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Sometimes we also have

- $S.\textbf{merge}(S')$: $S := S \cup S'$; $S' := \emptyset$.

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An addressable Priority Queue also supports:

- **handle S. insert(x)**: Adds element x to the data-structure, and returns a handle to the object for future reference.
- **S. delete(h)**: Deletes element specified through handle h.
- **S. decrease-key(h, k)**: Decreases the key of the element specified by handle h to k. Assumes that the key is at least k before the operation.

Dijkstra's Shortest Path Algorithm

Algorithm 1 Shortest-Path($G = (V, E, d), s \in V$)

```
Input: weighted graph G = (V, E, d); start vertex s;
Output: key-field of every node contains distance from s;
S.build(); // build empty priority queue
for all v ∈ V \ {s} do
    v.key ← ∞;
    h_v ← S.insert(v);
s.key ← 0; S.insert(s);
while S.is-empty() = false do
    v ← S.delete-min();
    for all x ∈ V s.t. (v, x) ∈ E do
        if x.key > v.key + d(v, x) then
            S.decrease-key(h_x, v.key + d(v, x));
            x.key ← v.key + d(v, x);
```

Prim's Minimum Spanning Tree Algorithm

Algorithm 2 Prim-MST$(G = (V, E, d), s \in V)$

Input: weighted graph $G = (V, E, d)$; start vertex s;
Output: pred-fields encode MST;
S.build(); // build empty priority queue
for all $v \in V \setminus \{s\}$ **do**
    $v.\text{key} \leftarrow \infty$;
    $h_v \leftarrow S$.insert(v);
$s.\text{key} \leftarrow 0$; S.insert(s);
while S.is-empty$() = \text{false}$ **do**
    $v \leftarrow S$.delete-min$()$;
    for all $x \in V$ s.t. $\{v, x\} \in E$ **do**
        if $x.\text{key} > d(v, x)$ **then**
            S.decrease-key$(h_x, d(v, x))$;
            $x.\text{key} \leftarrow d(v, x)$;
            $x.\text{pred} \leftarrow v$;

Analysis of Dijkstra and Prim

Both algorithms require:

- 1 build() operation
- $|V|$ insert() operations
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How good a running time can we obtain?

8 Priority Queues

Operation	*Binary Heap*	*BST*	*Binomial Heap*	*Fibonacci Heap* *
build	n	$n \log n$	$n \log n$	n
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n$ **	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
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Note that most applications use **build()** only to create an empty heap which then costs time 1.

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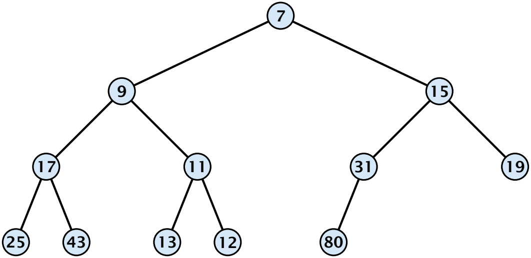
Fibonacci heaps only give an amortized guarantee.

8 Priority Queues

Using Binary Heaps, Prim and Dijkstra run in time $\mathcal{O}((|V| + |E|)\log|V|)$.

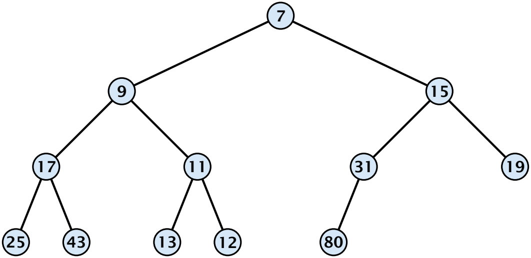
Using Fibonacci Heaps, Prim and Dijkstra run in time $\mathcal{O}(|V|\log|V| + |E|)$.

8.1 Binary Heaps



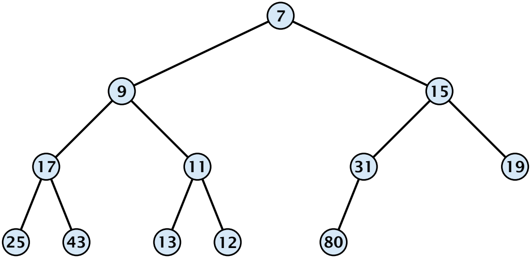
8.1 Binary Heaps

- Nearly complete binary tree; only the last level is not full, and this one is filled from left to right.



8.1 Binary Heaps

- Nearly complete binary tree; only the last level is not full, and this one is filled from left to right.
- Heap property: A node's key is not larger than the key of one of its children.



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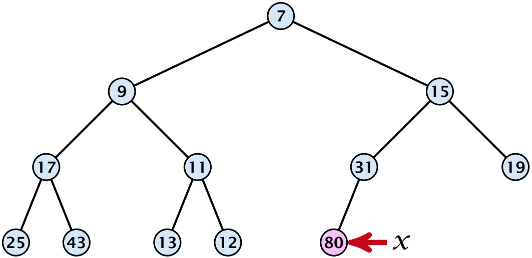

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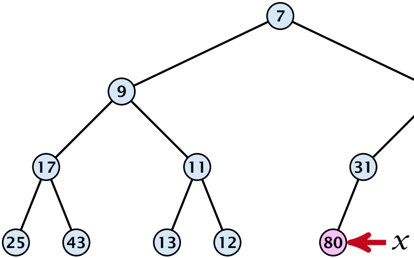
Maintain a pointer to the last element x.

- We can compute the predecessor of x (last element when x is deleted) in time $\mathcal{O}(\log n)$.



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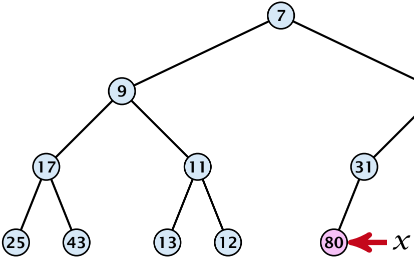

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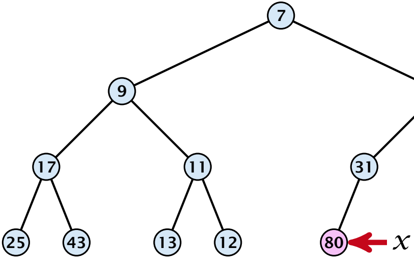

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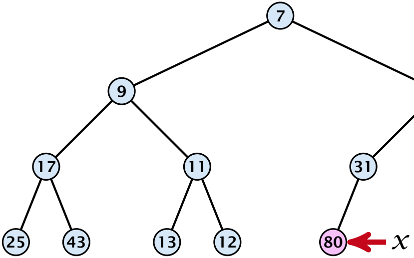

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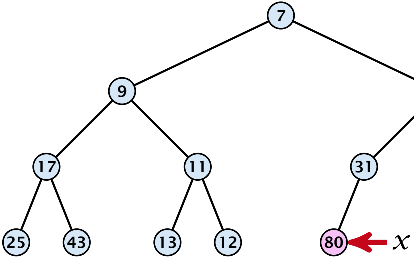

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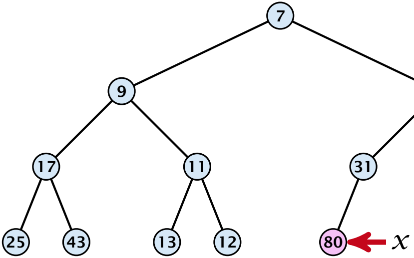

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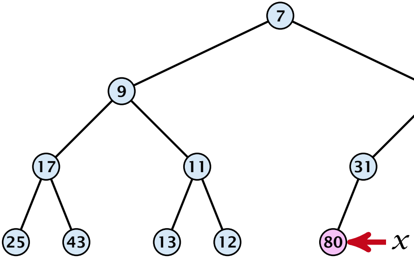
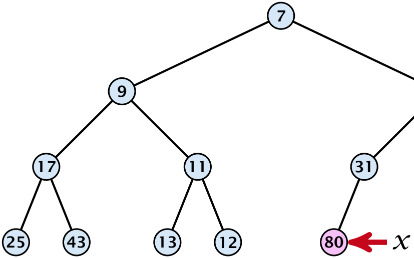
  insert a new element as a left child;



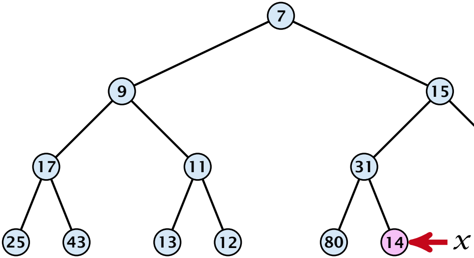
Insert

1. Insert element at successor of x.



Insert

1. Insert element at successor of x.
2. Exchange with parent until heap property is fulfilled.



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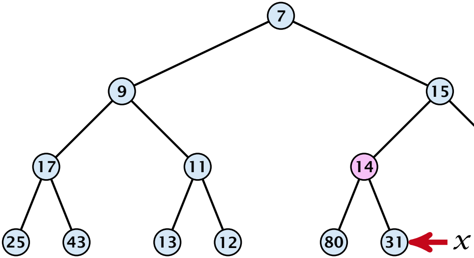

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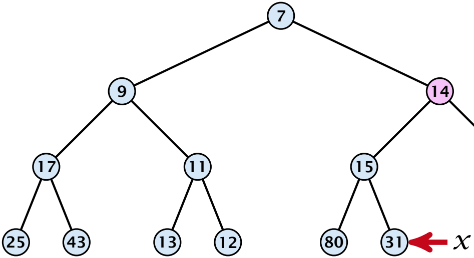

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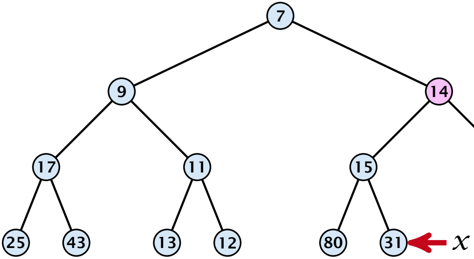
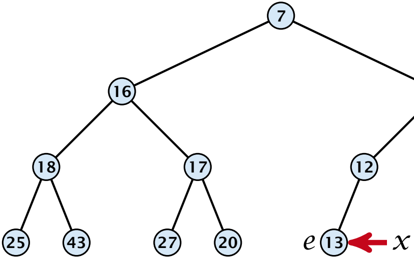


Note that an exchange can either be done by moving the data or by changing pointers. The latter method leads to an addressable priority queue.

Delete

1. Exchange the element to be deleted with the element e pointed to by x.



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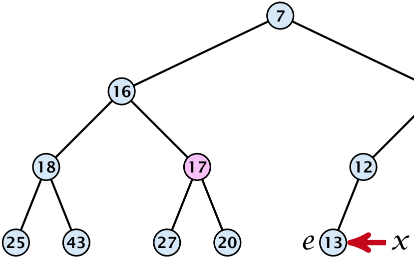
2. Restore the heap-property for the element e.



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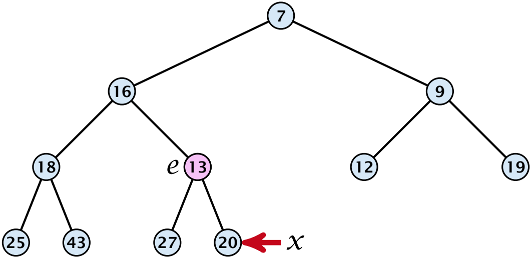

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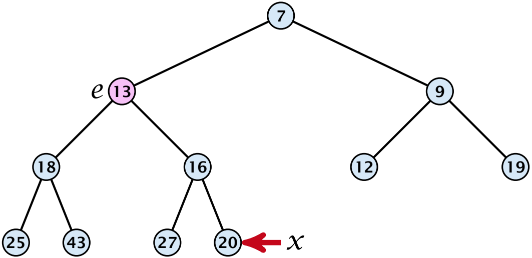

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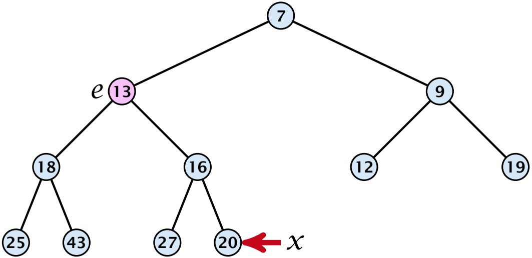


At its new position e may either travel up or down in the tree (but not both directions).

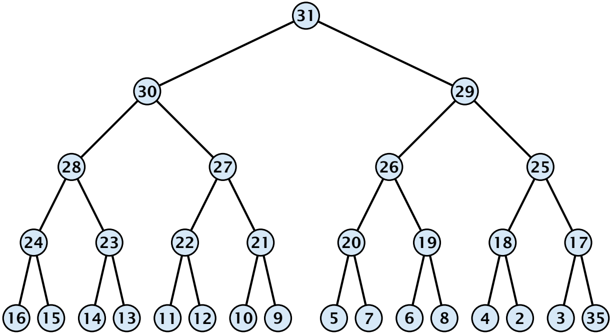
Binary Heaps

Operations:

- **minimum()**: return the root-element. Time $\mathcal{O}(1)$.
- **is-empty()**: check whether root-pointer is null. Time $\mathcal{O}(1)$.
- **insert(k)**: insert at successor of x and bubble up. Time $\mathcal{O}(\log n)$.
- **delete(h)**: swap with x and bubble up or sift-down. Time $\mathcal{O}(\log n)$.

