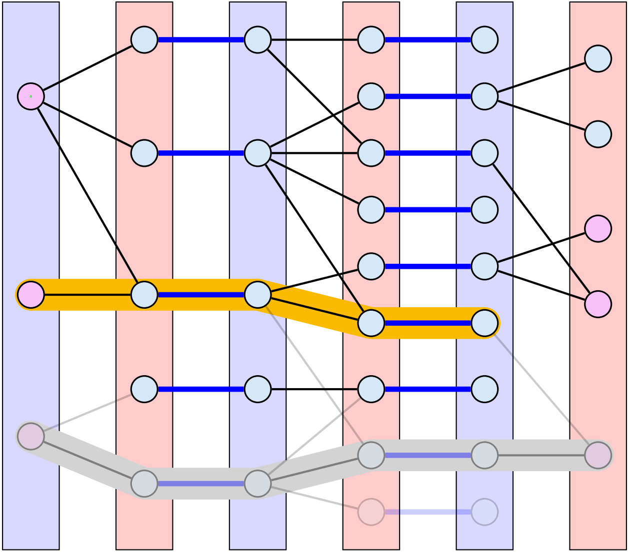

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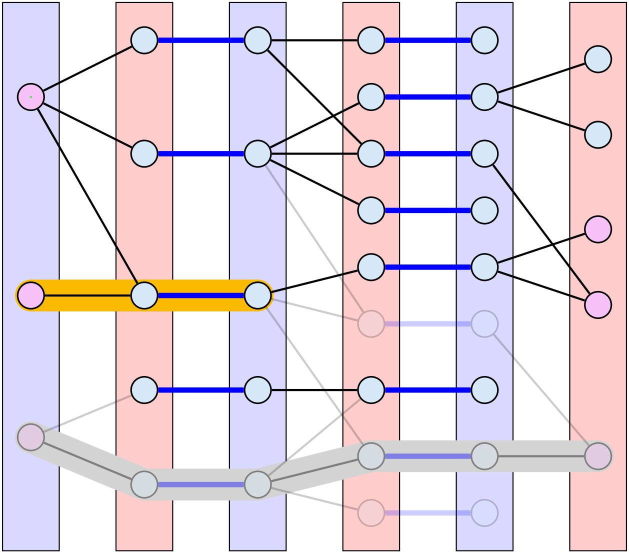

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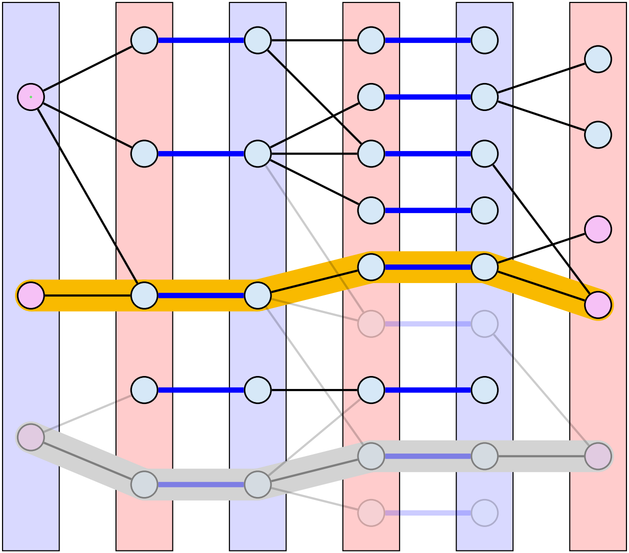

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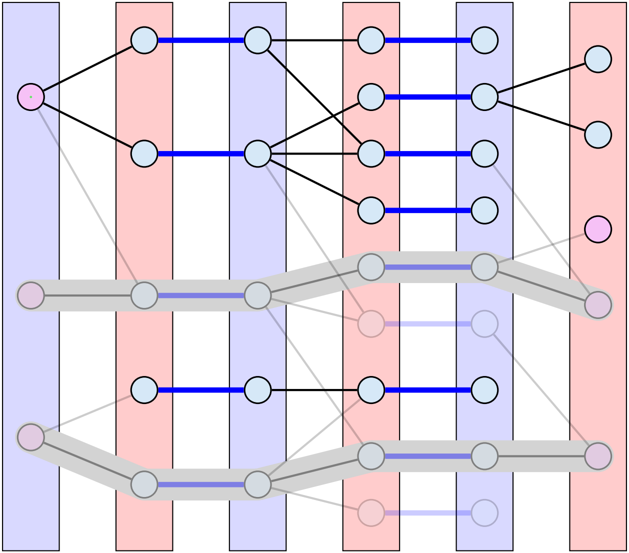

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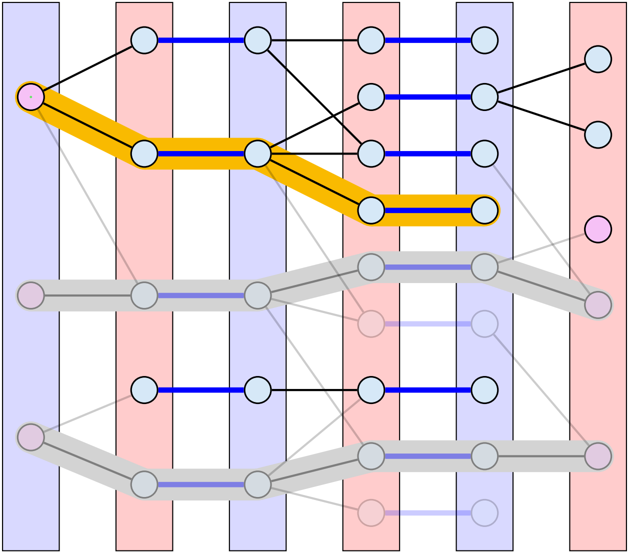

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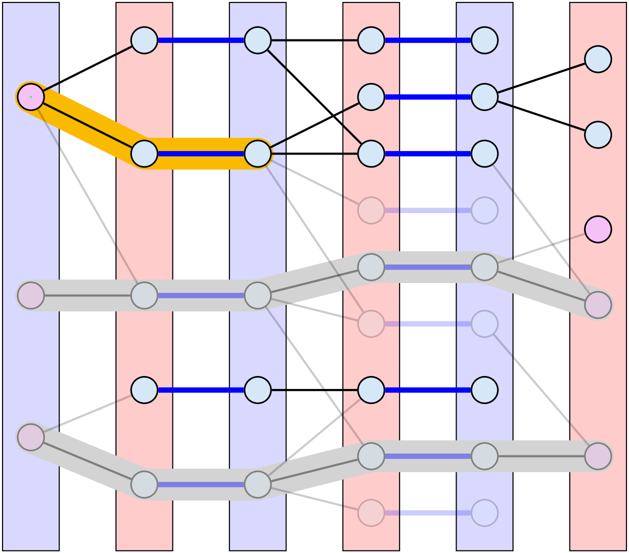

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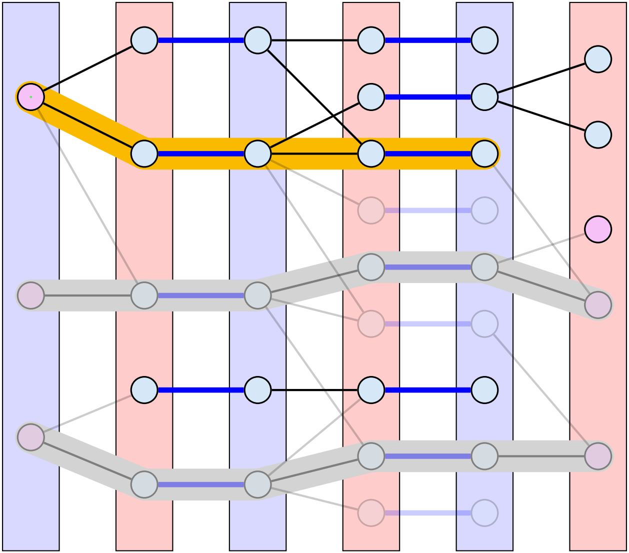

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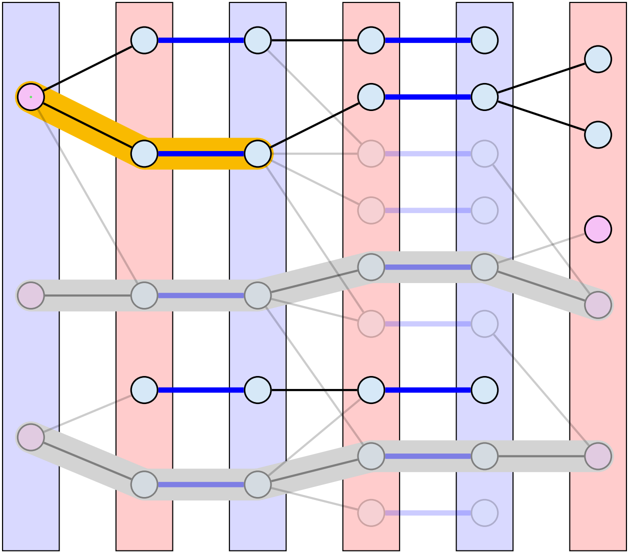

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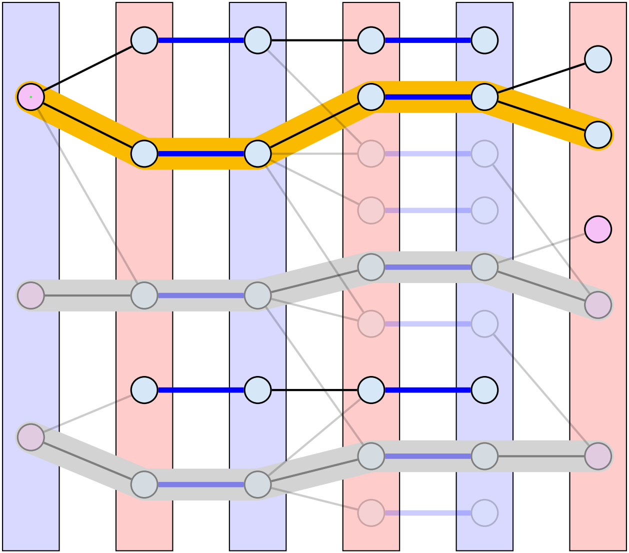

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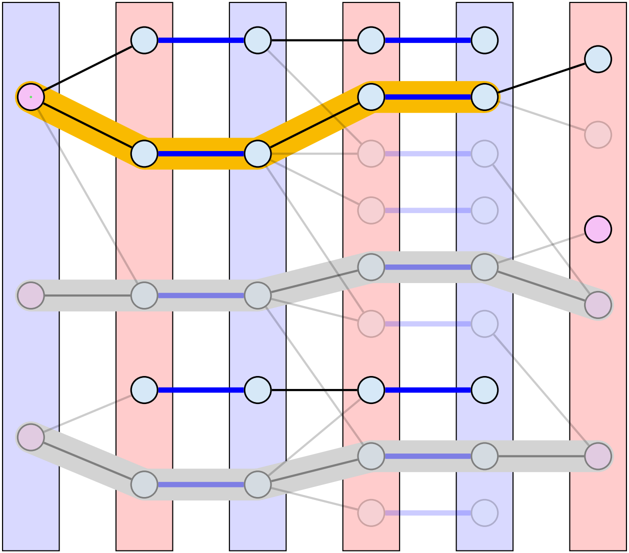

