Preflows

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An (s,t)-preflow is a function $f : E \mapsto \mathbb{R}^+$ that satisfies

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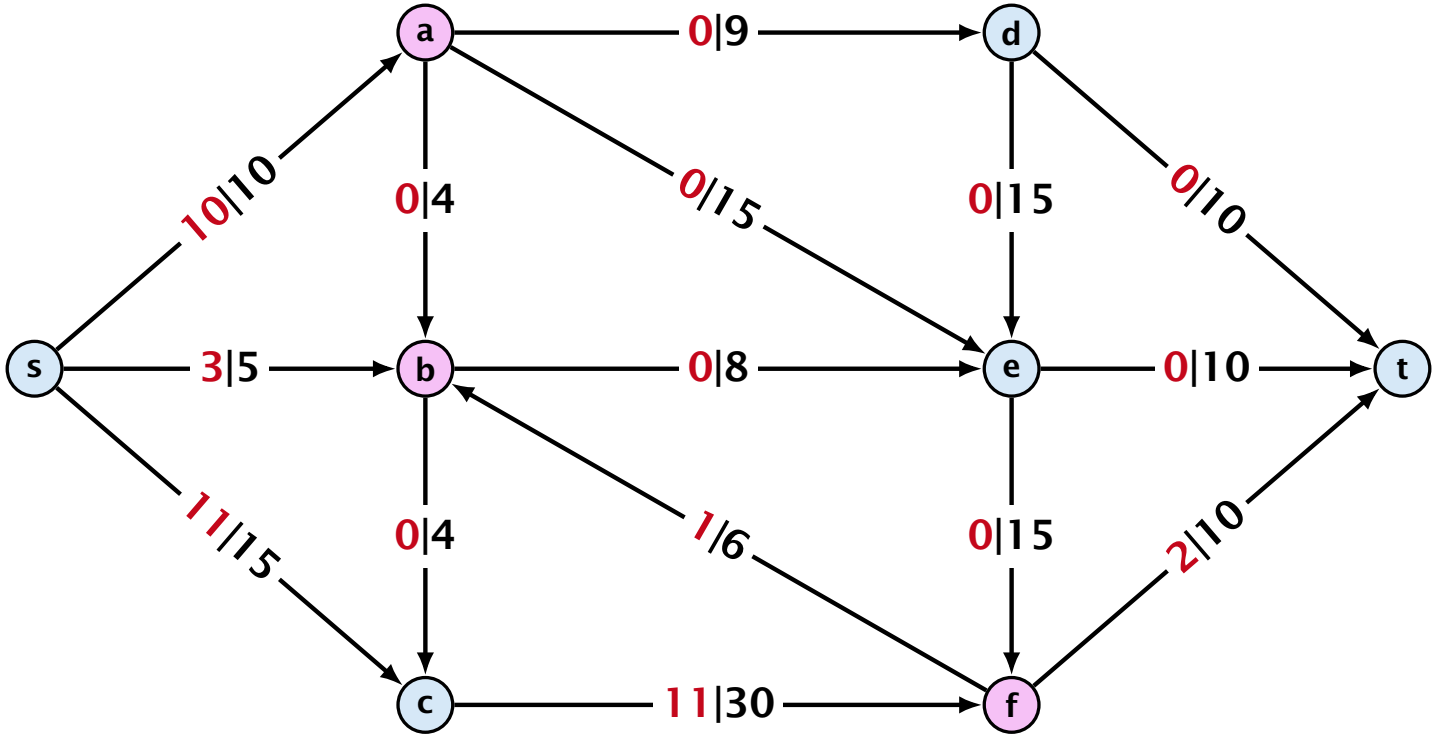
(capacity constraints)

2. For each $v \in V \setminus \{s,t\}$

$$\sum_{e \in \text{out}(v)} f(e) \leq \sum_{e \in \text{into}(v)} f(e) \ .$$

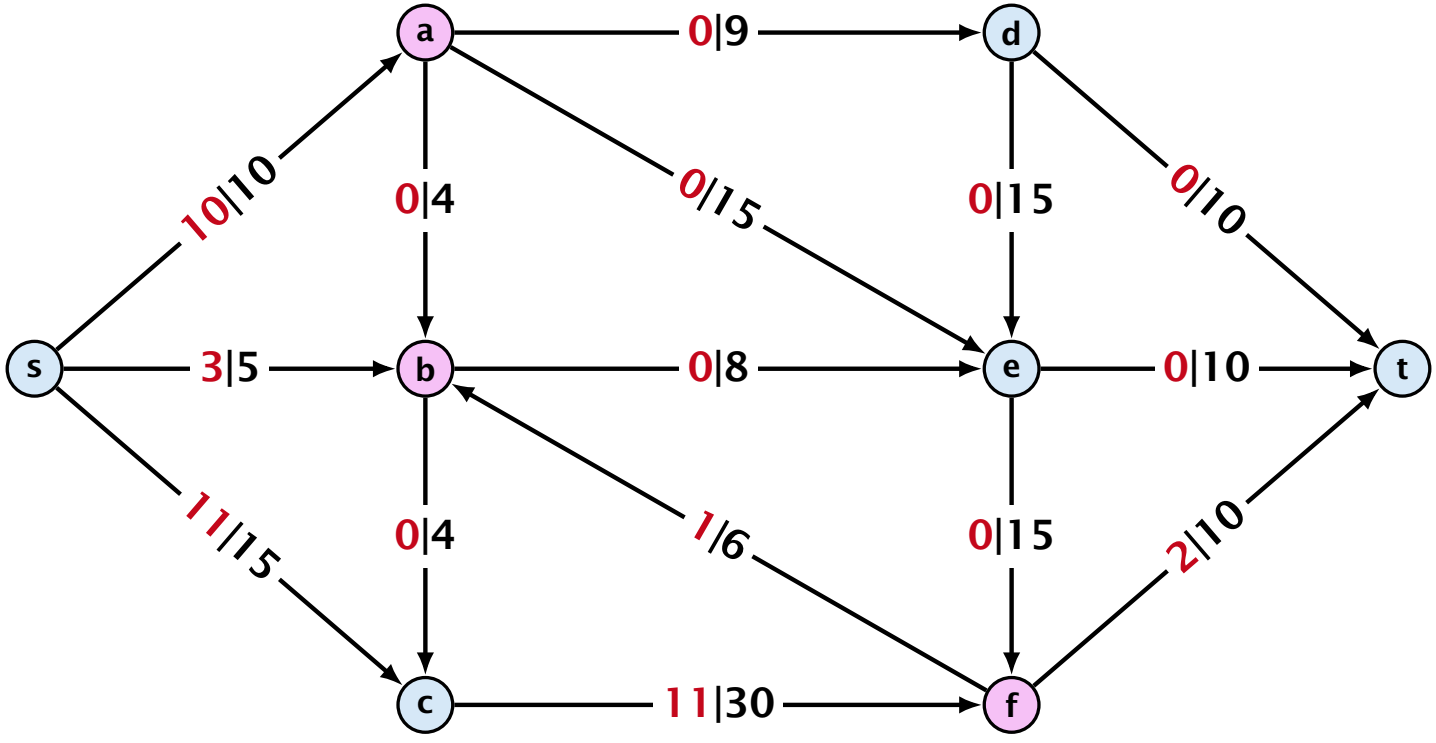
Preflows

Example 6



Preflows

Example 6



A node that has $\sum_{e \in \text{out}(v)} f(e) < \sum_{e \in \text{into}(v)} f(e)$ is called an active node.

Preflows

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Definition:

A labelling is a function $\ell : V \to \mathbb{N}$. It is valid for preflow f if

- $\ell(u) \leq \ell(v) + 1$ for all edges (u, v) in the residual graph G_f (only non-zero capacity edges!!!)

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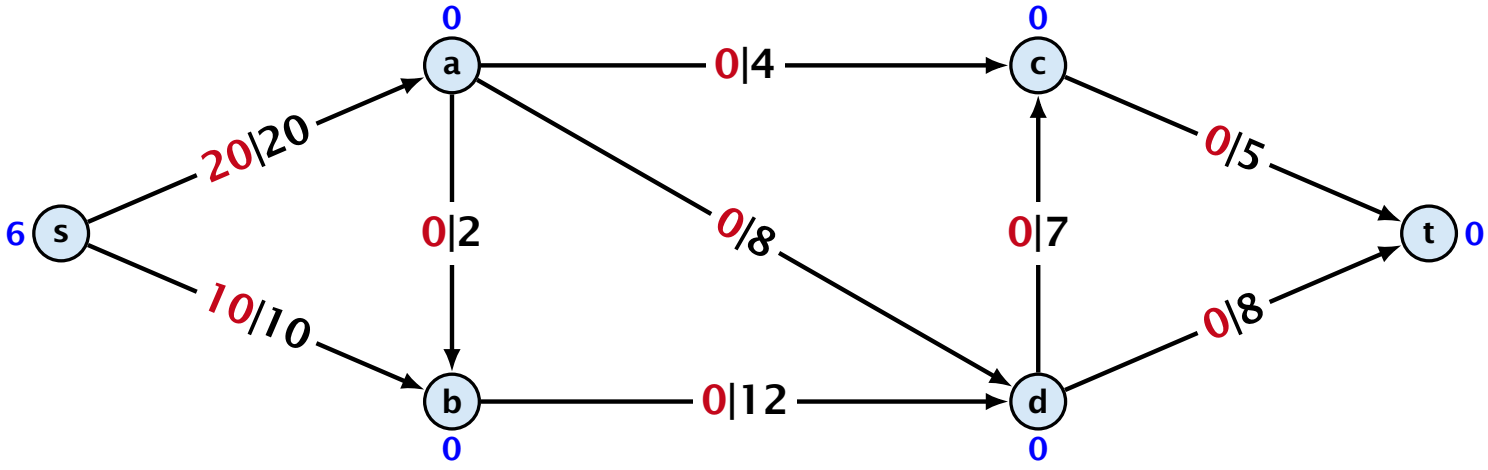
Intuition:

The labelling can be viewed as a height function. Whenever the height from node u to node v decreases by more than 1 (i.e., it goes very steep downhill from u to v), the corresponding edge must be saturated.

Preflows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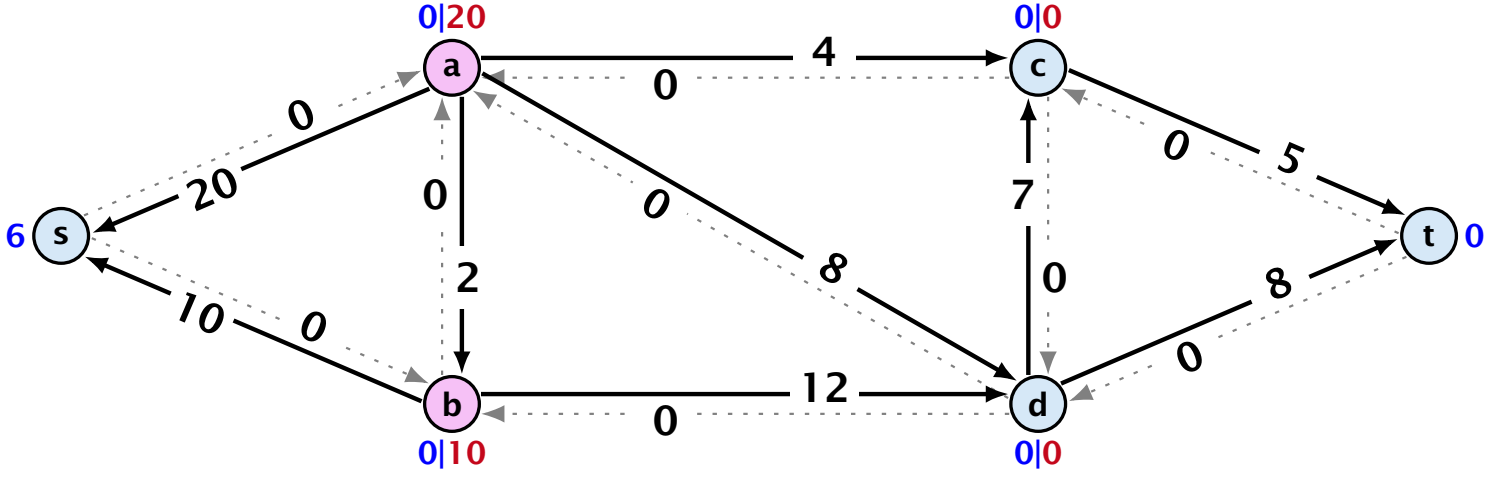
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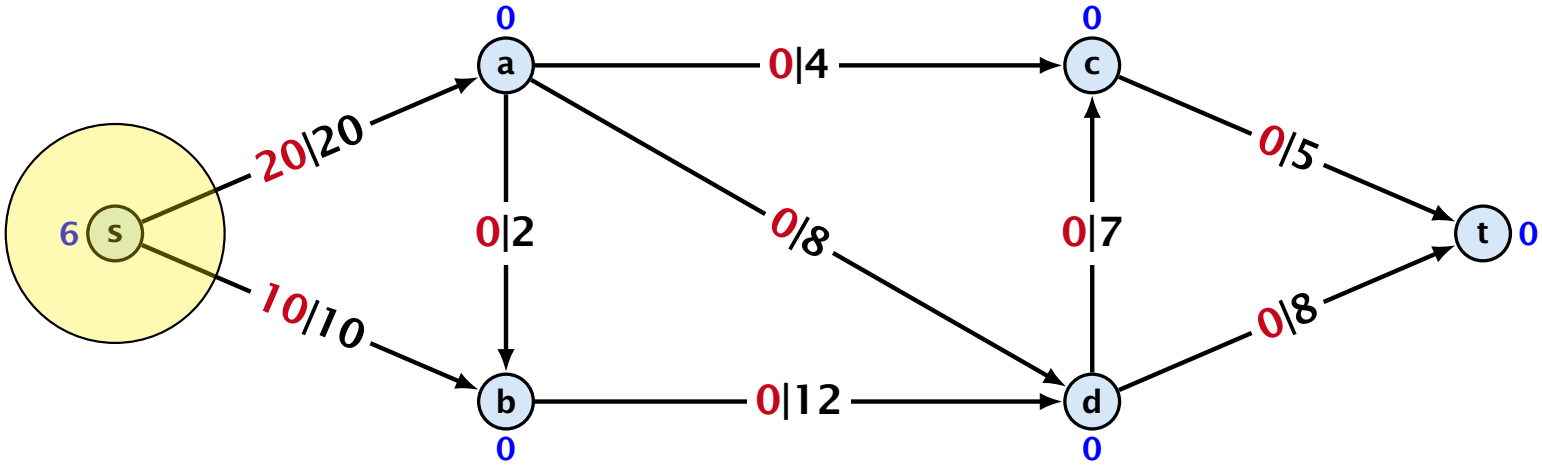
Preflows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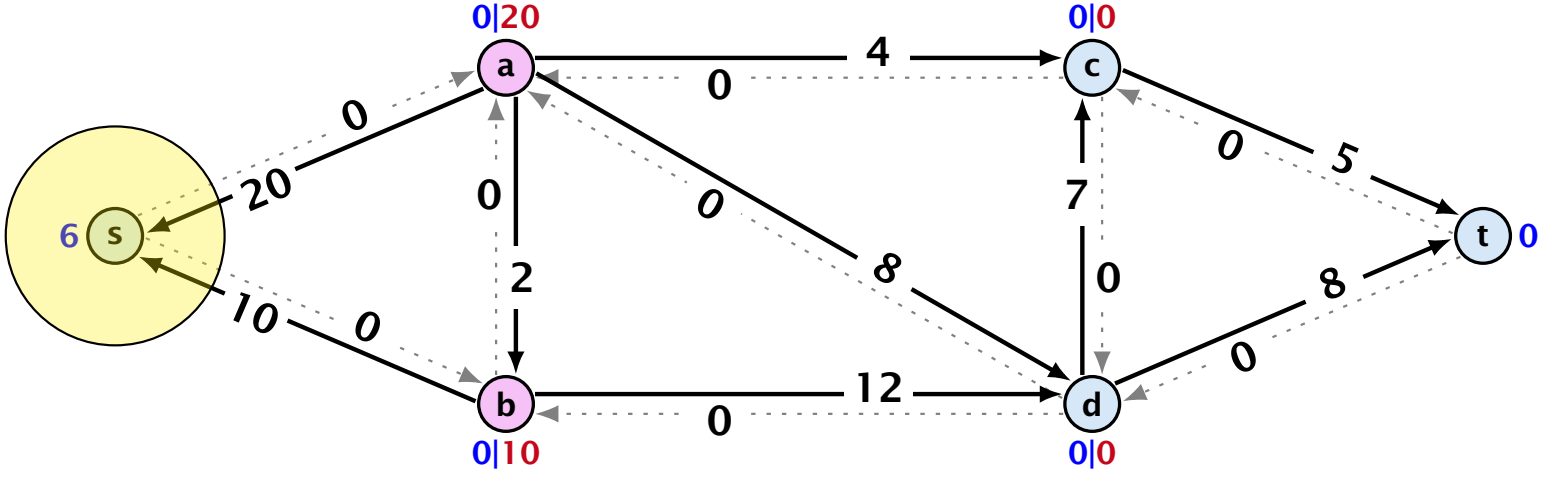
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Preflows

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Lemma 7

A preflow that has a valid labelling saturates a cut.

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Lemma 8

A flow that has a valid labelling is a maximum flow.

Push Relabel Algorithms

Push Relabel Algorithms

Idea:

- start with some preflow and some valid labelling

Note that this is somewhat dual to an augmenting path algorithm. The former maintains the property that it has a feasible flow. It successively changes this flow until it saturates some cut in which case we conclude that the flow is maximum. A preflow push algorithm maintains the property that it has a saturated cut. The preflow is changed iteratively until it fulfills conservation constraints in which case we can conclude that we have a maximum flow.

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- successively change the preflow while maintaining a valid labelling
- stop when you have a flow (i.e., no more active nodes)

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The push operation

Consider an active node u with excess flow $f(u) = \sum_{e \in \text{into}(u)} f(e) - \sum_{e \in \text{out}(u)} f(e)$ and suppose $e = (u, v)$ is an admissible arc with residual capacity $c_f(e)$.

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We can send flow $\min\{c_f(e), f(u)\}$ along e and obtain a new preflow. The old labelling is still valid (!!!).

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  the arc e is deleted from the residual graph

Note that a push-operation may be saturating **and** deactivating at the same time.

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- deactivating push: $\min\{f(u), c_f(e)\} = f(u)$
  the node u becomes inactive

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Push Relabel Algorithms

The relabel operation

Consider an active node u that does not have an outgoing admissible arc.

Increasing the label of u by 1 results in a valid labelling.

- Edges (w, u) incoming to u still fulfill their constraint $\ell(w) \leq \ell(u) + 1$.
- An outgoing edge (u, w) had $\ell(u) < \ell(w) + 1$ before since it was not admissible. Now: $\ell(u) \leq \ell(w) + 1$.

Push Relabel Algorithms

Intuition:

We want to send flow downwards, since the source has a height/label of n and the target a height/label of 0. If we see an active node u with an admissible arc we push the flow at u towards the other end-point that has a lower height/label. If we do not have an admissible arc but excess flow into u it should roughly mean that the level/height/label of u should rise. (If we consider the flow to be water then this would be natural.)

Note that the above intuition is very incorrect as the labels are integral, i.e., they cannot really be seen as the height of a node.

Reminder

- In a preflow nodes may not fulfill conservation constraints; a node may have more incoming flow than outgoing flow.
- Such a node is called active.
- A labelling is valid if for every edge (u, v) in the residual graph $\ell(u) \leq \ell(v) + 1$.
- An arc (u, v) in residual graph is admissible if $\ell(u) = \ell(v) + 1$.
- A saturating push along e pushes an amount of $c(e)$ flow along the edge, thereby saturating the edge (and making it dissappear from the residual graph).
- A deactivating push along $e = (u, v)$ pushes a flow of $f(u)$, where $f(u)$ is the excess flow of u. This makes u inactive.

