

12 – Dynamic Programming (2)

Matrix-chain Multiplication

Segmented Least Squares



Optimal substructure

Dynamic programming is typically applied to
optimization problems.

An optimal solution to the original problem contains
optimal solutions to smaller subproblems.

Matrix-chain multiplication

Given: Sequence (chain) $\langle A_1, A_2, \dots, A_n \rangle$ of matrices

Goal: Compute the product $A_1 \cdot A_2 \cdot \dots \cdot A_n$.

Problem: Parenthesize the product in a way that **minimizes the number of scalar multiplications.**

Definition: A product of matrices is *fully parenthesized* if it is either a **single matrix** or the product of two fully parenthesized matrix products, **surrounded by parentheses.**

Multiplying two matrices

$$A = (a_{ij})_{p \times q}, B = (b_{ij})_{q \times r}, A \cdot B = C = (c_{ij})_{p \times r},$$

$$c_{ij} = \sum_{k=1}^q a_{ik} b_{kj}.$$

Algorithm *Matrix-Mult*

Input: $(p \times q)$ matrix A , $(q \times r)$ matrix B

Output: $(p \times r)$ matrix $C = A \cdot B$

```
1 for  $i := 1$  to  $p$  do
2   for  $j := 1$  to  $r$  do
3      $C[i, j] := 0$ 
4     for  $k := 1$  to  $q$  do
5        $C[i, j] := C[i, j] + A[i, k] \cdot B[k, j]$ 
```

Number of multiplications and additions: $p \cdot q \cdot r$

Remark: Using this algorithm, multiplying two $(n \times n)$ matrices requires n^3 multiplications. This can also be done using $O(n^{2.376})$ multiplications.

Matrix-chain multiplication: Example

Computation of the product $A_1 A_2 A_3$, where

A_1 : (10 × 100) matrix

A_2 : (100 × 5) matrix

A_3 : (5 × 50) matrix

Parenthesization $((A_1 A_2) A_3)$ requires

$$A' = (A_1 A_2): 10 \cdot 100 \cdot 5 = 5000$$

$$A' A_3: 10 \cdot 5 \cdot 50 = 2500$$

Sum: 7500

Matrix-chain multiplication: Example

A_1 : (10×100) matrix

A_2 : (100×5) matrix

A_3 : (5×50) matrix

Parenthesization $(A_1 (A_2 A_3))$ requires

$$A'' = (A_2 A_3): 100 \cdot 5 \cdot 50 = 25000$$

$$A_1 A'': 10 \cdot 100 \cdot 50 = 50000$$

Sum: 75000

Fully parenthesized matrix products

All possible fully parenthesized matrix products of the chain $\langle A_1, A_2, A_3, A_4 \rangle$ are:

$$(A_1(A_2(A_3A_4)))$$

$$(A_1((A_2A_3)A_4))$$

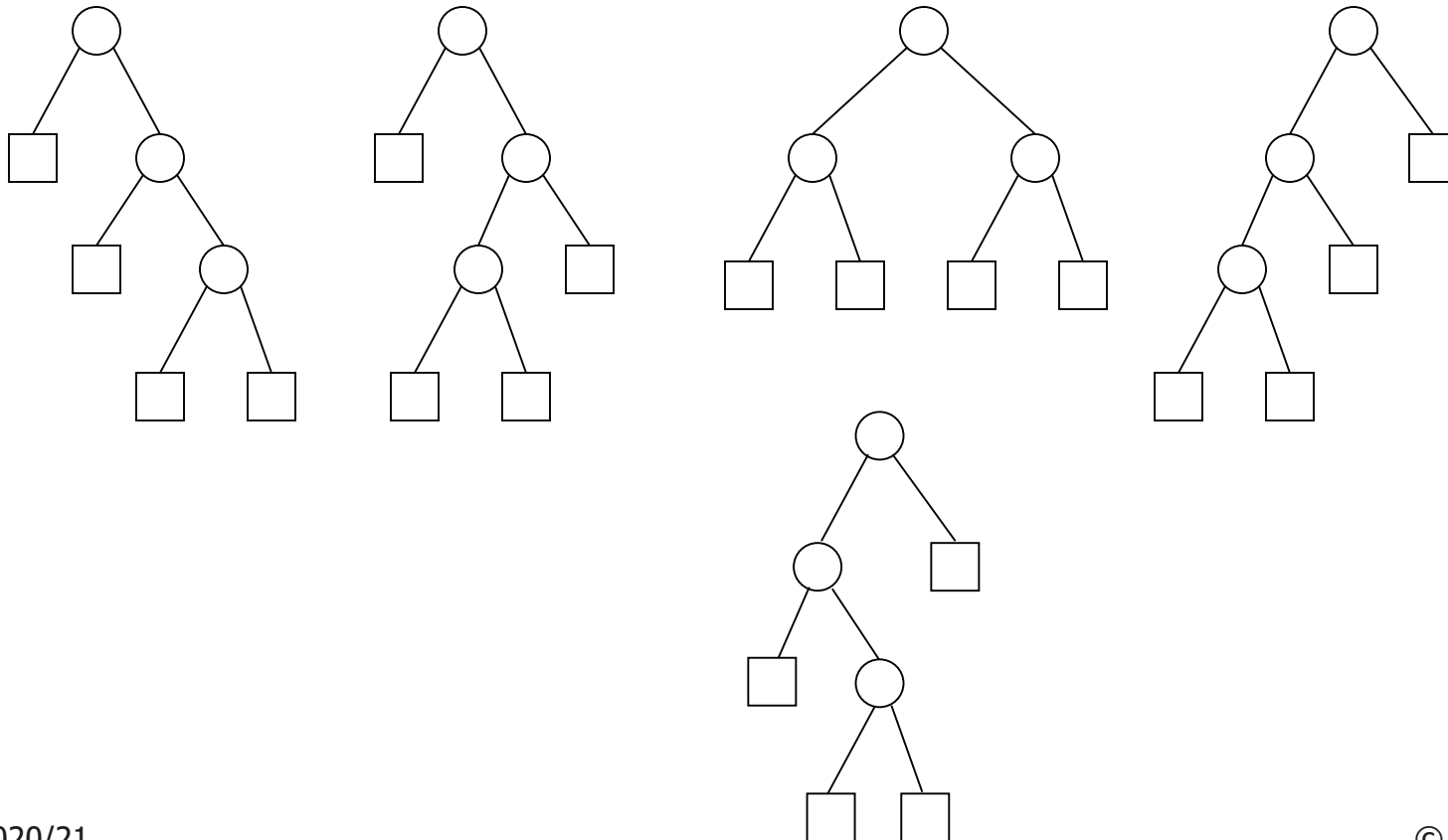
$$((A_1A_2)(A_3A_4))$$

$$((A_1(A_2A_3))A_4)$$

$$(((A_1A_2)A_3)A_4)$$

Number of different parenthesizations

Different parenthesizations correspond to different trees (selection).



Number of different parenthesizations

Let $P(n)$ be the number of alternative parenthesizations of the product $A_1 \cdots A_k A_{k+1} \cdots A_n$.

$$P(1) = 1$$

$$P(n) = \sum_{k=1}^{n-1} P(k)P(n-k) \quad \text{for } n \geq 2$$

$$P(n) = C_{n-1} \quad (n-1)\text{-st Catalan number}$$

$$C(n) = \frac{1}{n+1} \binom{2n}{n} \approx \frac{4^n}{n\sqrt{\pi n}} + O\left(\frac{4^n}{\sqrt{n^5}}\right)$$

Remark: Determining the optimal parenthesization by exhaustive search is not reasonable.

Matrix-chain multiplication

- Given:** Sequence $\langle A_1, A_2, \dots, A_n \rangle$ of matrices
Matrix A_i has dimension $p_{i-1} \times p_i$, for $i = 1, \dots, n$.
- Goal:** Parenthesize the product in a way that **minimizes the number of scalar multiplications**.

Structure of an optimal parenthesization

Subproblems $A_{i\dots j}$ $1 \leq i \leq j \leq n$

$$A_{i\dots j} = A_i \qquad A_{i\dots j} = (A_{i\dots k})(A_{k+1\dots j}) \quad i \leq k < j$$

Any optimal solution to the matrix-chain multiplication problem contains optimal solutions to subproblems. Determine an optimal solution recursively.