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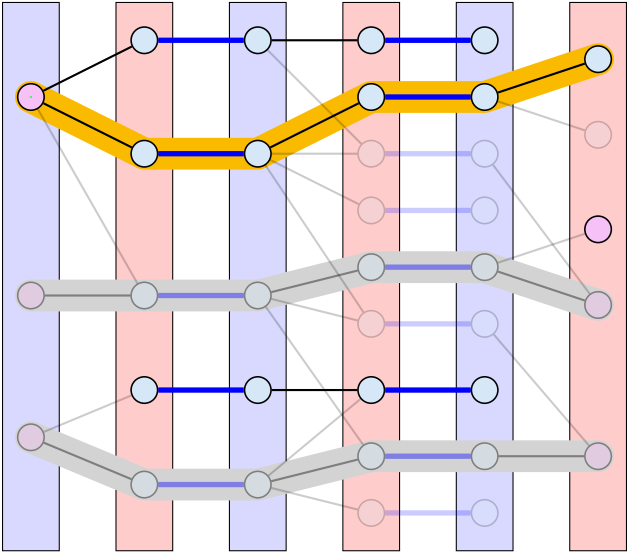

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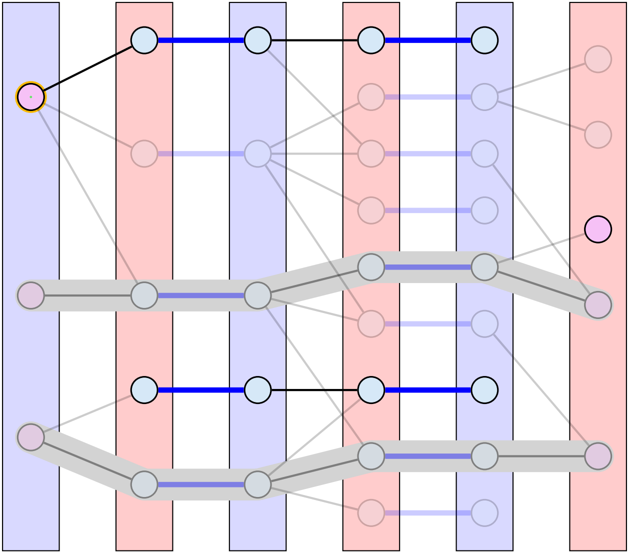

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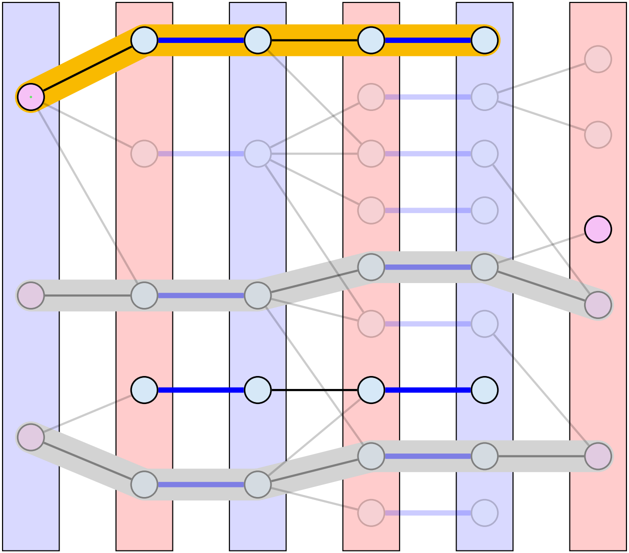

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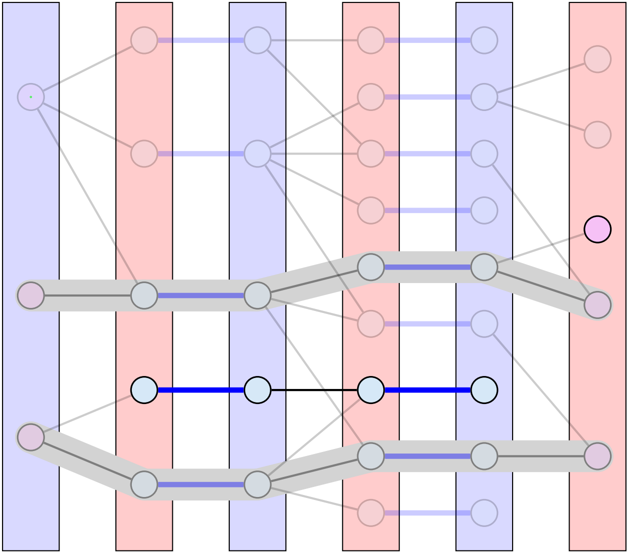

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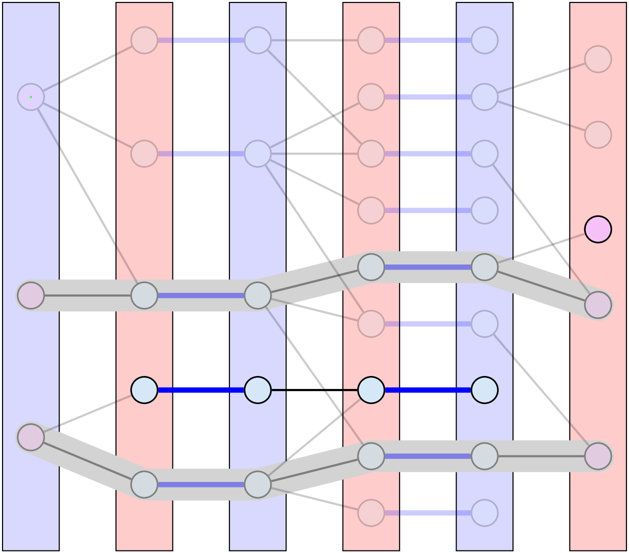

# Analysis: Shortest Augmenting Path for Flows

**cost for searches during a phase is $\mathcal{O}(mn)$**

- a search (successful or unsuccessful) takes time $\mathcal{O}(n)$
- a search deletes at least one edge from the level graph

**there are at most $n$ phases**

Time: $\mathcal{O}(mn^2)$.

# Analysis for Unit-capacity Simple Networks

**cost for searches during a phase is $\mathcal{O}(m)$**

- an edge/vertex is traversed at most twice

**need at most $\mathcal{O}(\sqrt{n})$ phases**

- after $\sqrt{n}$ phases there is a cut of size at most $\sqrt{n}$ in the residual graph
- hence at most $\sqrt{n}$ additional augmentations required

Time: $\mathcal{O}(m\sqrt{n})$.

# 21 Gomory Hu Trees

Given an undirected, weighted graph $G = (V, E, c)$ a cut-tree $T = (V, F, w)$ is a tree with edge-set $F$ and capacities $w$ that fulfills the following properties.

1. **Equivalent Flow Tree:** For any pair of vertices $s, t \in V$, $f(s, t)$ in $G$ is equal to $f_T(s, t)$.
2. **Cut Property:** A minimum $s$-$t$ cut in $T$ is also a minimum cut in $G$.

Here, $f(s, t)$ is the value of a maximum $s$-$t$ flow in $G$, and $f_T(s, t)$ is the corresponding value in $T$.

# Overview of the Algorithm

The algorithm maintains a partition of $V$, (sets $S_1, \ldots, S_t$), and a spanning tree $T$ on the vertex set $\{S_1, \ldots, S_t\}$.

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**In the end this gives a tree on the vertex set $V$.**

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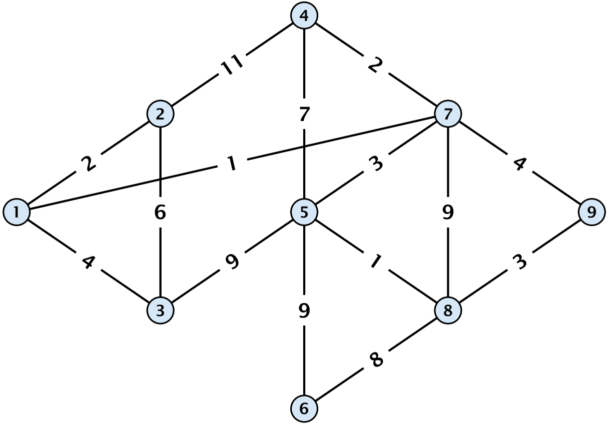
- Replace an edge $\{S_i, S_x\}$ by $\{S_i^a, S_x\}$ if $S_x \subset A$ and by $\{S_i^b, S_x\}$ if $S_x \subset B$.

# Example: Gomory-Hu Construction



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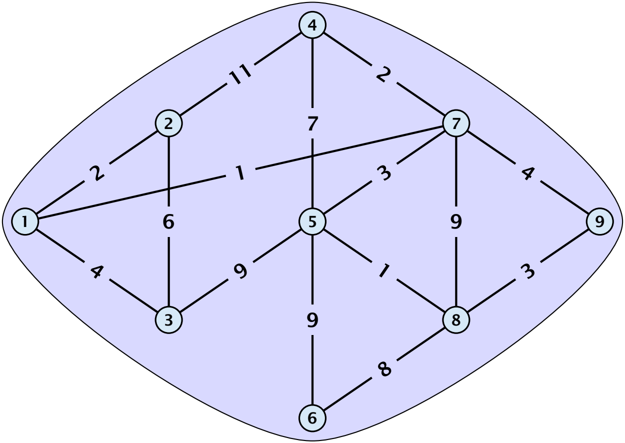

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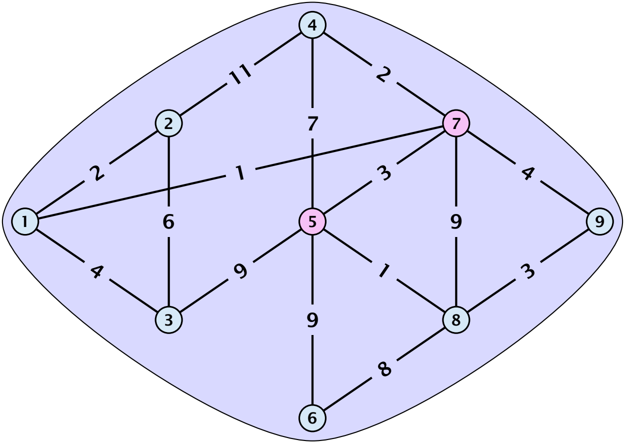

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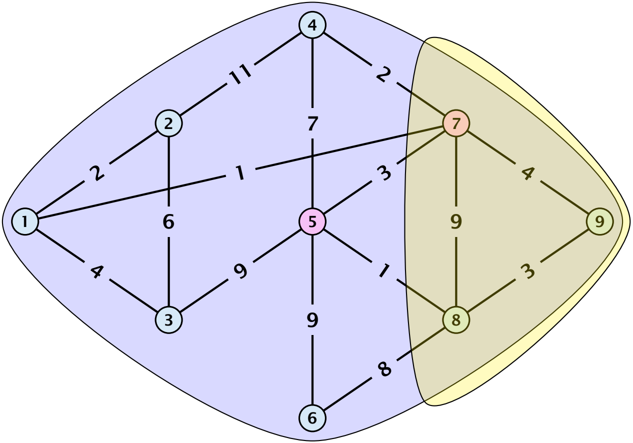

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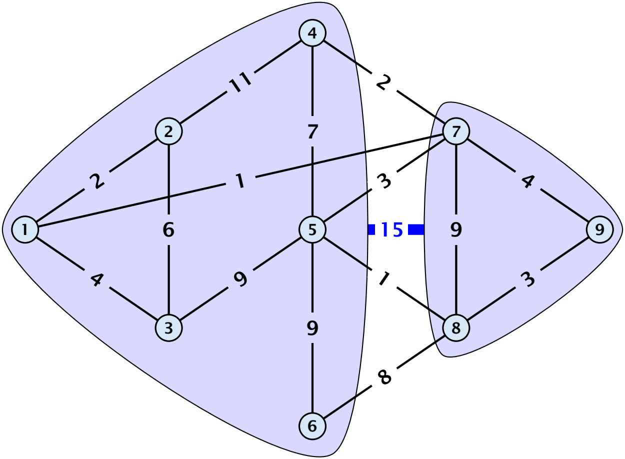

