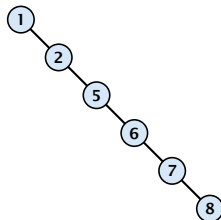
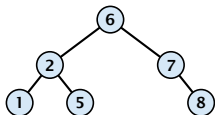


7.1 Binary Search Trees

An (internal) **binary search tree** stores the elements in a binary tree. Each tree-node corresponds to an element. All elements in the left sub-tree of a node v have a smaller key-value than $\text{key}[v]$ and elements in the right sub-tree have a larger-key value. We assume that all key-values are different.

(**External** Search Trees store objects only at leaf-vertices)

Examples:

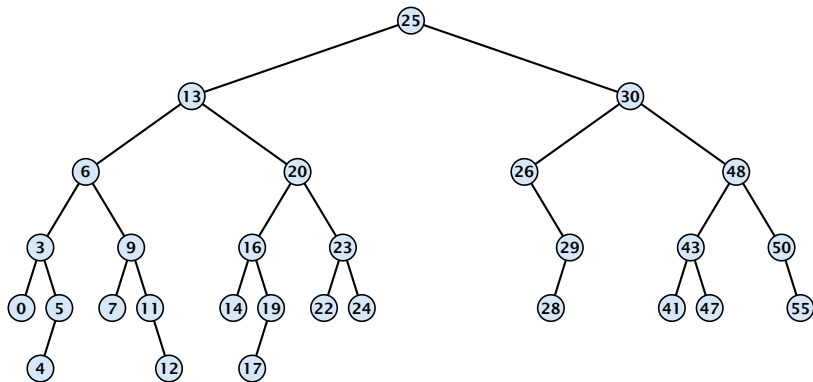


7.1 Binary Search Trees

We consider the following operations on binary search trees. Note that this is a super-set of the dictionary-operations.

- ▶ $T.\text{insert}(x)$
- ▶ $T.\text{delete}(x)$
- ▶ $T.\text{search}(k)$
- ▶ $T.\text{successor}(x)$
- ▶ $T.\text{predecessor}(x)$
- ▶ $T.\text{minimum}()$
- ▶ $T.\text{maximum}()$

Binary Search Trees: Searching

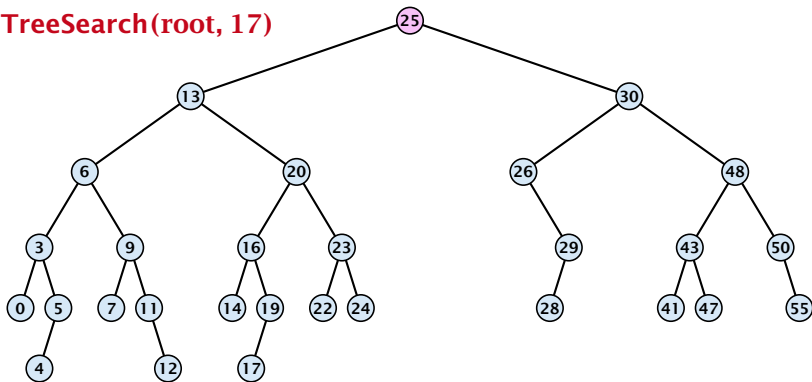


Algorithm 1 TreeSearch(x, k)

- 1: **if** $x = \text{null}$ **or** $k = \text{key}[x]$ **return** x
- 2: **if** $k < \text{key}[x]$ **return** TreeSearch(left[x], k)
- 3: **else return** TreeSearch(right[x], k)

Binary Search Trees: Searching

TreeSearch(root, 17)

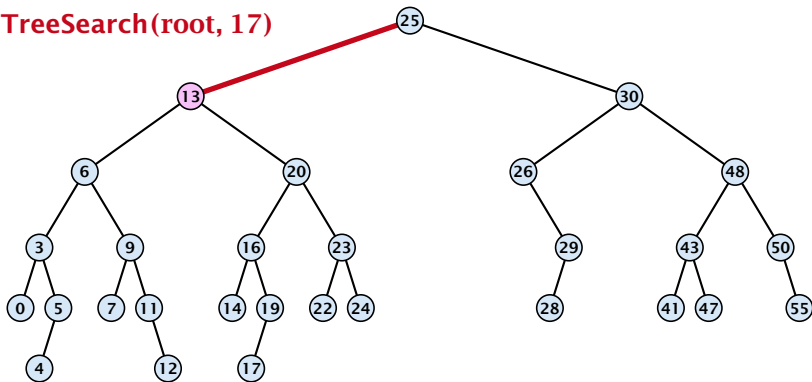


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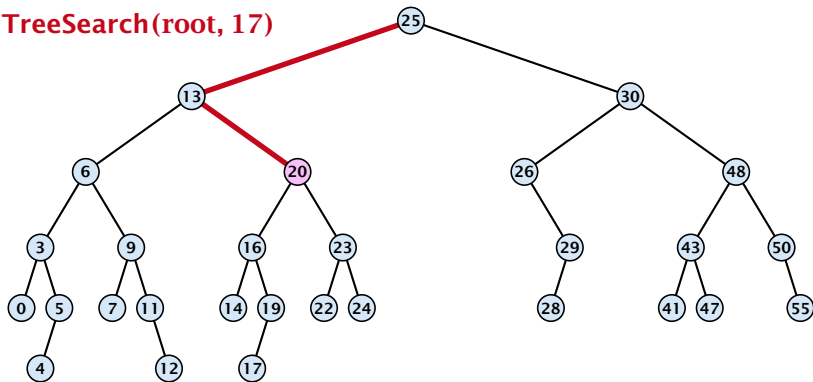


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Binary Search Trees: Searching

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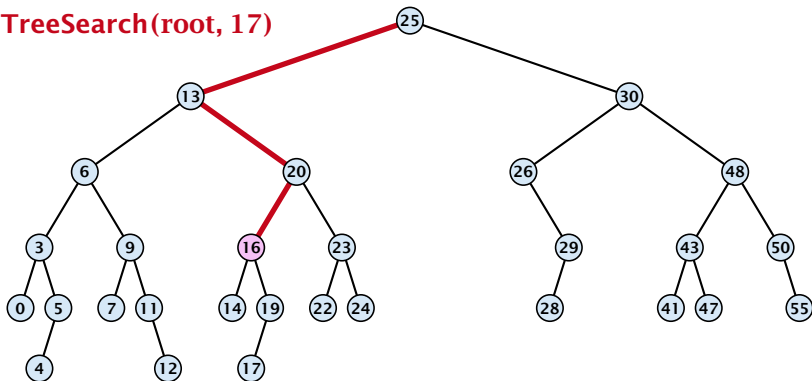