Let $m[i,j]$ be the **minimum number of operations** needed to compute the product $A_{i\dots j}$.

$$m[i,j] = 0 \quad \text{if } i = j$$

$$m[i,j] = \min_{i \leq k < j} \{m[i,k] + m[k+1,j] + p_{i-1}p_k p_j\} \quad \text{otherwise}$$

$s[i,j]$ = **optimal split value** k , i.e. the optimal parenthesization of $A_{i\dots j}$ splits the product between A_k and A_{k+1}

Recursive matrix-chain multiplication

Algorithm *RecMatChain*(p, i, j)

Input: Sequence $p = \langle p_0, p_1, \dots, p_n \rangle$,

where $(p_{i-1} \times p_i)$ is the dimension of matrix A_i

Invariant: *RecMatChain*(p, i, j) returns $m[i, j]$

```

1  if  $i = j$  then return 0;
2   $m[i, j] := \infty$ ;
3  for  $k := i$  to  $j - 1$  do
4       $m[i, j] := \min( m[i, j], p_{i-1} p_k p_j +$ 
                         $\text{RecMatChain}(p, i, k) +$ 
                         $\text{RecMatChain}(p, k+1, j) );$ 
5  return  $m[i, j]$ ;

```

Initial call: *RecMatChain*($p, 1, n$)

Recursive matrix-chain multiplication

Let $T(n)$ be the time taken by $\text{rec-mat-chain}(p, 1, n)$.

$$T(1) \geq 1$$

$$T(n) \geq 1 + \sum_{k=1}^{n-1} (T(k) + T(n-k) + 1)$$

$$\geq n + 2 \sum_{i=1}^{n-1} T(i)$$

$$\Rightarrow T(n) \geq 3^{n-1} \quad (\text{induction})$$

Exponential running time!

Solution using dynamic programming



Algorithm *DynMatChain*

Input: Sequence $p = \langle p_0, p_1, \dots, p_n \rangle$ ($p_{i-1} \times p_i$) the dimension of matrix A_i

Output: $m[1, n]$

```
1   $n := \text{length}(p) - 1;$ 
2  for  $i := 1$  to  $n$  do  $m[i, i] := 0;$ 
3  for  $l := 2$  to  $n$  do                                /*  $l = \text{length of the subproblem}$  */
4  for  $i := 1$  to  $n - l + 1$  do                          /*  $i$  is the left index */
5       $j := i + l - 1;$                                   /*  $j$  is the right index */
6       $m[i, j] := \infty;$ 
7      for  $k := i$  to  $j - 1$  do
8           $m[i, j] := \min( m[i, j], p_{i-1} p_k p_j + m[i, k] + m[k + 1, j] );$ 
9  return  $m[1, n];$ 
```