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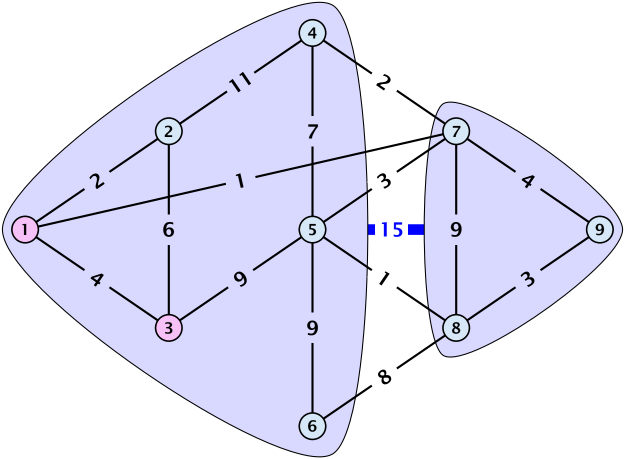

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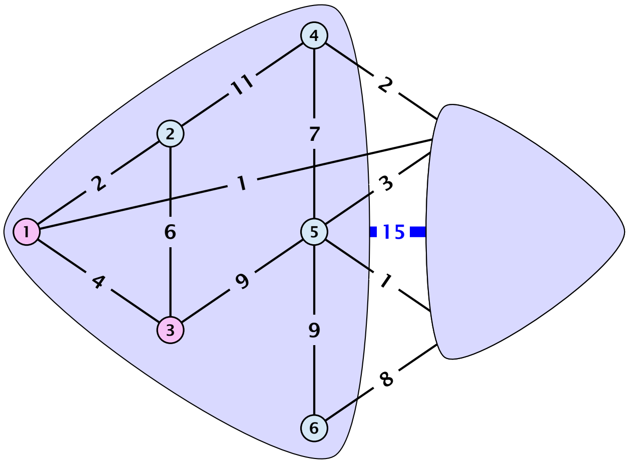

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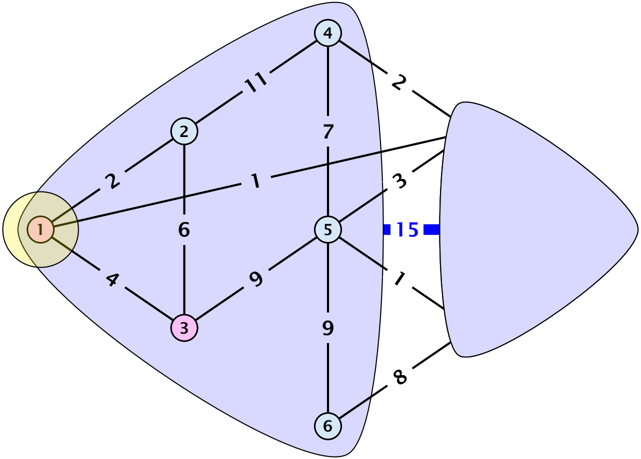

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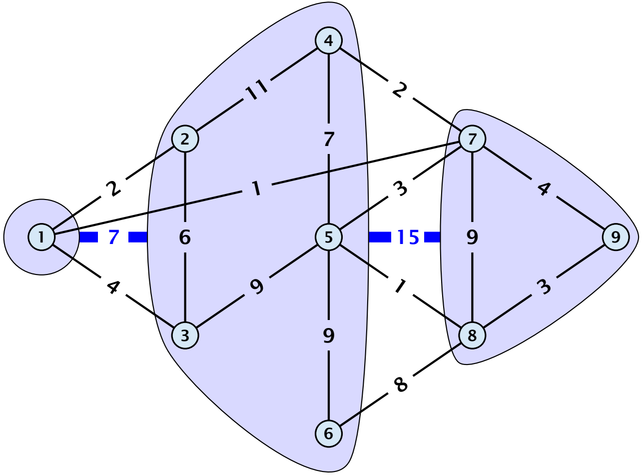

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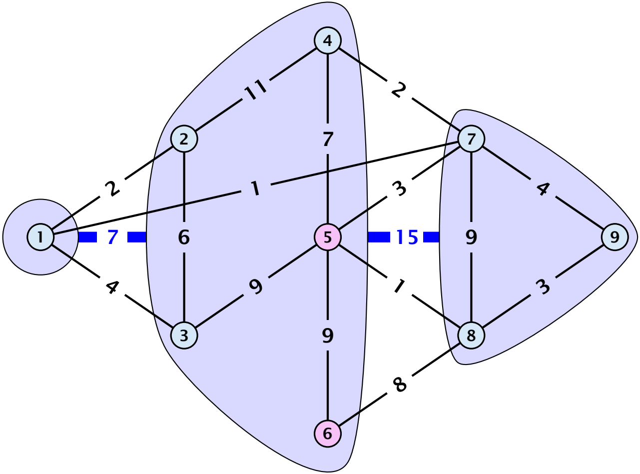

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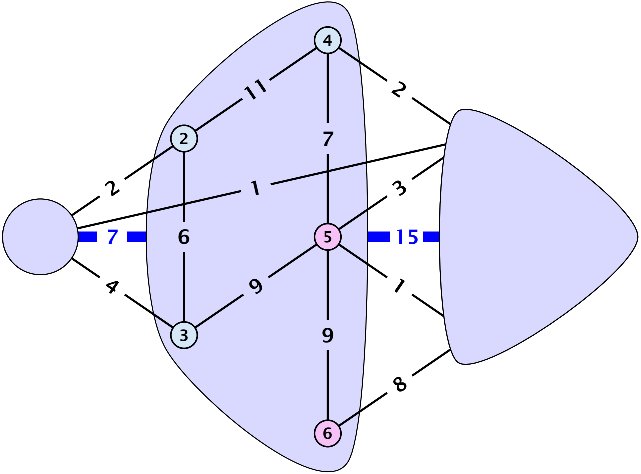

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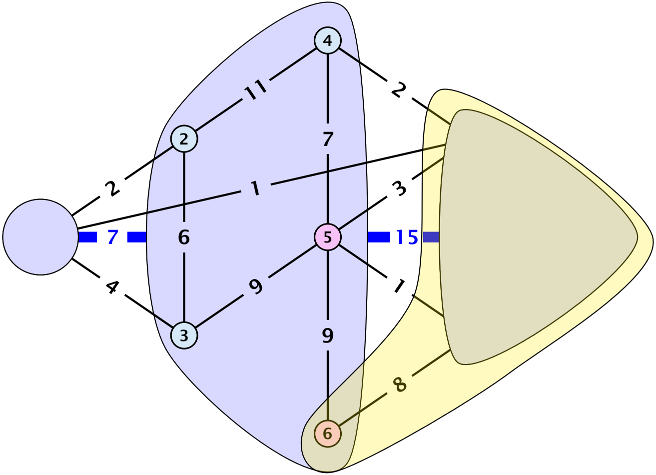

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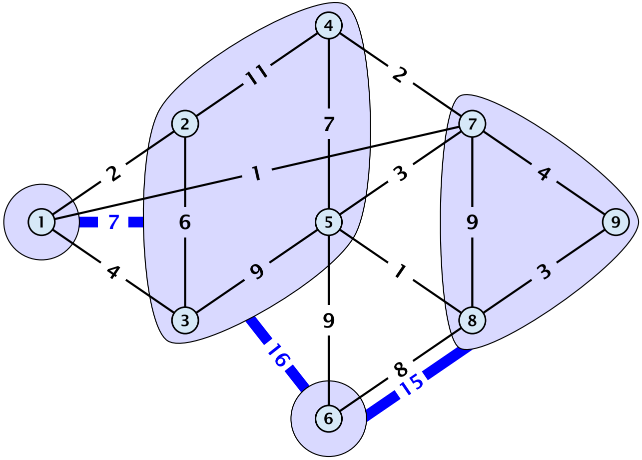

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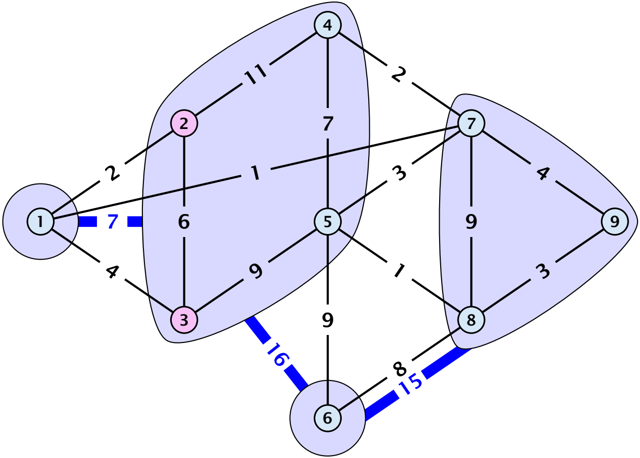

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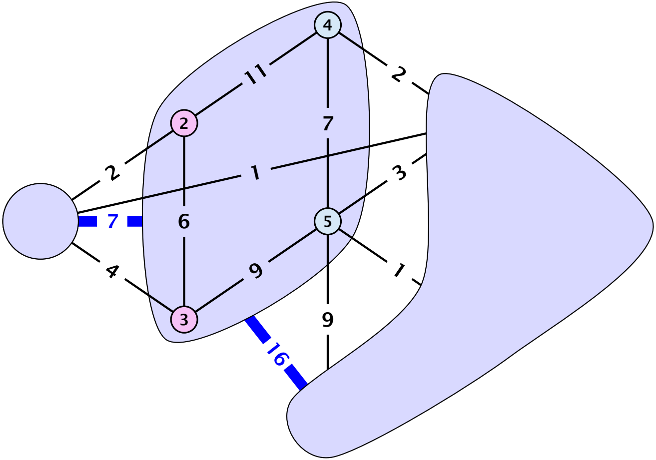

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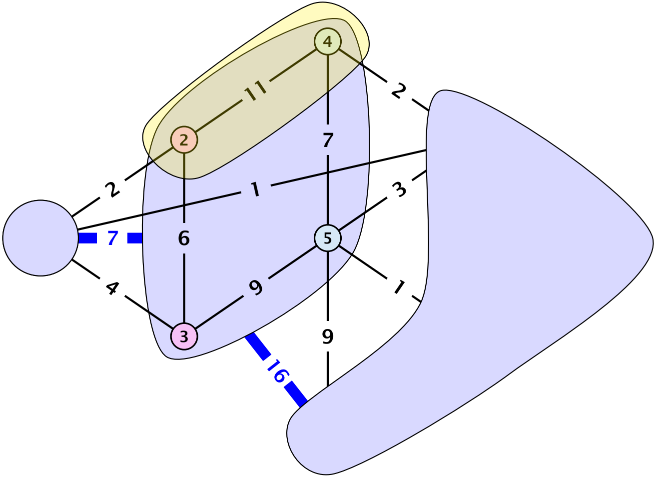

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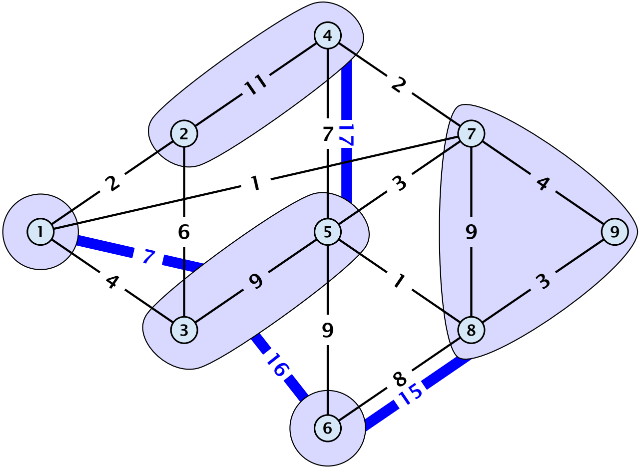

