7 Dictionary

Dictionary:

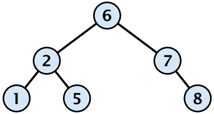
- $\boldsymbol{S}.\,\textbf{insert}(\boldsymbol{x})$: Insert an element x.
- $\boldsymbol{S}.\,\textbf{delete}(\boldsymbol{x})$: Delete the element pointed to by x.
- $\boldsymbol{S}.\,\textbf{search}(\boldsymbol{k})$: Return a pointer to an element e with $\text{key}[e] = k$ in S if it exists; otherwise return null.

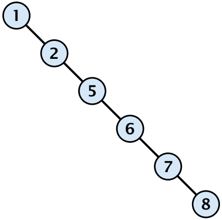
7.1 Binary Search Trees

An (internal) binary search tree stores the elements in a binary tree. Each tree-node corresponds to an element. All elements in the left sub-tree of a node v have a smaller key-value than $\mathrm{key}[v]$ and elements in the right sub-tree have a larger-key value. We assume that all key-values are different.

(External Search Trees store objects only at leaf-vertices)

Examples:



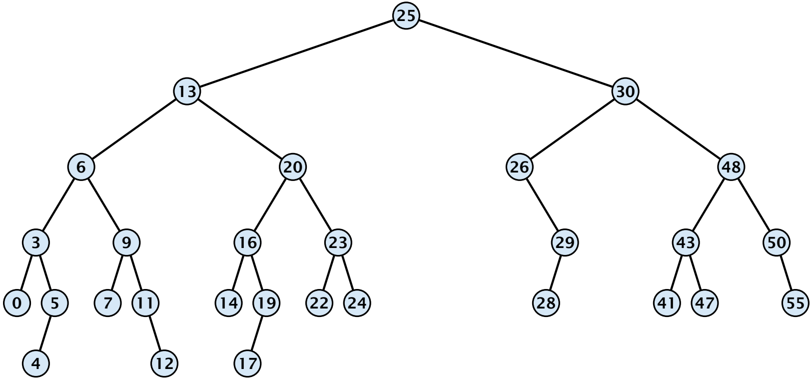


7.1 Binary Search Trees

We consider the following operations on binary search trees. Note that this is a super-set of the dictionary-operations.

- $T.\mathrm{insert}(x)$
- $T.\mathrm{delete}(x)$
- $T.\mathrm{search}(k)$
- $T.\mathrm{successor}(x)$
- $T.\mathrm{predecessor}(x)$
- $T.\mathrm{minimum}()$
- $T.\mathrm{maximum}()$

Binary Search Trees: Searching

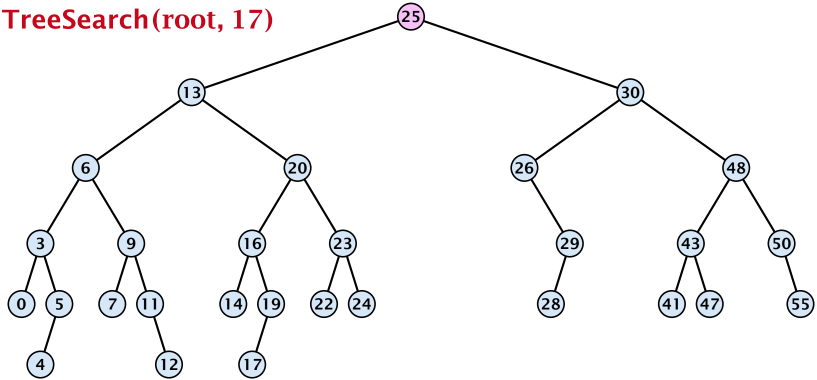


Algorithm 1 TreeSearch(x, k)

if $x = \text{null}$ **or** $k = \text{key}[x]$ **return** x
if $k < \text{key}[x]$ **return** TreeSearch$(\text{left}[x], k)$
else return TreeSearch$(\text{right}[x], k)$

Binary Search Trees: Searching

TreeSearch(root, 17)



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if x = null **or** k = key$[x]$ **return** x
if $k <$ key$[x]$ **return** TreeSearch(left$[x]$, k)
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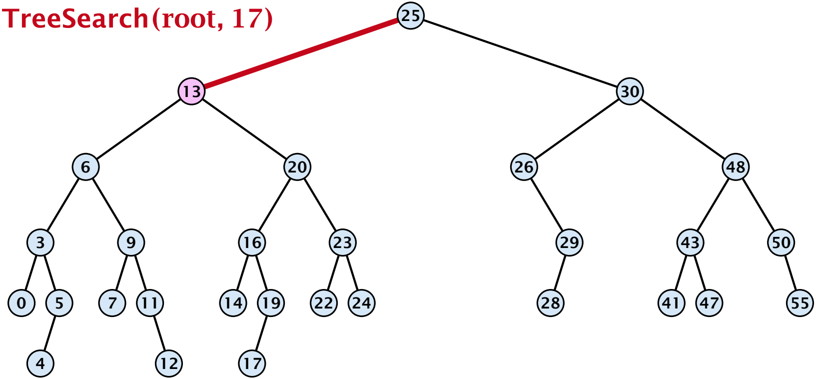
Binary Search Trees: Searching

TreeSearch(root, 17)



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Binary Search Trees: Searching