Example

$$A_1 \quad (30 \times 35)$$

$$A_4 \quad (5 \times 10)$$

$$A_2 \quad (35 \times 15)$$

$$A_5 \quad (10 \times 20)$$

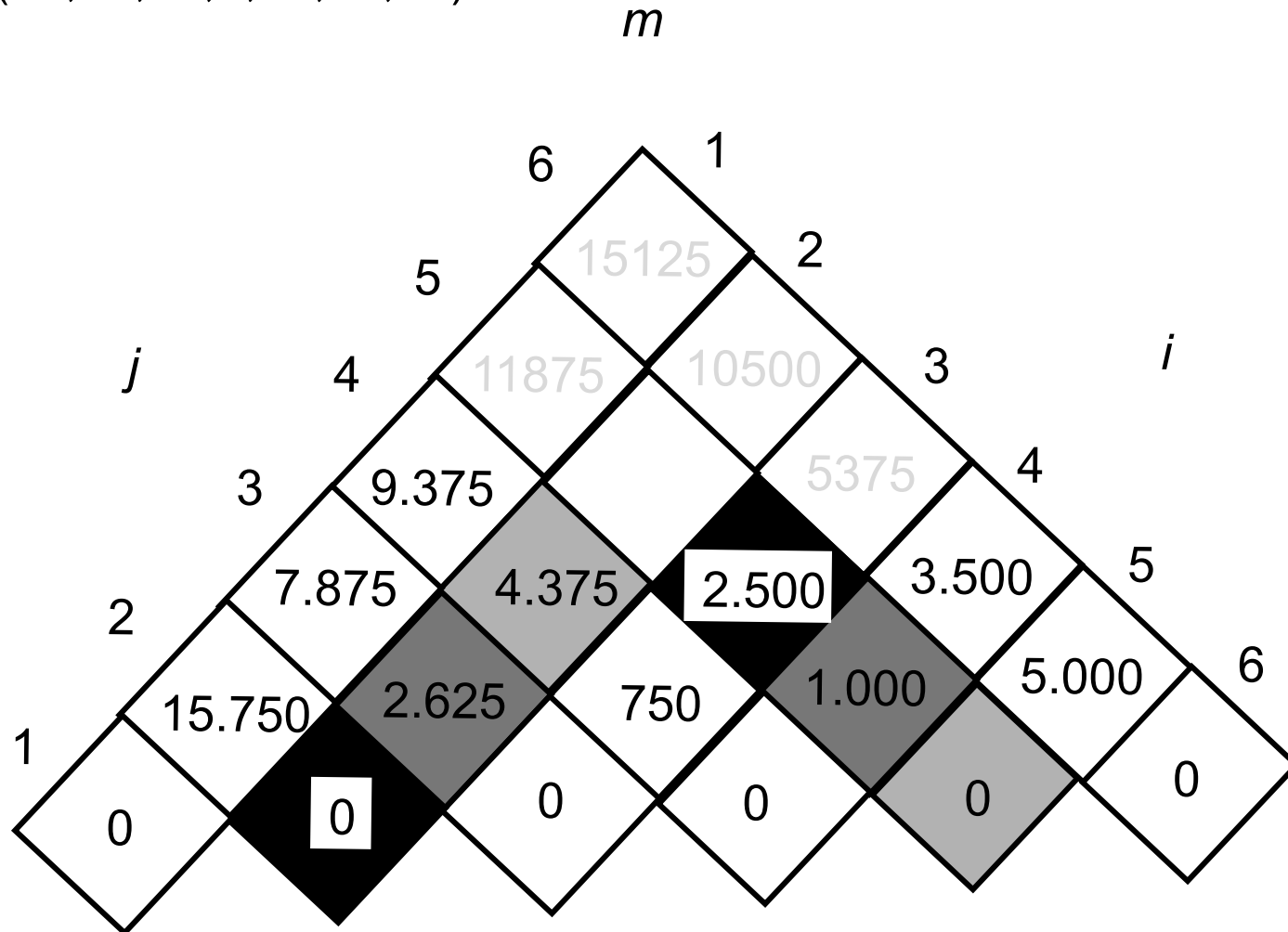
$$A_3 \quad (15 \times 5)$$

$$A_6 \quad (20 \times 25)$$

$$p = (30, 35, 15, 5, 10, 20, 25)$$

Example

$$p = (30, 35, 15, 5, 10, 20, 25)$$



Example



$$\begin{aligned} m[2,5] &= \min_{2 \leq k < 5} (m[2,k] + m[k+1,5] + p_1 p_k p_5) \\ &= \min \begin{cases} m[2,2] + m[3,5] + p_1 p_2 p_5 \\ m[2,3] + m[4,5] + p_1 p_3 p_5 \\ m[2,4] + m[5,5] + p_1 p_4 p_5 \end{cases} \\ &= \min \begin{cases} 0 + 2500 + 35 \cdot 15 \cdot 20 = 13000 \\ 2625 + 1000 + 35 \cdot 5 \cdot 20 = 7125 \\ 4375 + 0 + 35 \cdot 10 \cdot 20 = 11375 \end{cases} \\ &= 7125 \end{aligned}$$

Including optimal split values

Algorithm *DynMatChain*(p)

Input: Sequence $p = \langle p_0, p_1, \dots, p_n \rangle$, $(p_{i-1} \times p_i)$ the dimension of matrix A_i