Build Heap

We can build a heap in linear time:



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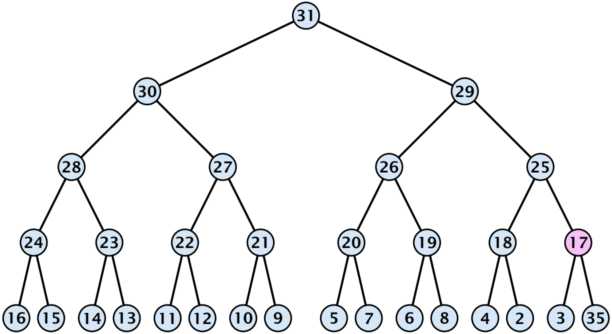

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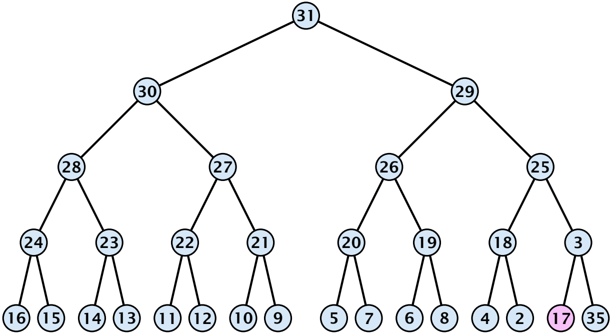

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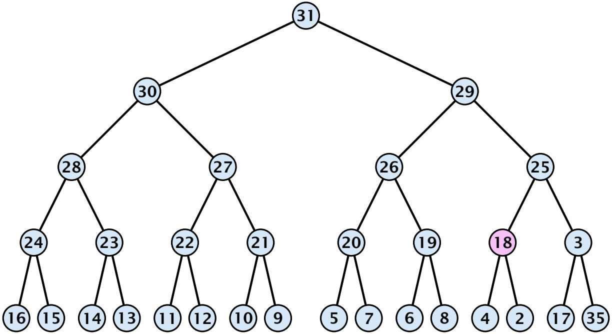

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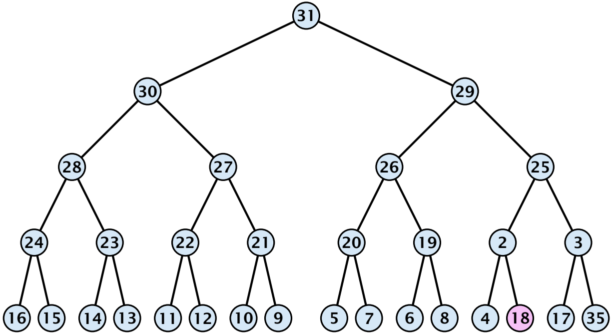

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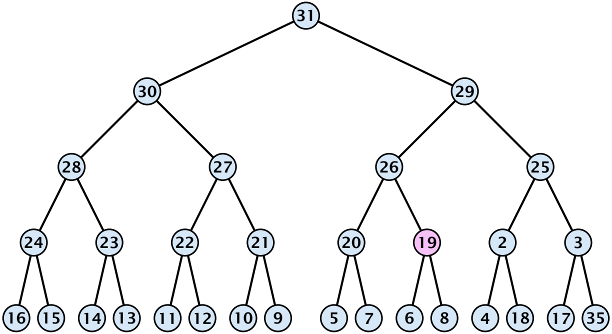

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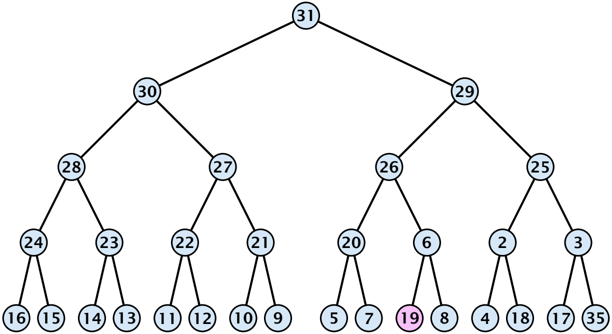

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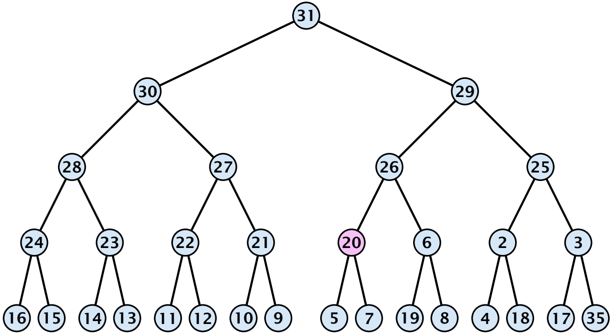

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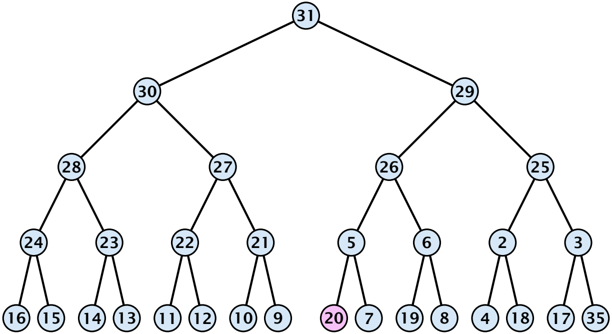

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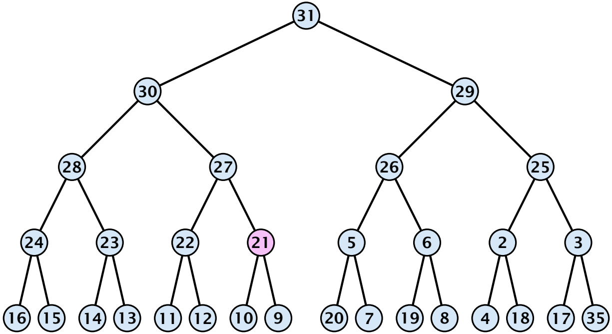

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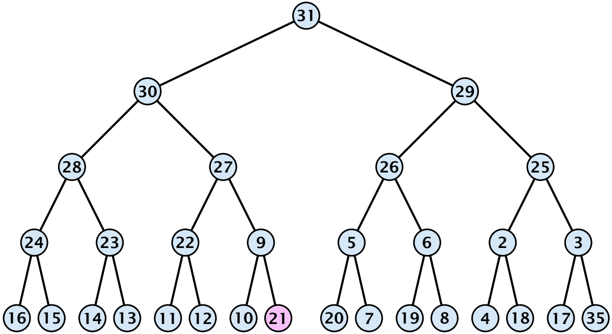

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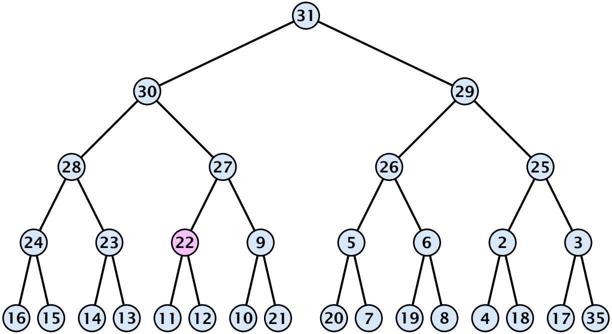

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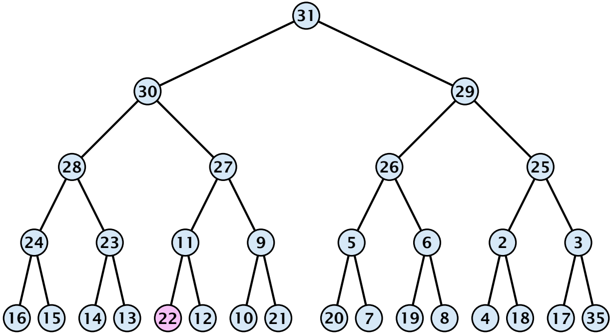

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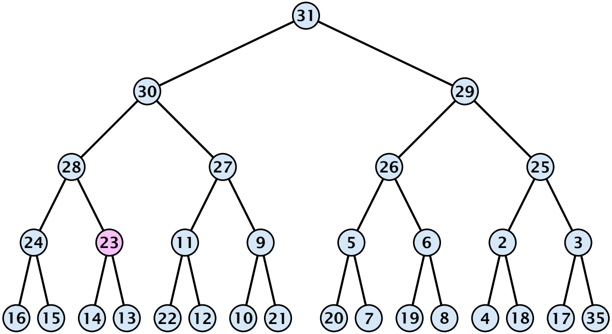

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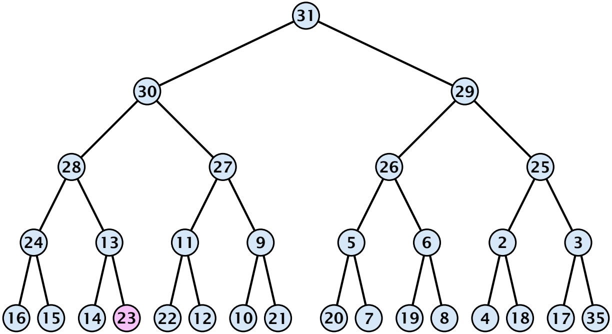

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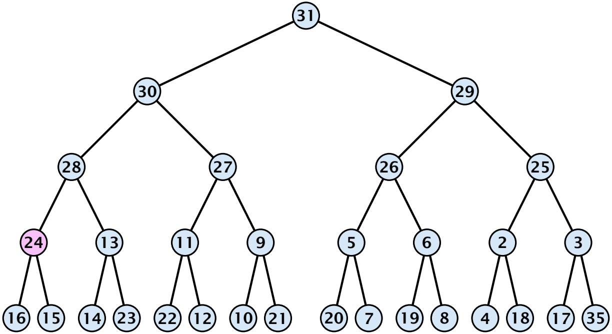

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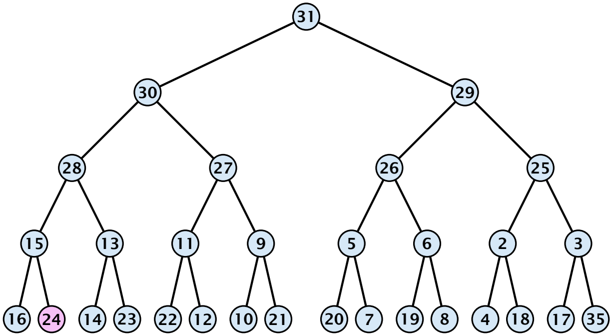

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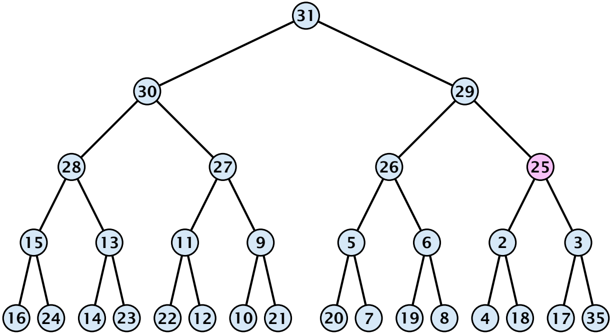

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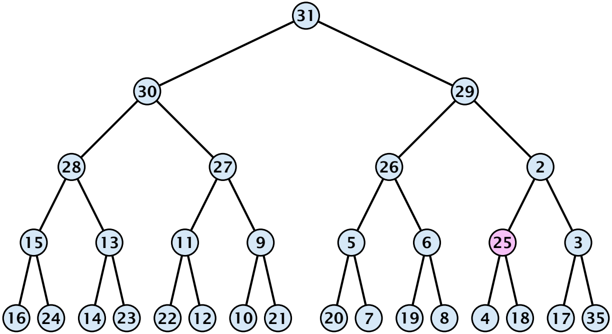

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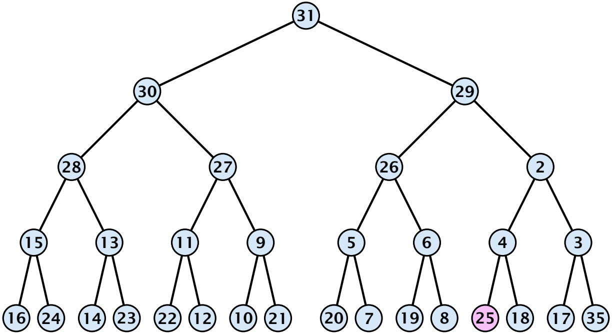

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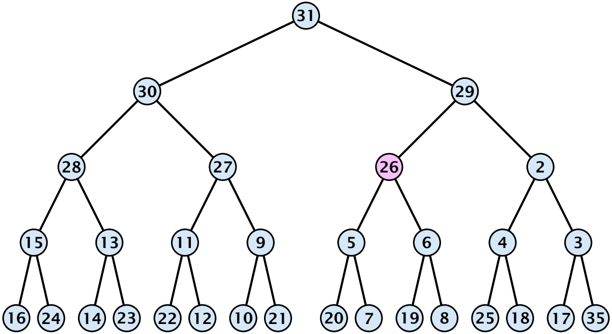

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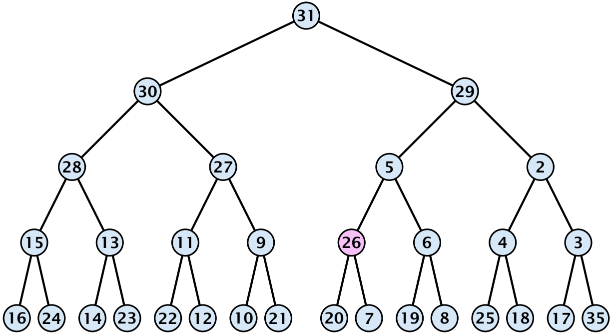