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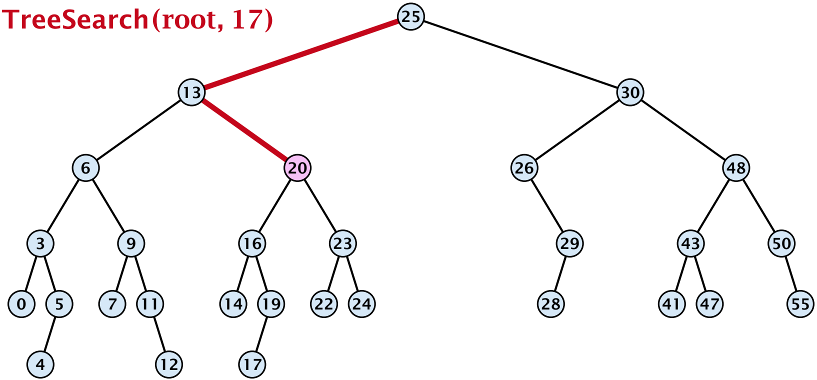

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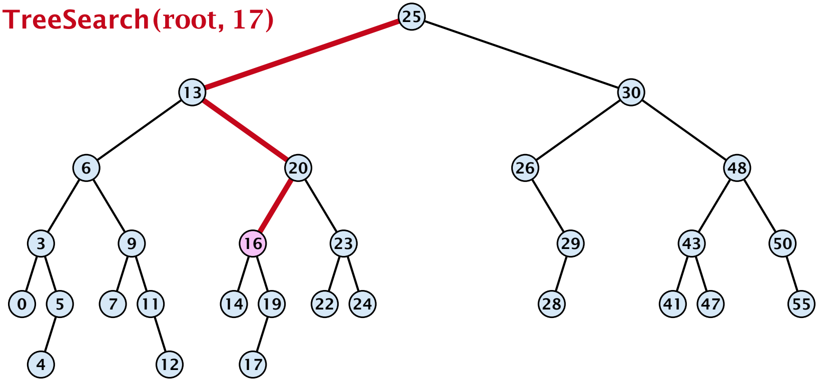
Binary Search Trees: Searching



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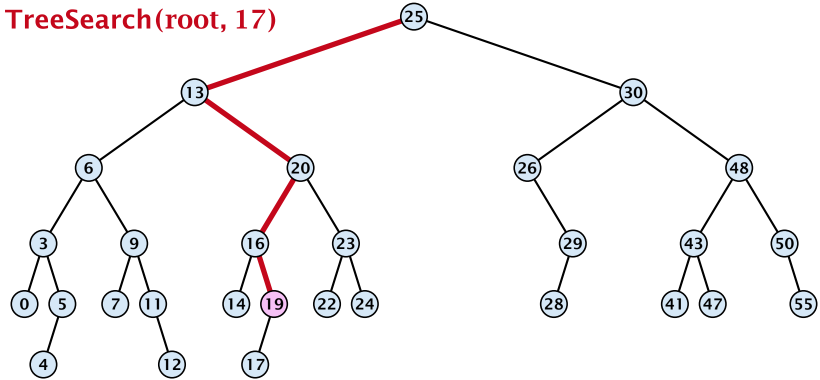
Binary Search Trees: Searching



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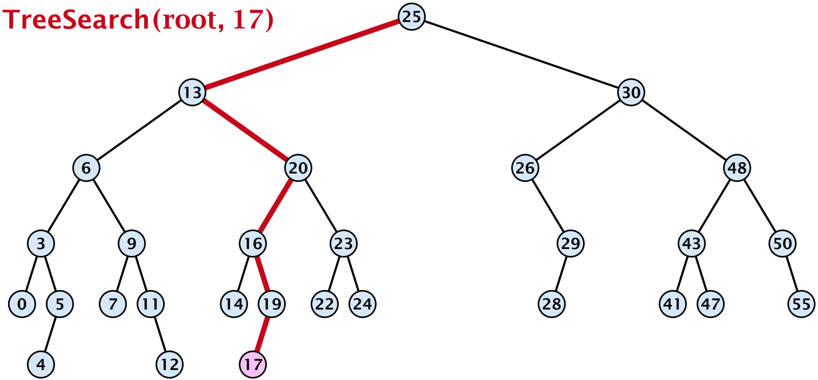
Binary Search Trees: Searching



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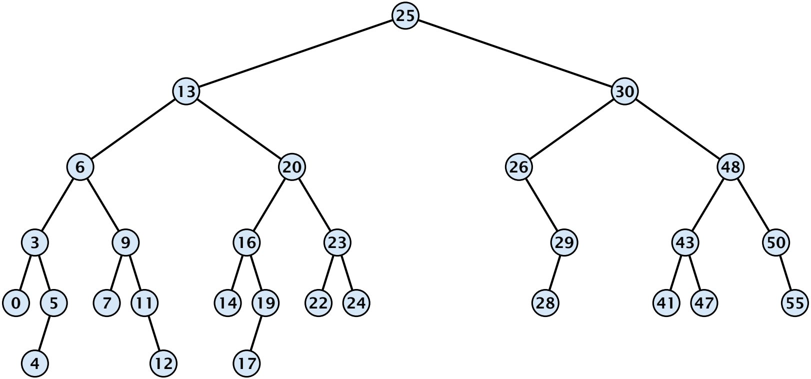
Binary Search Trees: Searching