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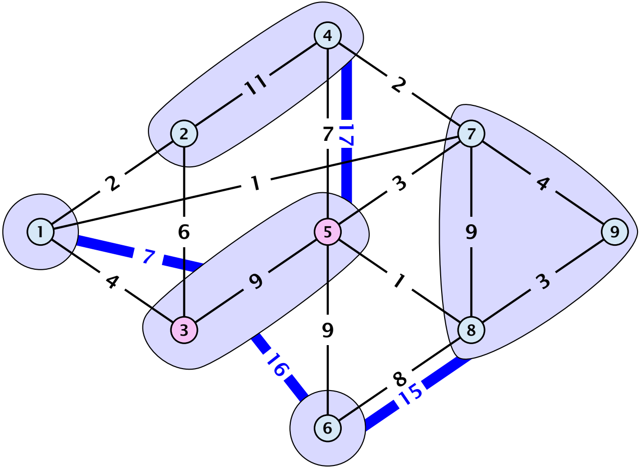

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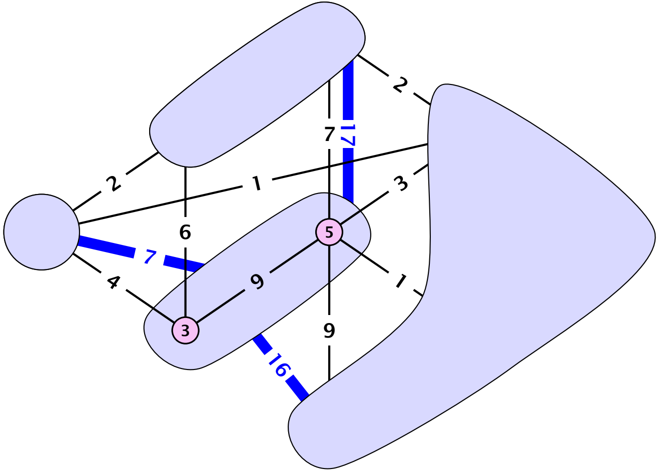

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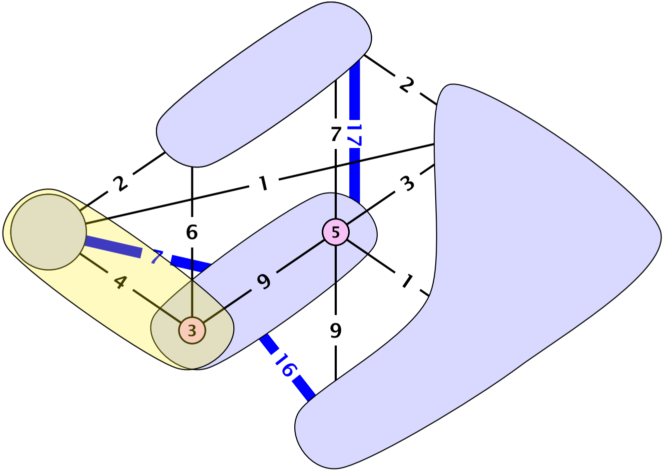

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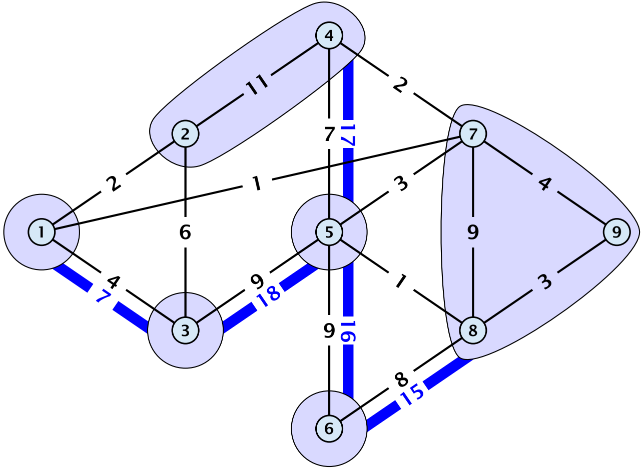

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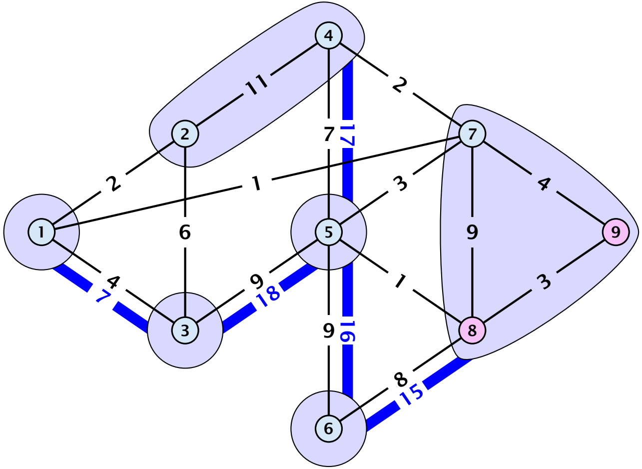

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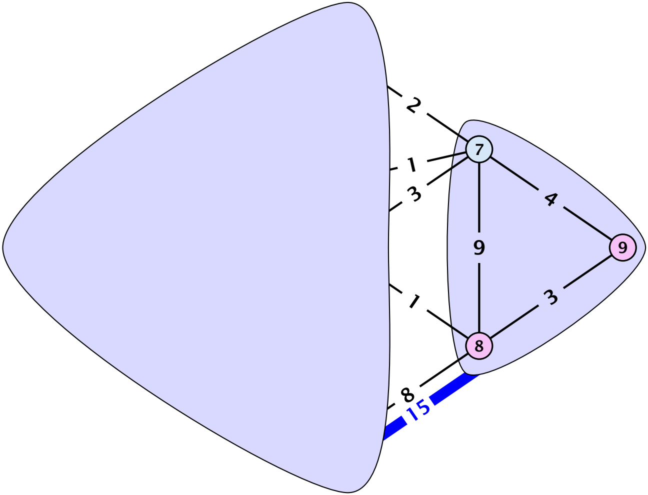

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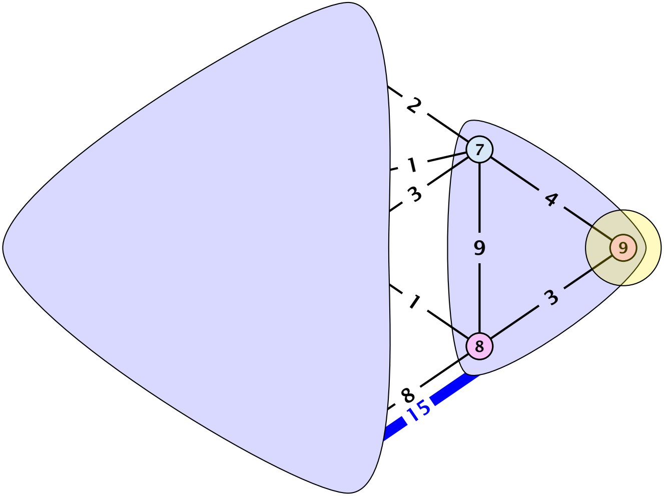

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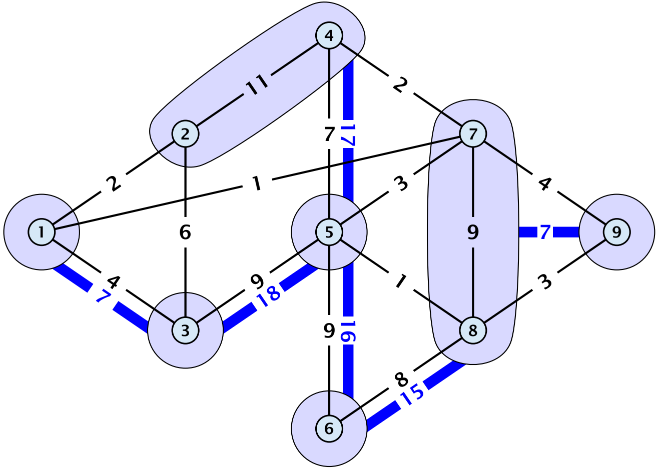

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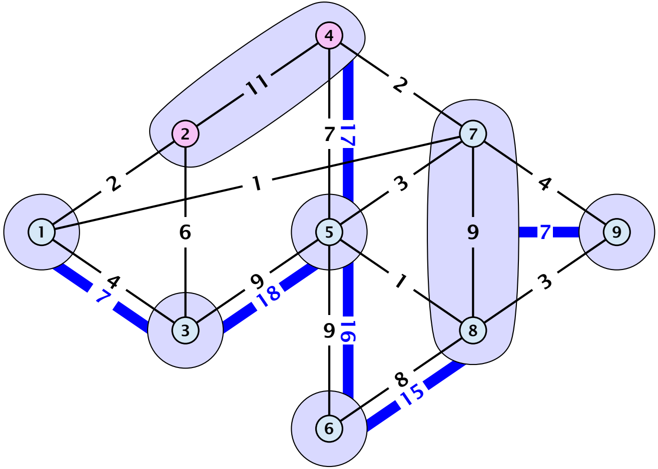

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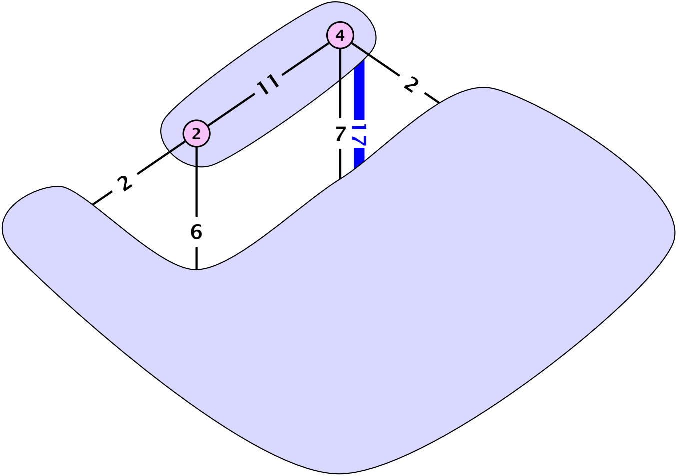

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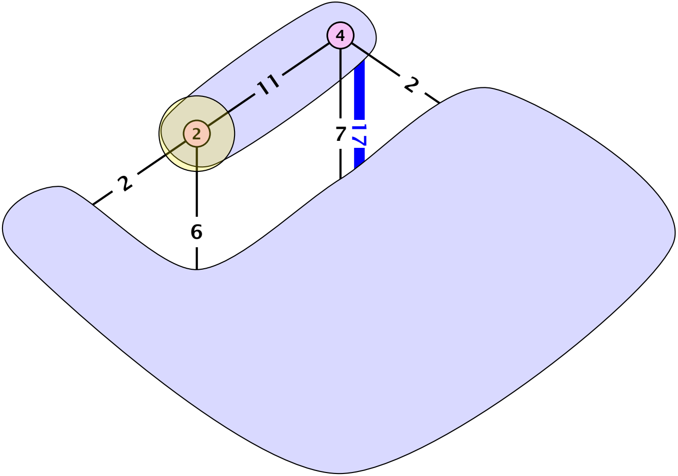

