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## Lemma 6

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- ▶ We can find the shortest augmenting paths in time  $\mathcal{O}(m)$  via BFS.
- ▶  $\mathcal{O}(m)$  augmentations for paths of exactly  $k < n$  edges.



## Shortest Augmenting Paths

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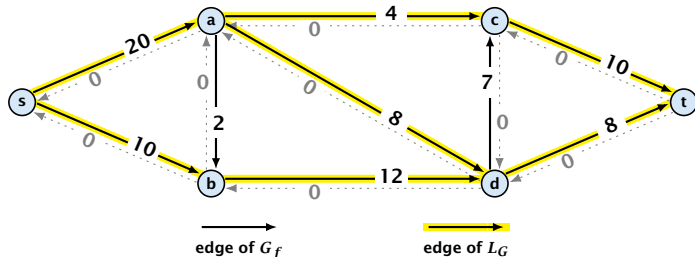
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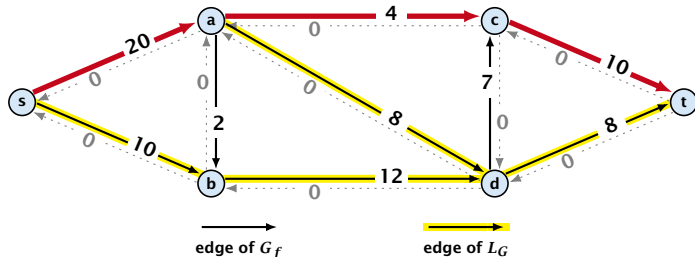


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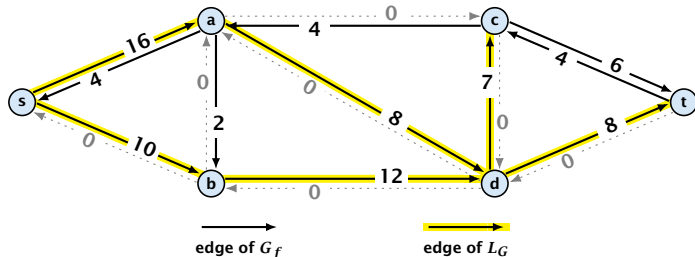


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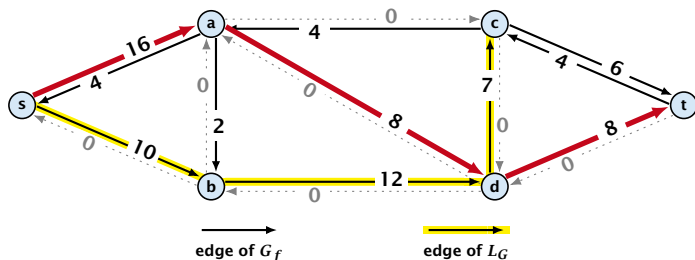


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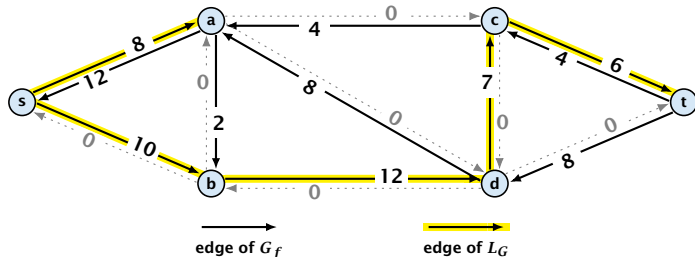


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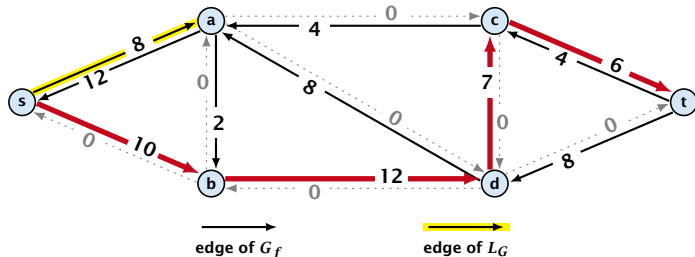


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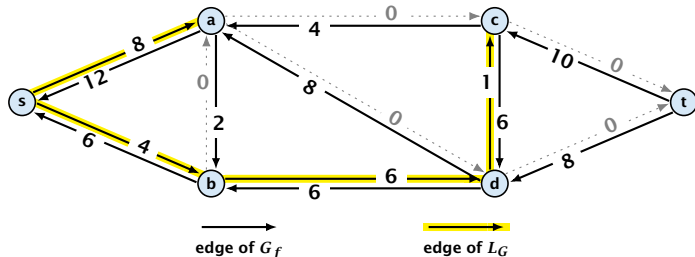


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In the following we assume that the residual graph  $G_f$  does not contain zero capacity edges.