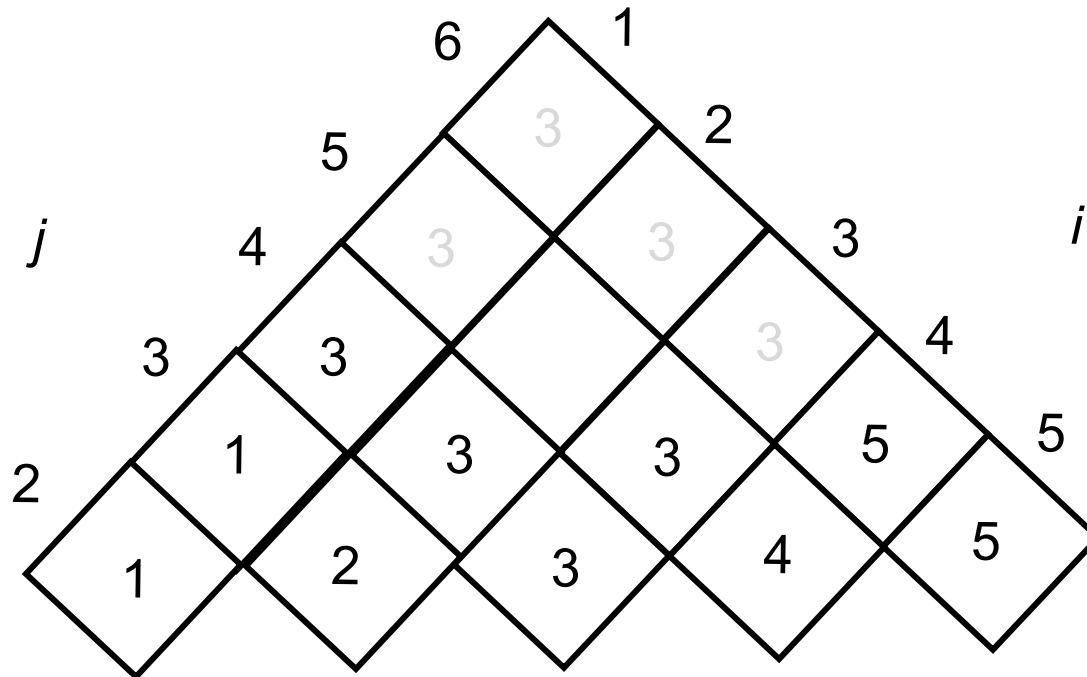
Output: $m[1, n]$ and a matrix $s[i, j]$ containing the optimal split values

```

1   $n := \text{length}(p) - 1;$ 
2  for  $i := 1$  to  $n$  do  $m[i, i] := 0$ 
3  for  $l := 2$  to  $n$  do
4    for  $i := 1$  to  $n - l + 1$  do
5       $j := i + l - 1;$ 
6       $m[i, j] := \infty;$ 
7      for  $k := i$  to  $j - 1$  do
8         $q := m[i, j];$ 
9         $m[i, j] := \min( m[i, j], p_{i-1} p_k p_j + m[i, k] + m[k + 1, j] );$ 
10       if  $m[i, j] < q$  then  $s[i, j] := k;$ 
11  return  $(m[1, n], s);$ 

```

Example of splitting values



Computation of an optimal parenthesization

Algorithm *OptParenthesization*

Input: Chain A of matrices, matrix s containing the optimal split values, two indices i and j

Output: An optimal parenthesization of $A_{i\dots j}$

```
1  if  $i < j$ 
2    then  $X := \text{OptParenthesization}(A, s, i, s[i, j]);$ 
3          $Y := \text{OptParenthesization}(A, s, s[i, j] + 1, j);$ 
4         return  $(X \cdot Y);$ 
5  else return  $A_i;$ 
```

Initial call: $\text{OptParenthesization}(A, s, 1, n)$

„*Memoization*“ for increasing the efficiency of a recursive solution:

Only the *first time* a subproblem is encountered, its **solution is computed** and then stored in a table. Each subsequent time that the subproblem is encountered, the value stored in the table is simply looked up and returned (without repeated computation!).

Memoized matrix-chain multiplication

Algorithm *MemMatChain*(p, i, j)

Invariant: *MemMatChain*(p, i, j) returns $m[i, j]$;
the value is correct if $m[i, j] < \infty$

```
1 if  $i = j$  then return 0;  
2 if  $m[i, j] < \infty$  then return  $m[i, j]$ ;  
3 for  $k := i$  to  $j - 1$  do  
4    $m[i, j] := \min( m[i, j], p_{i-1} p_k p_j +$   
      MemMatChain( $p, i, k$ ) +  
      MemMatChain( $p, k + 1, j$ ) );  
5 return  $m[i, j]$ ;
```

Memoized matrix-chain multiplication

Call:

```
1  $n := \text{length}(p) - 1;$   
2 for  $i := 1$  to  $n$  do  
3   for  $j := 1$  to  $n$  do  
4      $m[i, j] := \infty;$   
5 MemMatChain( $p, 1, n$ );
```

The computation of all entries $m[i, j]$ using *MemMatChain* takes $O(n^3)$ time.

