How to choose augmenting paths?

- We need to find paths efficiently.
- We want to guarantee a small number of iterations.

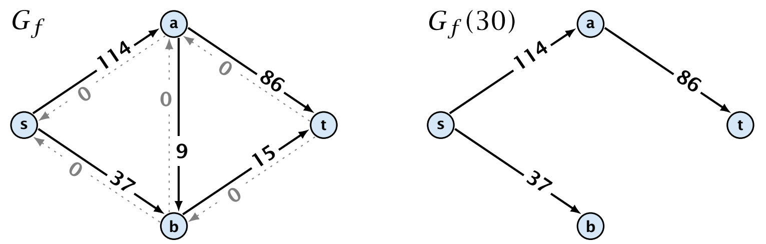
Several possibilities:

- Choose path with maximum bottleneck capacity.
- Choose path with sufficiently large bottleneck capacity.
- Choose the shortest augmenting path.

Capacity Scaling

Intuition:

- Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.
- Don't worry about finding the exact bottleneck.
- Maintain scaling parameter Δ.
- $G_f(\Delta)$ is a sub-graph of the residual graph G_f that contains only edges with capacity at least Δ.



Capacity Scaling

Algorithm 1 maxflow(G, s, t, c)

foreach $e \in E$ **do** $f_e \leftarrow 0$;
$\Delta \leftarrow 2^{\lceil \log_2 C \rceil}$
while $\Delta \geq 1$ **do**
  $G_f(\Delta) \leftarrow \Delta$-residual graph
  while there is augmenting path P in $G_f(\Delta)$ **do**
    $f \leftarrow$ augment(f, c, P)
    update$(G_f(\Delta))$
  $\Delta \leftarrow \Delta/2$
return f

Capacity Scaling

Assumption:
All capacities are integers between 1 and C.

Invariant:
All flows and capacities are/remain integral throughout the algorithm.

Correctness:
The algorithm computes a maxflow:

- because of integrality we have $G_f(1) = G_f$
- therefore after the last phase there are no augmenting paths anymore
- this means we have a maximum flow.

Capacity Scaling

Lemma 5

There are $\lceil \log C \rceil + 1$ *iterations over* Δ.

Proof: obvious.

Lemma 6

Let f *be the flow at the end of a* Δ*-phase. Then the maximum flow is smaller than* $\mathrm{val}(f) + m\Delta$.

Proof: less obvious, but simple:

- There must exist an s-t cut in $G_f(\Delta)$ of zero capacity.
- In G_f this cut can have capacity at most $m\Delta$.
- This gives me an upper bound on the flow that I can still add.

Capacity Scaling

Lemma 7

There are at most $2m$ *augmentations per scaling-phase.*

Proof:

- Let f be the flow at the end of the previous phase.
- $\mathrm{val}(f^*) \le \mathrm{val}(f) + 2m\Delta$
- Each augmentation increases flow by Δ.

Theorem 8

We need $\mathcal{O}(m \log C)$ *augmentations. The algorithm can be implemented in time* $\mathcal{O}(m^2 \log C)$.