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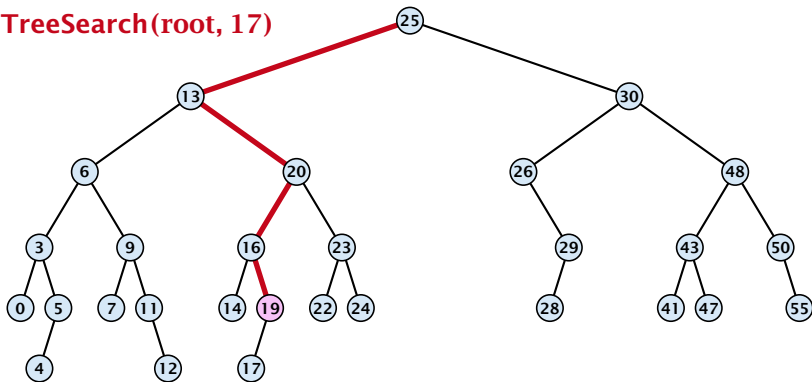


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Binary Search Trees: Searching

TreeSearch(root, 17)

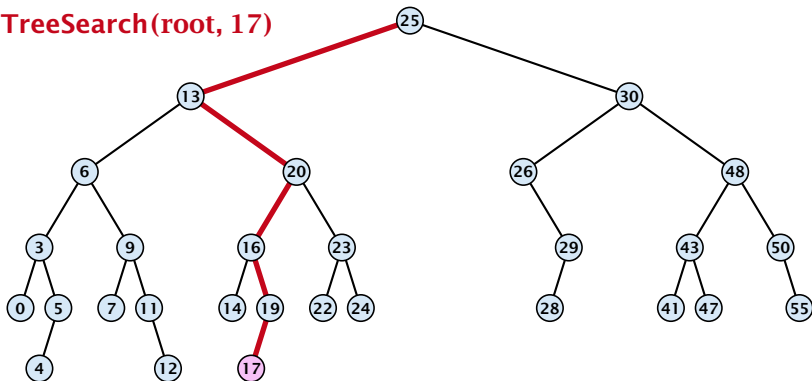


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Binary Search Trees: Searching

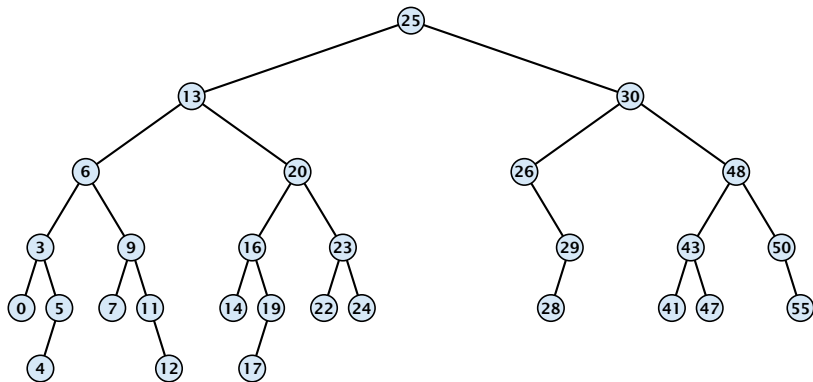
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Binary Search Trees: Searching

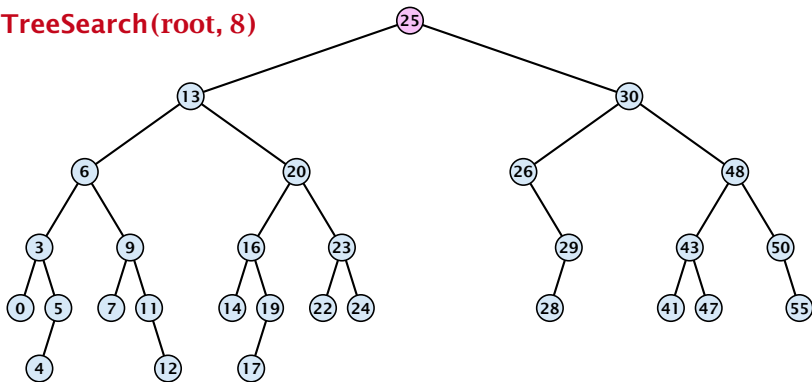


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Binary Search Trees: Searching

TreeSearch(root, 8)

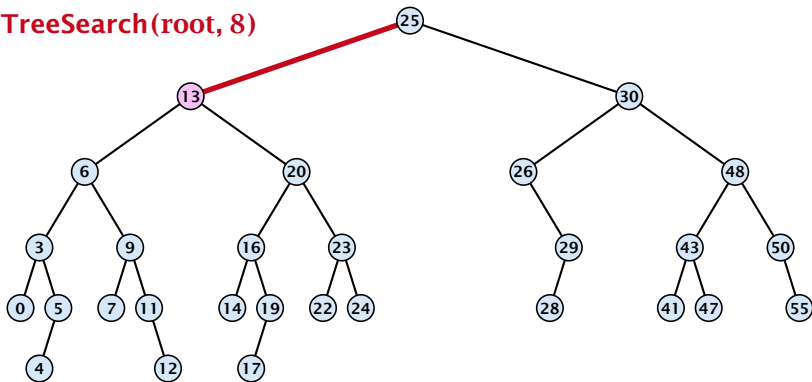


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Binary Search Trees: Searching

TreeSearch(root, 8)

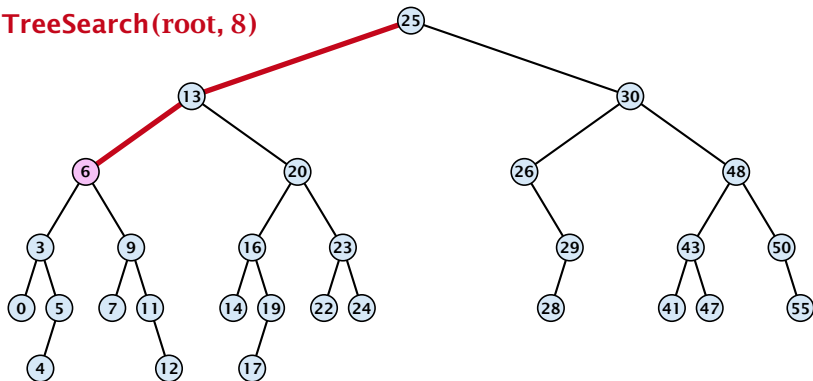


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Binary Search Trees: Searching