This means, we construct it in the usual sense and then delete edges of zero capacity.

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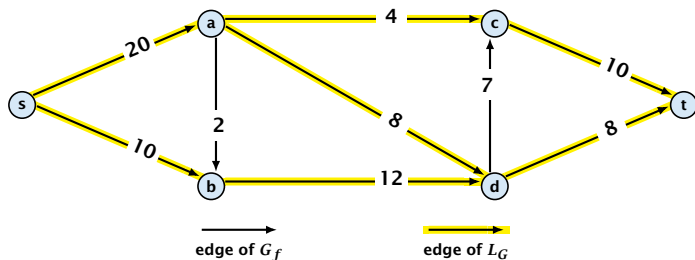
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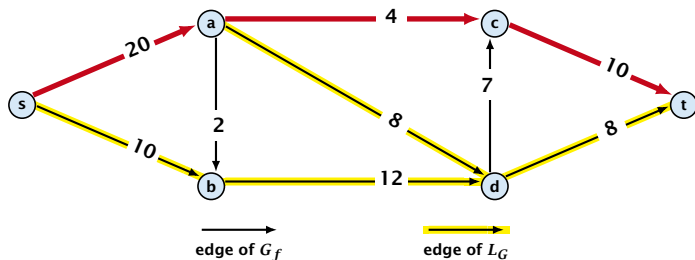
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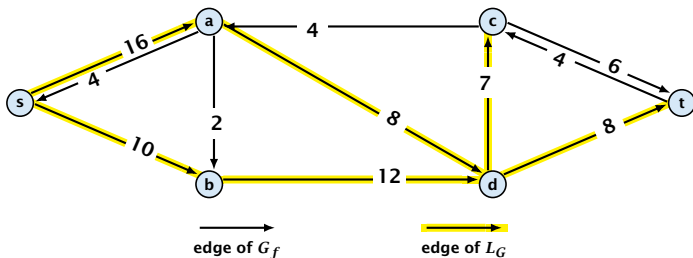
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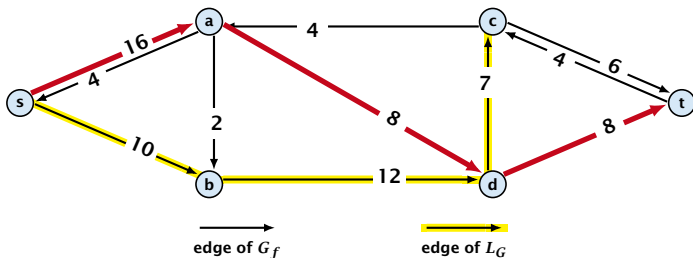
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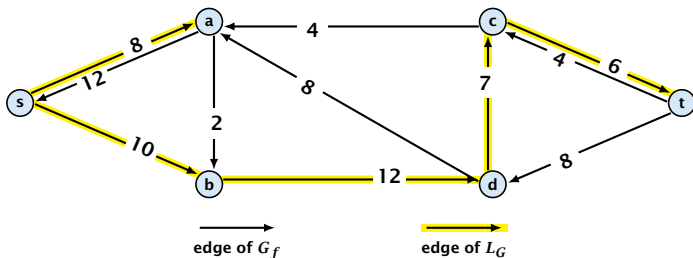
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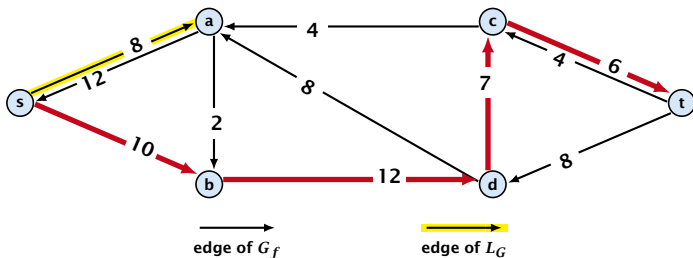
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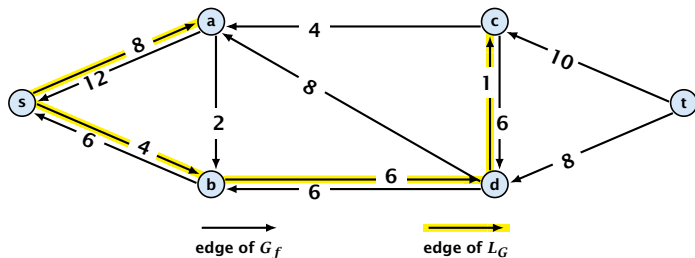
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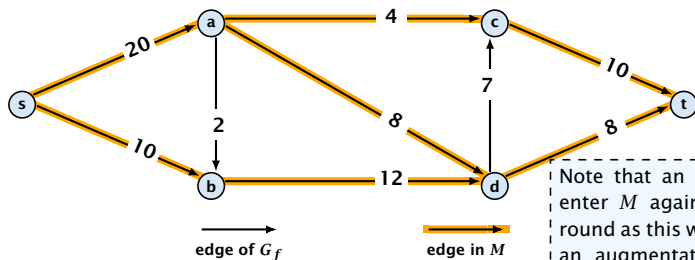
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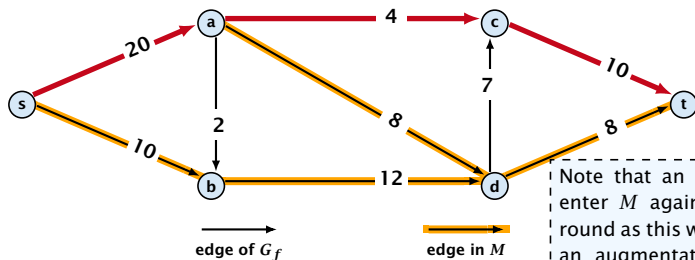
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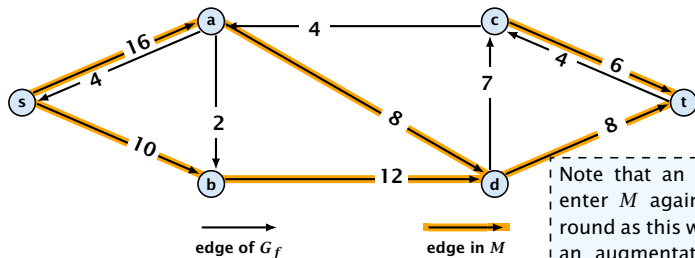