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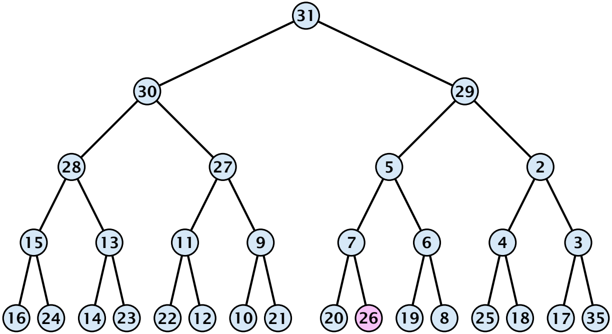

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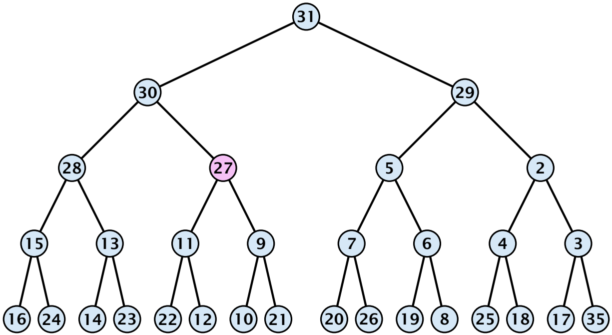

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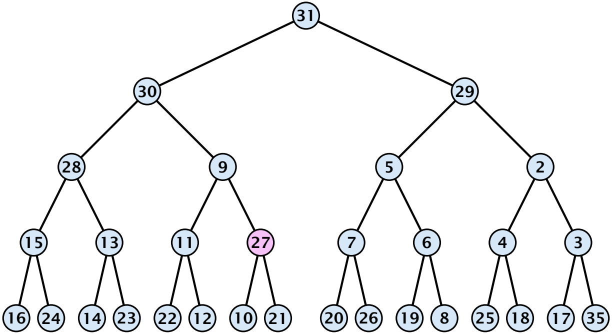

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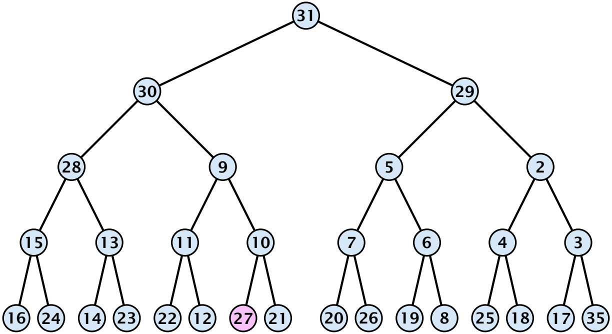

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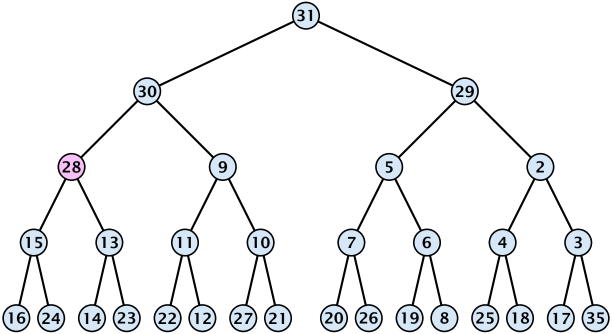

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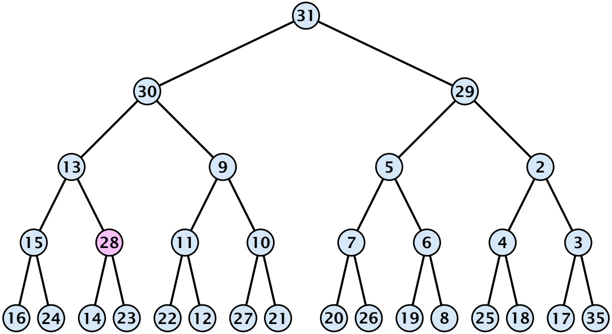

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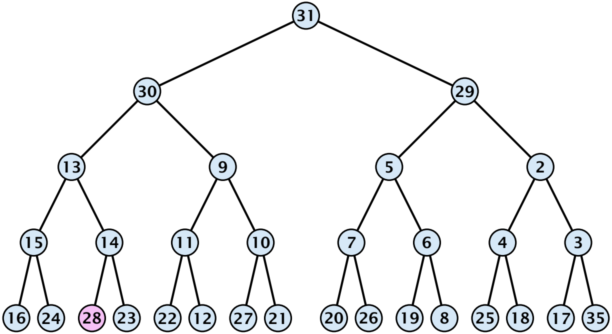

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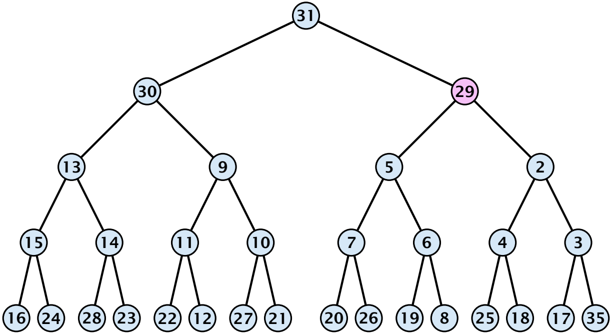

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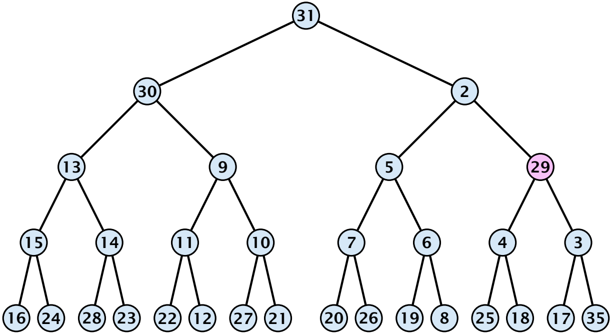

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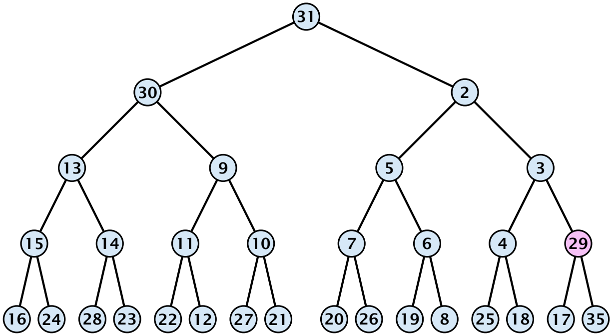

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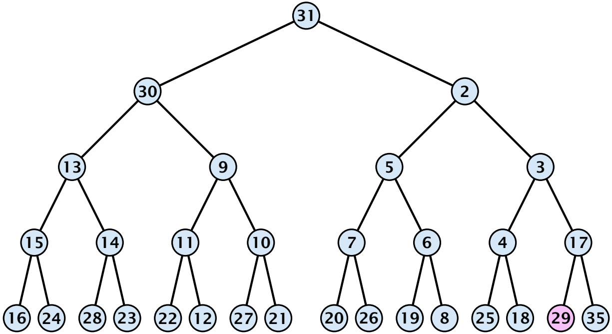

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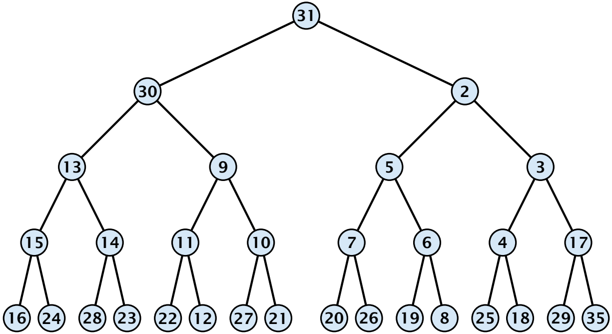

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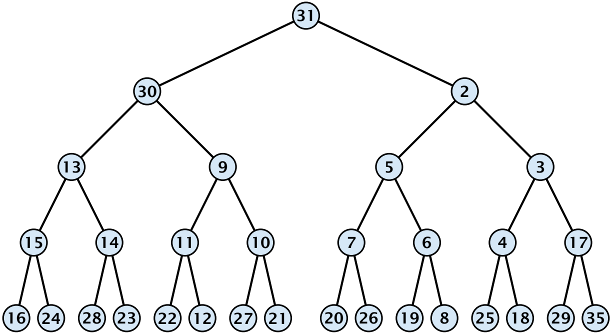

$$\sum_{\text{levels } \ell} 2^{\ell} \cdot (h - \ell) = \sum_i i 2^{h-i} = \mathcal{O}(2^h) = \mathcal{O}(n)$$

Binary Heaps

Operations:

- **minimum()**: Return the root-element. Time $\mathcal{O}(1)$.
- **is-empty()**: Check whether root-pointer is null. Time $\mathcal{O}(1)$.
- **insert(k)**: Insert at x and bubble up. Time $\mathcal{O}(\log n)$.
- **delete(h)**: Swap with x and bubble up or sift-down. Time $\mathcal{O}(\log n)$.
- **build($x_1, \dots, x_n$)**: Insert elements arbitrarily; then do sift-down operations starting with the lowest layer in the tree. Time $\mathcal{O}(n)$.

Binary Heaps

Binary Heaps

The standard implementation of binary heaps is via arrays. Let $A[0, \ldots, n-1]$ be an array

- The parent of i-th element is at position $\lfloor \frac{i-1}{2} \rfloor$.
- The left child of i-th element is at position $2i + 1$.
- The right child of i-th element is at position $2i + 2$.

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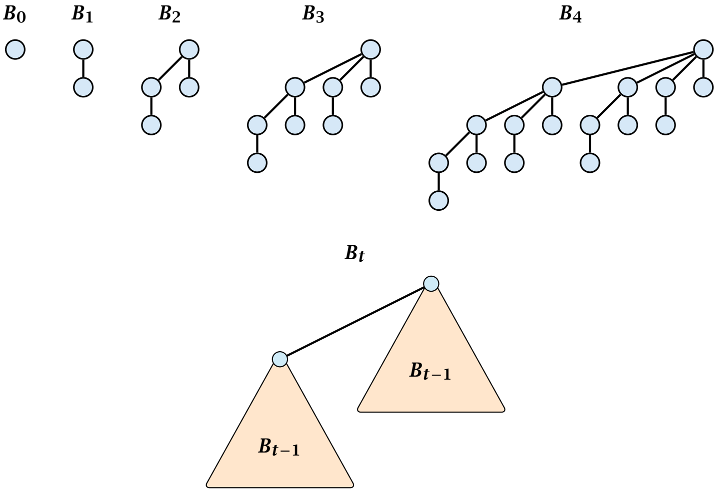
Finding the successor of x is much easier than in the description on the previous slide. Simply increase or decrease x.

The resulting binary heap is not addressable. The elements don't maintain their positions and therefore there are no stable handles.

8.2 Binomial Heaps

Operation	*Binary Heap*	*BST*	*Binomial Heap*	*Fibonacci Heap**
build	n	$n \log n$	$n \log n$	n
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n^{**}$	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	n	$n \log n$	**$\log n$**	1

Binomial Trees



Binomial Trees

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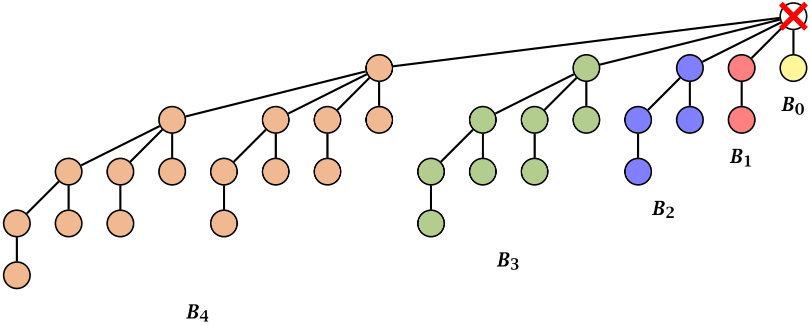
Binomial Trees

Properties of Binomial Trees

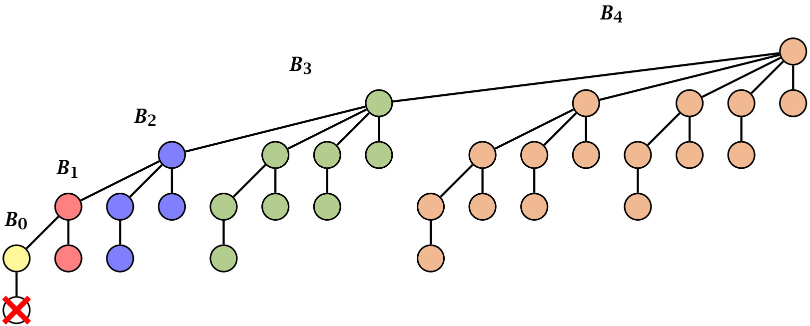
- B_k has 2^k nodes.
- B_k has height k.
- The root of B_k has degree k.
- B_k has $\binom{k}{\ell}$ nodes on level ℓ.
- Deleting the root of B_k gives trees $B_0, B_1, \ldots, B_{k-1}$.

Binomial Trees



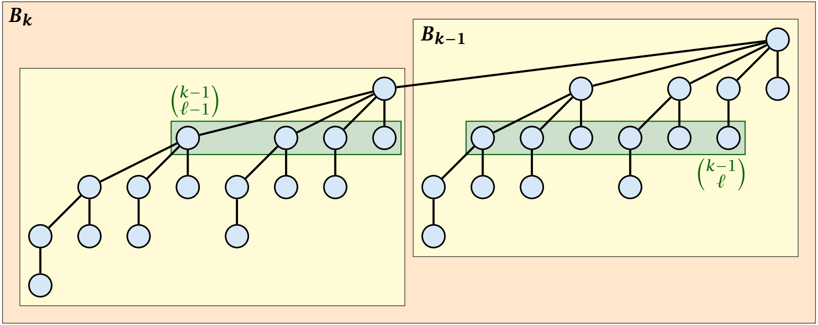
Deleting the root of B_5 leaves sub-trees B_4, B_3, B_2, B_1, and B_0.