$O(n^2)$ entries.

Each entry $m[i, j]$ is computed once.

Each entry $m[i, j]$ is looked up during the computation of $m[i', j']$ if $i' = i$ and $j' > j$ or $j' = j$ and $i' < i$.

Thus $m[i, j]$ is looked up during the computation of at most $2n$ entries.

1. There is an algorithm that determines an optimal parenthesization in time $O(n \log n)$.
2. There is a linear time algorithm that determines a parenthesization using at most $1.155 \cdot M_{opt}$ multiplications.

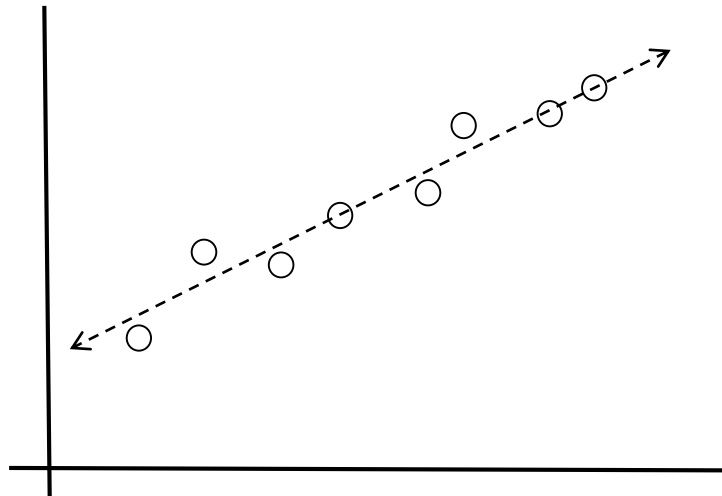
Segmented Least Squares

Problem in statistics and numerical analysis

$$P = \{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)\} \quad x_1 < x_2 < \dots < x_n$$

Find $L = ax + b$ that minimizes the error of L w.r.t. P .

$$\text{Error}(L, P) = \sum_{1 \leq i \leq n} (y_i - ax_i - b)^2$$



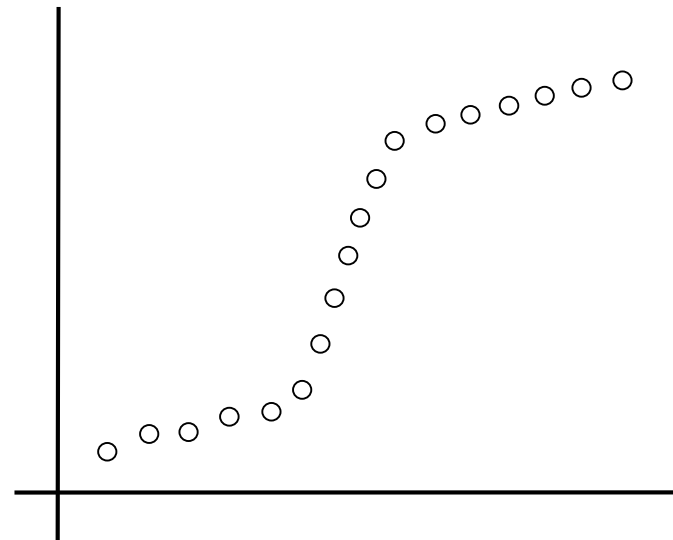
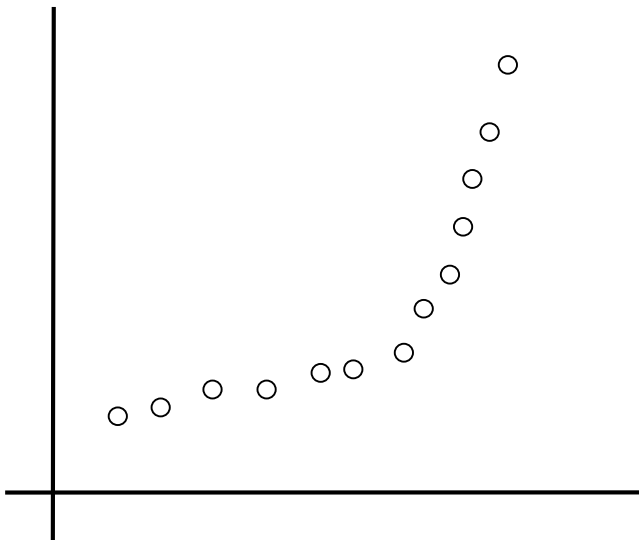
Solution

