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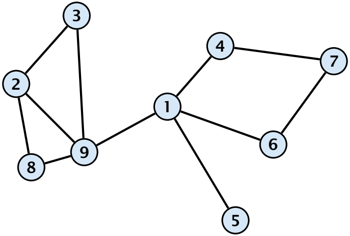

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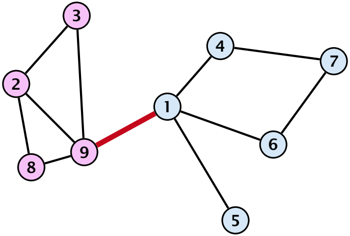

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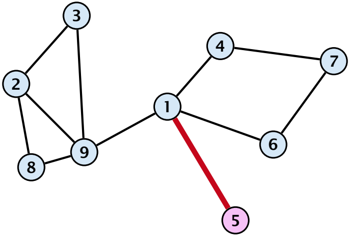

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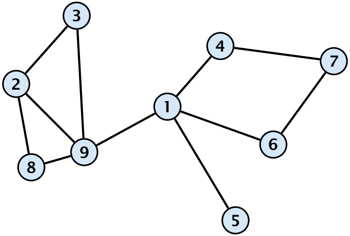

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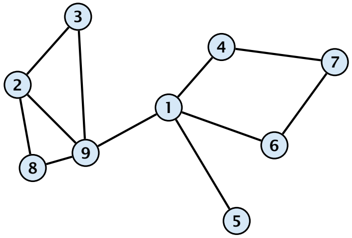

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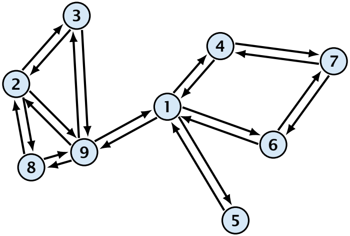

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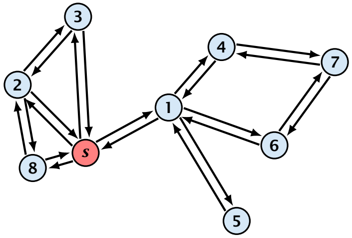

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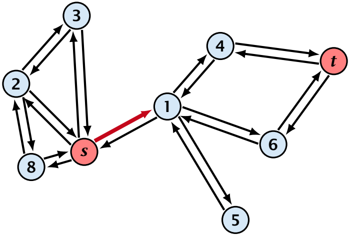

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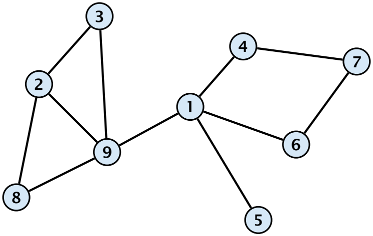

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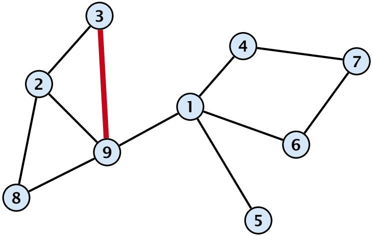

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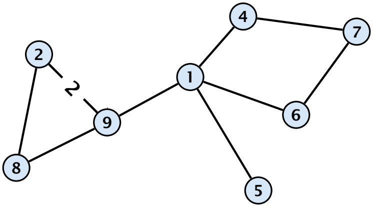

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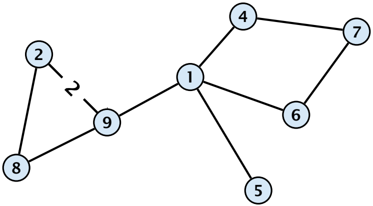

- Edge-contractions do no decrease the size of the mincut.

Edge Contractions

We can perform an edge-contraction in time $\mathcal{O}(n)$.

Randomized Mincut Algorithm

Algorithm 1 KargerMincut($G = (V, E, c)$)

for $i = 1 \to n - 2$ **do**
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- What is the probability that this algorithm returns a mincut?