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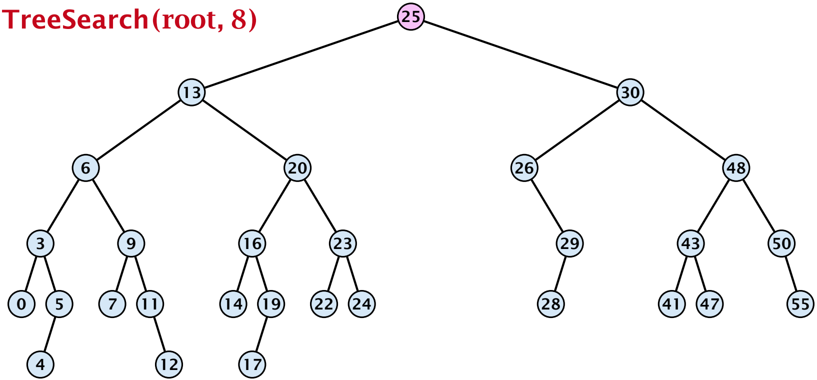
Binary Search Trees: Searching

TreeSearch(root, 8)



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Binary Search Trees: Searching

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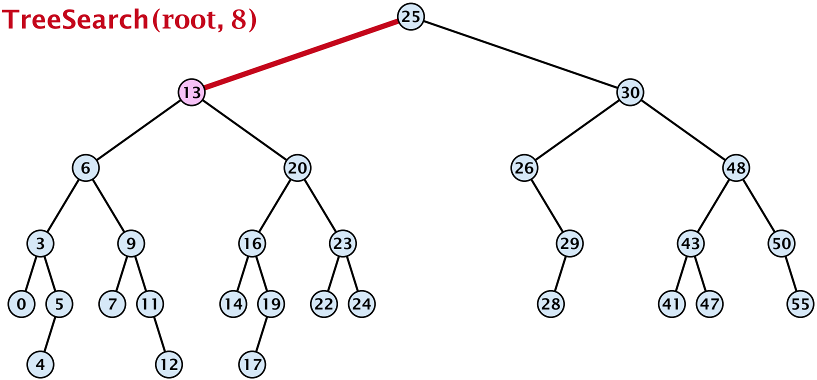

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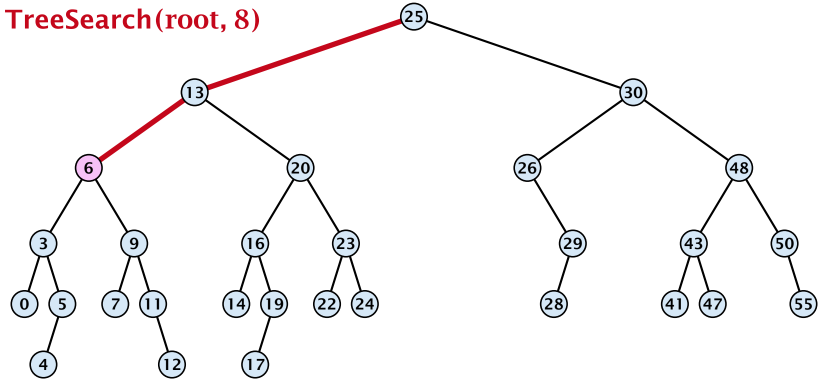

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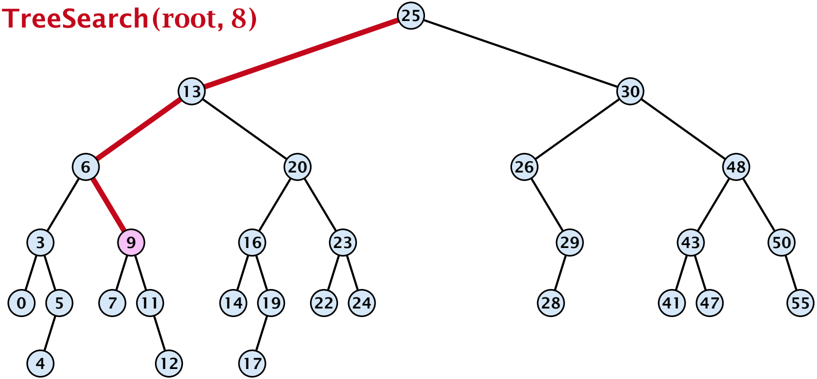