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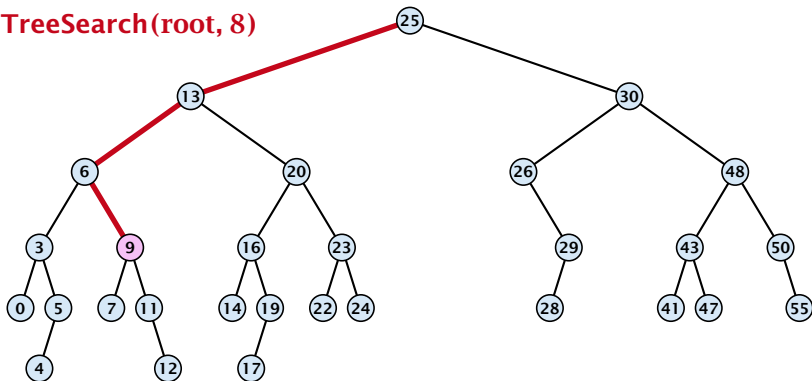


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Binary Search Trees: Searching

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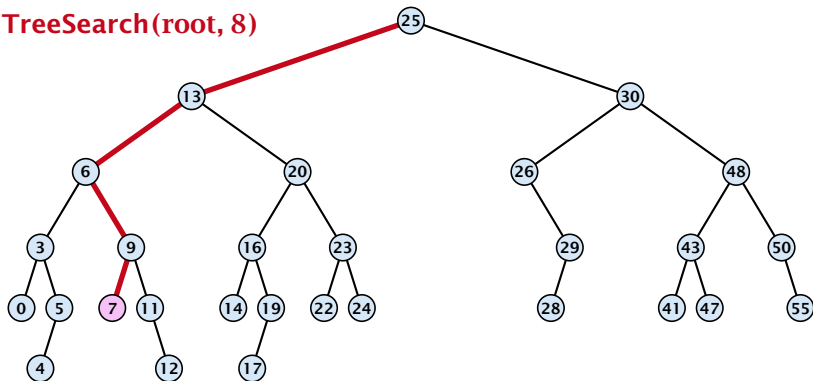


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Binary Search Trees: Searching

TreeSearch(root, 8)

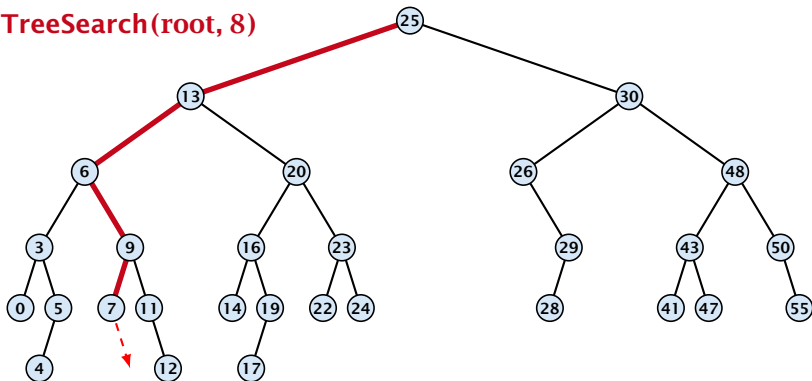


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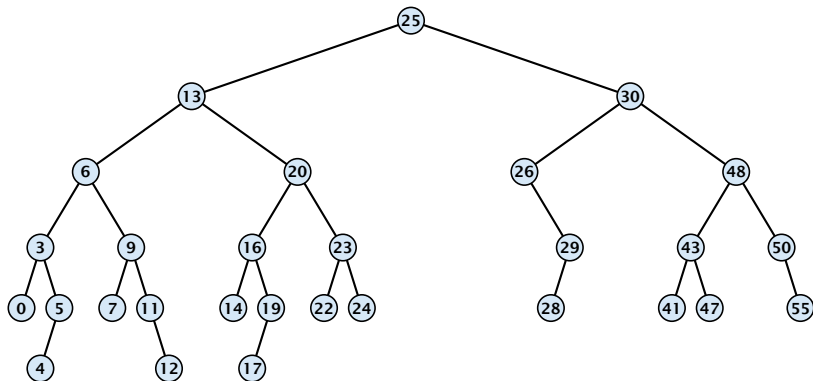
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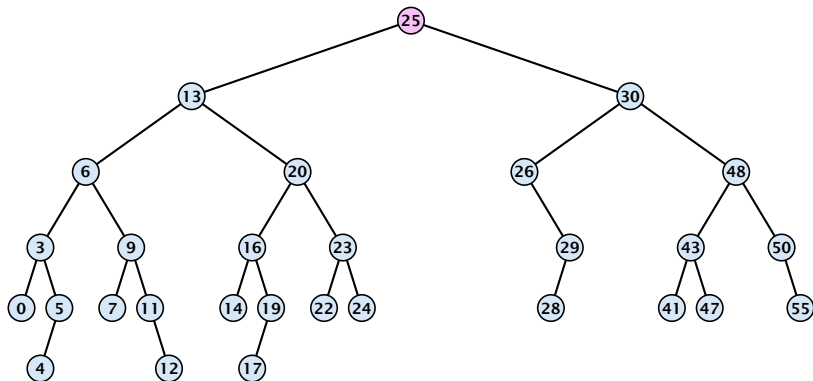
Binary Search Trees: Minimum



Algorithm 2 TreeMin(x)

- 1: **if** $x = \text{null}$ **or** $\text{left}[x] = \text{null}$ **return** x
- 2: **return** TreeMin($\text{left}[x]$)

