11 – Dynamic Programming (1)

Introduction

Weighted Interval Scheduling

Outline

- General approach, differences to a recursive solution
- Basic example: Computation of the Fibonacci numbers
- Weighted interval scheduling

Method of dynamic programming

Recursive approach: Solve a problem by solving several smaller analogous subproblems of the same type. Then combine these solutions to generate a solution to the original problem.

Drawback: Repeated computation of solutions

Dynamic-programming method: Once a subproblem has been solved, store its solution in a table so that it can be retrieved later by simple table lookup.

Example: Fibonacci numbers

$f(0) = 0$
$f(1) = 1$
$f(n) = f(n-1) + f(n-2)$, for $n \geq 2$

Remark:

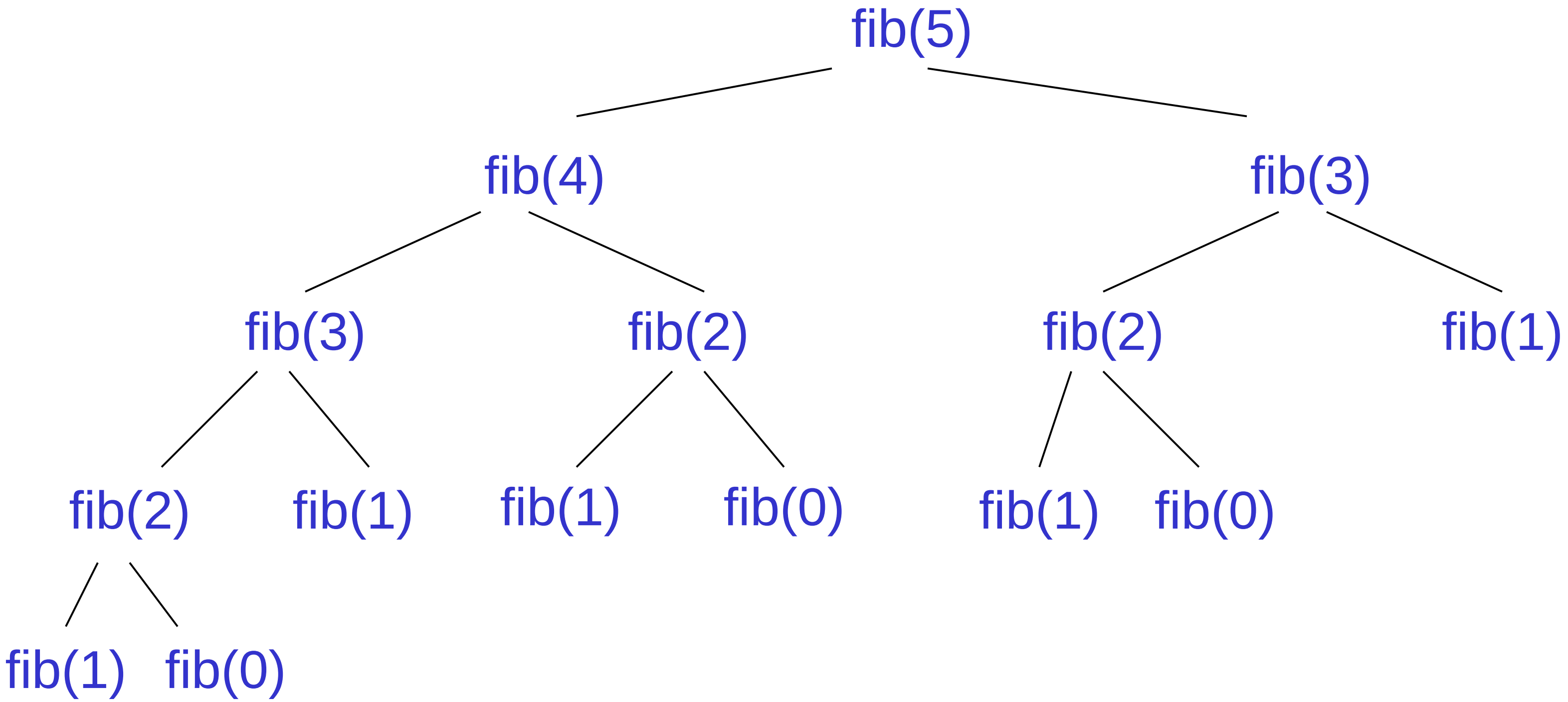
$$f(n) = \left\lfloor \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n \right\rceil$$