Binomial Trees



Deleting the leaf furthest from the root (in B_5) leaves a path that connects the roots of sub-trees B_4, B_3, B_2, B_1, and B_0.

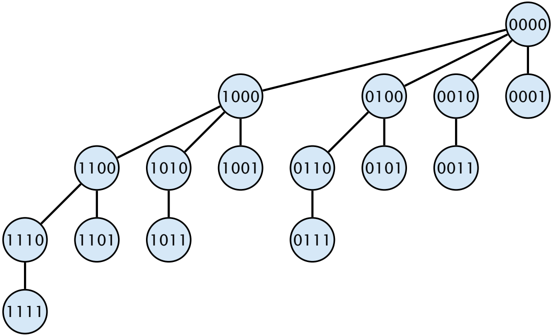
Binomial Trees



The number of nodes on level ℓ in tree B_k is therefore

$$\binom{k-1}{\ell-1} + \binom{k-1}{\ell} = \binom{k}{\ell}$$

Binomial Trees



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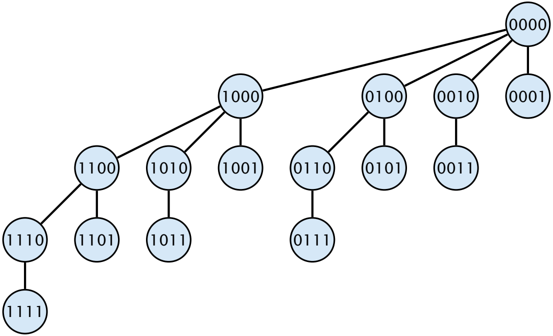

The binomial tree B_k is a sub-graph of the hypercube H_k.

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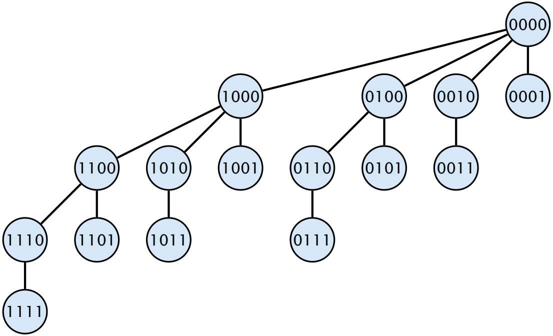

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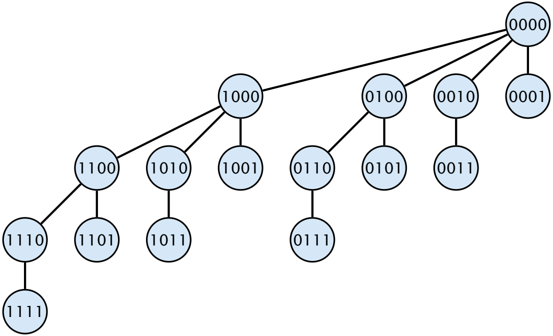
The parent of a node with label $b_k, \ldots, b_1$ is obtained by setting the least significant 1-bit to 0.

Binomial Trees



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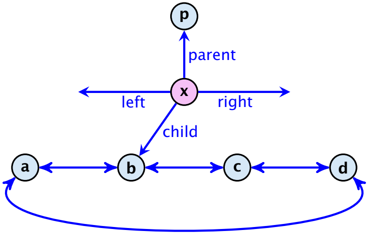
The ℓ-th level contains nodes that have ℓ 1's in their label.

8.2 Binomial Heaps

How do we implement trees with non-constant degree?

- The children of a node are arranged in a circular linked list.



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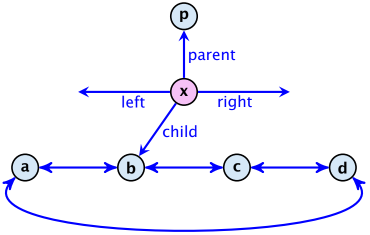

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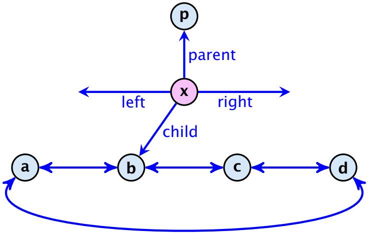

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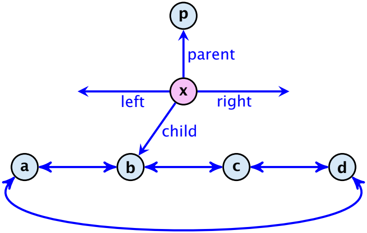
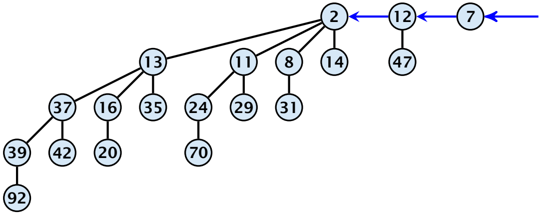
- The children of a node are arranged in a circular linked list.
- A child-pointer points to an arbitrary node within the list.
- A parent-pointer points to the parent node.
- Pointers $x.\mathrm{left}$ and $x.\mathrm{right}$ point to the left and right sibling of x (if x does not have siblings then $x.\mathrm{left} = x.\mathrm{right} = x$).



8.2 Binomial Heaps

- Given a pointer to a node x we can splice out the sub-tree rooted at x in constant time.
- We can add a child-tree T to a node x in constant time if we are given a pointer to x and a pointer to the root of T.

Binomial Heap



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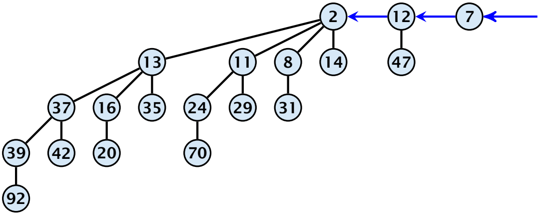

In a binomial heap the keys are arranged in a collection of binomial trees.

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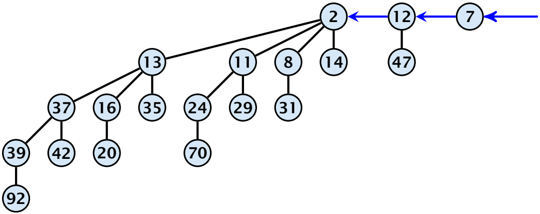

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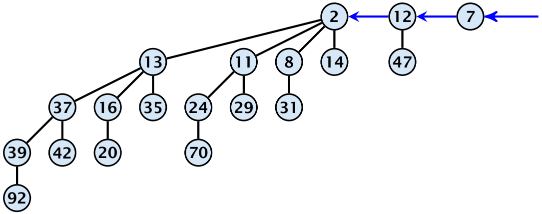

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Every tree fulfills the heap-property

There is at most one tree for every dimension/order. For example the above heap contains trees B_0, B_1, and B_4.

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Let $B_{k_1}, B_{k_2}, B_{k_3}, k_i < k_{i+1}$ denote the binomial trees in the collection and recall that every tree may be contained at most once.

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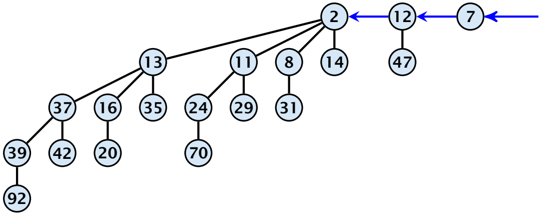
Then $n = \sum_i 2^{k_i}$ must hold. But since the k_i are all distinct this means that the k_i define the non-zero bit-positions in the binary representation of n.

Binomial Heap

Properties of a heap with n keys:



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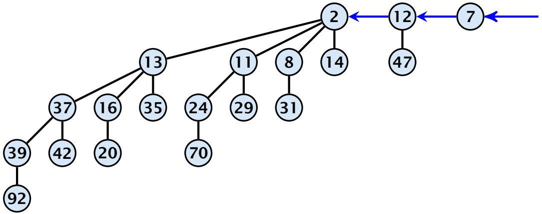

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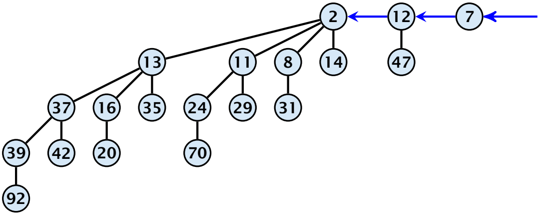

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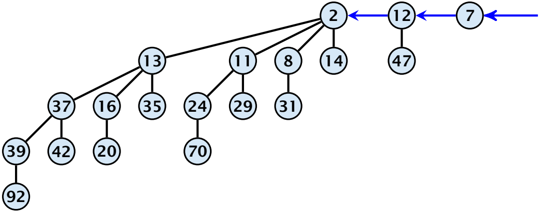

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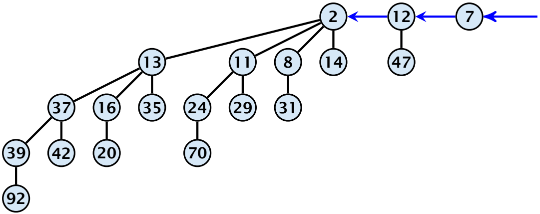

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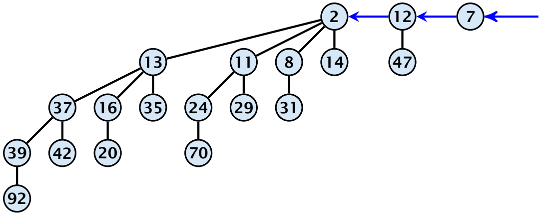

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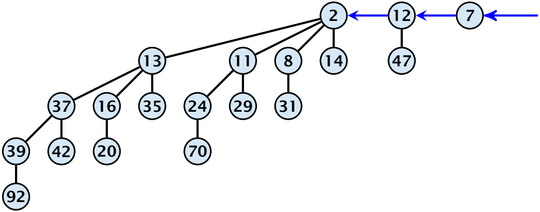
- Hence, at most $\lfloor \log n \rfloor + 1$ trees.
- The minimum must be contained in one of the roots.
- The height of the largest tree is at most $\lfloor \log n \rfloor$.
- The trees are stored in a single-linked list; ordered by dimension/size.



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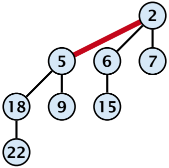

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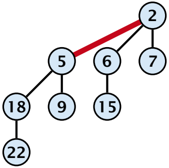
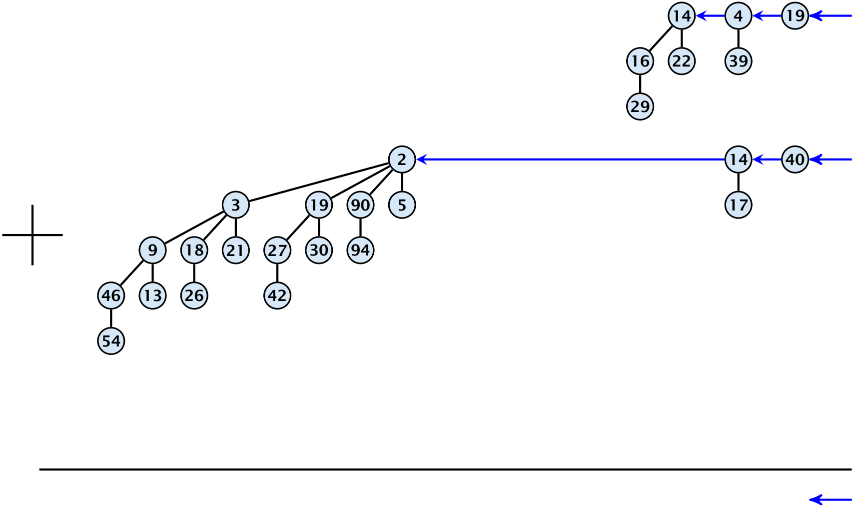


For more trees the technique is analogous to binary addition.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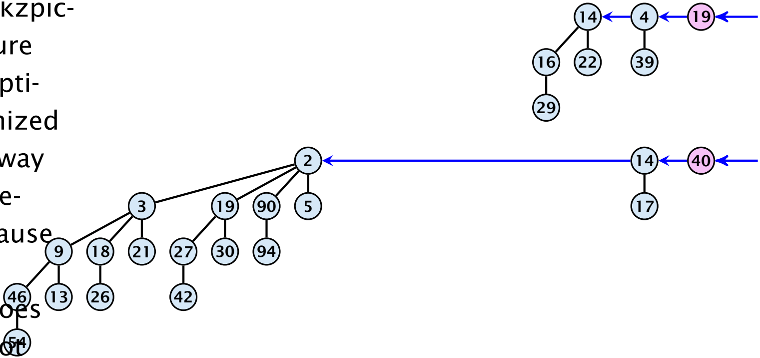