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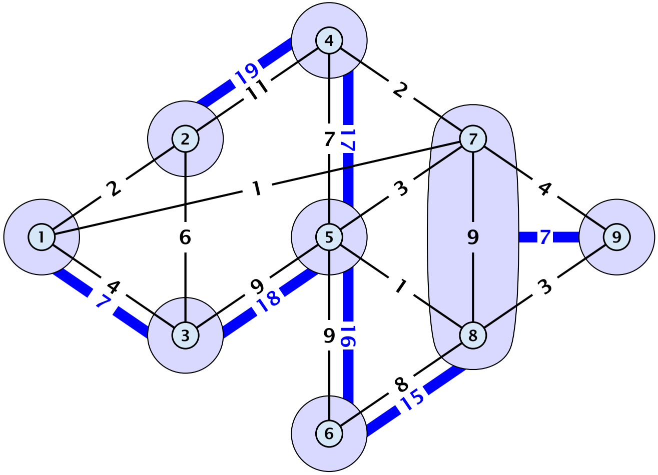

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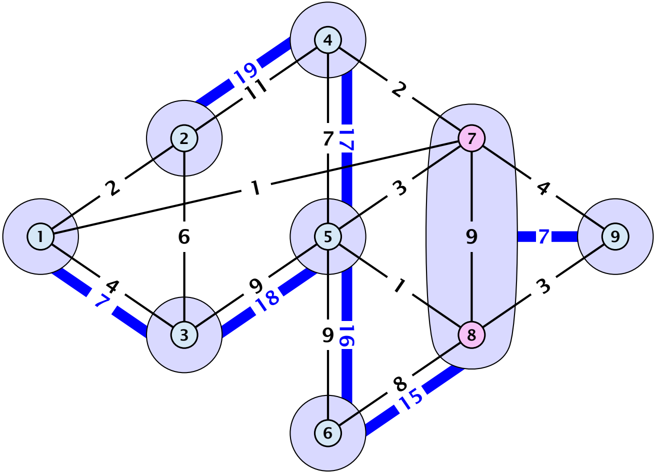

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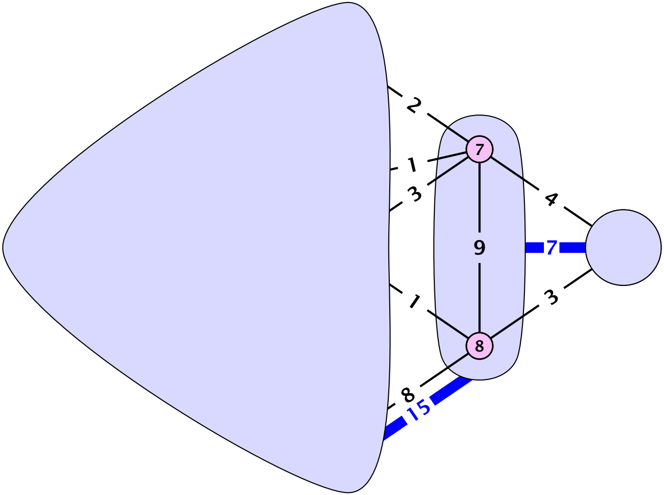

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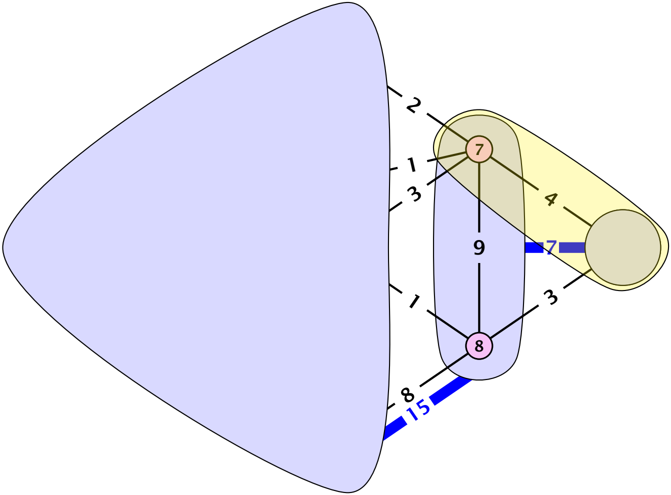

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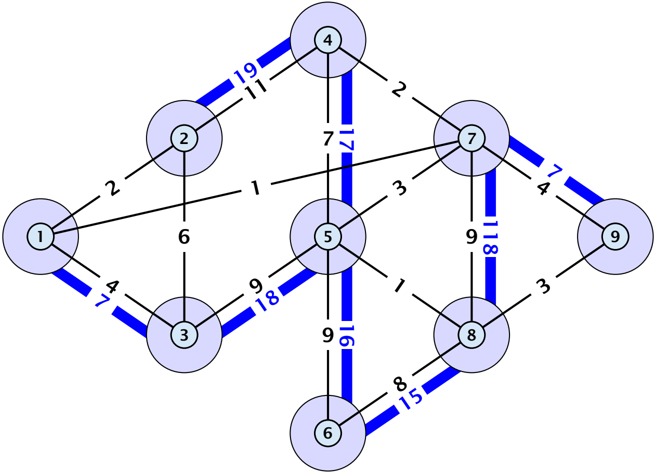

# Analysis

**Lemma 17**
*For nodes* $s, t, x \in V$ *we have* $f(s,t) \geq \min\{f(s,x), f(x,t)\}$

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$f(s,t) \geq \min\{f(s,x_1), f(x_1,x_2), \ldots, f(x_{k-1},x_k), f(x_k,t)\}$

**Lemma 19**

*Let $S$ be some minimum $r$-$s$ cut for some nodes $r, s \in V$ ($s \in S$), and let $v, w \in S$. Then there is a minimum $v$-$w$-cut $T$ with $T \subset S$.*

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**Proof:** Let $X$ be a minimum $v$-$w$ cut with $X \cap S \neq \emptyset$ and $X \cap (V \setminus S) \neq \emptyset$.

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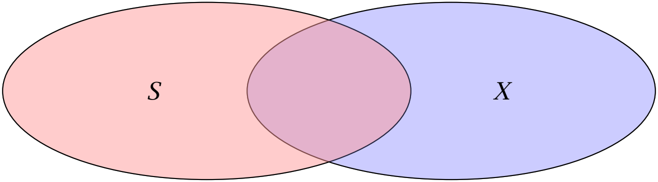


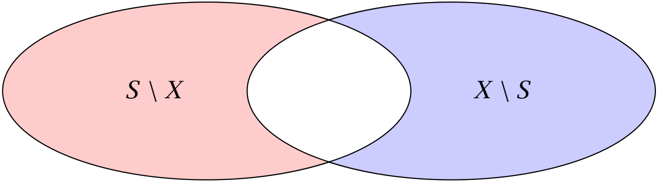

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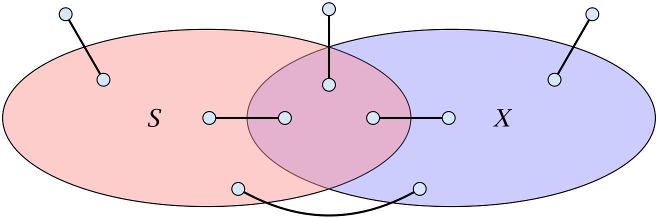


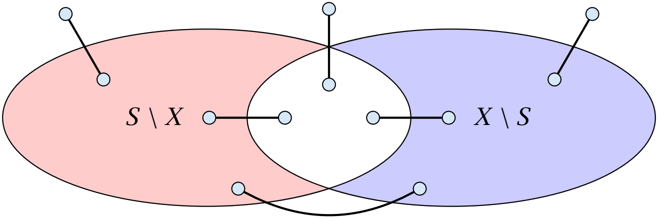

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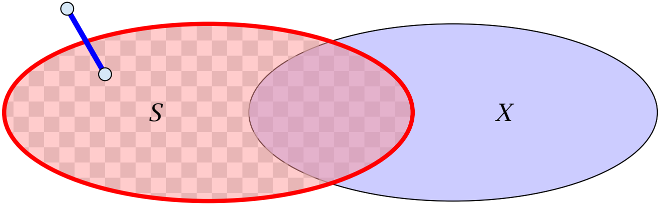


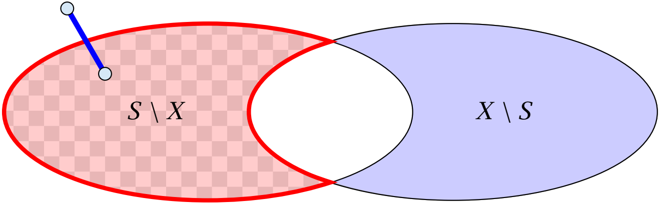

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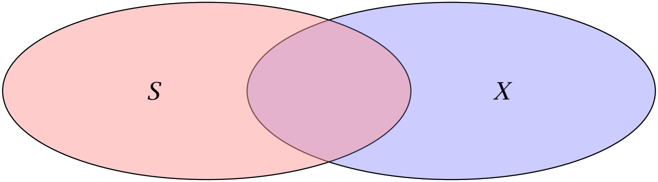


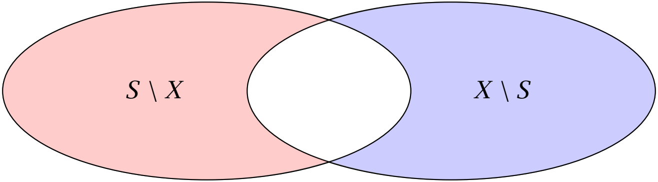

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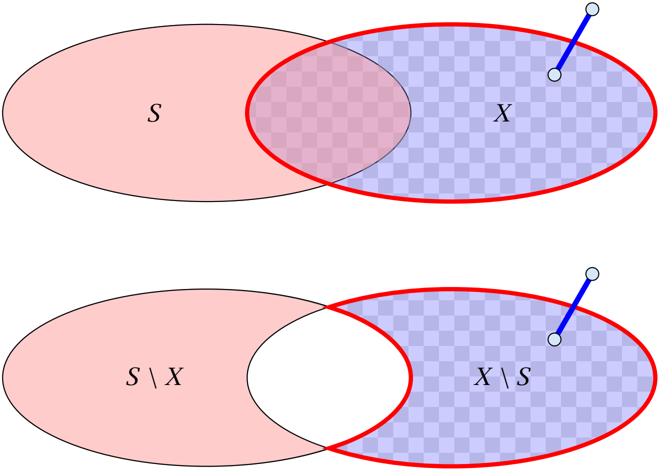

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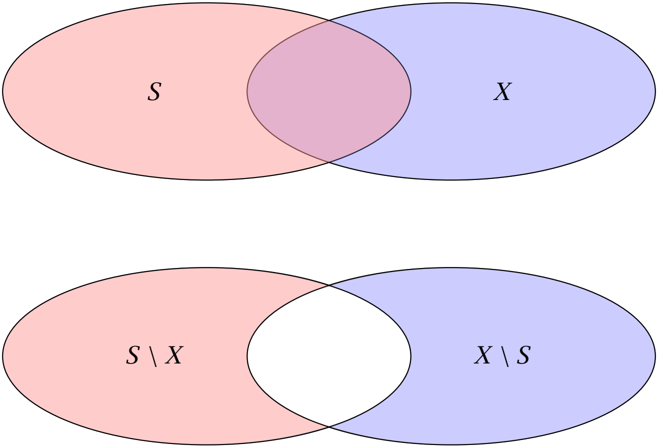