Line $L = ax+b$ where

$$a = \frac{n \sum_i x_i y_i - \left(\sum_i x_i \right) \left(\sum_i y_i \right)}{n \sum_i x_i^2 - \left(\sum_i x_i \right)^2}$$

$$b = \frac{\sum_i y_i - a \sum_i x_i}{n}$$

One line might not suffice



Segmented Least Squares

Problem: $P = \{p_1, p_2, \dots, p_n\}$ where
 $p_i = (x_i, y_i)$ for $i = 1, \dots, n$ $x_1 < x_2 < \dots < x_n$

A **segment** is a subset $\{p_i, \dots, p_j\}$ where $i \leq j$

Partition P into segments. **Penalty** is the sum of

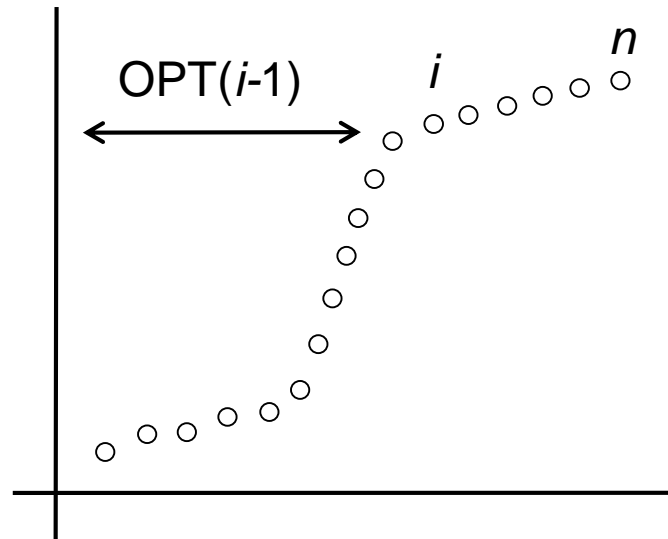
- $C \cdot \#$ segments where $C > 0$
- For each segment, the error of the optimum line through it.

Goal: Find partition of minimum penalty.

Dynamic programming approach

Last segment is $\{p_i, \dots, p_n\}$ for some $1 \leq i \leq n$.

The remaining segments are an optimal solution for $\{p_1, \dots, p_{i-1}\}$.



Dynamic programming approach

$\text{OPT}(j)$ = value of an optimal solution for $\{p_1, \dots, p_j\}$ $1 \leq j \leq n$

$\text{OPT}(0) := 0$

$e_{i,j}$ = minimum error of any line through $\{p_i, \dots, p_j\}$ $1 \leq i \leq j \leq n$

$$\text{OPT}(n) = \min_{1 \leq i \leq n} (e_{i,n} + C + \text{OPT}(i-1))$$

$$\text{OPT}(j) = \min_{1 \leq i \leq j} (e_{i,j} + C + \text{OPT}(i-1))$$

Dynamic programming algorithm

Array $m[0..n]$ contains the values of the optimal solutions.

Algorithm SegmentedLeastSquares(n)

```
1   $m[0] := 0;$ 
2  for all pairs  $i \leq j$  do
3      Compute  $e_{i,j}$  for segment  $\{p_i, \dots, p_j\};$ 
4  for  $j := 1$  to  $n$  do
5       $m[j] := \min_{1 \leq i \leq j} (e_{i,j} + C + m[i-1]);$ 
```

Running time: $O(n^2)$

Computing a solution

Algorithm FindSegments(j)

- 1 **if** $j = 0$ **then**
- 2 Output nothing;
- 3 **else**
- 4 Find an i that minimizes $e_{i,j} + C + m[i-1]$;
- 5 Output segment $\{p_i, \dots, p_j\}$ and the result of FindSegments($i-1$);
- 6 **endif**;