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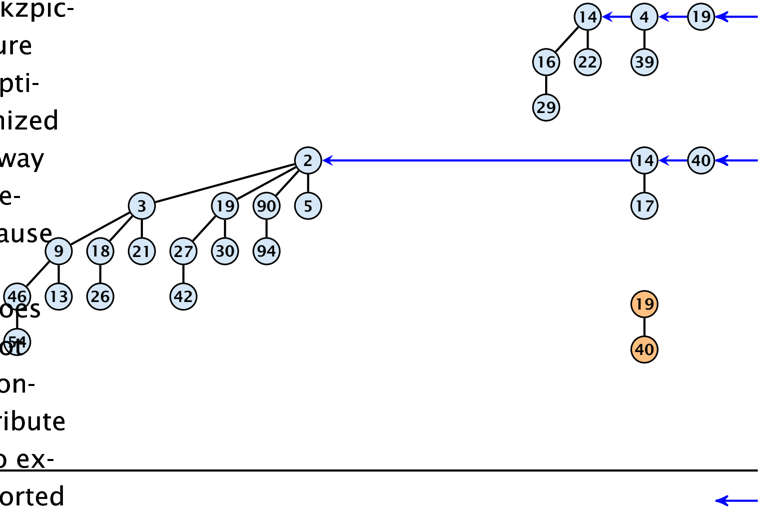

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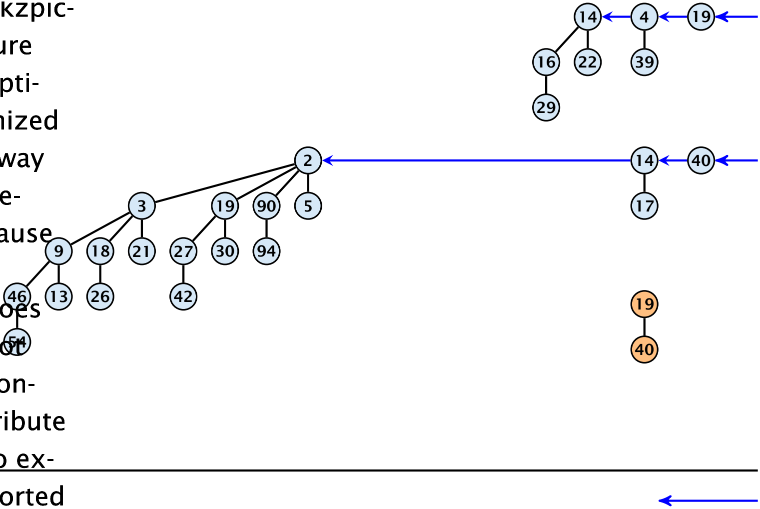

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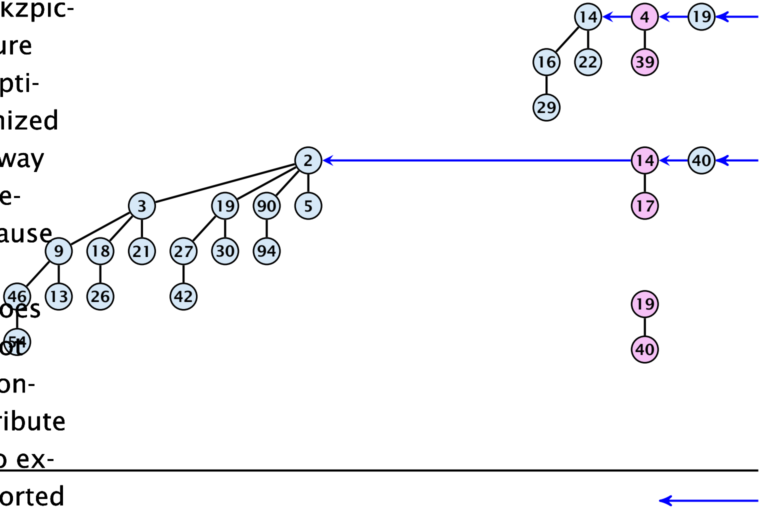

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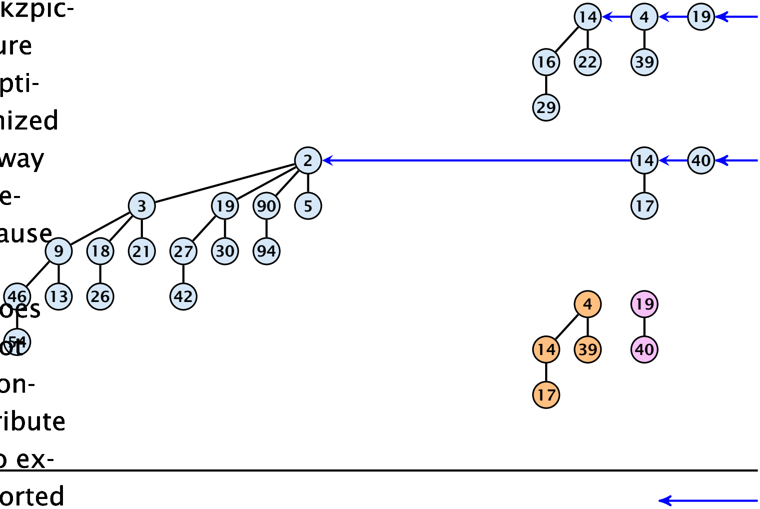

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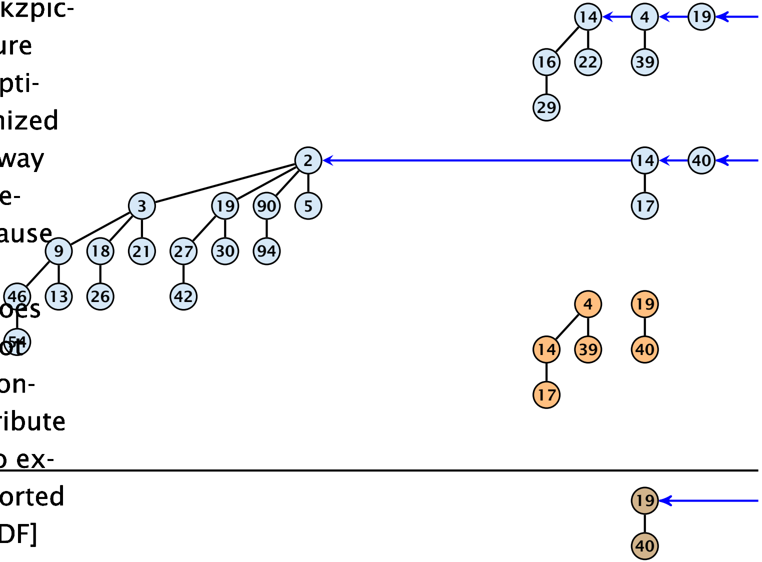

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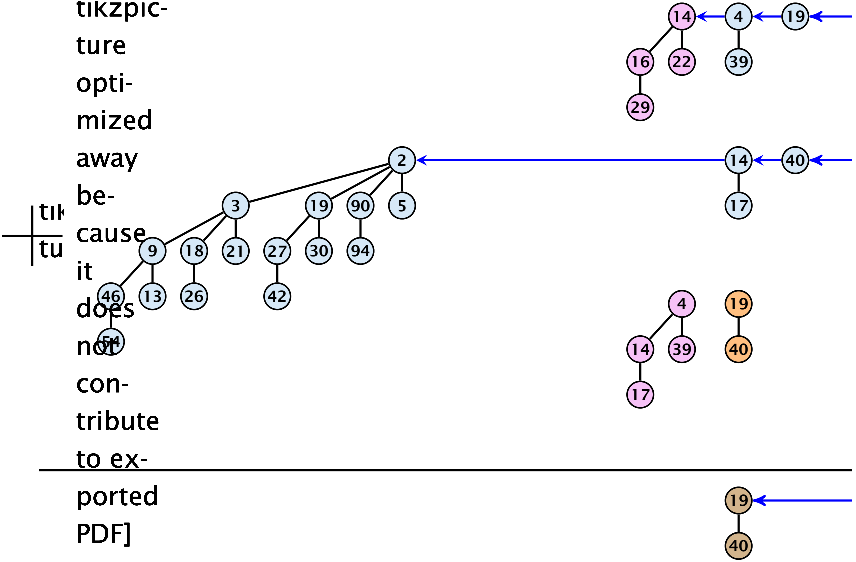

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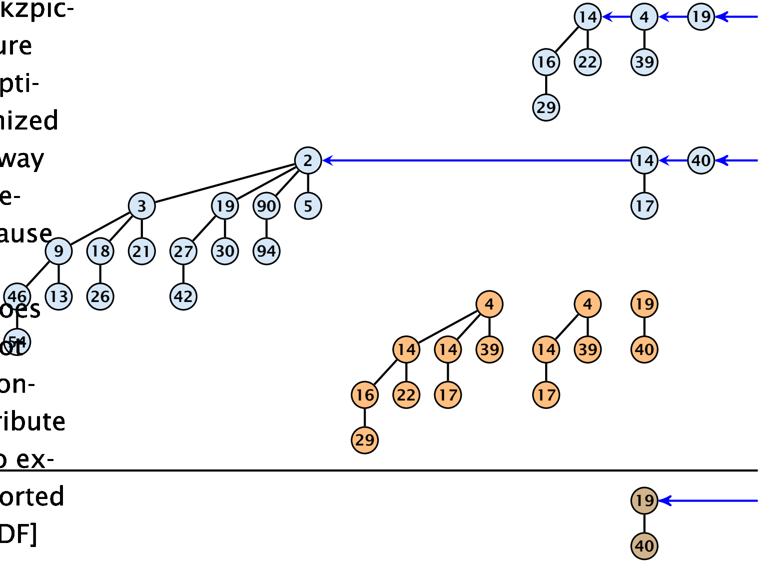

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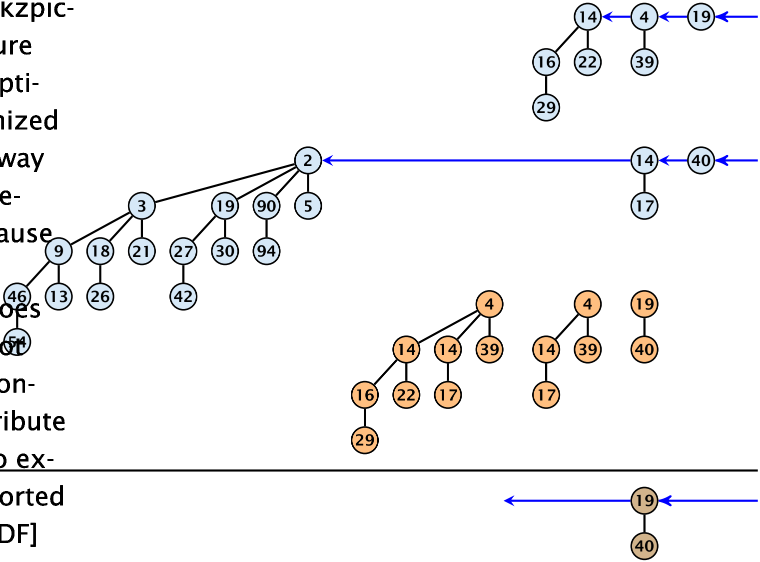

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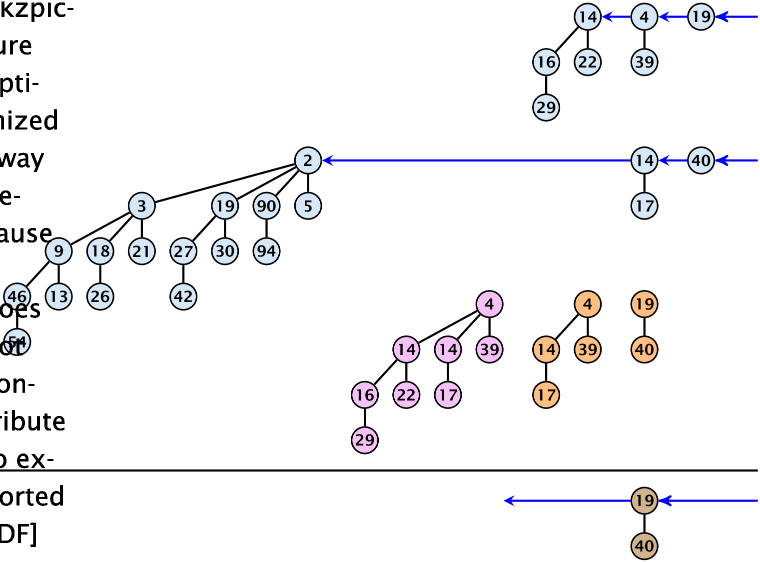

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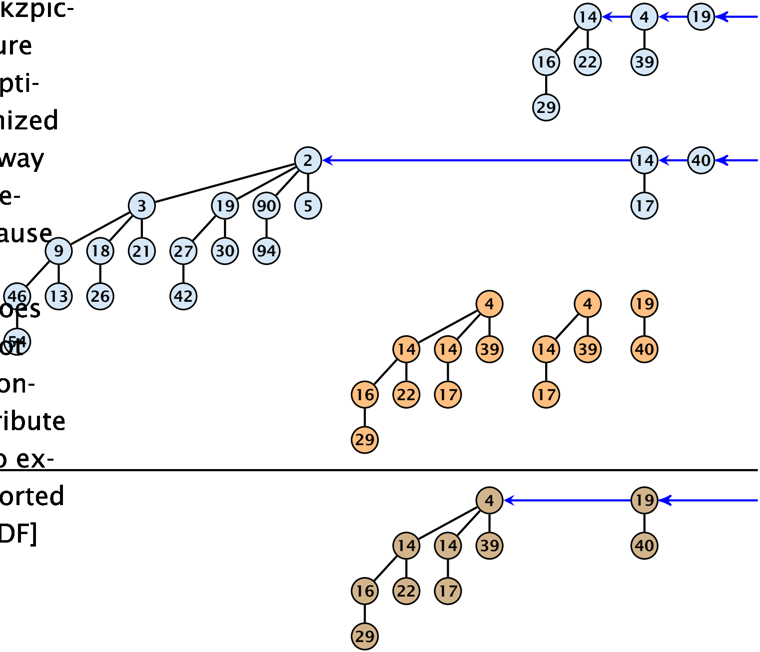

