Technische Universität München
Fakultät für Informatik
Lehrstuhl für Algorithmen und Komplexität
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Assignment 12
January 13, 2020

Efficient Algorithms and Data Structures I

*Deadline: January 20, 10:15 am in the **Efficient Algorithms** mailbox.*

Homework 1 (6 Points)

Archibald Anderson must coordinate the traffic of football fans from Munich central to the football stadium. He studies the effect of road blocks on the ability to transport the fans. He models the situation using a flow network. His manager asks him to determine the *most vital edge*, whose deletion causes the largest decrease in the maximum flow value.

Let $f : E \to \mathbb{N}$ be a maximum flow in the network. Either prove or disprove (using a counterexample) the following claims

1. A most vital edge is an edge with maximum capacity.
2. A most vital edge is an edge with maximum value of $f(e)$.
3. A most vital edge is an edge whose flow $f(e)$ equals the maximum value of $f(e')$ among edges e' belonging to some minimum cut.
4. An edge that does not belong to some minimum cut cannot be a most vital edge.
5. A network might contain several most vital edges.

Homework 2 (4 Points)

Let f be a flow in a network, and let α be a real number. The scalar flow product, denoted $\alpha \cdot f$, is a function from $V \times V$ to R defined by

$$(\alpha \cdot f)(u,v) = \alpha \cdot f(u,v)$$

Prove that the feasible flows in a network form a convex set. That is, show that if f_1 and f_2 are flows, then so is $\alpha f_1 + (1-\alpha)f_2$ for $0 \leq \alpha \leq 1$.

Homework 3 (4 Points)

Prove or disprove the following statements.

(a) There is a constant $c > 1$ so that for any $n \geq 4$ there exists a network on n nodes with a single maximum flow and at least c^n minimum cuts.

(b) There is a constant $c > 1$ so that for any $n \geq 4$ there exists a network on n nodes with a single minimum cut and at least c^n integral maximum flows.

Homework 4 (6 Points)

The department has s students $S_1,\dots,S_s$ looking for a thesis topic. There are t open thesis topics $T_1,\dots,T_t$. For each thesis topic, there is at least one student qualified for it and each student is qualified for at least one topic. Each topic is offered by exactly one of the r research groups $R_1,\dots,R_r$.
Each student must pick a thesis topic he is qualified for. No two students may pick the same topic and research group i can take care of at most u_i students.
Given full information about students, topics and research groups, is there a way to distribute the thesis topics?

(a) Show how to formulate the above problem as a Maximum Flow Problem. Explain the different elements of your construction.

(b) Given an integral maximum flow in your network, show how to determine whether a distribution scheme as desired is possible. Also explain how to obtain a valid distribution scheme, if one exists from a maximum flow in your network. Finally, show that given distribution scheme, you can construct a corresponding flow in the network.

Tutorial Exercise 1

The edge connectivity of an undirected graph is the minimum number of edges that must be removed to disconnect the graph.

(a) Show that the edge connectivity of any n-node graph is at most $n-1$.

(b) Show how to determine the edge connectivity of an undirected graph G of n vertices by running a black-box maximum-flow algorithm on at most n flow networks.

Tutorial Exercise 2

Consider a 0-1 matrix A with n rows and m columns. We refer to a row or a column of the matrix A as a line. We say that a set of 1's in the matrix A is independent if no two of them appear in the same line. We also say that a set of lines in the matrix is a *cover* of A if they cover all the 1's in the matrix. Using the max-flow min-cut theorem, show that the maximum number of independent 1's equals the minimum number of lines in a cover.

Tutorial Exercise 3

Show that there always exists a sequence of at most m augmenting paths that compute the maximum flow in a network of m edges.

> Shortly after the "iron curtain" fell in 1990, an American and a Russian, who had both worked on the development of weapons, met. The American asked: "When you developed the Bomb, how were you able to perform such an enormous amount of computing with your weak computers?" The Russian responded: "We used better algorithms."
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> – Y. Dinitz