Push Relabel Algorithms

Algorithm 1 maxflow(G, s, t, c)

```
1: find initial preflow f
2: while there is active node u do
3:     if there is admiss. arc e out of u then
4:         push(G, e, f, c)
5:     else
6:         relabel(u)
7: return f
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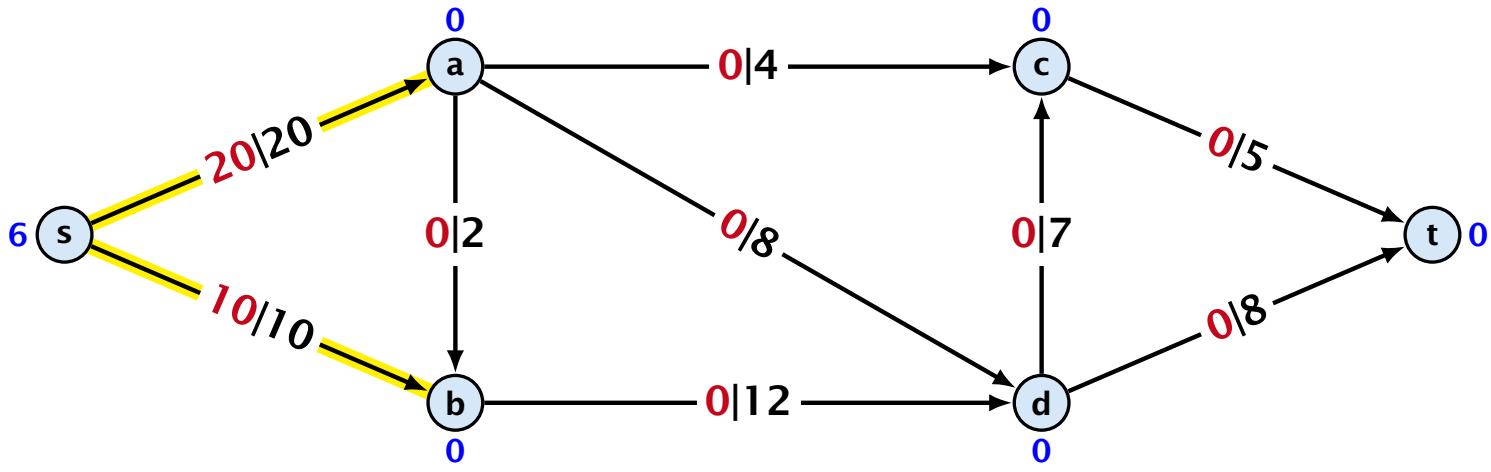
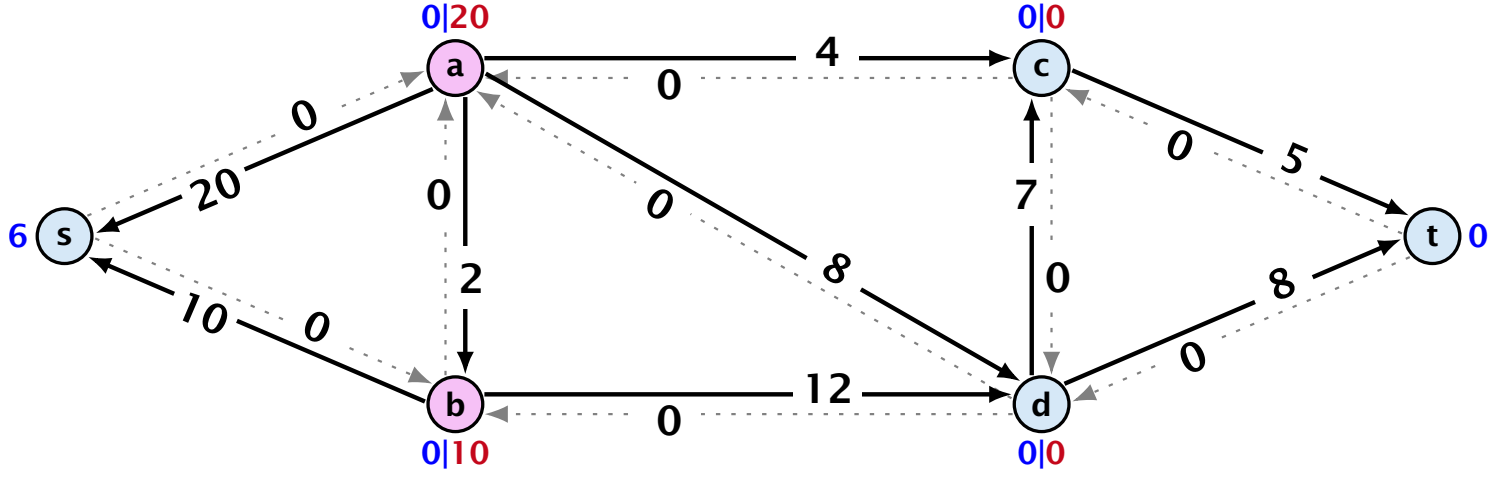
In the following example we always stick to the same active node u until it becomes inactive but this is not required.

Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.





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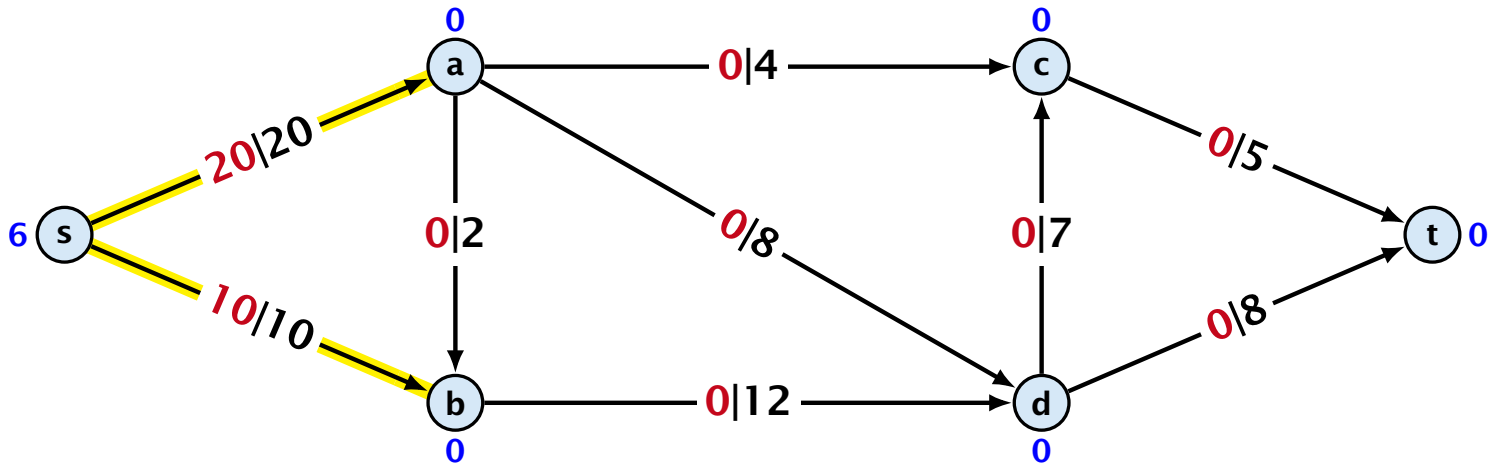

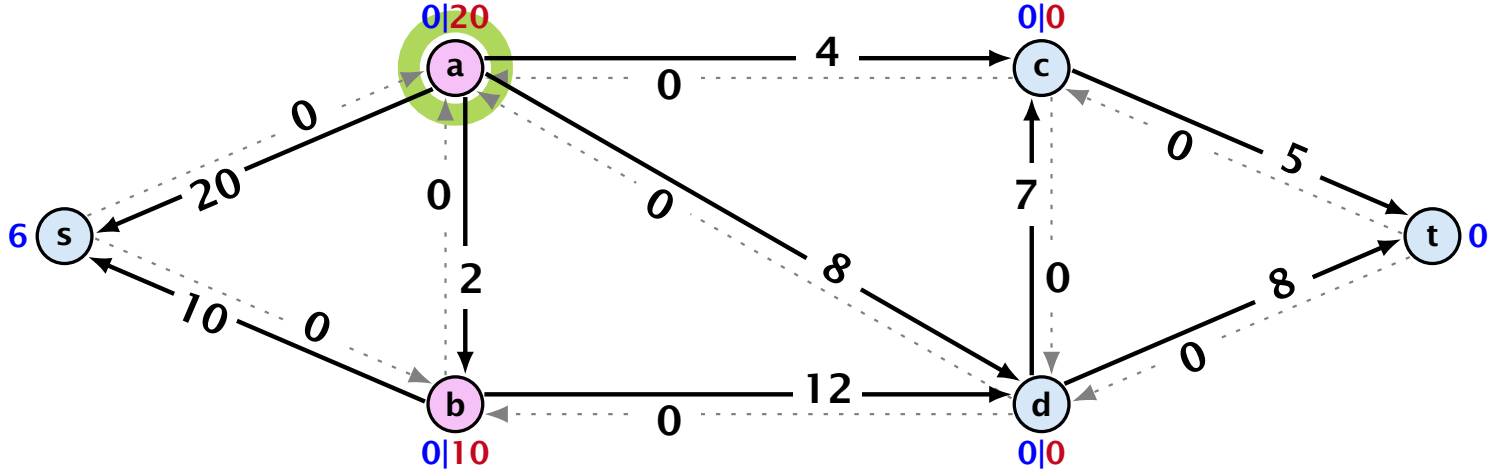

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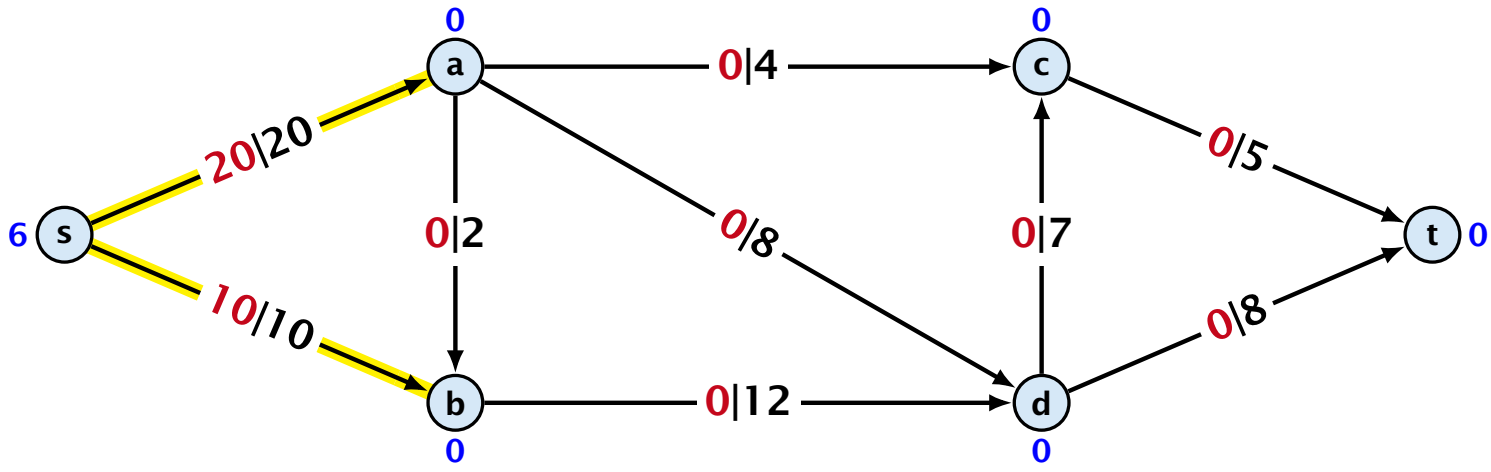
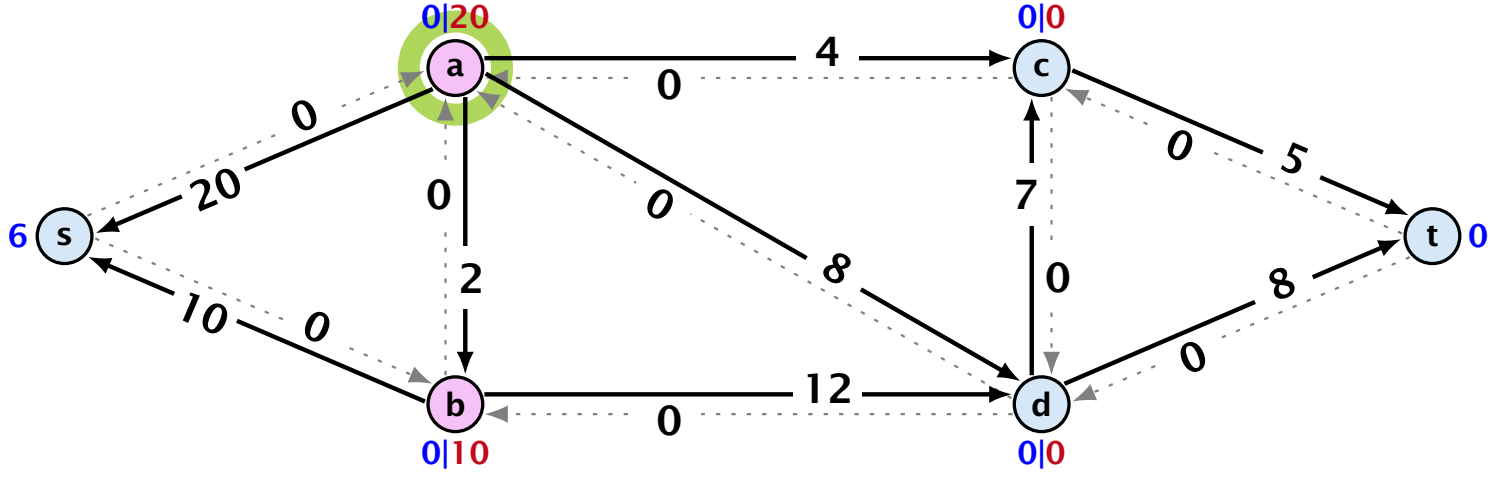


relabel to 1



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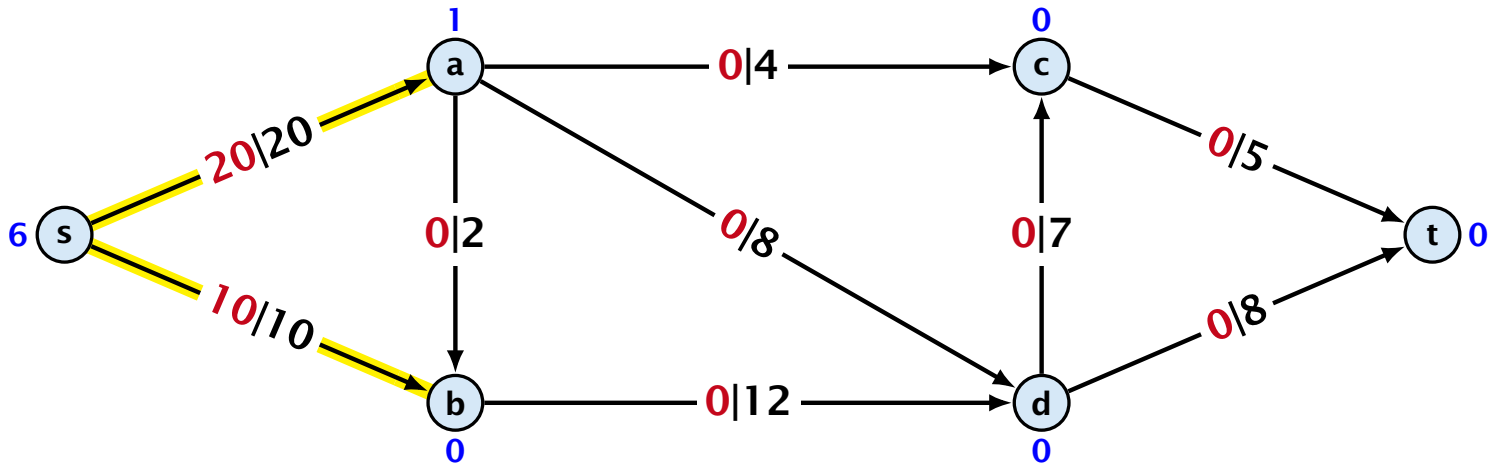

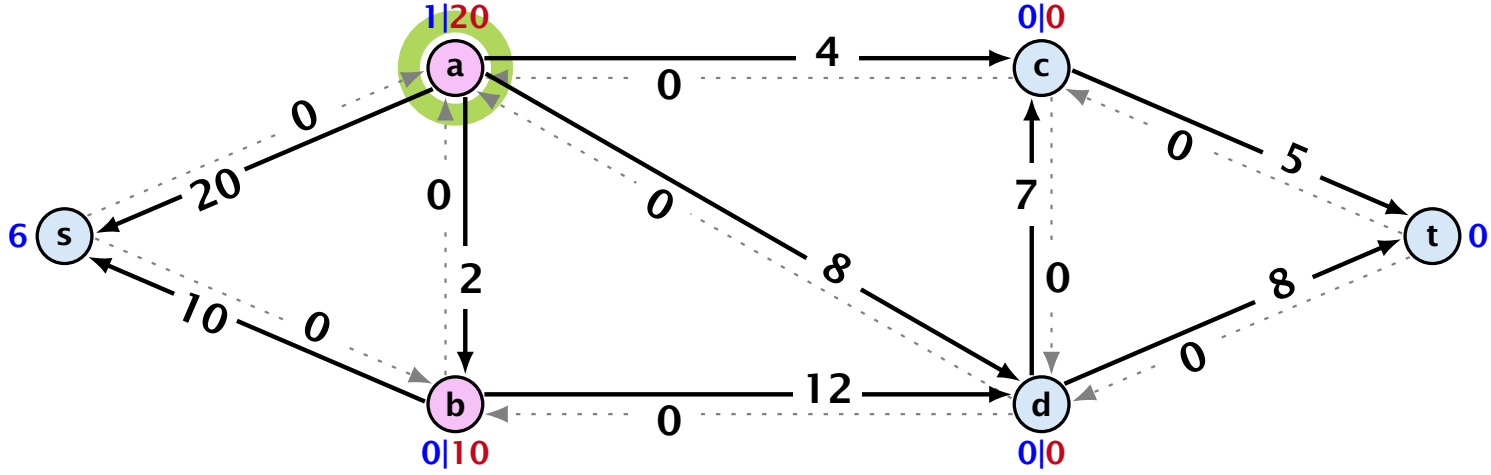

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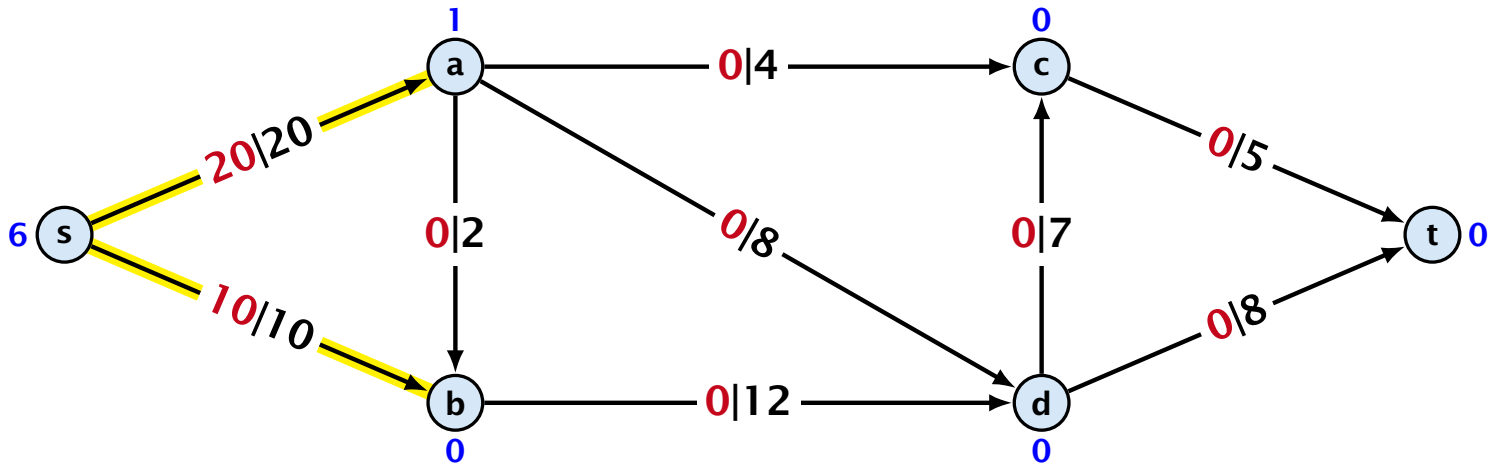
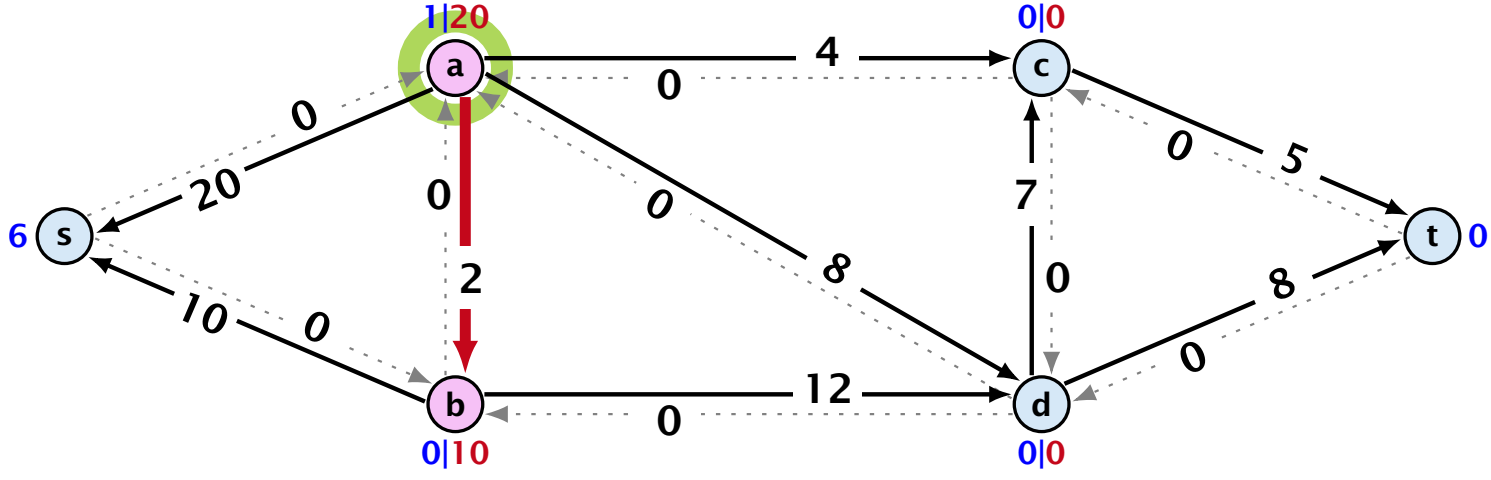


saturating push



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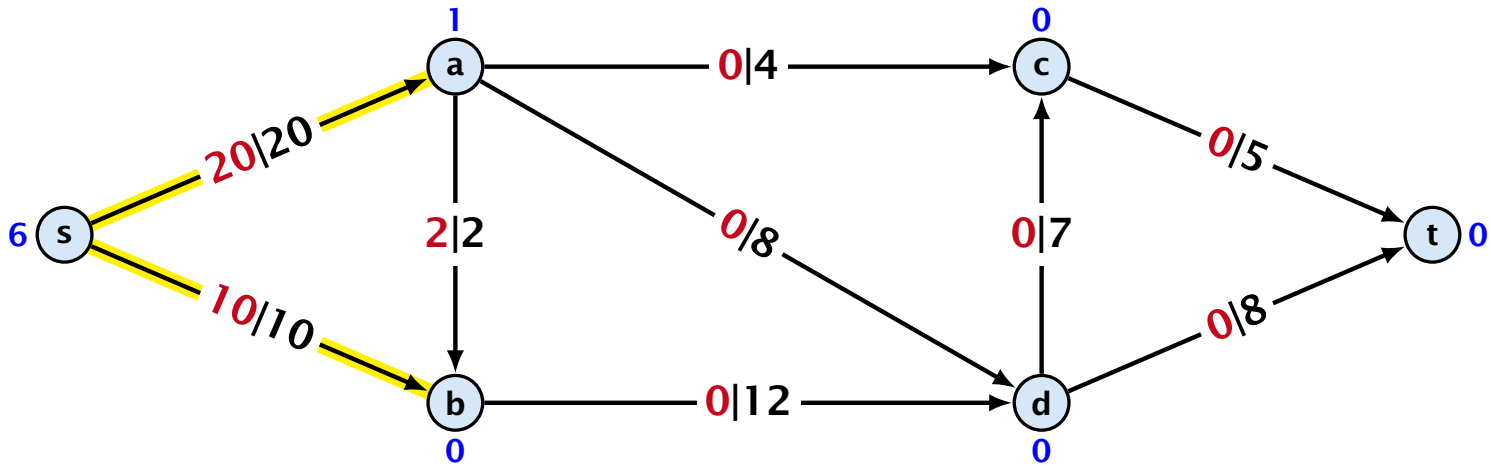

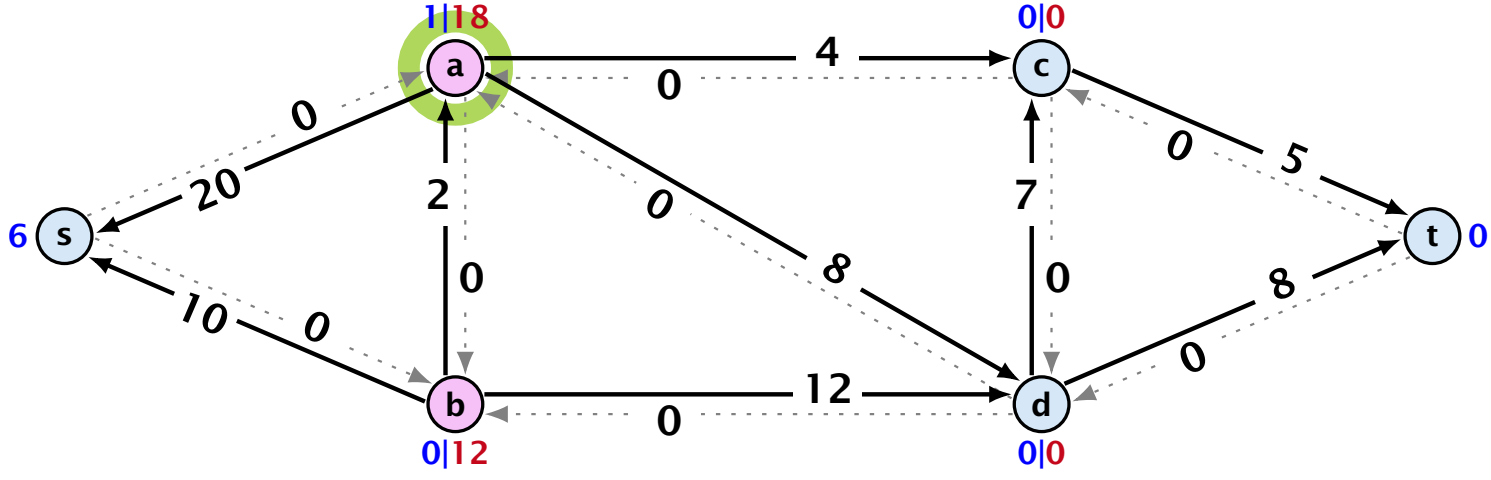

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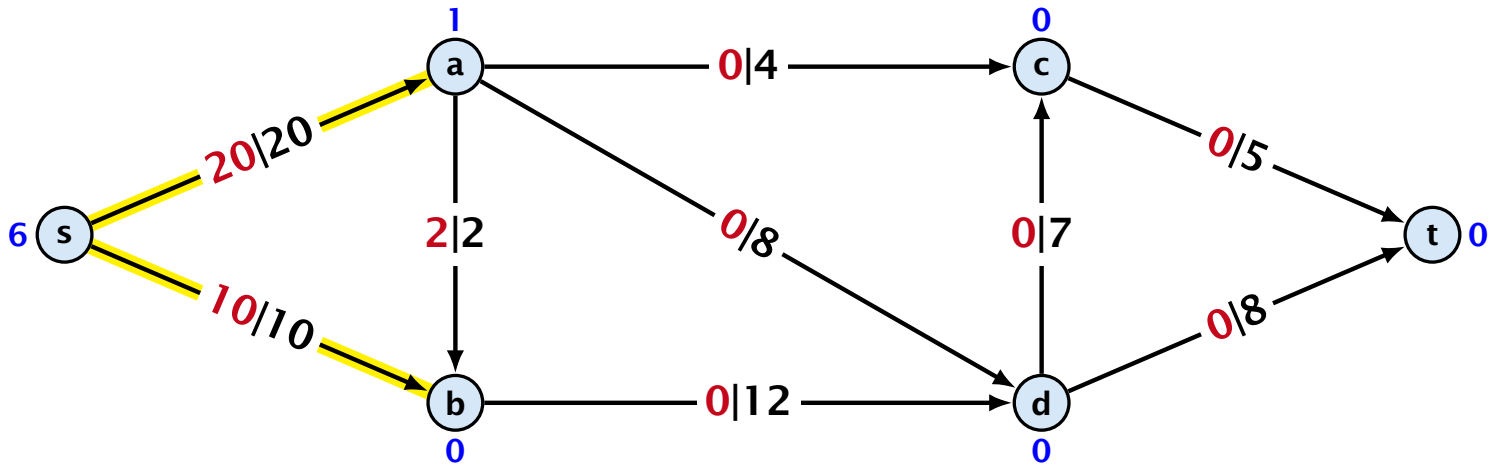

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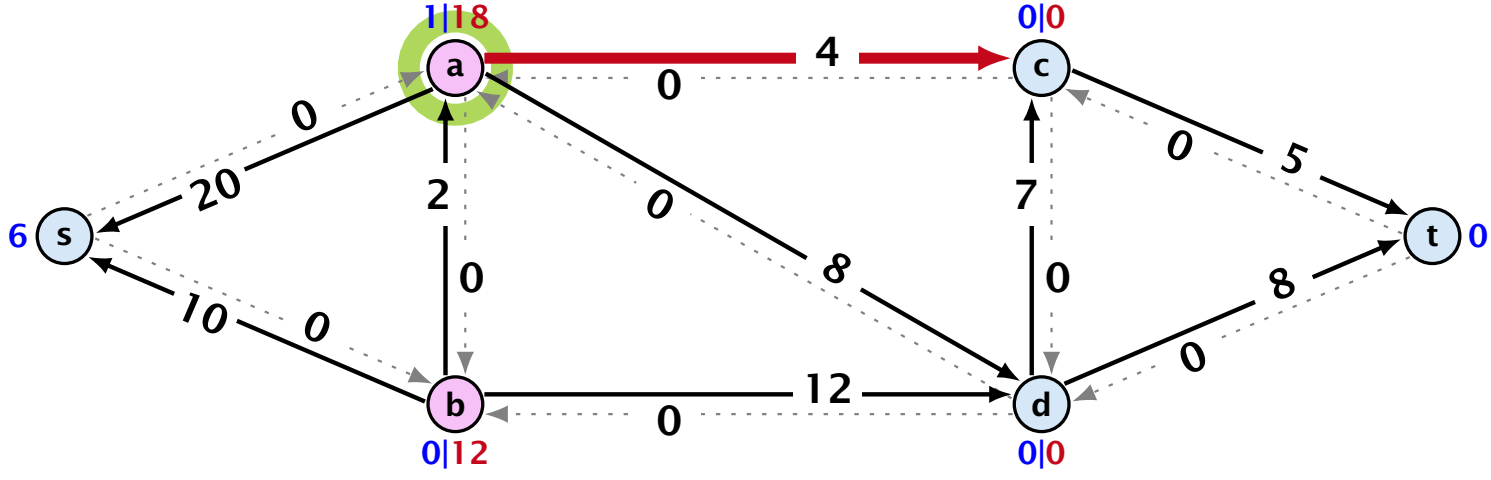

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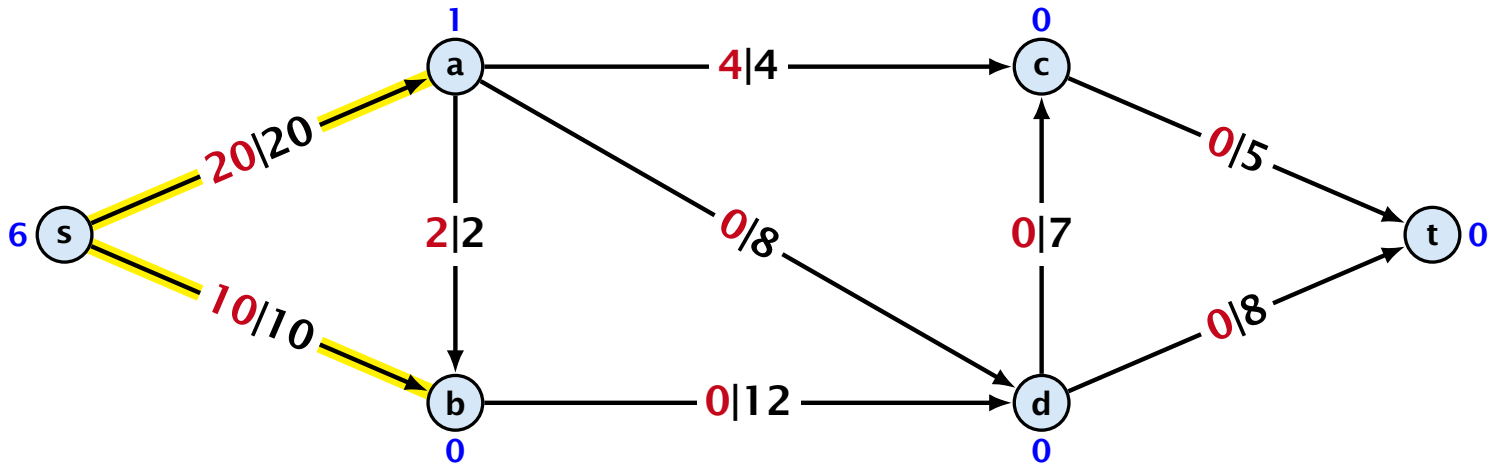

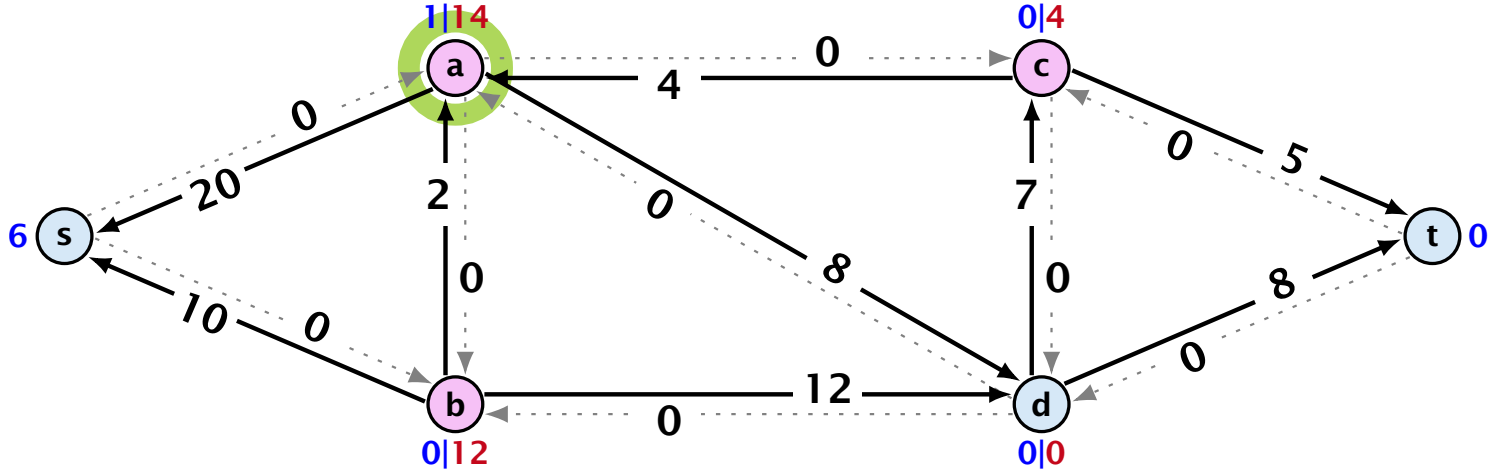

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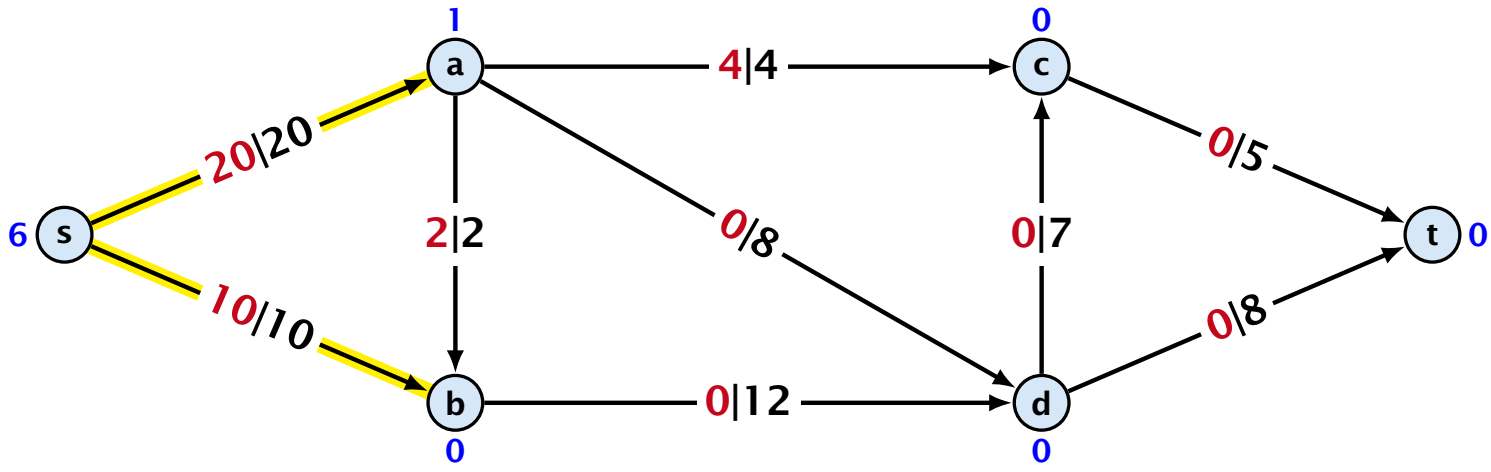

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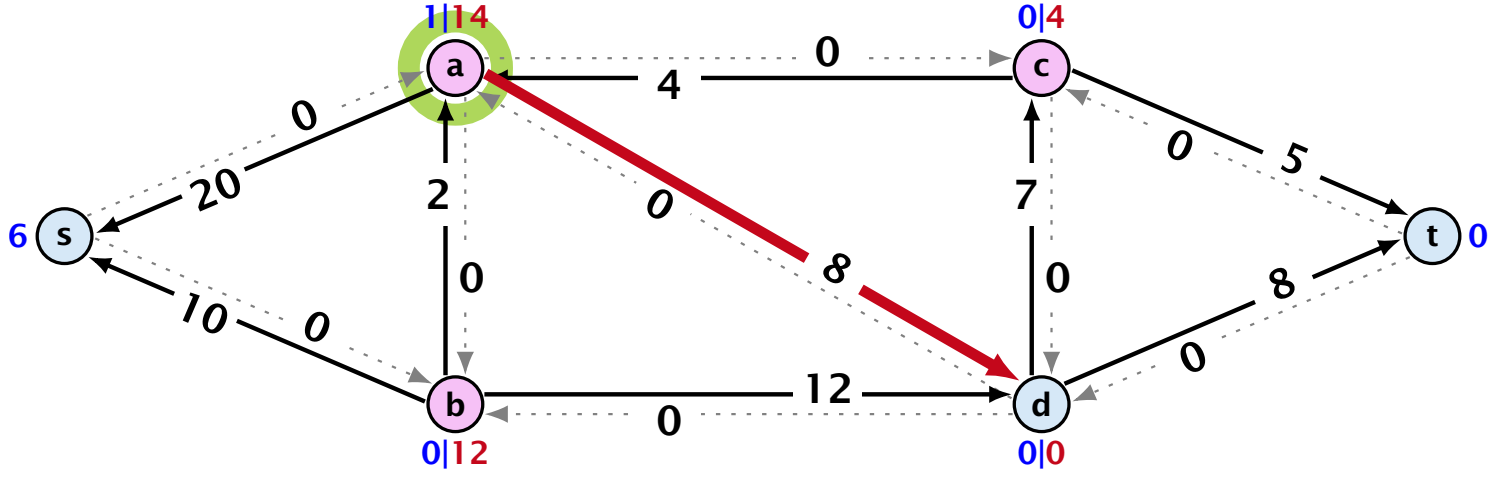

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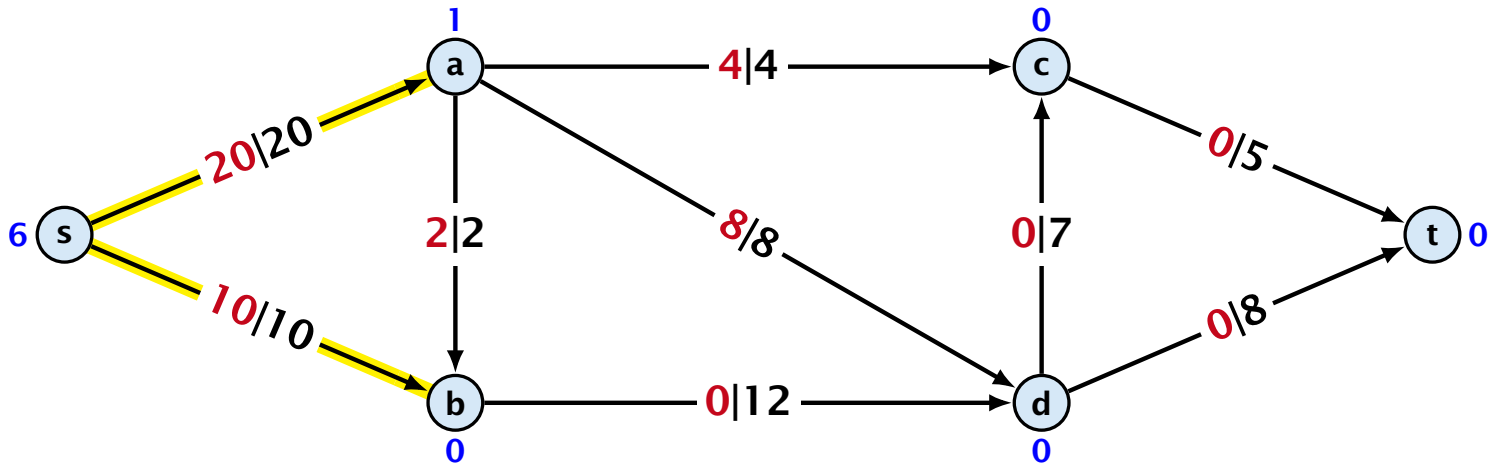

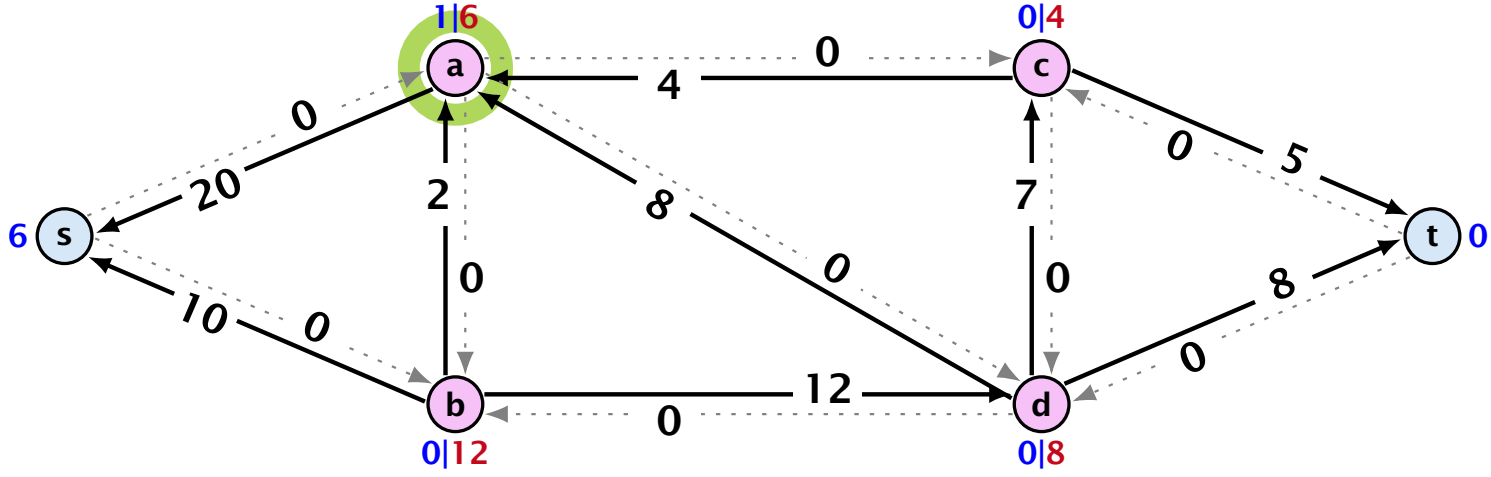

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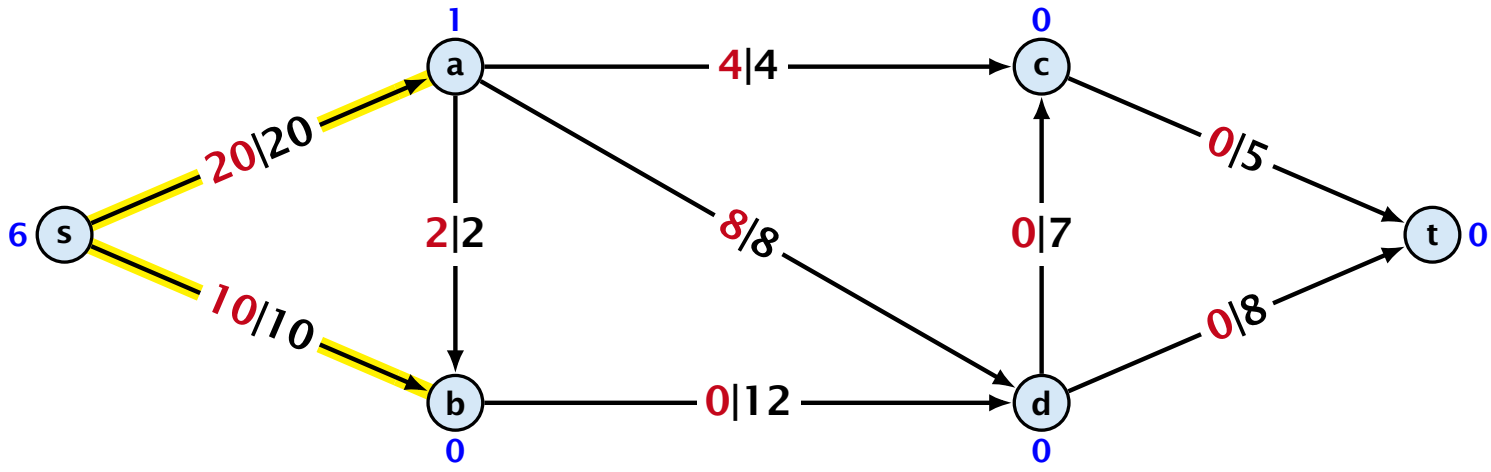
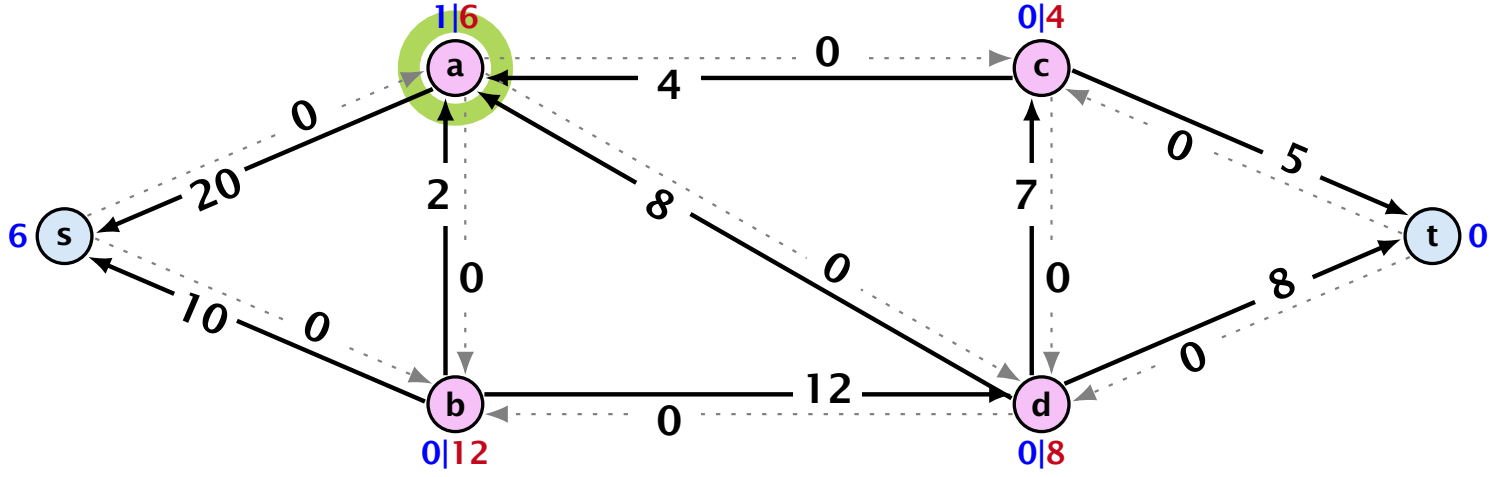


relabel to 7



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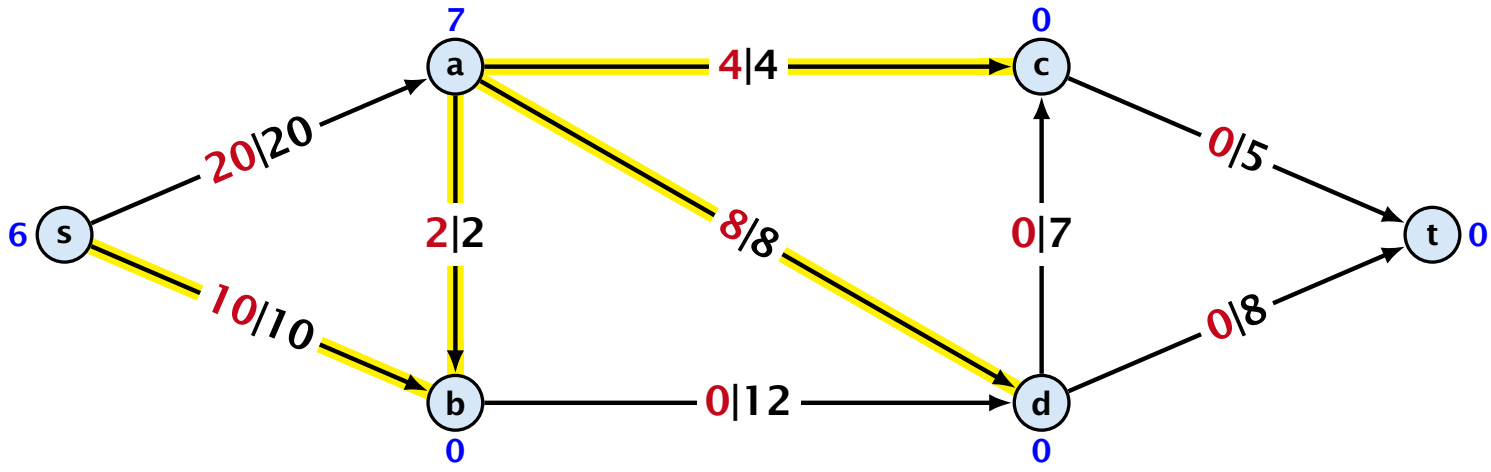
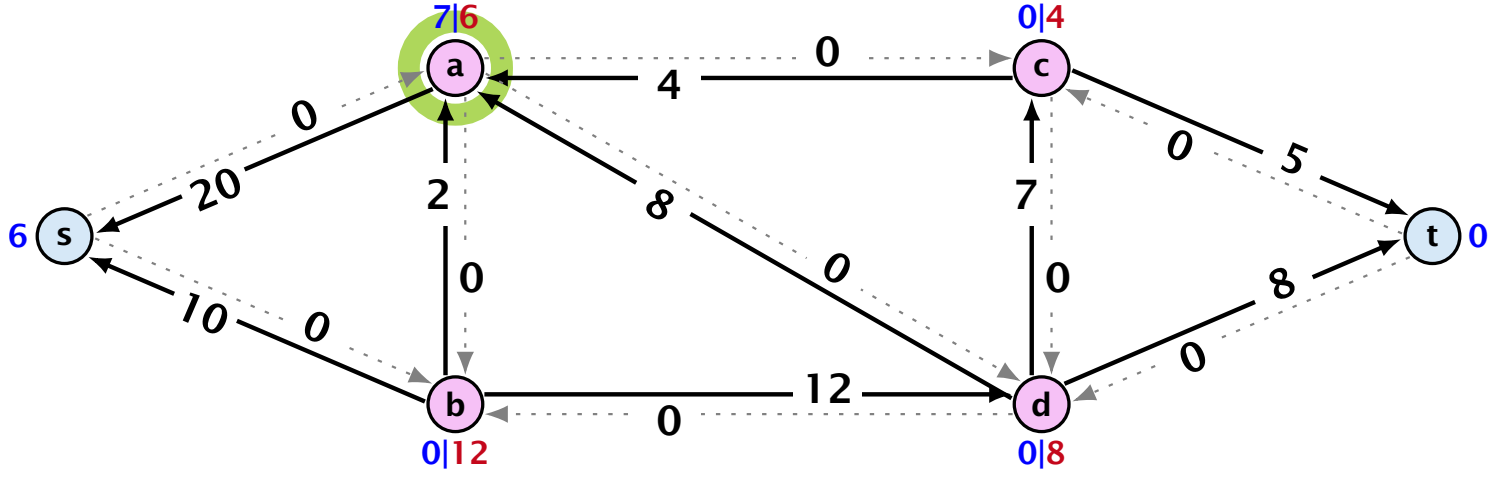
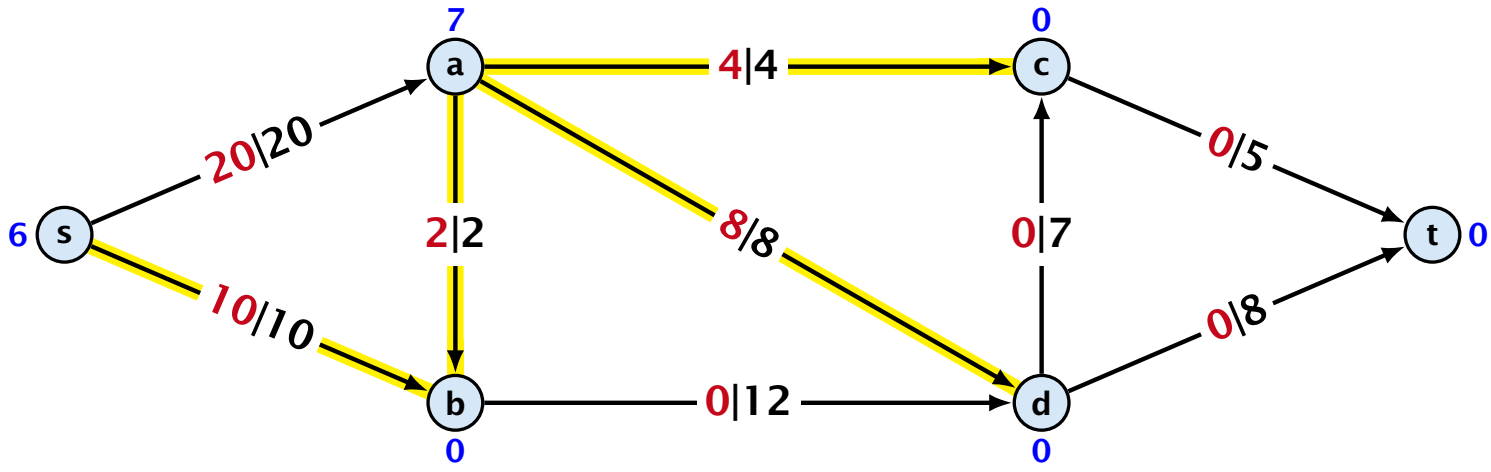


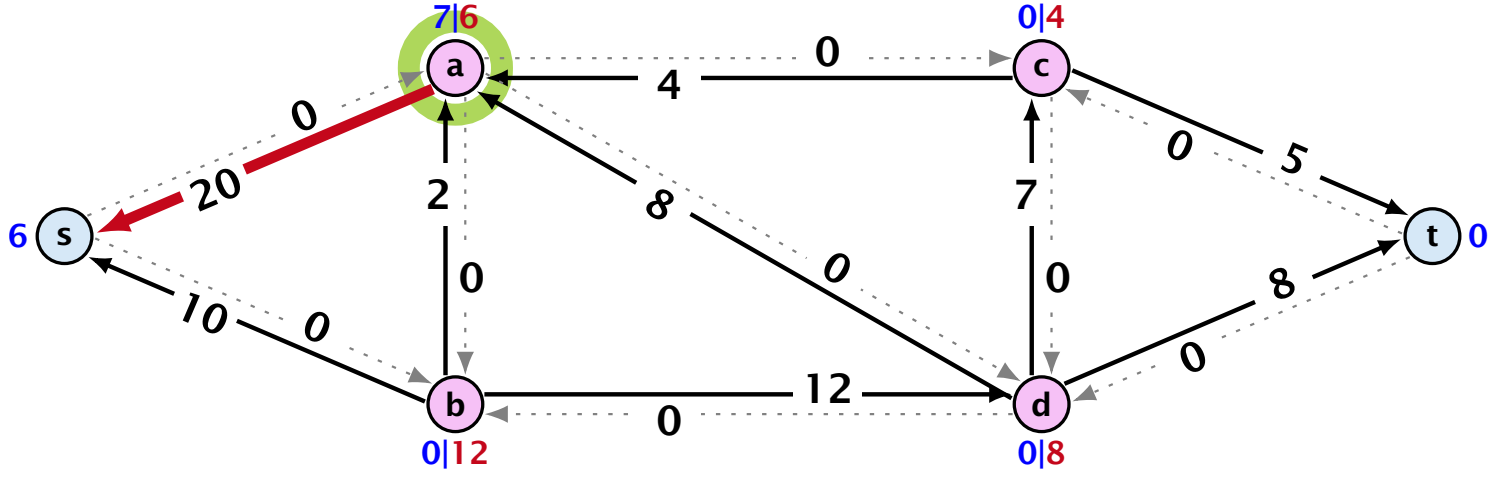


Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.



deactivating push



Preflow Push

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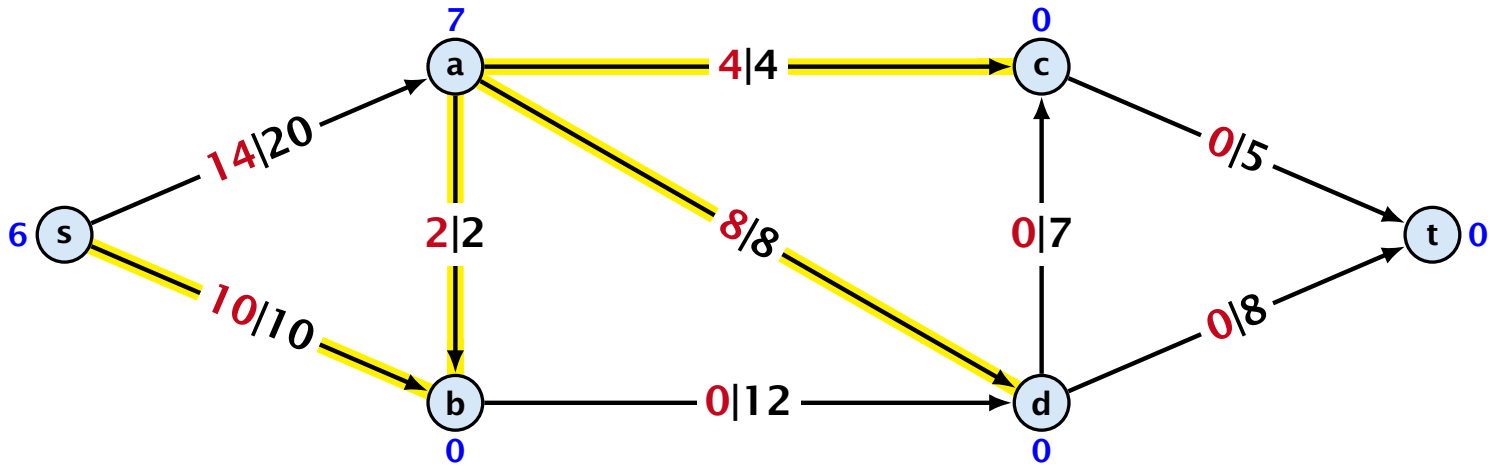

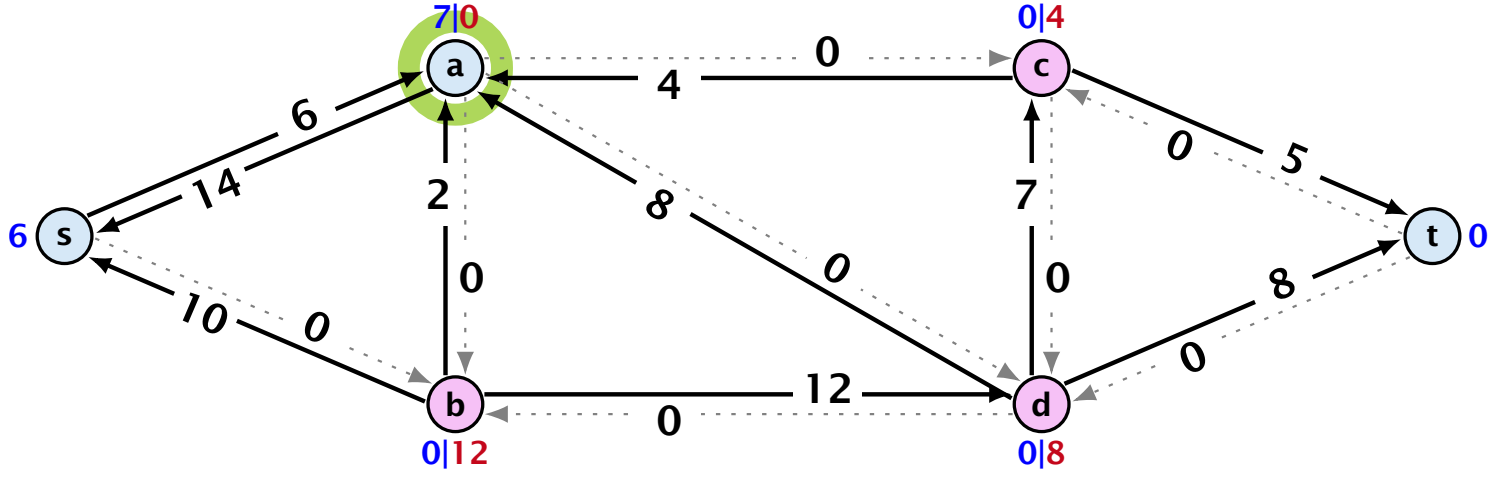

Preflow Push

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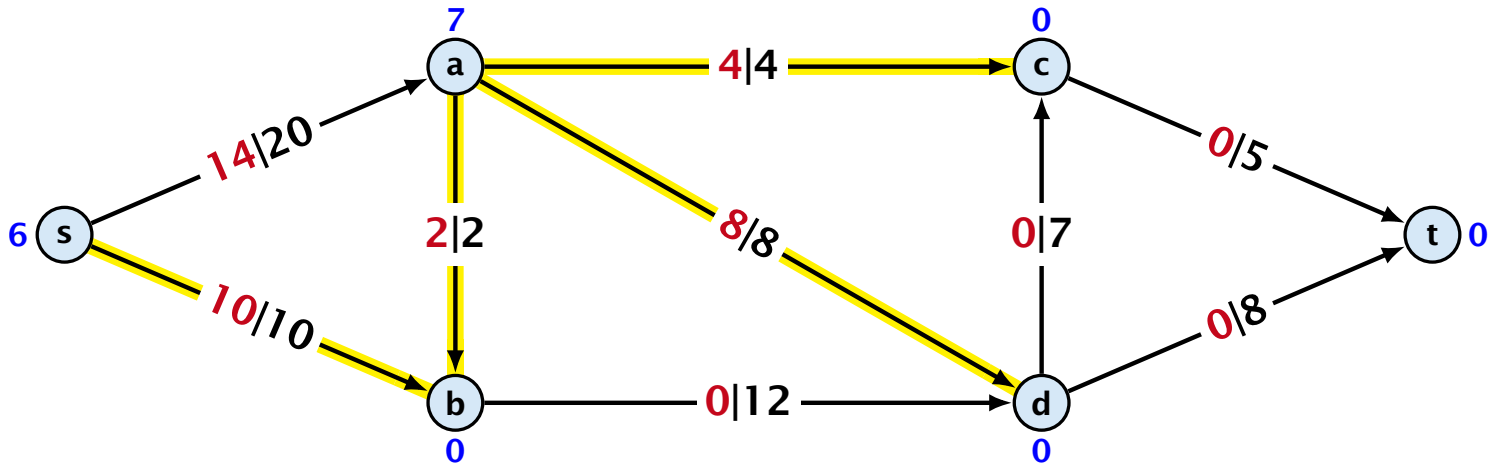

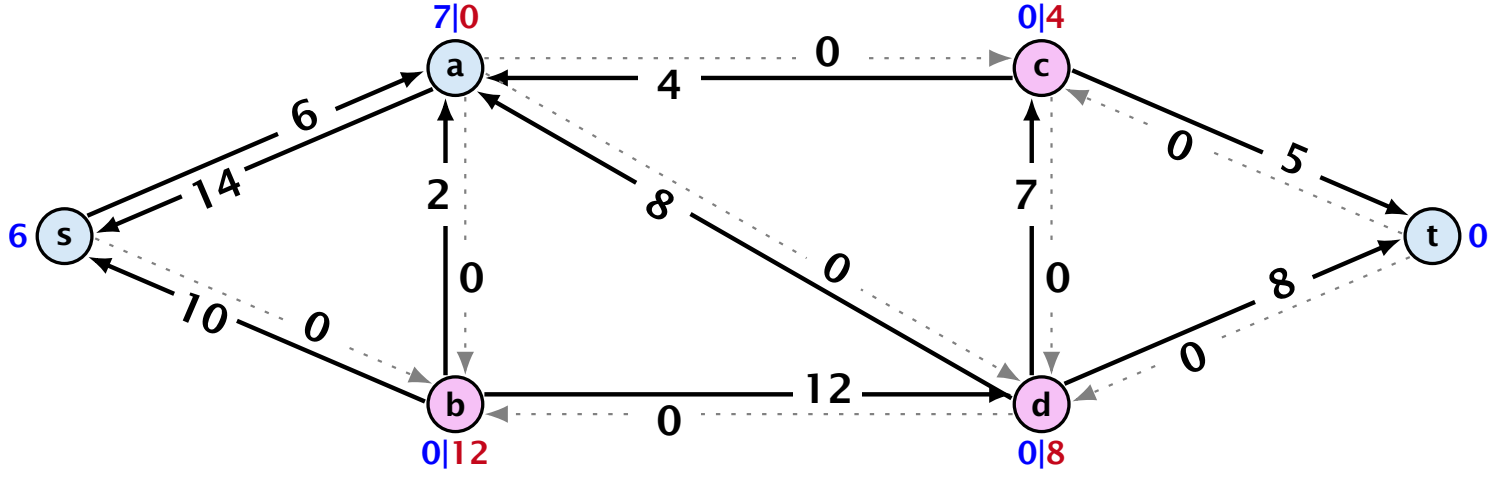

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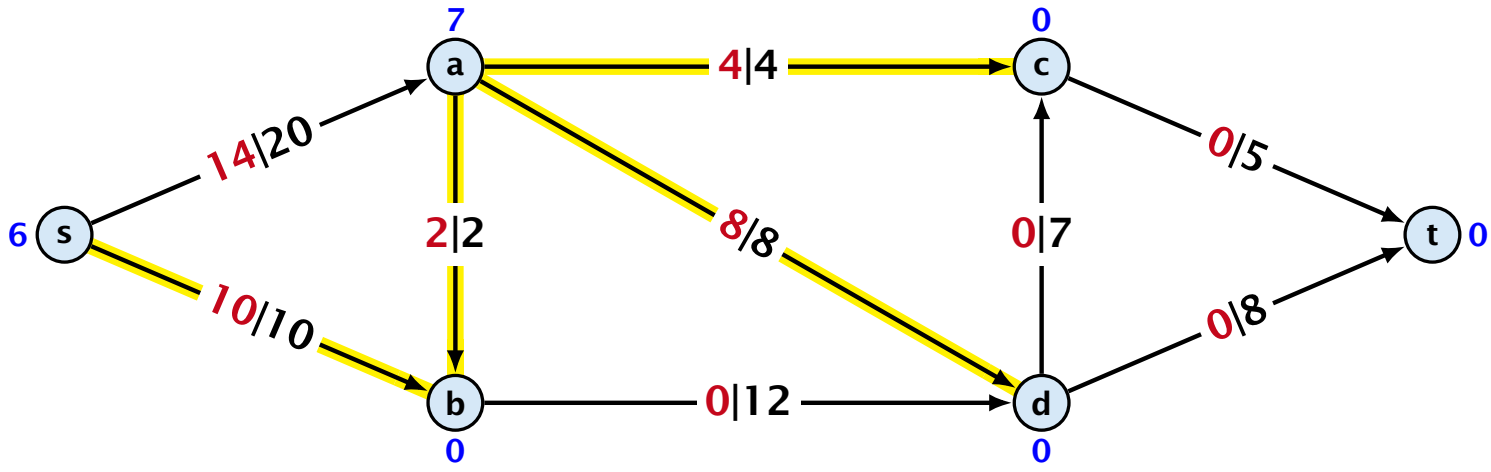

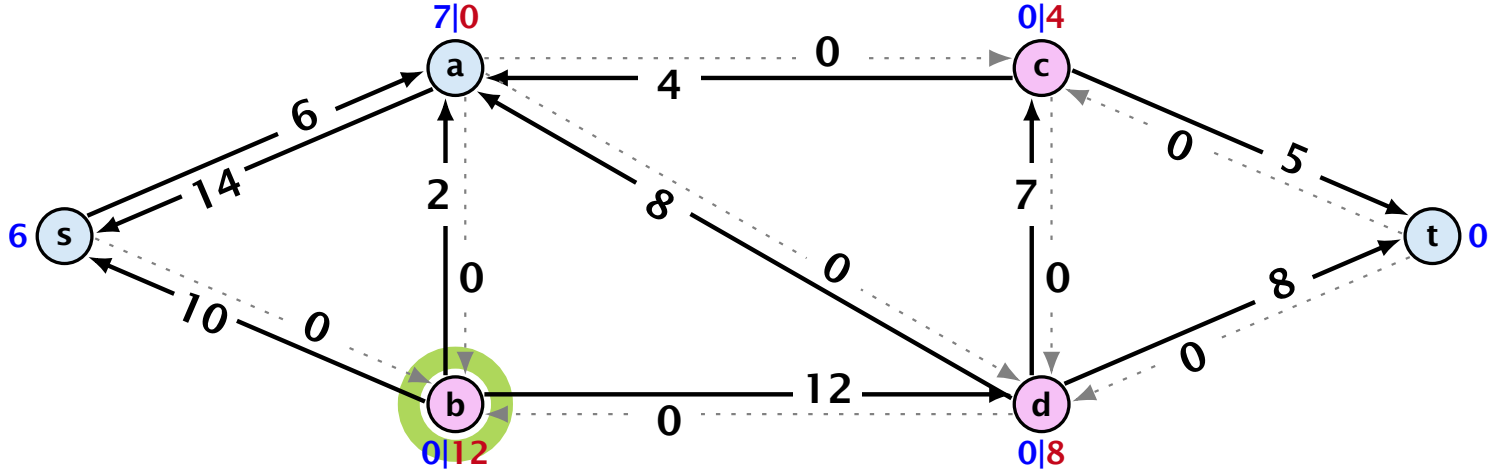

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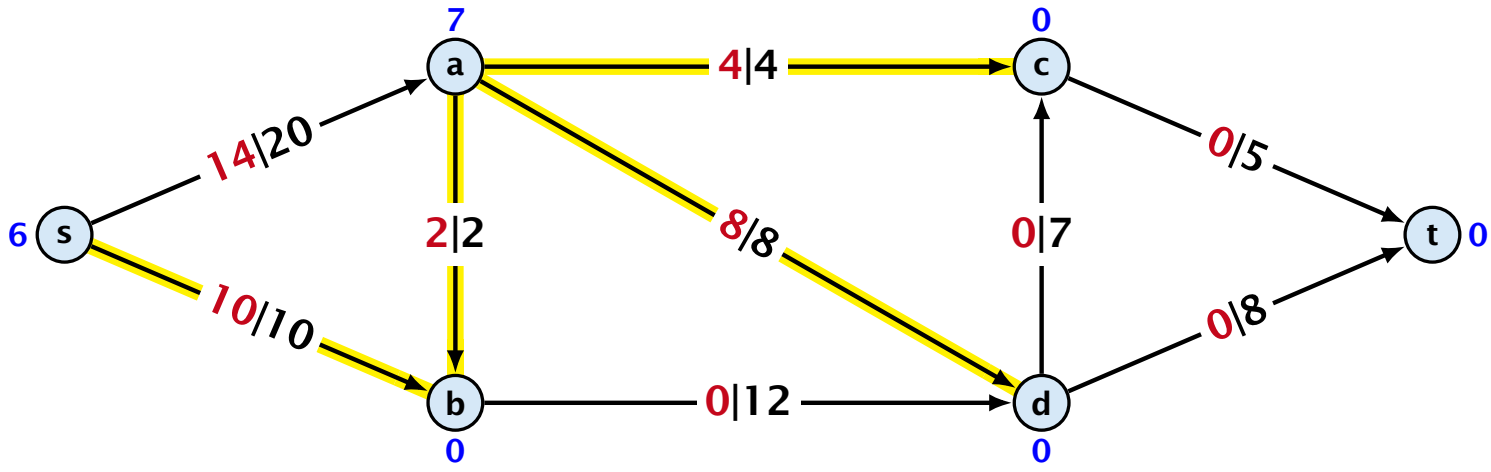
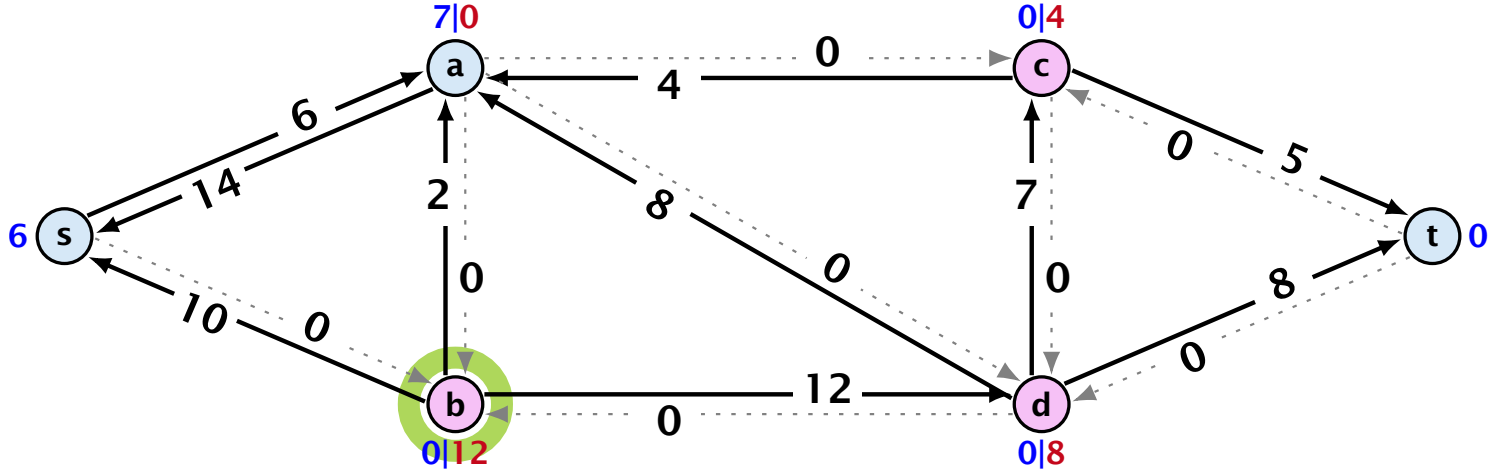


relabel to 1



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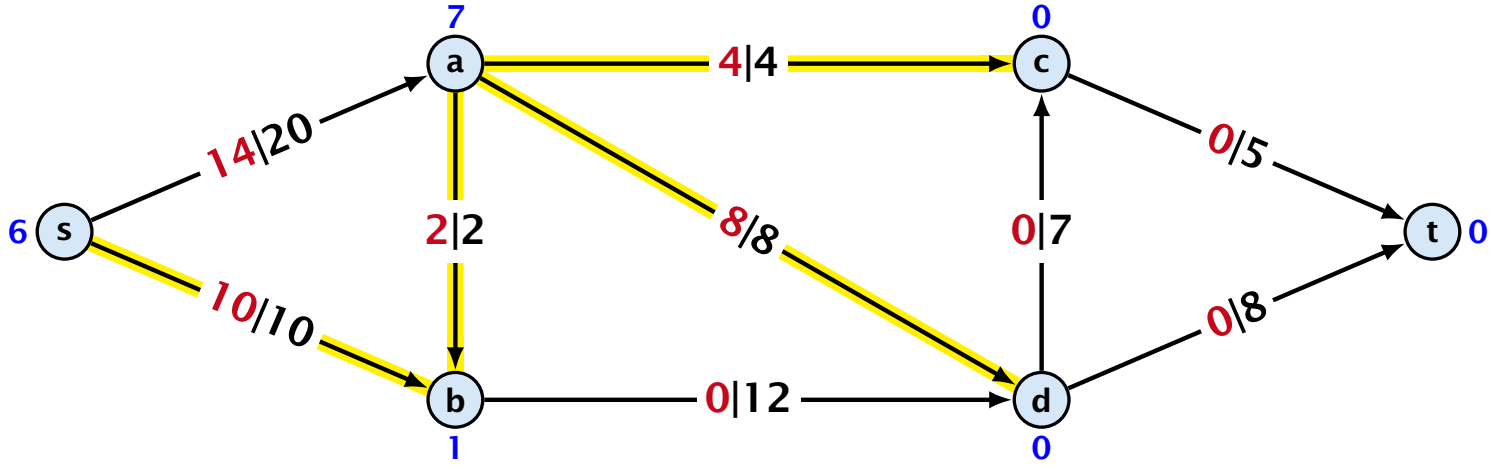

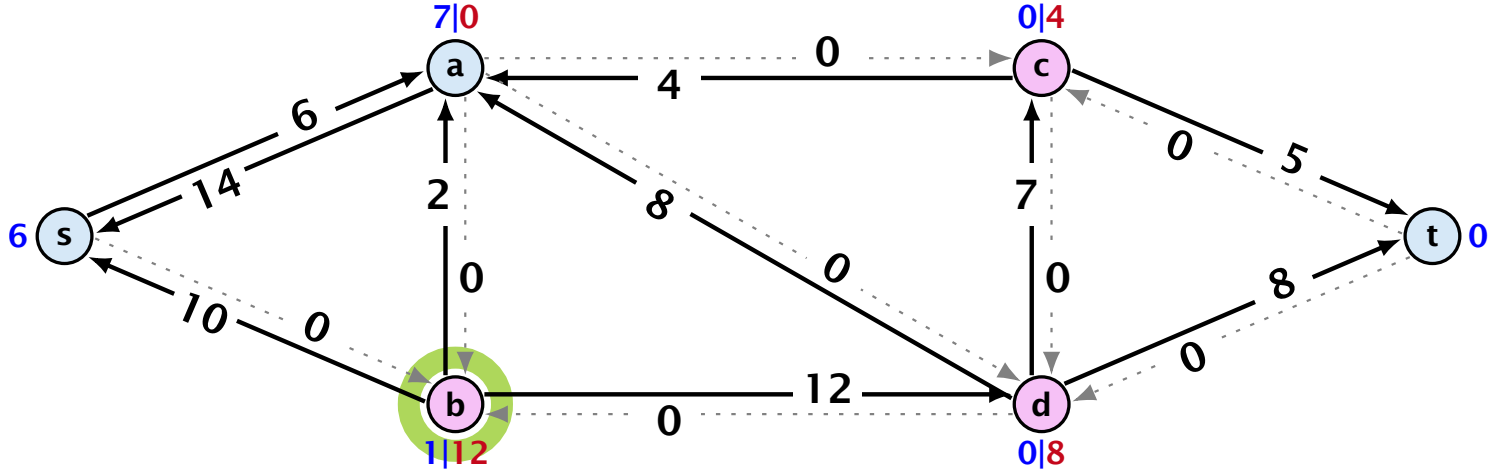

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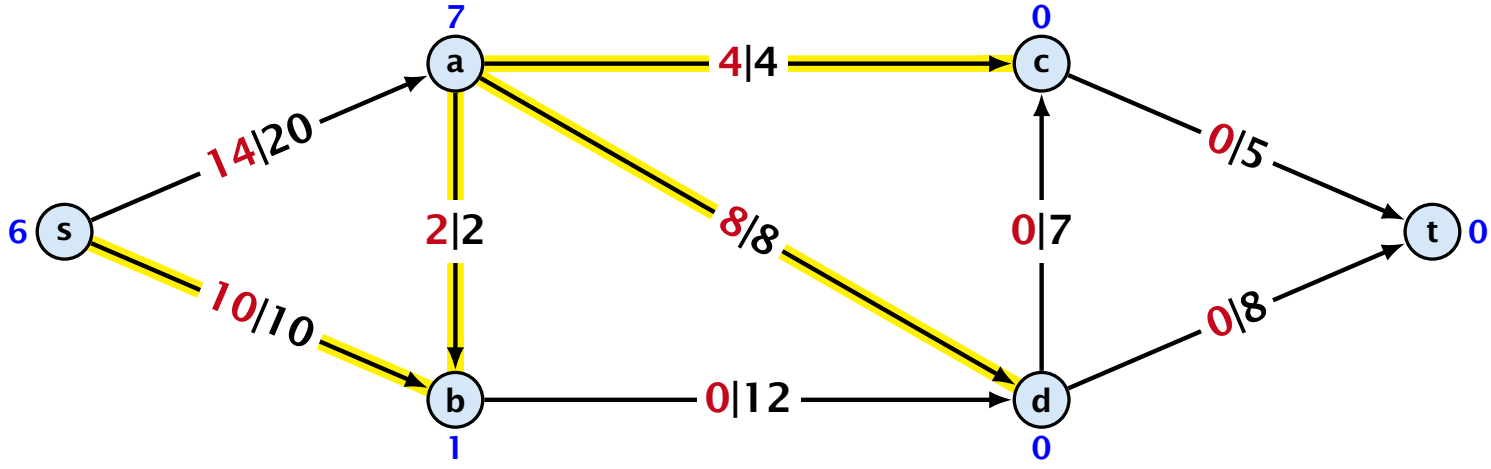
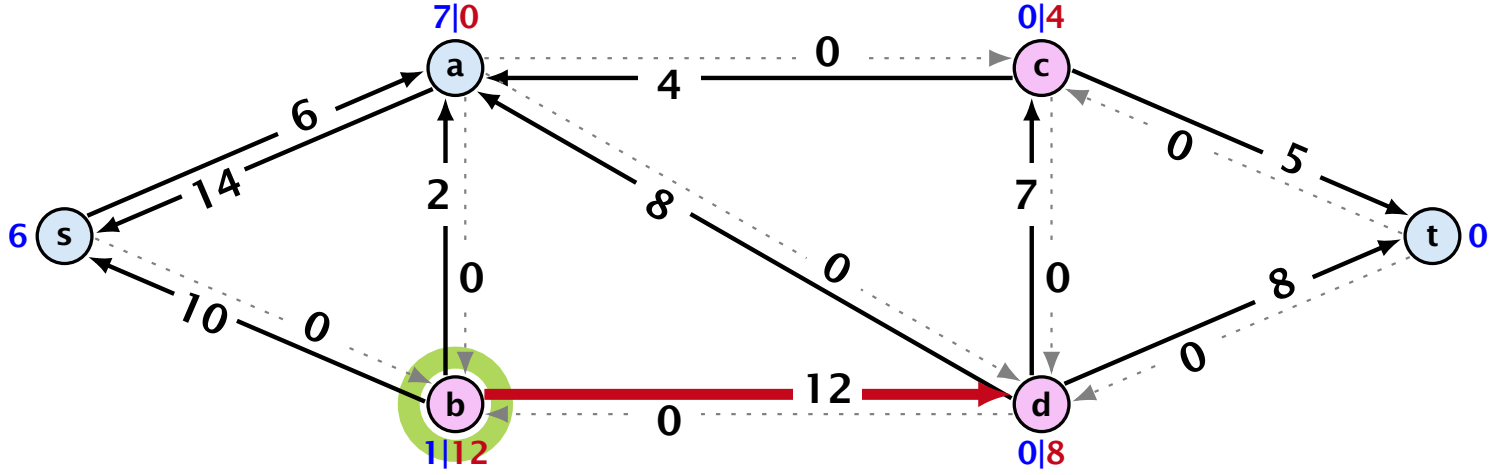


saturating and deactivating push



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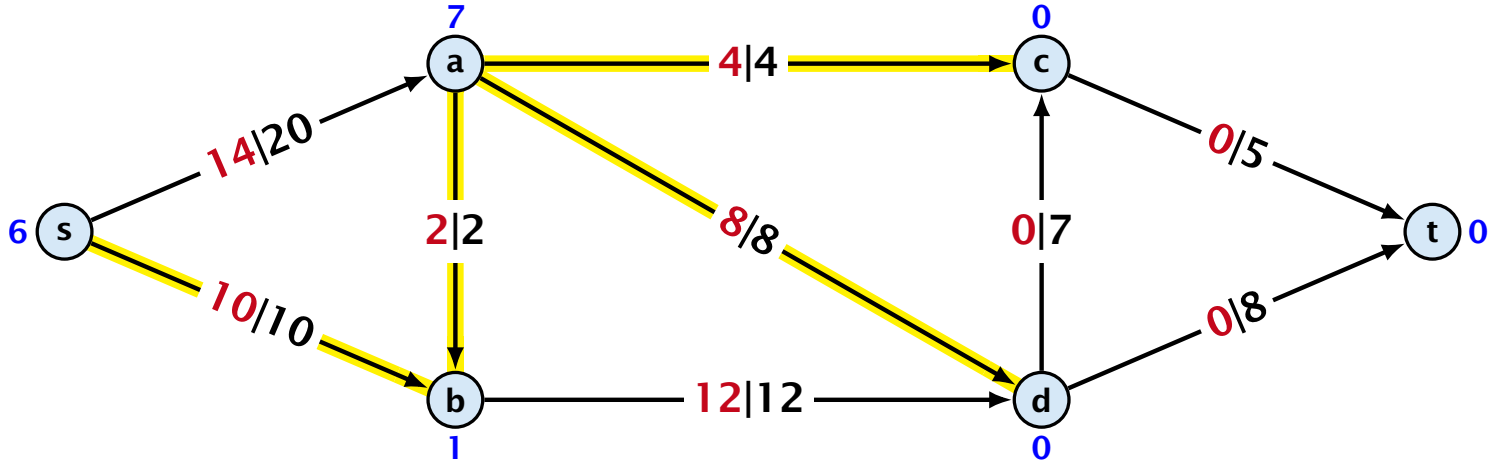

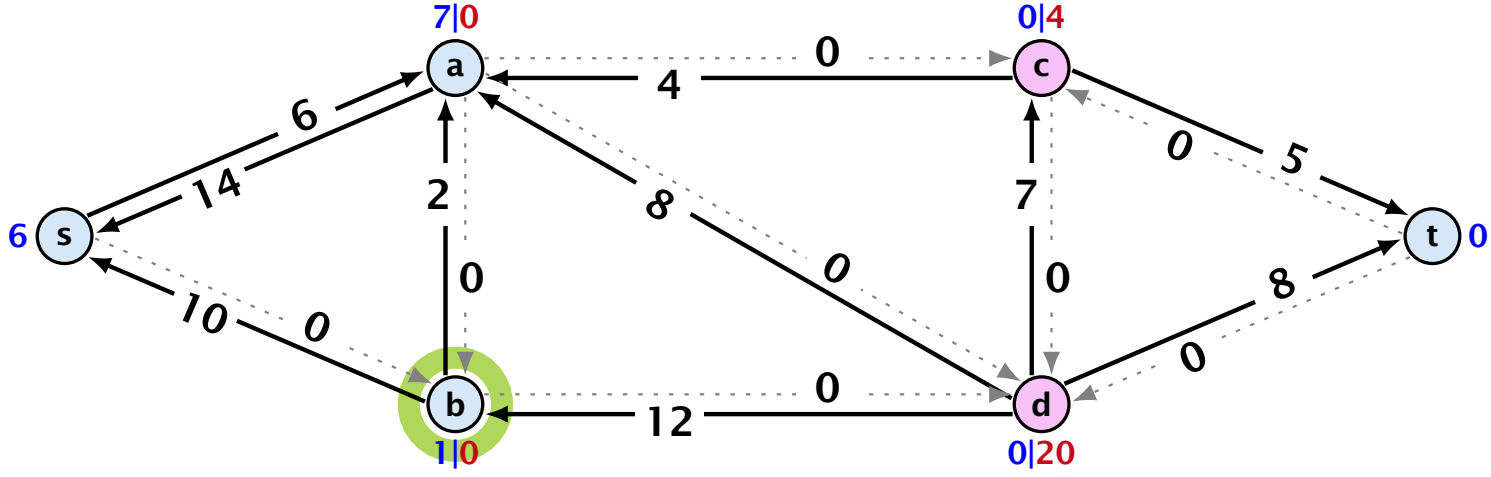

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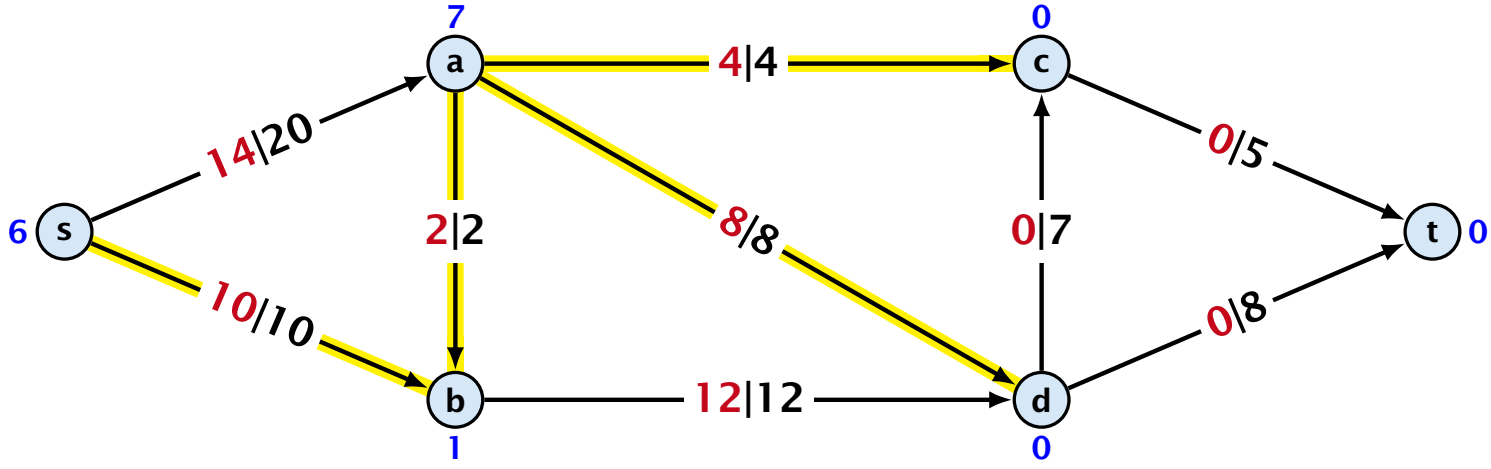

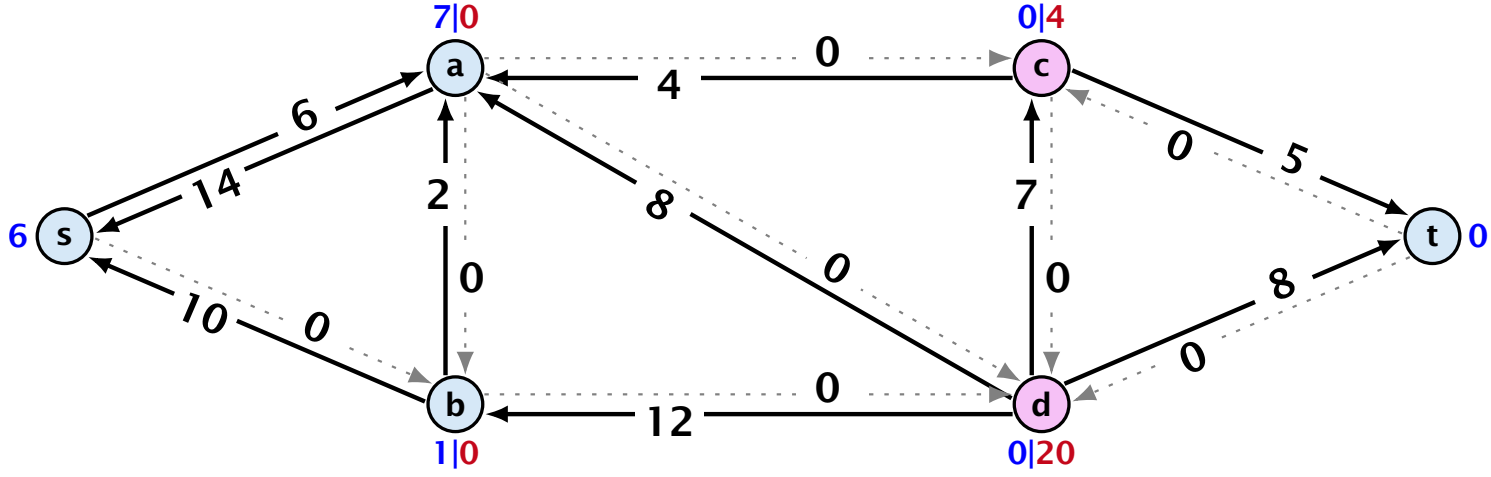

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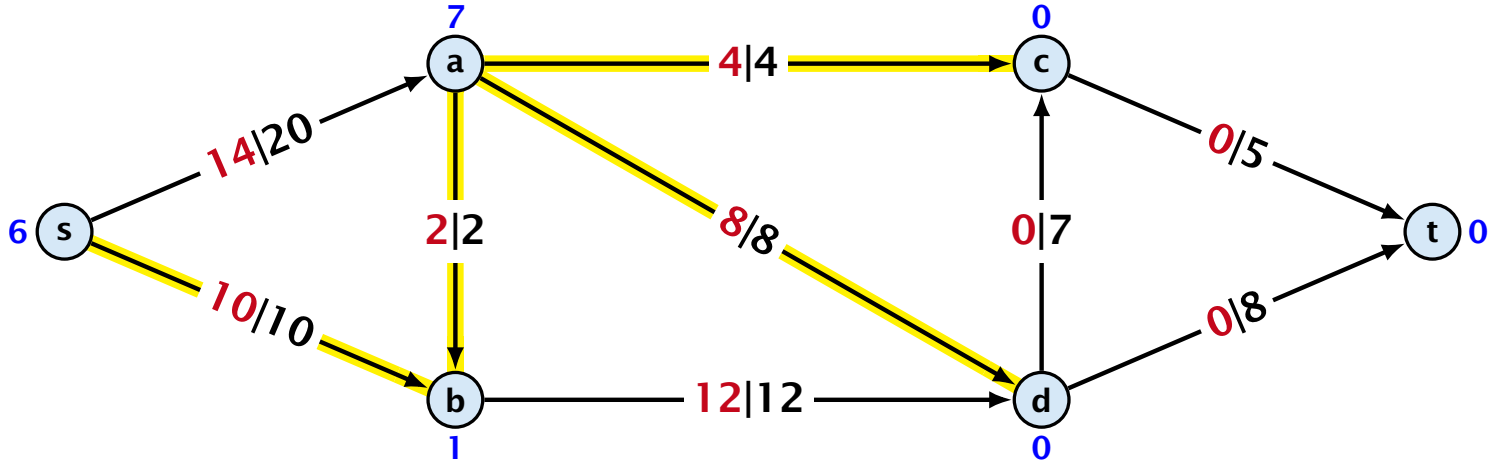

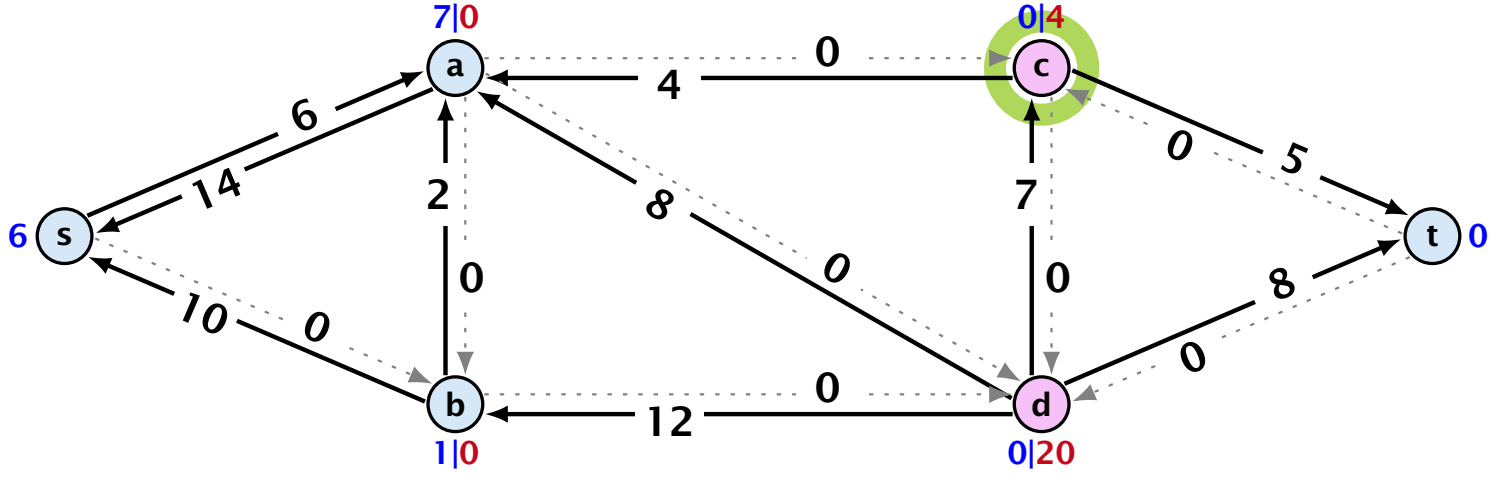

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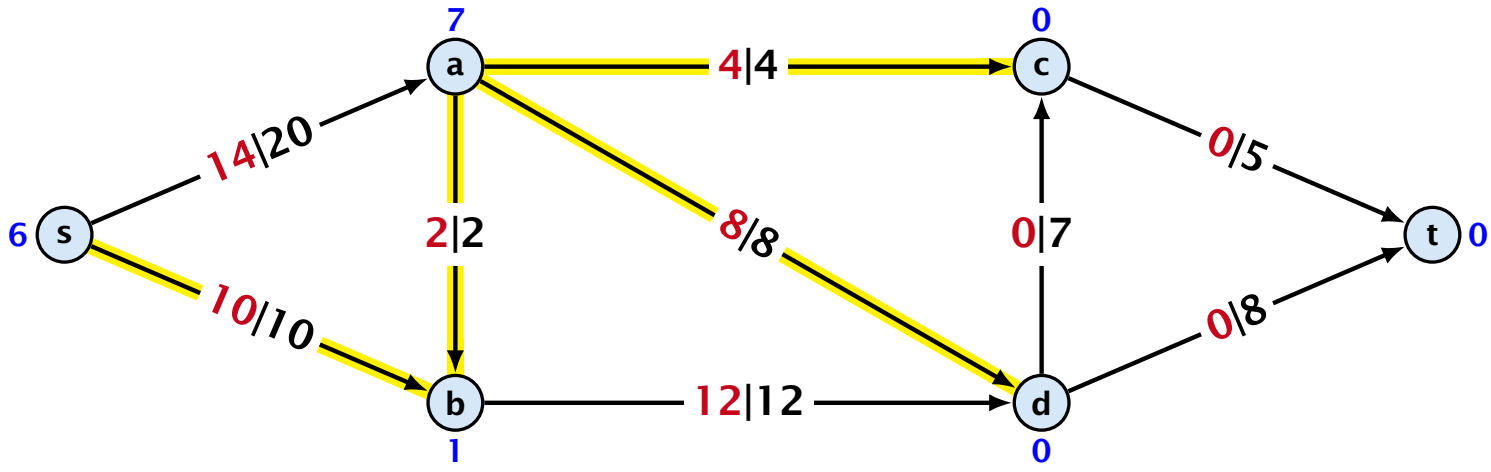
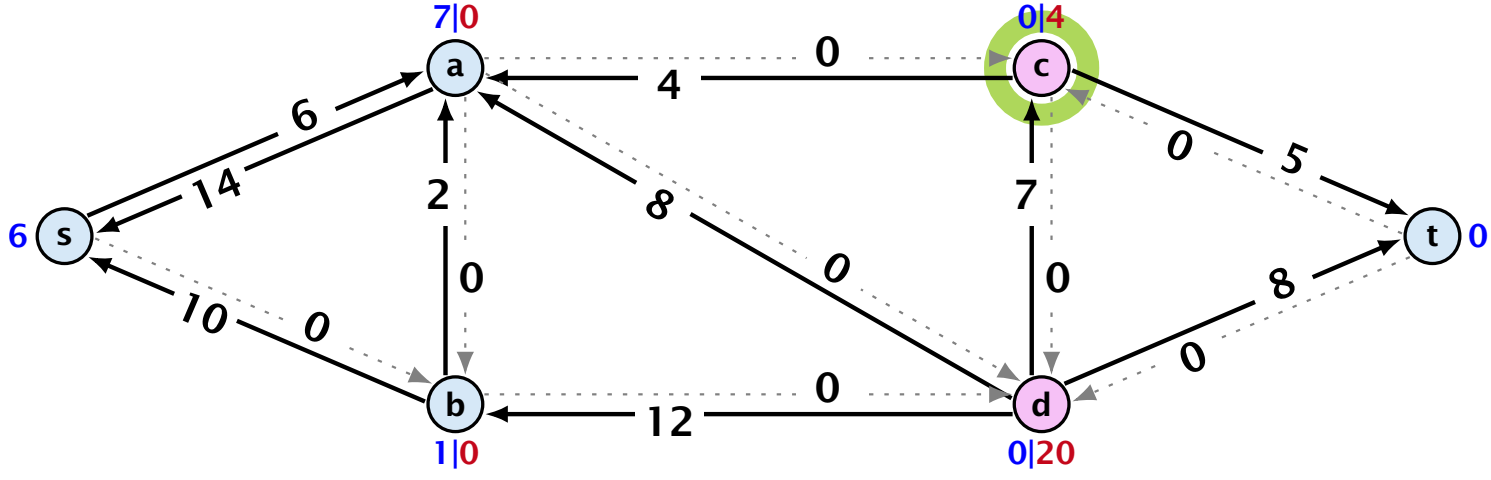