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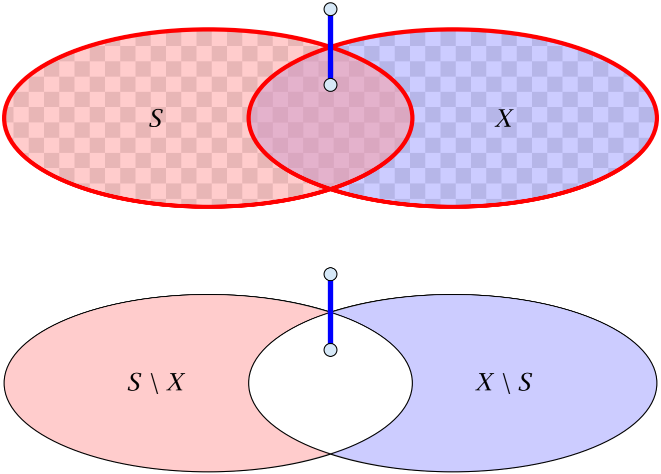

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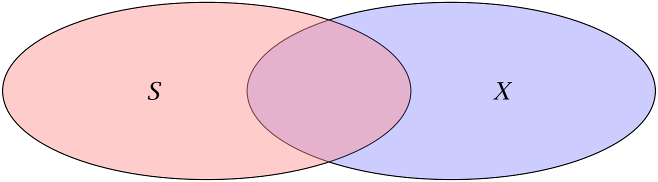


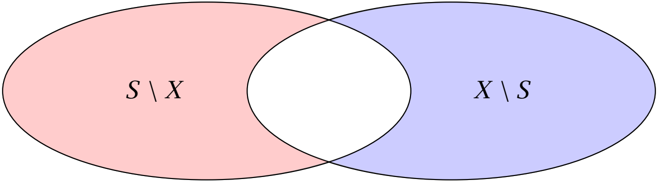

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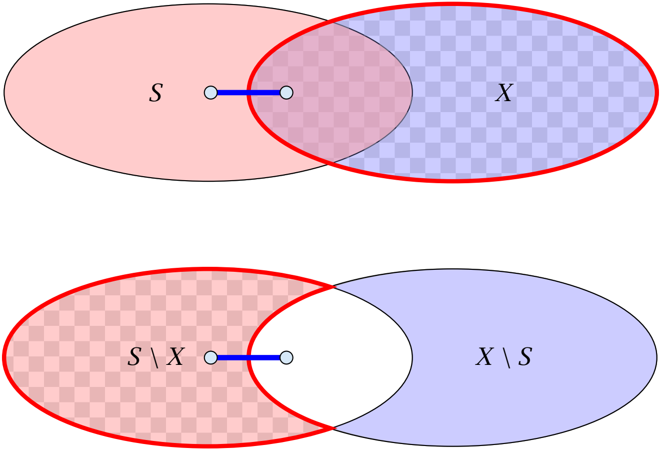

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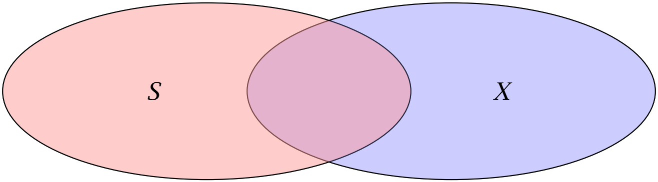


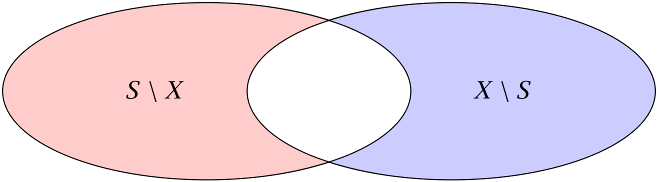

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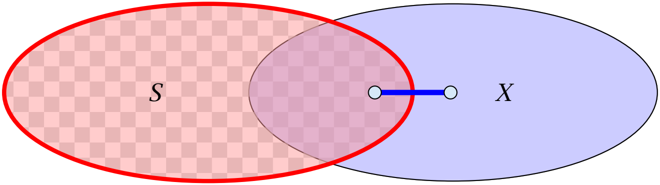


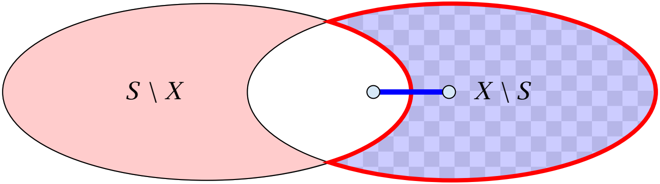

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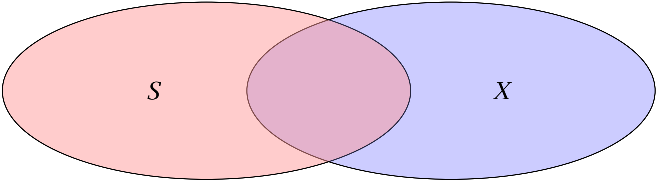


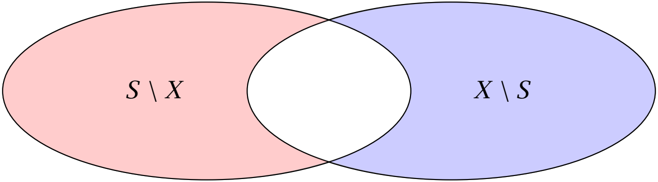

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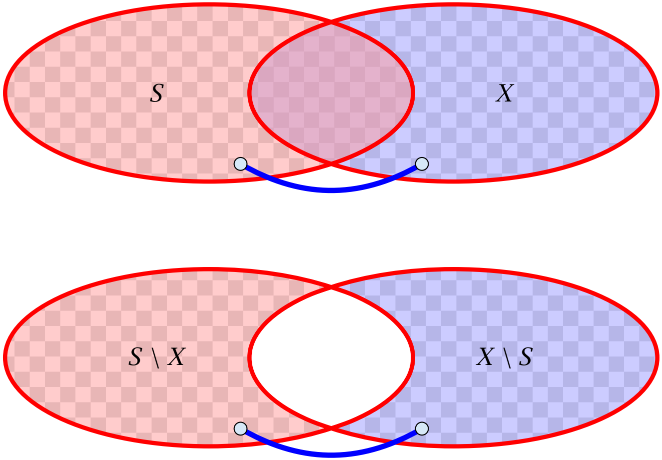

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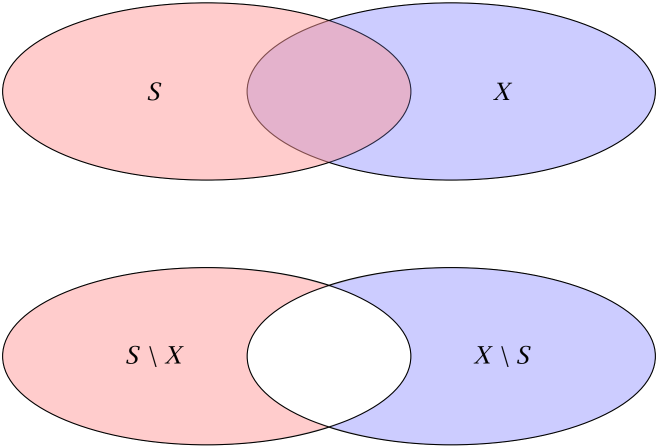

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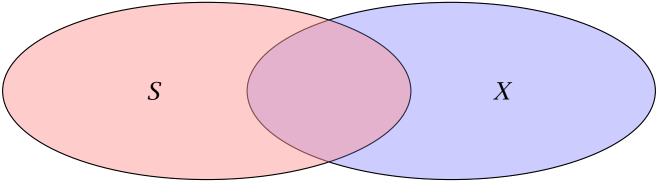


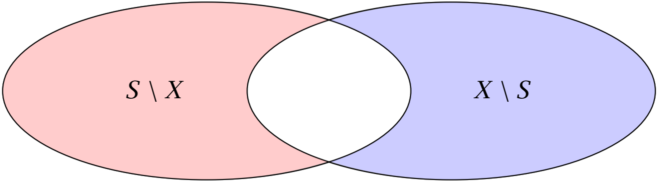

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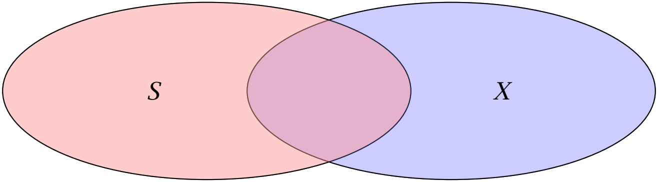


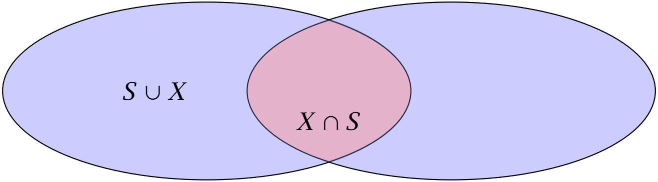

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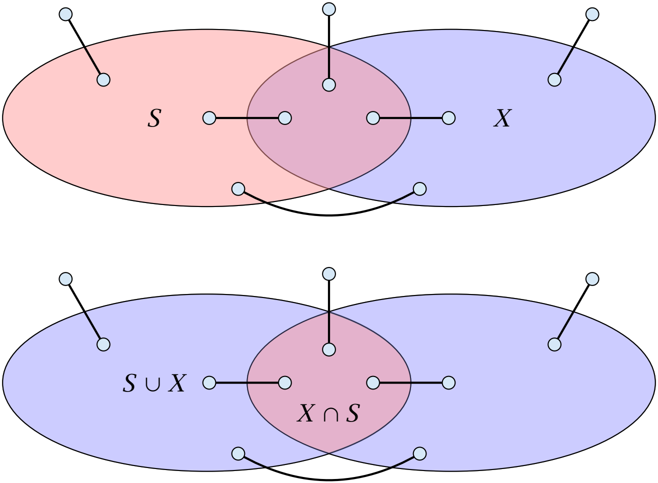

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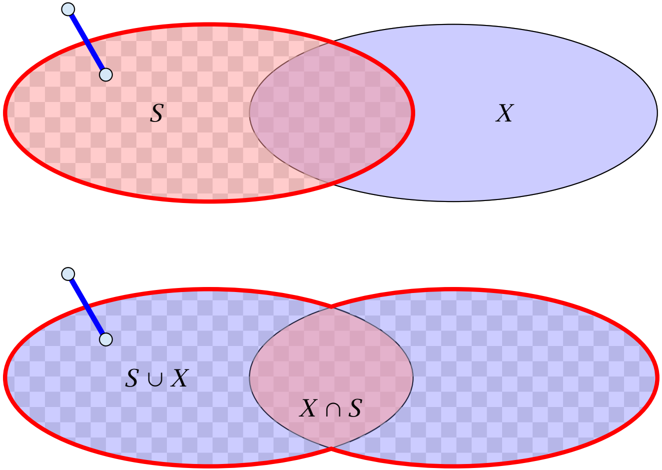

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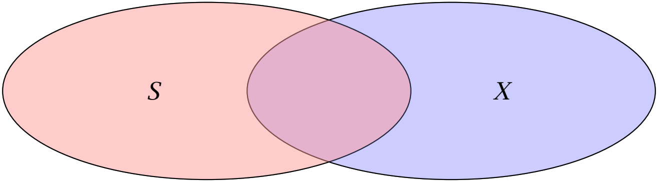