Straightforward implementation:

procedure *fib* (*n* : *integer*) : *integer*
if (*n* = 0) or (*n* =1)
    then return *n*;
    else return *fib*(*n* – 1) + *fib*(*n* – 2);

Fibonacci numbers

Recursion tree:



Repeated computation!

Dynamic programming

Approach:

1. Recursively define problem P.

2. Determine a set T consisting of all subproblems that have to be solved during the computation of a solution to P.

3. Find an order $T_0, ..., T_k$ of the subproblems in T such that during the computation of a solution to T_i only subproblems T_j with $j < i$ arise.

4. Solve $T_0,...,T_k$ in this order and store the solutions.

Fibonacci numbers

1. Recursive definition of the Fibonacci numbers, based on the standard equation.

2. $T = \{f(0),\dots,f(n)\}$

3. $T_i = f(i), \quad i = 0,\dots,n$

4. Computation of $fib(i)$, for $i \geq 2$, only requires the results of the last two subproblems $fib(i-1)$ and $fib(i-2)$.

Fibonacci numbers

Computation by dynamic programming, version 1:

```
procedure fib(n : integer) : integer
F[0] := 0; F[1] := 1;
for k := 2 to n do
        F[k] := F[k-1] + F[k-2];
return F[n];
```

Fibonacci numbers

Computation by dynamic programming, version 2:

procedure *fib* (*n* : *integer*) : *integer*

F(*secondlast*) := 0; *F*(*last*) :=1;

for *k* := 2 **to** *n* **do**

        F(*current*) := *F*(*last*) + *F*(*secondlast*);

        F(*secondlast*) := *F*(*last*);

        F(*last*) := *F*(*current*);

if $n \leq 1$ **then return** *n* **else return** *F*(*current*);

Linear running time, constant space requirement!

Recursive computation using memoization

Compute each number exactly once, store it in an array $F[0...n]$:

procedure *fib* (n : *integer*) : *integer*
$F[0] := 0$; $F[1] := 1$;
for $i := 2$ **to** n **do**
    $F[i] := \infty$;
return *lookupfib*(n);

The procedure *lookupfib* is defined as follows:

procedure *lookupfib*(k : *integer*) : *integer*
if $F[k] < \infty$
    then return $F[k]$;
    else $F[k] :=$ *lookupfib*($k - 1$) + *lookupfib*($k - 2$);
        return $F[k]$;

Weighted interval scheduling

Problem: Set $S = \{1,\ldots,n\}$ of n requests for a resource.

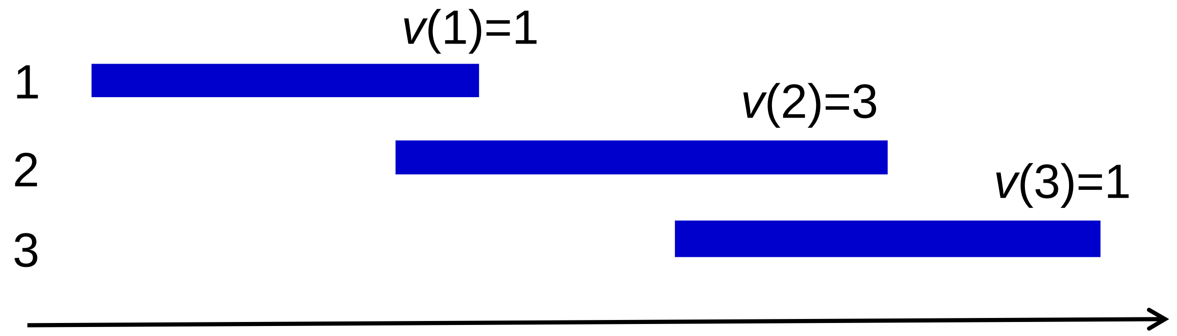
Request i: $[s(i), f(i))$ $s(i)$ = start time $f(i)$ = finish time

$v(i)$ = value/weight

Two requests are compatible if they do not overlap.