relabel to 1



Preflow Push

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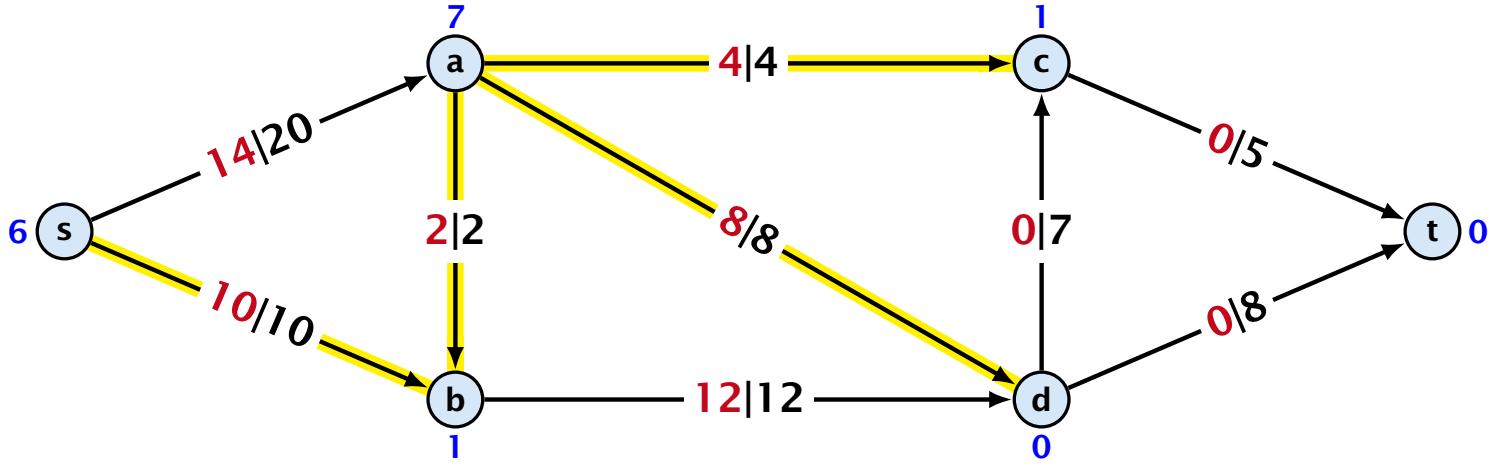

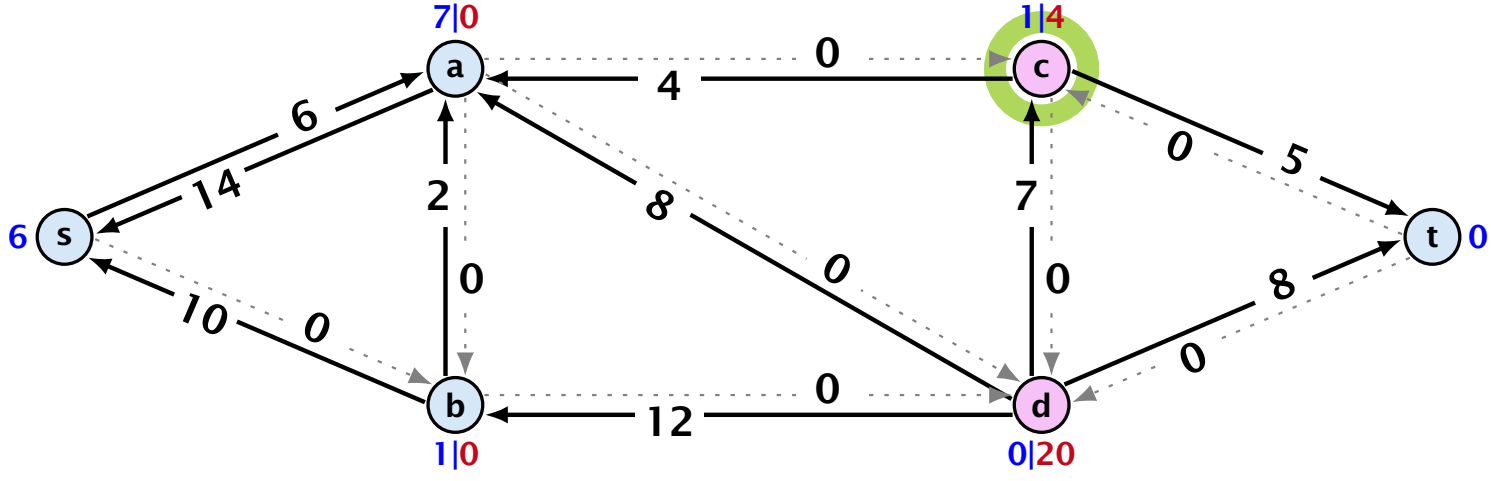

Preflow Push

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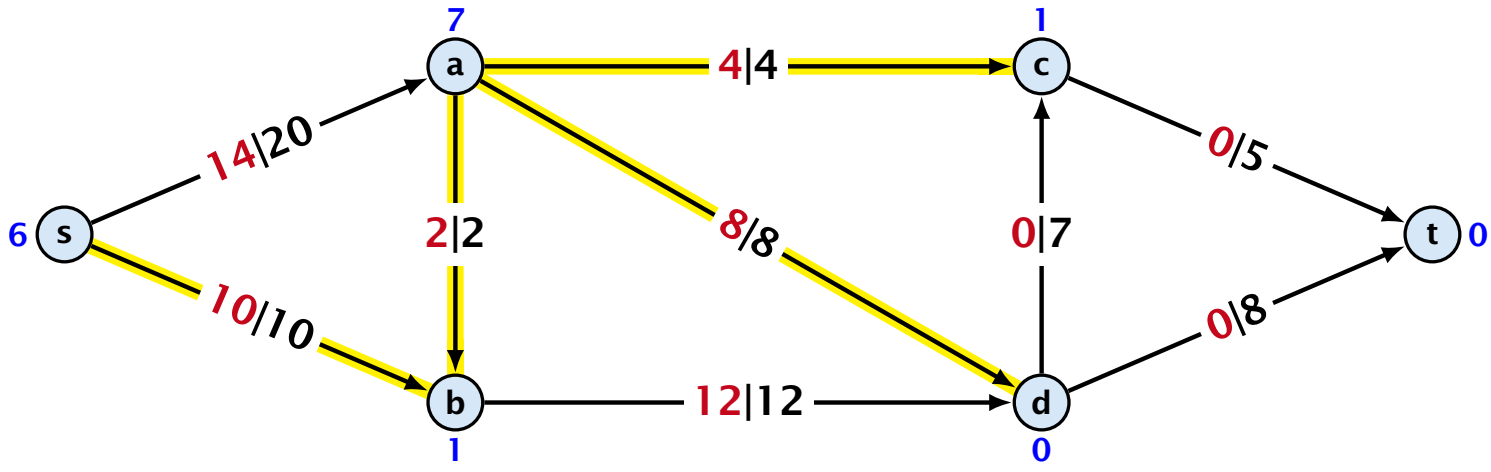
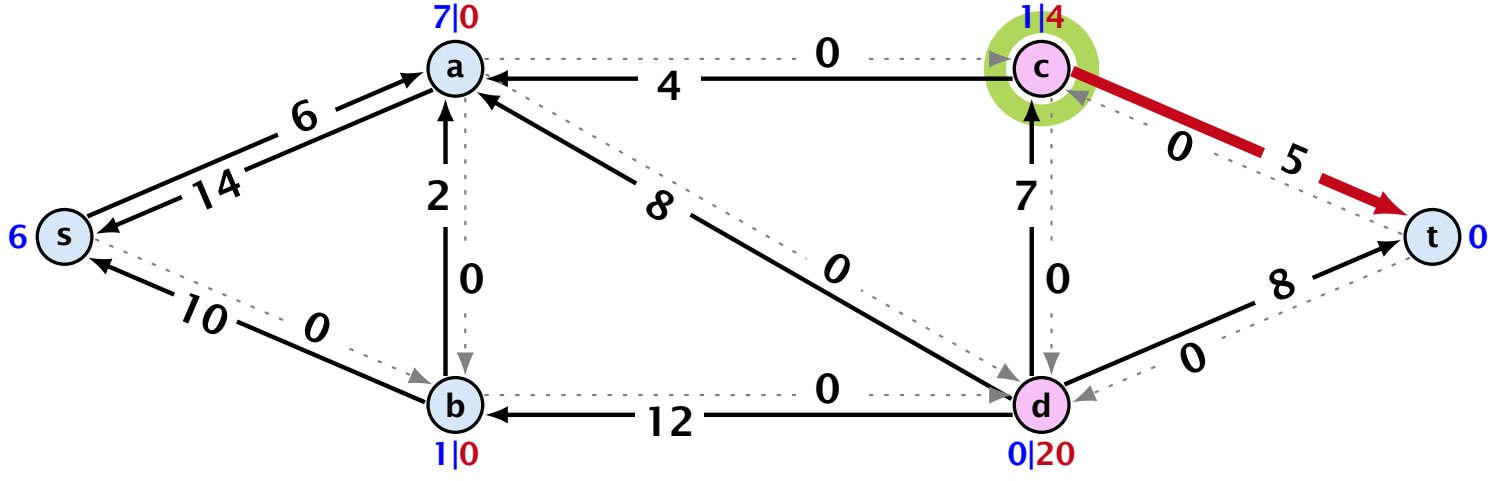


deactivating push



Preflow Push

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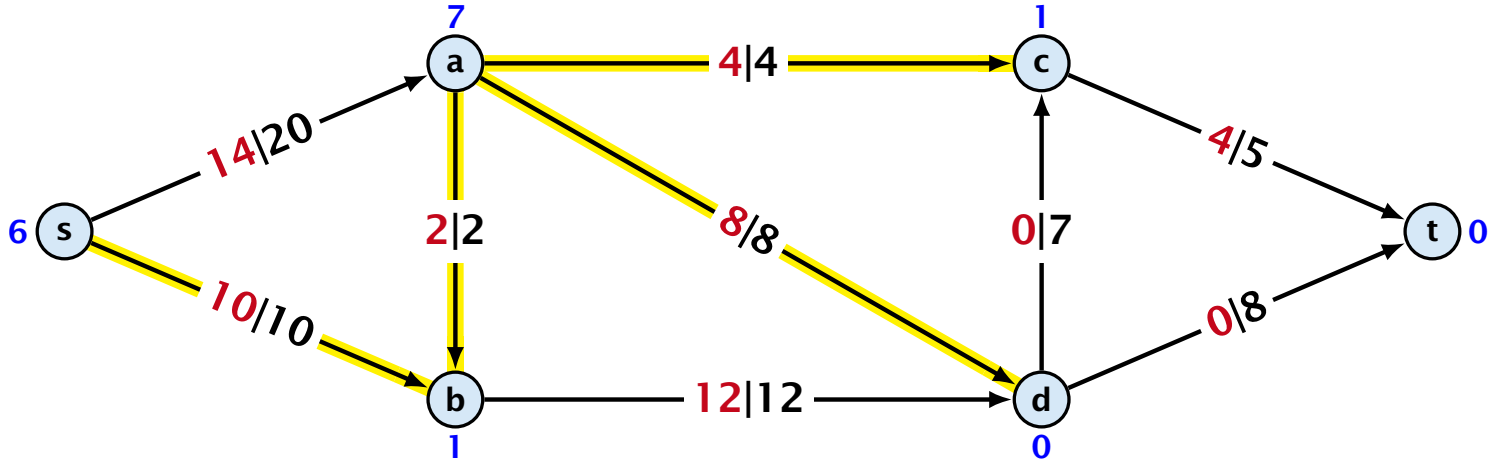

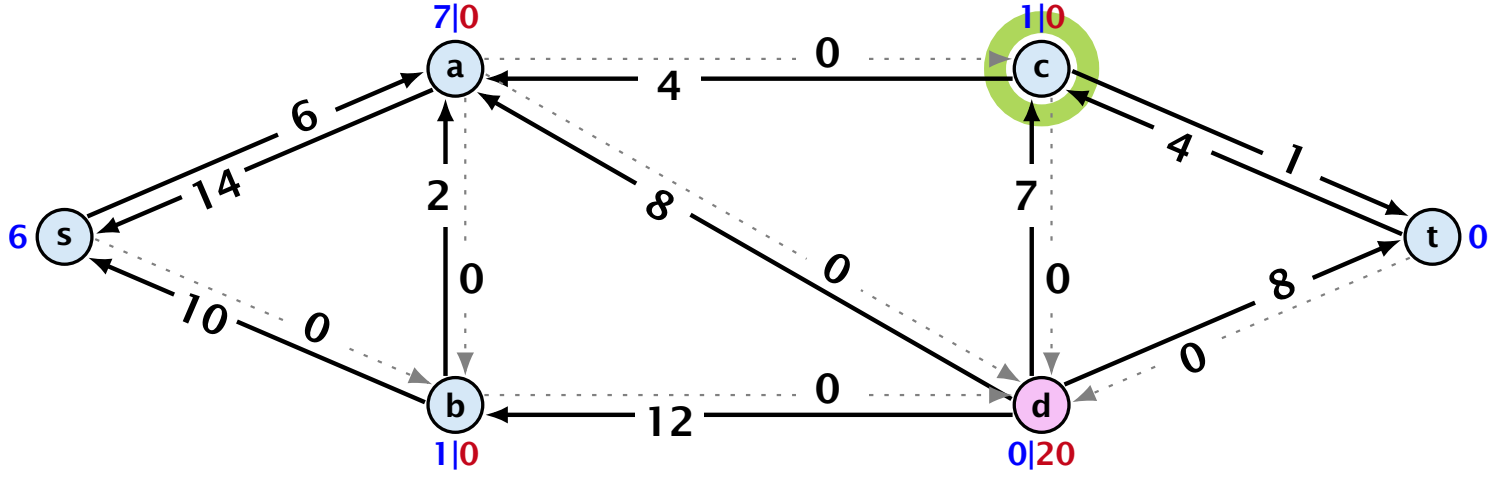

Preflow Push

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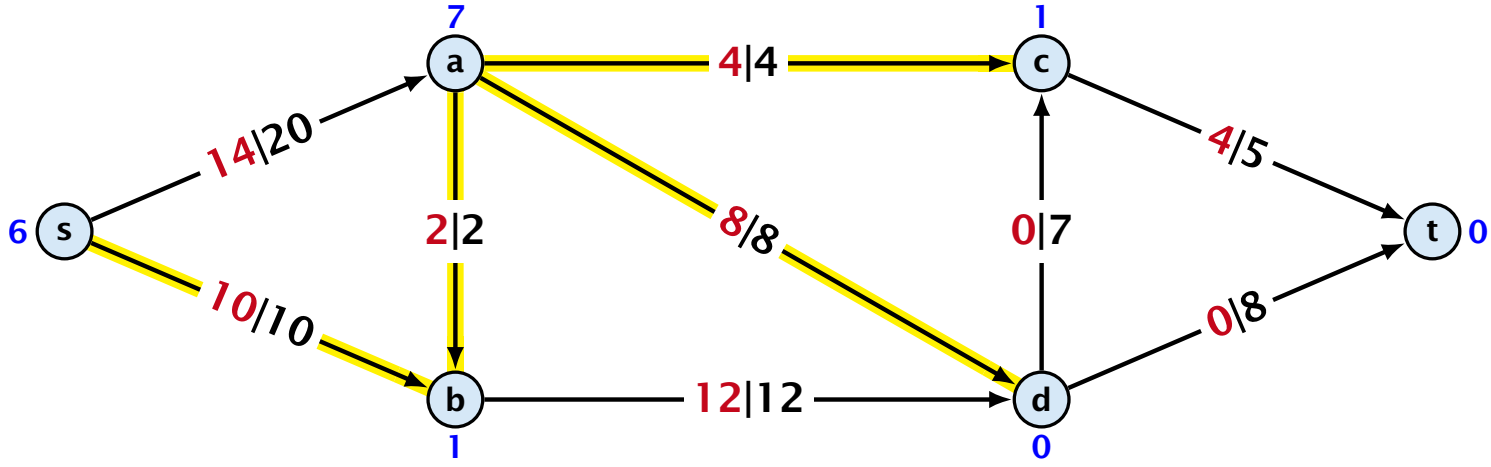

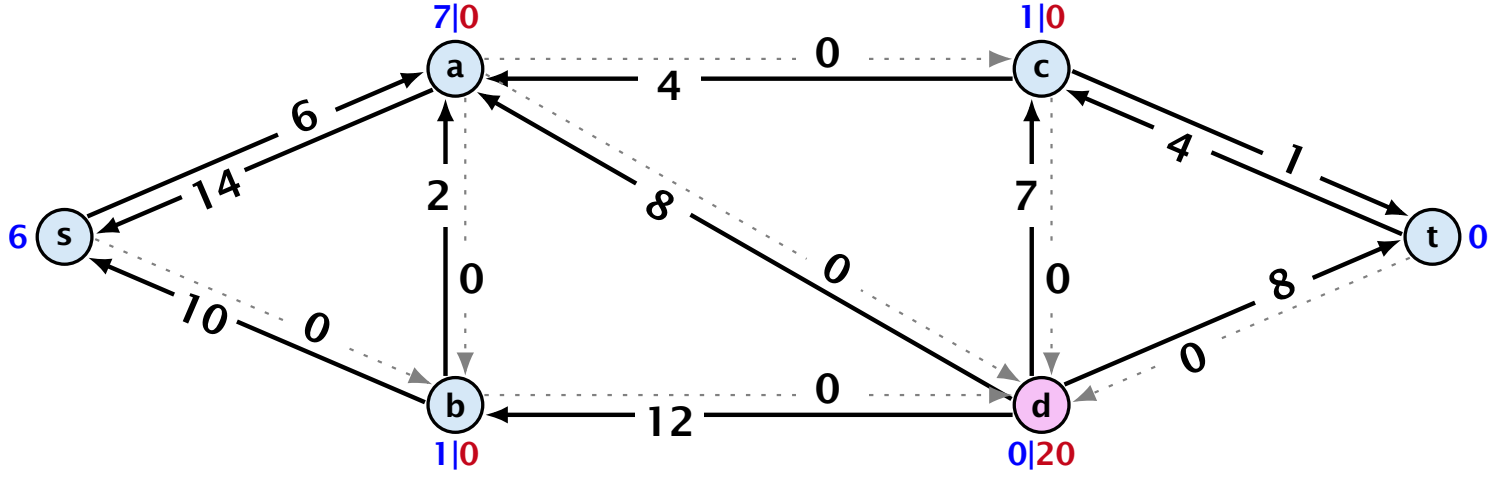

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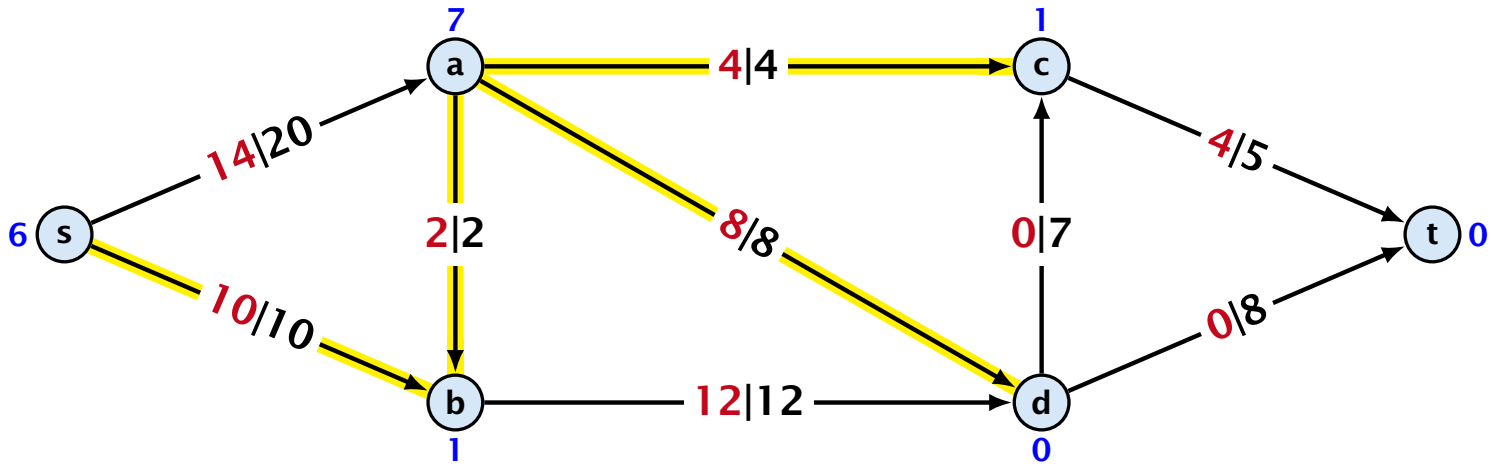

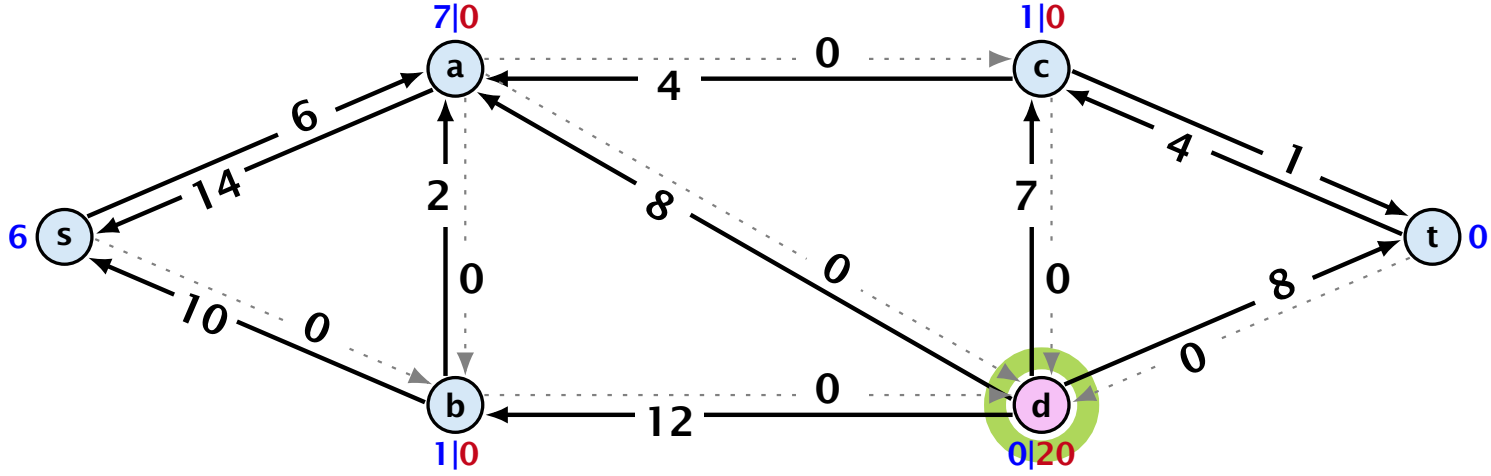

Preflow Push

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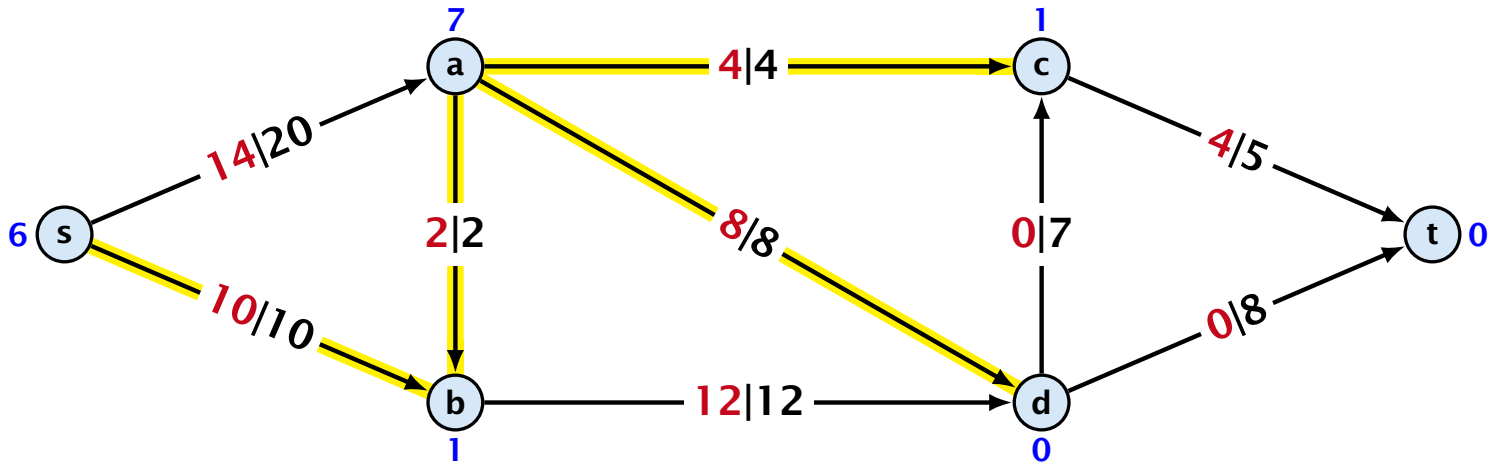
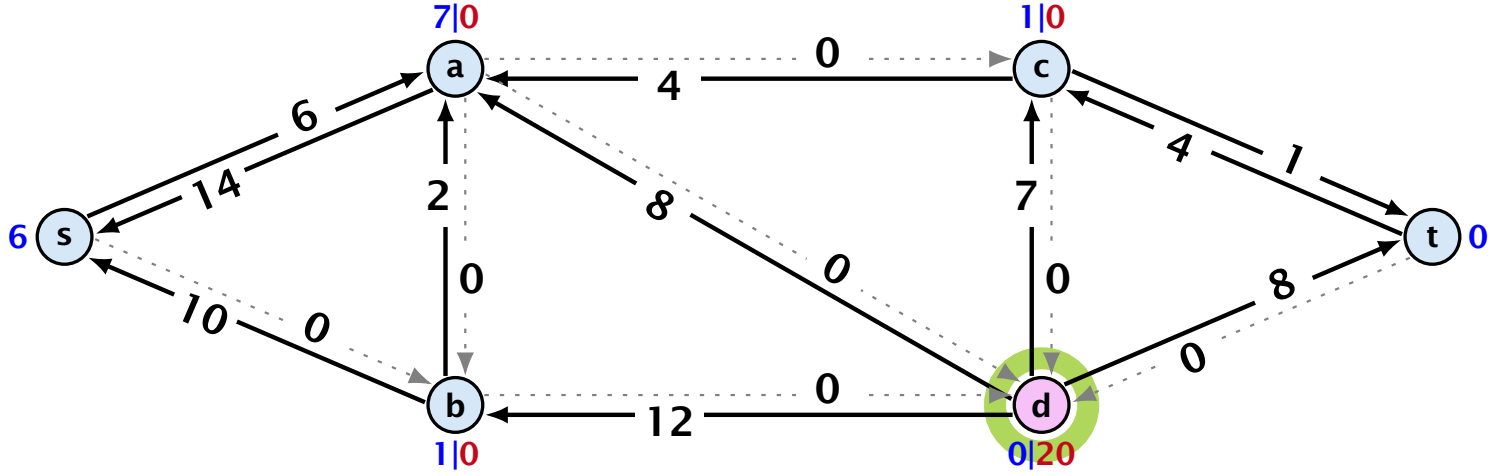


relabel to 1



Preflow Push

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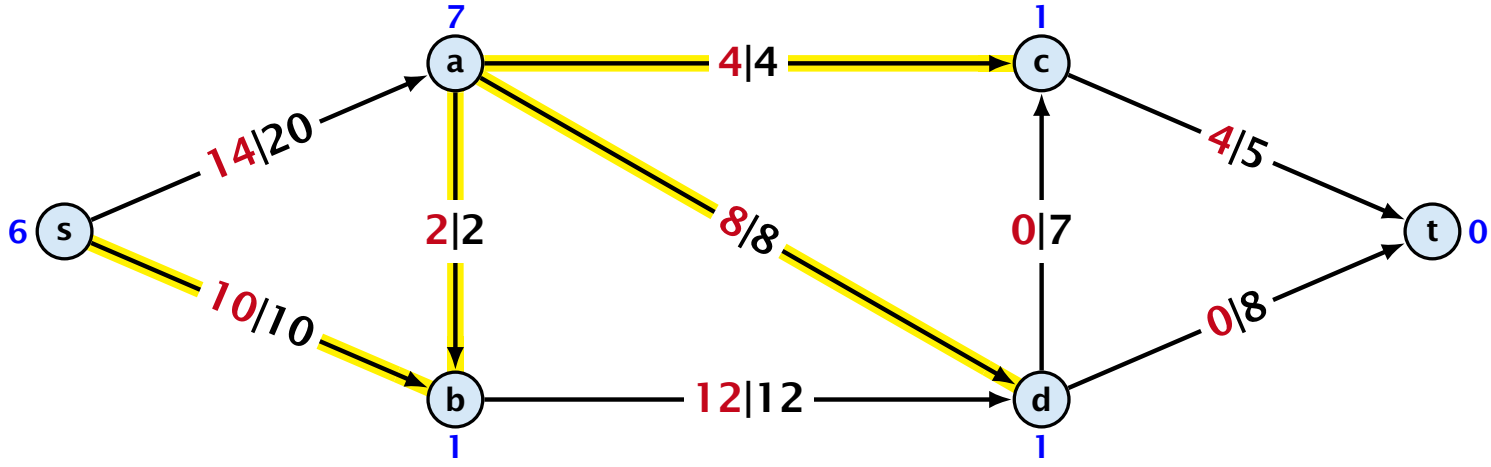

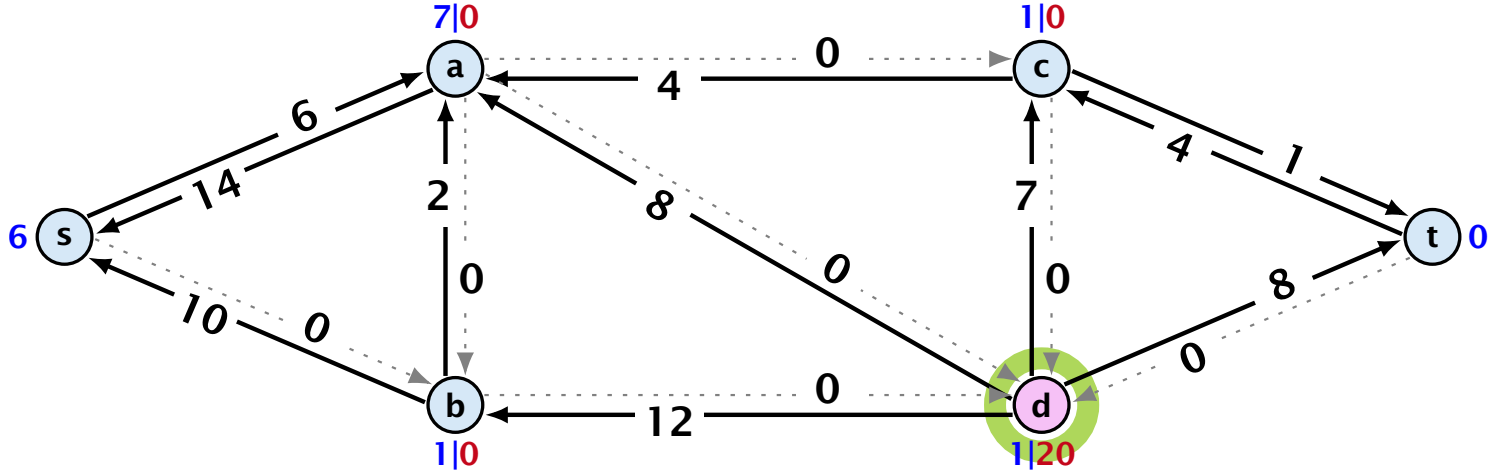

Preflow Push

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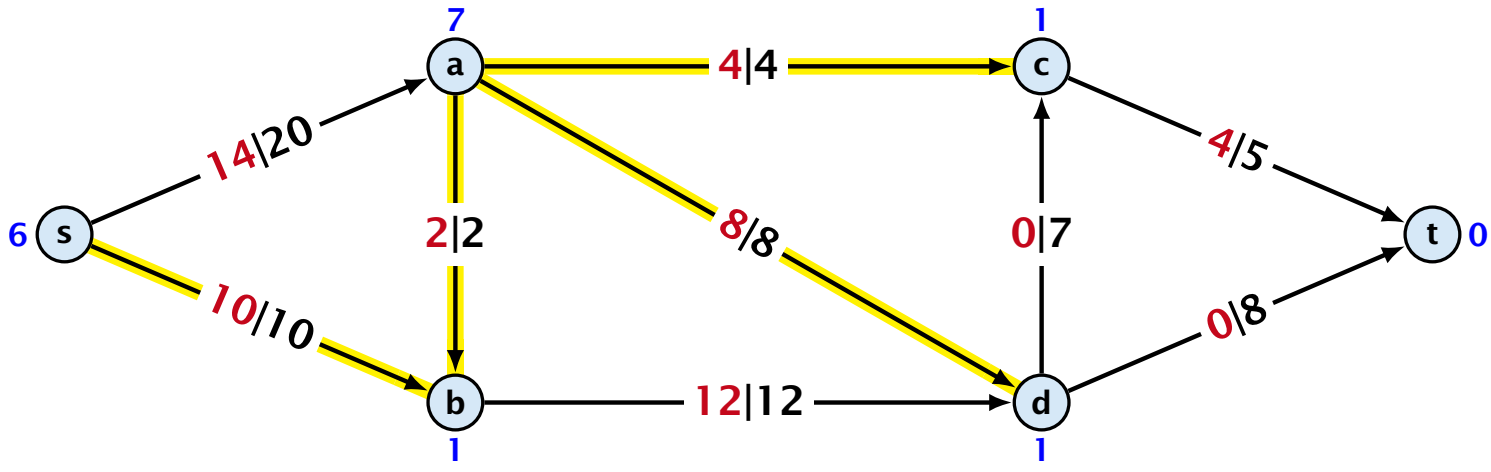
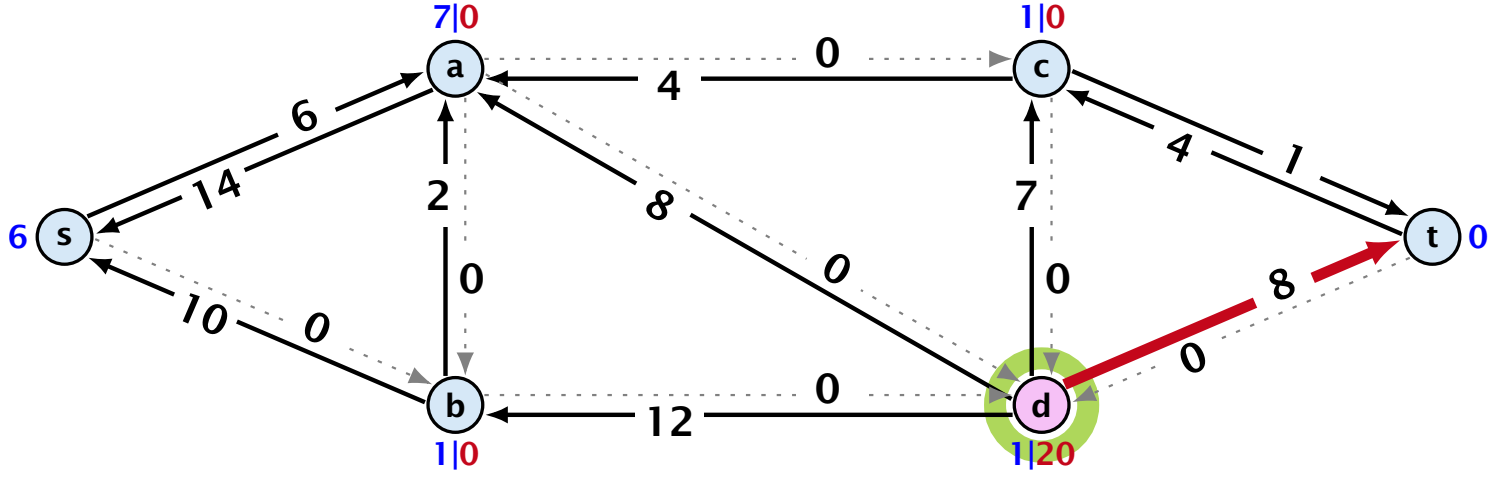


saturating push



Preflow Push

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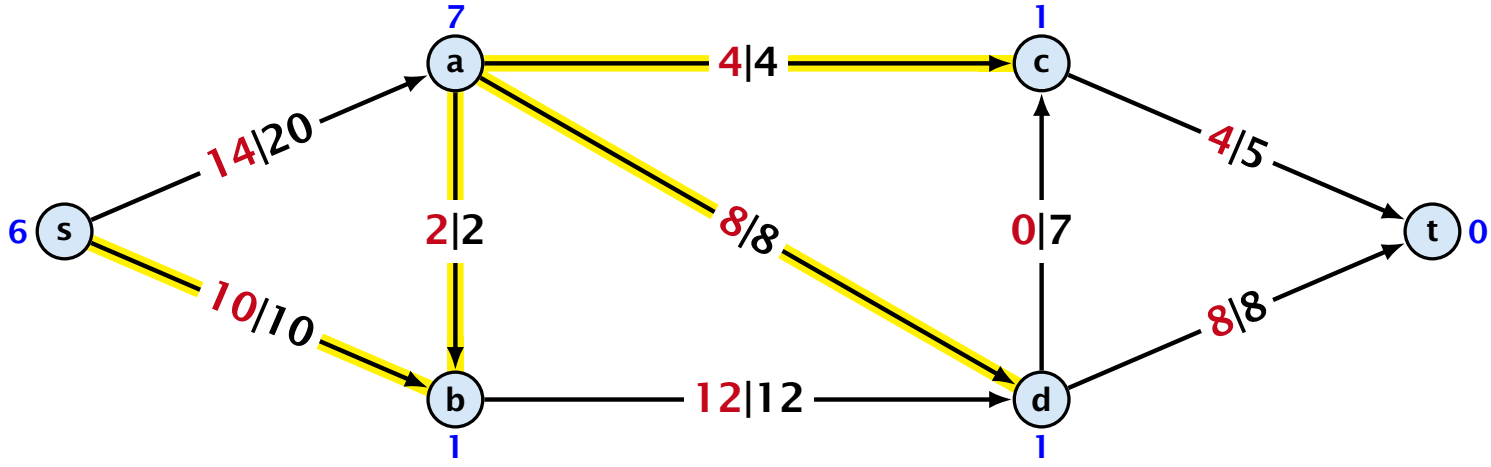

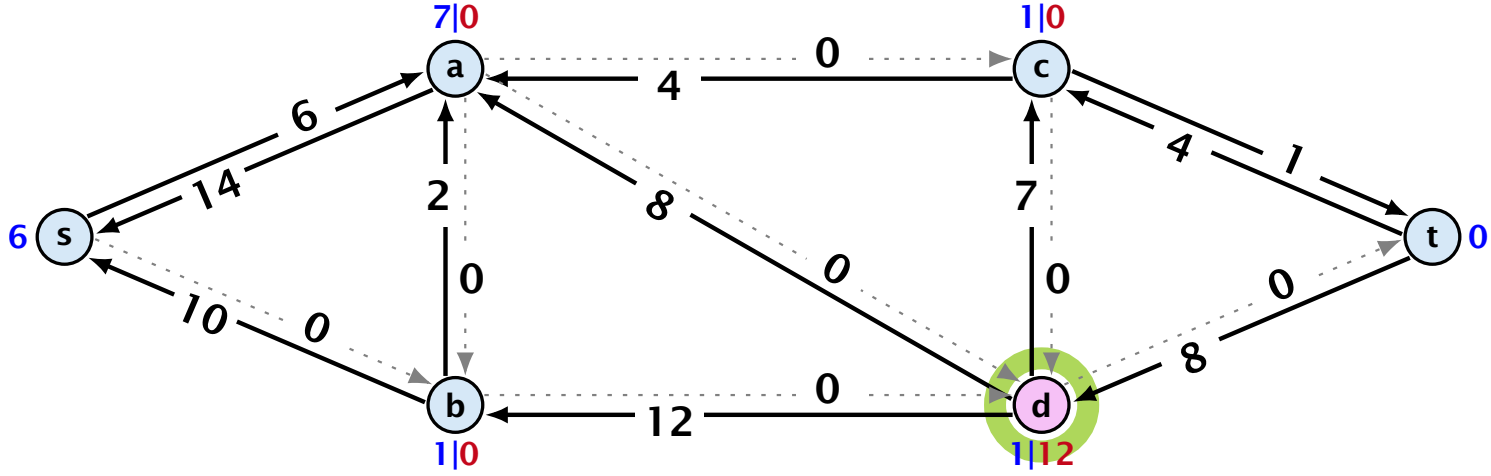

Preflow Push

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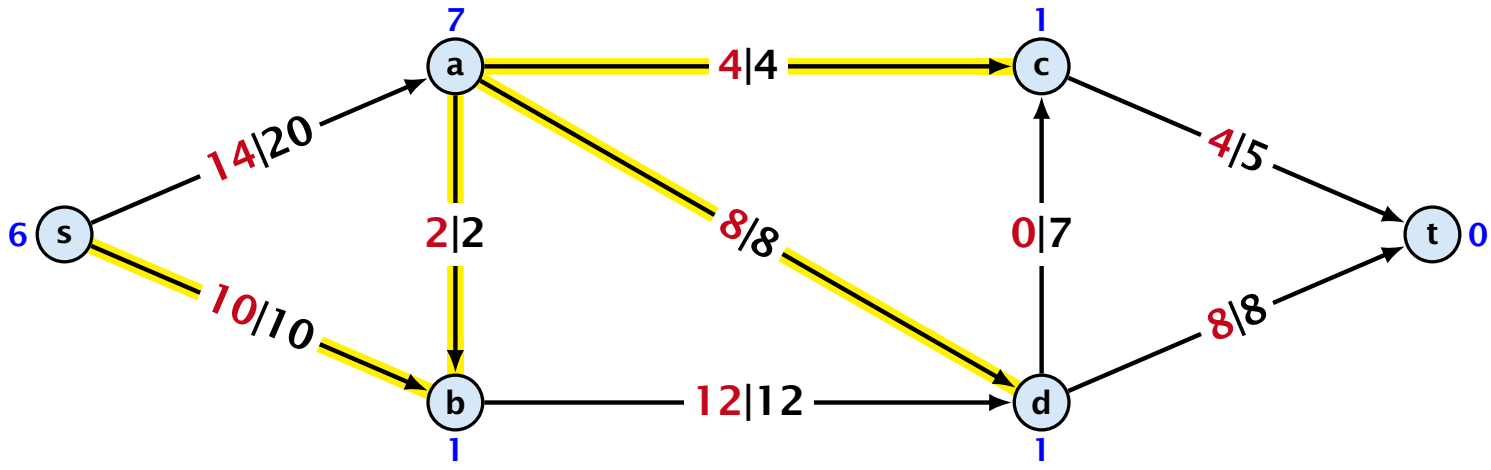
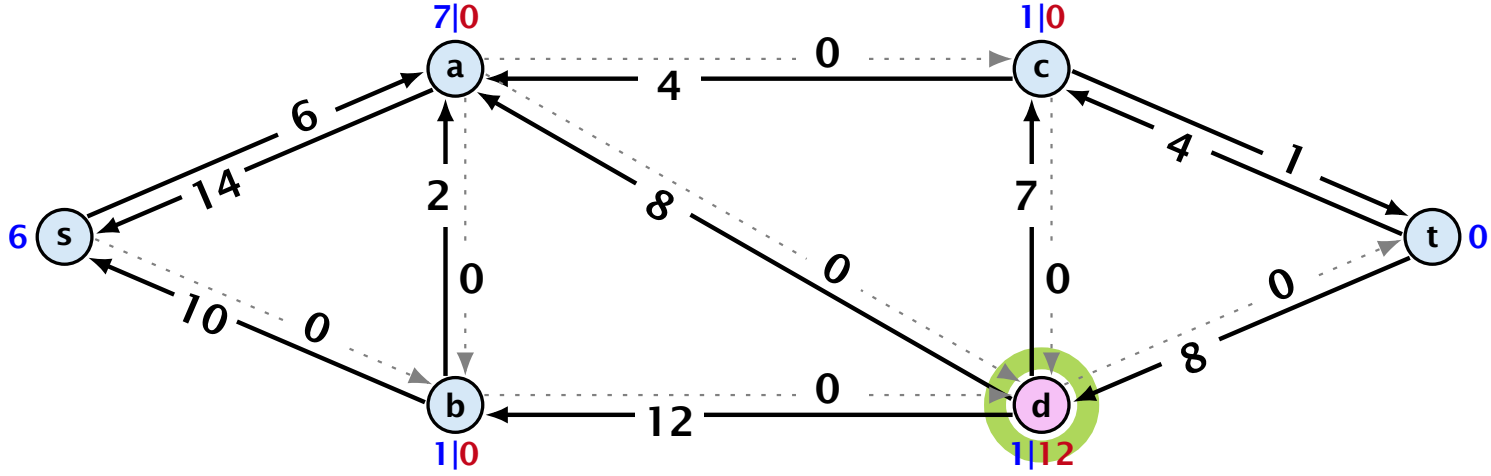


relabel to 2



Preflow Push

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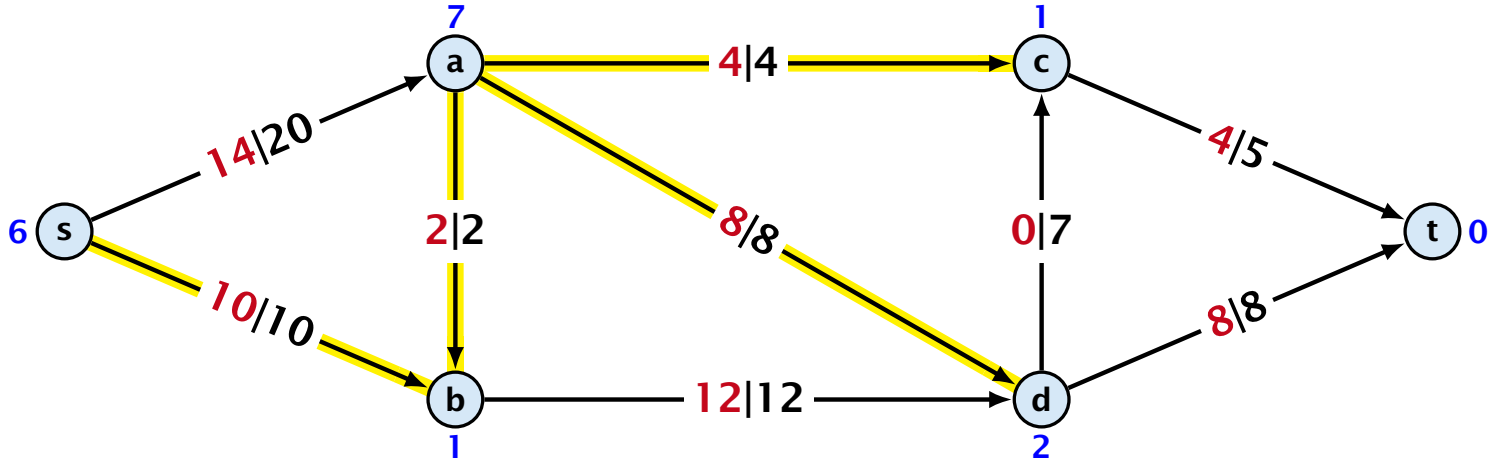

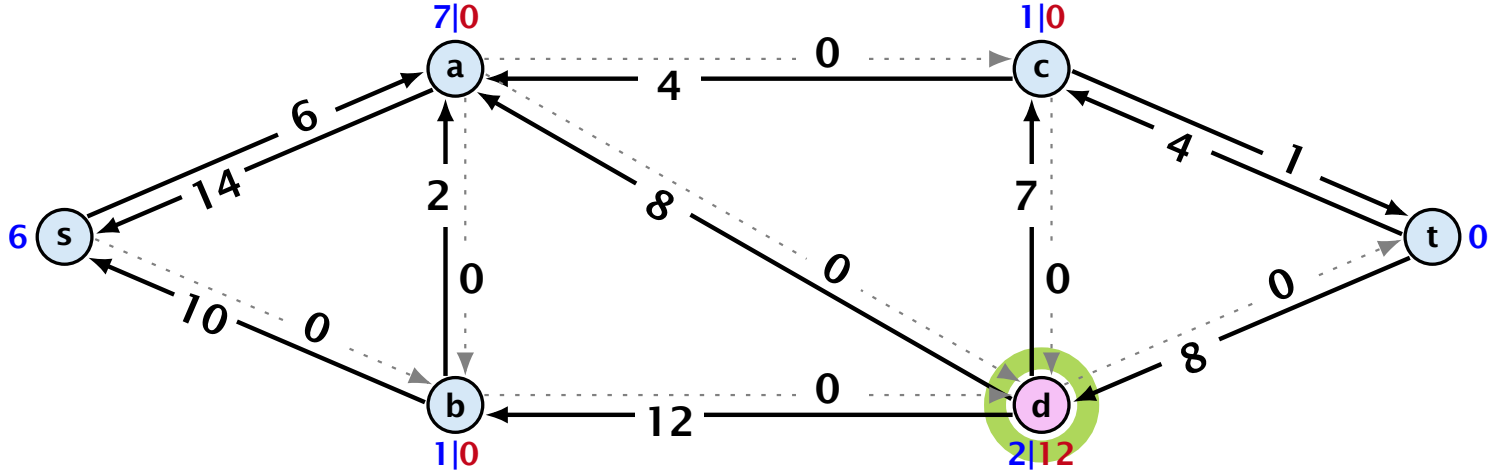

Preflow Push

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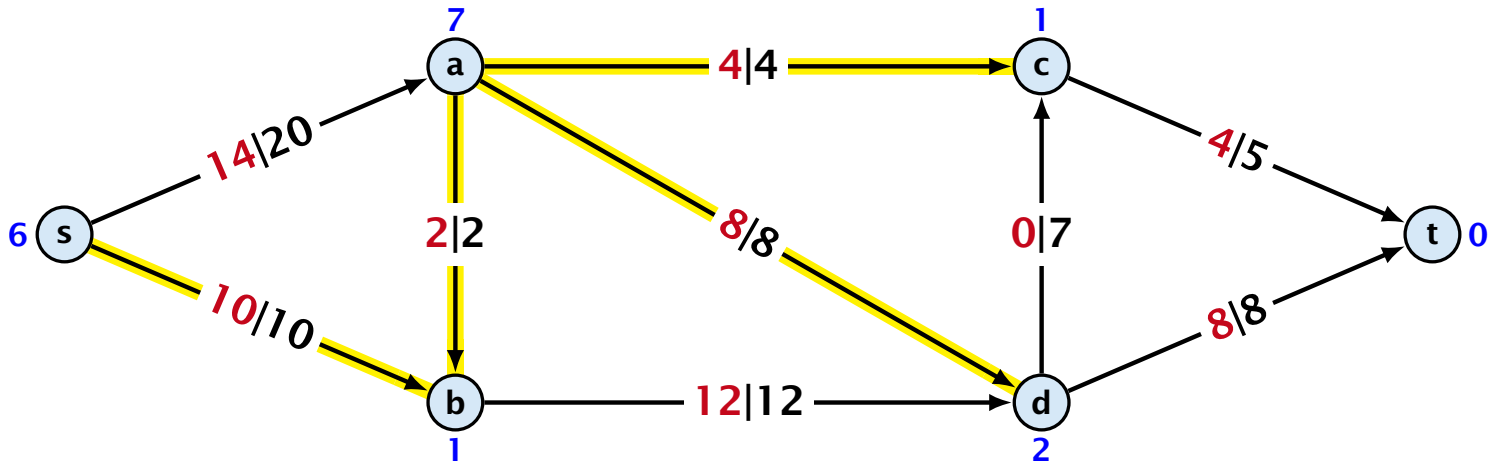
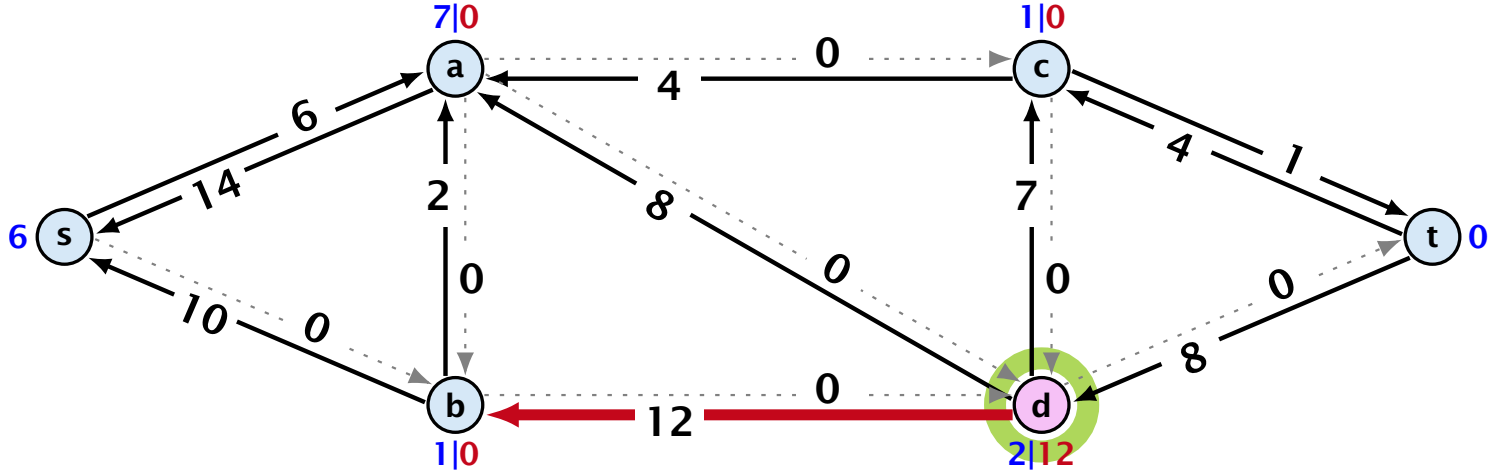


saturating and deactivating push



Preflow Push

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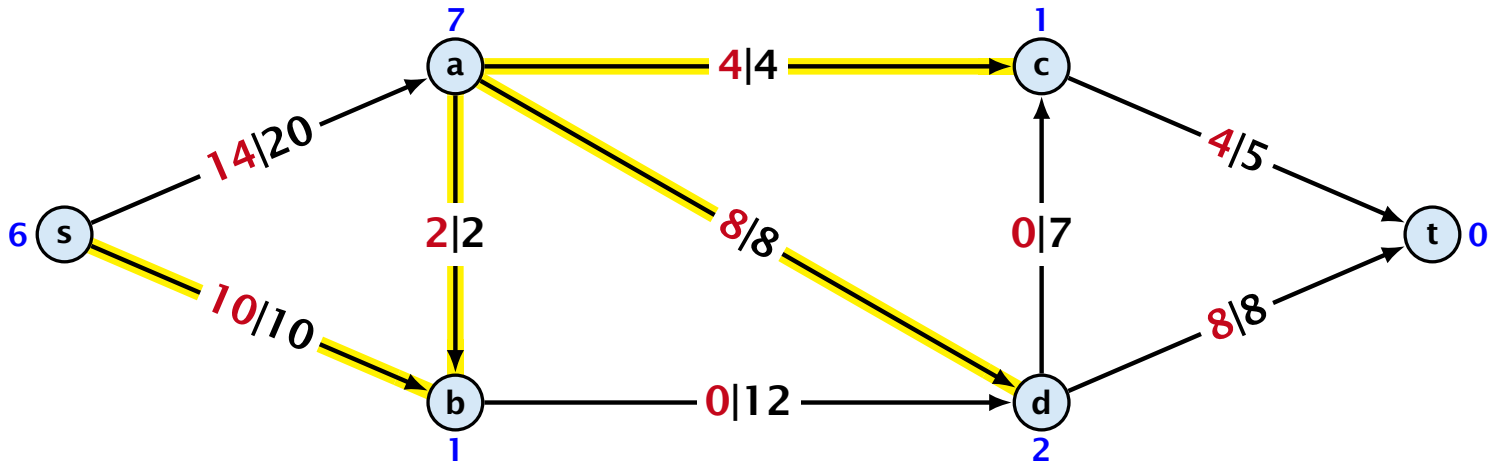

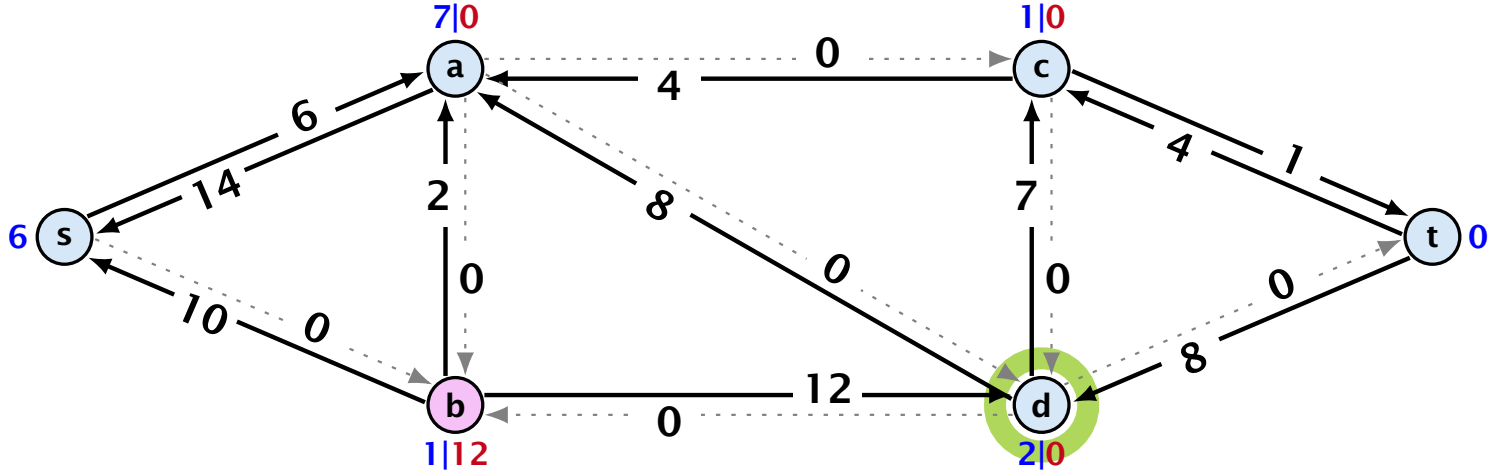

Preflow Push

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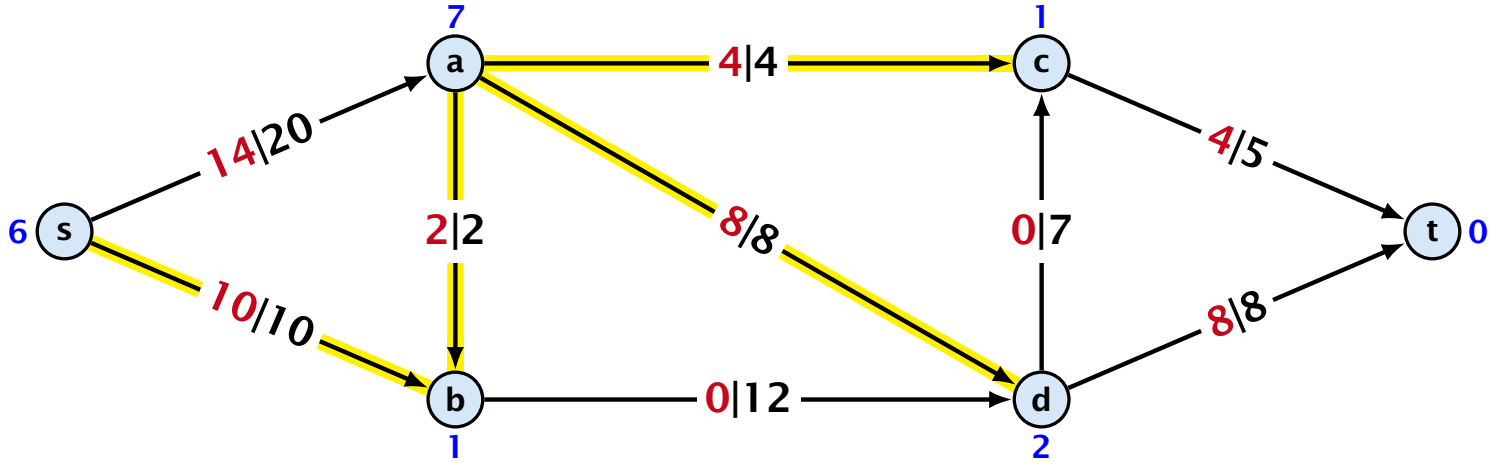

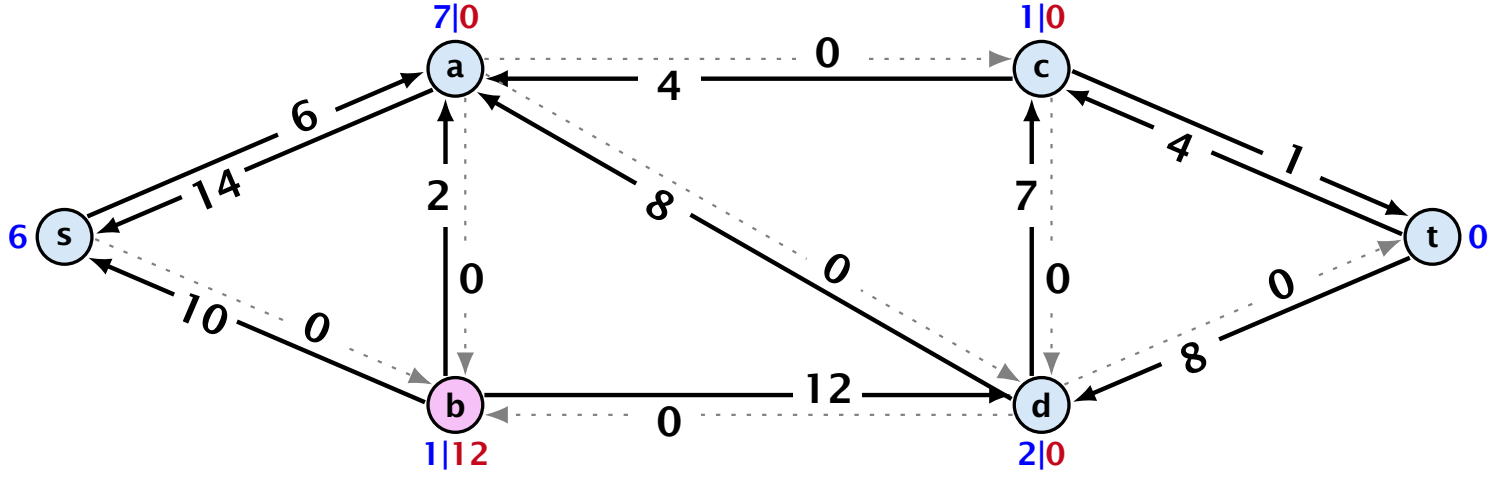

Preflow Push

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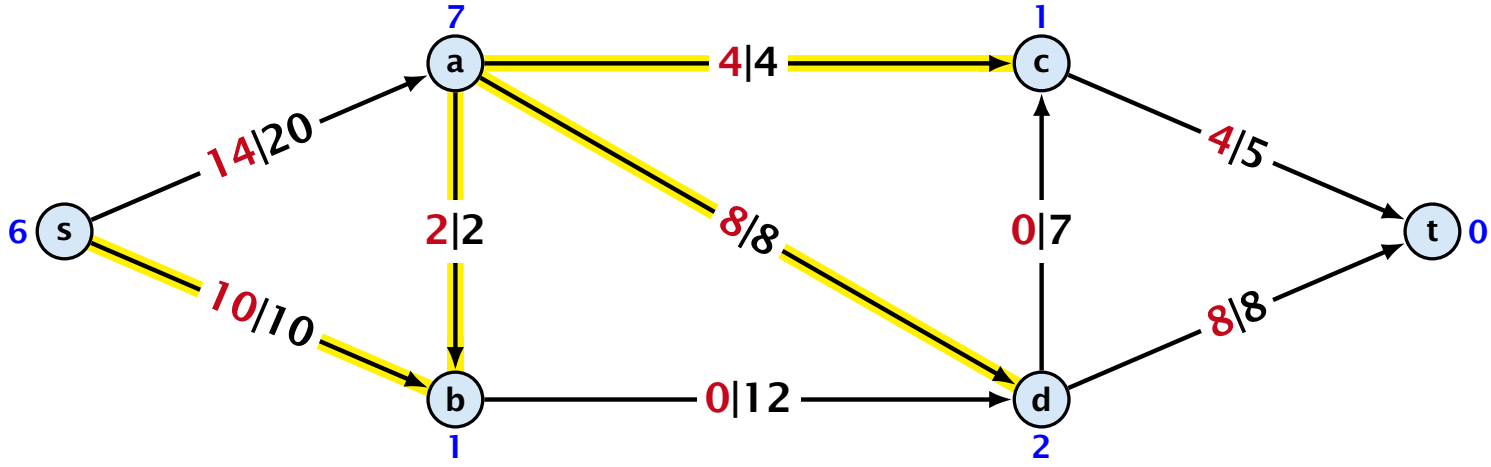

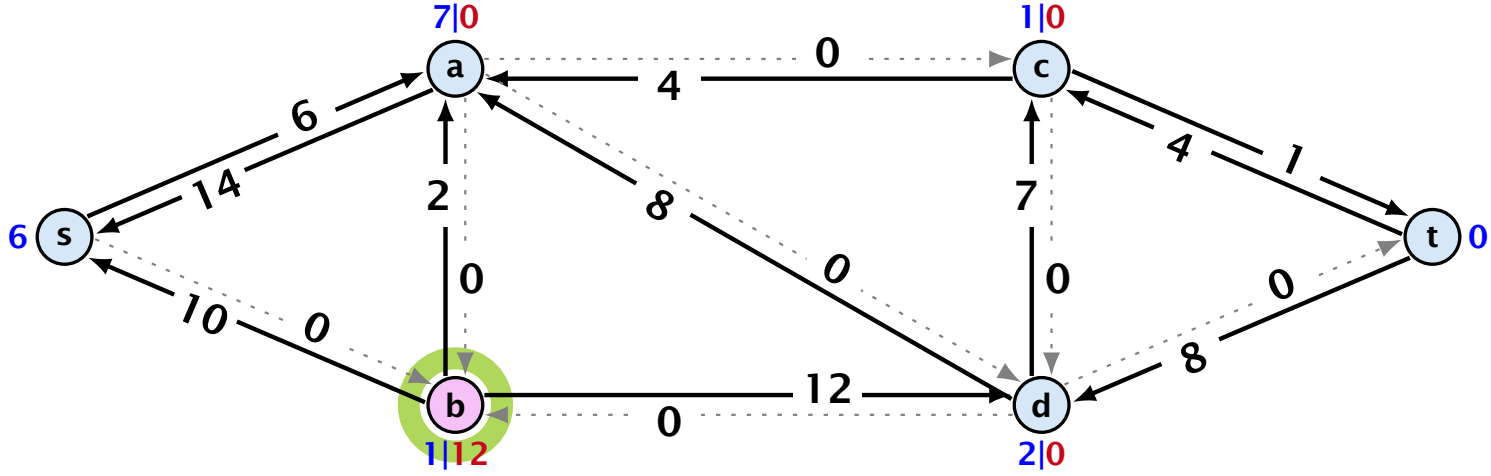

Preflow Push

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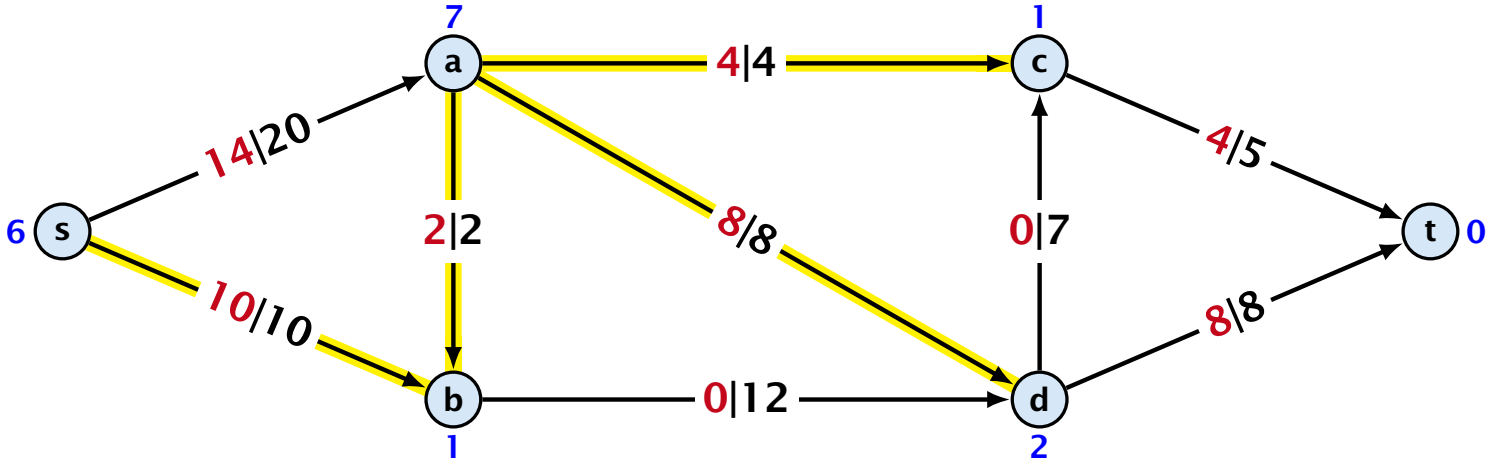
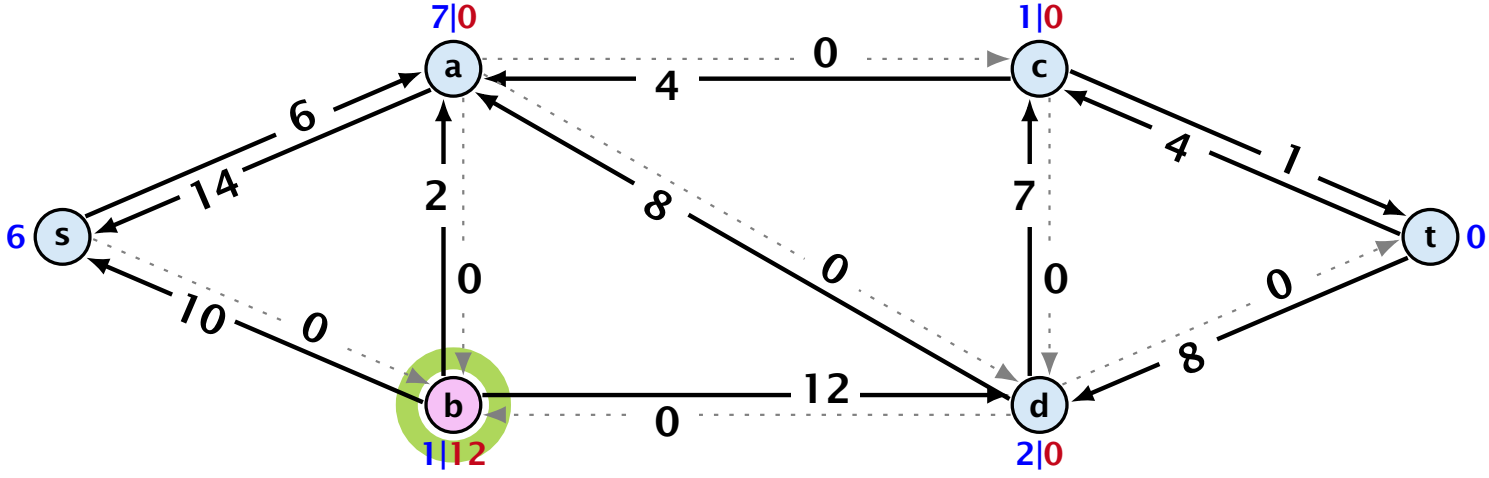


relabel to 3



Preflow Push

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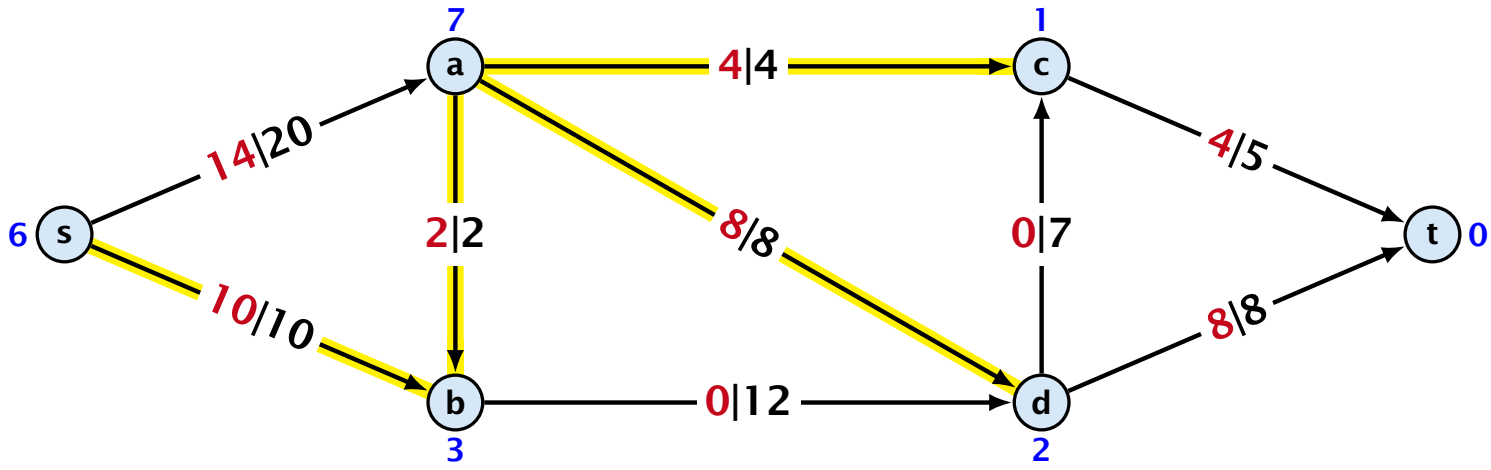
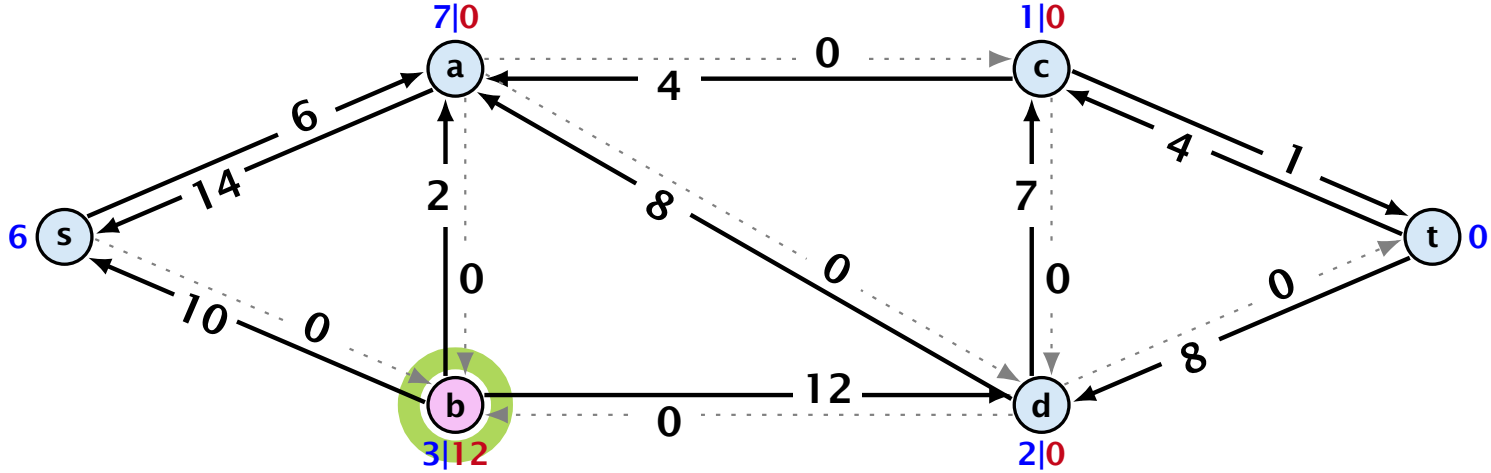
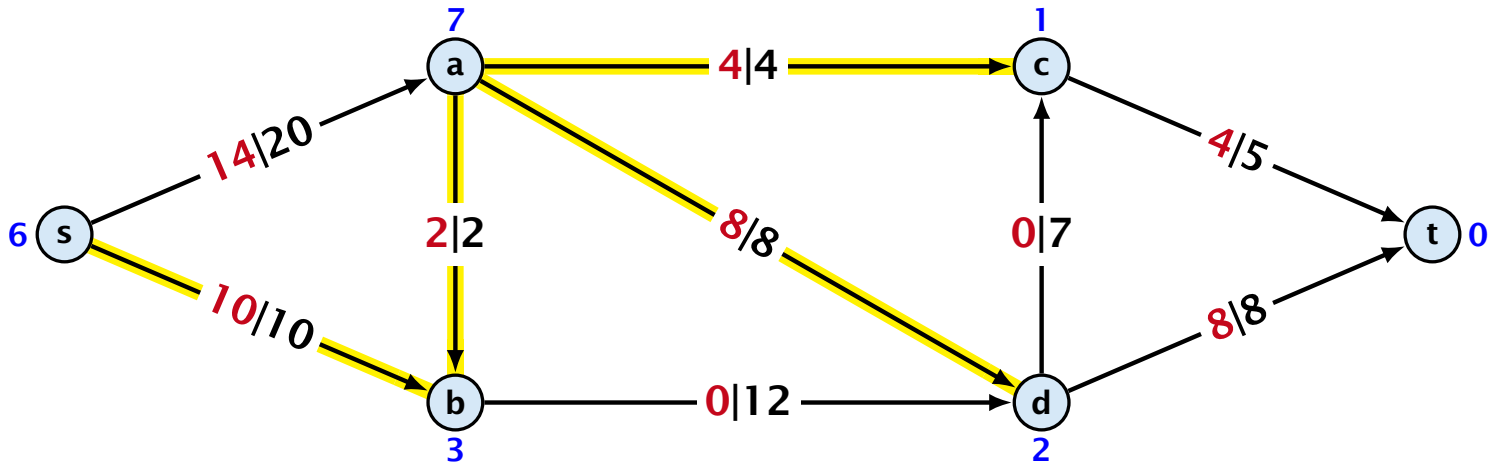


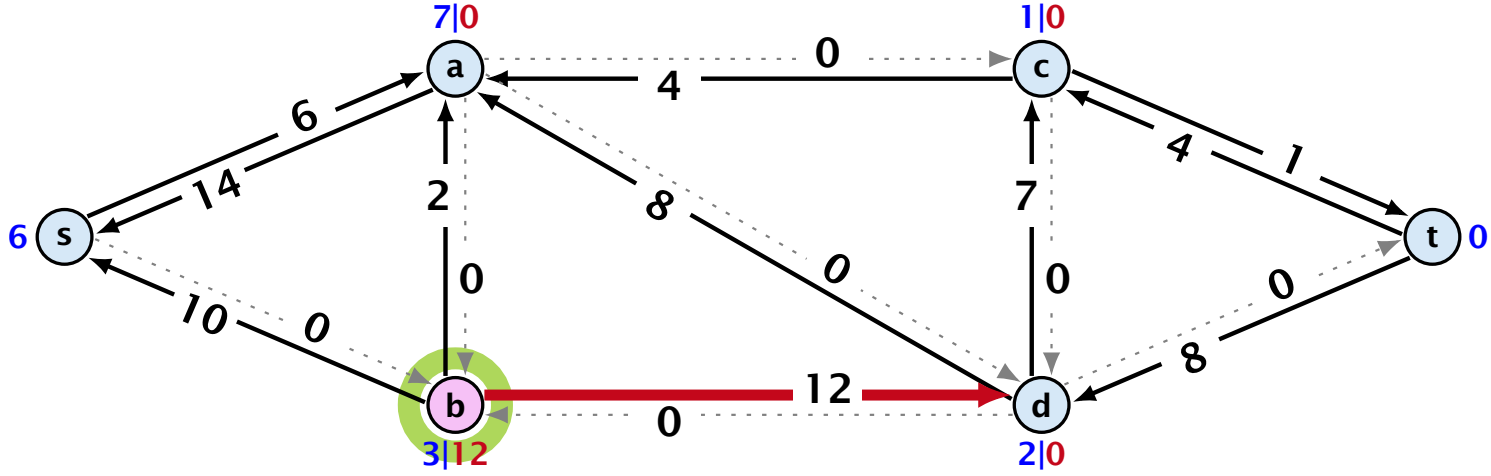


Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.



saturating and deactivating push



Preflow Push

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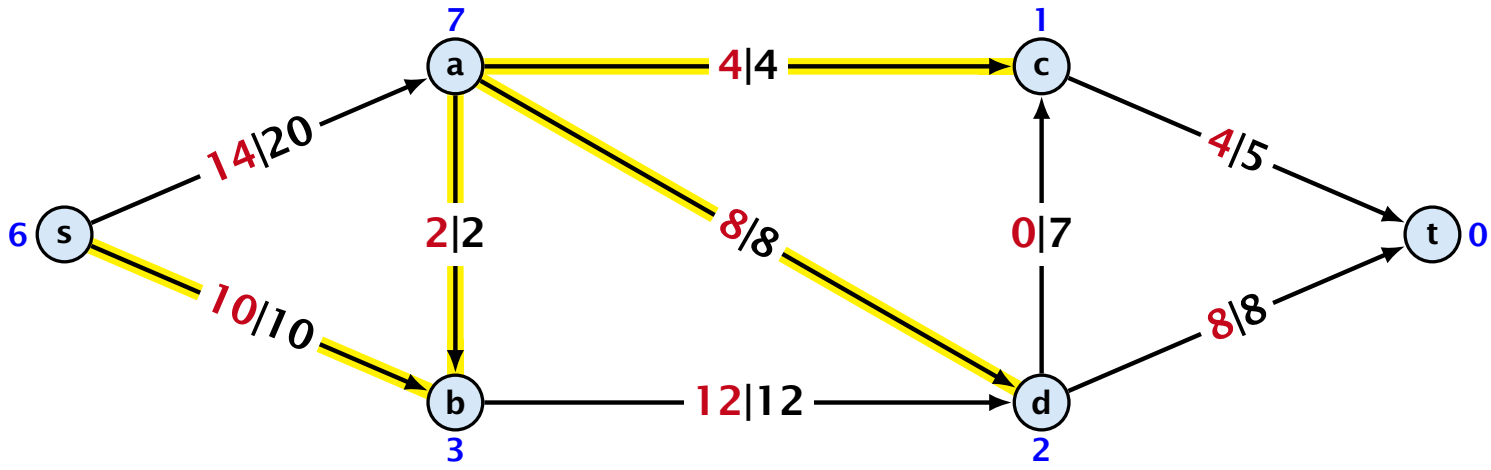

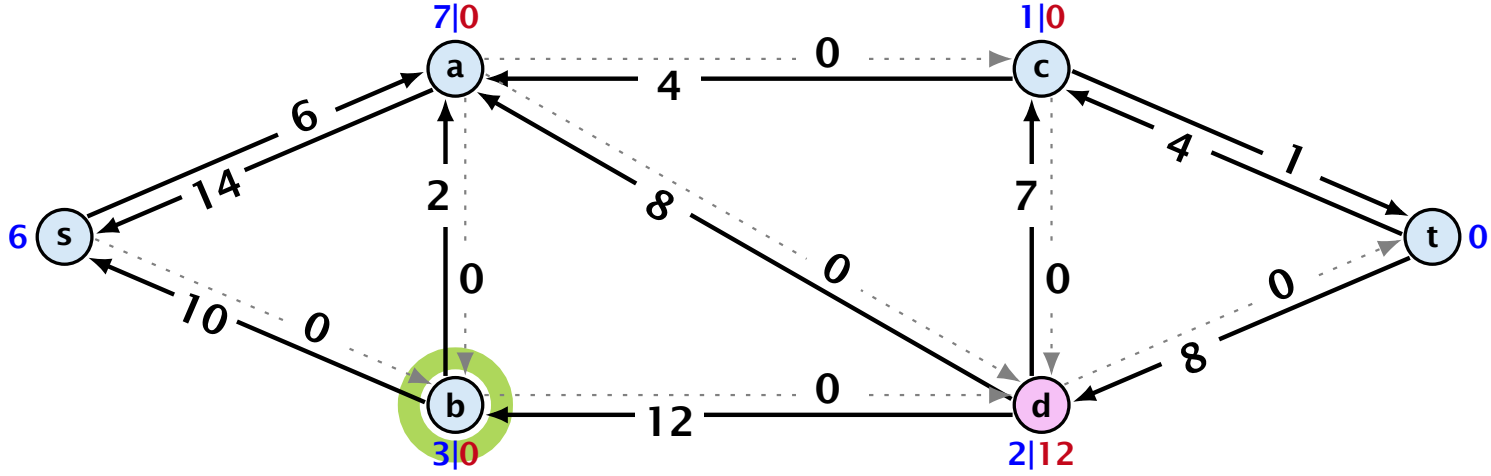

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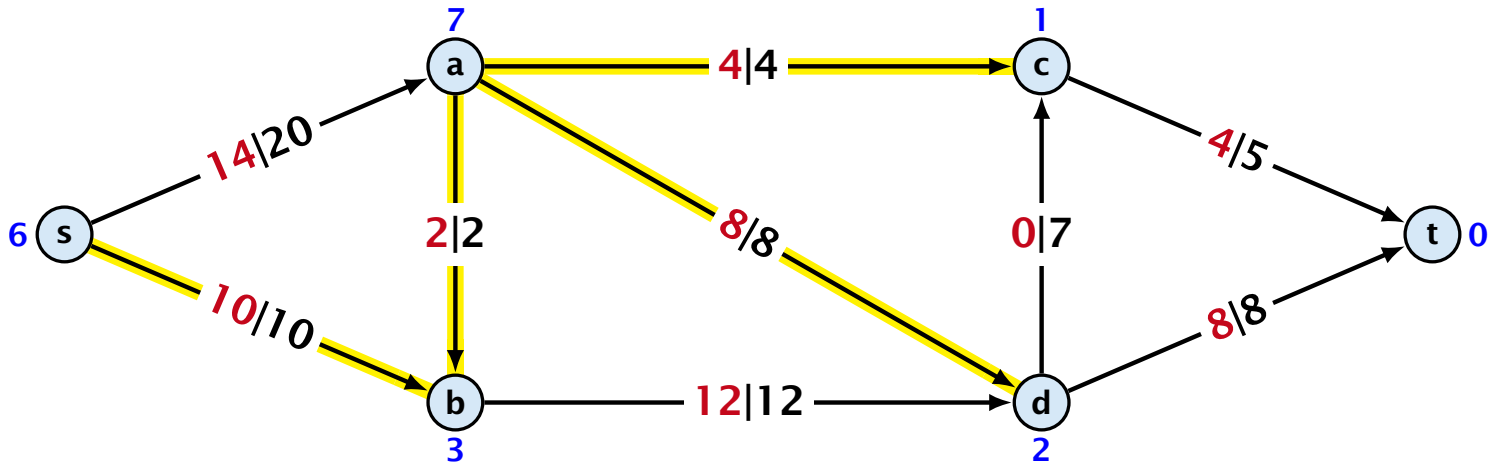

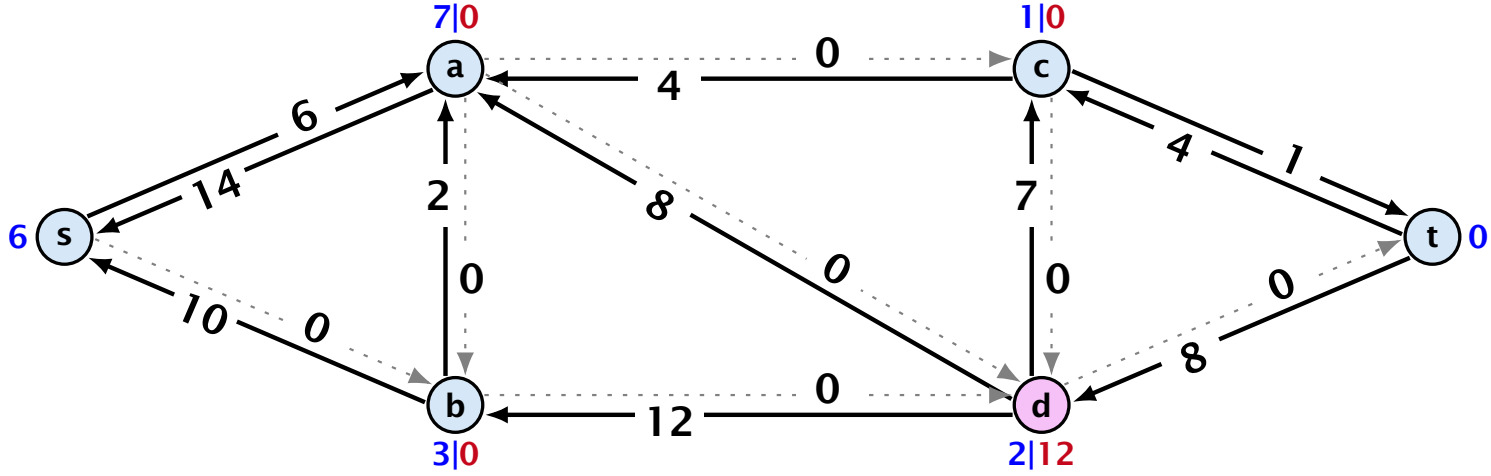

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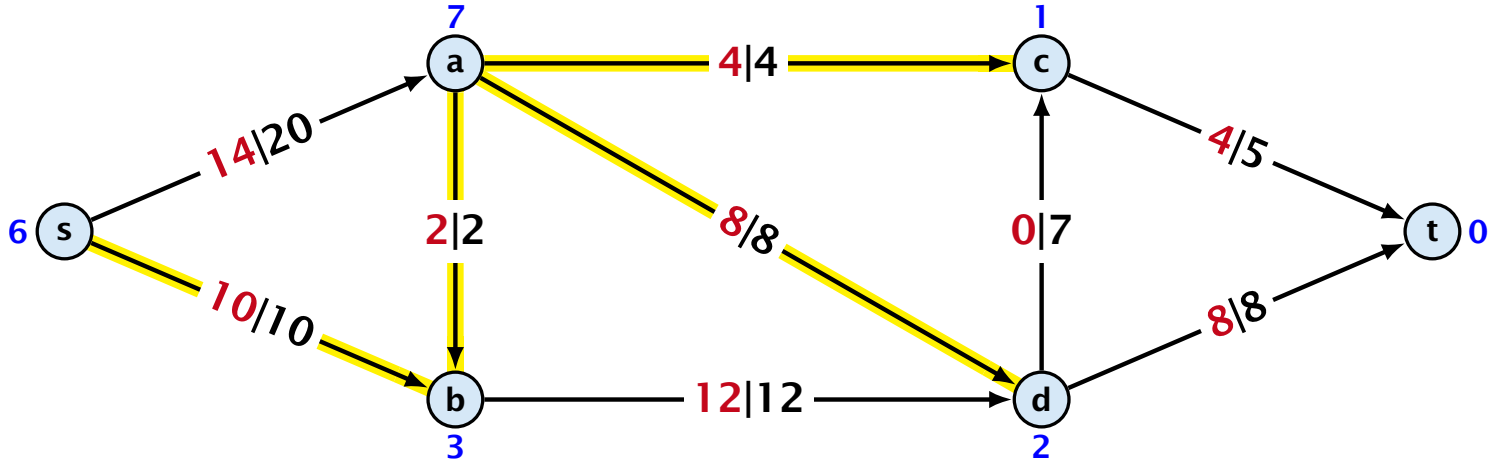

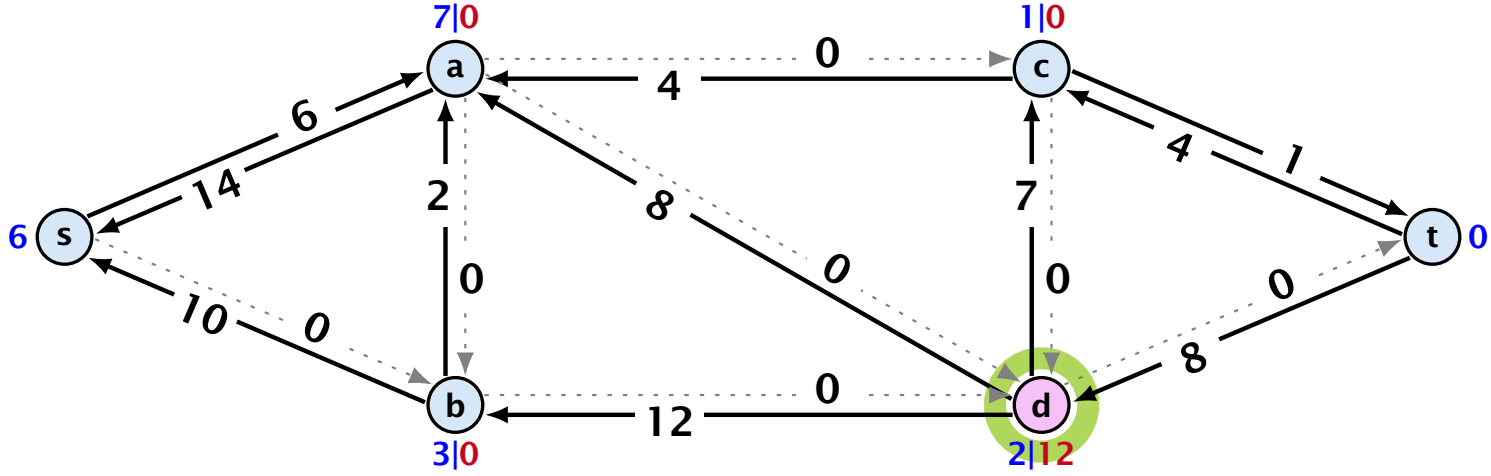

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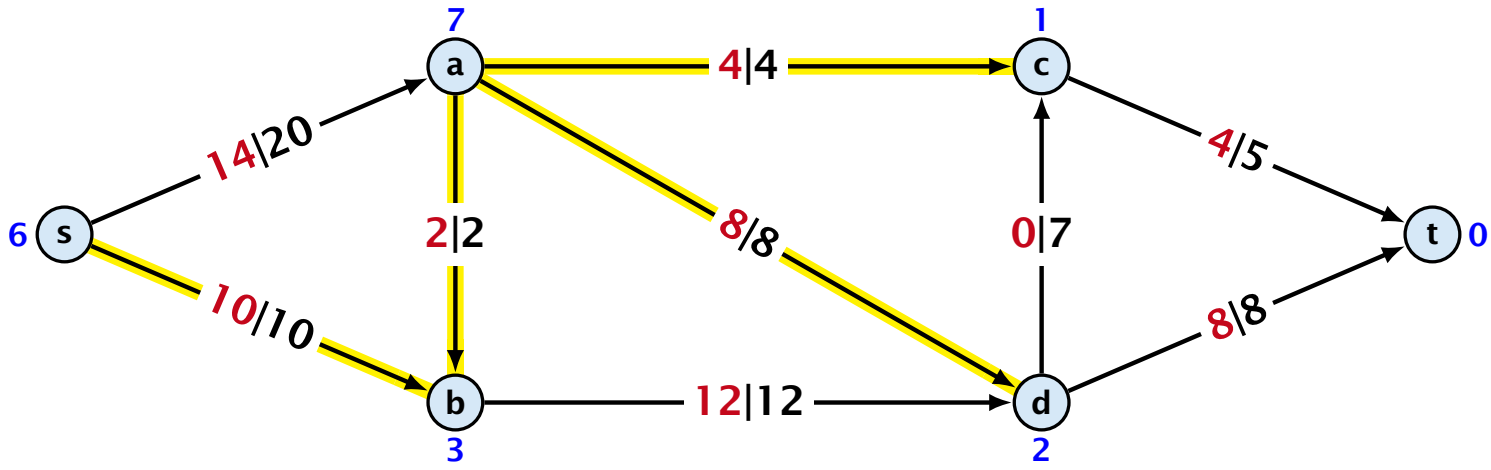
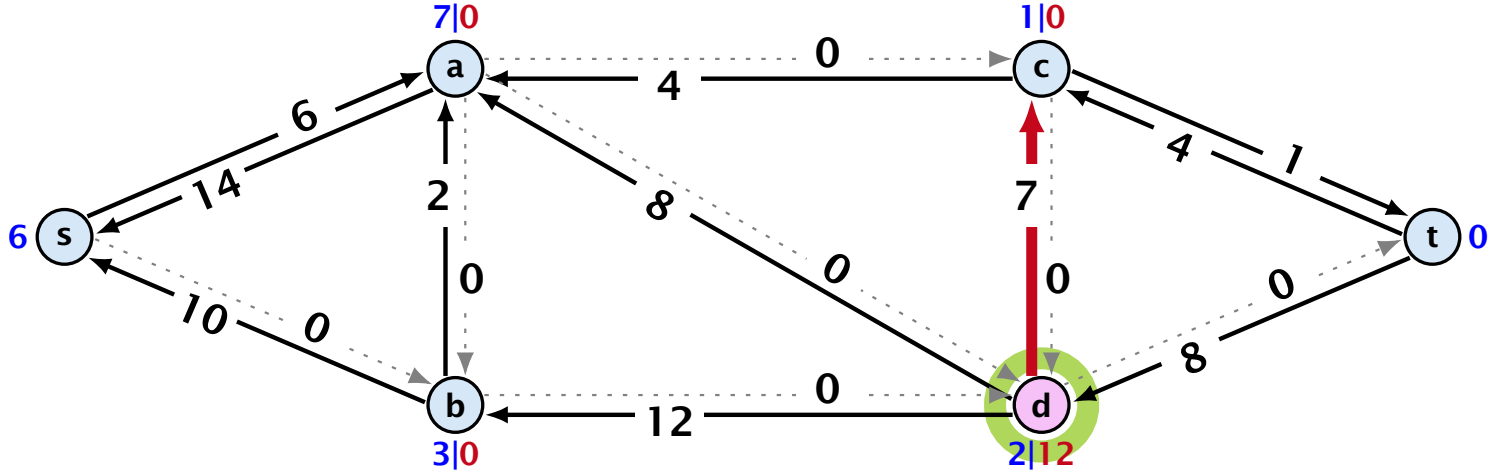


saturating push



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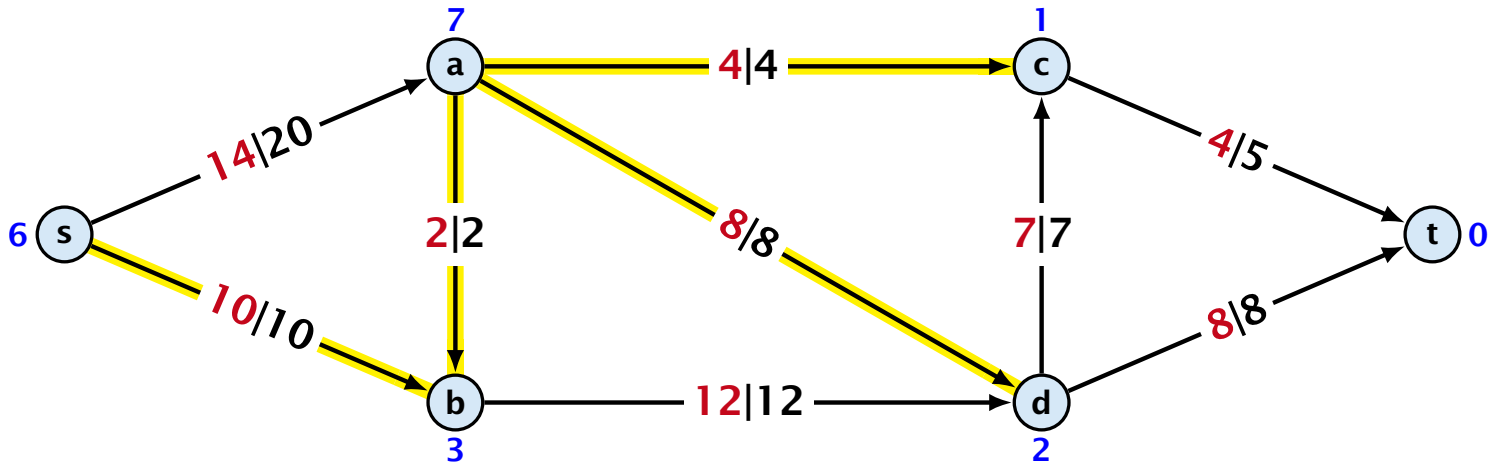

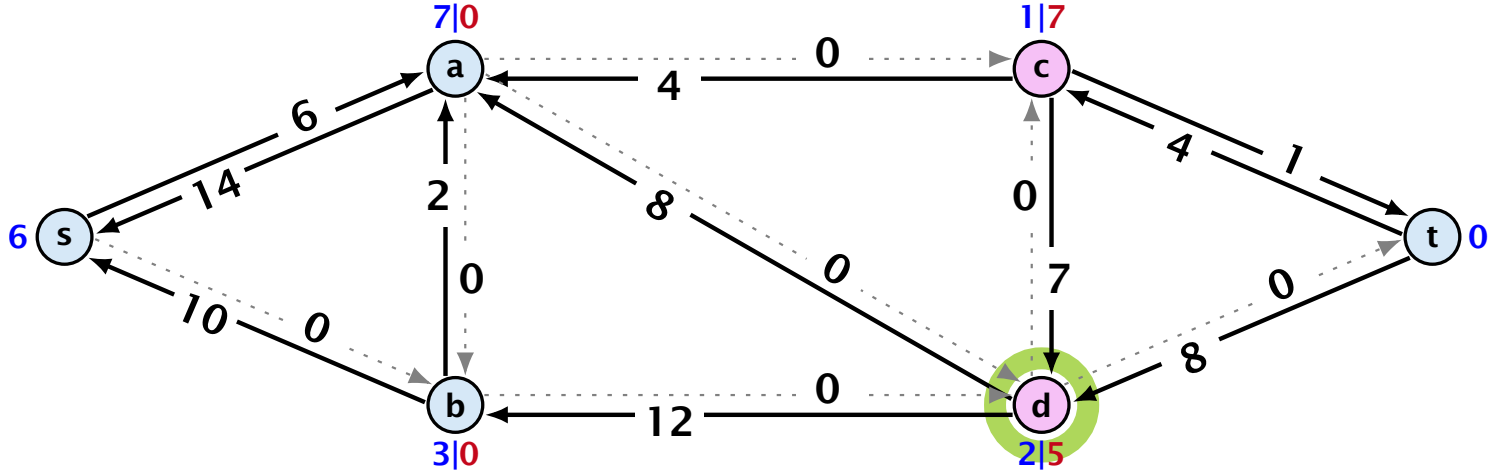

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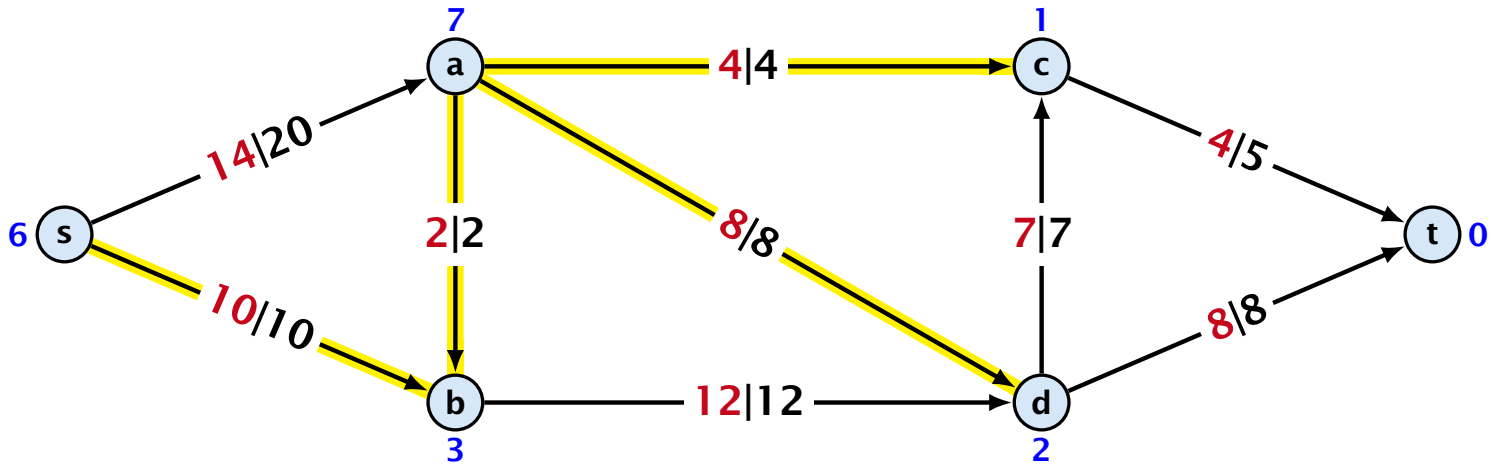
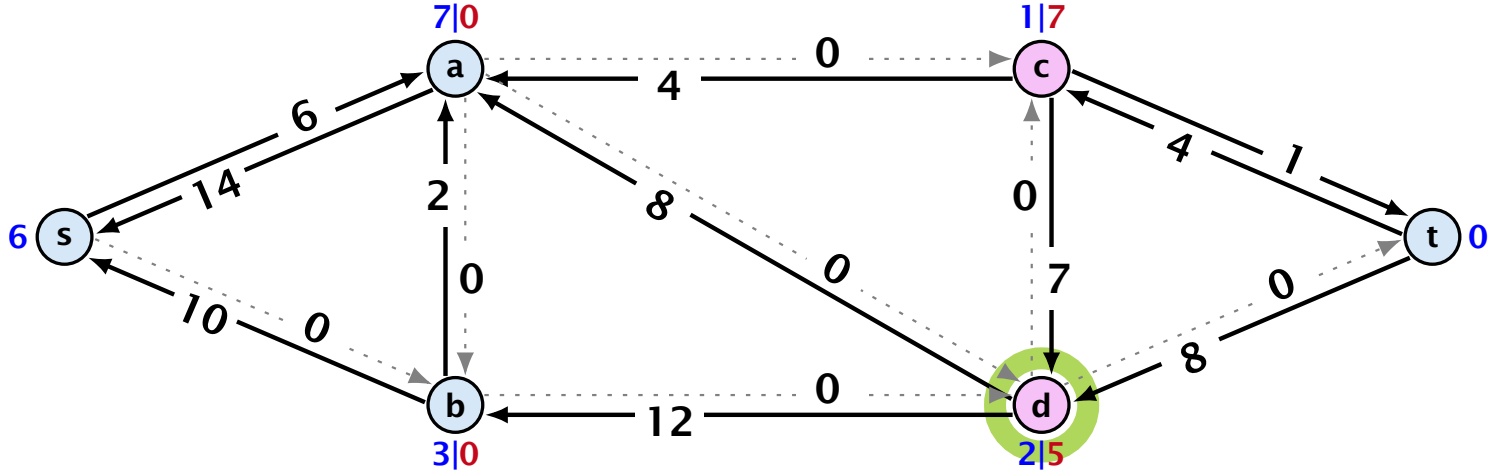


relabel to 4



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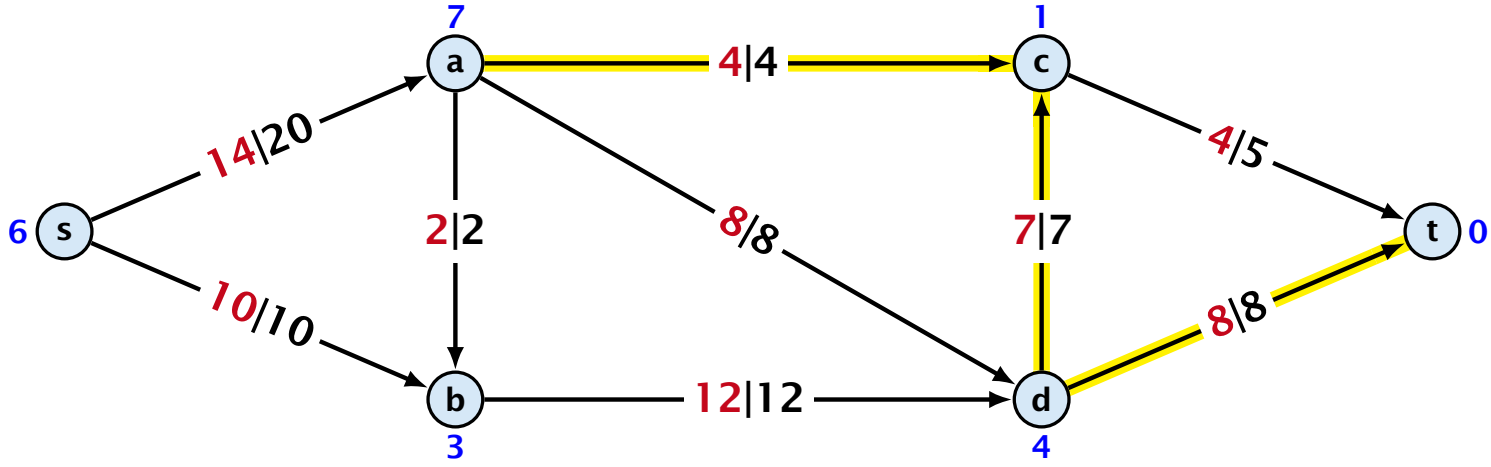