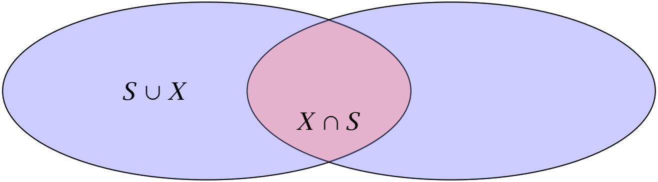

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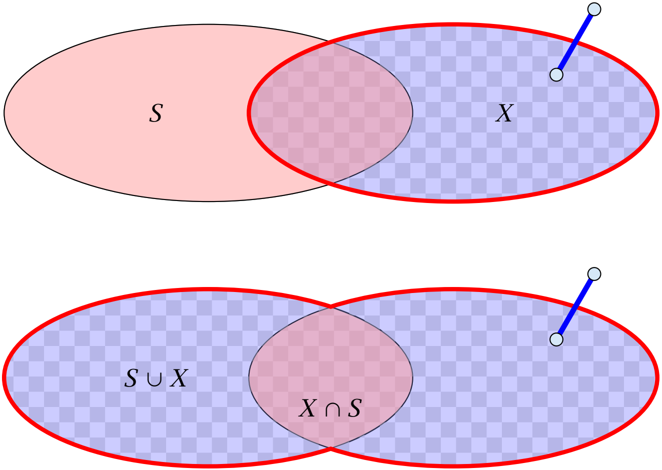

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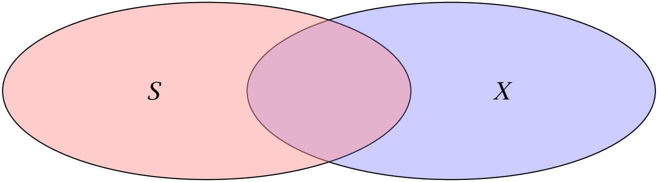


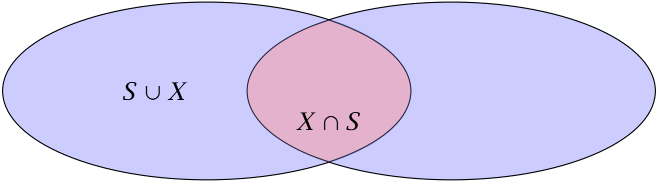

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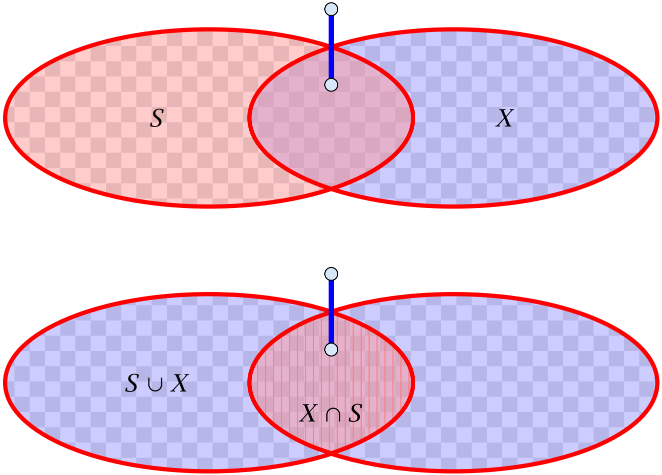

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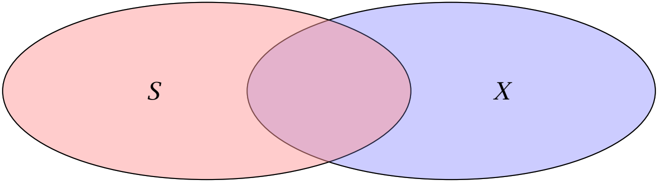


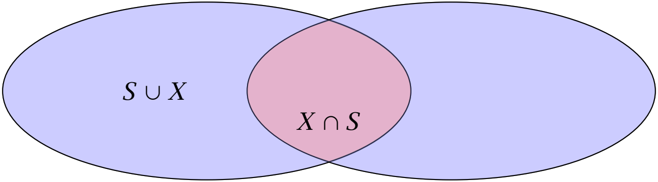

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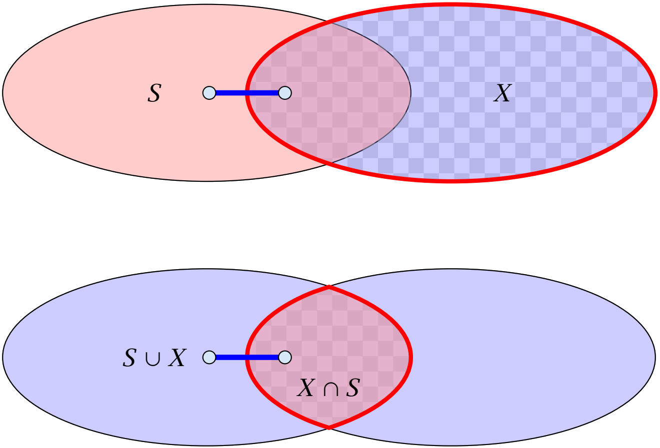

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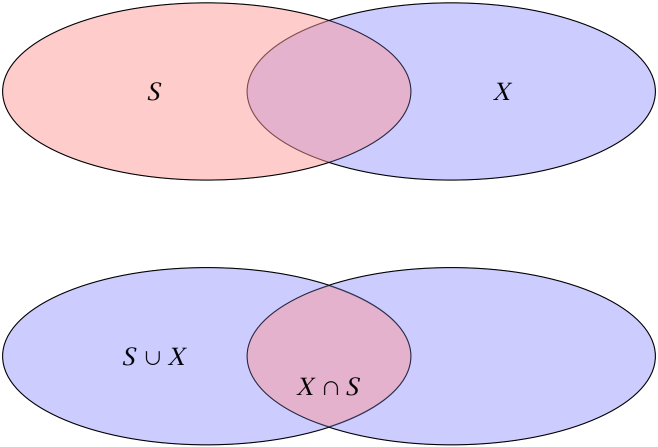

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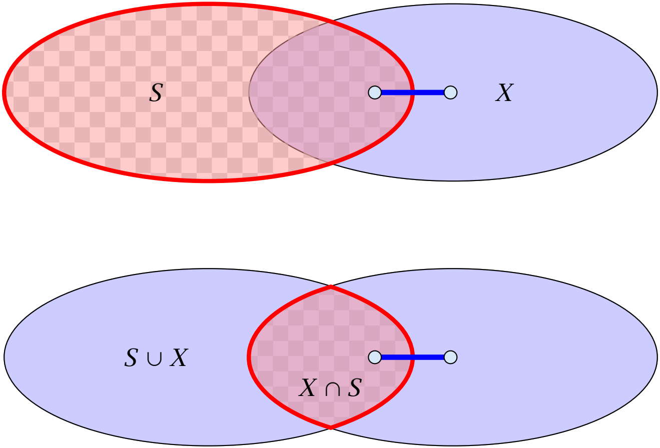

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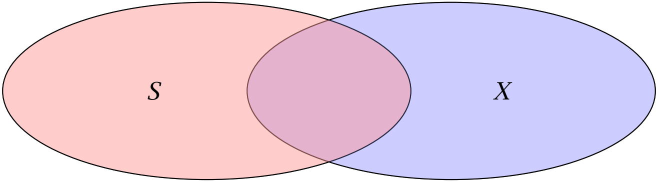


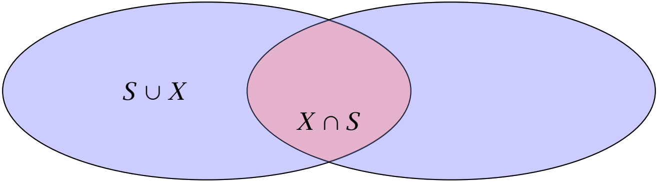

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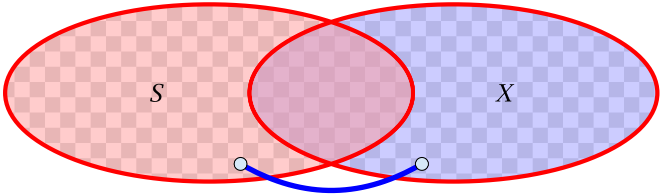


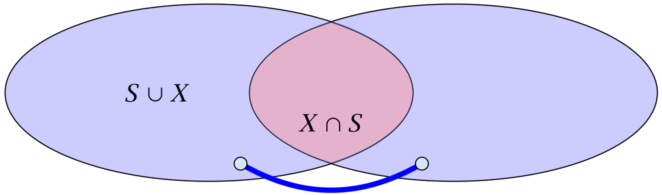

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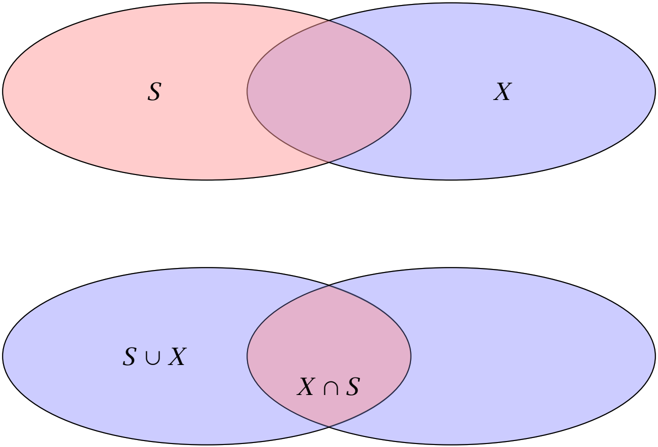

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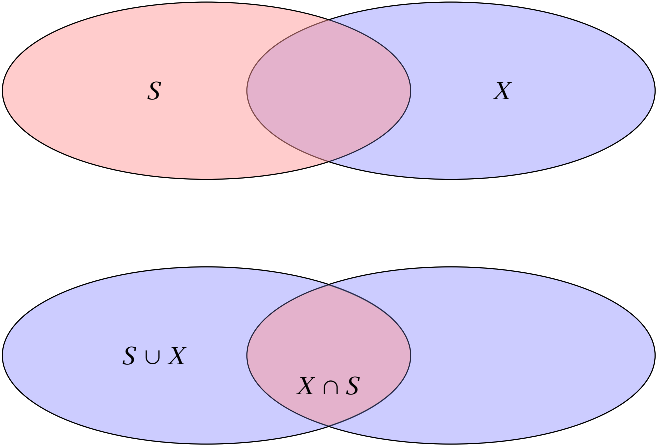

# Analysis

Lemma 19 tells us that if we have a graph $G = (V, E)$ and we contract a subset $X \subset V$ that corresponds to some mincut, then the value of $f(s,t)$ does not change for two nodes $s, t \notin X$.

We will show (later) that the connected components that we contract during a split-operation each correspond to some mincut and, hence, $f_H(s,t) = f(s,t)$, where $f_H(s,t)$ is the value of a minimum $s$-$t$ mincut in graph $H$.

# Analysis

**Invariant [existence of representatives]:**
For any edge $\{S_i, S_j\}$ in $T$, there are vertices $a \in S_i$ and $b \in S_j$ such that $w(S_i, S_j) = f(a, b)$ and the cut defined by edge $\{S_i, S_j\}$ is a minimum $a$-$b$ cut in $G$.

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- Since by the invariant this edge induces an $s$-$t$ cut with capacity $f(x_j, x_{j+1})$ we get $f(s,t) \le f(x_j, x_{j+1}) = f_T(s,t)$.

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- Since, we can send a flow of value $f(x_j, x_{j+1})$ btw. $s$ and $t$, this is an $s$-$t$ mincut (cut property).

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After the split we have to choose representatives for all edges. For the new edge $\{S_i^a, S_i^b\}$ with capacity $w(S_i^a, S_i^b) = f_H(a, b)$ we can simply choose $a$ and $b$ as representatives.

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If $s \in S_i^a$ we can keep $x$ and $s$ as representatives.

Otherwise, we choose $x$ and $a$ as representatives. We need to show that $f(x, a) = f(x, s)$.