Goal: Select $S \subseteq \{1,\ldots,n\}$ of mutually compatible requests so as to maximize $\Sigma_{i \in S}\, v(i)$.

Greedy* (Earliest Deadline First) is not optimal.



Predecessor function

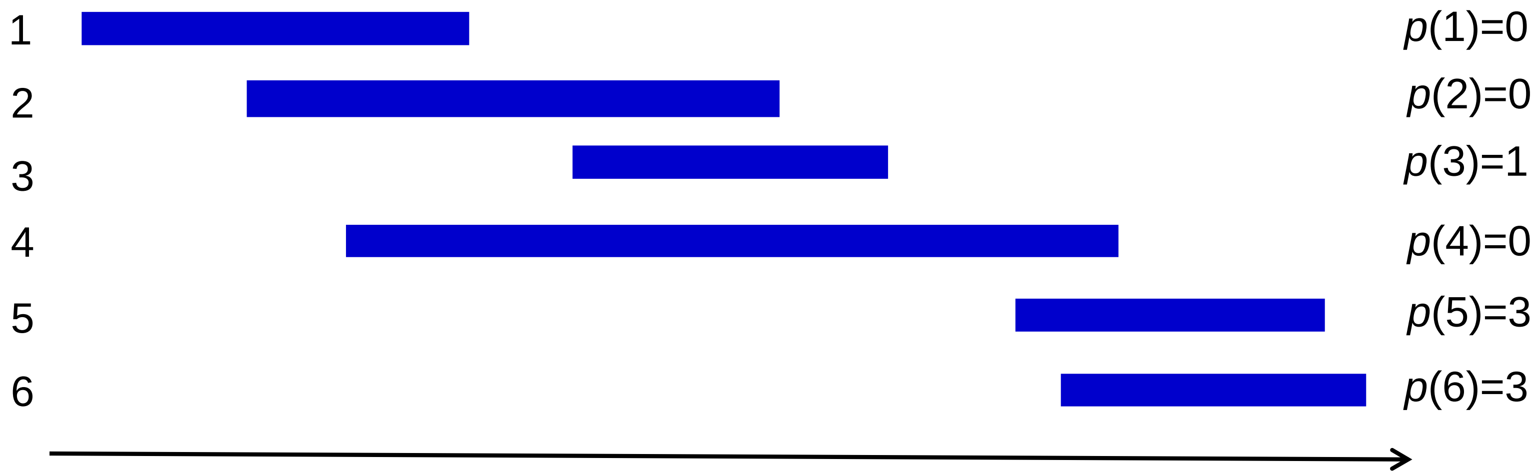
In the following, requests are numbered such that

$$f(1) \leq f(2) \leq f(3) \leq \ldots \leq f(n).$$

For j =1,…,n

$p(j)$ = largest $i<j$ such that requests i and j do not overlap

$p(j)$ = 0 if no request $i<j$ is disjoint from j



Dynamic programming approach

O = optimal subset of requests

- $n \notin O$: O is an optimal subset of $\{1,\ldots,n-1\}$
- $n \in O$: remaining requests in O are an optimal subset of $\{1,\ldots,p(n)\}$

For $j = 1,\ldots,n$

O_j = optimal subset of requests from $\{1,\ldots,j\}$

OPT(j) = value of an optimal solution   OPT(0) := 0

- $j \notin O_j$: O_j is an optimal subset of $\{1,\ldots,j-1\}$
- $j \in O_j$: remaining requests in O_j are an optimal subset of $\{1,\ldots,p(j)\}$