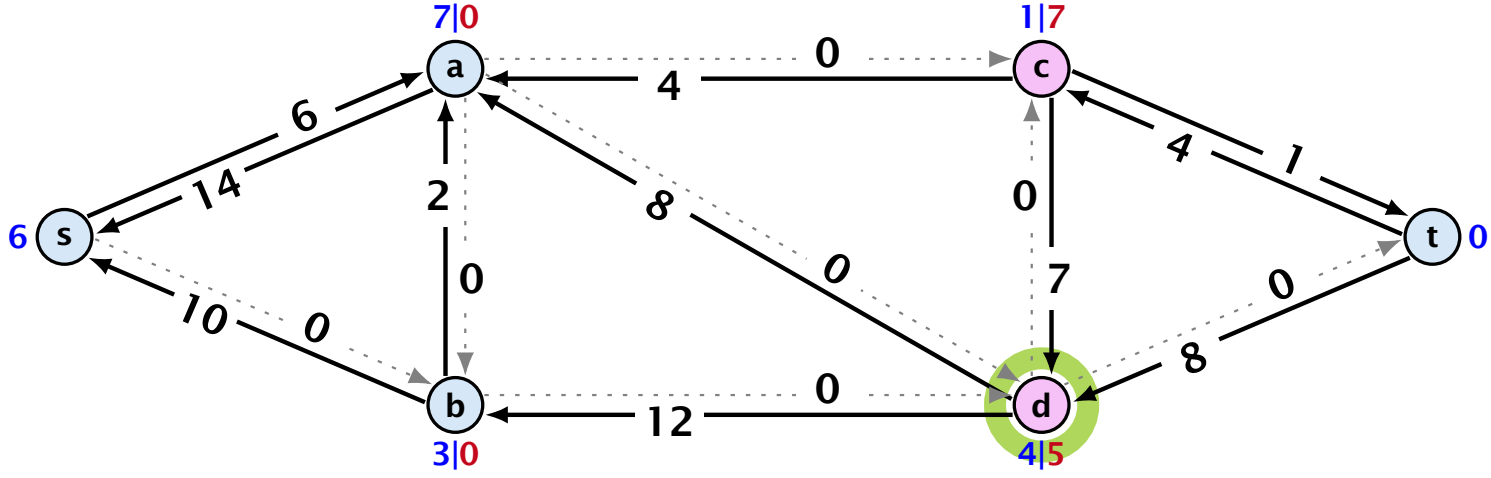

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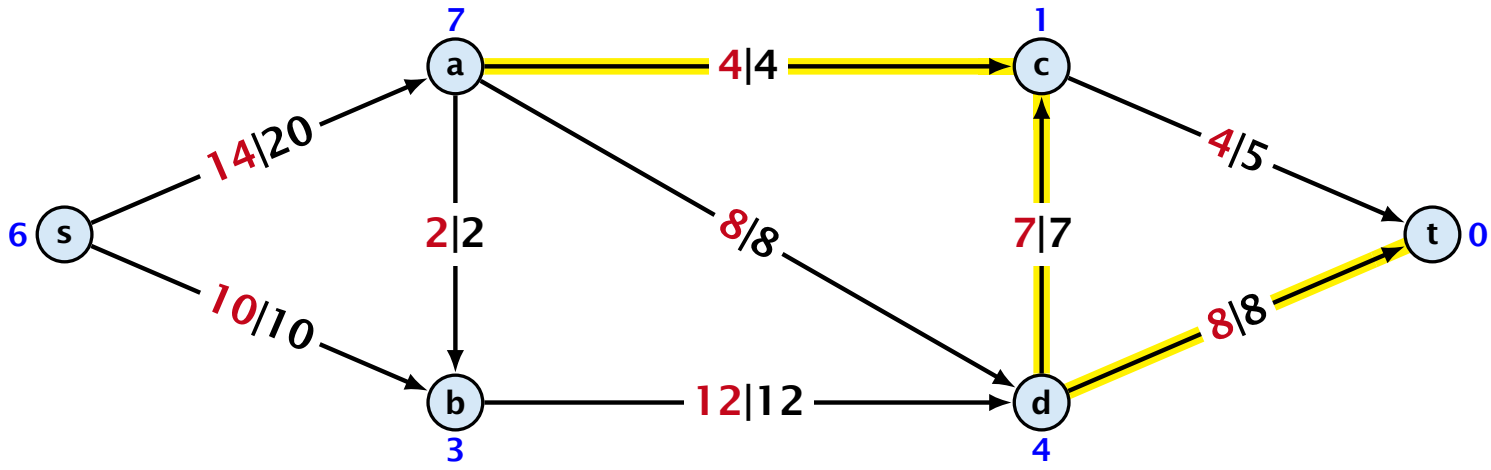
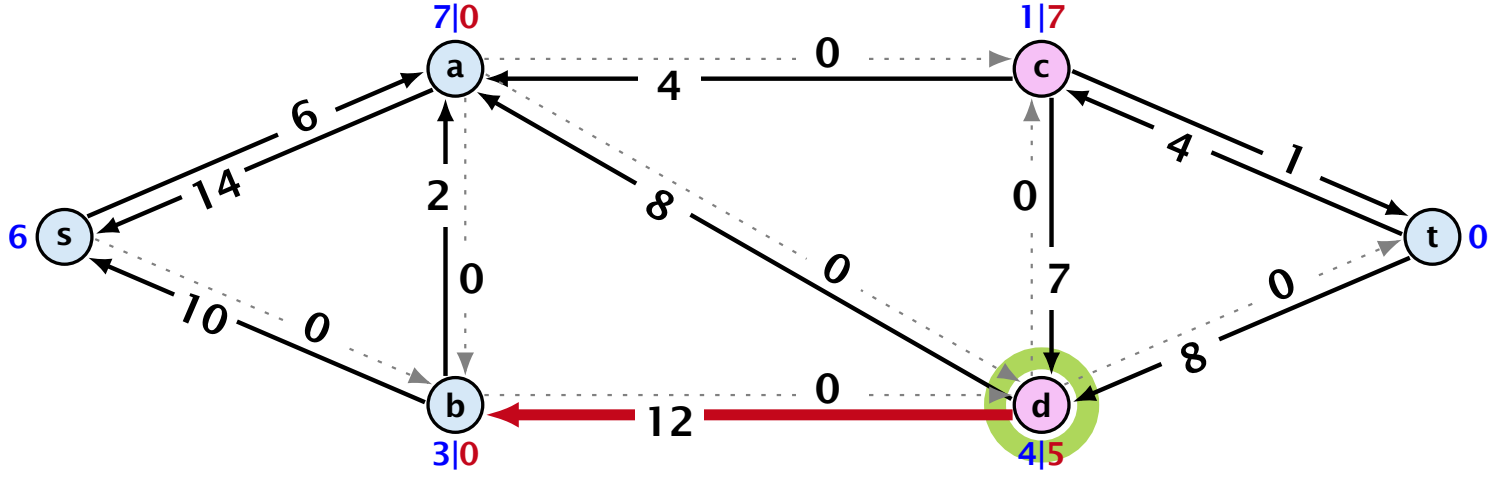


deactivating push



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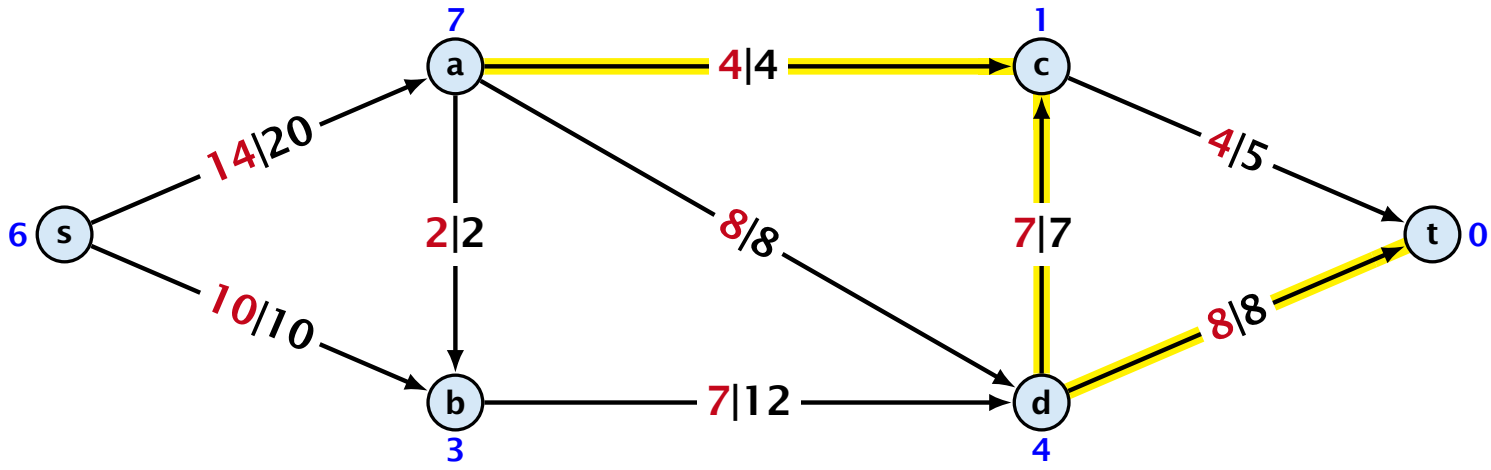

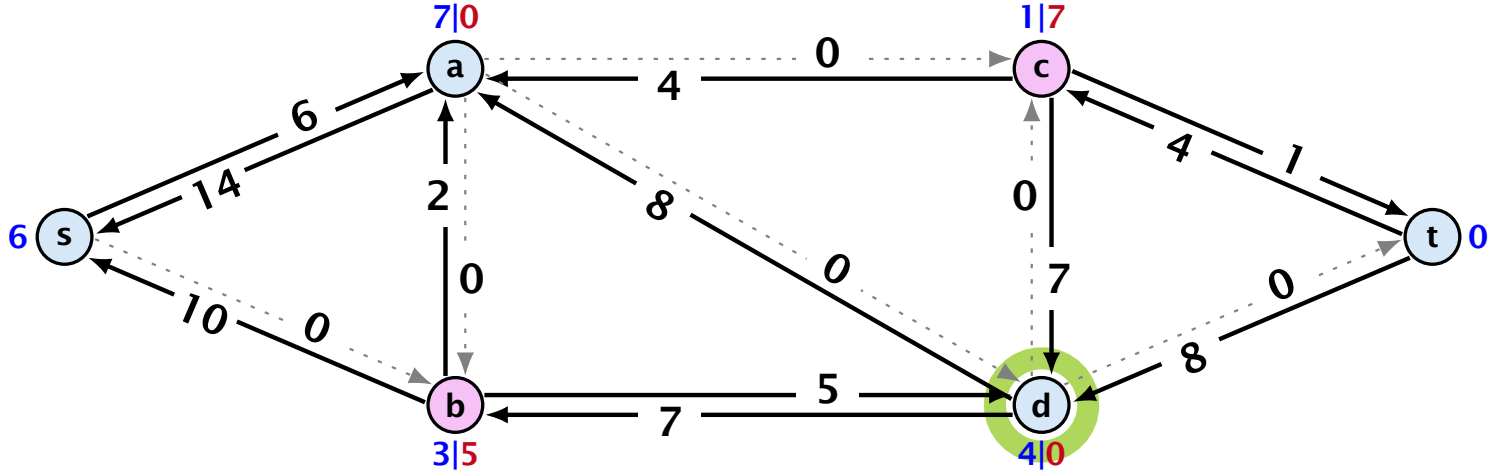

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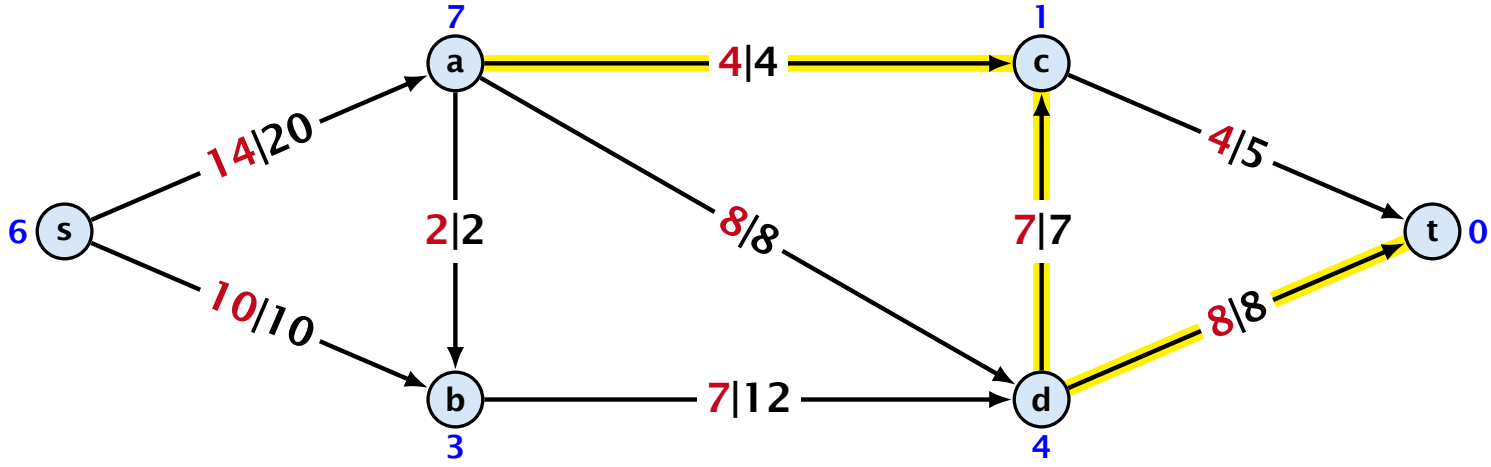

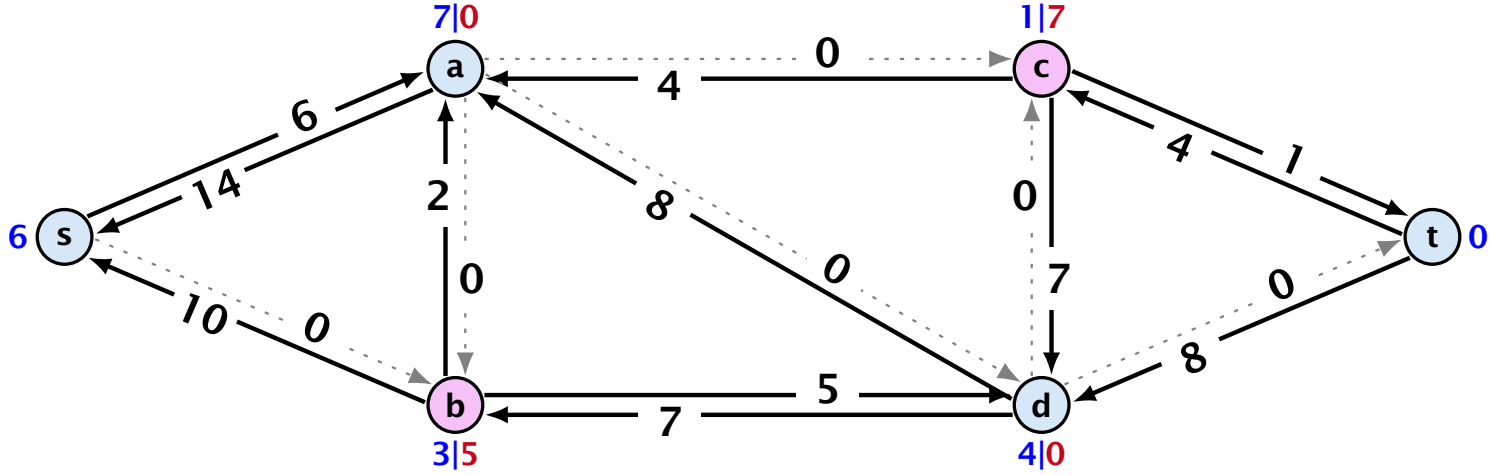

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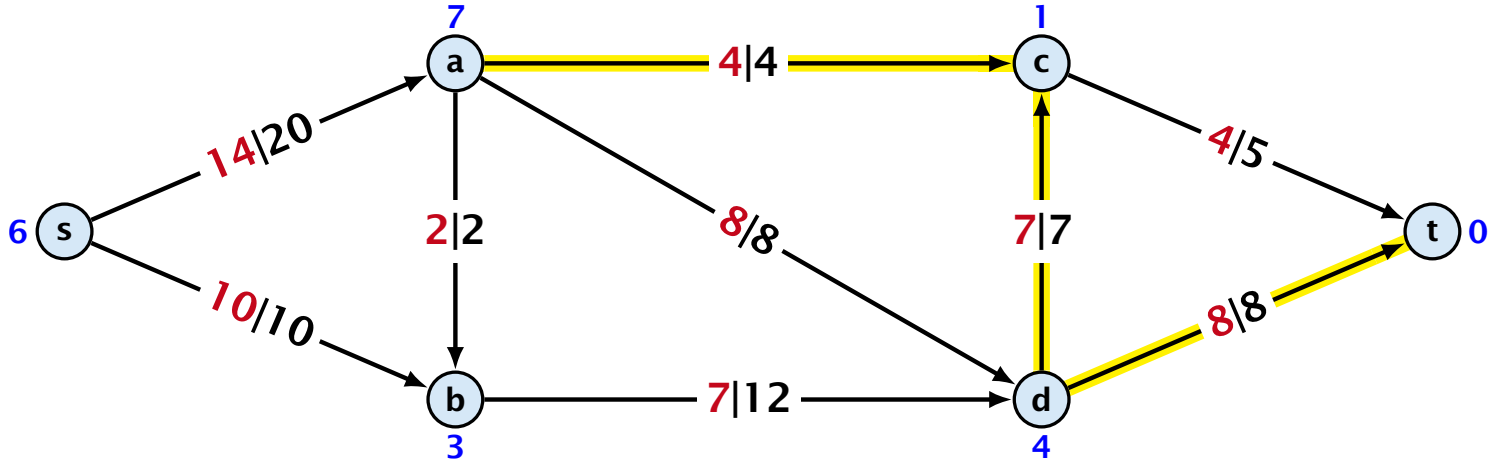

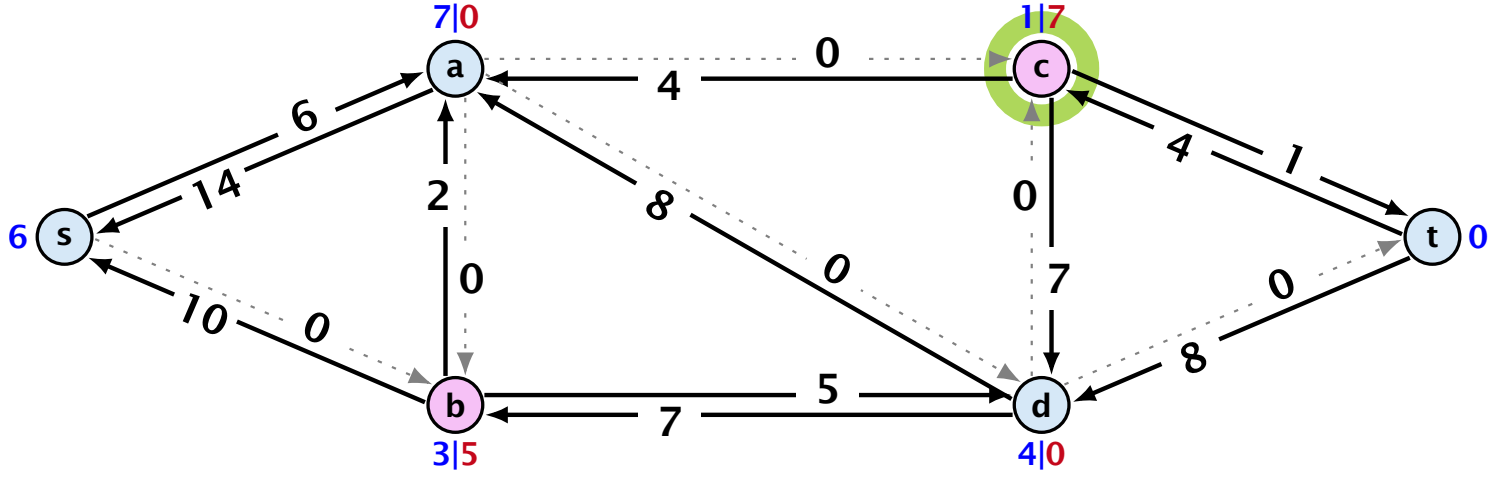

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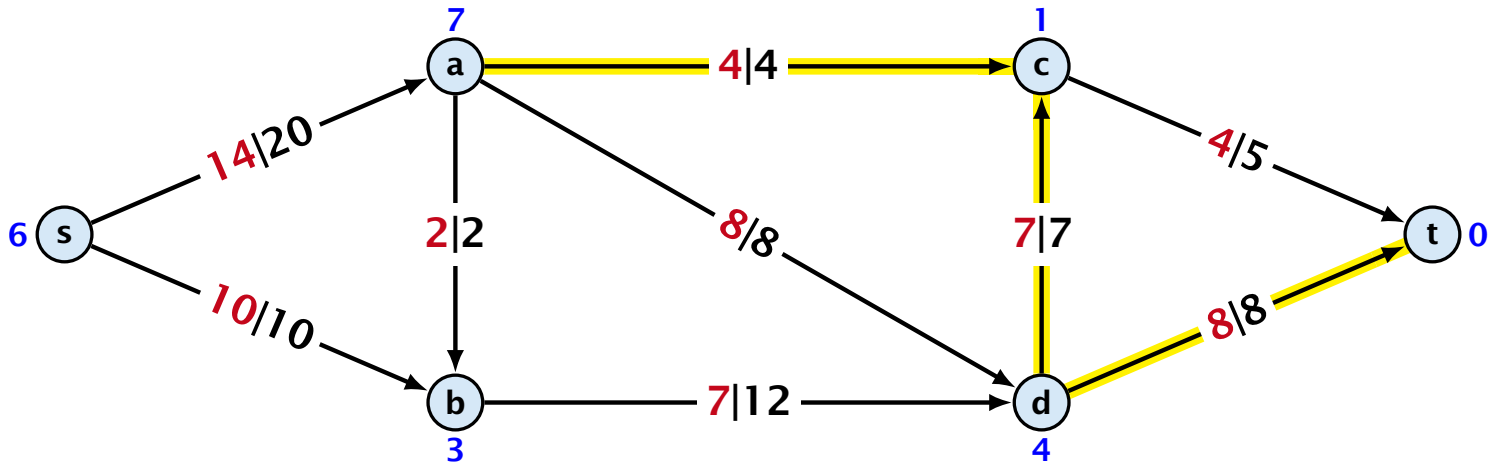
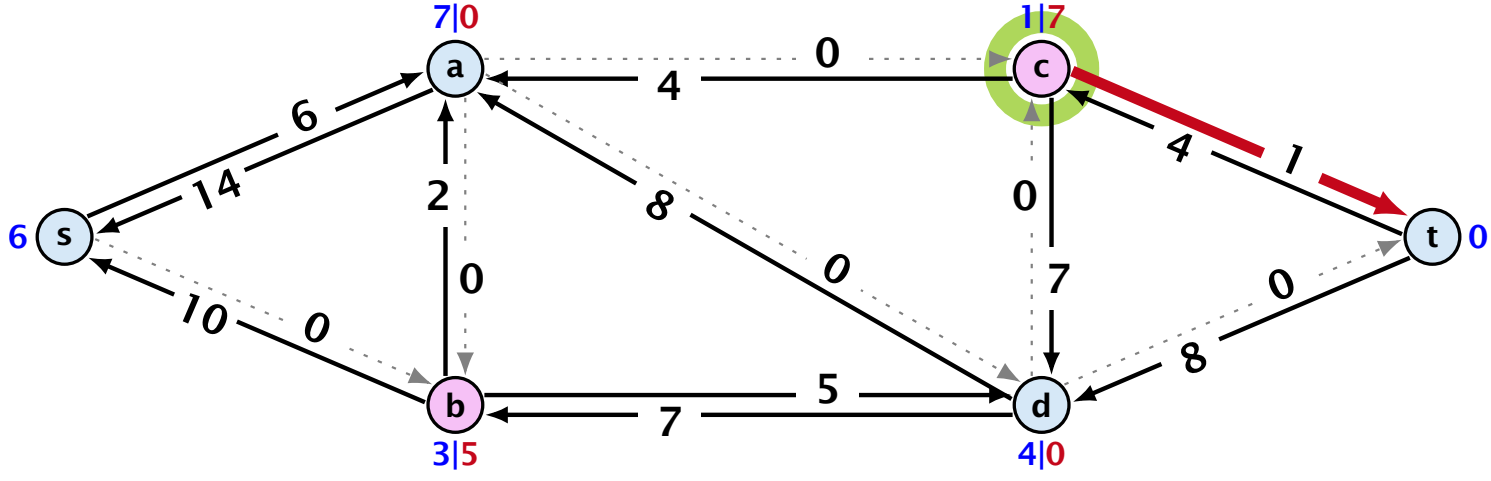


saturating push



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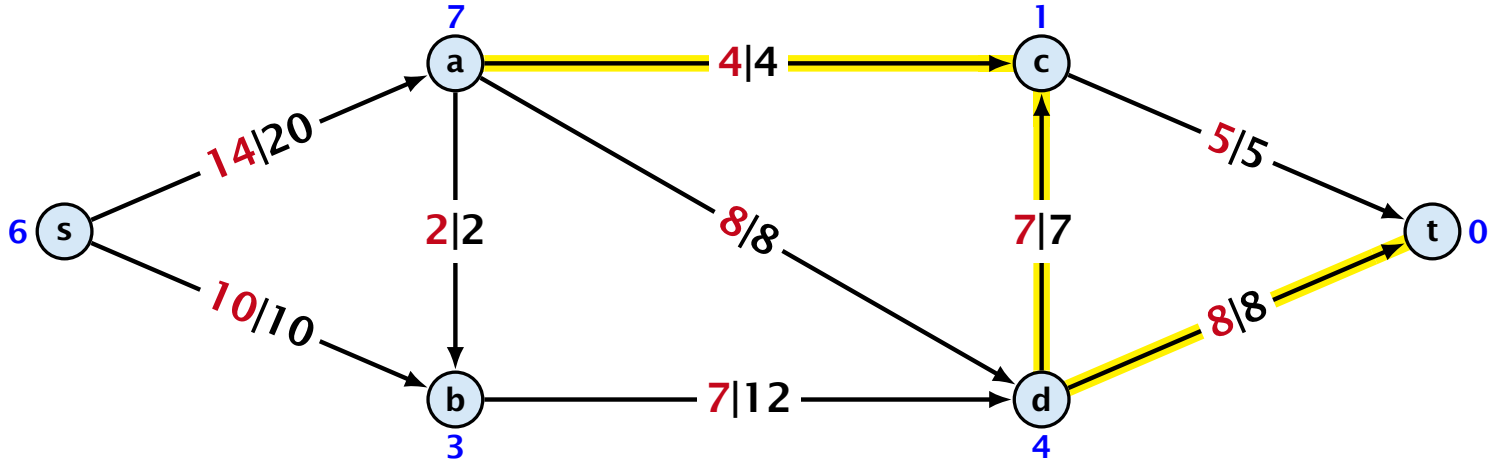

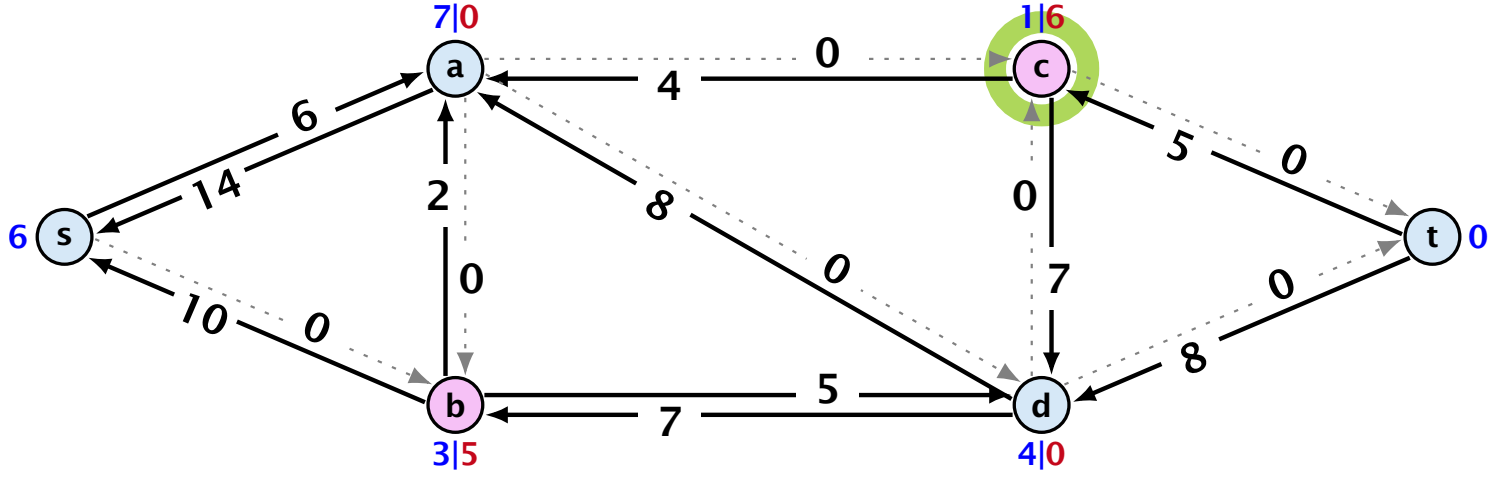

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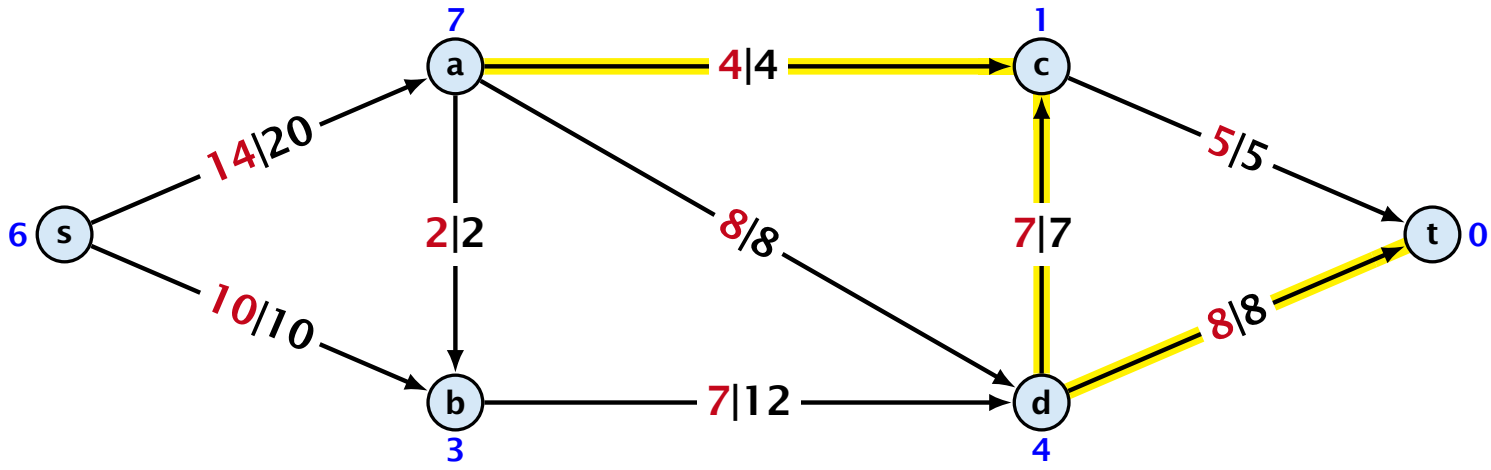
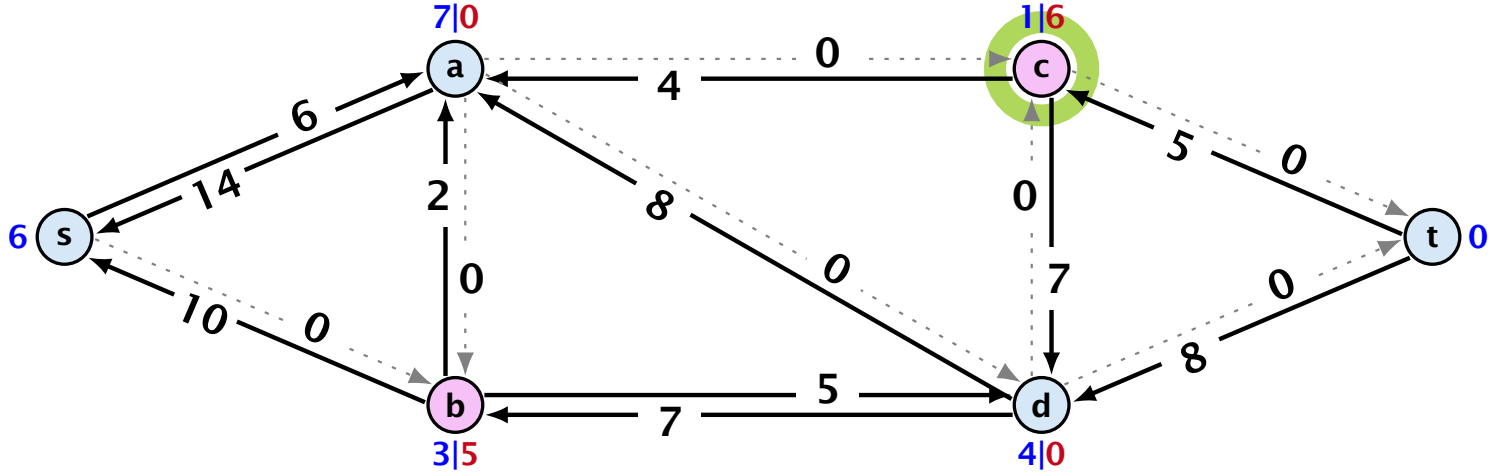


relabel to 5



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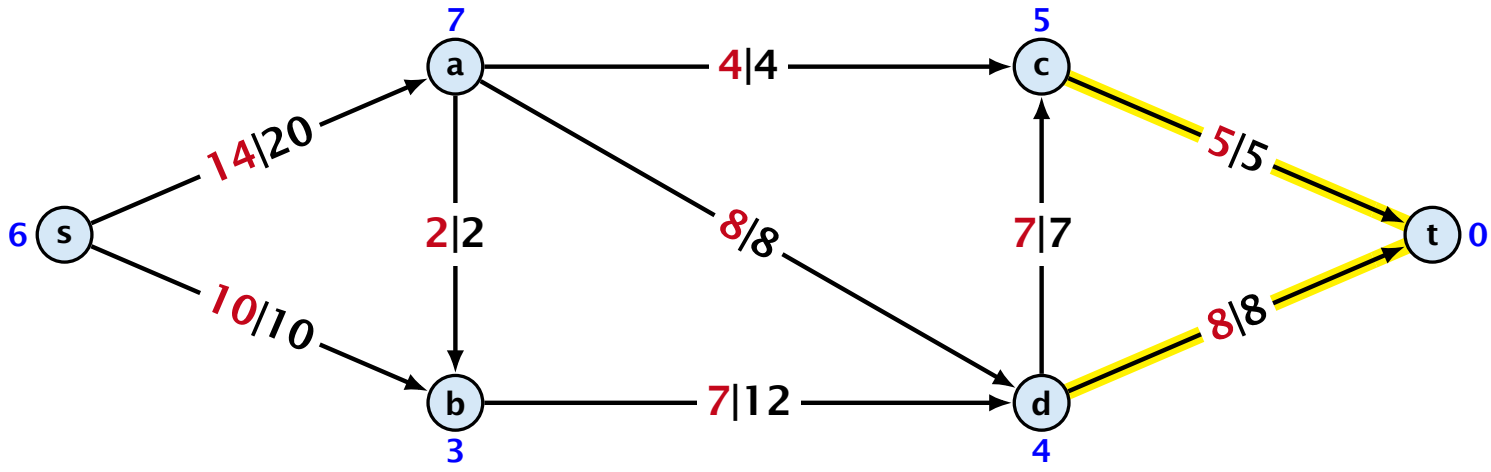

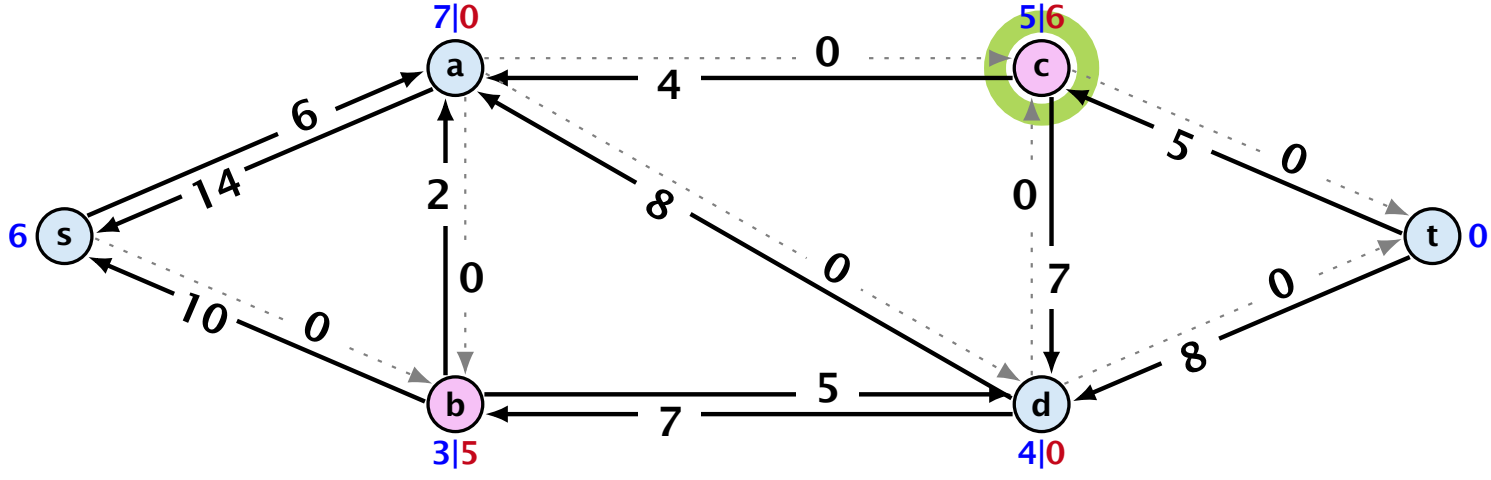

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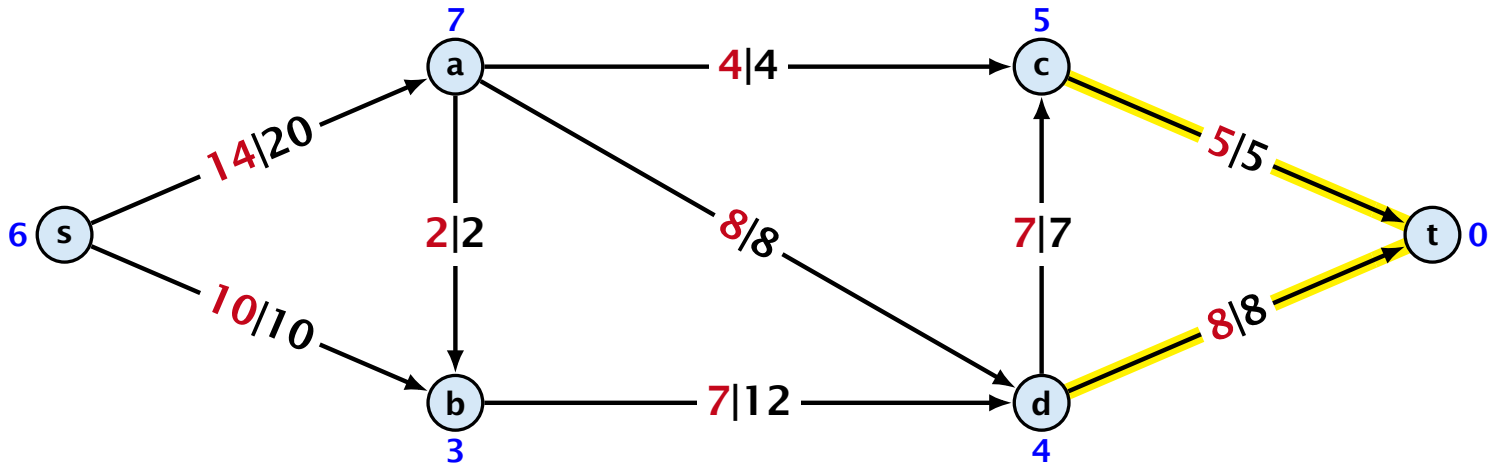
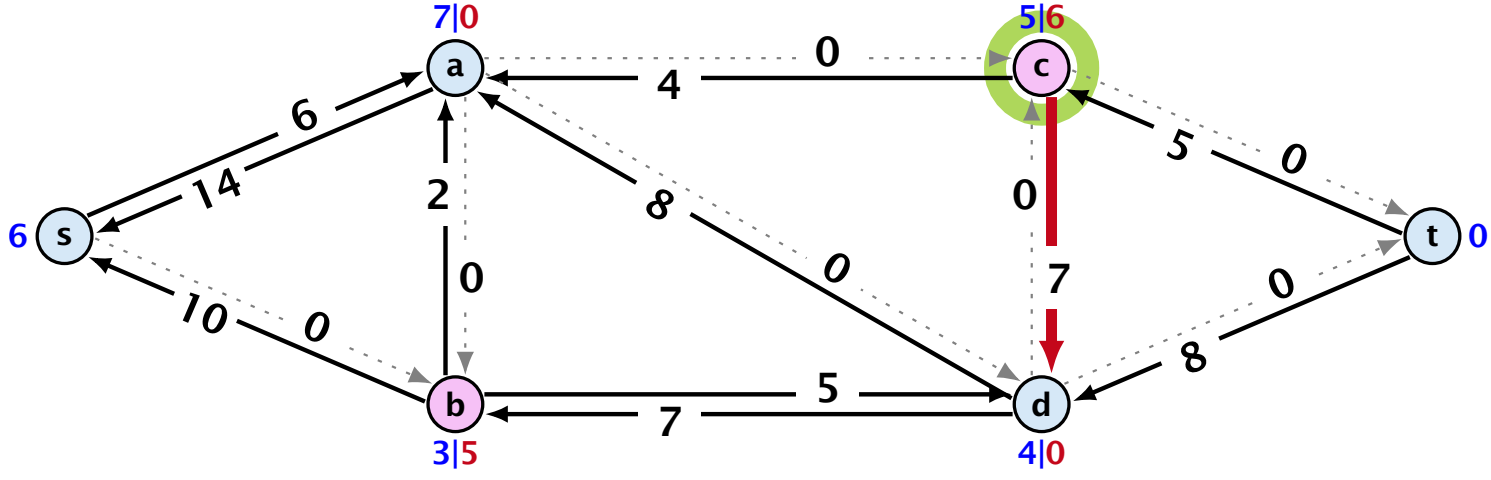


deactivating push



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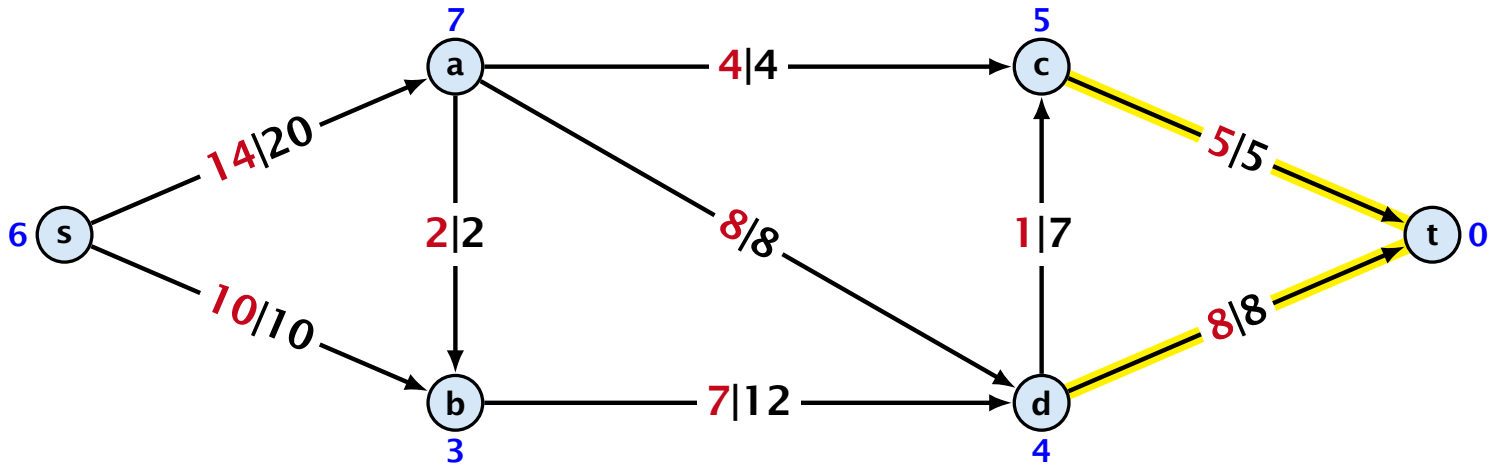

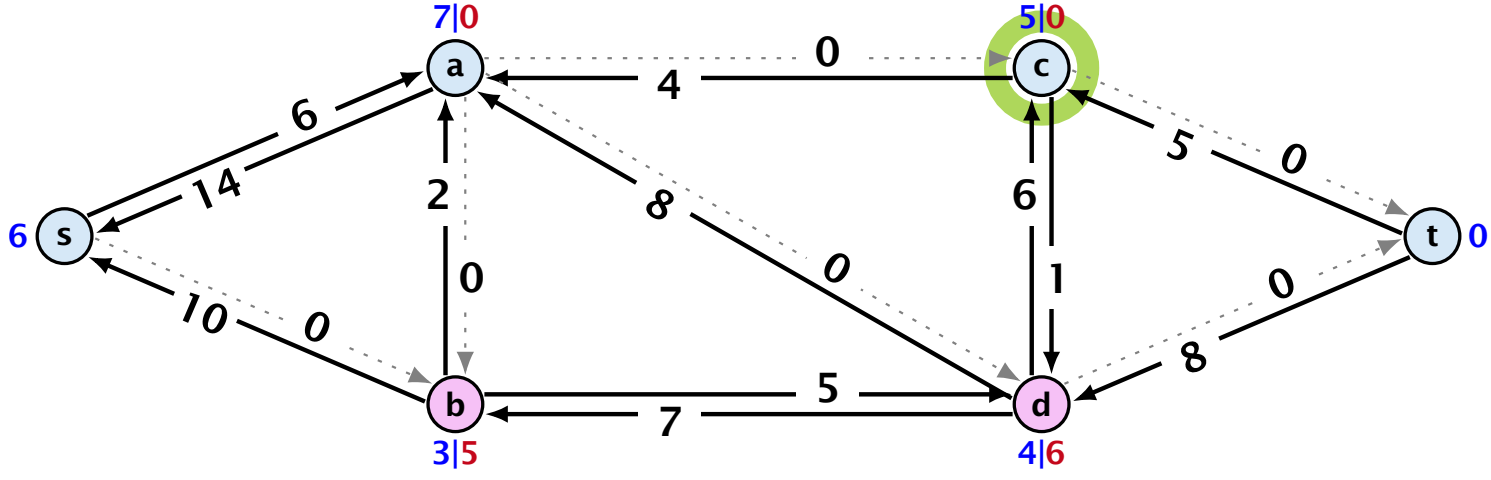

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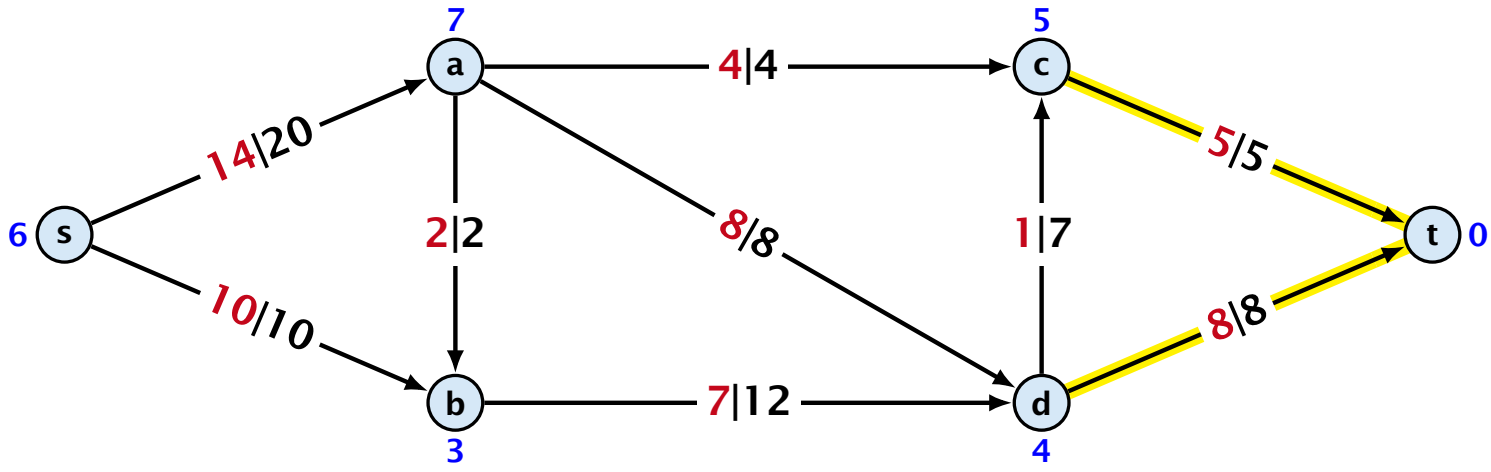

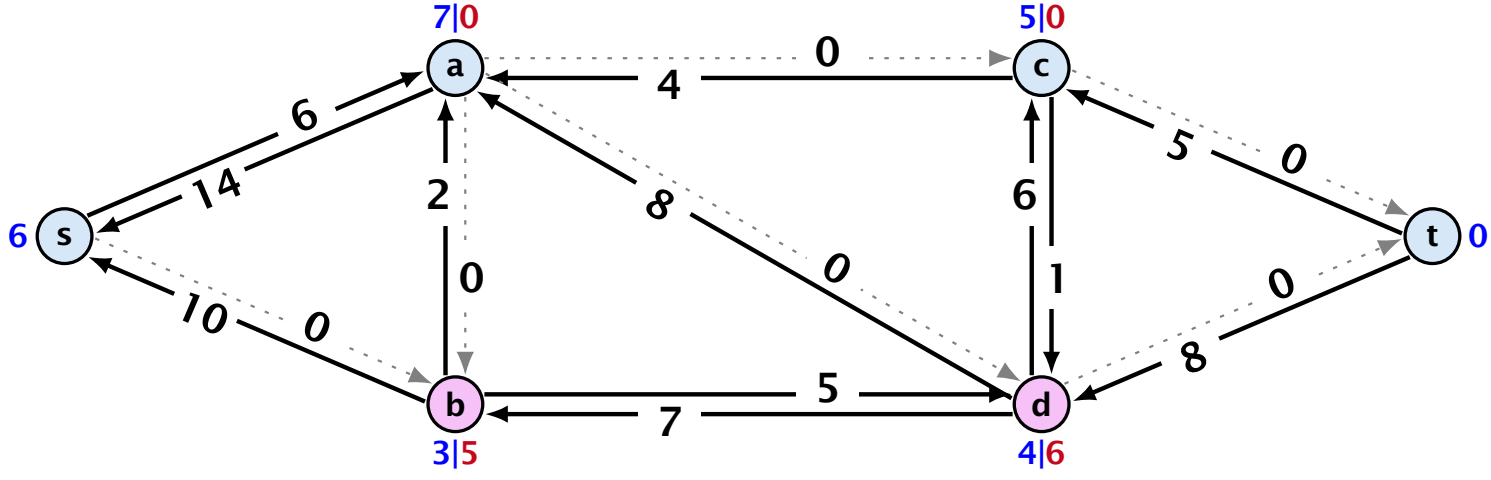

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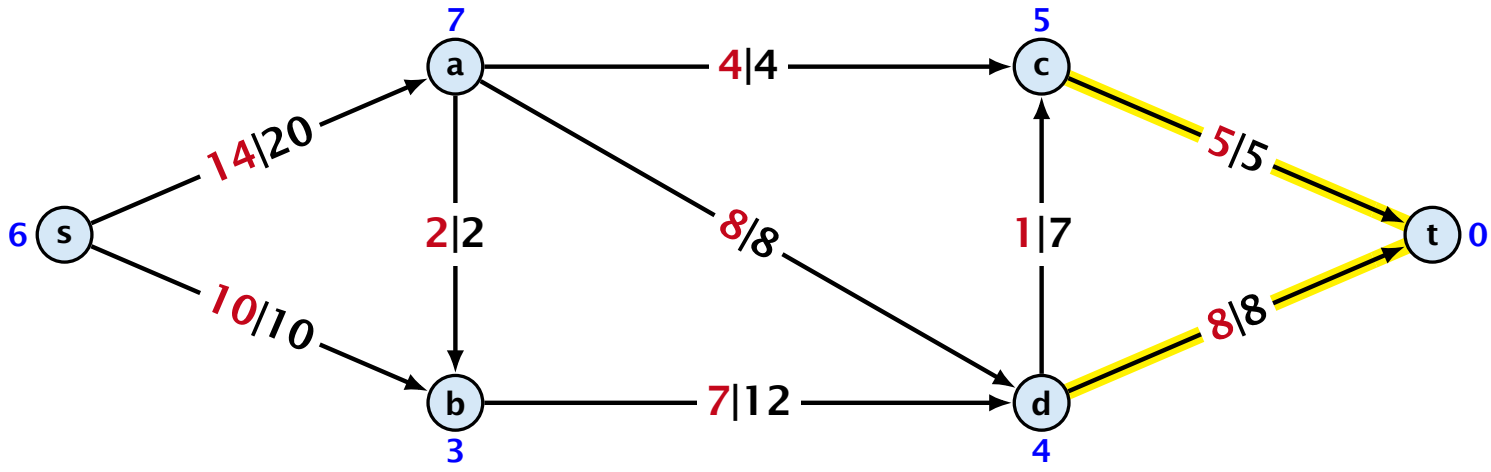

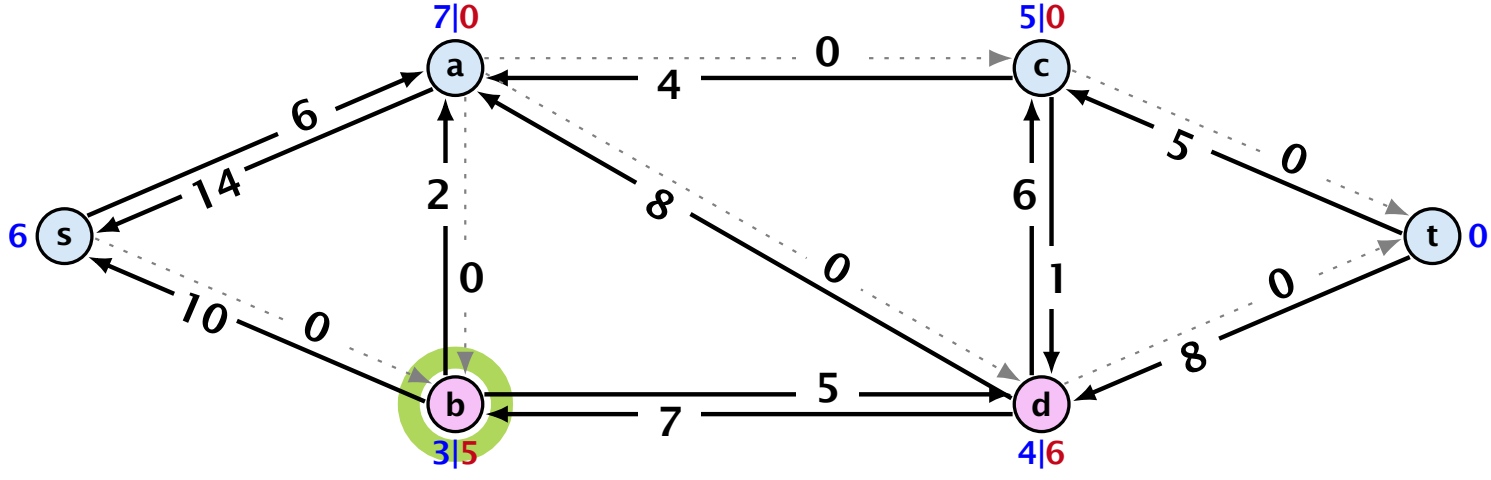

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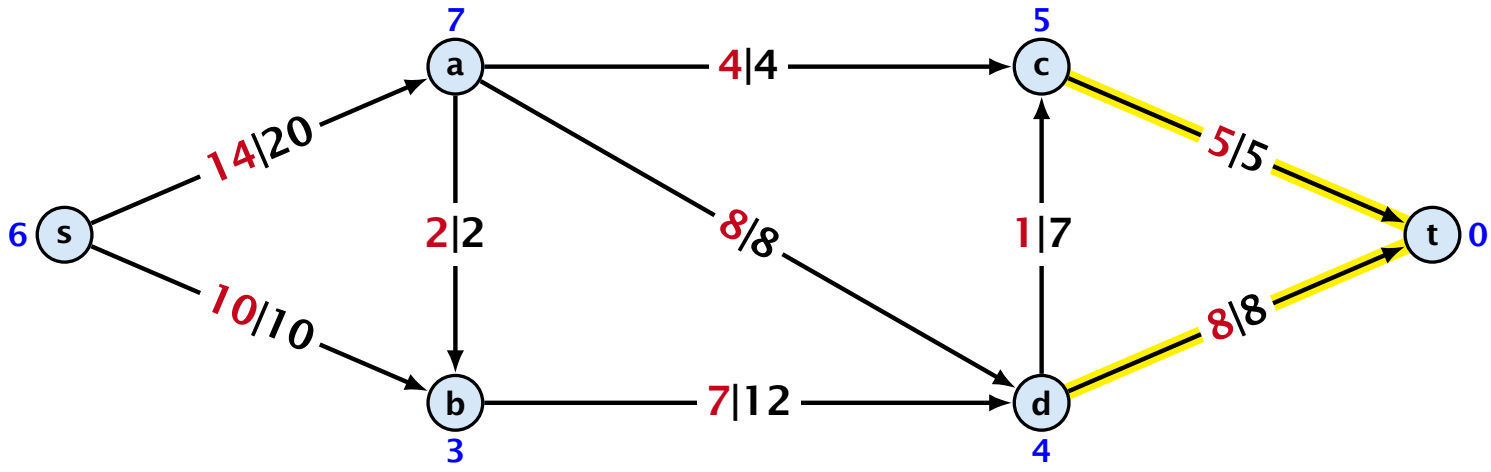
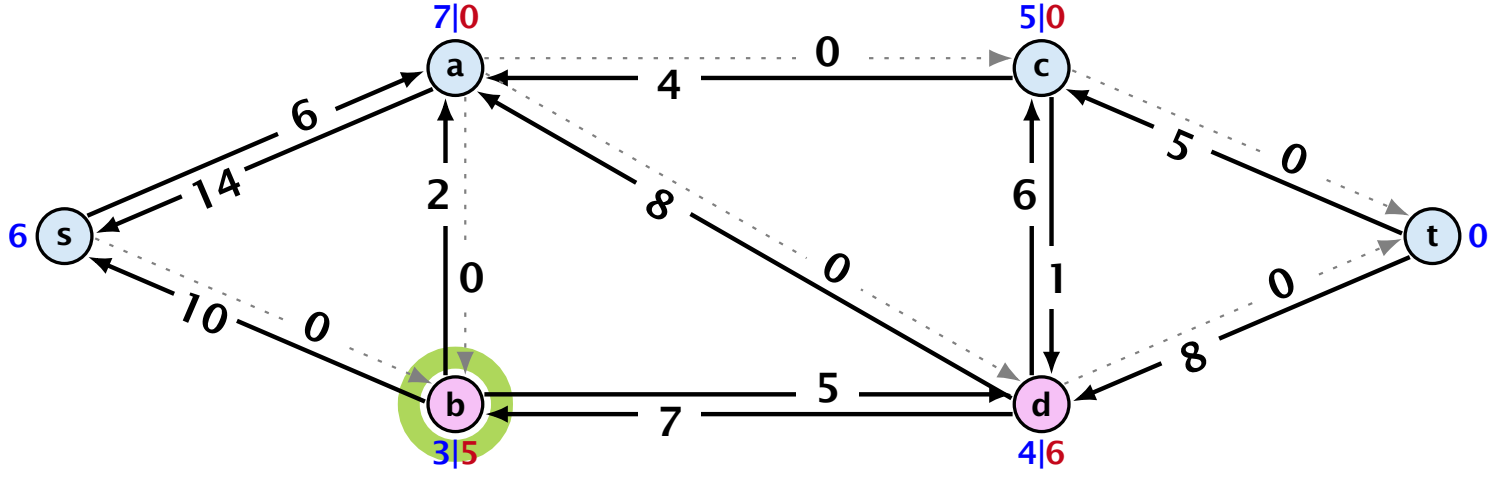


relabel to 5



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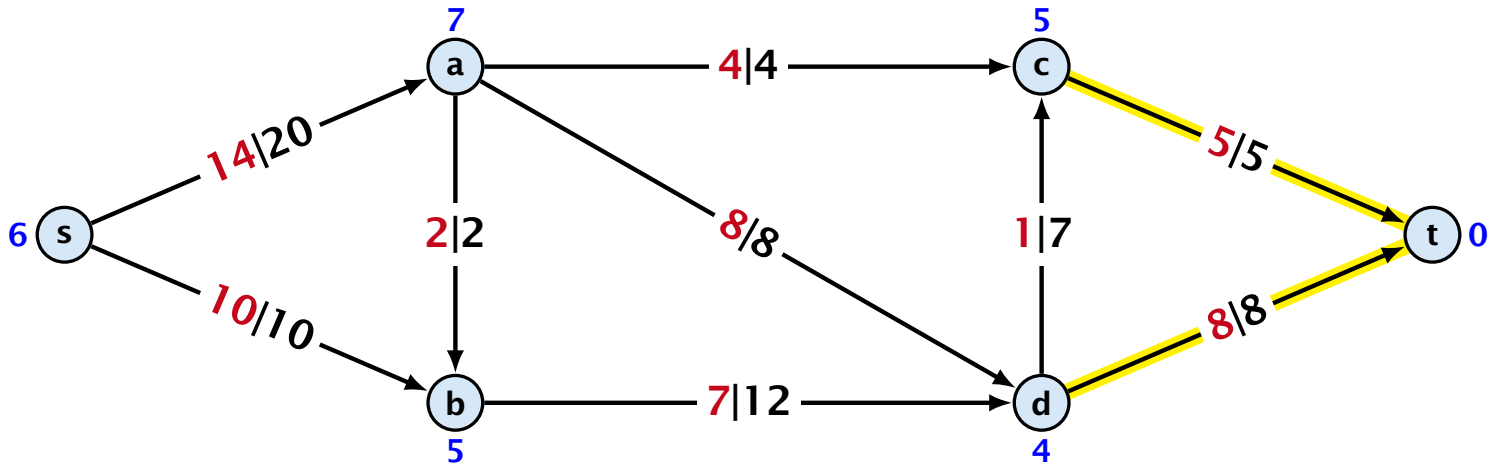

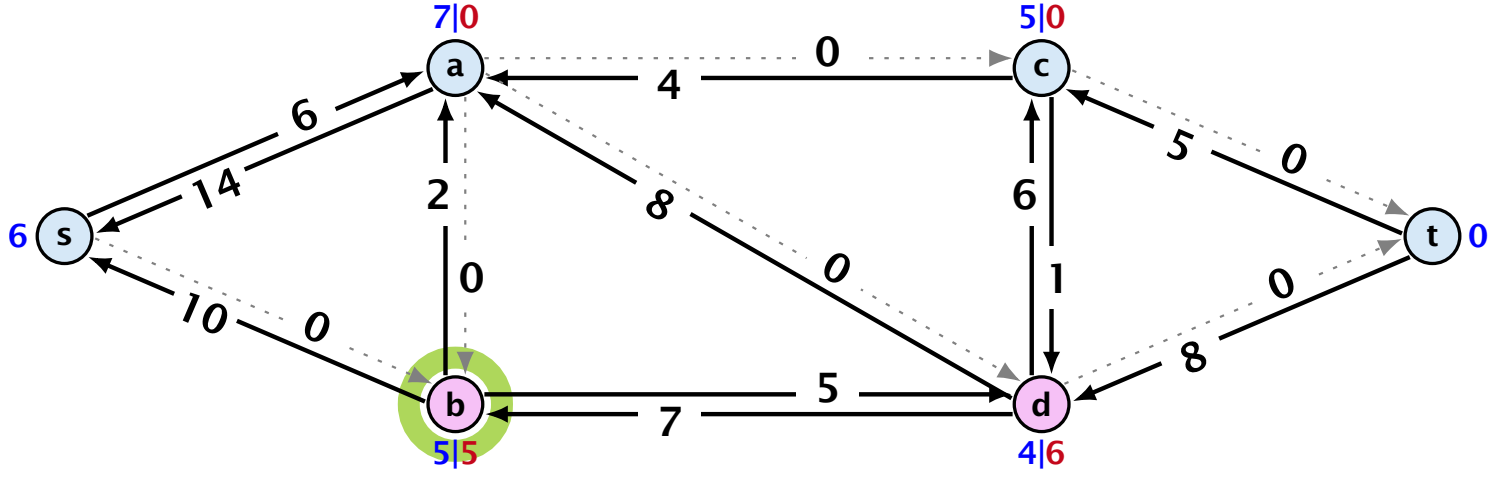

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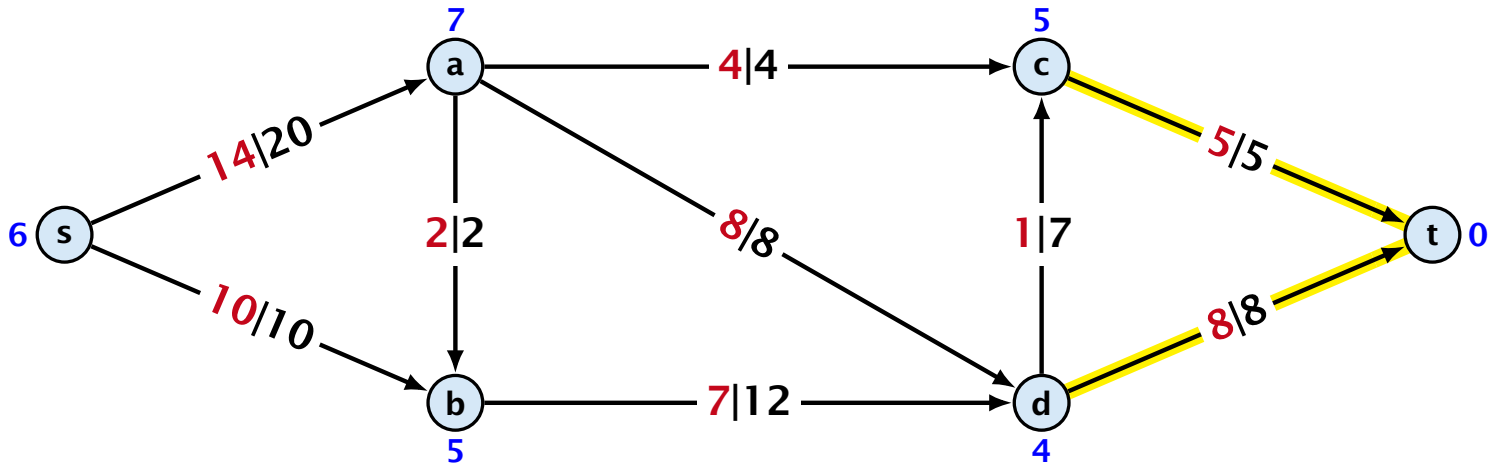
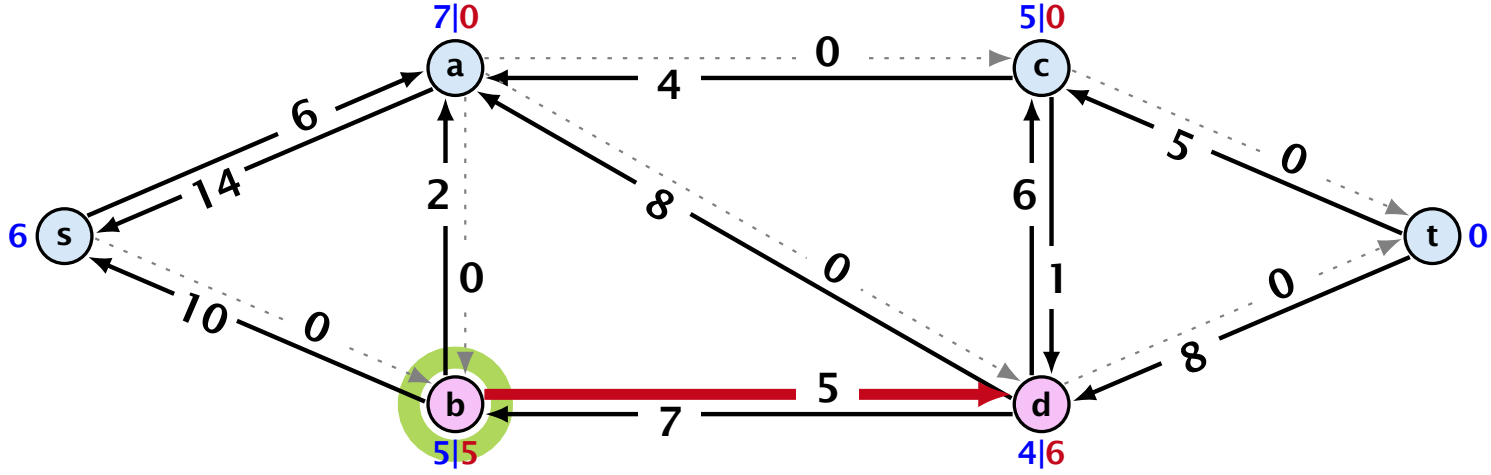


saturating and deactivating push



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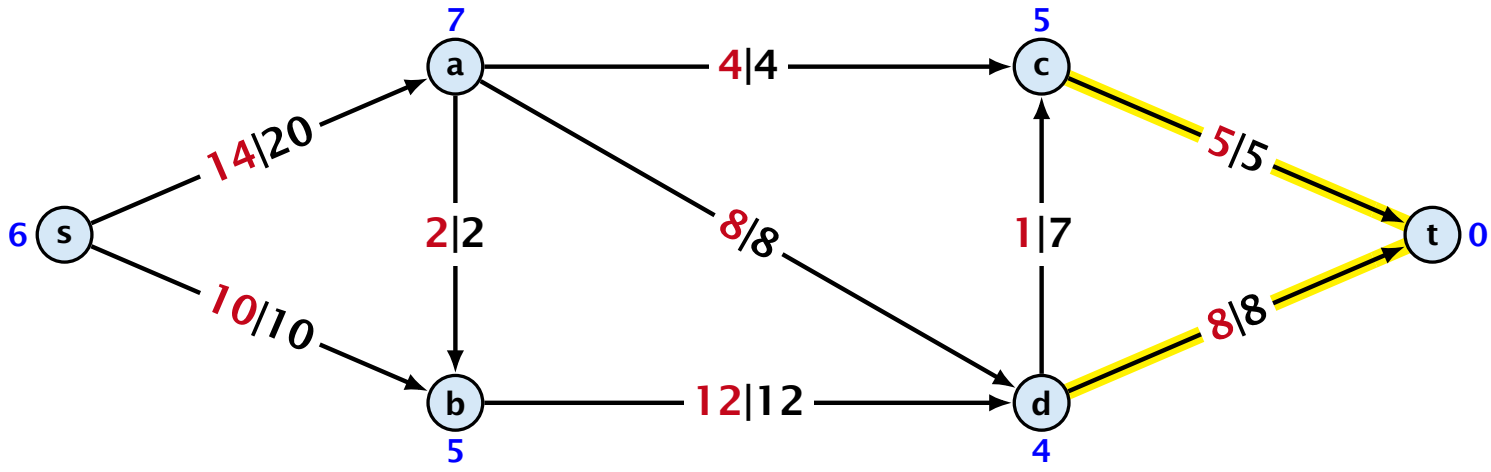

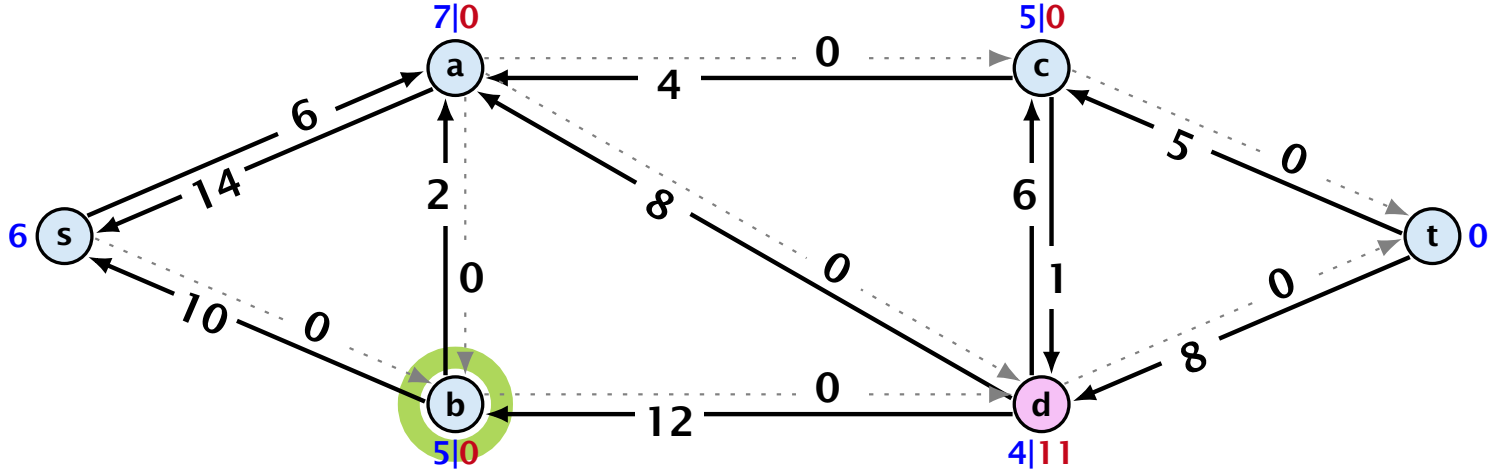

Preflow Push

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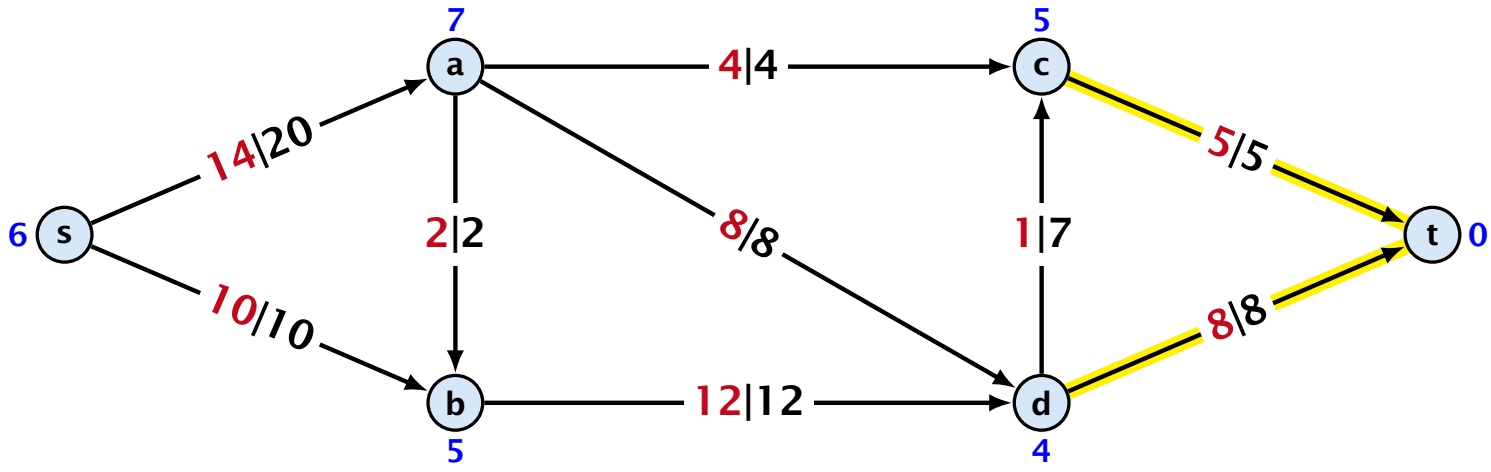

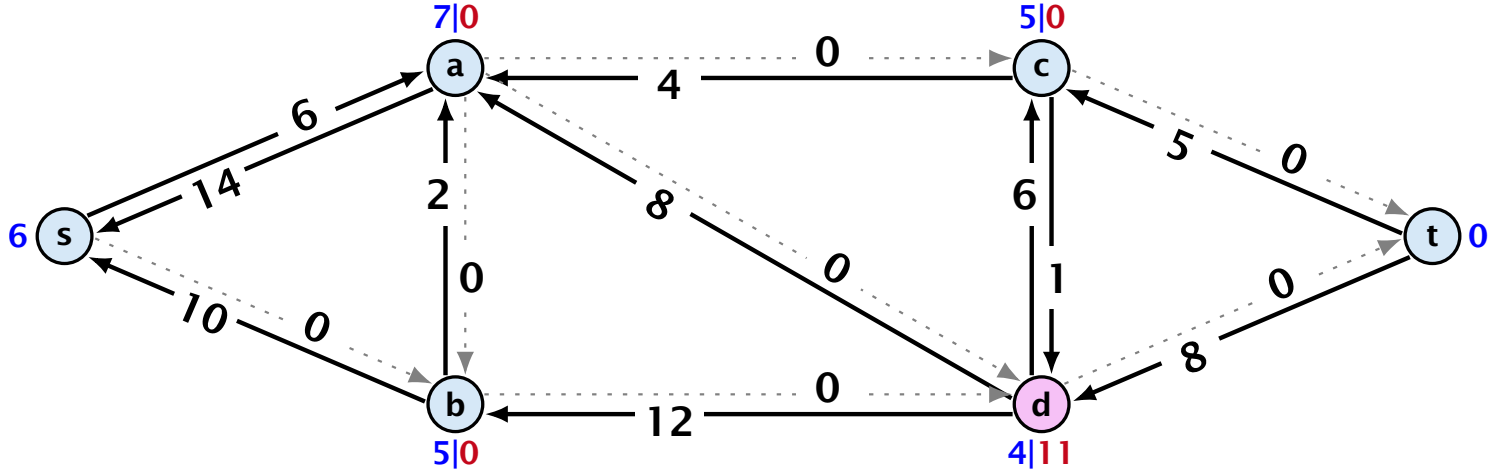

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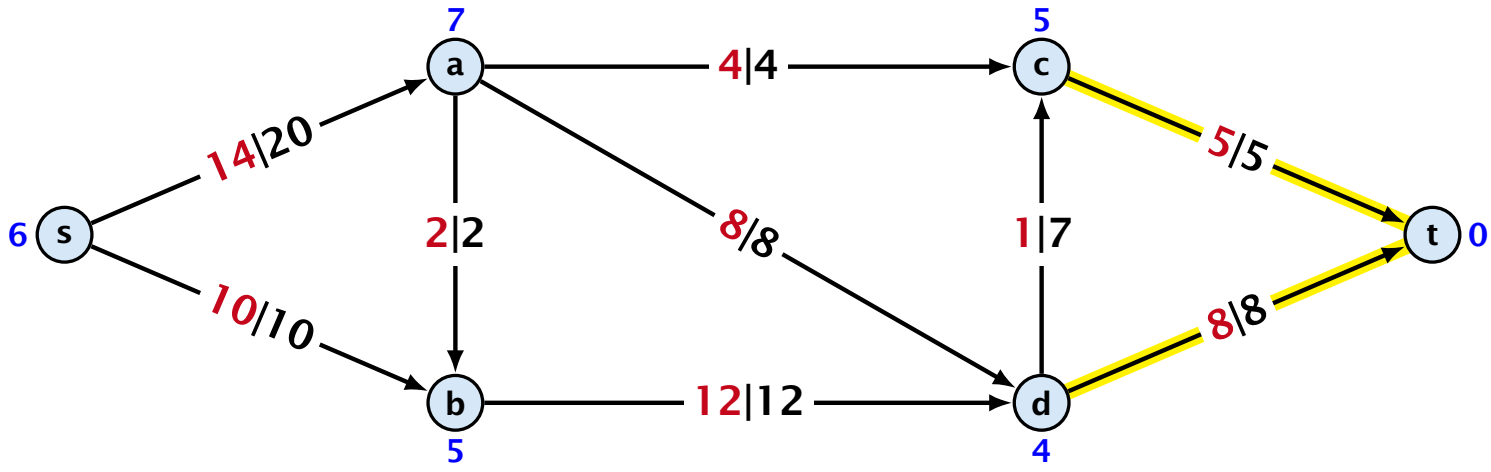

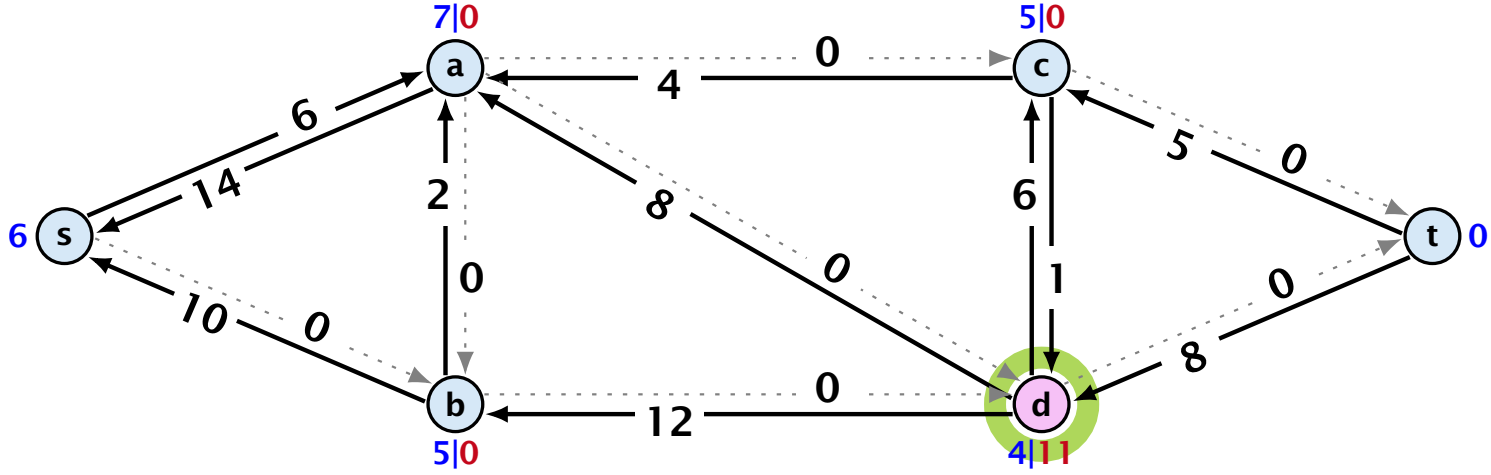

Preflow Push

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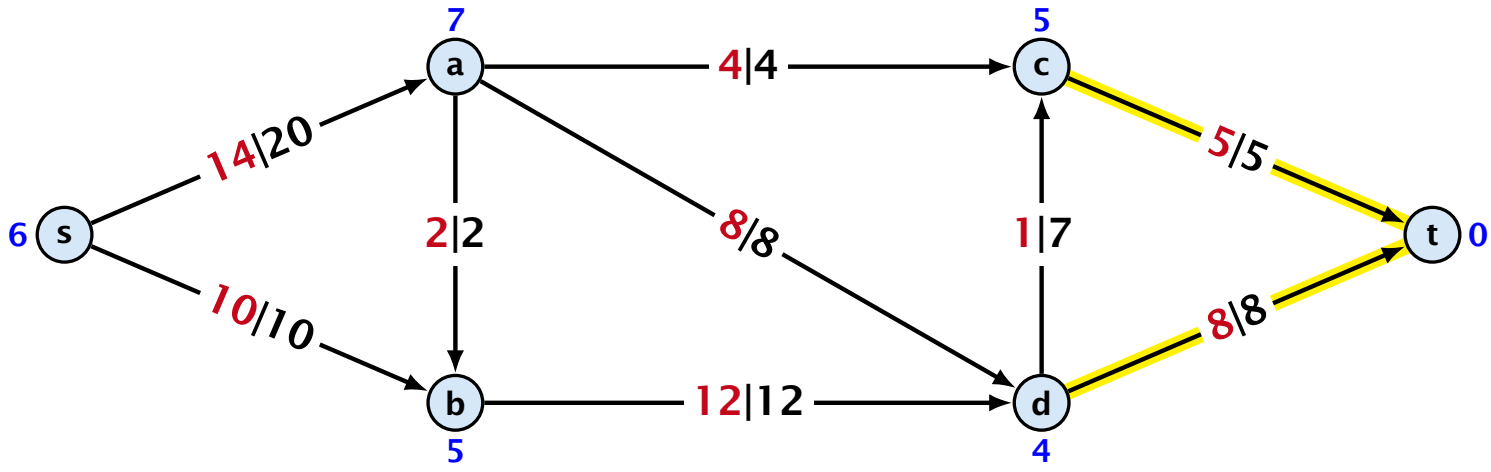
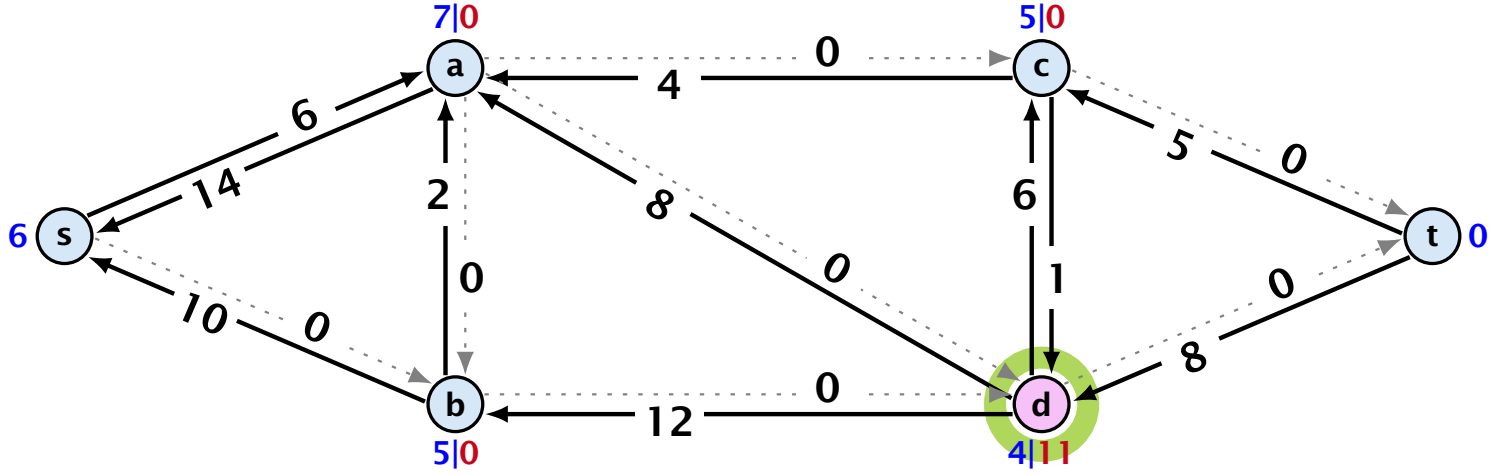


relabel to 6



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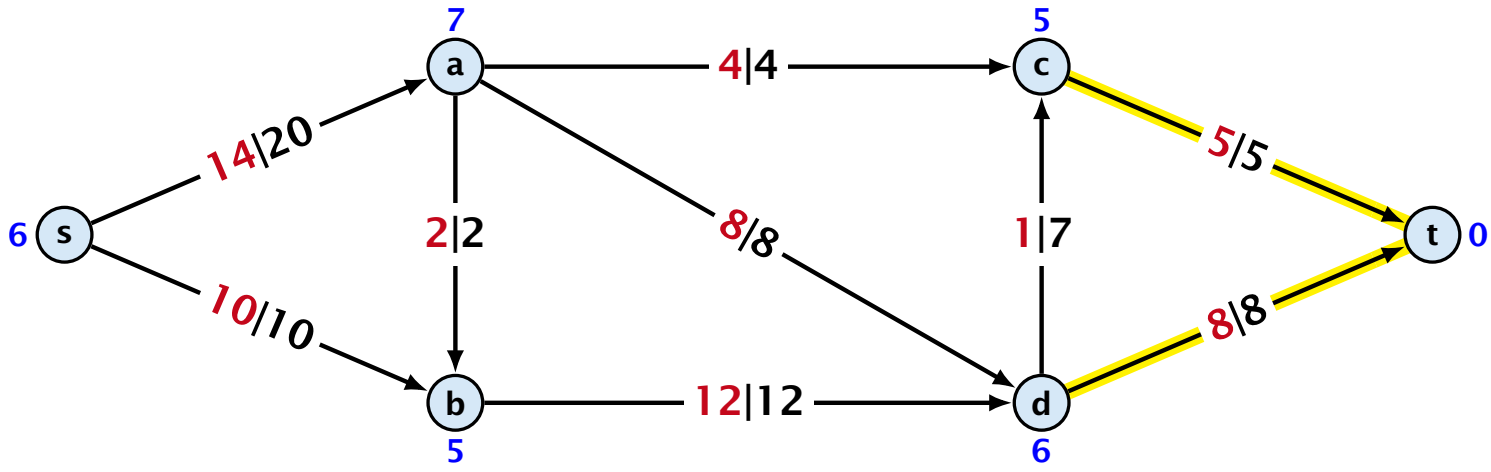

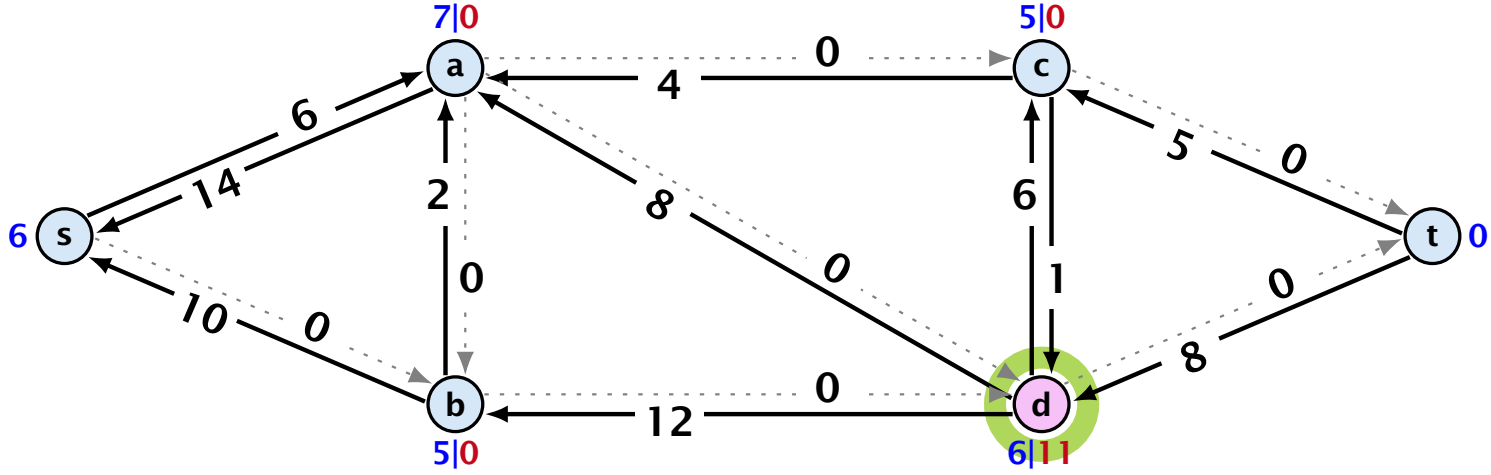

Preflow Push

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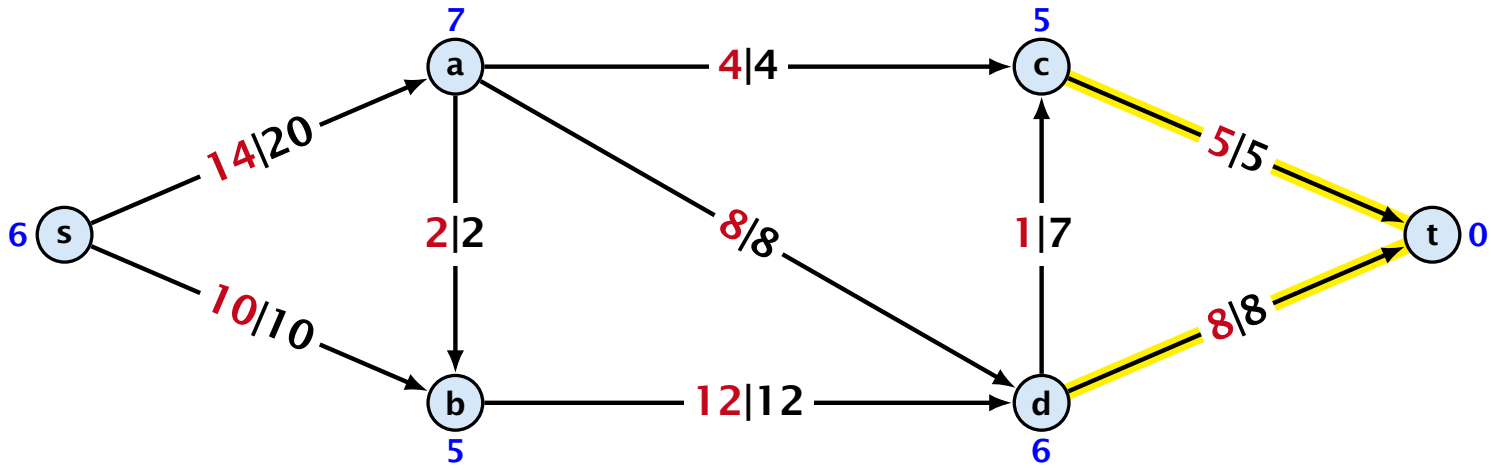
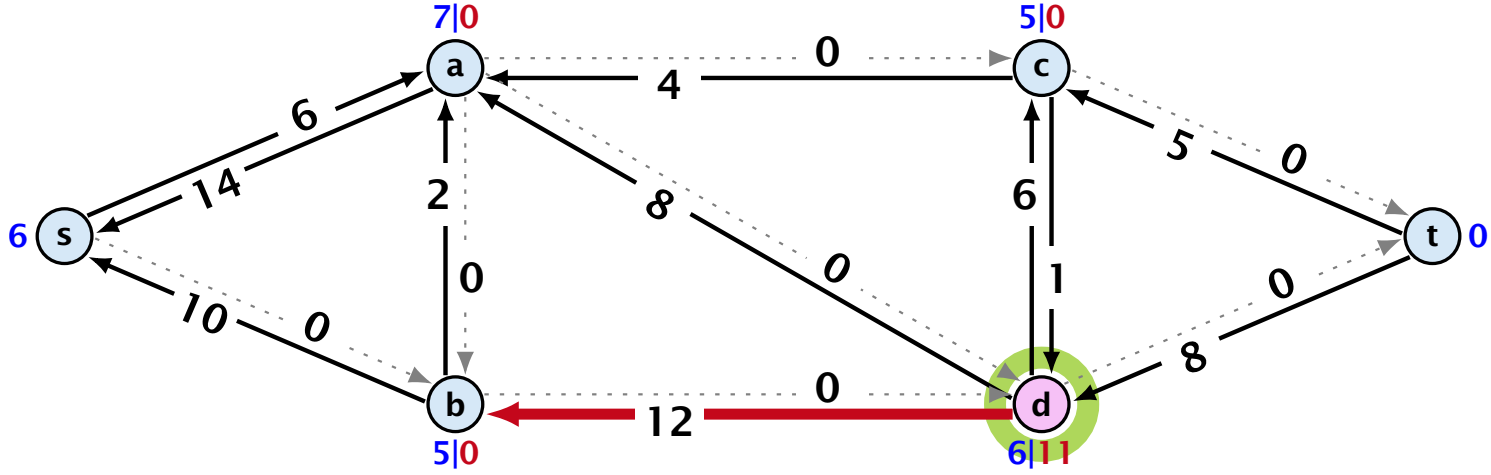


deactivating push



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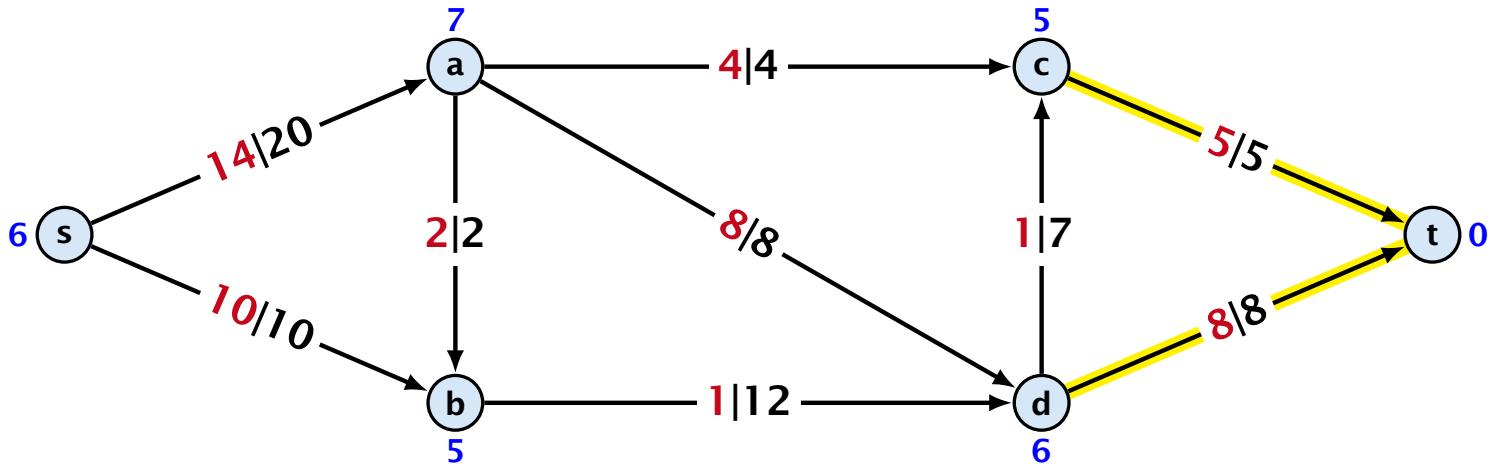

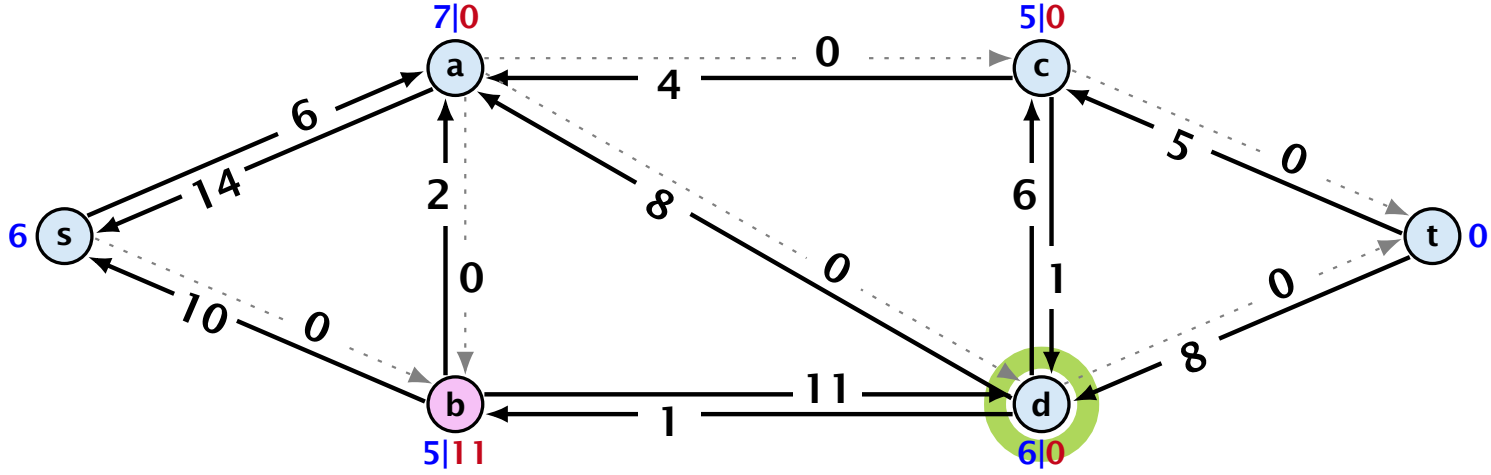

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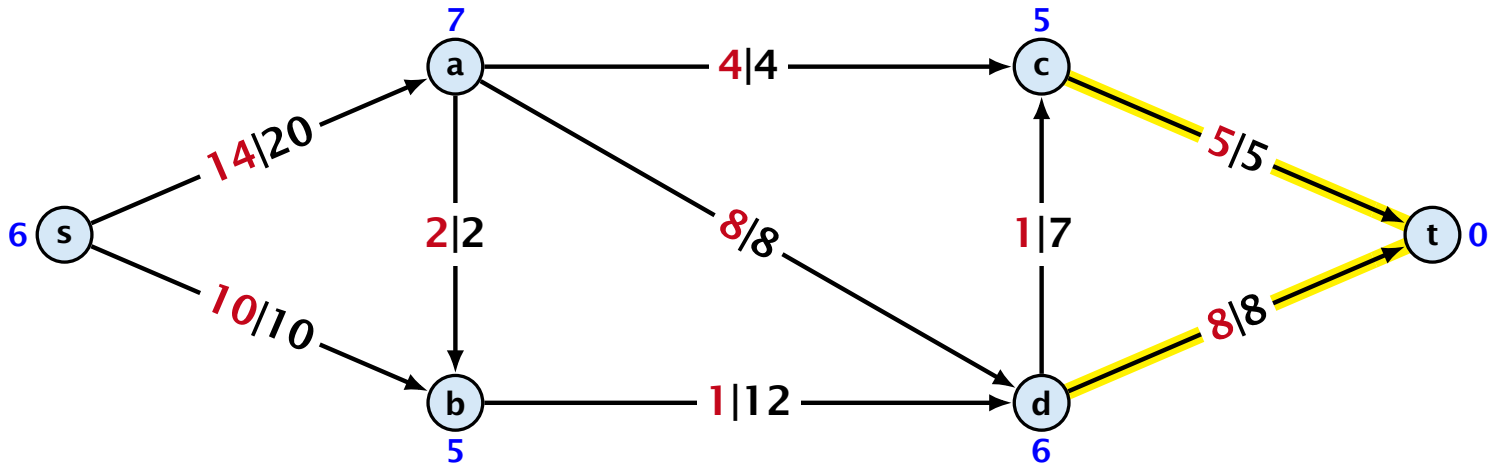

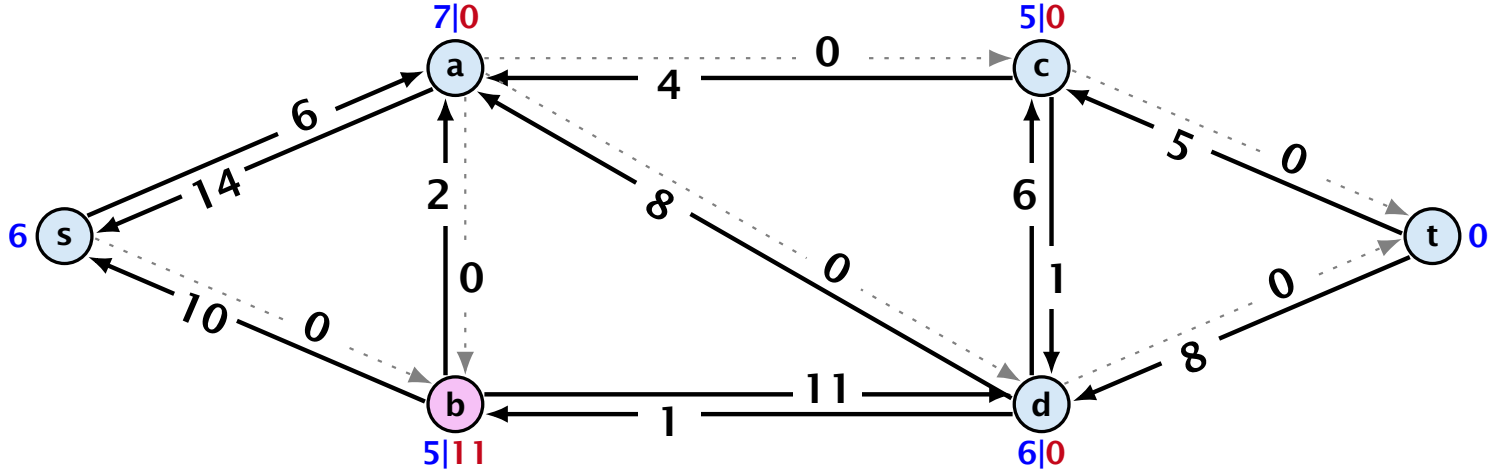

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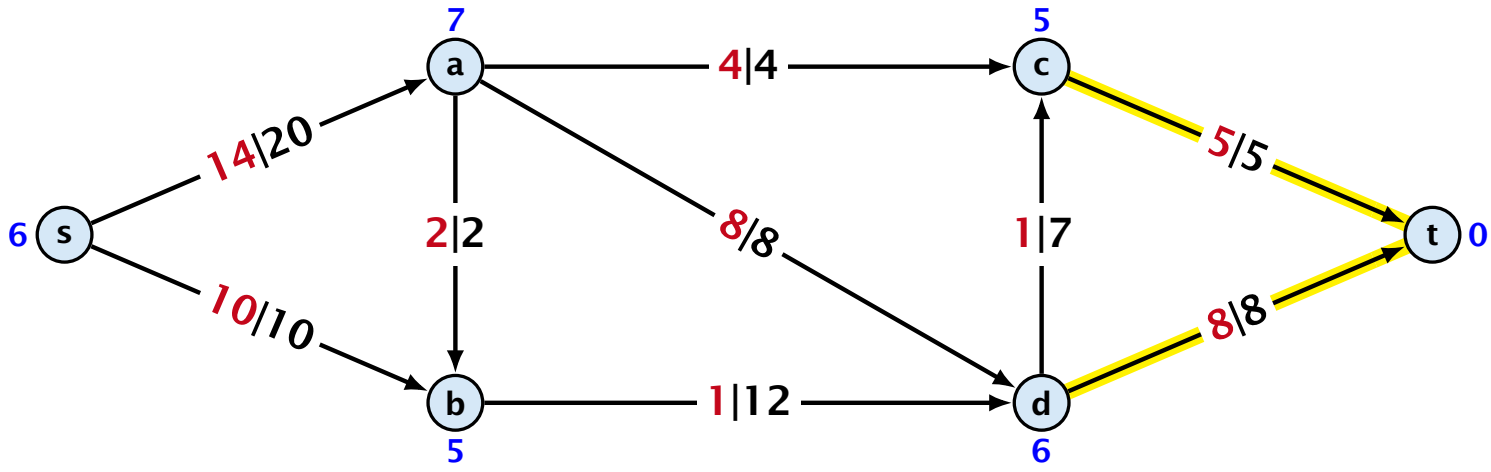

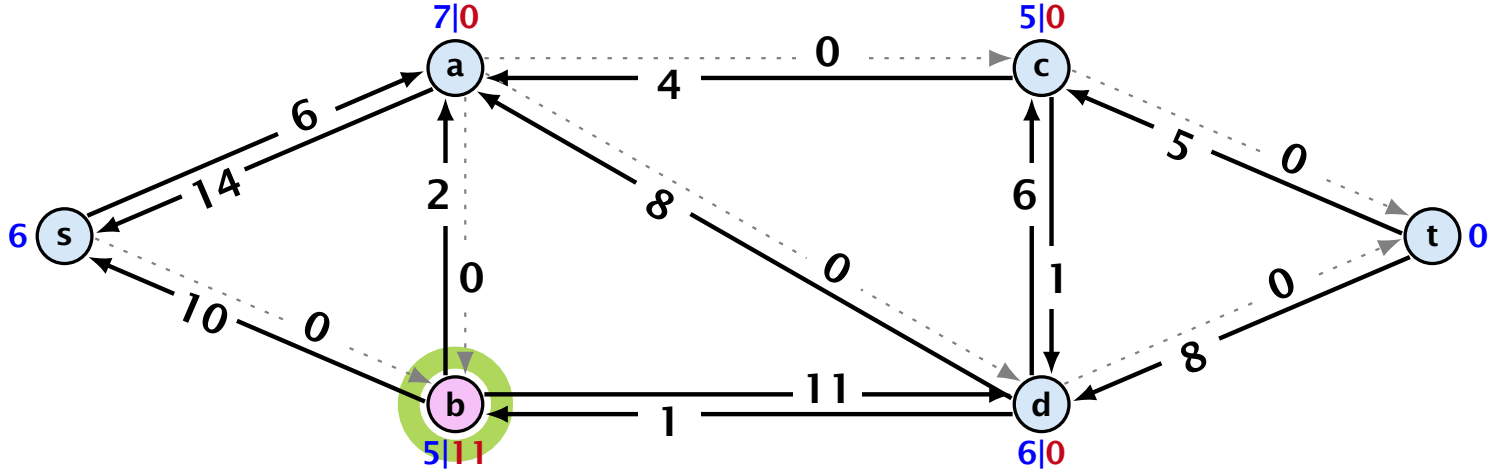

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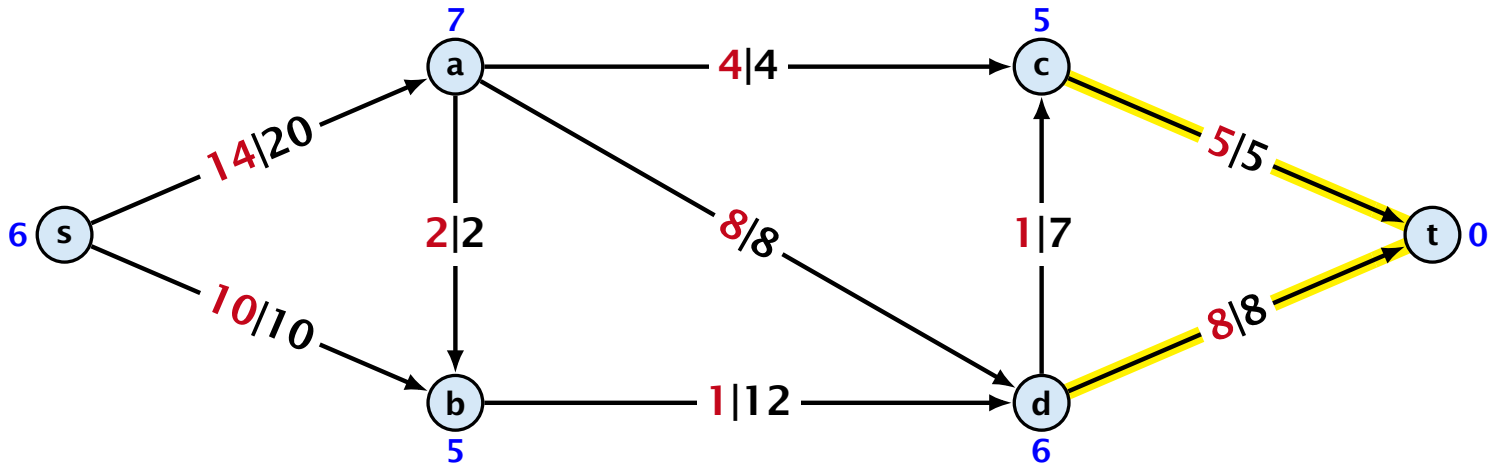
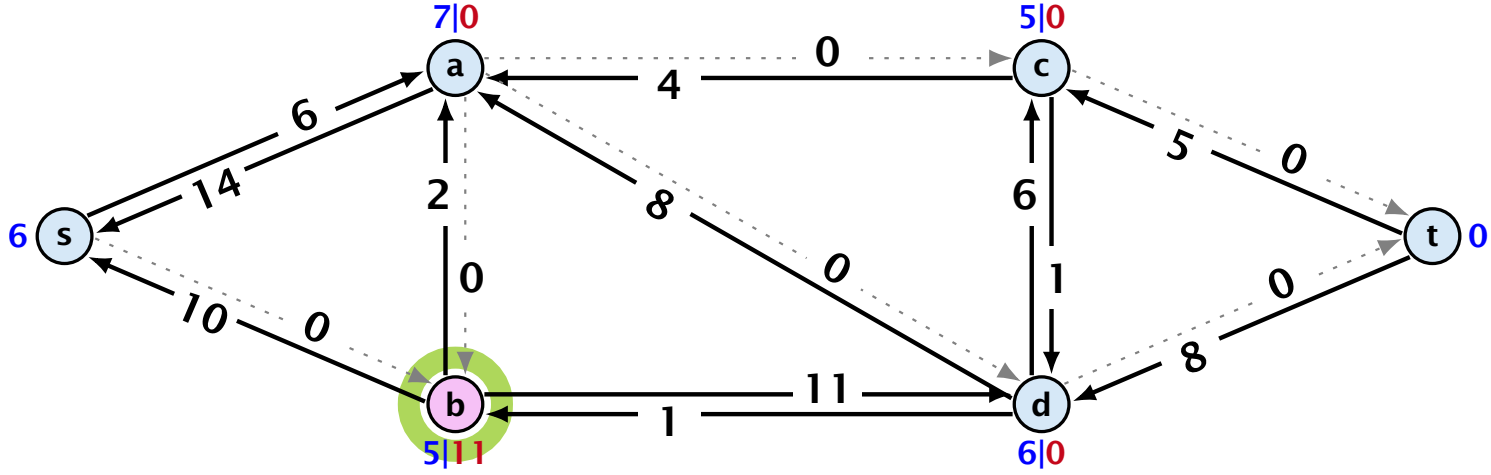


relabel to 7



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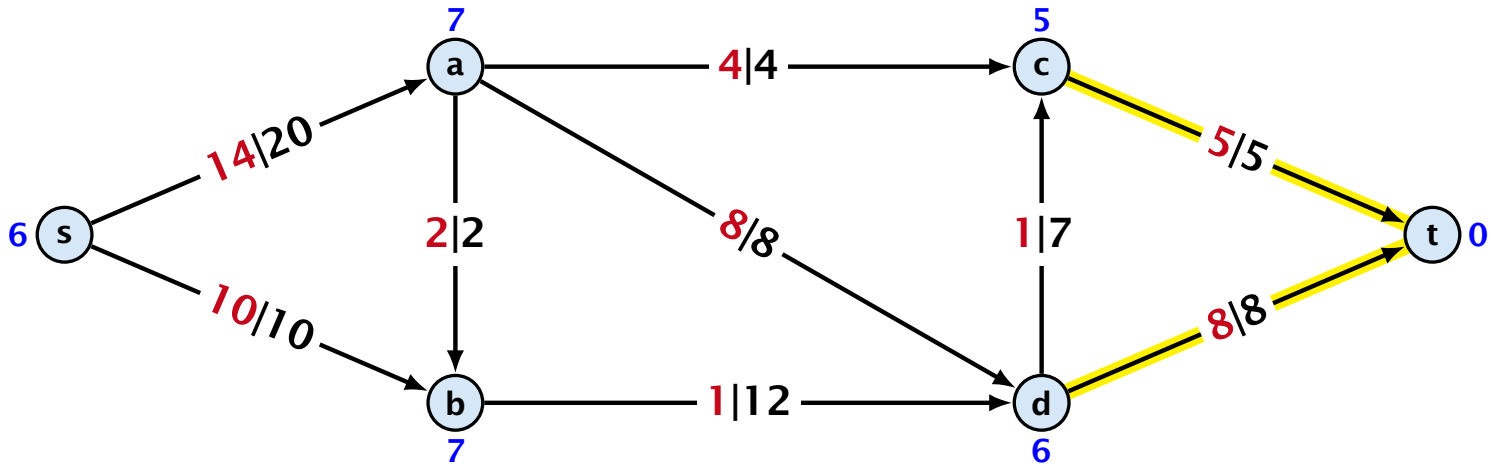

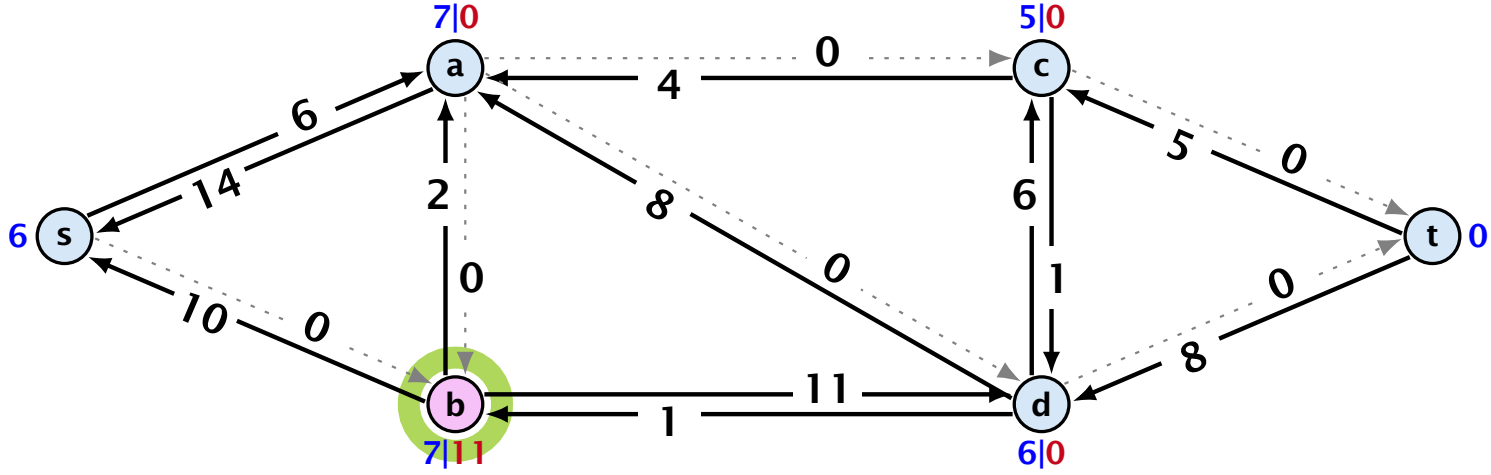