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The set $B$ forms a mincut separating $a$ from $b$. Contracting all nodes in this set gives a new graph $G'$ where the set $B$ is represented by node $v_B$. Because of Lemma 19 we know that $f'(x,a) = f(x,a)$ as $x, a \notin B$.

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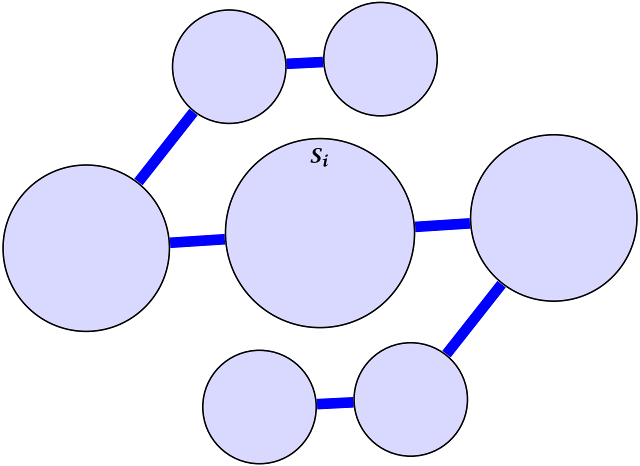
Since $s \in B$ we have $f'(v_B,x) \geq f(s,x)$.

Also, $f'(a,v_B) \geq f(a,b) \geq f(x,s)$ since the $a$-$b$ cut that splits $S_i$ into $S_i^a$ and $S_i^b$ also separates $s$ and $x$.

# Analysis



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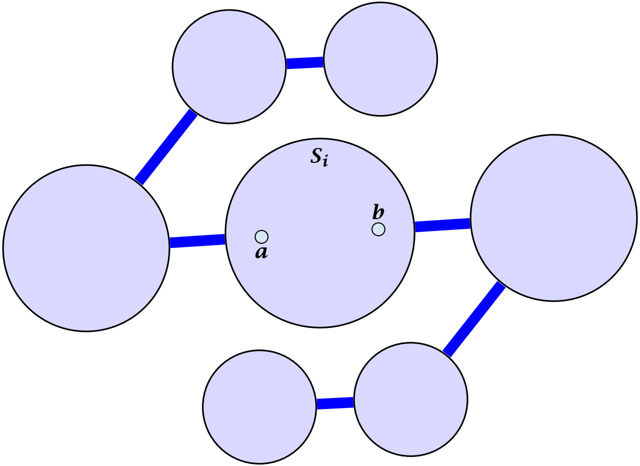

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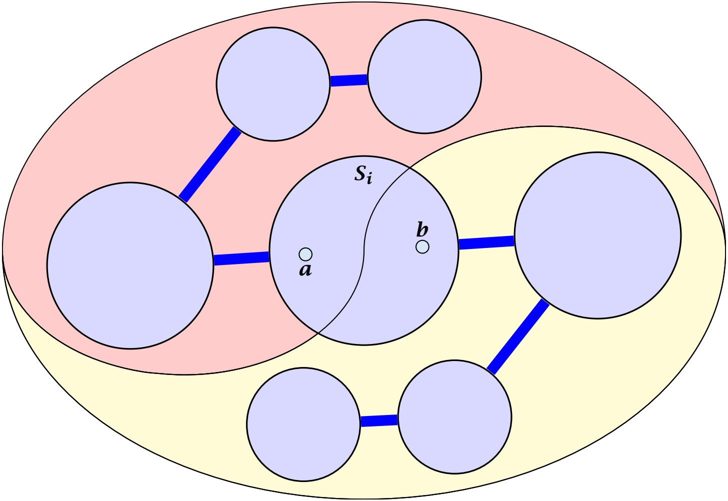

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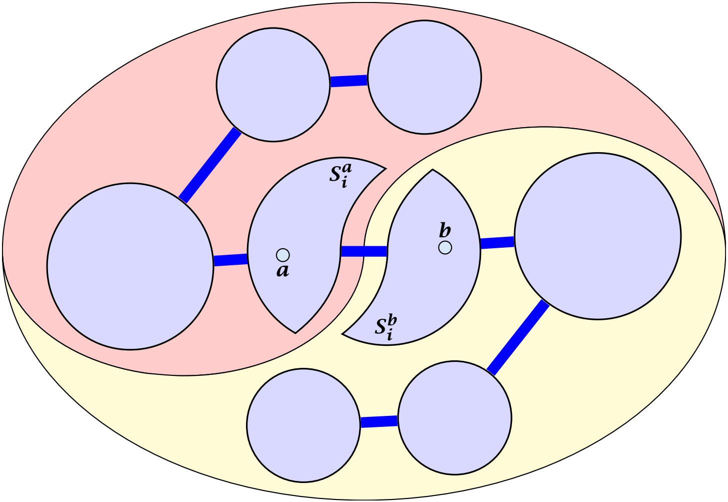

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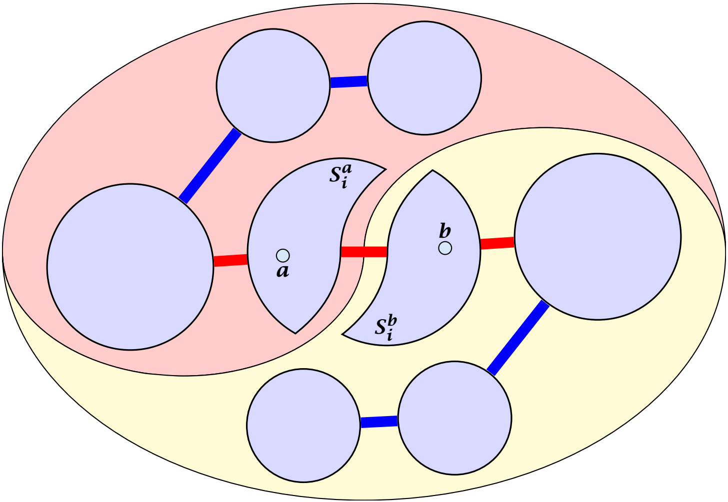

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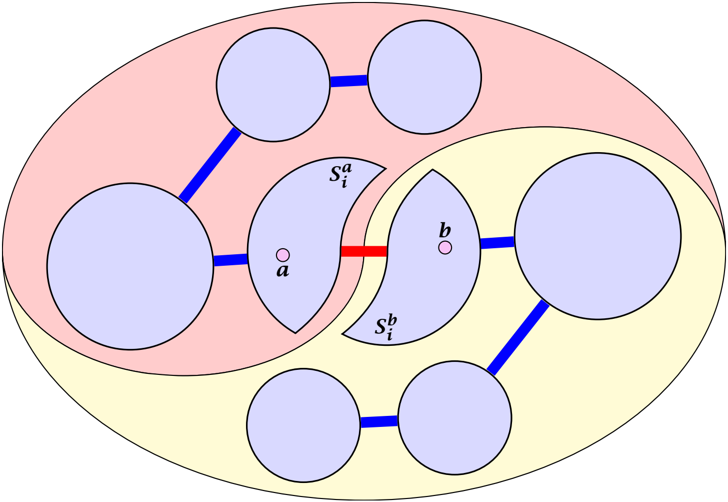

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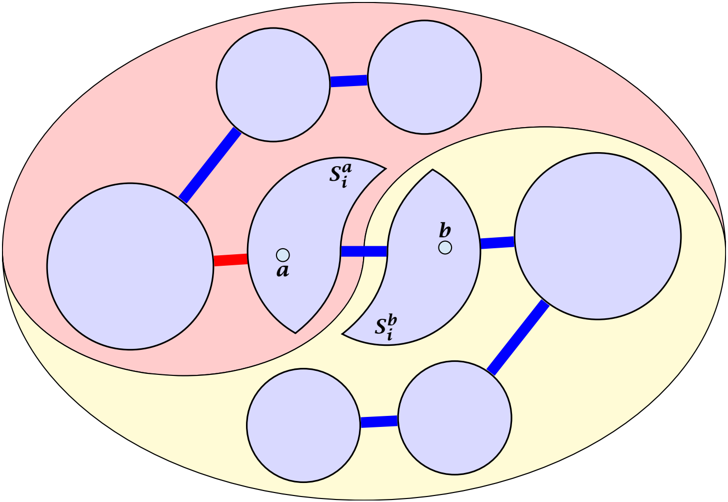

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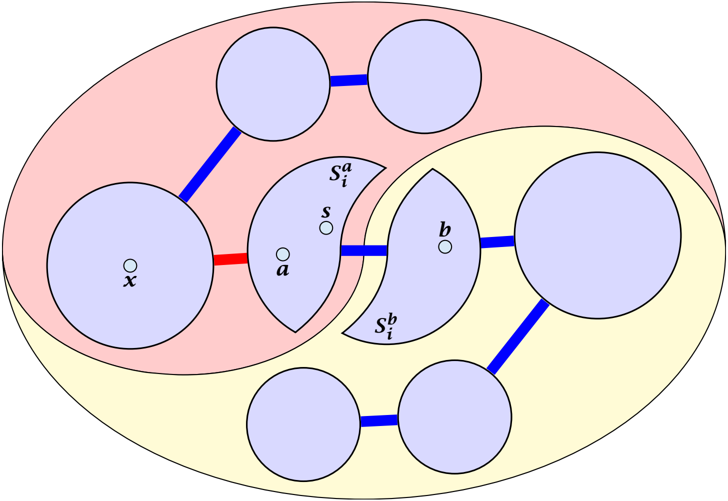

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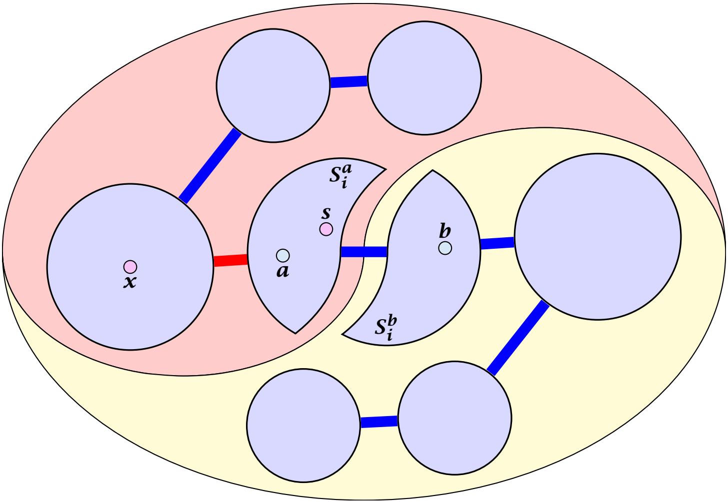

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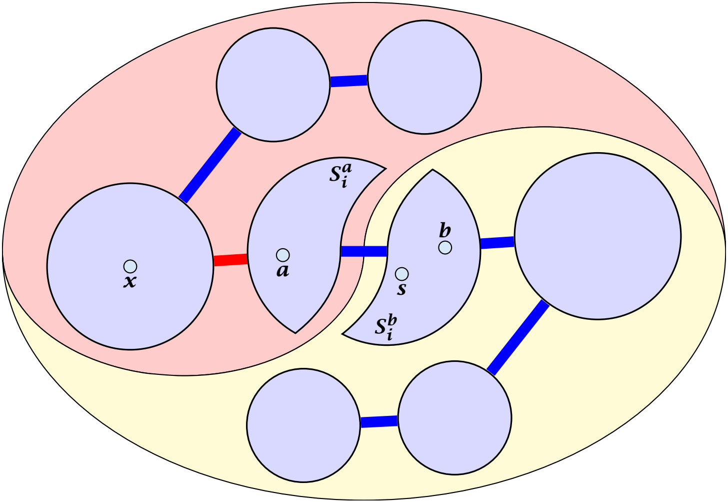

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