Dynamic programming approach

For $j = 1,\ldots,n$

O_j = optimal subset of requests from $\{1,\ldots,j\}$

OPT(j) = value of an optimal solution   OPT(0) := 0

- $j \notin O_j$: O_j is an optimal subset of $\{1,\ldots,j\text{-}1\}$
- $j \in O_j$: remaining requests in O_j are an optimal subset of $\{1,\ldots,p(j)\}$

$$\text{OPT}(j) = \max\{\, v(j) + \text{OPT}(p(j))\,,\ \text{OPT}(j\text{-}1)\,\}$$

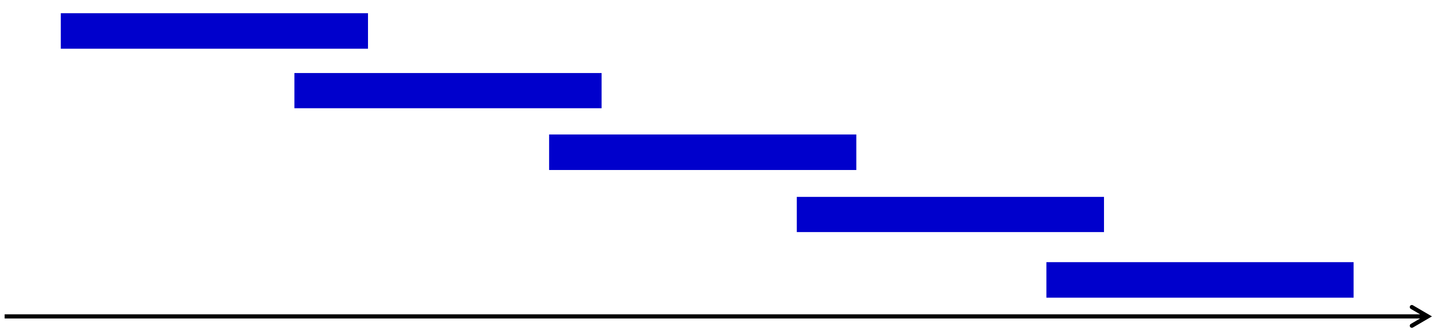
Request j belongs to an optimal solution for $\{1,\ldots,j\}$ if and only if