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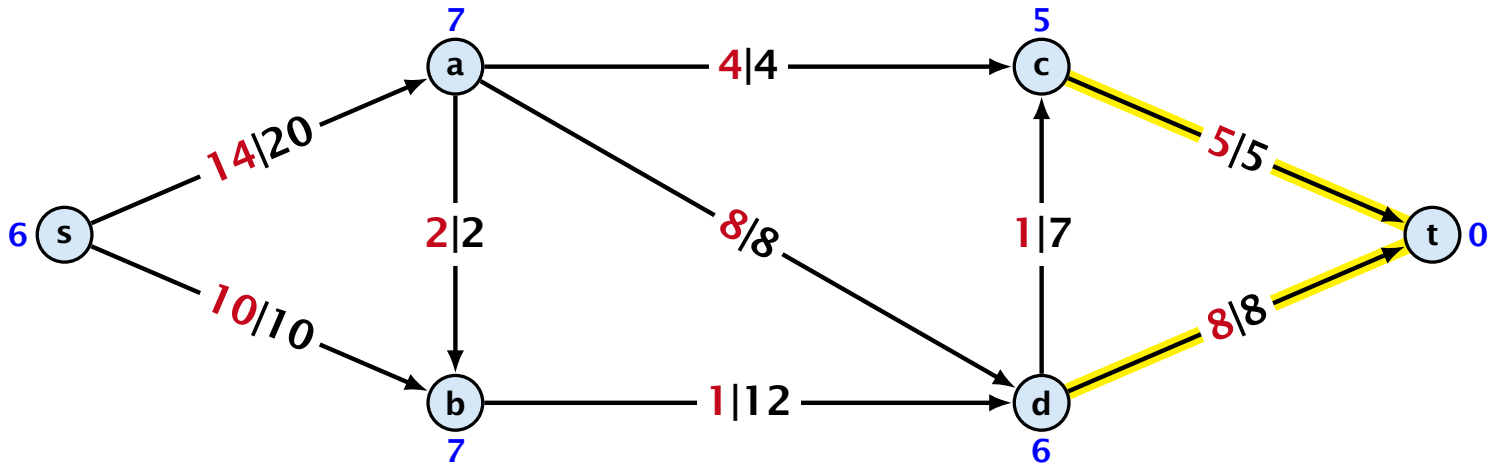
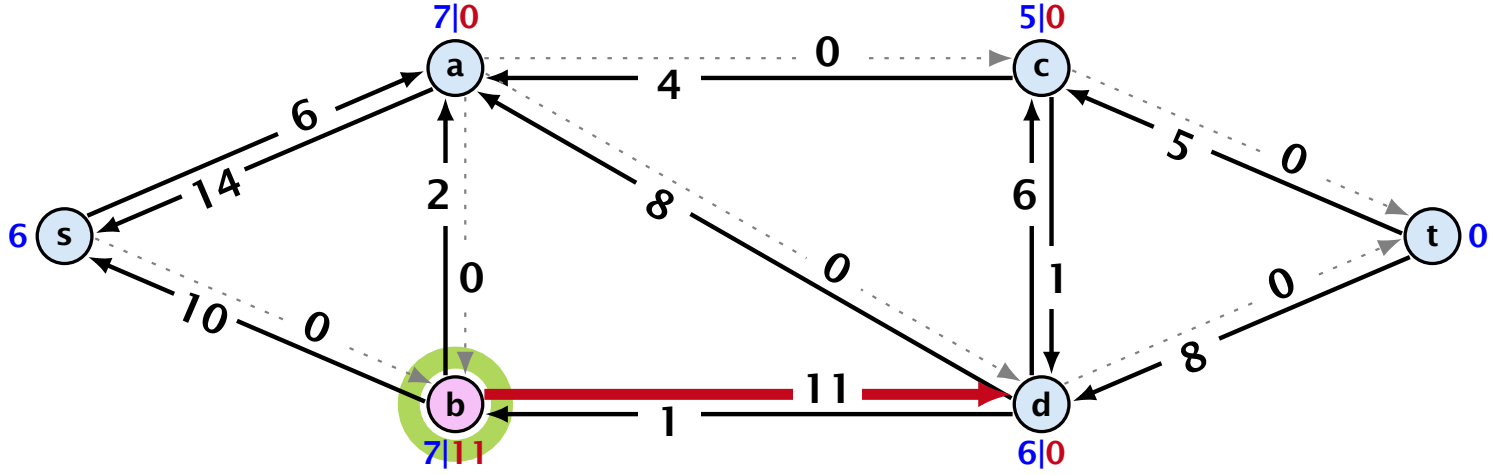


saturating and deactivating push



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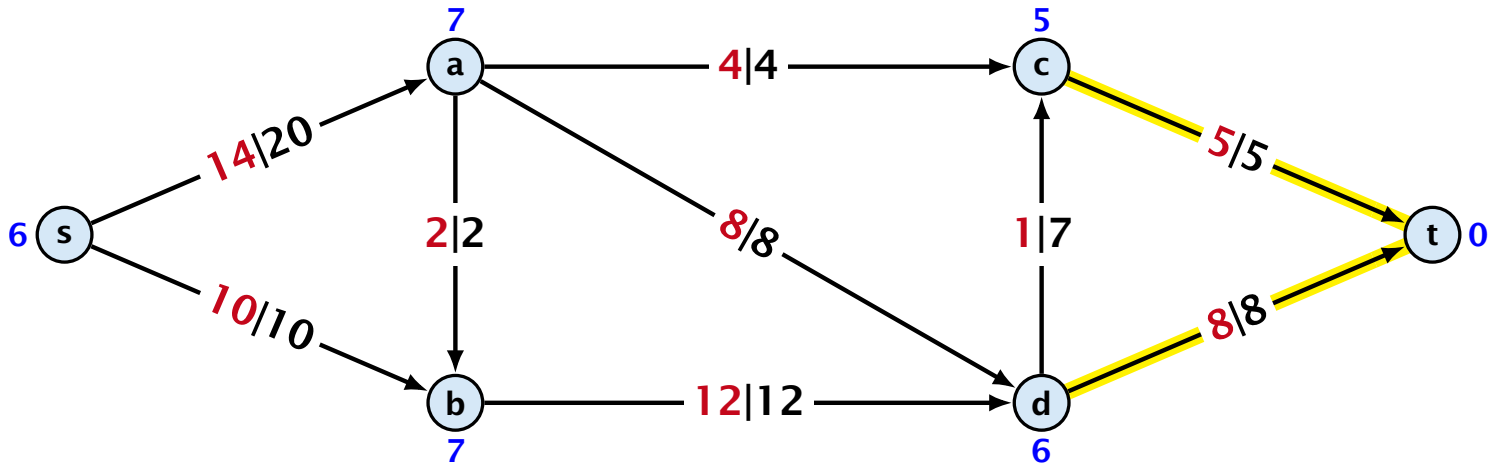

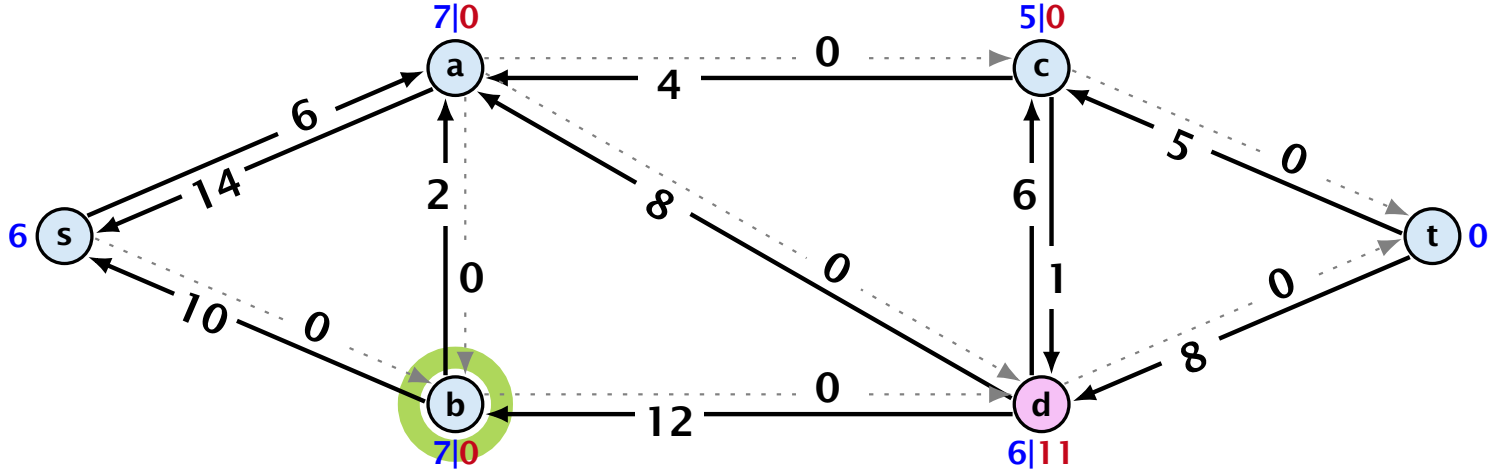

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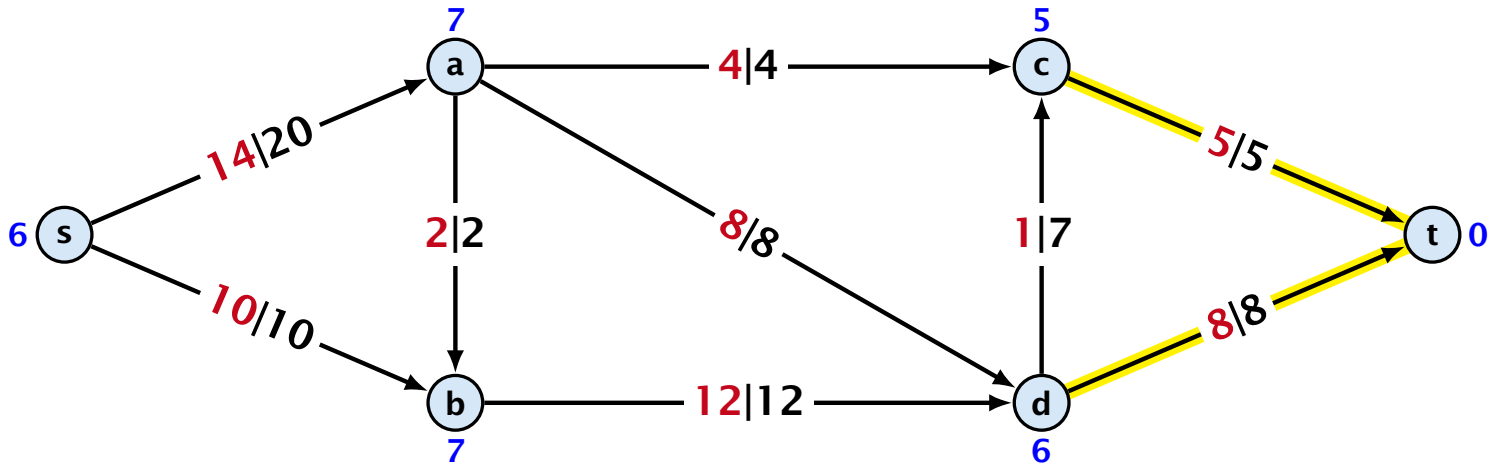

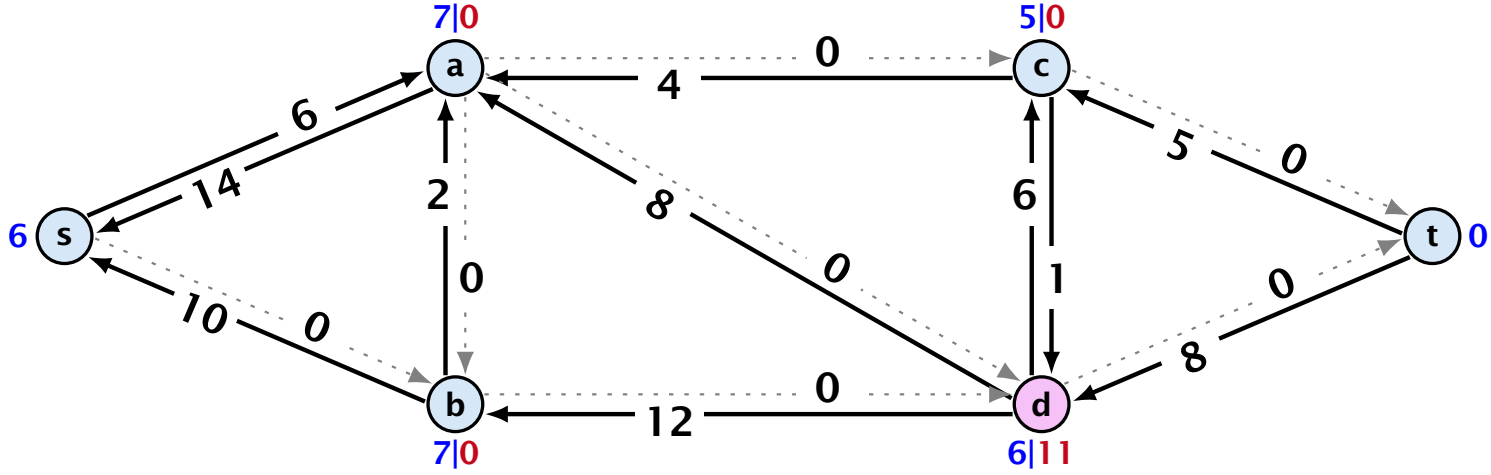

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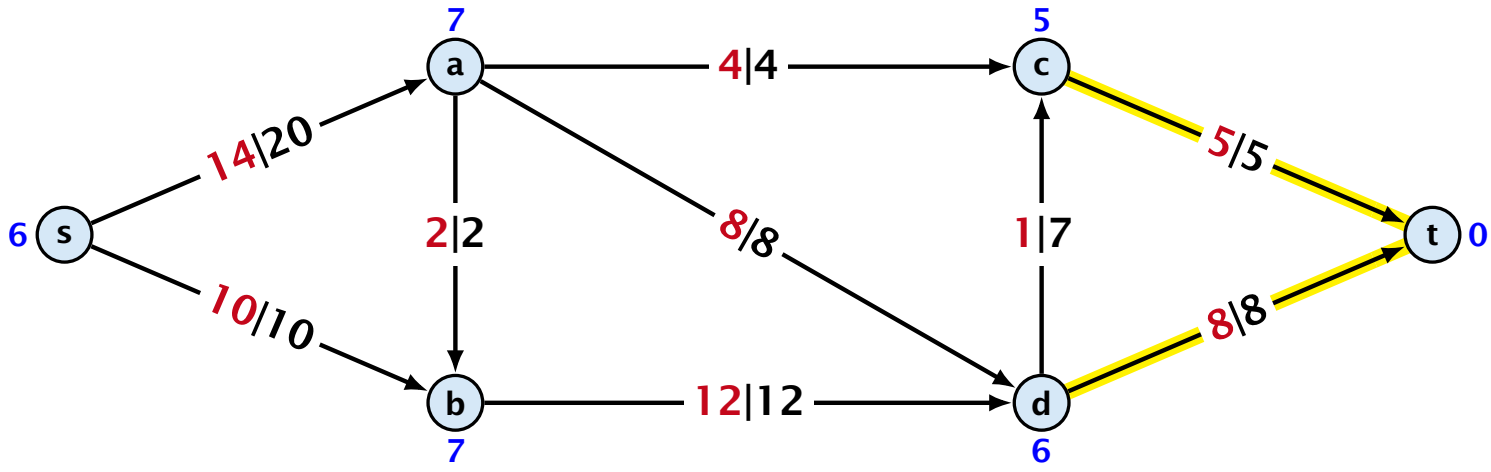

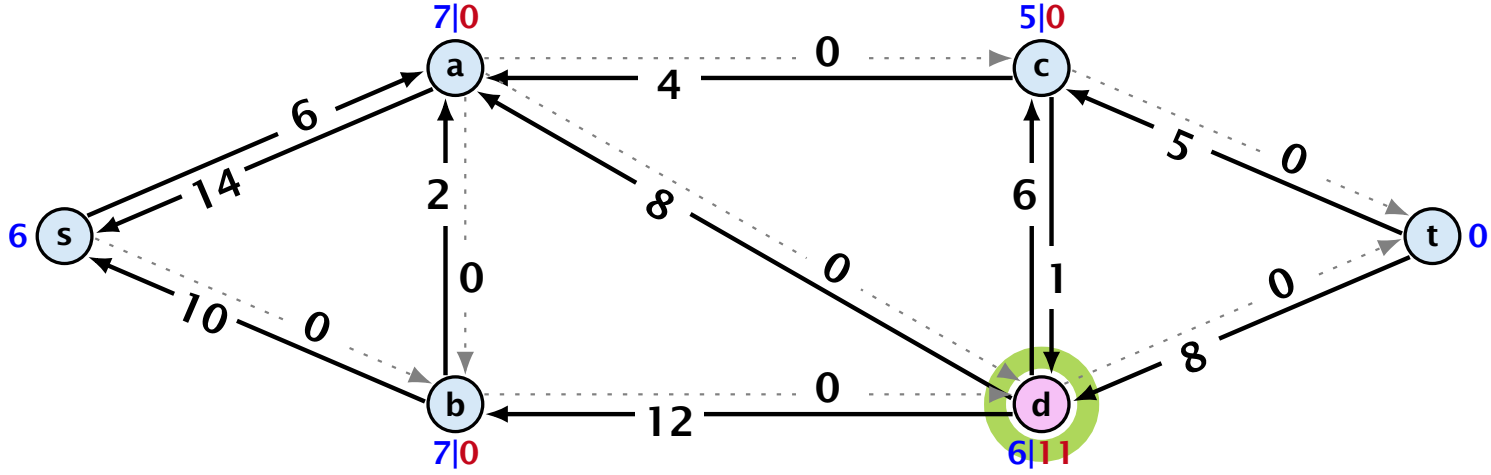

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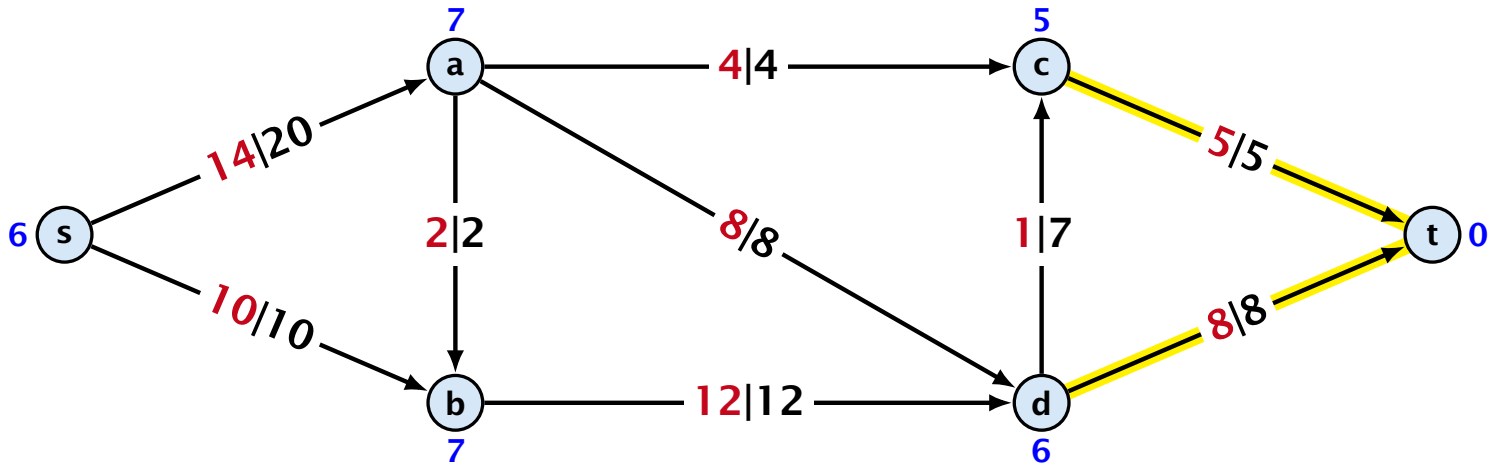
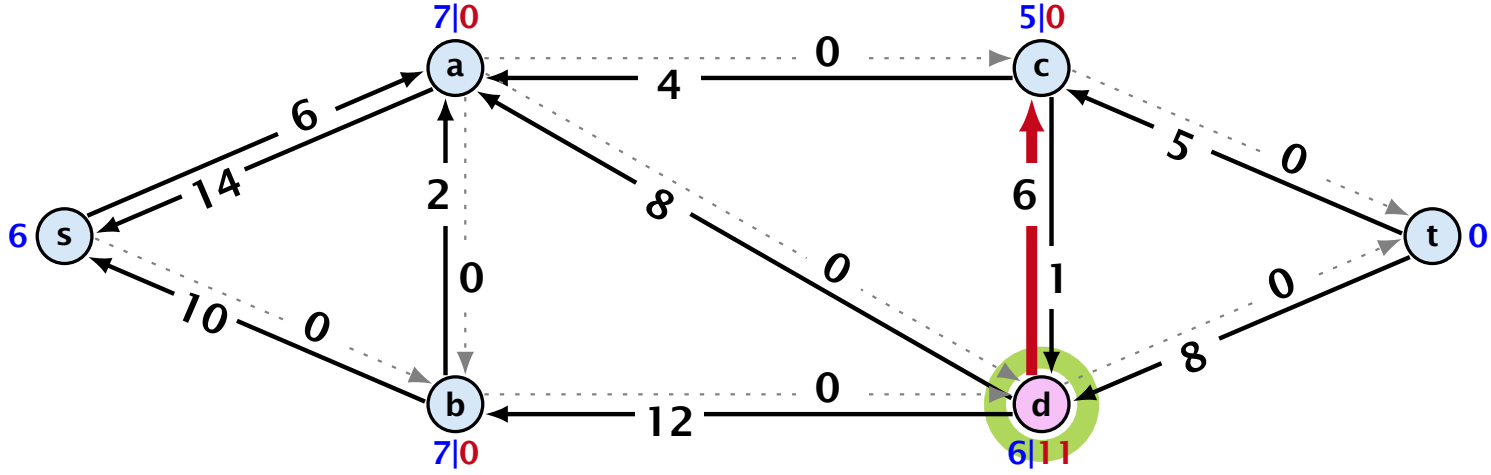


saturating push



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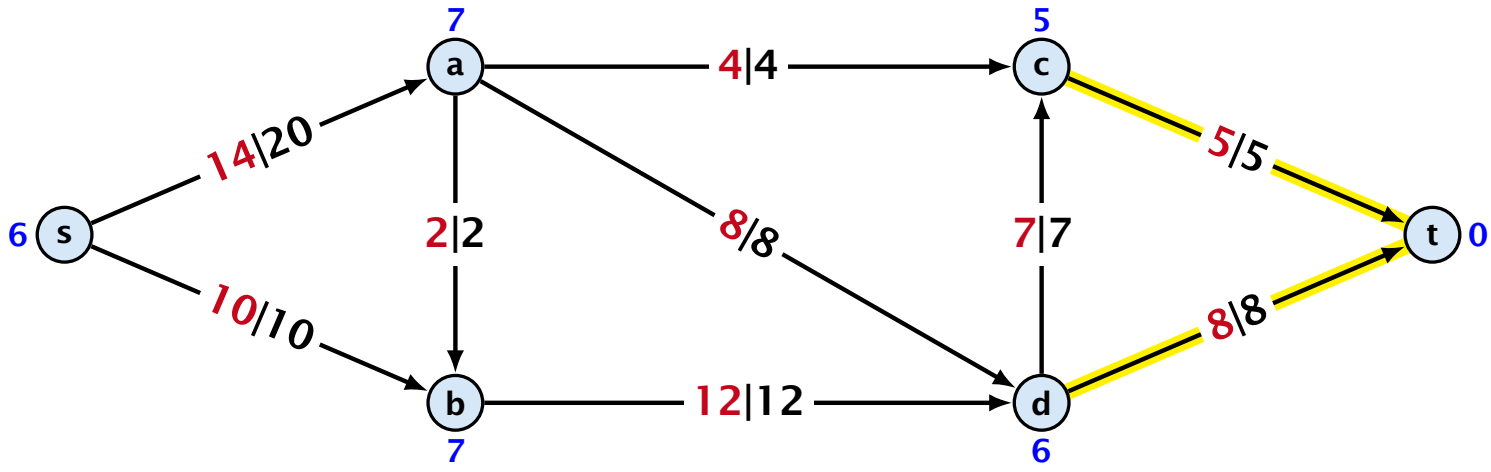

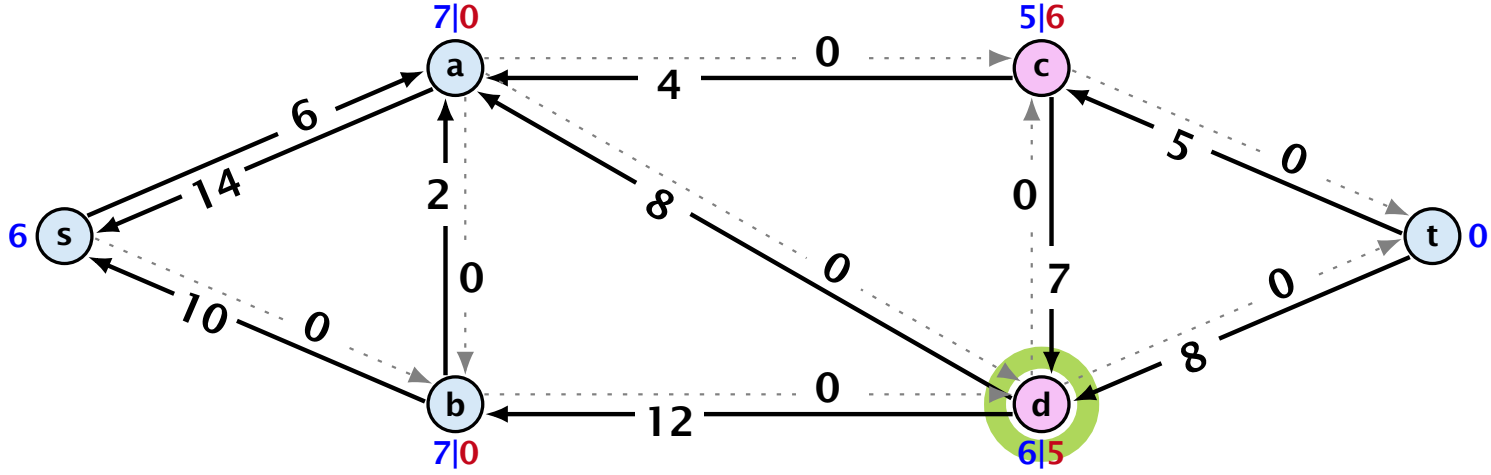

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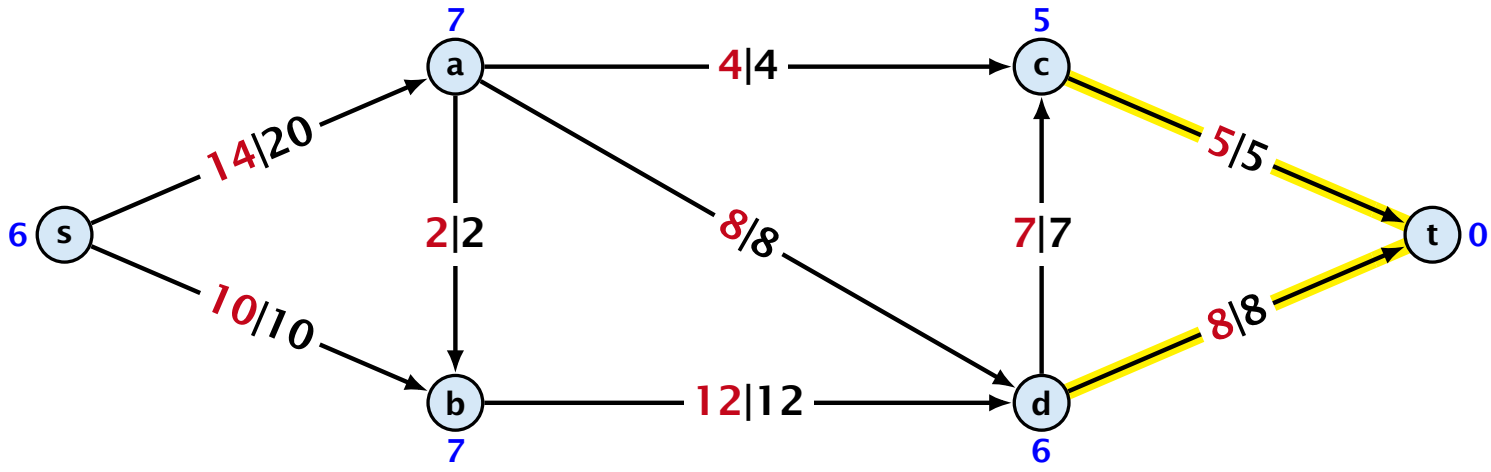
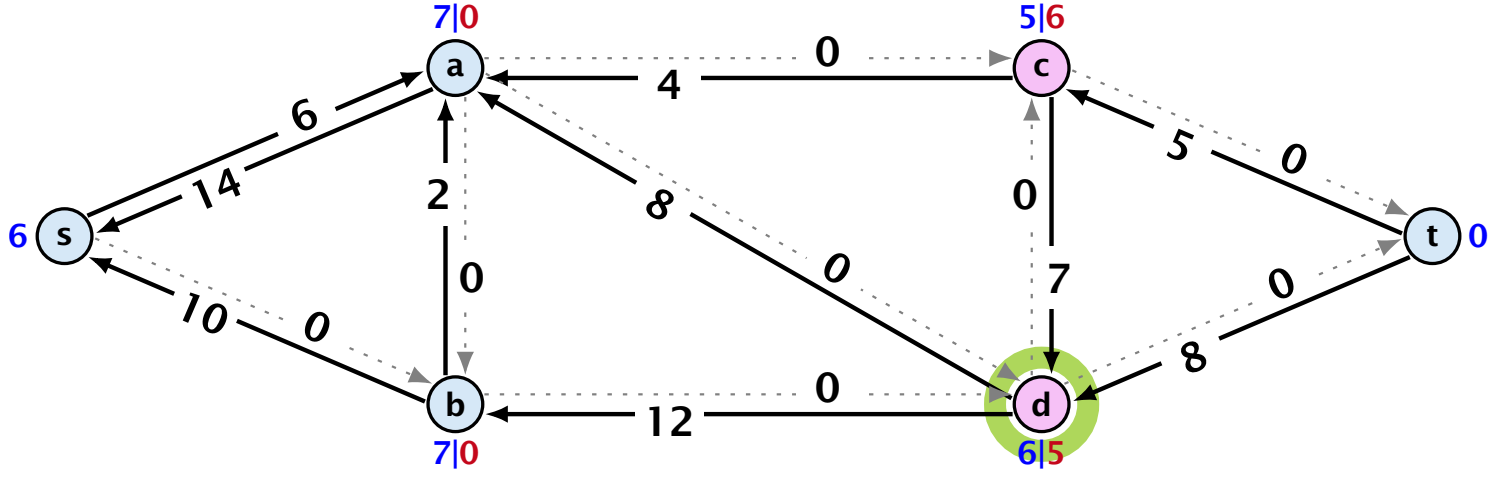


relabel to 8



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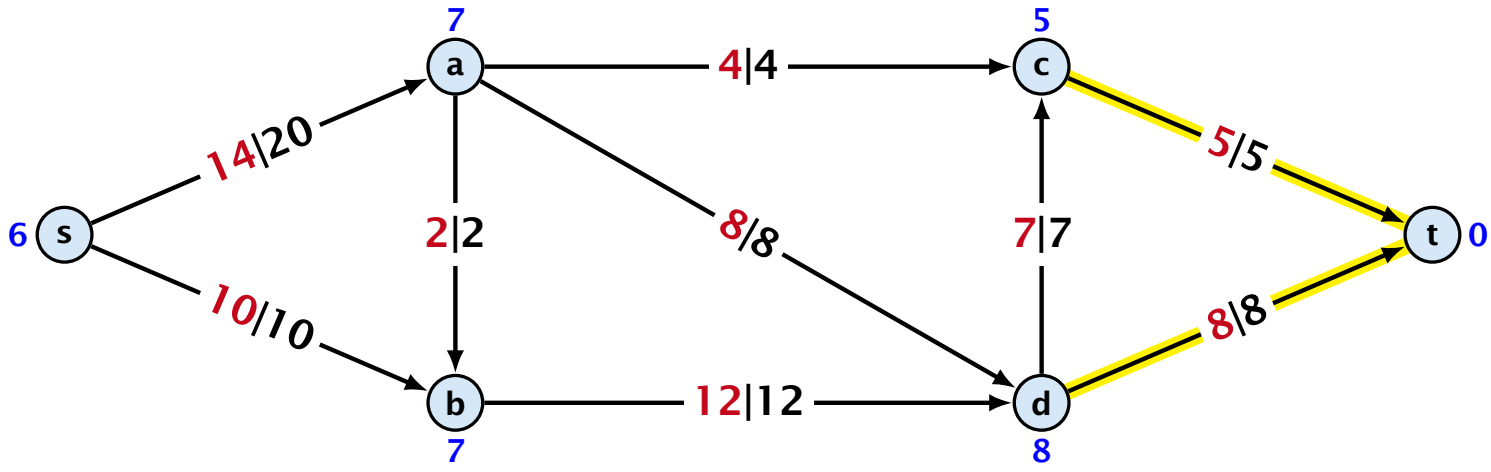

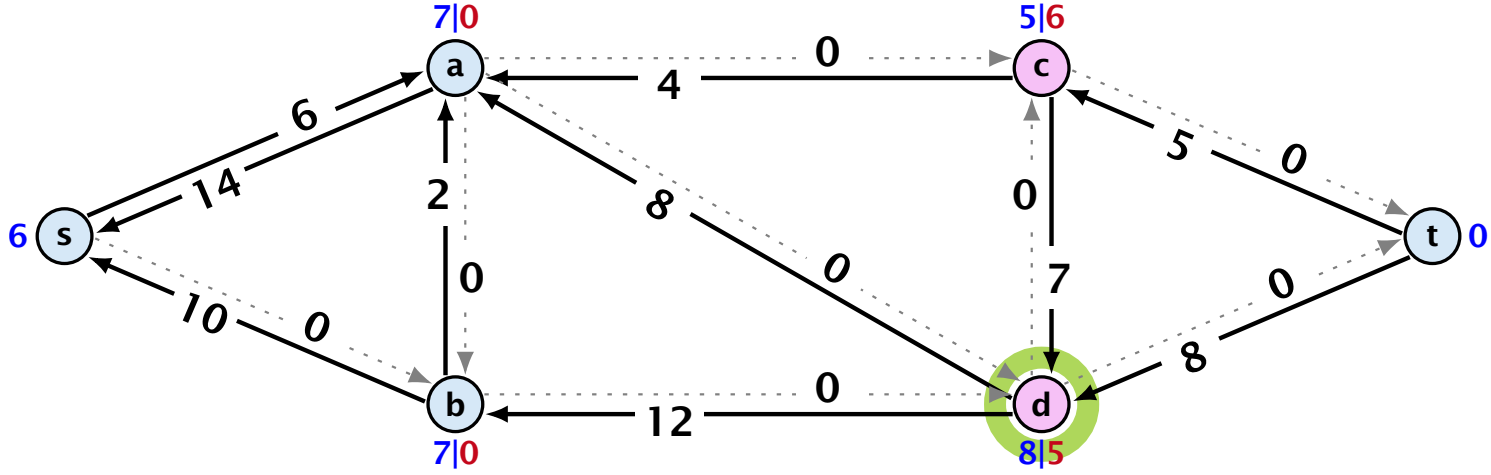

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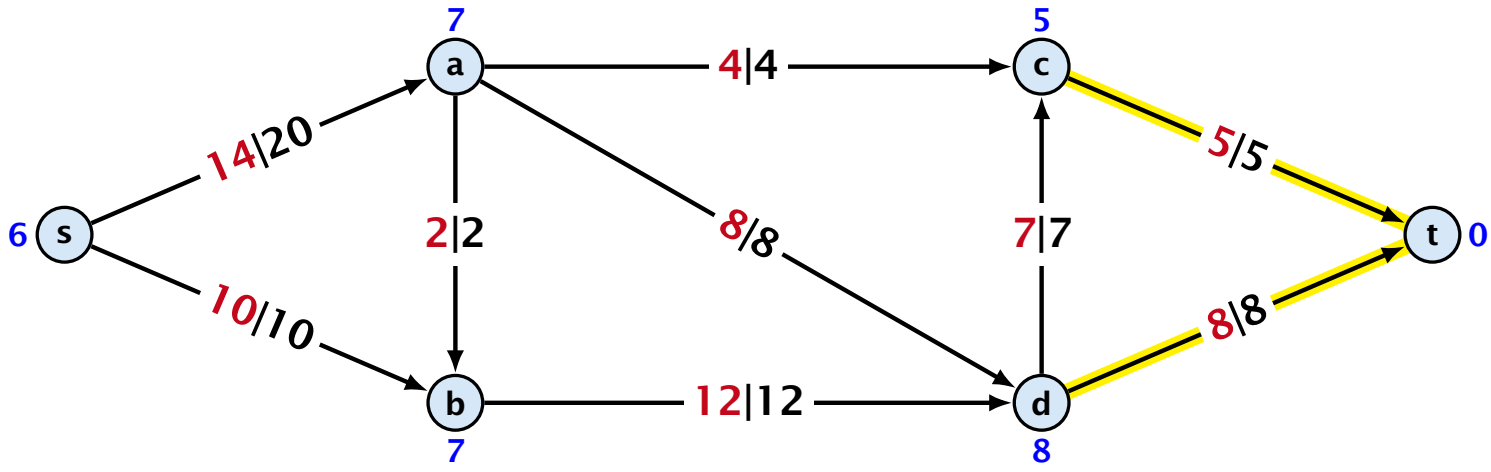
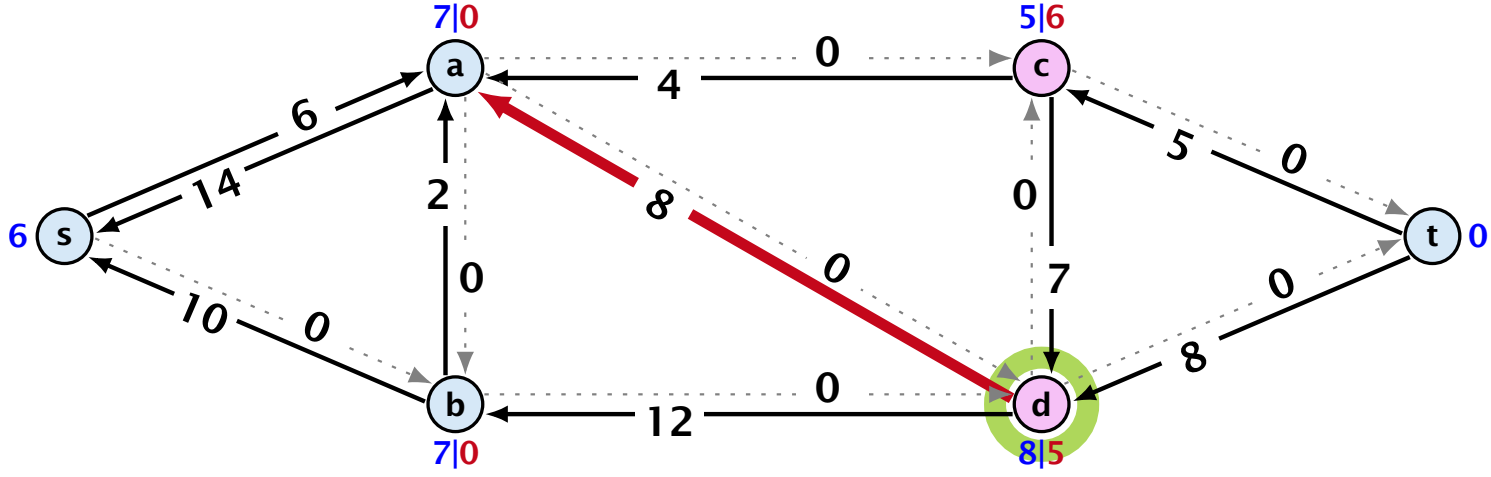


deactivating push



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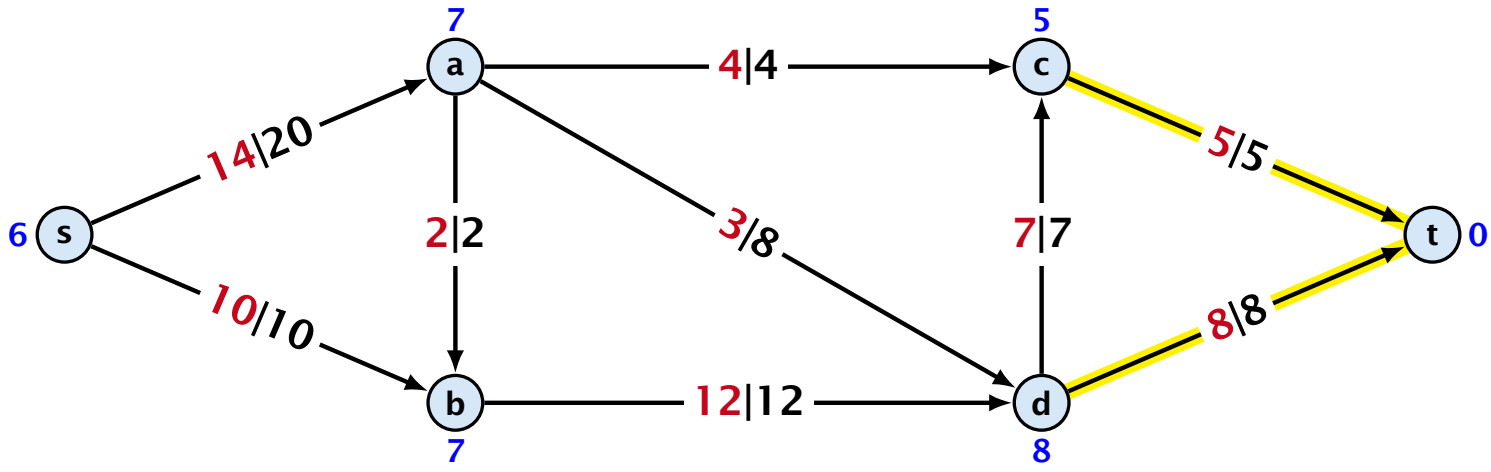

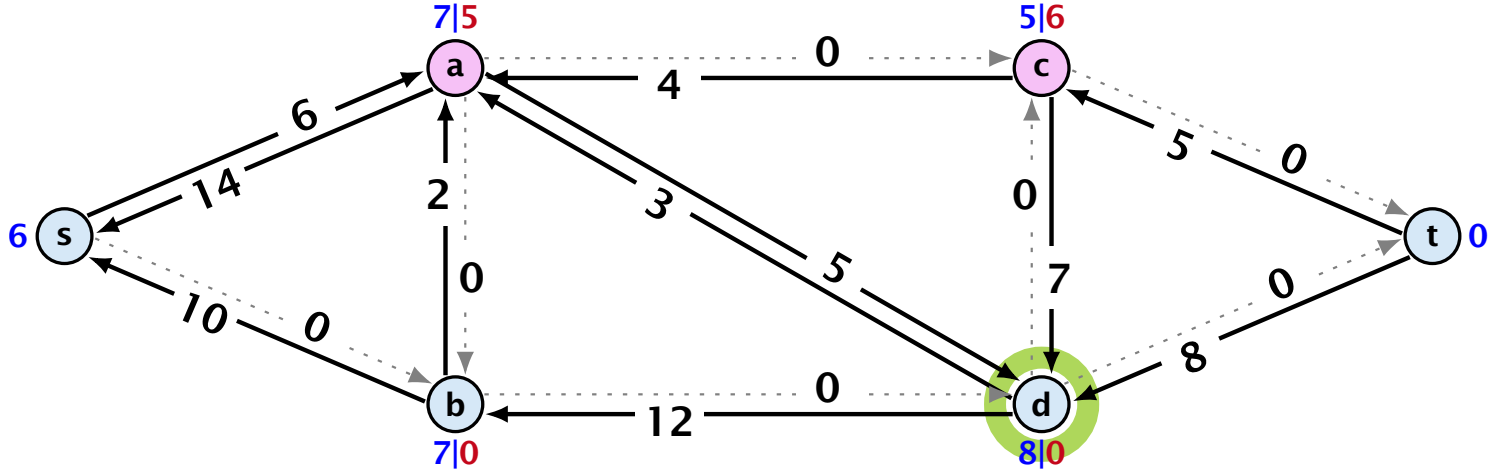

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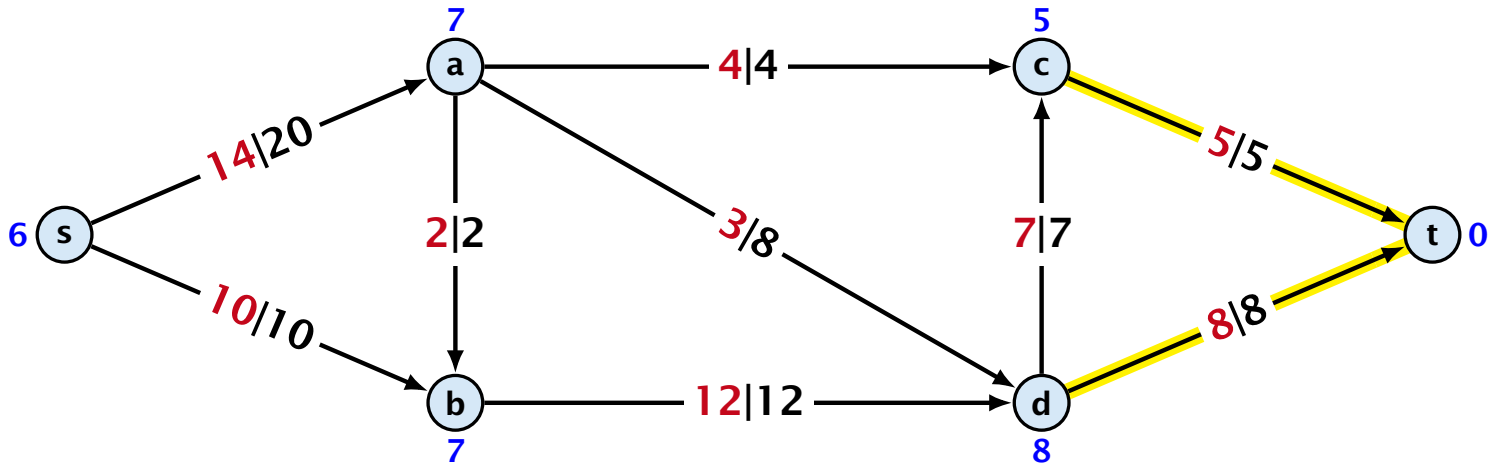

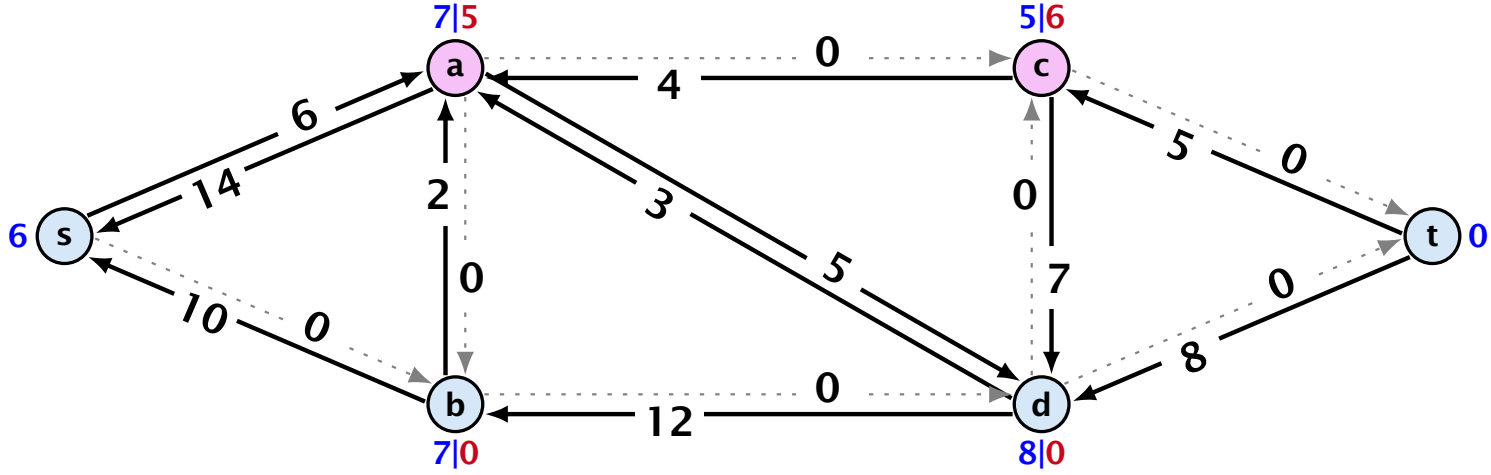

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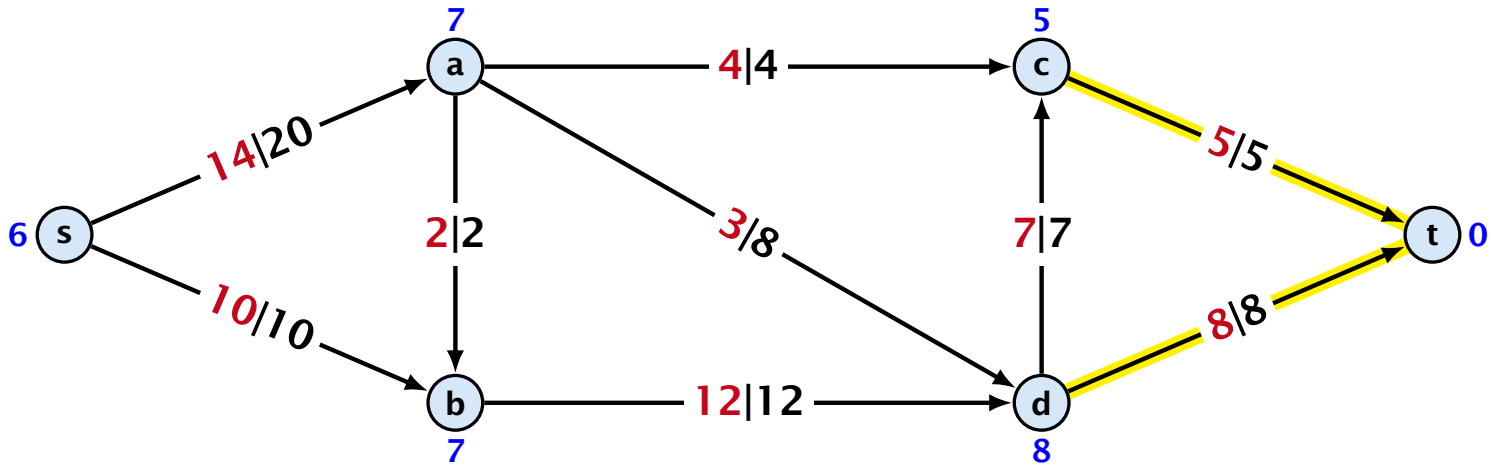

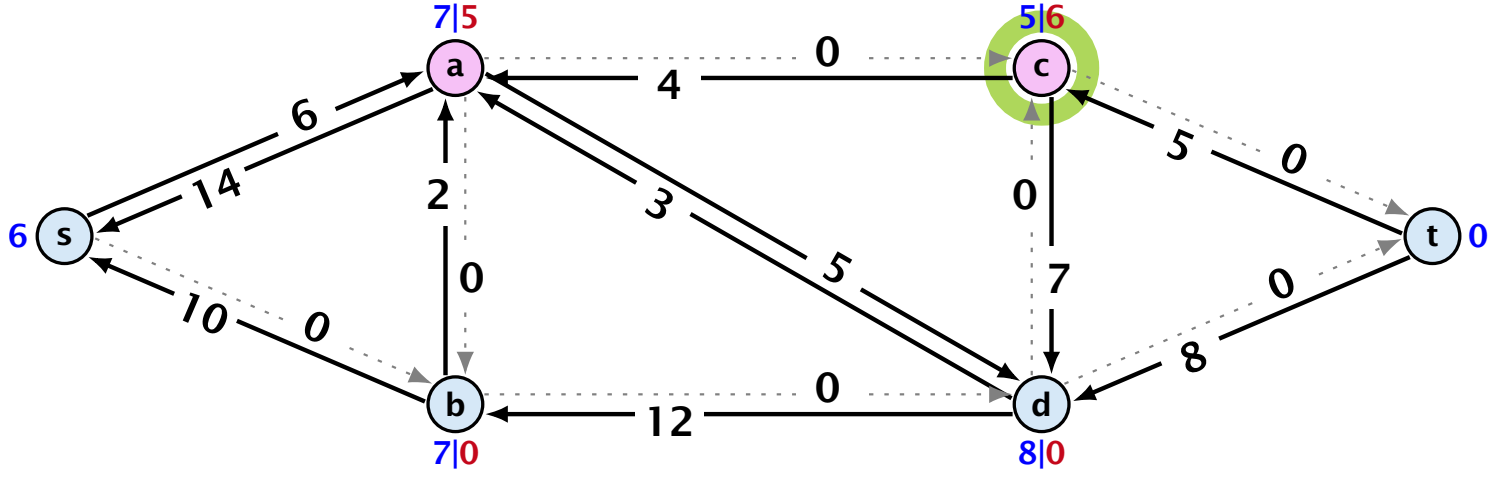

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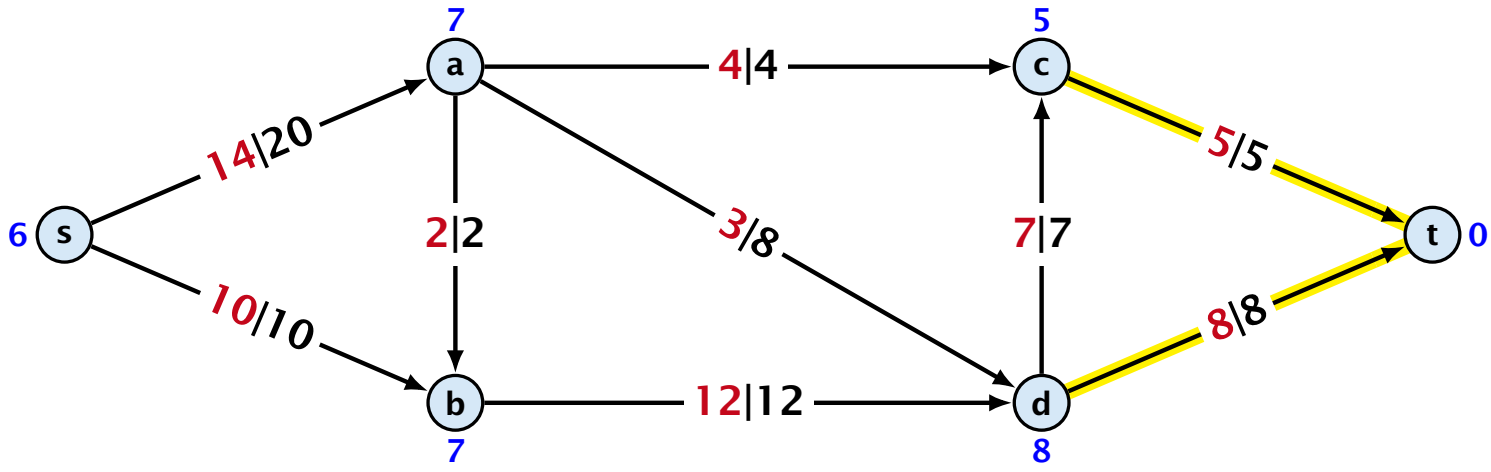
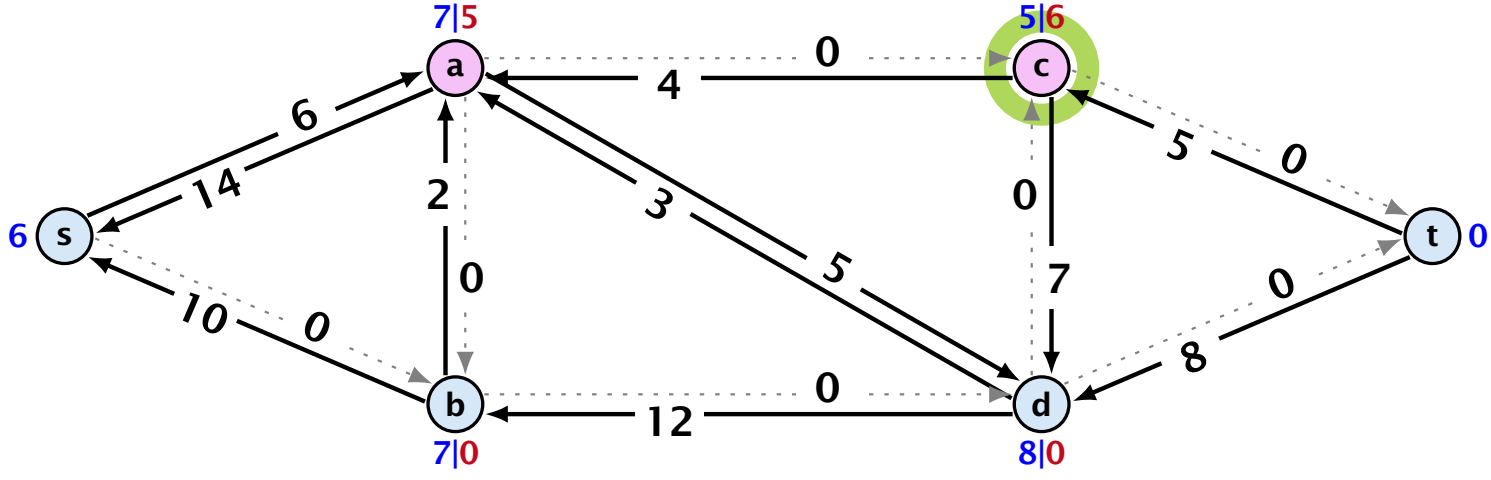


relabel to 8



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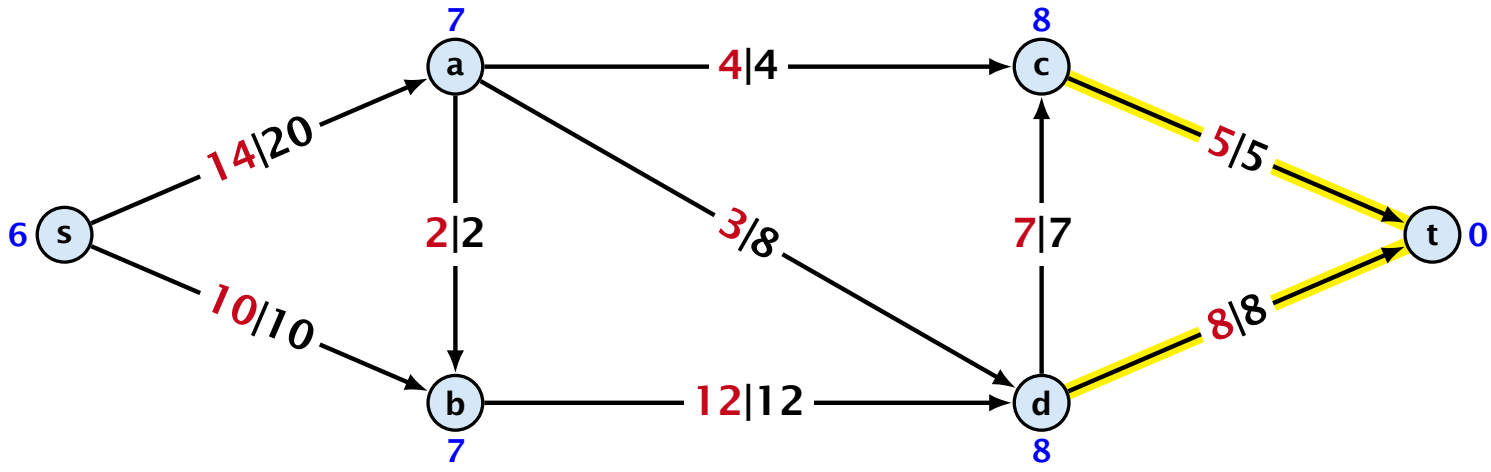

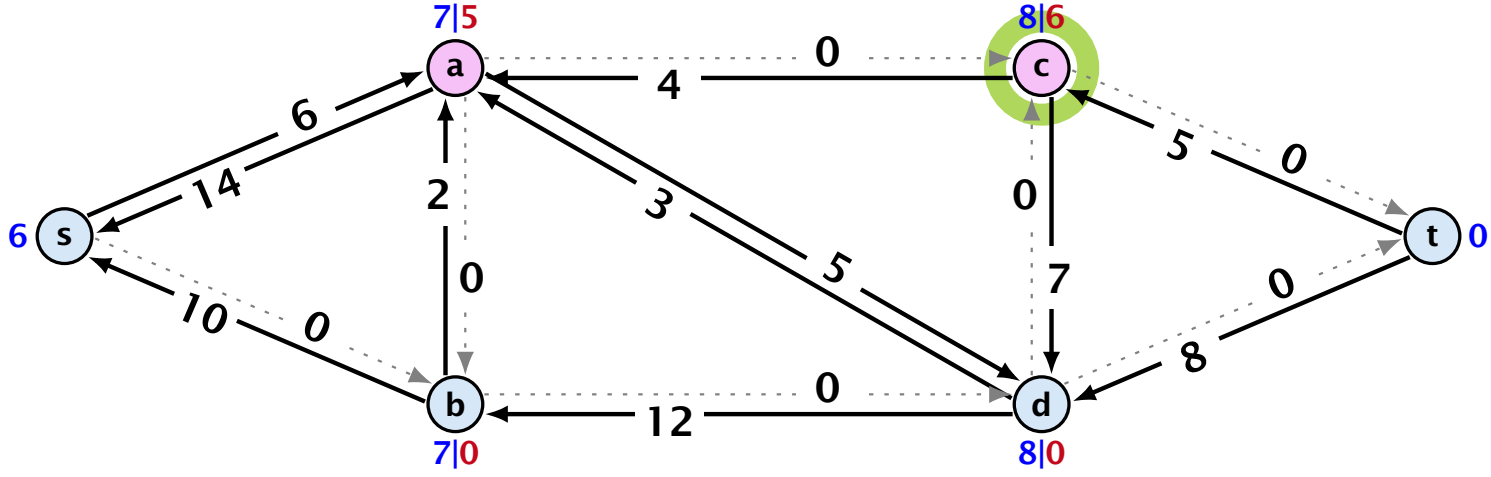

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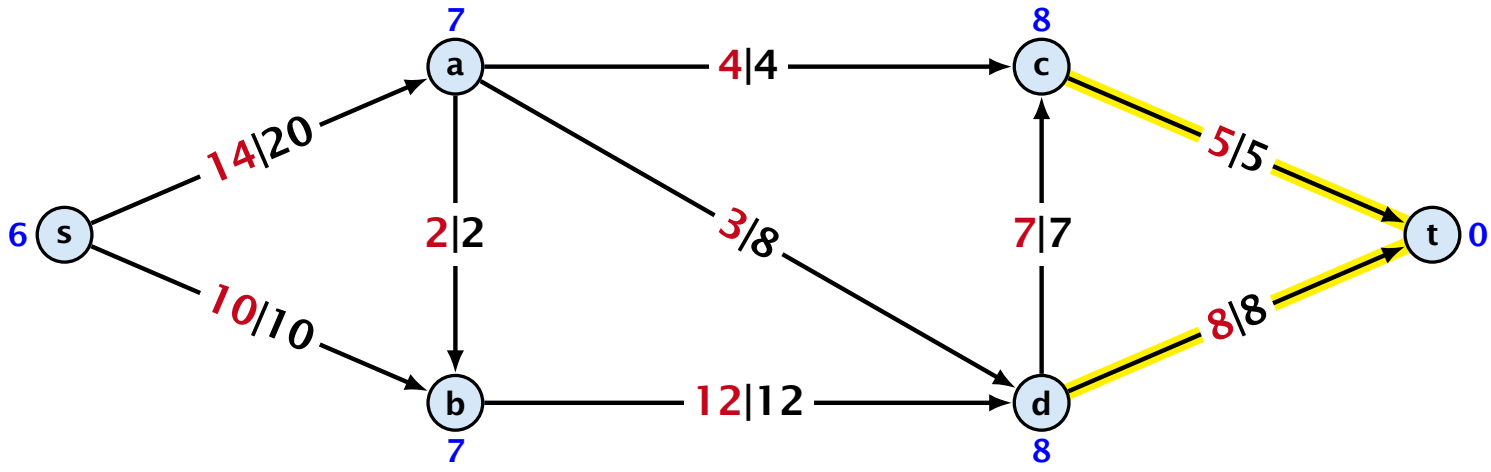
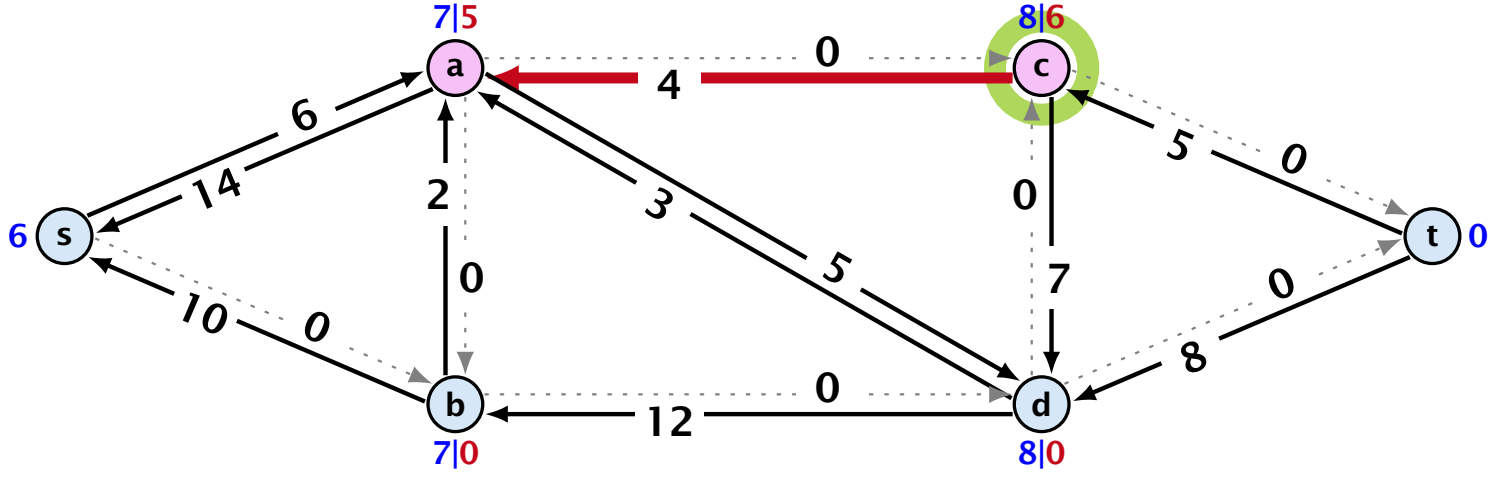


saturating push



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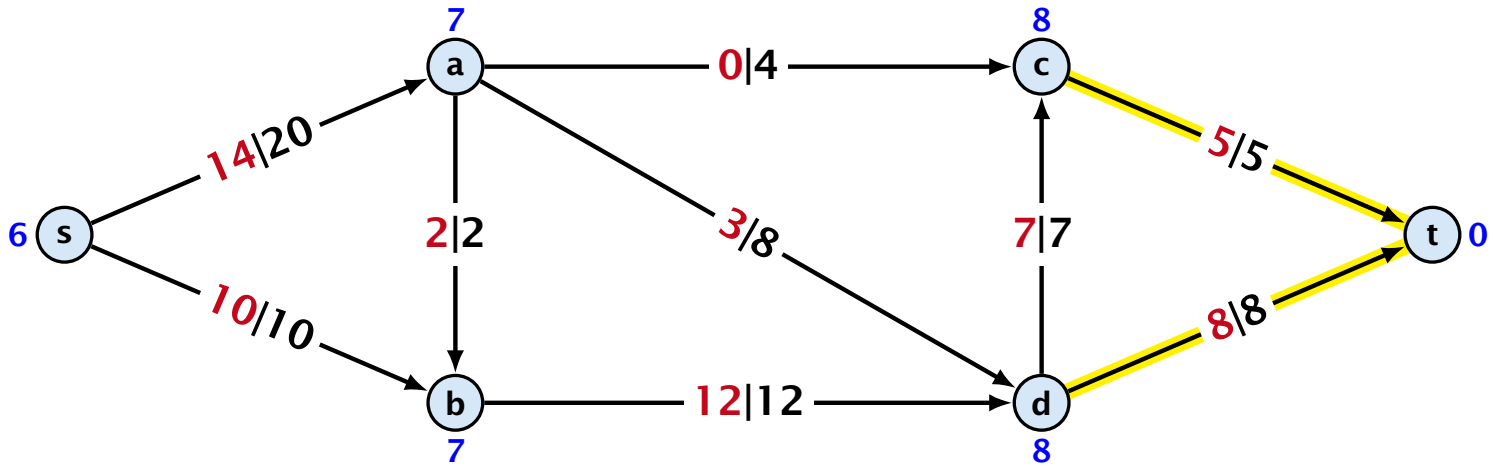

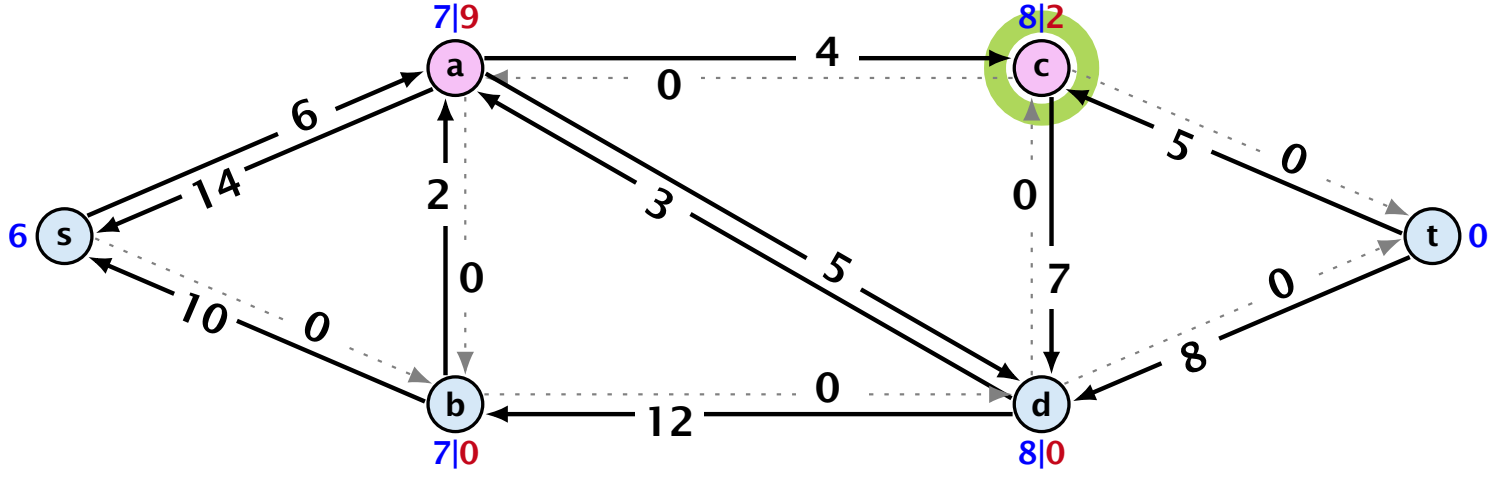

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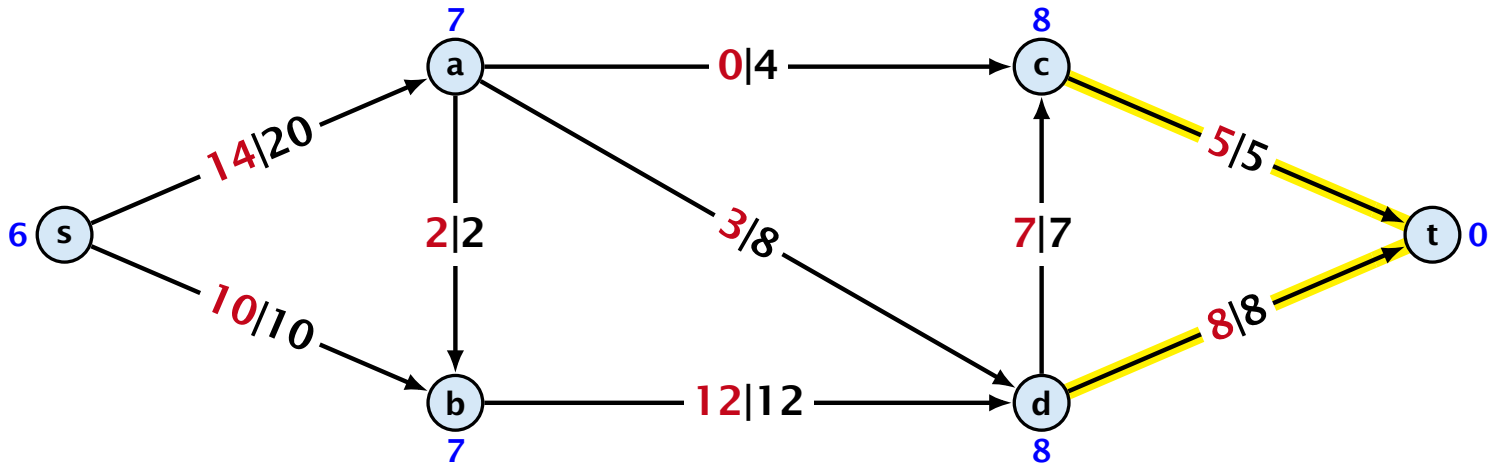
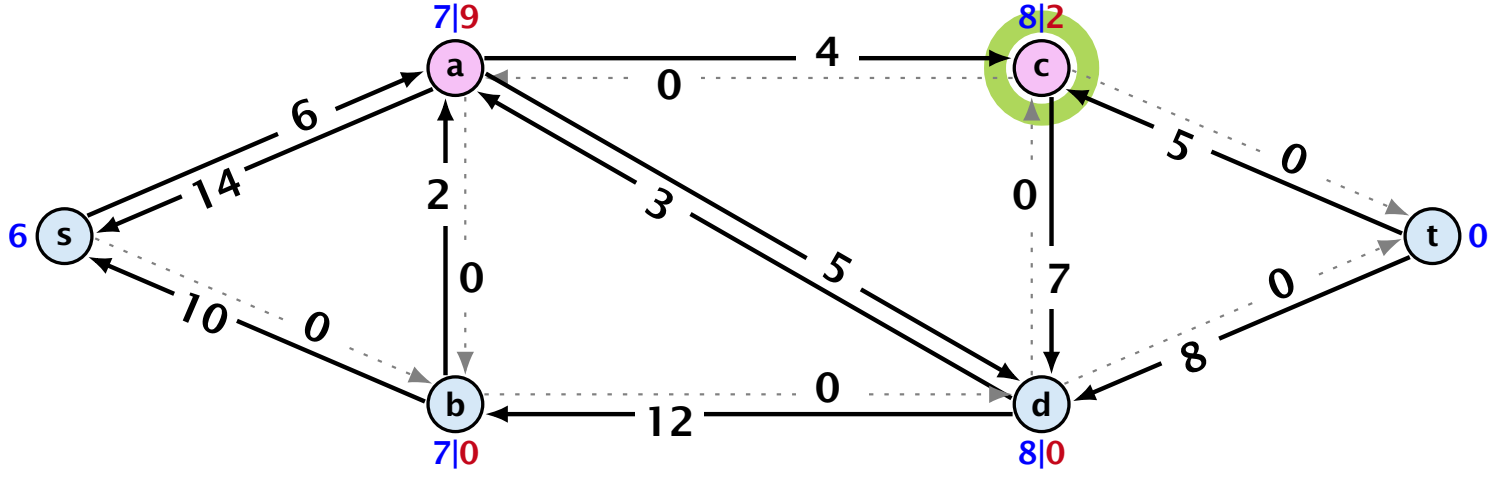


relabel to 9



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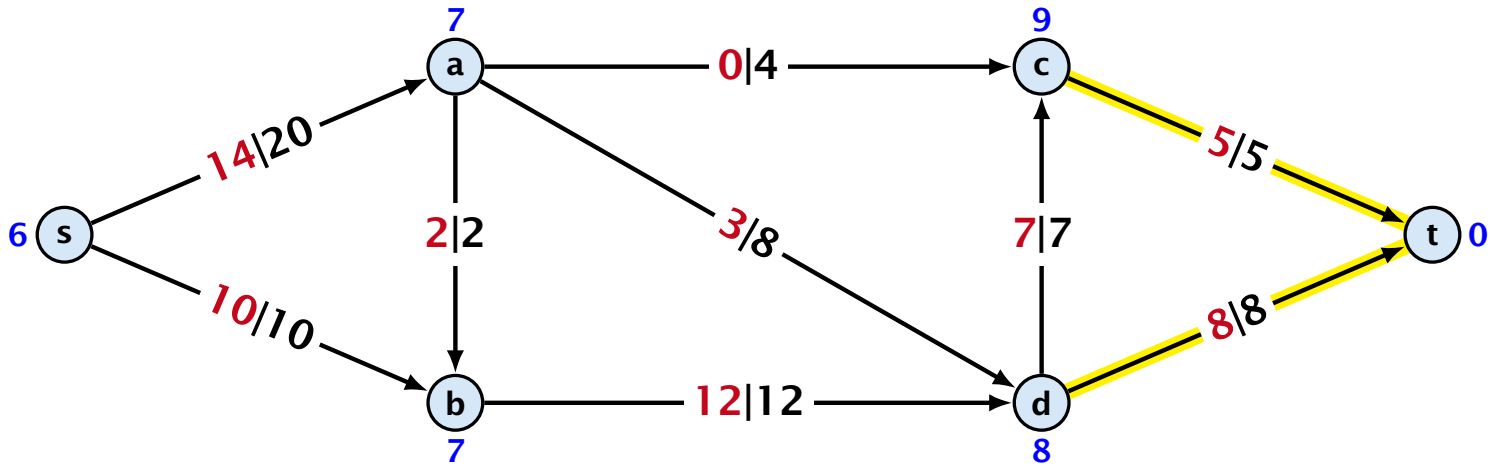

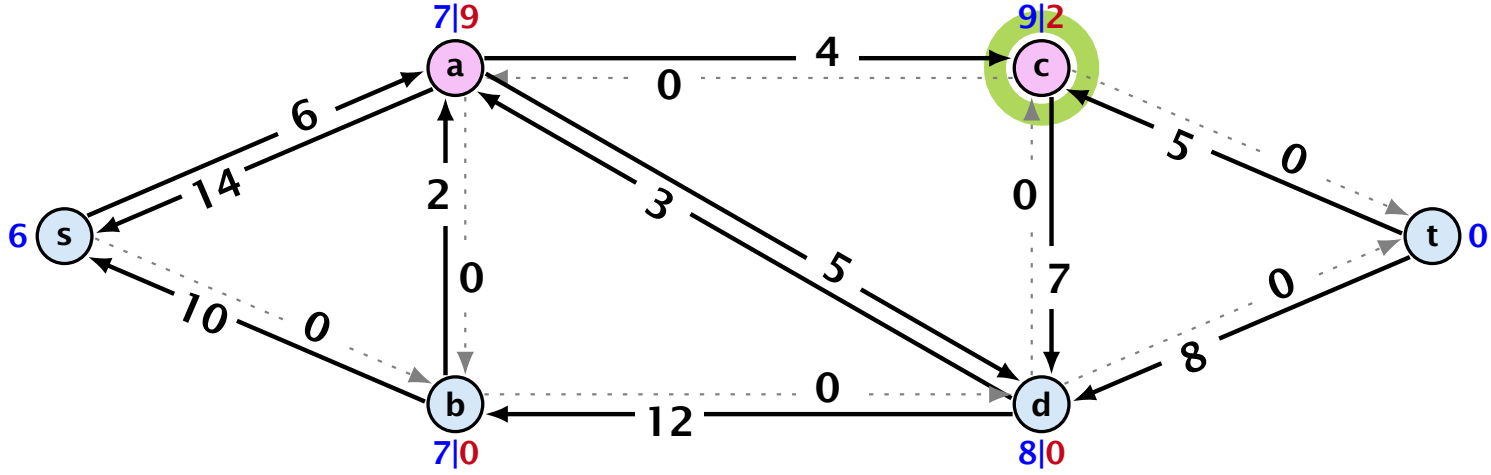

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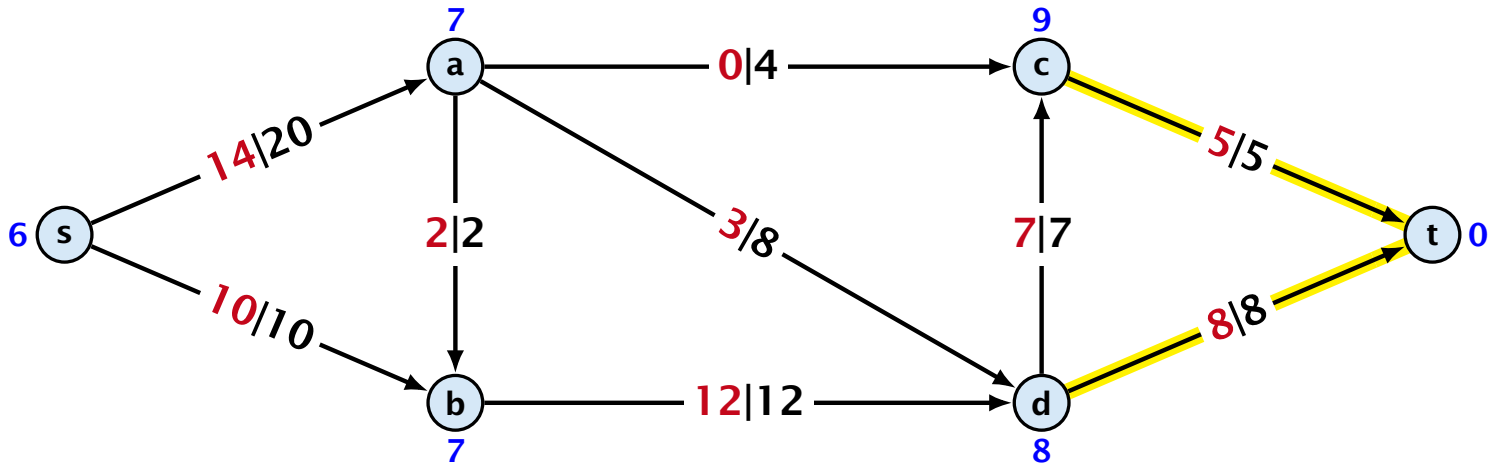
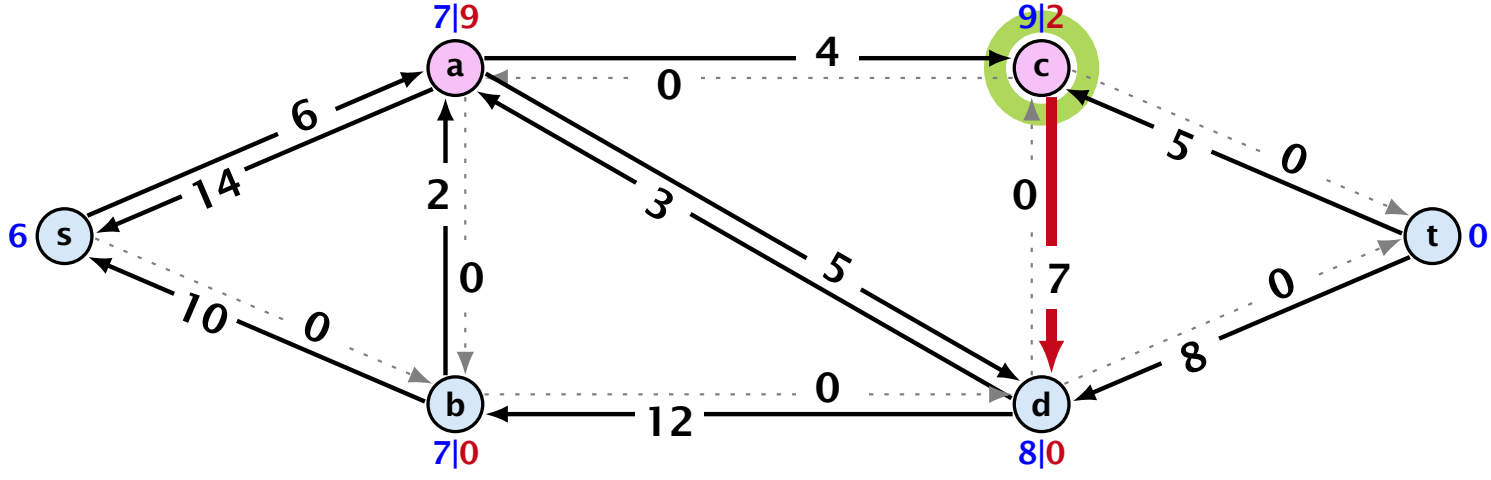


deactivating push



Preflow Push

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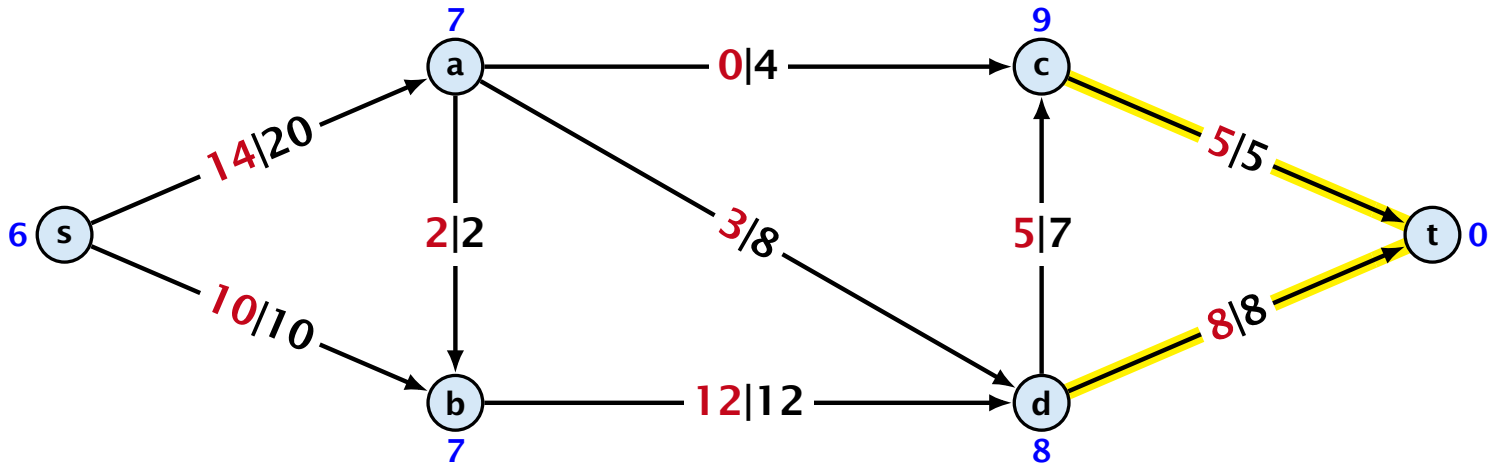

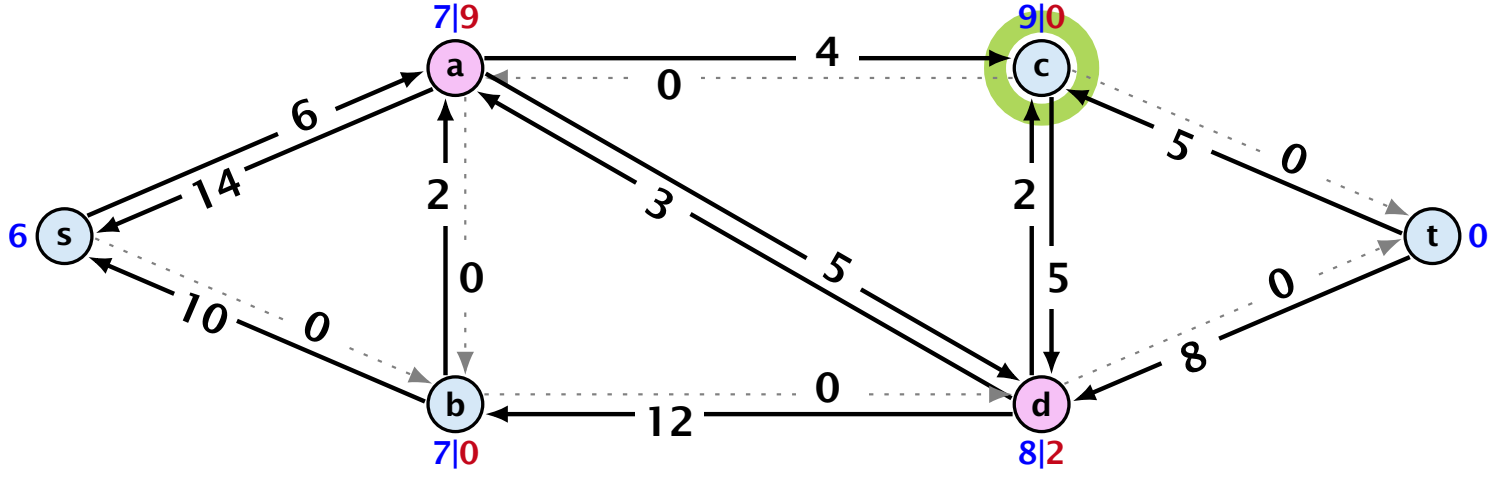

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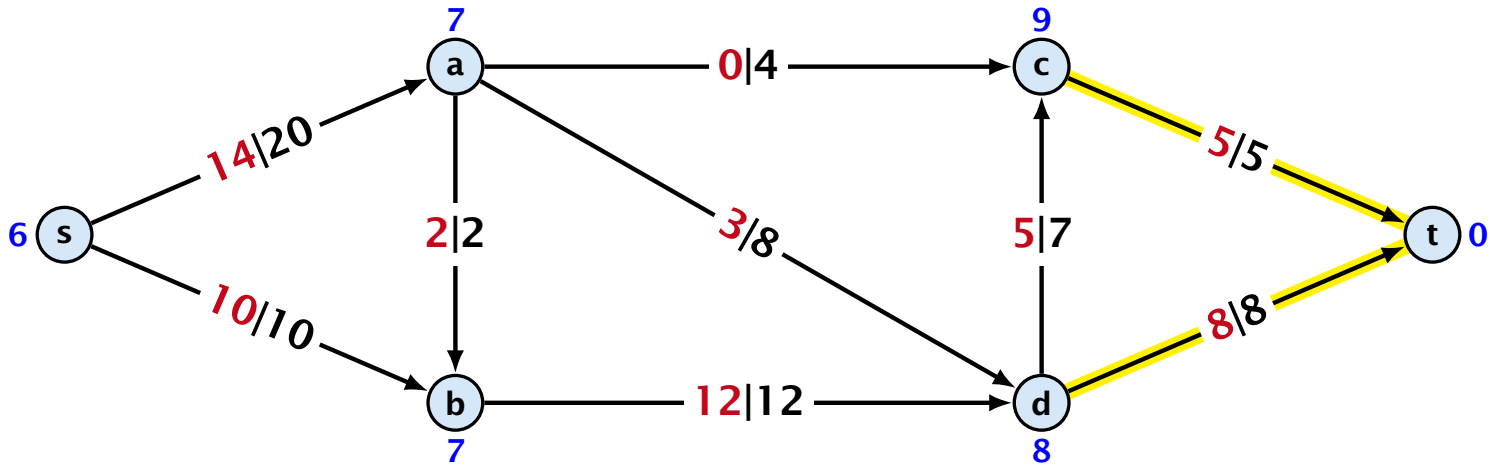

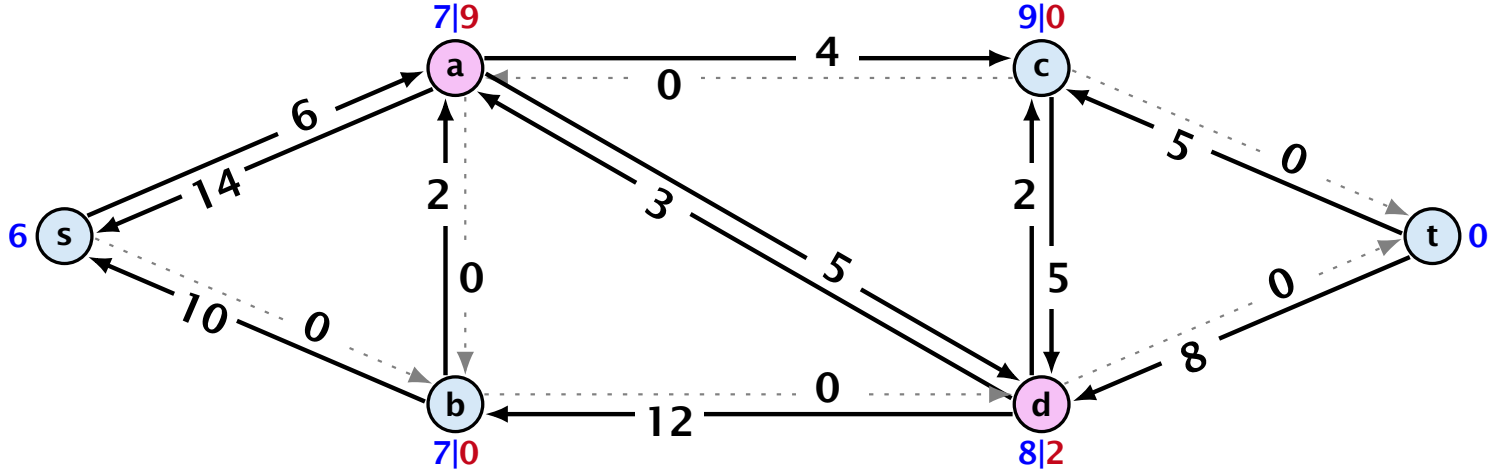

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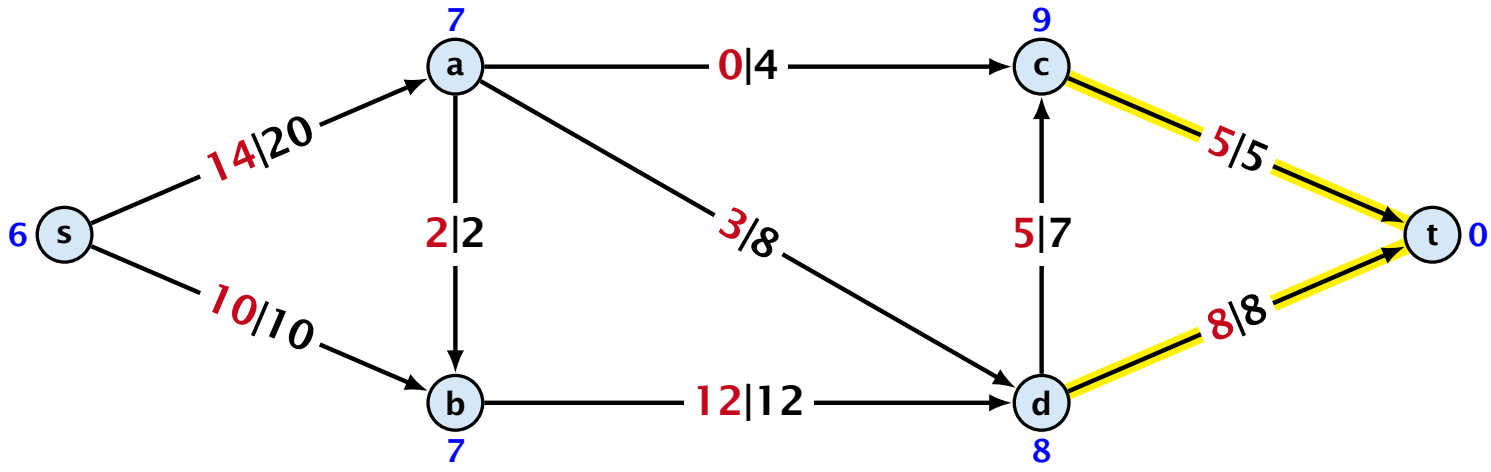
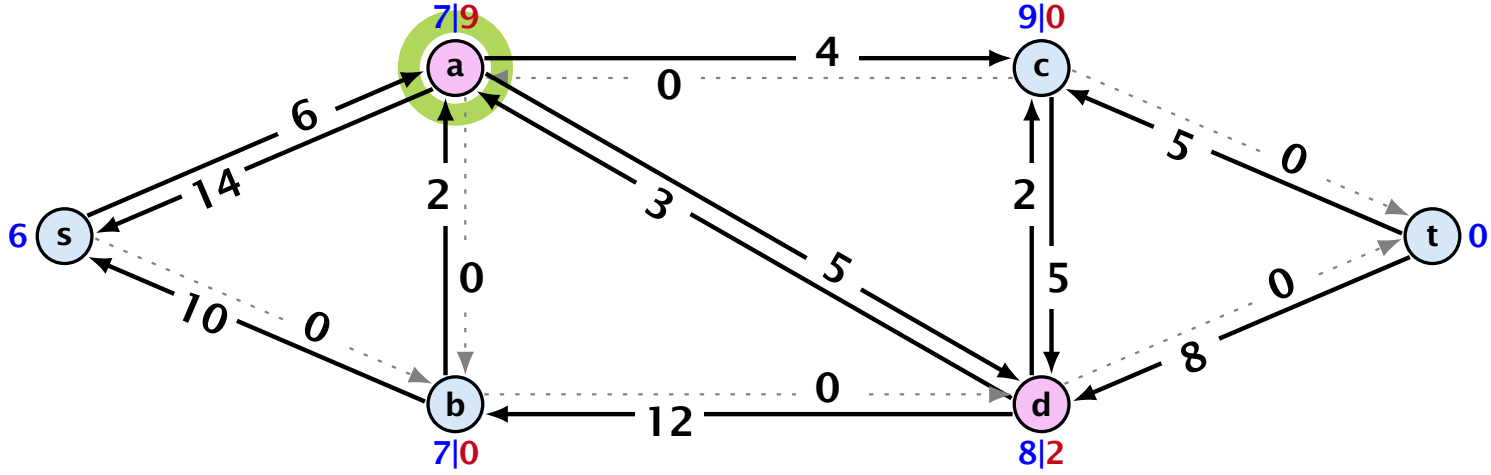
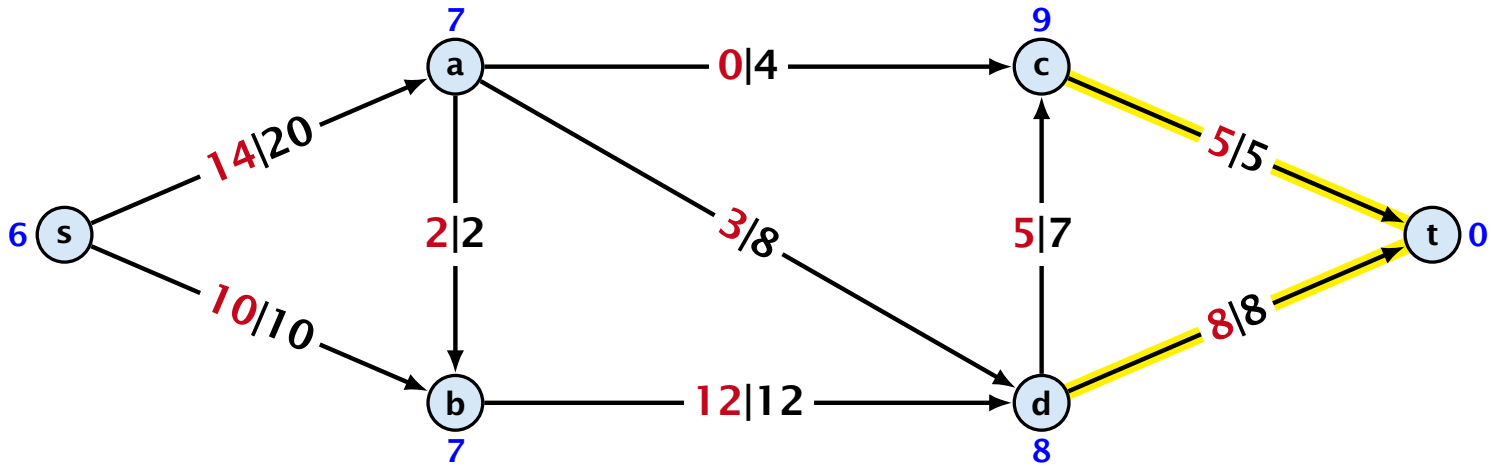


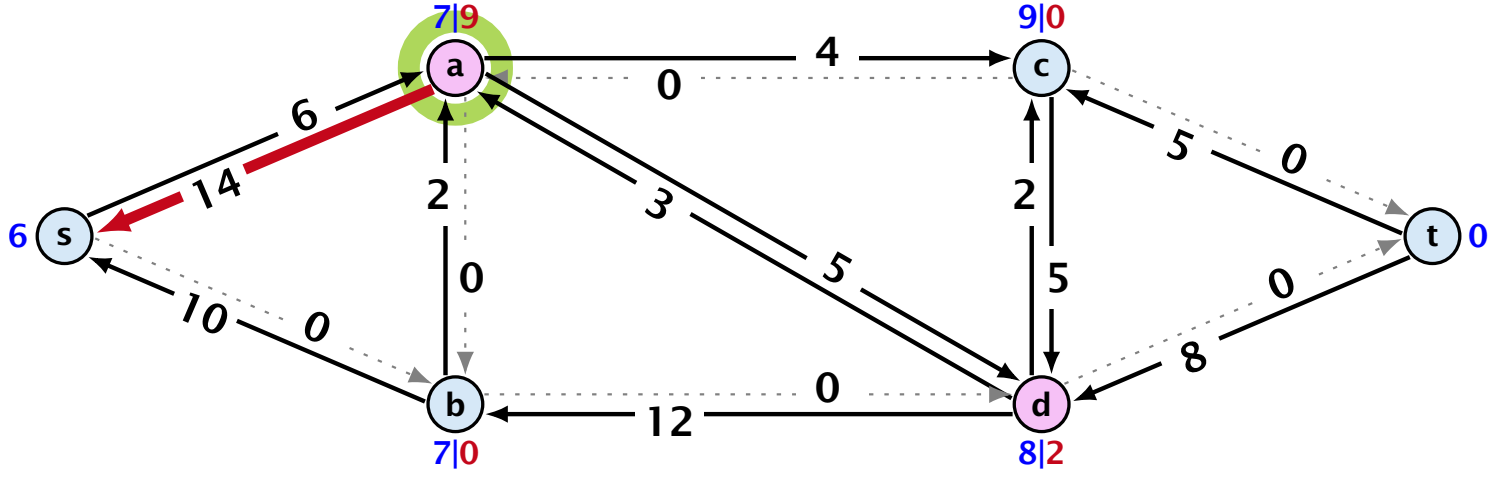


Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.



deactivating push



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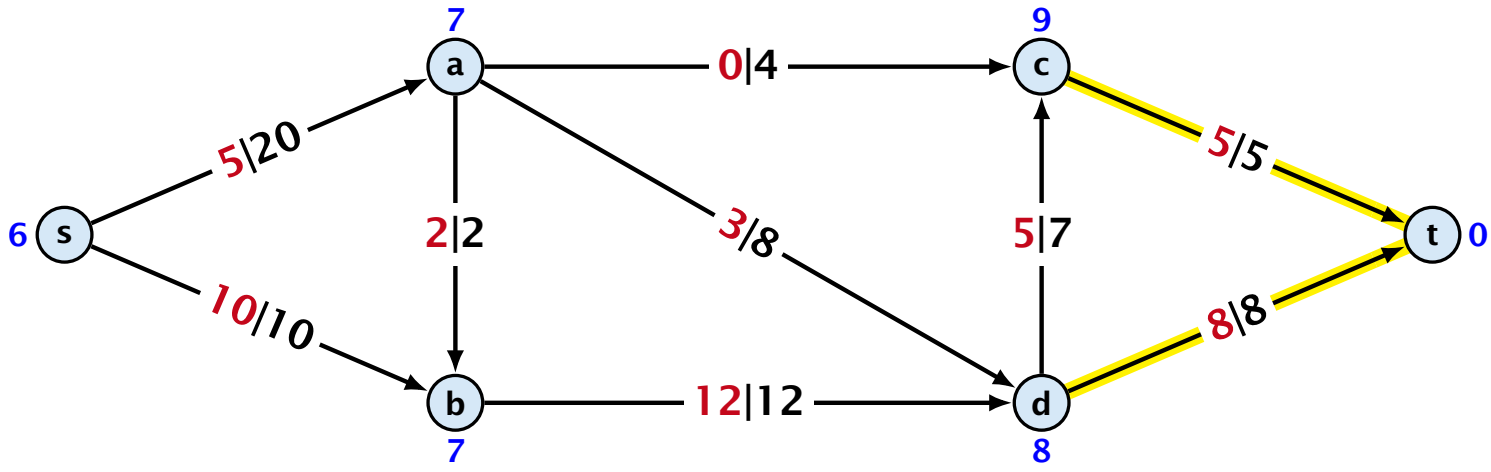

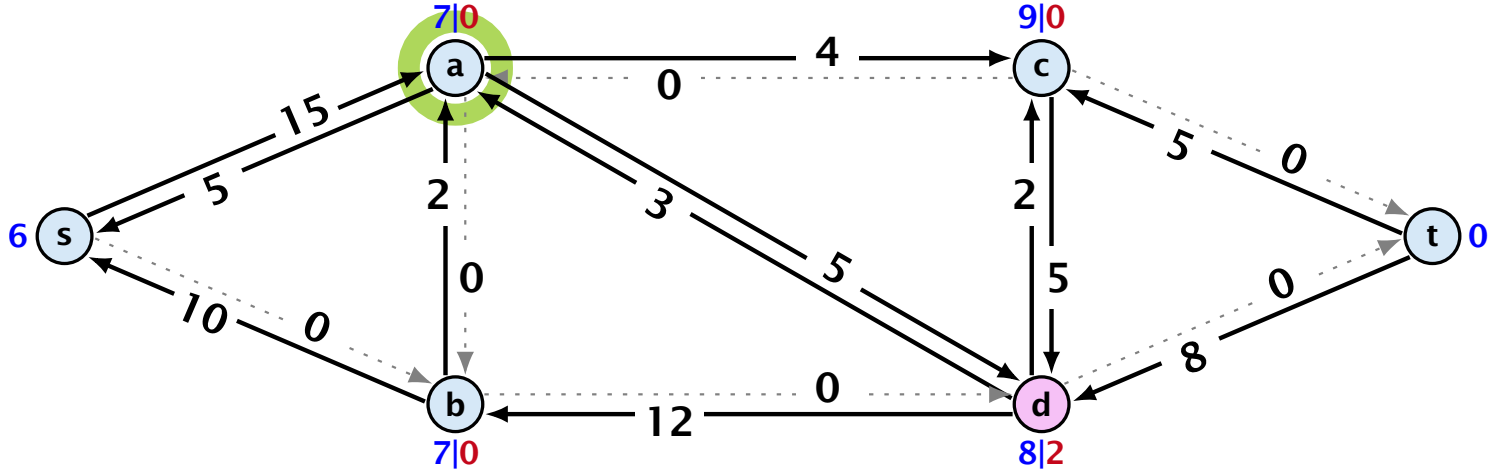

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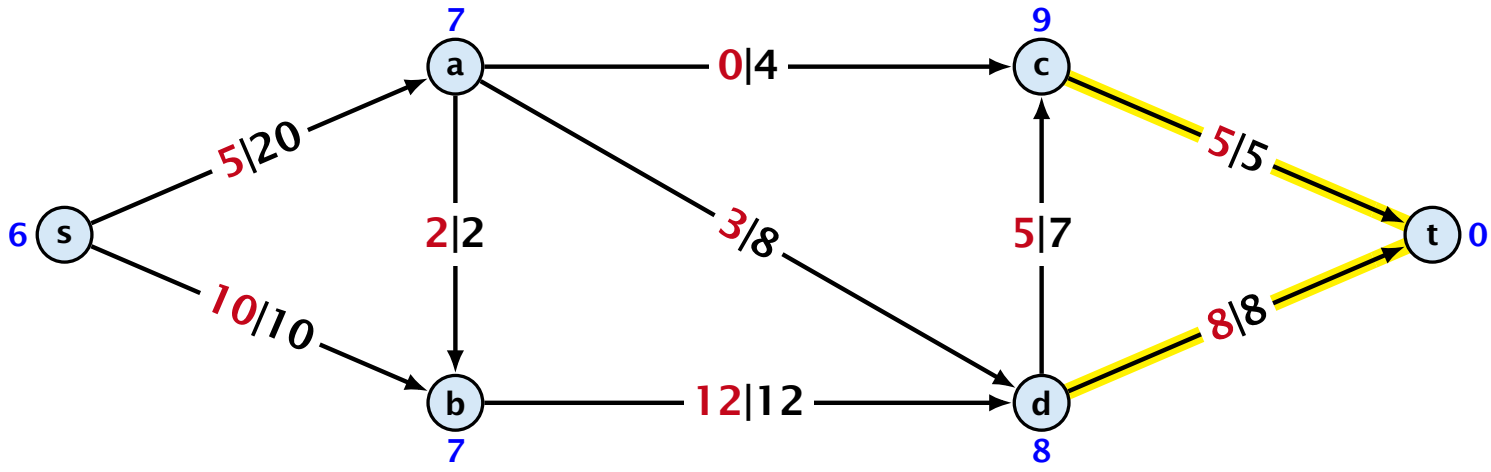

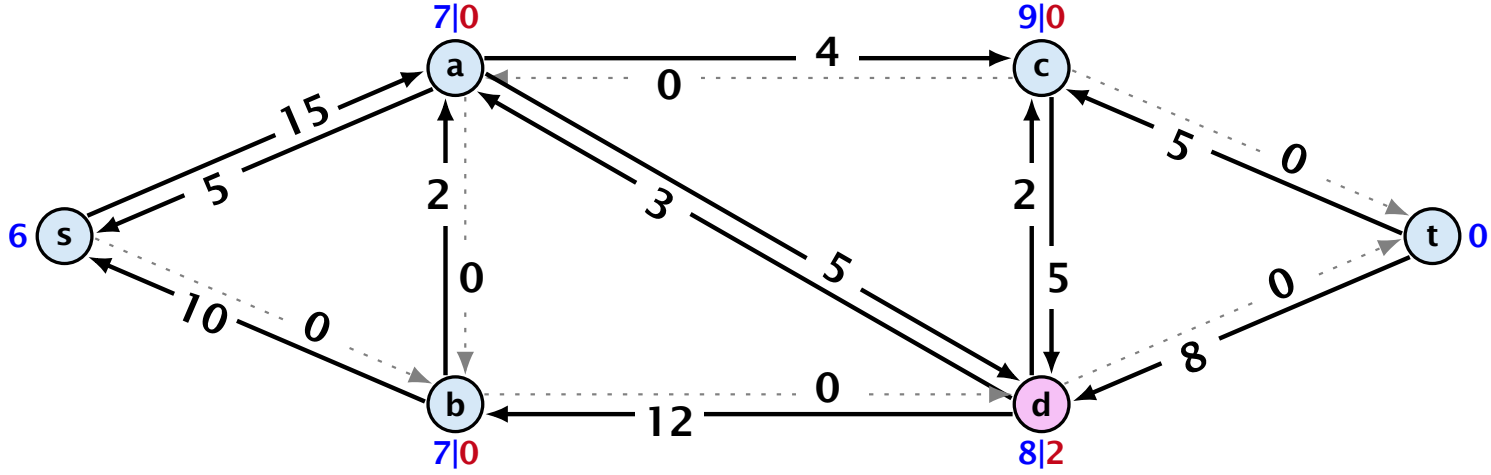

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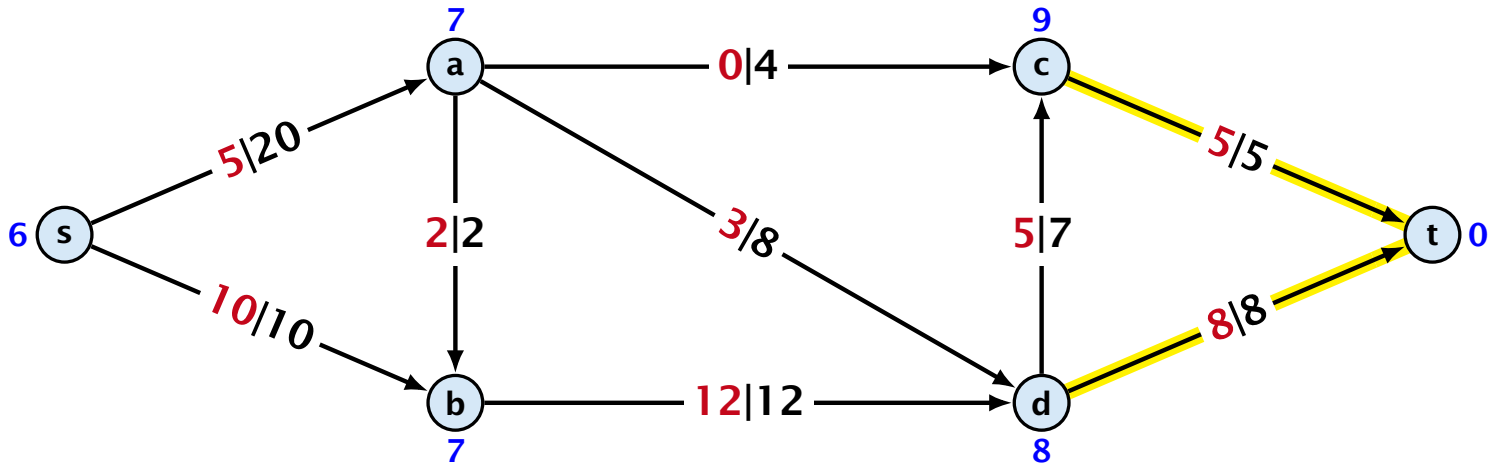

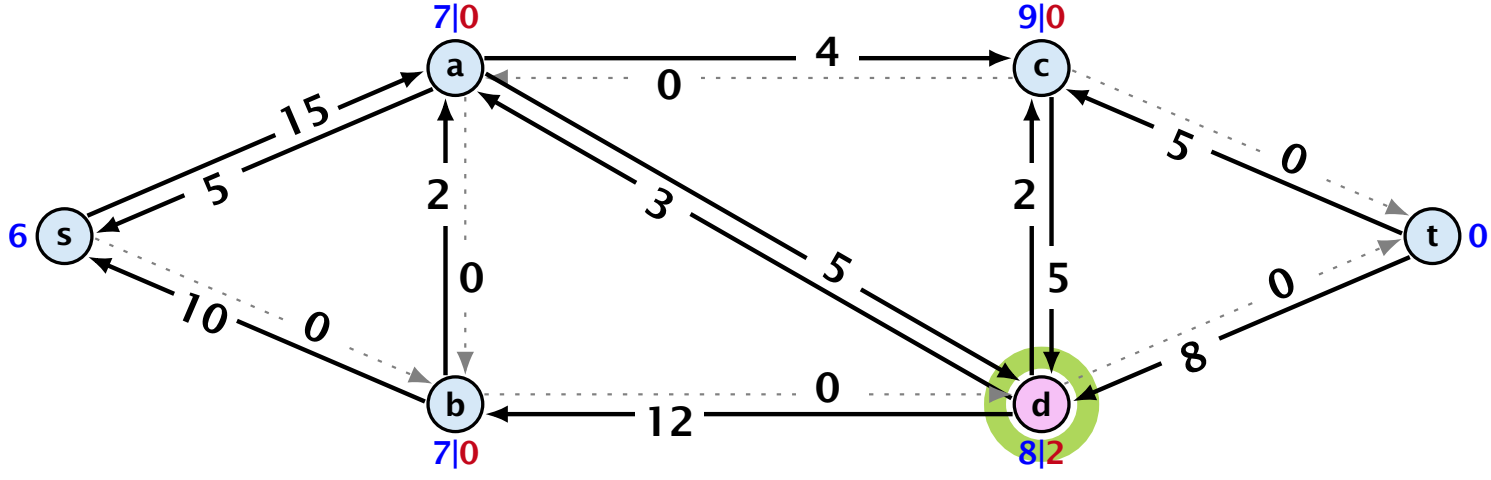

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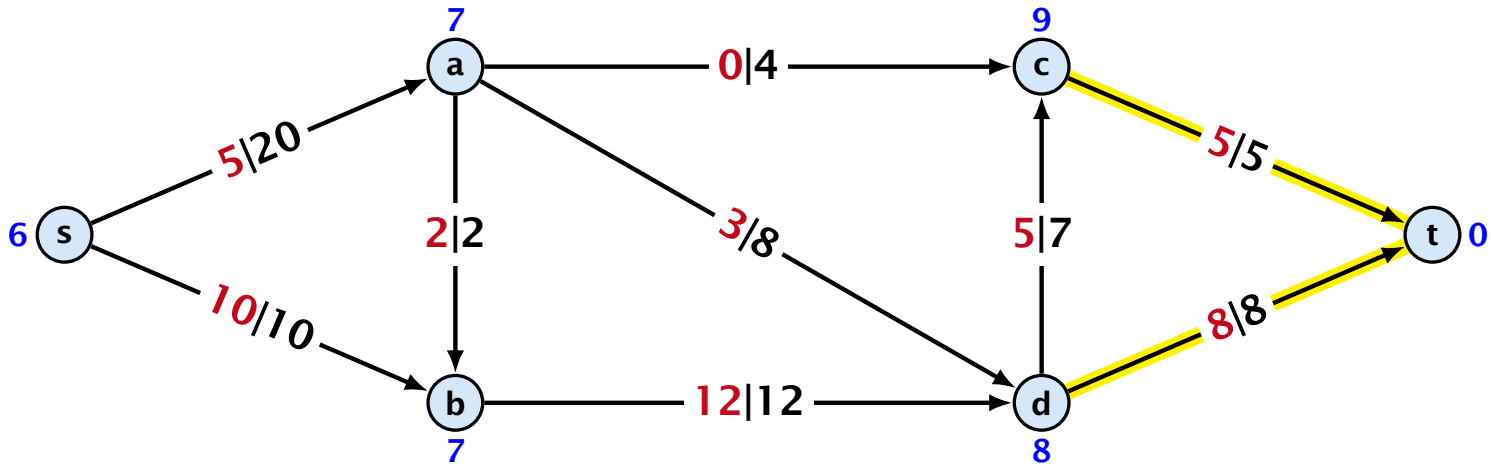