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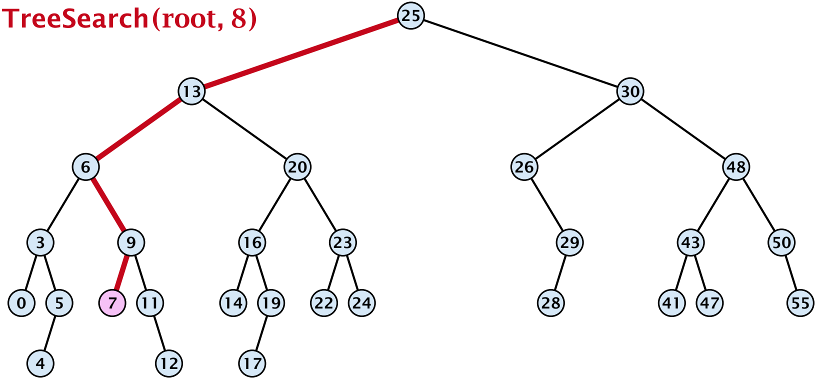
```

Binary Search Trees: Searching

TreeSearch(root, 8)



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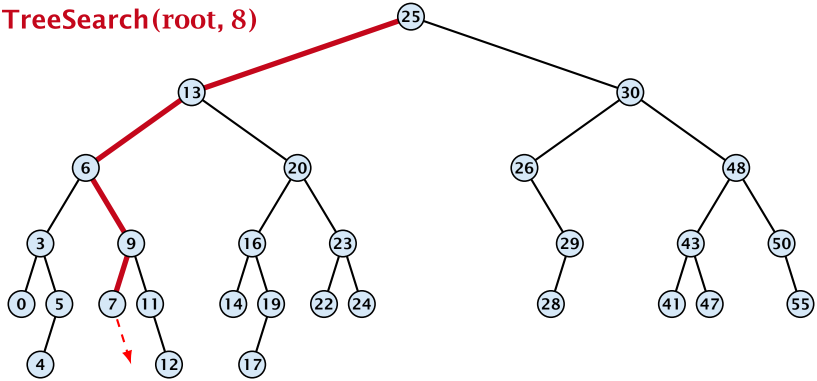

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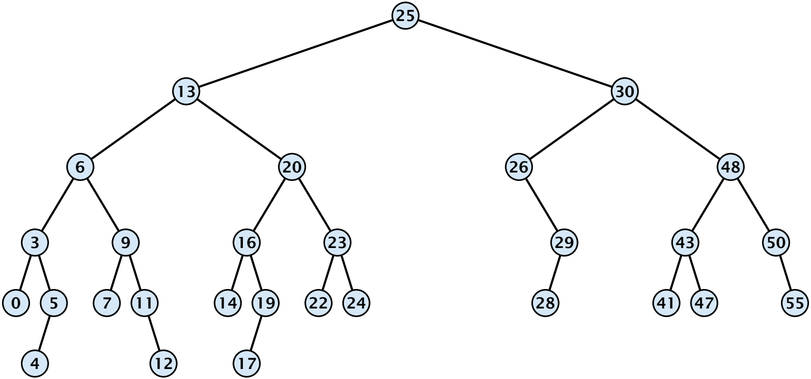
Binary Search Trees: Minimum

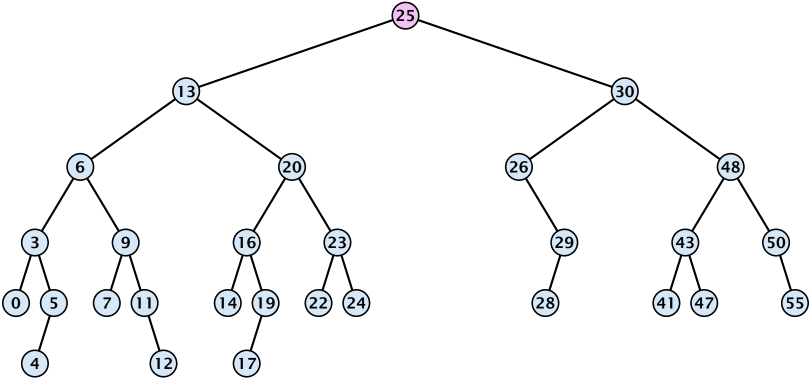


Algorithm 2 TreeMin(x)

1: **if** $x = \text{null}$ **or** $\text{left}[x] = \text{null}$ **return** x

2: **return** TreeMin$(\text{left}[x])$

Binary Search Trees: Minimum