$$v(j) + \text{OPT}(p(j)) \geq \text{OPT}(j\text{-}1).$$

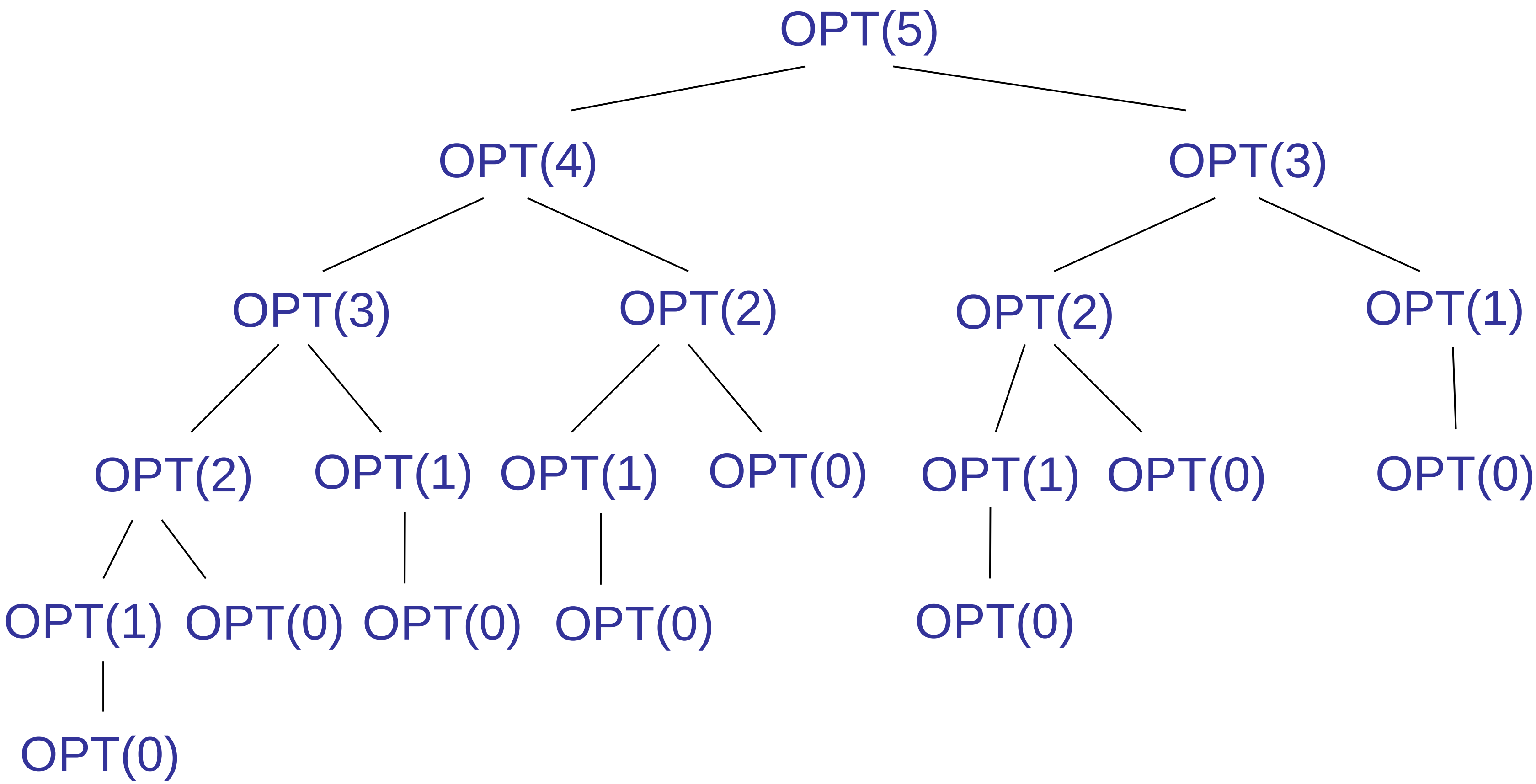
Straightforward implementation

Assume that values $p(j)$, for $j=1,\ldots,n$, have been computed.

procedure ComputeOpt(j : *integer*)
if $j = 0$
  then return 0;
  else return $\max\{v(j) + \mathrm{OPT}(p(j)) , \mathrm{OPT}(j-1)\}$;



Instance taking exponential time



Iterative solution

Array M[0..*n*] contains the values of the optimal solutions.

procedure ComputeOpt(*n* : *integer*)

M[0] := 0;

for *j* := 1 **to** *n* **do**

    M[*j*] := max{*v*(*j*) + M[*p*(*j*)] , M[*j*-1]};

endfor;

Running time: O(*n*)

Recursion using memoization

```
procedure ComputeOpt(j : integer)
if j = 0 then
    return 0;
else if M[j] is not empty then
    return M[j];
else
    M[j] := max{v(j) + ComputeOPT(p(j)) , ComputeOpt(j-1)};
    return M[j];
endif;
```

Proposition: The running time of ComputeOpt(n) is O(n) if the requests are sorted in order of non-decreasing finish times and the values $p(j)$, $1 \le j \le n$, are computed.

Proof: The running time is a constant times the number of recursive calls to ComputeOpt. Two calls are issued whenever a new array entry is filled. Hence there are a total of at most $2n$ calls.

Computing a solution

procedure FindSolution(j : *integer*)

```
if j = 0 then
    Output nothing;
else if v(j) + M[p(j)] ≥ M[j-1] then
    Output j together with the result of FindSolution(p(j));
else
    Output the result of FindSolution(j-1);
endif;
```

FindSolution calls itself only on strictly smaller values. Therefore FindSolution(n) issues less than n recursive calls and the running time is $O(n)$.