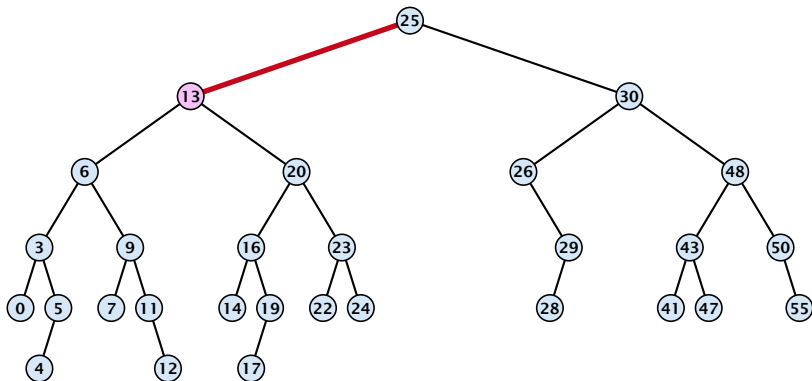
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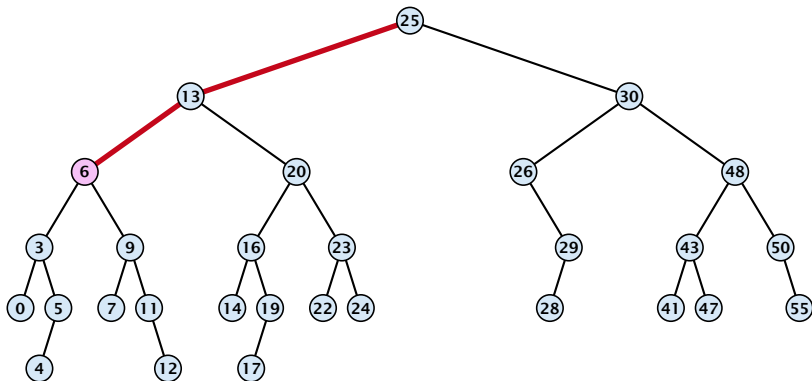
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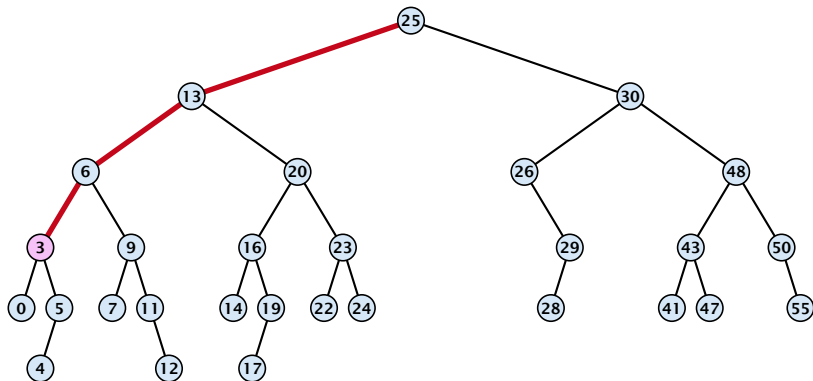
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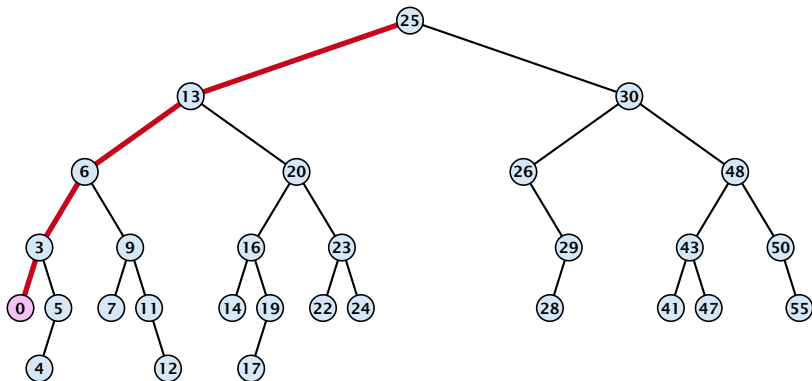
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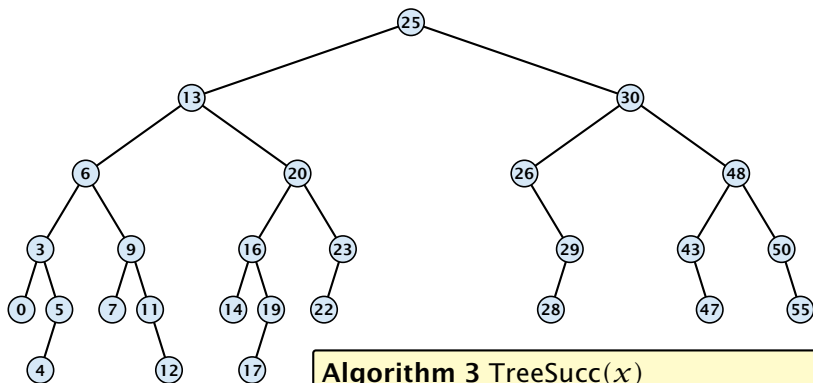
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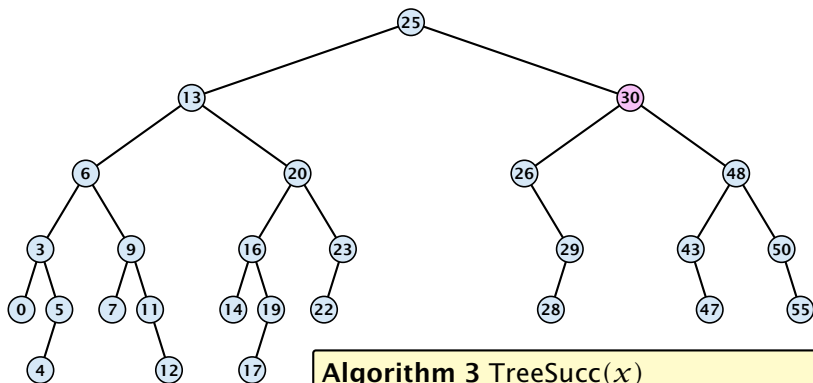
Binary Search Trees: Successor



Algorithm 3 TreeSucc(x)

- 1: **if** $\text{right}[x] \neq \text{null}$ **return** $\text{TreeMin}(\text{right}[x])$
- 2: $y \leftarrow \text{parent}[x]$
- 3: **while** $y \neq \text{null}$ **and** $x = \text{right}[y]$ **do**
- 4: $x \leftarrow y; y \leftarrow \text{parent}[x]$
- 5: **return** y ;