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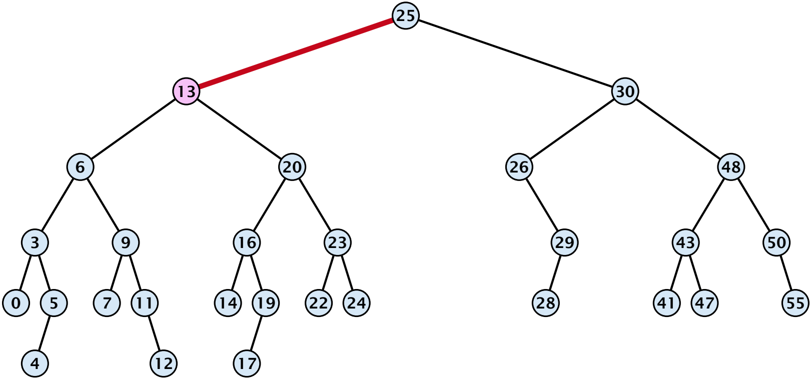
2: **return** TreeMin($\text{left}[x]$)

Binary Search Trees: Minimum



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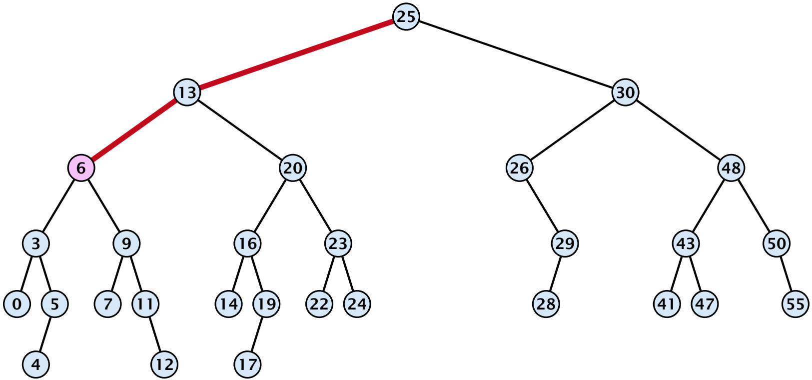

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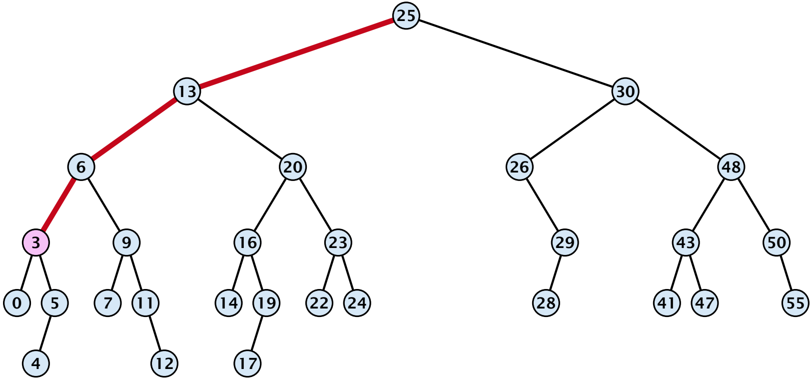

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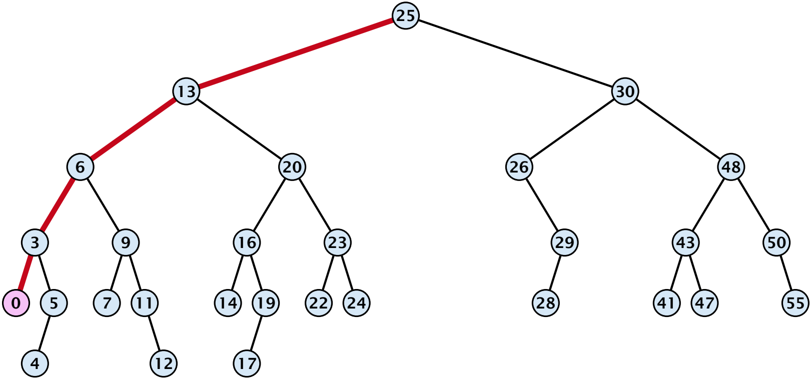
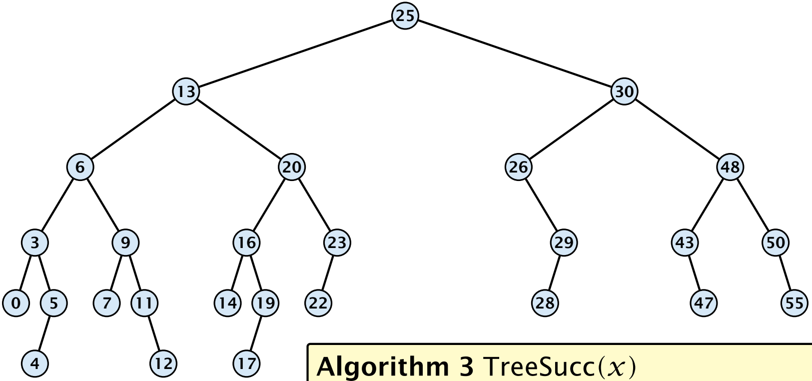


Algorithm 2 TreeMin(x)

1: **if** x = null **or** left[x] = null **return** x

2: **return** TreeMin(left[x])

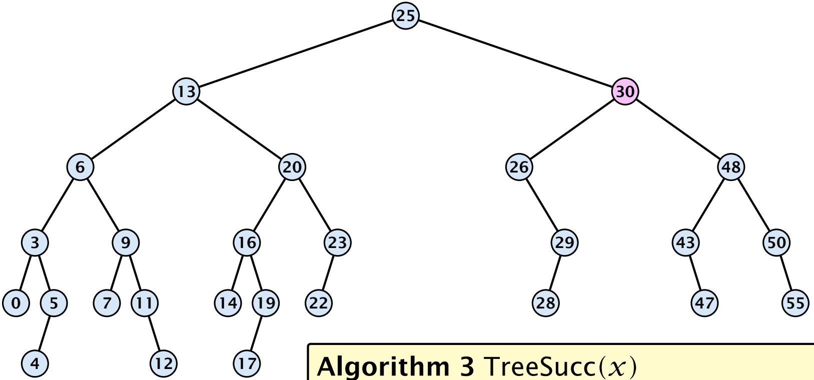
Binary Search Trees: Successor