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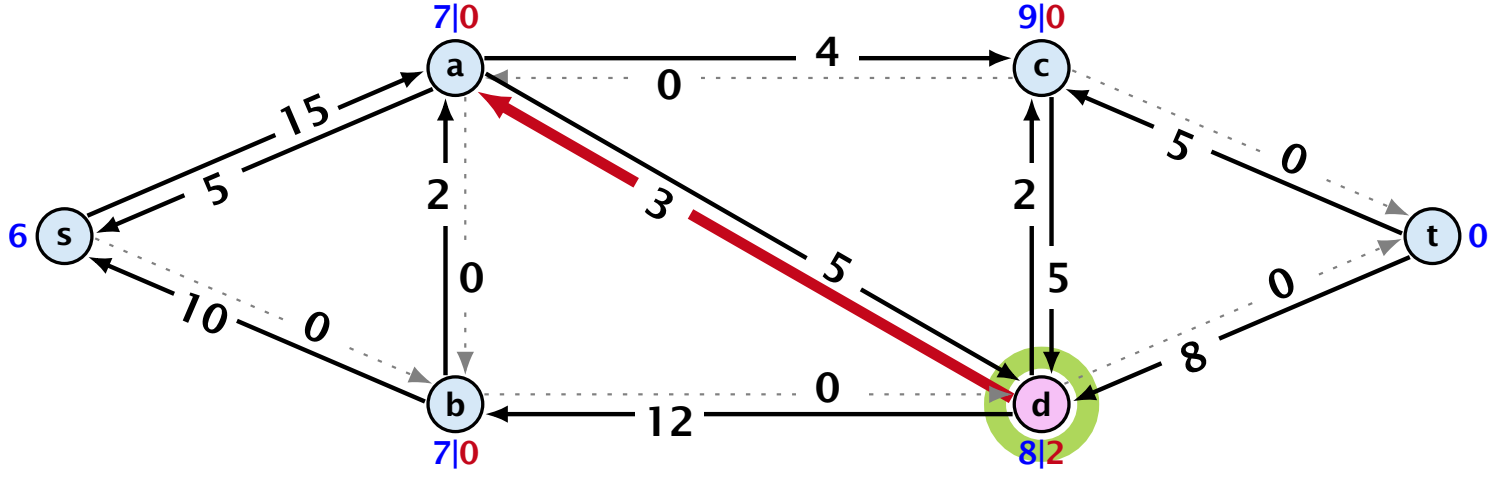

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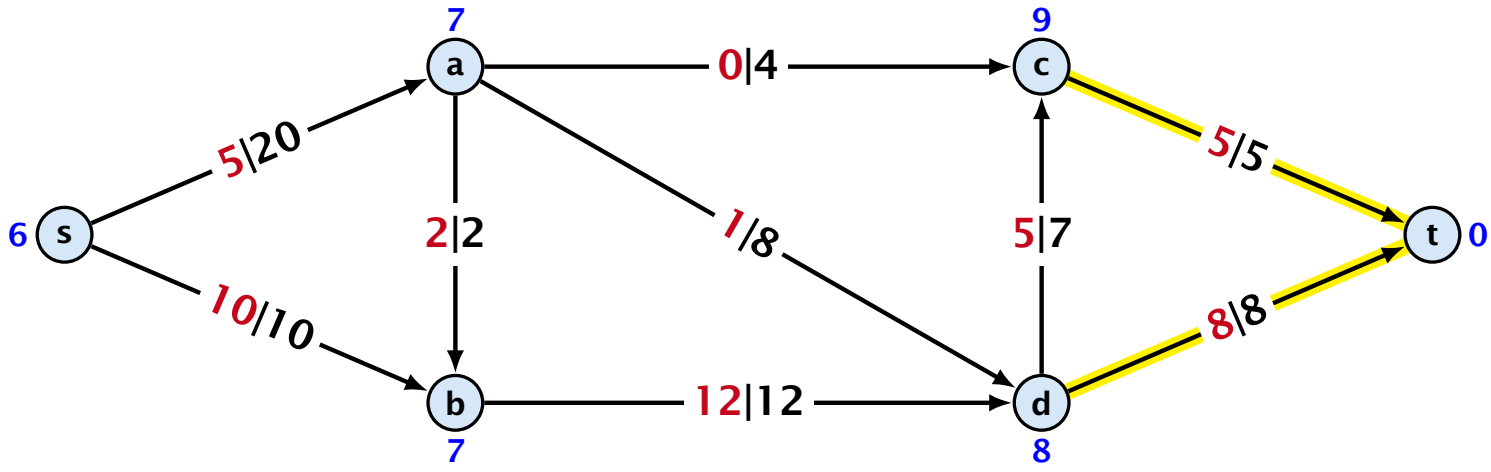

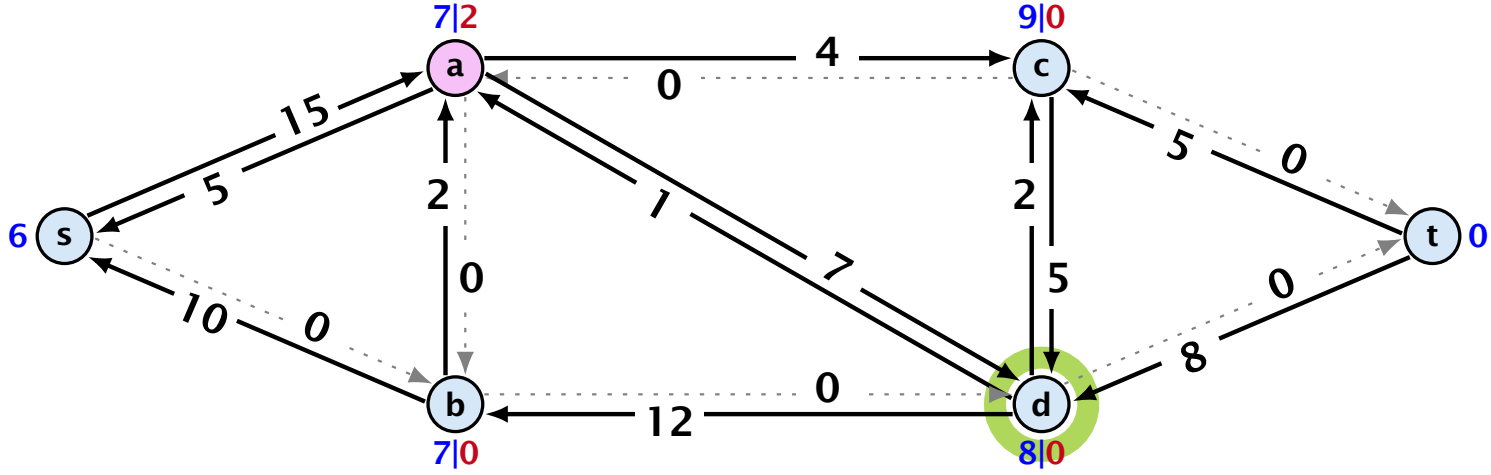

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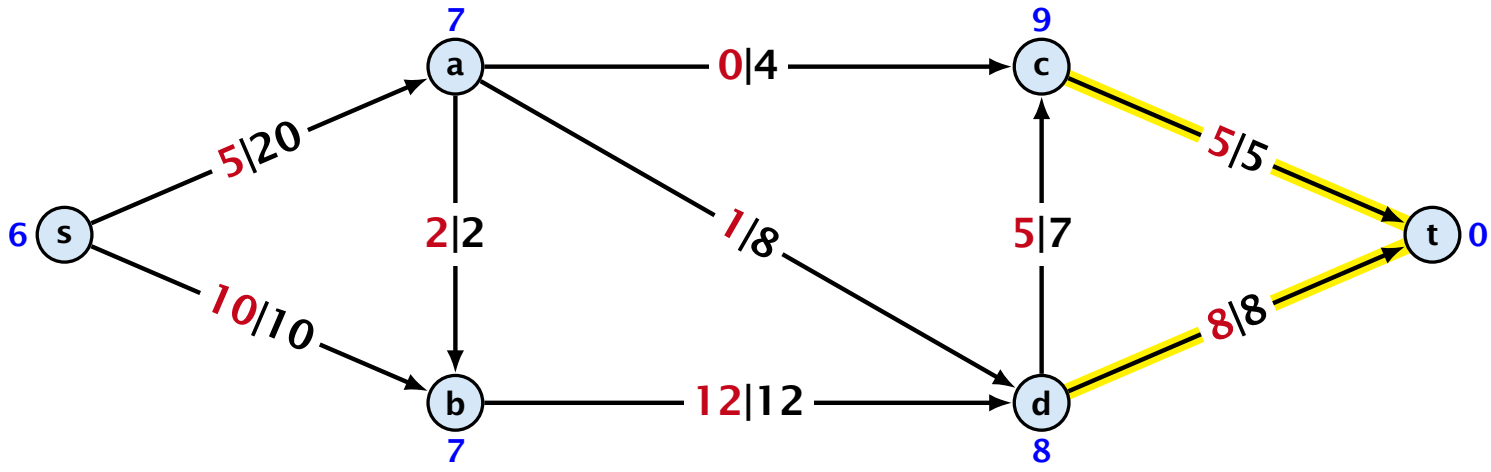

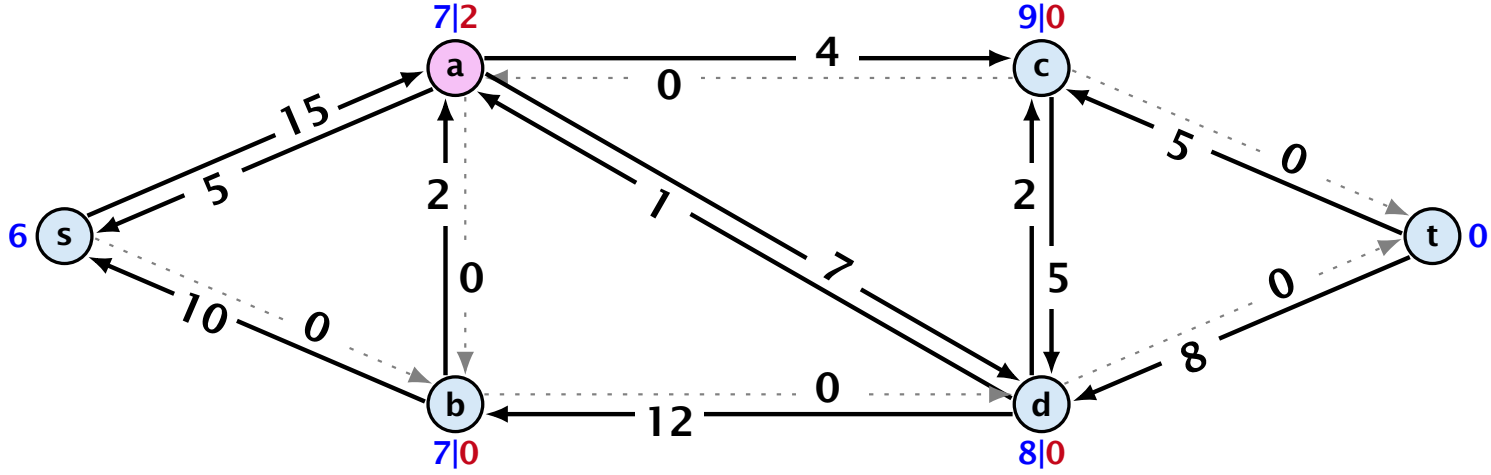

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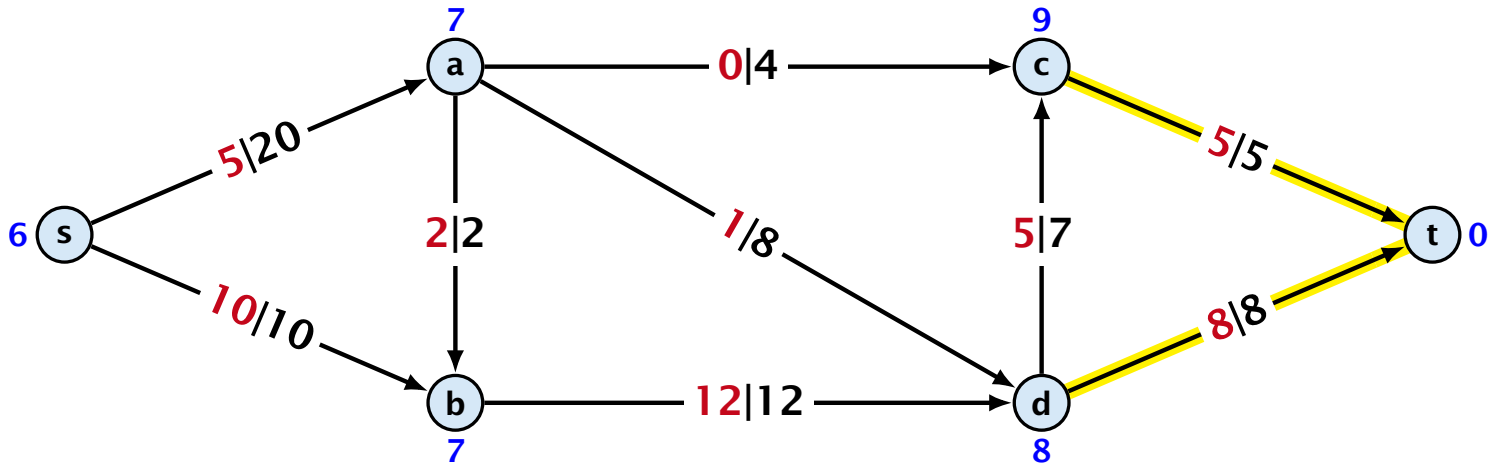

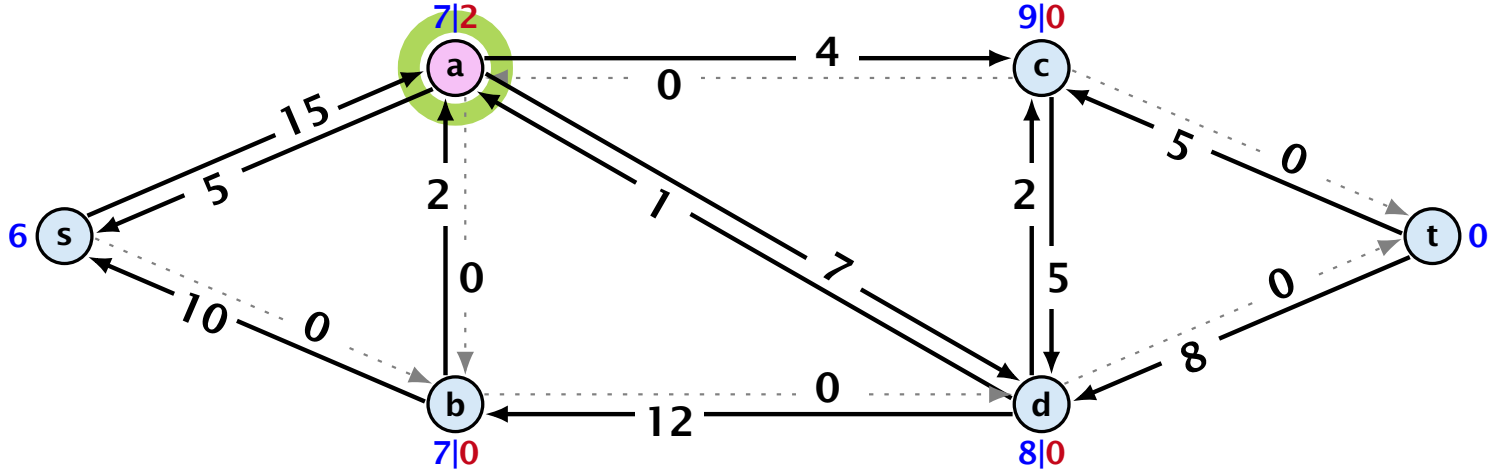

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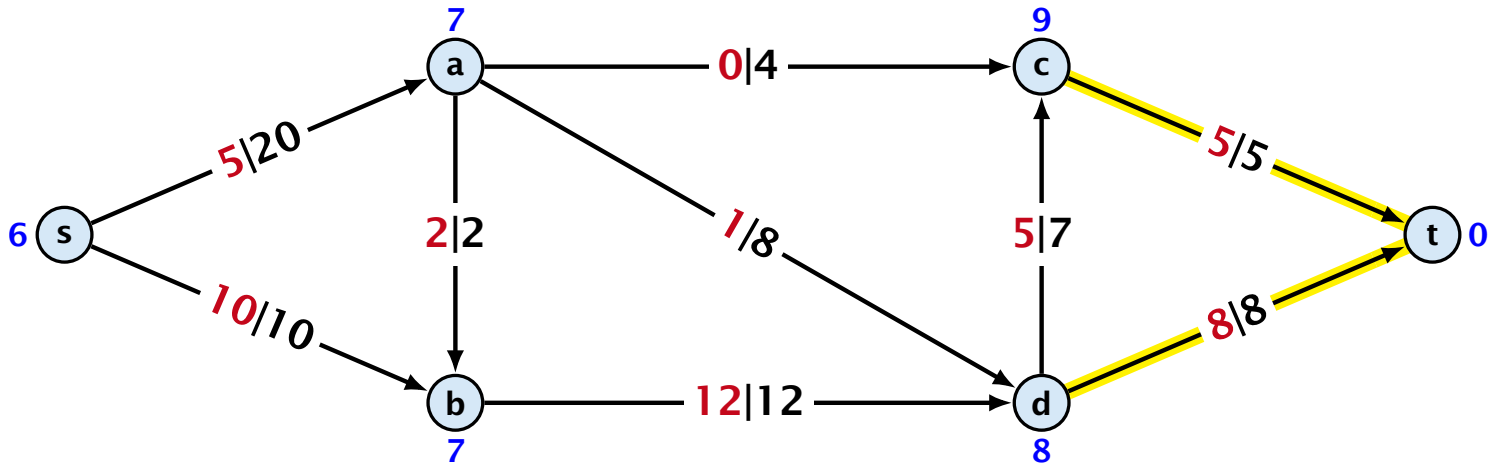

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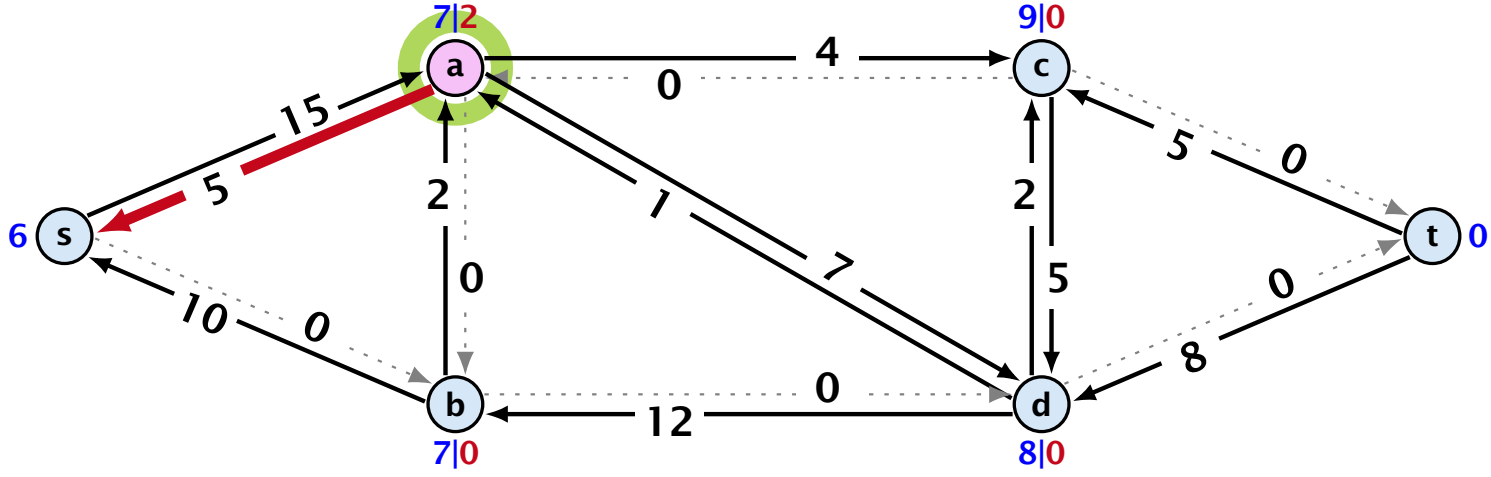

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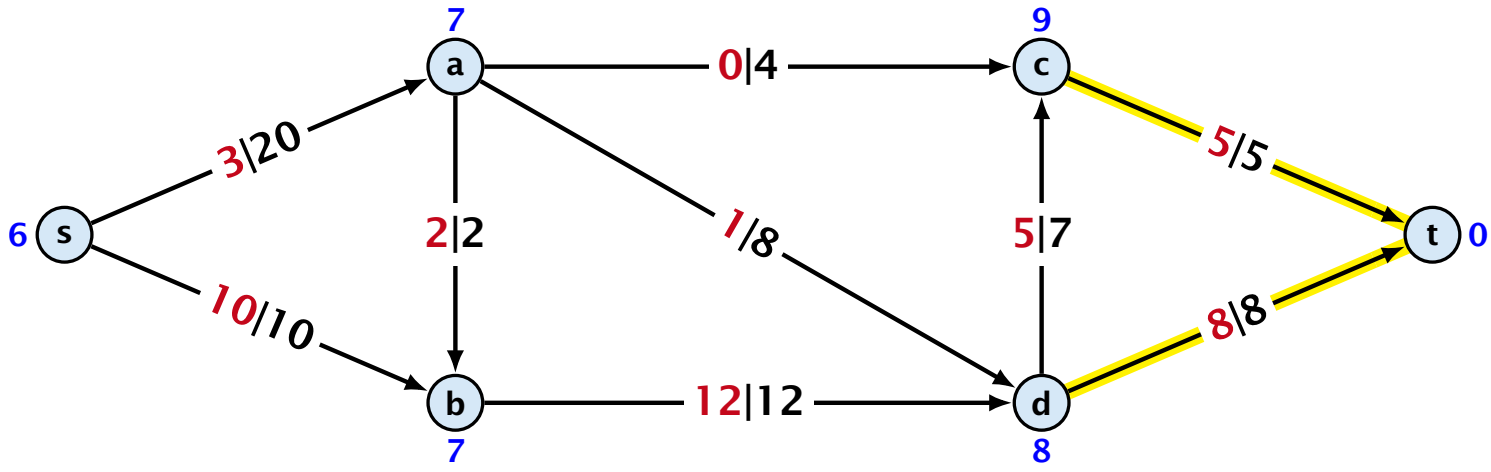

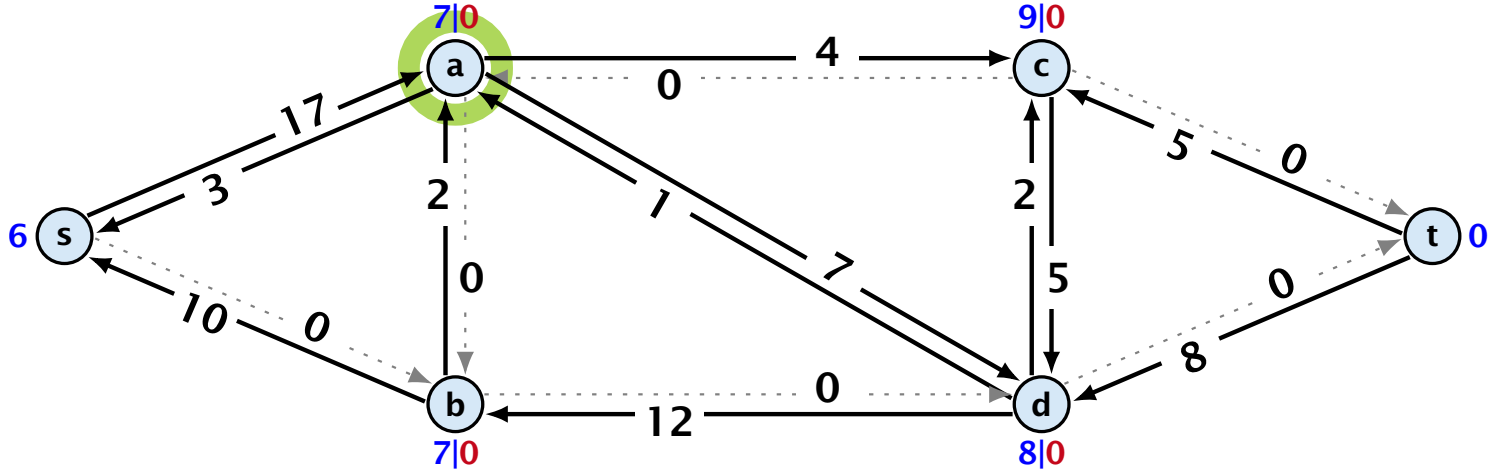

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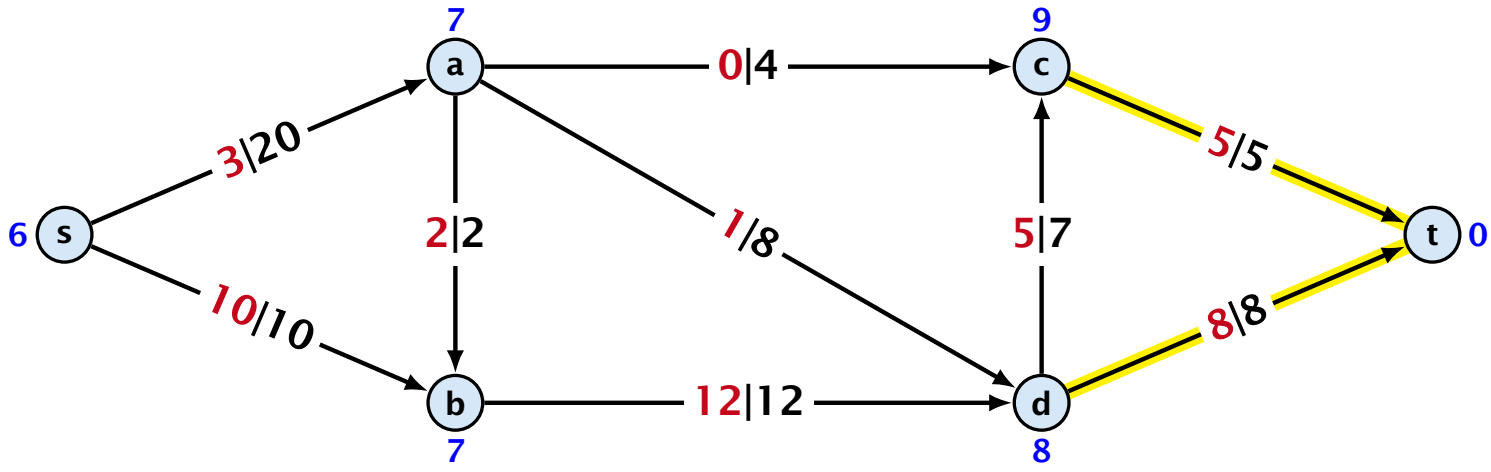

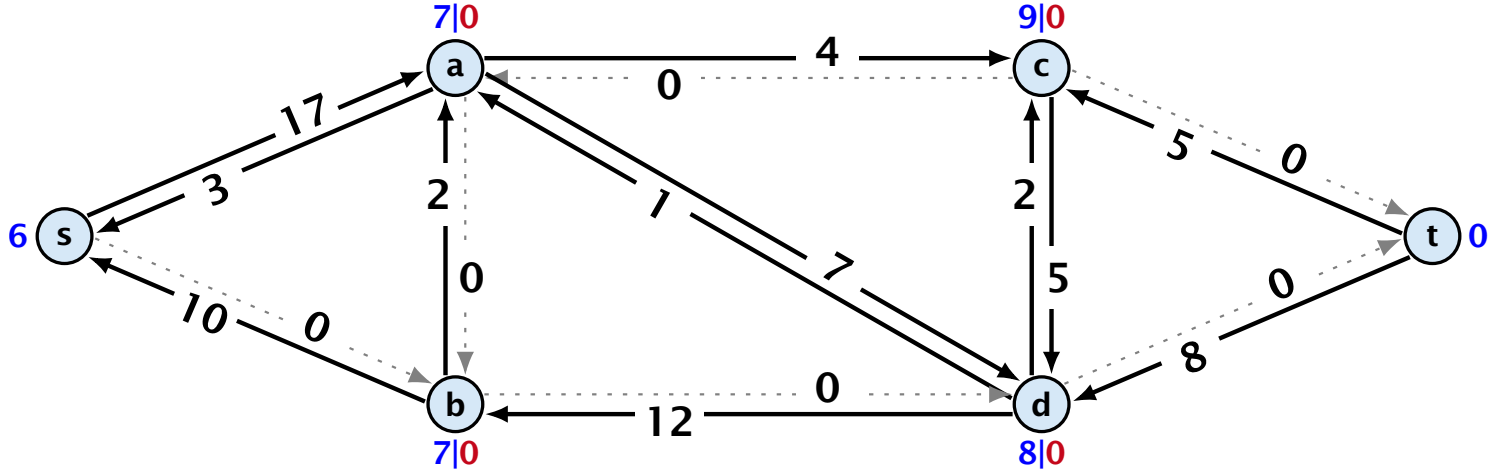

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- Let $f(B) = \sum_{v \in B} f(v)$ be the excess flow of all nodes in B.

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

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Hence, the excess flow $f(b)$ must be 0 for every node $b \in B$.

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- Since the label of v is at most $2n - 1$, there are at most n pushes along (u, v).

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- A relabel increases Φ by at most 1.

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The number of deactivating pushes performed is at most $\mathcal{O}(n^2 m)$.

Proof.

- Define a potential function $\Phi(f) = \sum_{\text{active nodes } v} \ell(v)$
- A saturating push increases Φ by $\leq 2n$ (when the target node becomes active it may contribute at most $2n$ to the sum).
- A relabel increases Φ by at most 1.
- A deactivating push decreases Φ by at least 1 as the node that is pushed from becomes inactive and has a label that is strictly larger than the target.

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- Hence,

$$\begin{aligned}\#\text{deactivating\_pushes} &\leq \#\text{relabels} + 2n \cdot \#\text{saturating\_pushes} \\ &\leq \mathcal{O}(n^2 m) \ .\end{aligned}$$

Analysis

Theorem 14

There is an implementation of the generic push relabel algorithm with running time $\mathcal{O}(n^2 m)$.

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For every node maintain a list of admissible edges starting at that node. Further maintain a list of active nodes.

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A push along an edge (u, v) can be performed in constant time

- check whether edge (v, u) needs to be added to G_f

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- check whether (u, v) needs to be deleted (saturating push)
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A relabel at a node u can be performed in time $\mathcal{O}(n)$

- check for all outgoing edges if they become admissible

Analysis

Proof:

For every node maintain a list of admissible edges starting at that node. Further maintain a list of active nodes.

A push along an edge (u, v) can be performed in constant time

- check whether edge (v, u) needs to be added to G_f
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- check whether u becomes inactive and has to be deleted from the set of active nodes

A relabel at a node u can be performed in time $\mathcal{O}(n)$

- check for all outgoing edges if they become admissible
- check for all incoming edges if they become non-admissible

Analysis

For special variants of push relabel algorithms we organize the neighbours of a node into a linked list (possible neighbours in the residual graph G_f). Then we use the discharge-operation:

Algorithm 2 discharge(u)