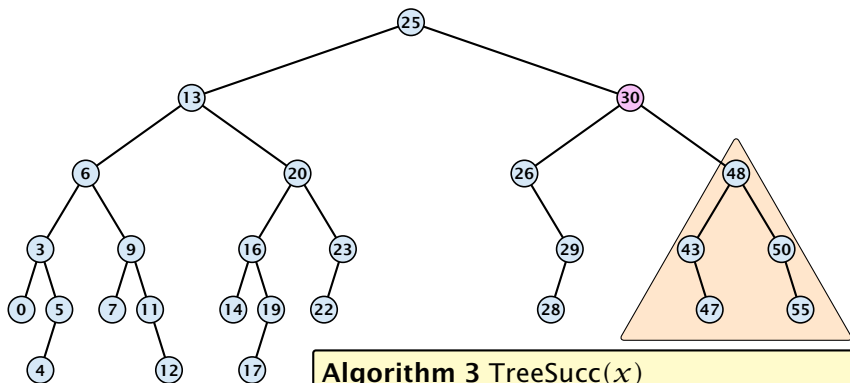
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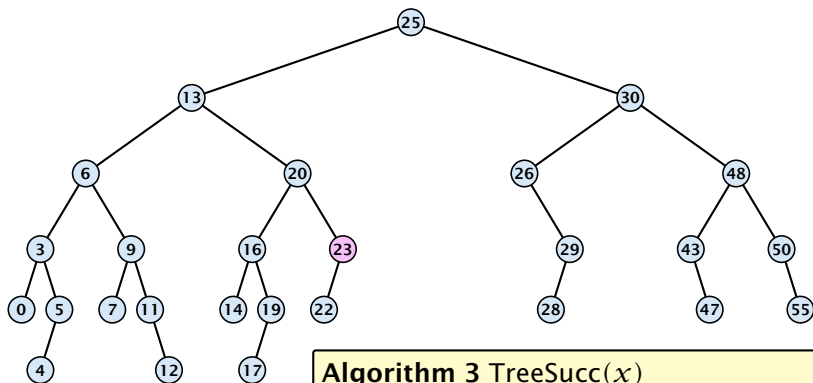
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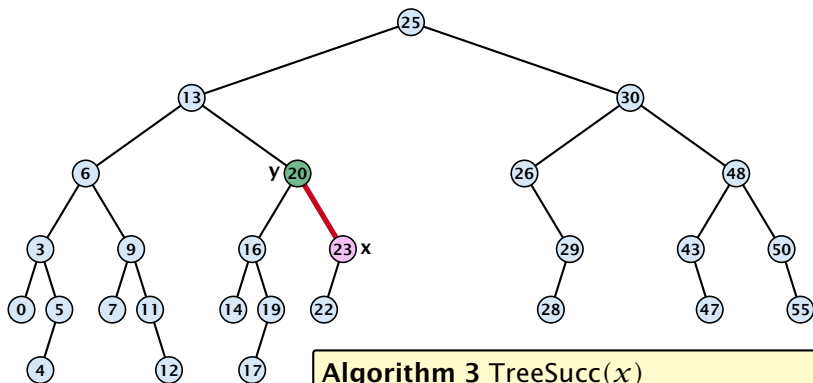
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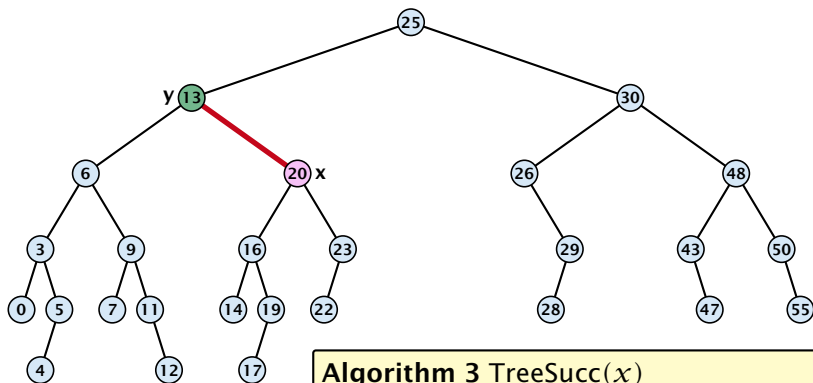
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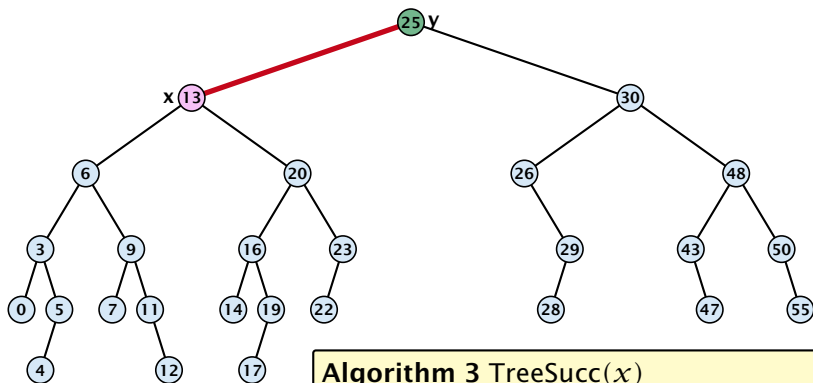
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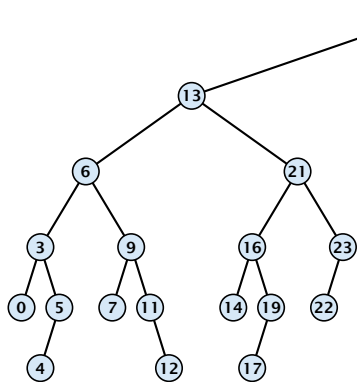
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Binary Search Trees: Insert