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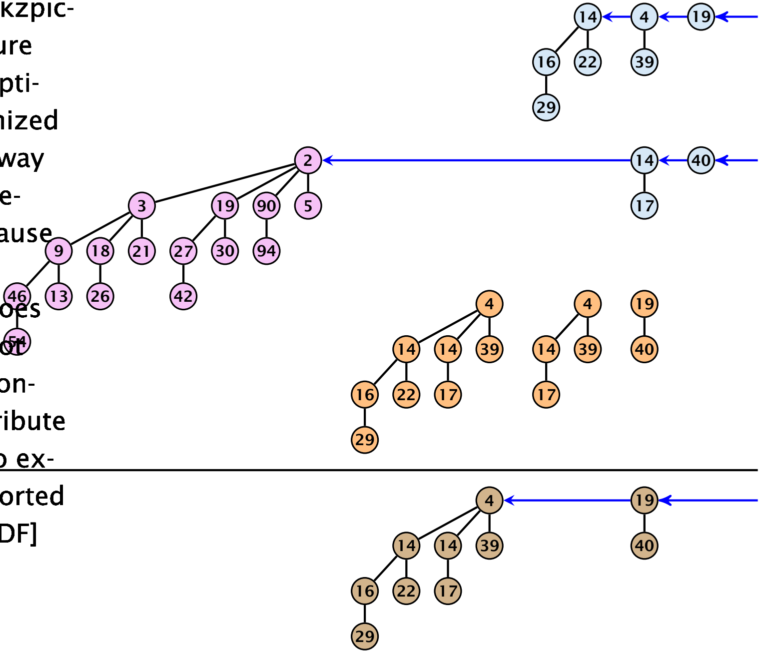

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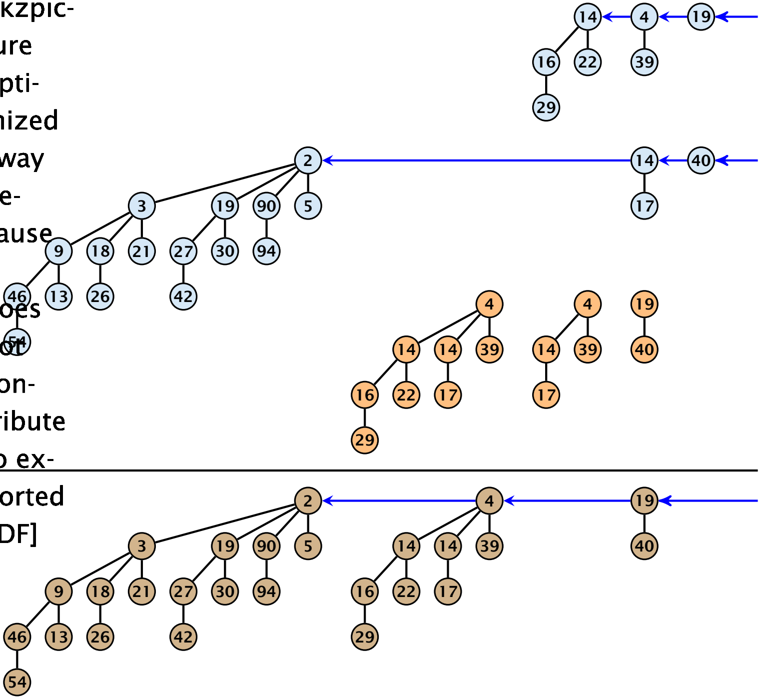

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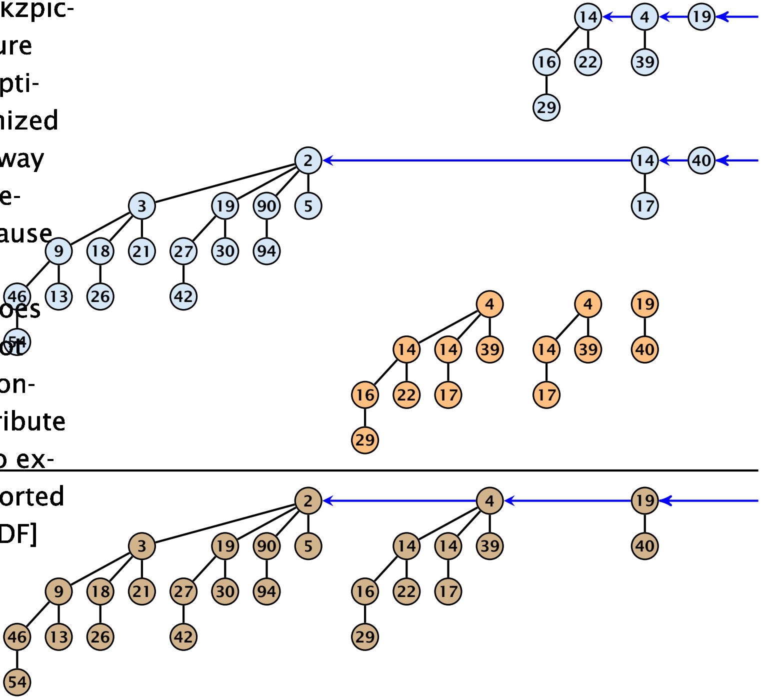

8.2 Binomial Heaps

$S_1.\mathbf{merge}(S_2)$:

- Analogous to binary addition.

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- Time is proportional to the number of trees in both heaps.

8.2 Binomial Heaps

$\boldsymbol{S_1.\mathrm{merge}(S_2)}$:

- Analogous to binary addition.
- Time is proportional to the number of trees in both heaps.
- Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

All other operations can be reduced to $\mathrm{merge}()$.

$\boldsymbol{S}.\,\mathbf{insert}(\boldsymbol{x})$**:**

- Create a new heap S' that contains just the element x.

8.2 Binomial Heaps

All other operations can be reduced to $\mathrm{merge}()$.

$S.\mathbf{insert}(x)$:

- Create a new heap S' that contains just the element x.
- Execute $S.\mathrm{merge}(S')$.

8.2 Binomial Heaps

All other operations can be reduced to $\text{merge}()$.

$S.\text{insert}(x)$:

- Create a new heap S' that contains just the element x.
- Execute $S.\text{merge}(S')$.
- Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

S. **minimum():**

- Find the minimum key-value among all roots.
- Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

S. delete-min():

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- Find the minimum key-value among all roots.
- Remove the corresponding tree $T_{\min}$ from the heap.
- Create a new heap S' that contains the trees obtained from $T_{\min}$ after deleting the root (note that these are just $\mathcal{O}(\log n)$ trees).

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- Compute $S.\mathrm{merge}(S')$.

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- Create a new heap S' that contains the trees obtained from $T_{\min}$ after deleting the root (note that these are just $\mathcal{O}(\log n)$ trees).
- Compute $S.\mathrm{merge}(S')$.
- Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

S. decrease-key(handle h):

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- Bubble the element up in the tree until the heap property is fulfilled.

8.2 Binomial Heaps

S. decrease-key(handle h):

- Decrease the key of the element pointed to by h.
- Bubble the element up in the tree until the heap property is fulfilled.
- Time: $\mathcal{O}(\log n)$ since the trees have height $\mathcal{O}(\log n)$.

S. delete(handle h):

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- Execute $S.\text{decrease-key}(h, -\infty)$.

8.2 Binomial Heaps

S. delete(handle h):

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- Execute $S.\text{delete-min}()$.

8.2 Binomial Heaps

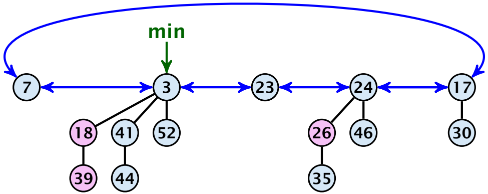
S. delete(handle h):

- Execute $S.\text{decrease-key}(h, -\infty)$.
- Execute $S.\text{delete-min}()$.
- Time: $\mathcal{O}(\log n)$.

8.3 Fibonacci Heaps

Collection of trees that fulfill the heap property.

Structure is much more relaxed than binomial heaps.



8.3 Fibonacci Heaps

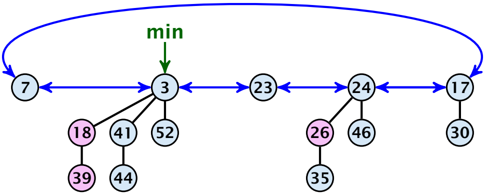
Additional implementation details:

- Every node x stores its degree in a field $x.\mathrm{degree}$. Note that this can be updated in constant time when adding a child to x.
- Every node stores a boolean value $x.\mathrm{marked}$ that specifies whether x is marked or not.

8.3 Fibonacci Heaps

The potential function:

- $t(S)$ denotes the number of trees in the heap.
- $m(S)$ denotes the number of marked nodes.
- We use the potential function $\Phi(S) = t(S) + 2m(S)$.



The potential is $\Phi(S) = 5 + 2 \cdot 3 = 11$.

We assume that one unit of potential can pay for a constant amount of work, where the constant is chosen “big enough” (to take care of the constants that occur).

To make this more explicit we use c to denote the amount of work that a unit of potential can pay for.

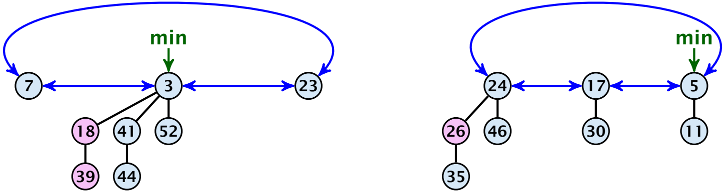
S. **minimum()**

- Access through the min-pointer.
- Actual cost $\mathcal{O}(1)$.
- No change in potential.
- Amortized cost $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

S. merge(S')

- Merge the root lists.
- Adjust the min-pointer



8.3 Fibonacci Heaps

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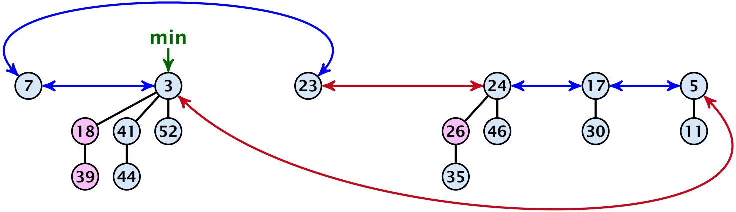

Running time:

- Actual cost $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

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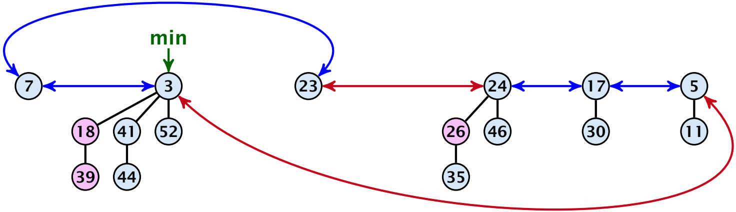
- Adjust the min-pointer



Running time:

- Actual cost $\mathcal{O}(1)$.
- No change in potential.

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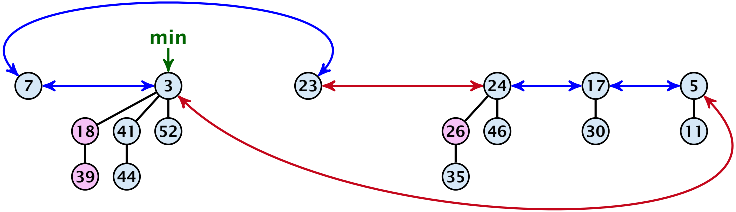


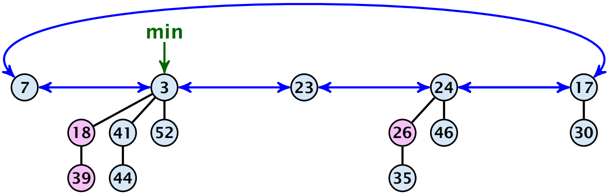
Running time:

- Actual cost $\mathcal{O}(1)$.
- No change in potential.
- Hence, amortized cost is $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

$S.$ insert(x)

- Create a new tree containing x.
- Insert x into the root-list.
- Update min-pointer, if necessary.



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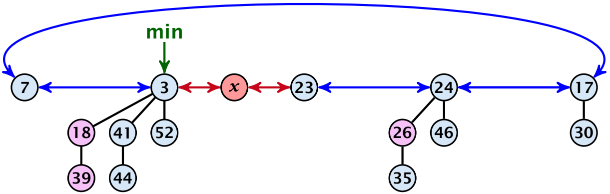

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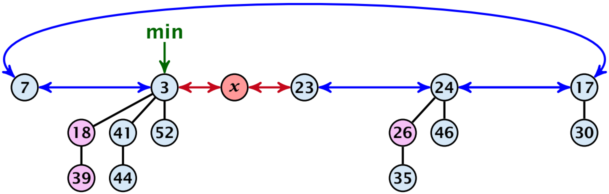
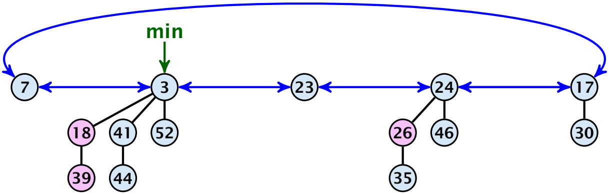


Running time:

- Actual cost $\mathcal{O}(1)$.
- Change in potential is $+1$.
- Amortized cost is $c + \mathcal{O}(1) = \mathcal{O}(1)$.