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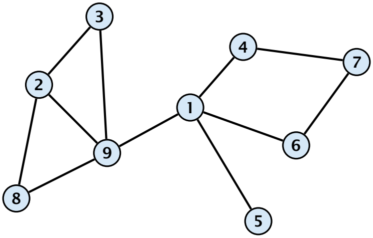

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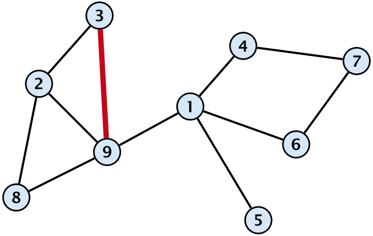

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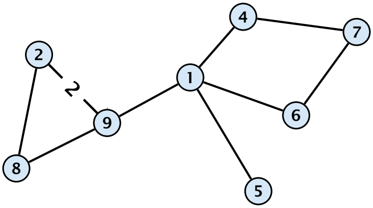

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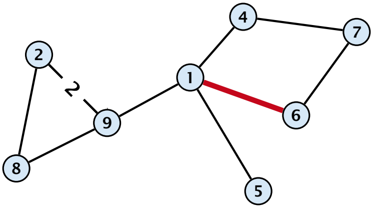

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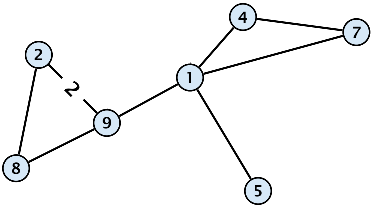

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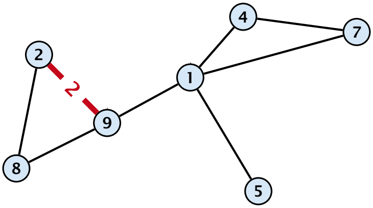

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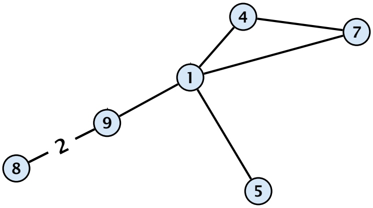

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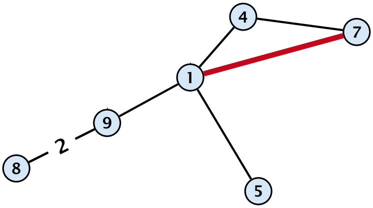

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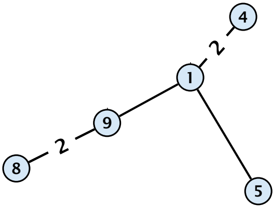

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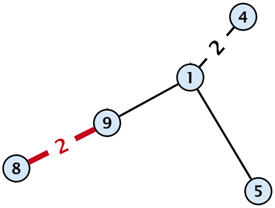

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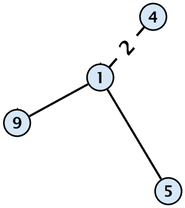

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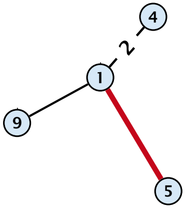

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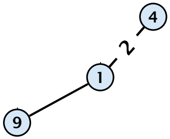

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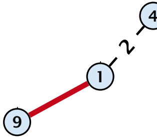

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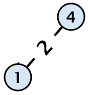

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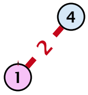

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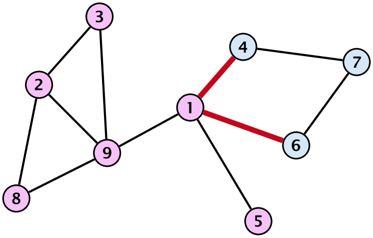

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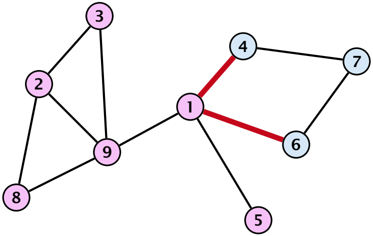

What is the probability that this algorithm returns a mincut?

Analysis

What is the probability that a given mincut A is still possible after round i?

- It is still possible to obtain cut A in the end if so far no edge in $(A, V \setminus A)$ has been contracted.

Analysis

What is the probability that we select an edge from A in iteration i?

$n - i + 1$ is the number of nodes in graph $G_{n-i+1} = (V_{n-i+1}, E_{n-i+1})$, the graph at the start of iteration i.

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- Hence, the probability of choosing an edge from the cut is at most $\min / c(E) \leq 2/(n - i + 1)$.

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Choosing $t=2$ gives that with probability $1/\binom{n}{2}$ the algorithm computes a mincut.

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Theorem 7

The randomized mincut algorithm computes an optimal cut with high probability. The total running time is $\mathcal{O}(n^4\log n)$.

Improved Algorithm

Algorithm 2 RecursiveMincut$(G = (V, E, c))$

for $i = 1 \to n - n/\sqrt{2}$ **do**
    choose $e \in E$ randomly with probability $c(e)/c(E)$
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if $|V| = 2$ **return** cut-value;
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Note that the above implementation only works for very special values of n.