Algorithm 3 TreeSucc(x)

1: **if** right[x] $\neq$ null **return** TreeMin(right[x])
2: $y \leftarrow$ parent[x]
3: **while** $y \neq$ null **and** $x =$ right[y] **do**
4: $\quad x \leftarrow y; y \leftarrow$ parent[x]
5: **return** y;

Binary Search Trees: Successor



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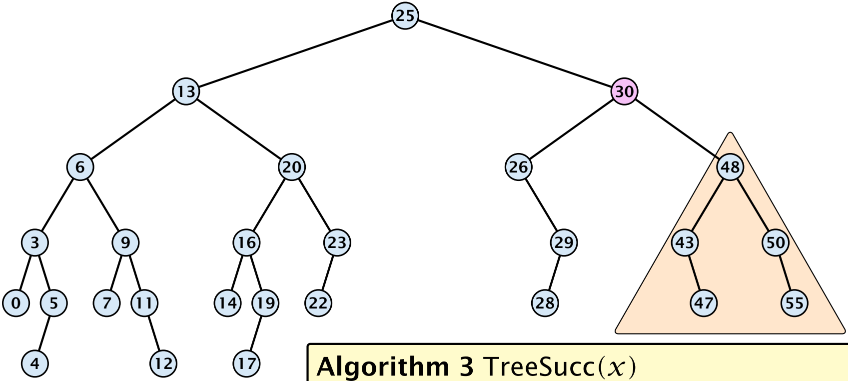

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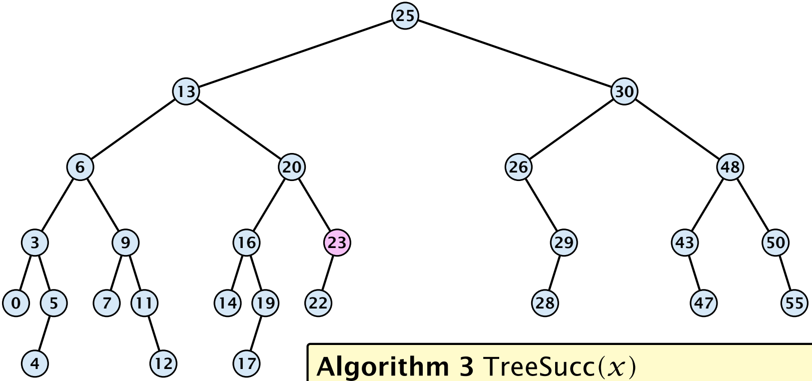

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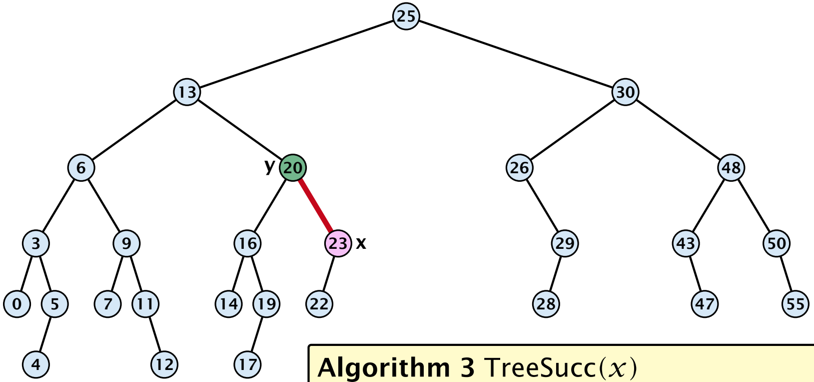
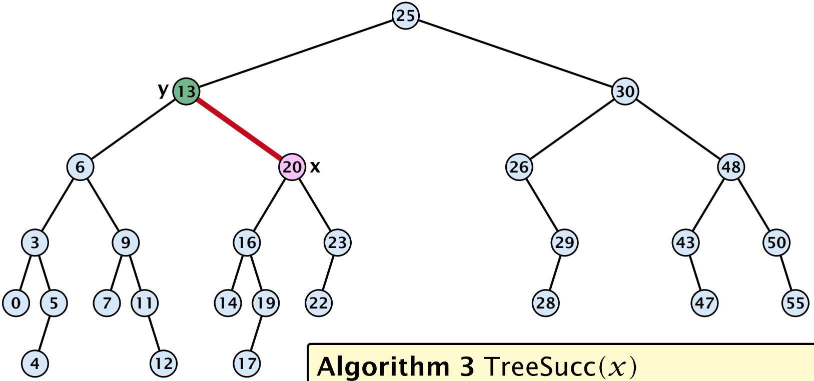


Algorithm 3 TreeSucc(x)