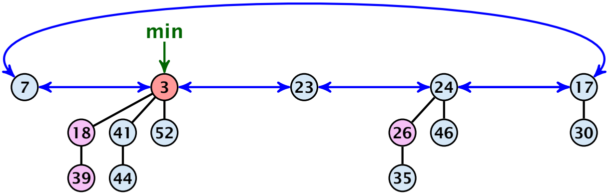
8.3 Fibonacci Heaps

S. delete-min(x)



S. **delete-min**(x)

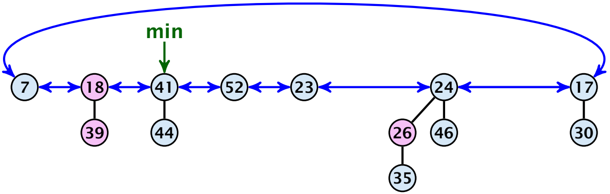
- Delete minimum; add child-trees to heap; time: $D(\min) \cdot \mathcal{O}(1)$.



8.3 Fibonacci Heaps

S. delete-min(x)

- Delete minimum; add child-trees to heap; time: $D(\min) \cdot \mathcal{O}(1)$.
- Update min-pointer; time: $(t + D(\min)) \cdot \mathcal{O}(1)$.



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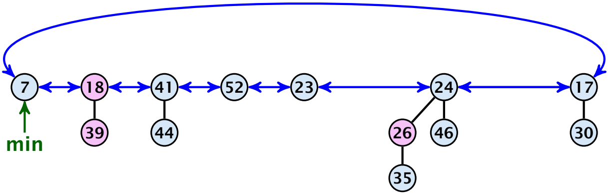

8.3 Fibonacci Heaps

S. delete-min(x)

- Delete minimum; add child-trees to heap; time: $D(\min) \cdot \mathcal{O}(1)$.
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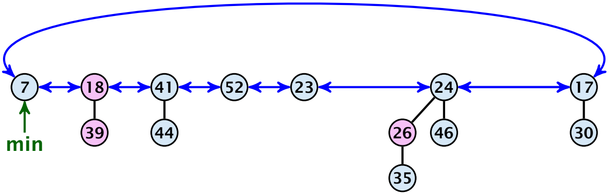
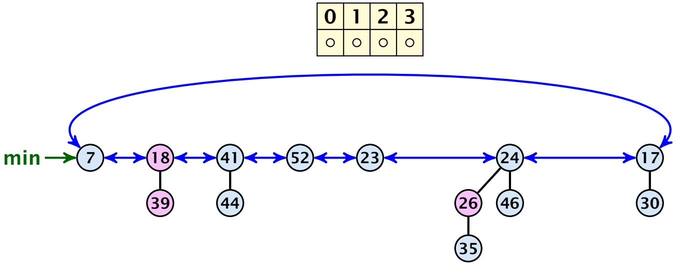


- Consolidate root-list so that no roots have the same degree. Time $t \cdot \mathcal{O}(1)$ (see next slide).

8.3 Fibonacci Heaps

Consolidate:



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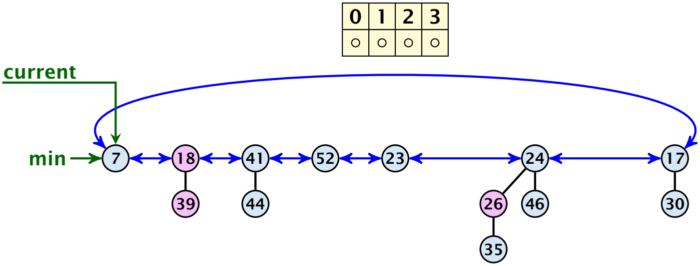

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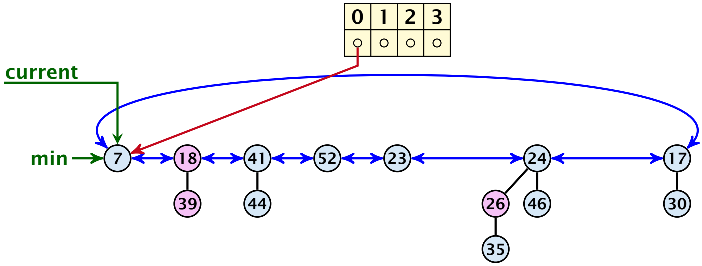

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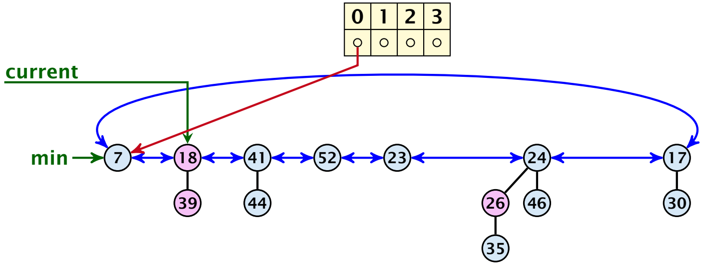

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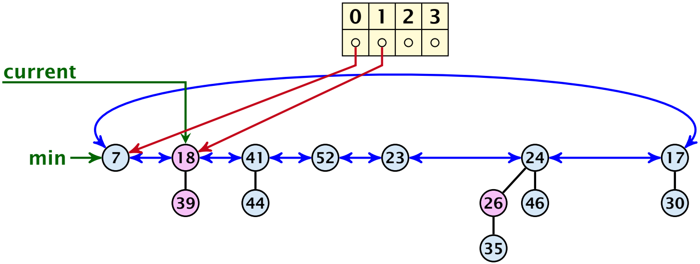

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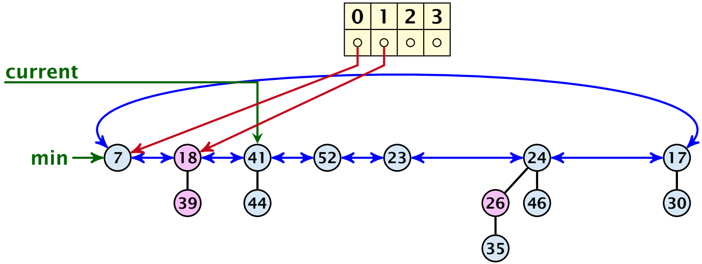

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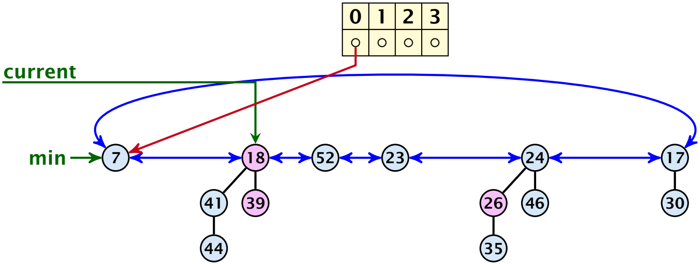

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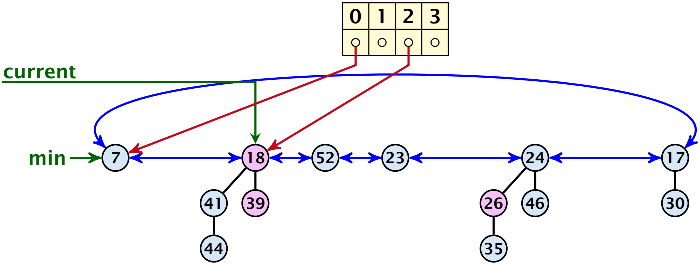

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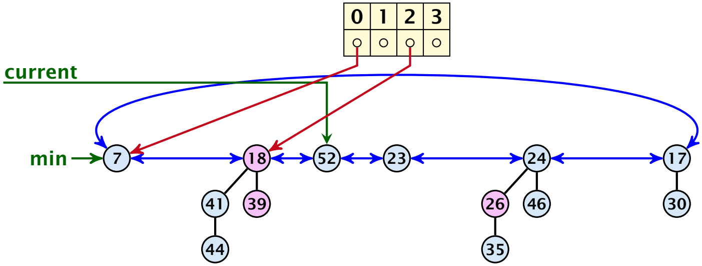

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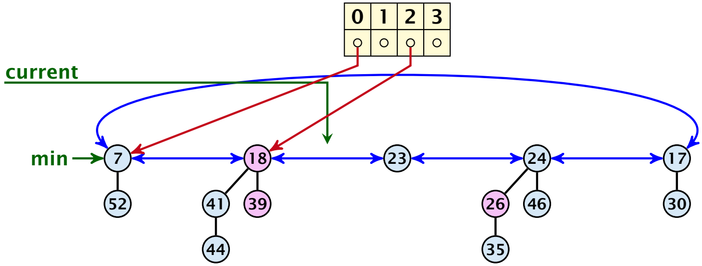

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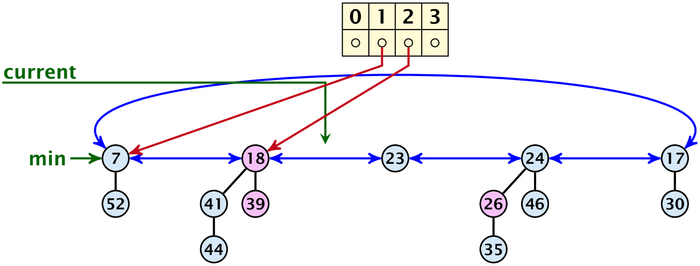

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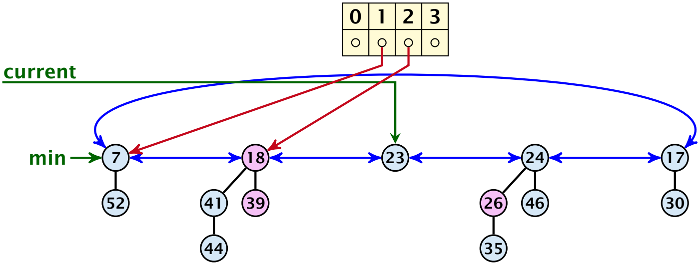

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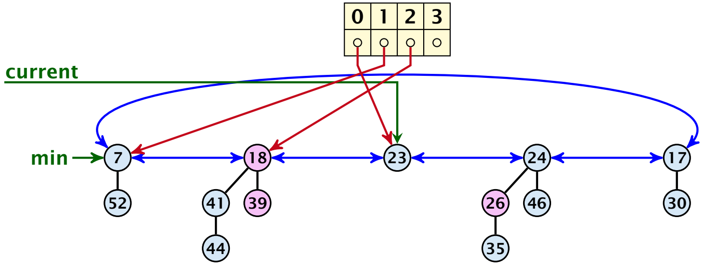

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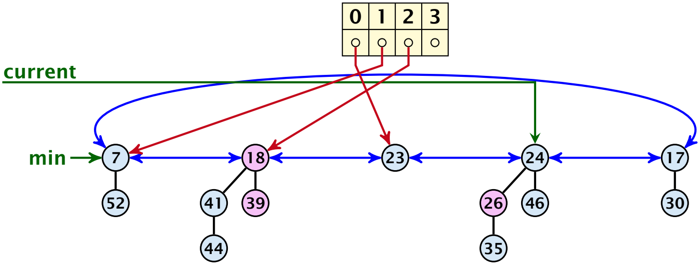

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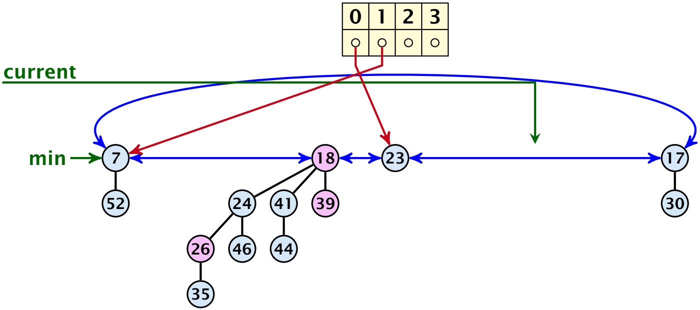

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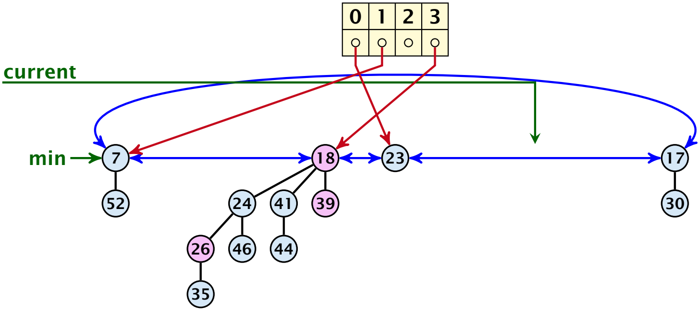

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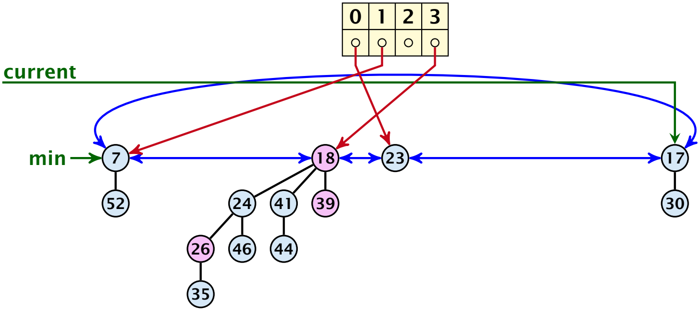

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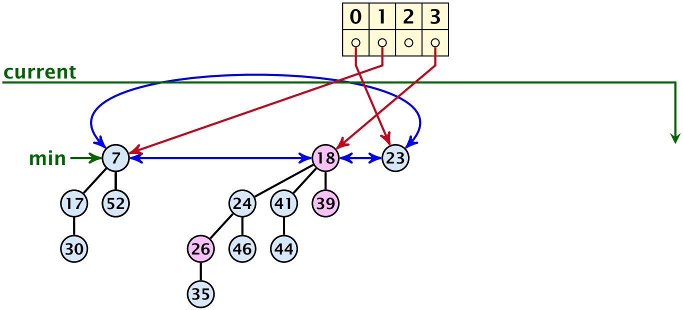

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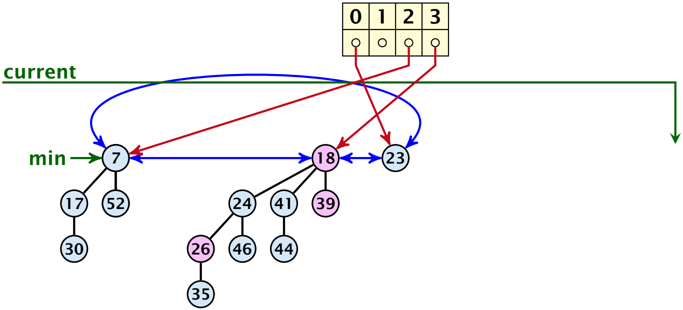

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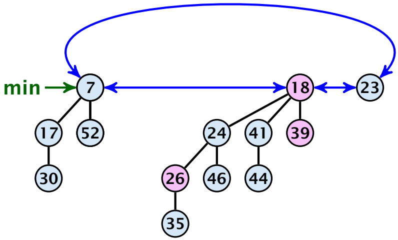

8.3 Fibonacci Heaps

Actual cost for delete-min()

- At most $D_n + t$ elements in root-list before consolidate.