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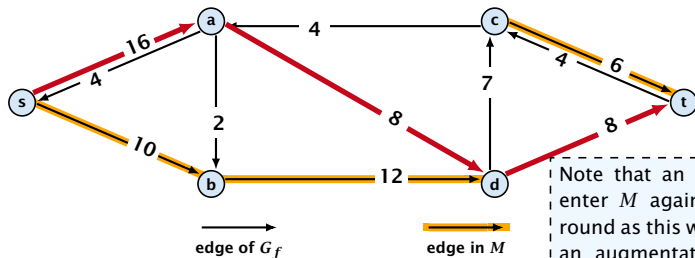
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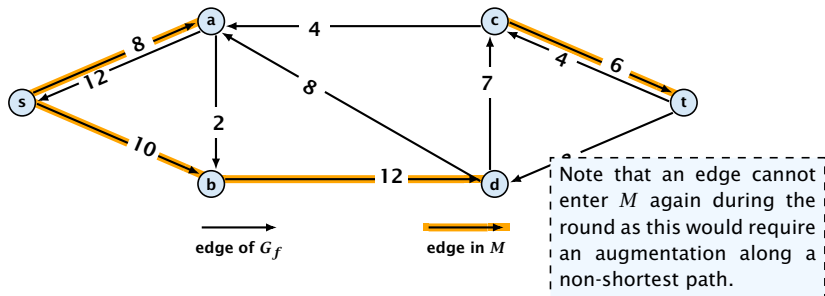
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### **Note:**

There always exists a set of  $m$  augmentations that gives a maximum flow (why?).



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However, we can improve the running time to  $\mathcal{O}(mn^2)$  by improving the running time for finding an augmenting path (currently we assume  $\mathcal{O}(m)$  per augmentation for this).

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Note that  $M$  is not the set of edges of the level graph but a subset of level-graph edges.

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You can delete incoming edges of  $v$  from  $M$ .

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There are at most  $n$  phases. Hence, total cost is  $\mathcal{O}(mn^2)$ .