```
1: while u is active do
2:     v ← u.current-neighbour
3:     if v = null then
4:         relabel(u)
5:         u.current-neighbour ← u.neighbour-list-head
6:     else
7:         if (u, v) admissible then push(u, v)
8:         else u.current-neighbour ← v.next-in-list
```

Note that $u.\textit{current-neighbour}$ is a global variable. It is only changed within the discharge routine, but keeps its value between consecutive calls to discharge.

Lemma 15

If $v = \text{null}$ in Line 3, then there is no outgoing admissible edge from u.

In order for e to become admissible the other end-point say v has to push flow to u (so that the edge (u, v) re-appears in the residual graph). For this the label of v needs to be larger than the label of u. Then in order to make (u, v) admissible the label of u has to increase.

Proof.

- While pushing from u the current-neighbour pointer is only advanced if the current edge is not admissible.
- The only thing that could make the edge admissible again would be a relabel at u.
- If we reach the end of the list ($v = \text{null}$) all edges are not admissible. □

This shows that discharge(u) is correct, and that we can perform a relabel in Line 4.

13.2 Relabel to Front

Algorithm 1 relabel-to-front(G, s, t)

```
 1: initialize preflow
 2: initialize node list L containing V \ {s, t} in any order
 3: foreach u ∈ V \ {s, t} do
 4:     u.current-neighbour ← u.neighbour-list-head
 5: u ← L.head
 6: while u ≠ null do
 7:     old-height ← ℓ(u)
 8:     discharge(u)
 9:     if ℓ(u) > old-height then // relabel happened
10:         move u to the front of L
11:     u ← u.next
```

13.2 Relabel to Front

Lemma 16 (Invariant)

In Line 6 of the relabel-to-front algorithm the following invariant holds.

1. *The sequence L is topologically sorted w.r.t. the set of admissible edges; this means for an admissible edge (x, y) the node x appears before y in sequence L.*
2. *No node before u in the list L is active.*

Proof:

- Initialization:
  1. In the beginning s has label $n \geq 2$, and all other nodes have label 0. Hence, no edge is admissible, which means that any ordering L is permitted.
  2. We start with u being the head of the list; hence no node before u can be active
- Maintenance:
  1.
     - Pushes do no create any new admissible edges. Therefore, if discharge() does not relabel u, L is still topologically sorted.
     - After relabeling, u cannot have admissible incoming edges as such an edge (x, u) would have had a difference $\ell(x) - \ell(u) \geq 2$ before the re-labeling (such edges do not exist in the residual graph).
       Hence, moving u to the front does not violate the sorting property for any edge; however it fixes this property for all admissible edges leaving u that were generated by the relabeling.

13.2 Relabel to Front

Proof:

- Maintenance:
  2. If we do a relabel there is nothing to prove because the only node before u' (u in the next iteration) will be the current u; the discharge(u) operation only terminates when u is not active anymore.

     For the case that we do not relabel, observe that the only way a predecessor could be active is that we push flow to it via an admissible arc. However, all admissible arc point to successors of u.

Note that the invariant means that for $u = \text{null}$ we have a preflow with a valid labelling that does not have active nodes. This means we have a maximum flow.

13.2 Relabel to Front

Lemma 17

There are at most $\mathcal{O}(n^3)$ calls to discharge(u).

Every discharge operation without a relabel advances u (the current node within list L). Hence, if we have n discharge operations without a relabel we have $u = \text{null}$ and the algorithm terminates.

Therefore, the number of calls to discharge is at most $n(\#relabels + 1) = \mathcal{O}(n^3)$.

13.2 Relabel to Front

Lemma 18

The cost for all relabel-operations is only $\mathcal{O}(n^2)$.

A relabel-operation at a node is constant time (increasing the label and resetting $u.\mathit{current\text{-}neighbour}$). In total we have $\mathcal{O}(n^2)$ relabel-operations.

13.2 Relabel to Front

Recall that a saturating push operation ($\min\{c_f(e), f(u)\} = c_f(e)$) can also be a deactivating push operation ($\min\{c_f(e), f(u)\} = f(u)$).

Lemma 19

*The cost for all saturating push-operations that are **not** deactivating is only $\mathcal{O}(mn)$.*

Note that such a push-operation leaves the node u active but makes the edge e disappear from the residual graph. Therefore the push-operation is immediately followed by an increase of the pointer $u.\mathit{current\text{-}neighbour}$.
This pointer can traverse the neighbour-list at most $\mathcal{O}(n)$ times (upper bound on number of relabels) and the neighbour-list has only $\mathit{degree}(u) + 1$ many entries ($+1$ for null-entry).

13.2 Relabel to Front

Lemma 20
The cost for all deactivating push-operations is only $\mathcal{O}(n^3)$.

A deactivating push-operation takes constant time and ends the current call to discharge(). Hence, there are only $\mathcal{O}(n^3)$ such operations.

Theorem 21
The push-relabel algorithm with the rule relabel-to-front takes time $\mathcal{O}(n^3)$.

13.3 Highest Label

Algorithm 1 highest-label(G, s, t)