```
if right[x] ≠ null return TreeMin(right[x])
y ← parent[x]
while y ≠ null and x = right[y] do
    x ← y; y ← parent[x]
return y;
```

Binary Search Trees: Successor



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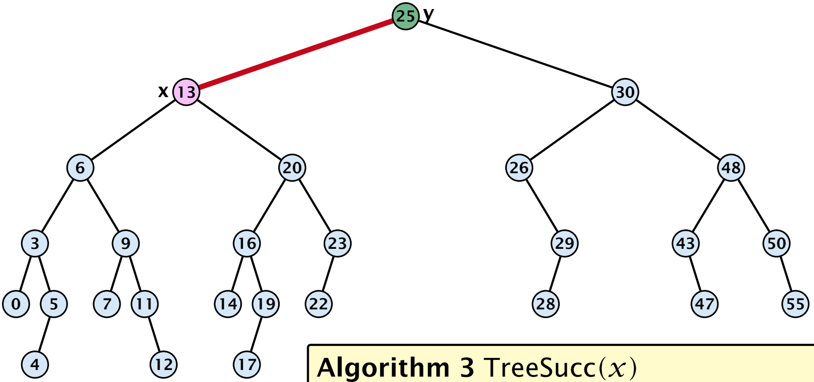

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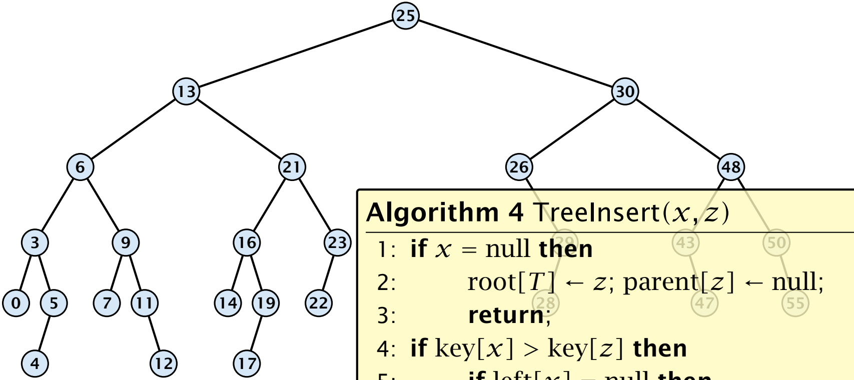
Binary Search Trees: Insert



Algorithm 4 TreeInsert(x, z)

---

if $x = \text{null}$ **then**
  $\text{root}[T] \leftarrow z$; $\text{parent}[z] \leftarrow \text{null}$;
  return;
if $\text{key}[x] > \text{key}[z]$ **then**
  if $\text{left}[x] = \text{null}$ **then**
    $\text{left}[x] \leftarrow z$; $\text{parent}[z] \leftarrow x$;
  else TreeInsert$(\text{left}[x], z)$;
else
  if $\text{right}[x] = \text{null}$ **then**
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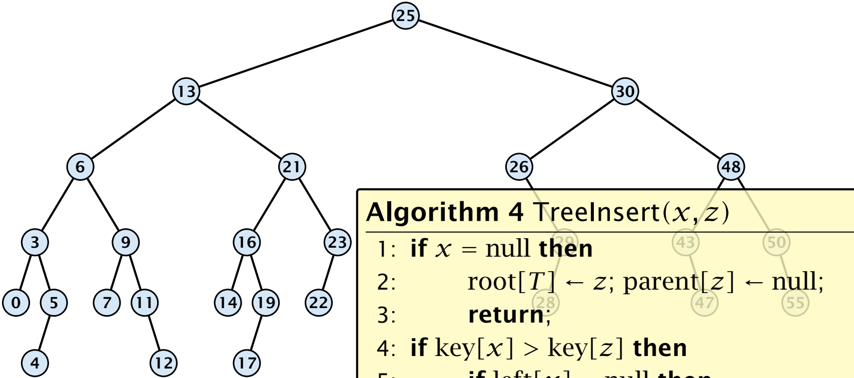
Binary Search Trees: Insert

Insert element **not** in the tree.



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if x = null **then**
    root$[T] \leftarrow z$; parent$[z] \leftarrow$ null;
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if key$[x] >$ key$[z]$ **then**
    if left$[x]$ = null **then**
        left$[x] \leftarrow z$; parent$[z] \leftarrow x$;
    else TreeInsert(left$[x], z)$;
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    if right$[x]$ = null **then**
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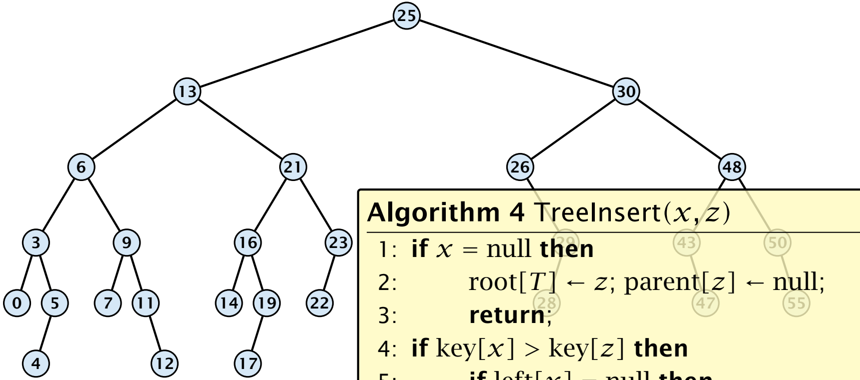

Search for z. At some point the search stops at a null-pointer. This is the place to insert z.

Algorithm 4 TreeInsert(x, z)