8.3 Fibonacci Heaps

Actual cost for delete-min()

- At most $D_n + t$ elements in root-list before consolidate.
- Actual cost for a delete-min is at most $\mathcal{O}(1) \cdot (D_n + t)$. Hence, there exists c_1 s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

8.3 Fibonacci Heaps

Actual cost for delete-min()

- At most $D_n + t$ elements in root-list before consolidate.
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  Hence, there exists c_1 s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

Amortized cost for delete-min()

- $t' \leq D_n + 1$ as degrees are different after consolidating.

8.3 Fibonacci Heaps

Actual cost for delete-min()

- At most $D_n + t$ elements in root-list before consolidate.
- Actual cost for a delete-min is at most $\mathcal{O}(1) \cdot (D_n + t)$.
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\begin{aligned}
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for $c \geq c_1$.

If the input trees of the consolidation procedure are binomial trees (for example only singleton vertices) then the output will be a set of distinct binomial trees, and, hence, the Fibonacci heap will be (more or less) a Binomial heap right after the consolidation.

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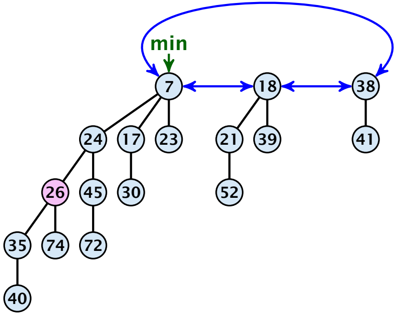
If we do not have delete or decrease-key operations then $D_n \leq \log n$.

Fibonacci Heaps: decrease-key(handle h, v)



Case 1: decrease-key does not violate heap-property

- Just decrease the key-value of element referenced by h. Nothing else to do.

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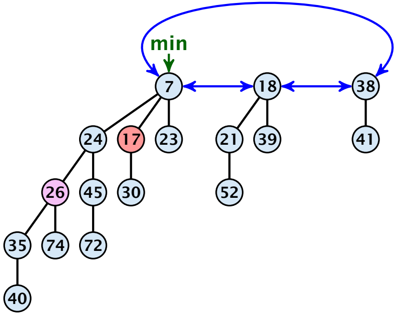

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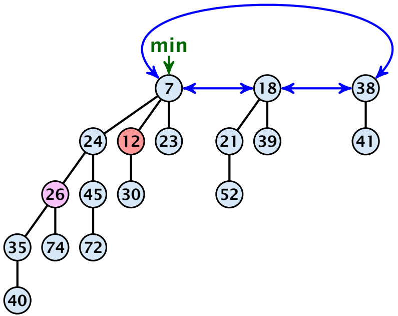

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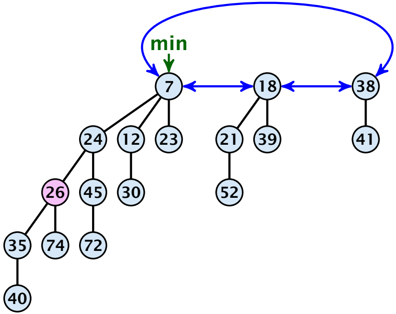

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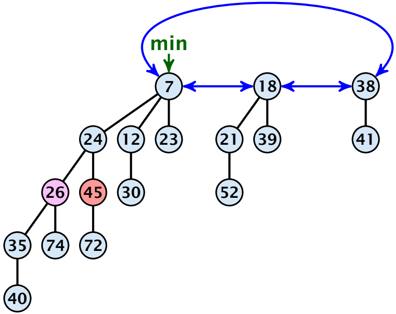

Case 2: heap-property is violated, but parent is not marked

- Decrease key-value of element x reference by h.
- If the heap-property is violated, cut the parent edge of x, and make x into a root.
- Adjust min-pointers, if necessary.
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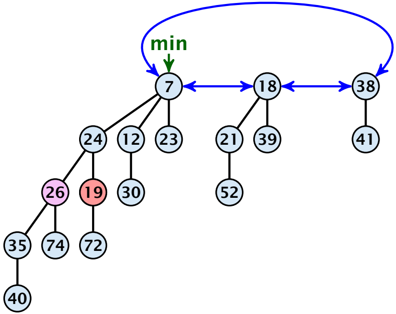

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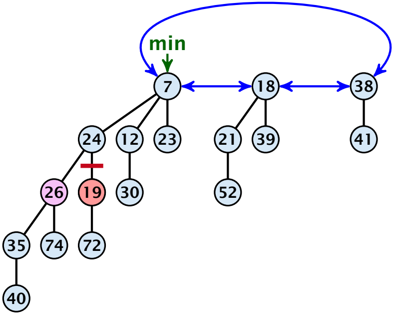

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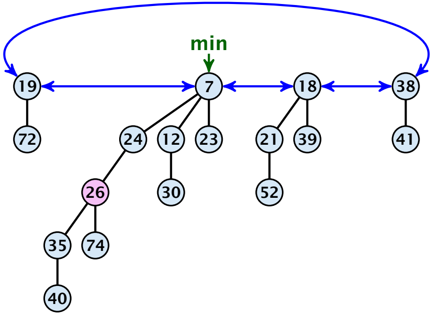

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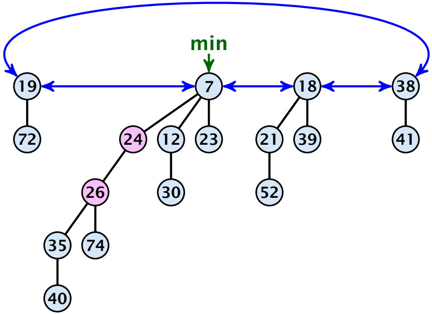

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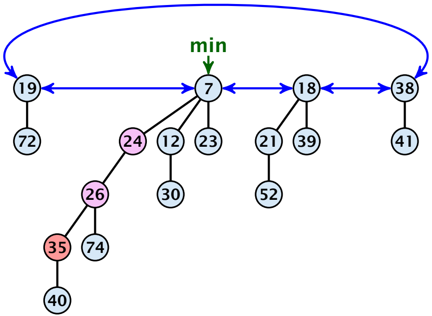

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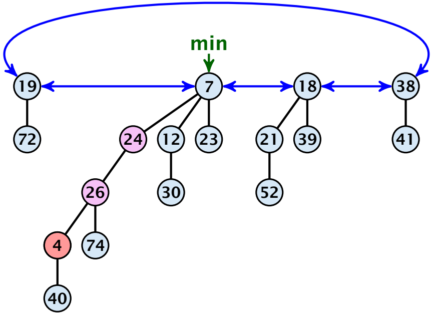

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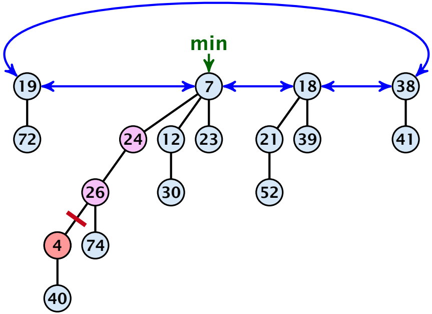

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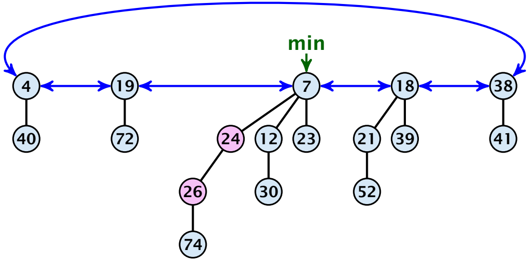

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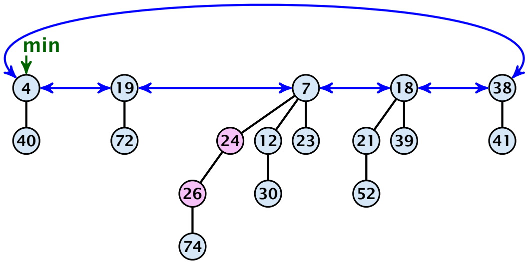

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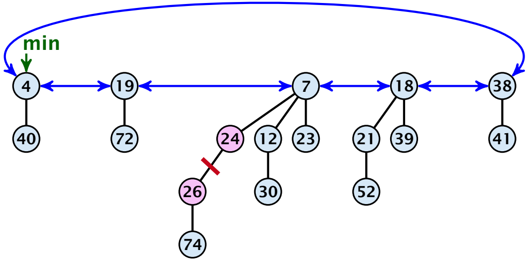

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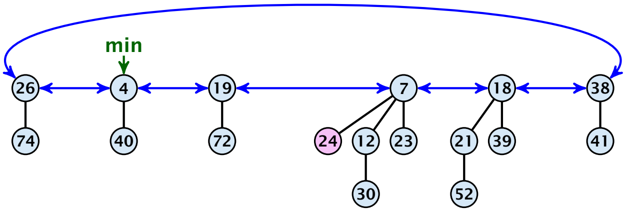

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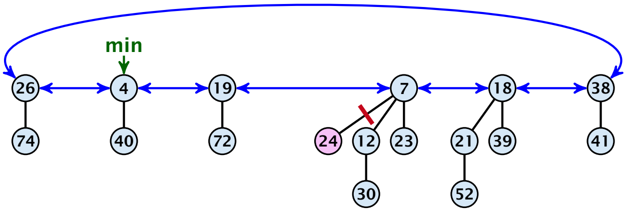

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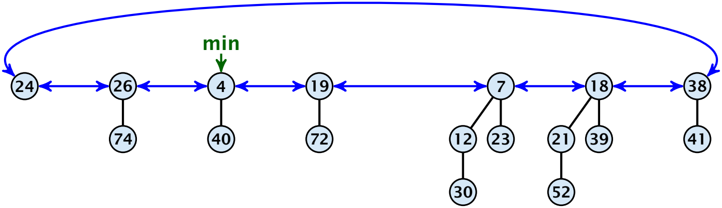

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- Execute the following:

  $p \leftarrow \text{parent}[x]$;
  while (p is marked)
      $pp \leftarrow \text{parent}[p]$;
      cut of p; make it into a root; unmark it;
      $p \leftarrow pp$;
  if p is unmarked and not a root mark it;

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  $$c_2(\ell+1)+c(4-\ell) \leq (c_2-c)\ell+4c+c_2 = \mathcal{O}(1)\,,$$

  if $c \geq c_2$.

Delete node

$H.\mathbf{delete}(x)$:

- decrease value of x to $-\infty$.
- delete-min.

Amortized cost: $\mathcal{O}(D_n)$

- $\mathcal{O}(1)$ for decrease-key.
- $\mathcal{O}(D_n)$ for delete-min.

Lemma 1

Let x be a node with degree k and let $y_1, \ldots, y_k$ denote the children of x in the order that they were linked to x. Then

$$\mathrm{degree}(y_i) \geq \begin{cases} 0 & \textit{if } i = 1 \\ i - 2 & \textit{if } i > 1 \end{cases}$$

Proof

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- Therefore, $\text{degree}(y_i) \geq i - 2$.

8.3 Fibonacci Heaps

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$$

8.3 Fibonacci Heaps

$\phi = \frac{1}{2}(1+\sqrt{5})$ denotes the *golden ratio*. Note that $\phi^2 = 1 + \phi$.

Definition 2

Consider the following non-standard Fibonacci type sequence:

$$F_k = \begin{cases} 1 & \text{if } k = 0 \\ 2 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

Facts:

1. $F_k \geq \phi^k$.
2. For $k \geq 2$: $F_k = 2 + \sum_{i=0}^{k-2} F_i$.

The above facts can be easily proved by induction. From this it follows that $s_k \geq F_k \geq \phi^k$, which gives that the maximum degree in a Fibonacci heap is logarithmic.

k=0: $1 = F_0 \geq \Phi^0 = 1$

k=1: $2 = F_1 \geq \Phi^1 \approx 1.61$

k-2,k-1→ k: $F_k = F_{k-1} + F_{k-2} \geq \Phi^{k-1} + \Phi^{k-2} = \Phi^{k-2}\overbrace{(\Phi + 1)}^{\Phi^2} = \Phi^k$

k=2: $3 = F_2 = 2 + 1 = 2 + F_0$

k-1→ k: $F_k = F_{k-1} + F_{k-2} = 2 + \sum_{i=0}^{k-3} F_i + F_{k-2} = 2 + \sum_{i=0}^{k-2} F_i$