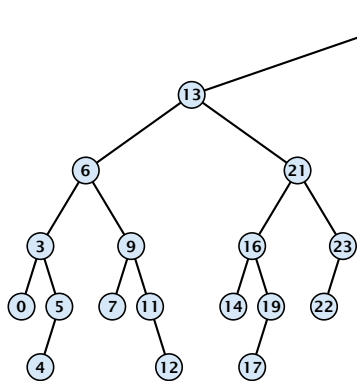


Algorithm 4 TreeInsert(x, z)

```
1: if  $x = \text{null}$  then  
2:    $\text{root}[T] \leftarrow z$ ;  $\text{parent}[z] \leftarrow \text{null}$ ;  
3:   return;  
4: if  $\text{key}[x] > \text{key}[z]$  then  
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Binary Search Trees: Insert

Insert element **not** in the tree.

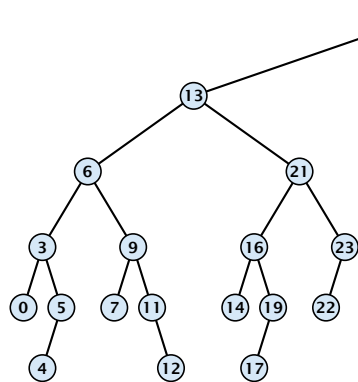


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Search for z . At some point the search stops at a null-pointer. This is the place to insert z .

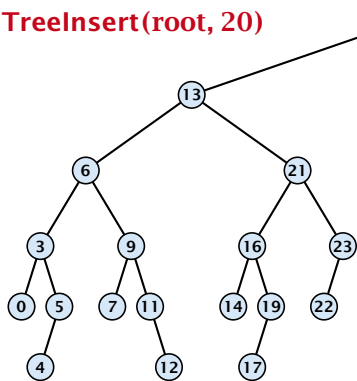
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Binary Search Trees: Insert

Insert element **not** in the tree.

TreeInsert(root, 20)



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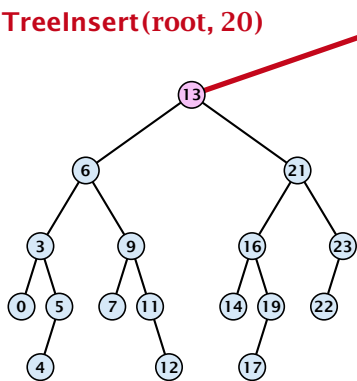
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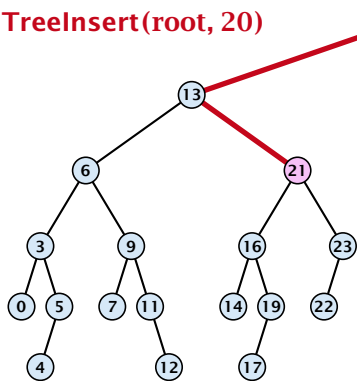
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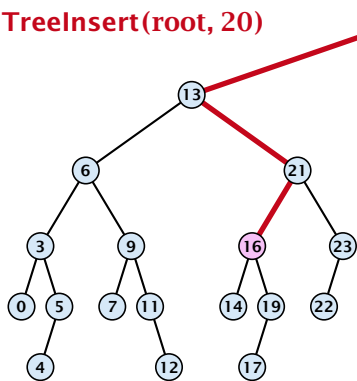
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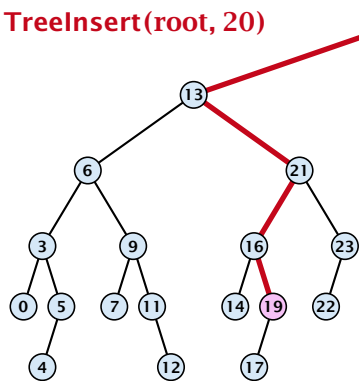
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5:   if  $\text{left}[x] = \text{null}$  then  
6:      $\text{left}[x] \leftarrow z$ ;  $\text{parent}[z] \leftarrow x$ ;  
7:   else TreeInsert( $\text{left}[x], z$ );  
8: else  
9:   if  $\text{right}[x] = \text{null}$  then  
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```