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- This gives $T(n) = \mathcal{O}(n^2 \log n)$.

Note that the above implementation only works for very special values of n.

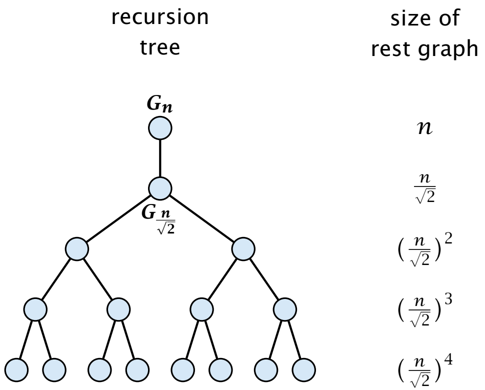
Probability of Success

The probability of not contracting an edge from the mincut during one iteration through the for-loop is at least

$$\frac{t(t-1)}{n(n-1)} \geq \frac{t^2}{n^2} = \frac{1}{2} \ ,$$

as $t = \frac{n}{\sqrt{2}}$.

Probability of Success



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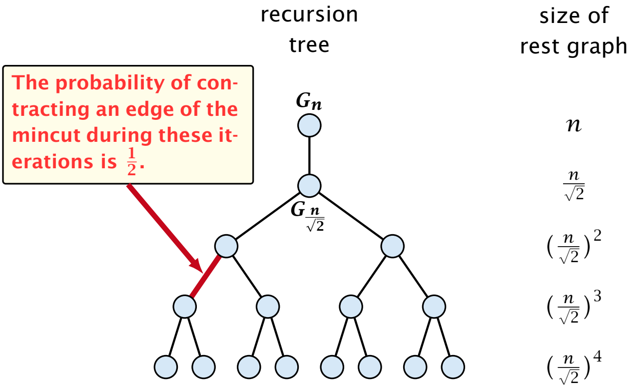

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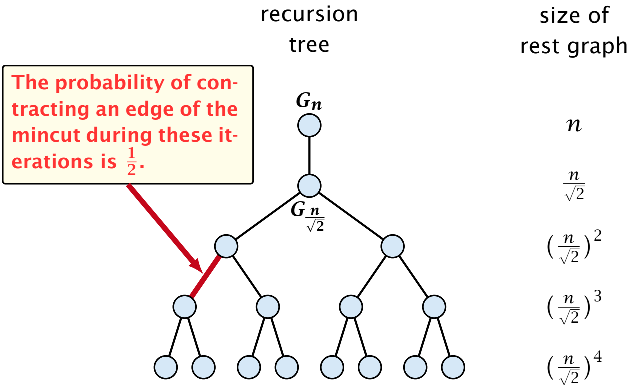

We can estimate the success probability by using the following game on the recursion tree. Delete every edge with probability $\frac{1}{2}$. If in the end you have a path from the root to at least one leaf node you are successful.

Probability of Success

Let for an edge e in the recursion tree, $h(e)$ denote the height (distance to leaf level) of the parent-node of e (end-point that is higher up in the tree). Let h denote the height of the root node.

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Call an edge e alive if there exists a path from the parent-node of e to a descendant leaf, after we randomly deleted edges. Note that an edge can only be alive if it hasn't been deleted.

Probability of Success

Let for an edge e in the recursion tree, $h(e)$ denote the height (distance to leaf level) of the parent-node of e (end-point that is higher up in the tree). Let h denote the height of the root node.

Call an edge e alive if there exists a path from the parent-node of e to a descendant leaf, after we randomly deleted edges. Note that an edge can only be alive if it hasn't been deleted.

Lemma 8

The probability that an edge e is alive is at least $\frac{1}{h(e)+1}$.

Probability of Success

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$$p_d = \frac{1}{2}\left(2p_{d-1} - p_{d-1}^2\right) \quad \boxed{\Pr[A \vee B] = \Pr[A] + \Pr[B] - \Pr[A \wedge B]}$$

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$$
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&= p_{d-1} - \frac{p_{d-1}^2}{2}
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Lemma 9

One run of the algorithm can be performed in time $\mathcal{O}(n^2 \log n)$ *and has a success probability of* $\Omega(\frac{1}{\log n})$.

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One run of the algorithm can be performed in time $\mathcal{O}(n^2 \log n)$ *and has a success probability of* $\Omega(\frac{1}{\log n})$.

Doing $\Theta(\log^2 n)$ *runs gives that the algorithm succeeds with high probability. The total running time is* $\mathcal{O}(n^2 \log^3 n)$.