```
1: initialize preflow
2: foreach u ∈ V \ {s, t} do
3:     u.current-neighbour ← u.neighbour-list-head
4: while ∃ active node u do
5:     select active node u with highest label
6:     discharge(u)
```

13.3 Highest Label

Lemma 22

When using highest label the number of deactivating pushes is only $\mathcal{O}(n^3)$.

A push from a node on level ℓ can only "activate" nodes on levels strictly less than ℓ.

This means, after a deactivating push from u a relabel is required to make u active again.

Hence, after n deactivating pushes without an intermediate relabel there are no active nodes left.

Therefore, the number of deactivating pushes is at most $n(\#relabels + 1) = \mathcal{O}(n^3)$.

13.3 Highest Label

Since a discharge-operation is terminated by a deactivating push this gives an upper bound of $\mathcal{O}(n^3)$ on the number of discharge-operations.

The cost for relabels and saturating pushes can be estimated in exactly the same way as in the case of the generic push-relabel algorithm.

Question:
How do we find the next node for a discharge operation?

13.3 Highest Label

Maintain lists L_i, $i \in \{0, \ldots, 2n\}$, where list L_i contains active nodes with label i (maintaining these lists induces only constant additional cost for every push-operation and for every relabel-operation).

After a discharge operation terminated for a node u with label k, traverse the lists $L_k, L_{k-1}, \ldots, L_0$, (in that order) until you find a non-empty list.

Unless the last (deactivating) push was to s or t the list $k-1$ must be non-empty (i.e., the search takes constant time).

13.3 Highest Label

Hence, the total time required for searching for active nodes is at most

$$\mathcal{O}(n^3) + n(\#\textit{deactivating-pushes-to-}s\textit{-or-}t)$$

Lemma 23

The number of deactivating pushes to s or t is at most $\mathcal{O}(n^2)$.

With this lemma we get

Theorem 24

The push-relabel algorithm with the rule highest-label takes time $\mathcal{O}(n^3)$.

13.3 Highest Label

Proof of the Lemma.

- We only show that the number of pushes to the source is at most $\mathcal{O}(n^2)$. A similar argument holds for the target.
- After a node v (which must have $\ell(v) = n + 1$) made a deactivating push to the source there needs to be another node whose label is increased from $\leq n + 1$ to $n + 2$ before v can become active again.
- This happens for every push that v makes to the source. Since, every node can pass the threshold $n + 2$ at most once, v can make at most n pushes to the source.
- As this holds for every node the total number of pushes to the source is at most $\mathcal{O}(n^2)$.