Binary Search Trees: Insert

Insert element **not** in the tree.

TreeInsert(root, 20)



Search for **z**. At some point the search stops at a null-pointer. This is the place to insert **z**.

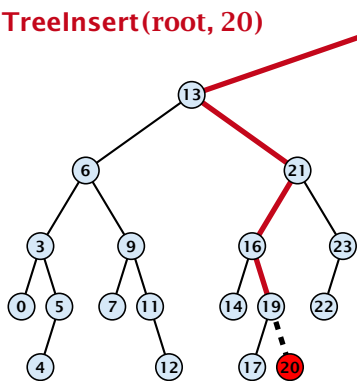
Algorithm 4 TreeInsert(x, z)

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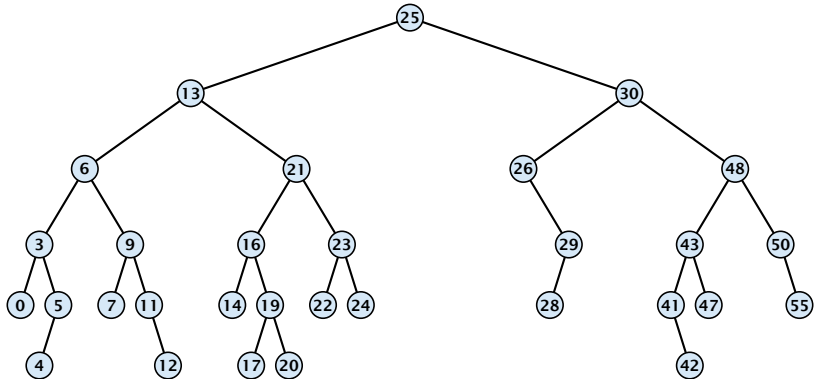


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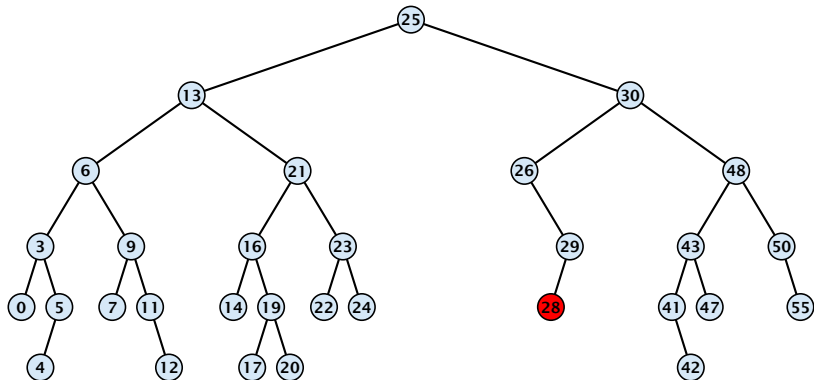
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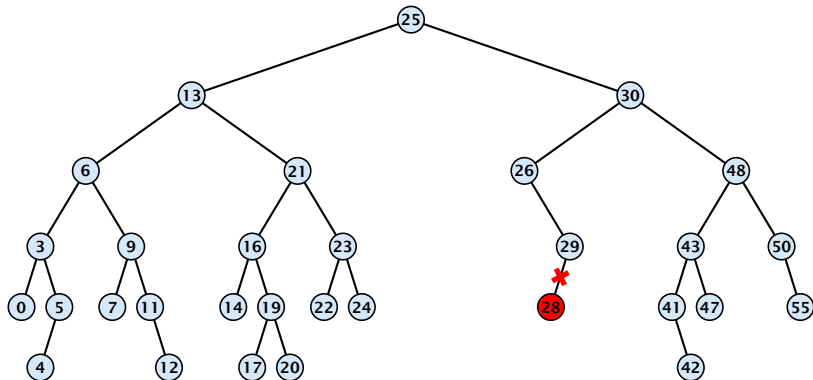


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Element does not have any children

- Simply go to the parent and set the corresponding pointer to **null**.

Binary Search Trees: Delete

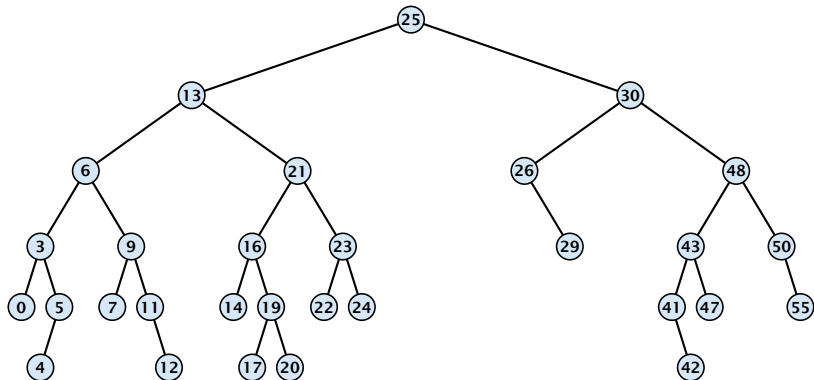


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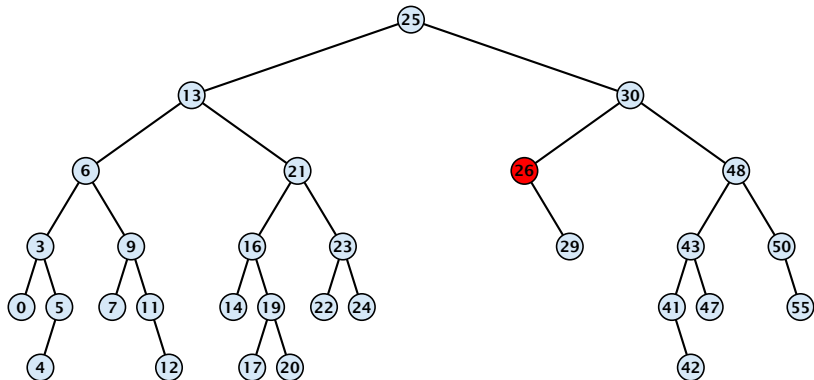


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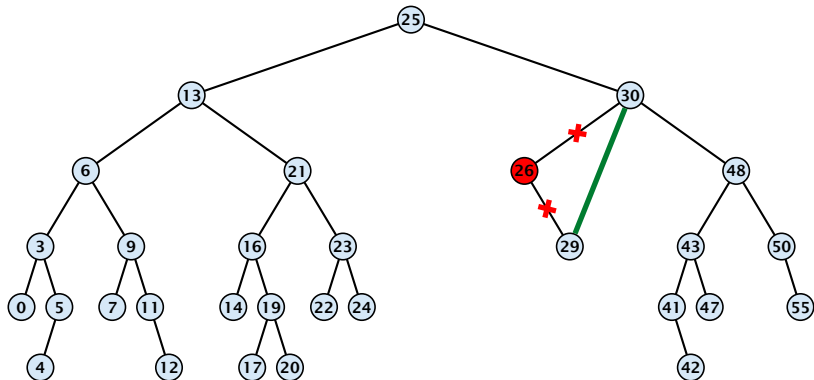


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Element has exactly one child

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Binary Search Trees: Delete

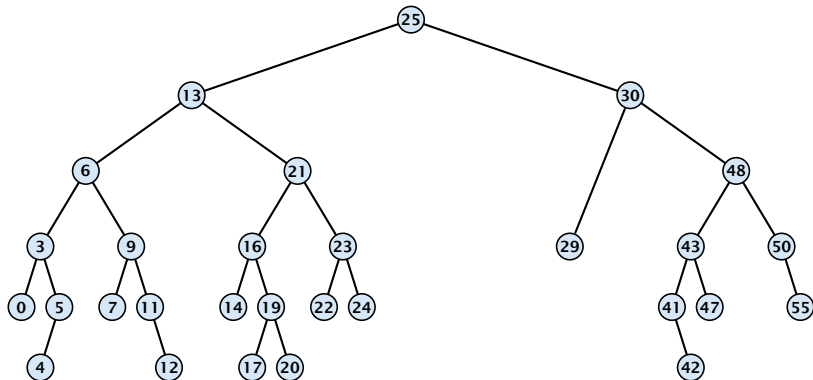


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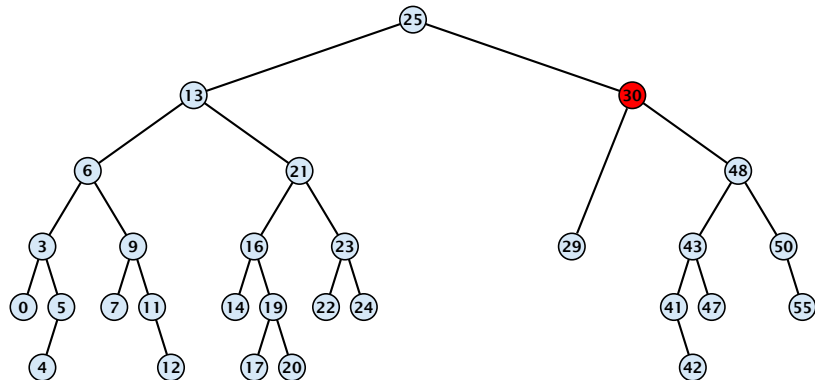


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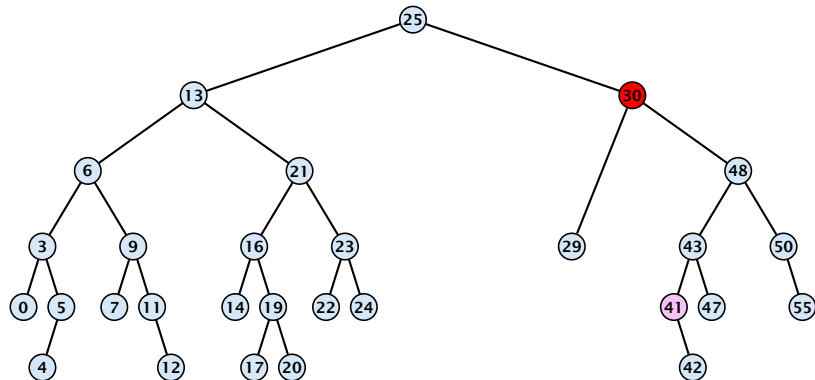


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Element has two children

- ▶ Find the successor of the element
- ▶ Splice successor out of the tree
- ▶ Replace content of element by content of successor

Binary Search Trees: Delete

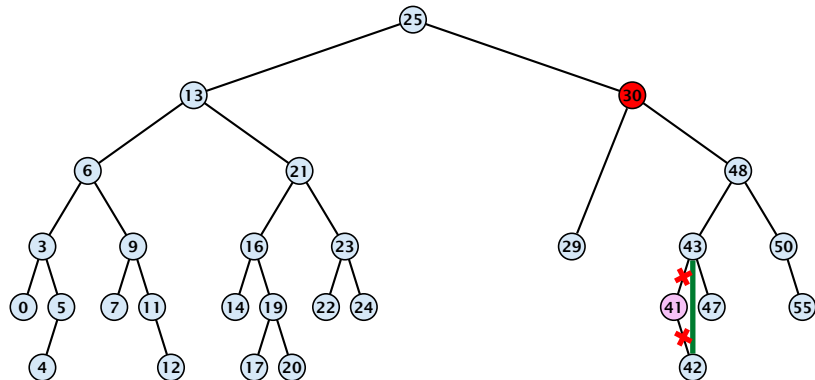


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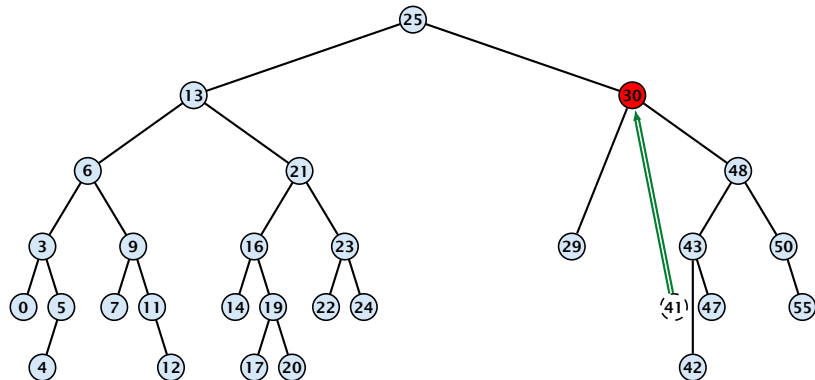


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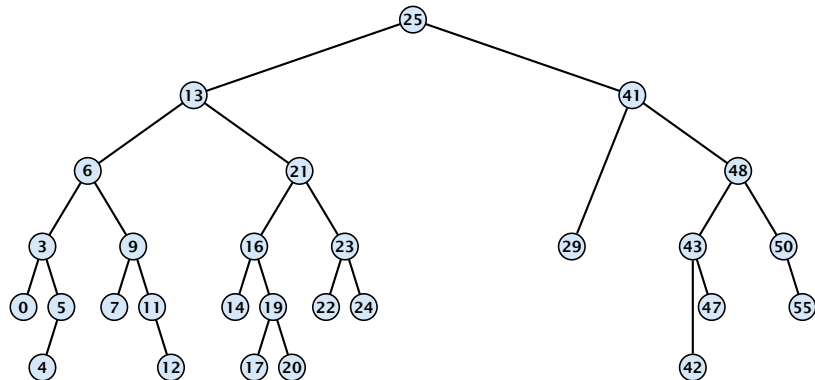


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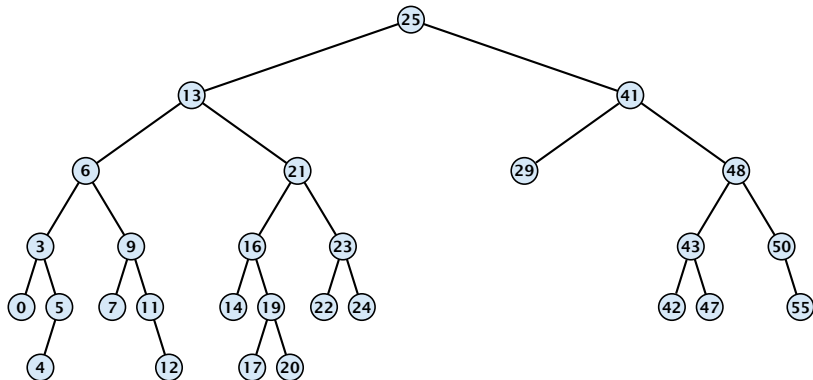


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Binary Search Trees: Delete

Algorithm 9 TreeDelete(z)

```
1: if left[ $z$ ] = null or right[ $z$ ] = null
2:   then  $y \leftarrow z$  else  $y \leftarrow \text{TreeSucc}(z)$ ;    select  $y$  to splice out
3:   if left[ $y$ ]  $\neq$  null
4:     then  $x \leftarrow \text{left}[y]$  else  $x \leftarrow \text{right}[y]$ ;  $x$  is child of  $y$  (or null)
5:   if  $x \neq \text{null}$  then parent[ $x$ ]  $\leftarrow$  parent[ $y$ ];    parent[ $x$ ] is correct
6:   if parent[ $y$ ] = null then
7:     root[ $T$ ]  $\leftarrow x$ 
8:   else
9:     if  $y = \text{left}[\text{parent}[y]]$  then
10:      left[parent[ $y$ ]]  $\leftarrow x$ 
11:     else
12:      right[parent[ $y$ ]]  $\leftarrow x$ 
13:   if  $y \neq z$  then copy  $y$ -data to  $z$ 
```

} fix pointer to x

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AVL-trees, Red-black trees, Scapegoat trees, 2-3 trees, B-trees, AA trees, Treaps

similar: SPLAY trees.