```
if x = null then
    root[T] ← z; parent[z] ← null;
    return;
if key[x] > key[z] then
    if left[x] = null then
        left[x] ← z; parent[z] ← x;
    else TreeInsert(left[x], z);
else
    if right[x] = null then
        right[x] ← z; parent[z] ← x;
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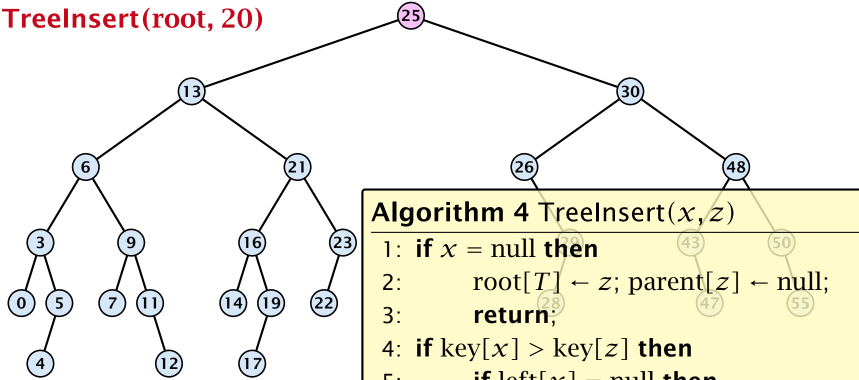
```

Binary Search Trees: Insert

Insert element **not** in the tree.

TreeInsert(root, 20)



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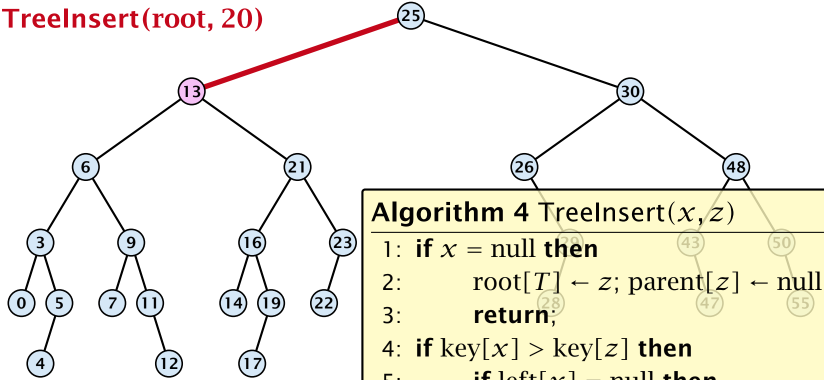

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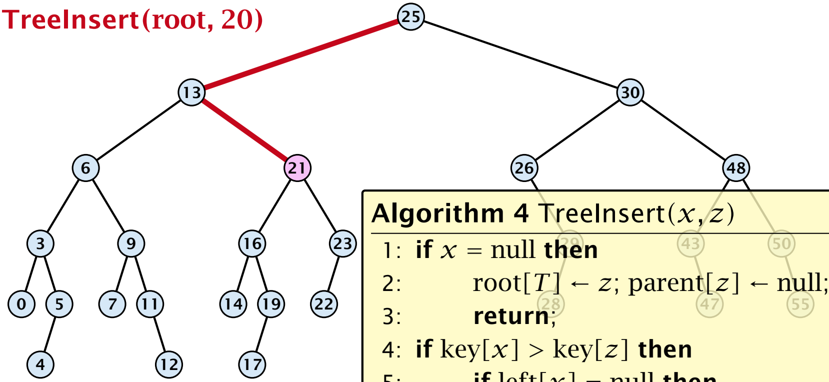

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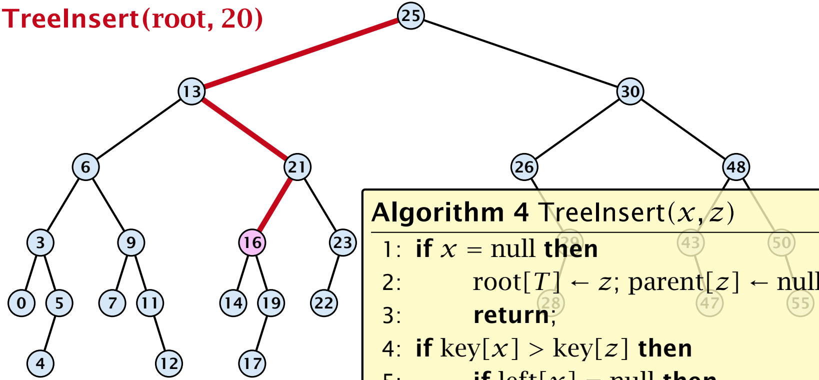

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    return;
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    else TreeInsert(left[x], z);
else
    if right[x] = null then
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```

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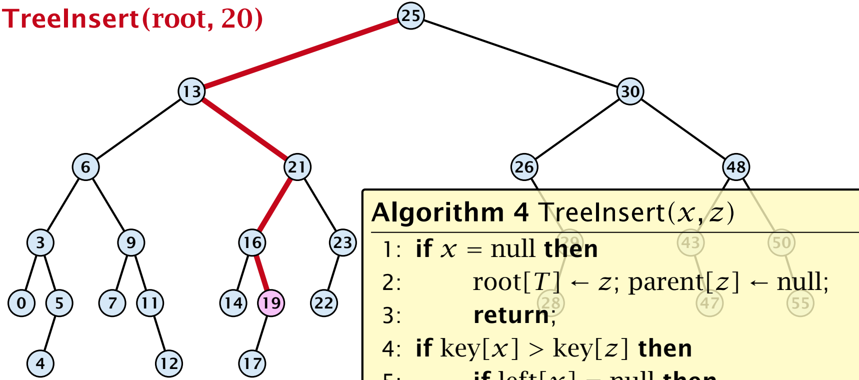

Search for z. At some point the search stops at a null-pointer. This is the place to insert z.

Algorithm 4 TreeInsert(x, z)

if x = null **then**
  root$[T] \leftarrow z$; parent$[z] \leftarrow$ null;
  return;
if key$[x] >$ key$[z]$ **then**
  if left$[x]$ = null **then**
    left$[x] \leftarrow z$; parent$[z] \leftarrow x$;
  else TreeInsert(left$[x], z$);
else
  if right$[x]$ = null **then**
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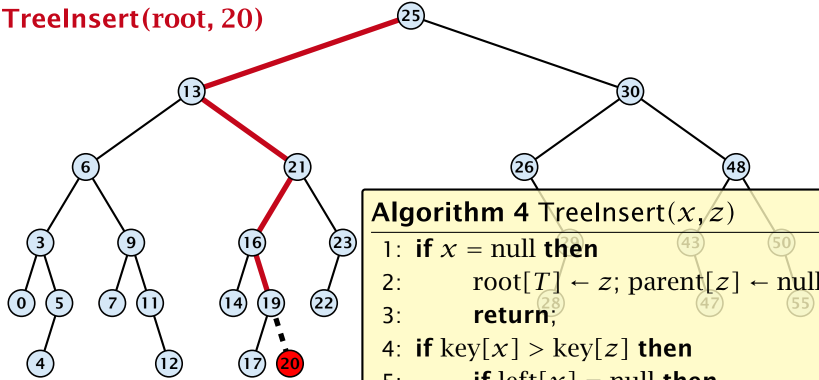
Binary Search Trees: Insert

Insert element **not** in the tree.

TreeInsert(root, 20)



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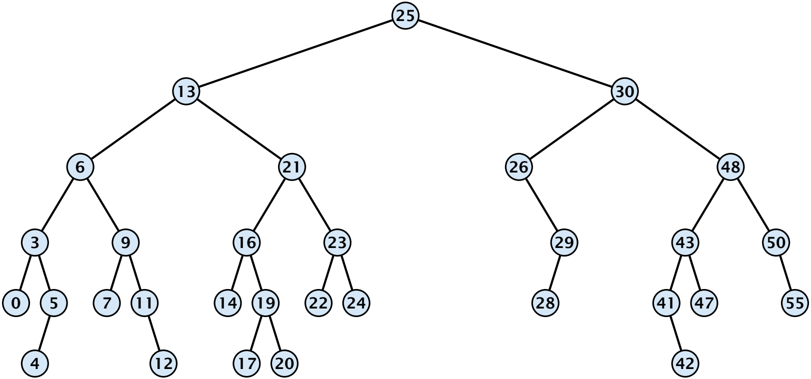

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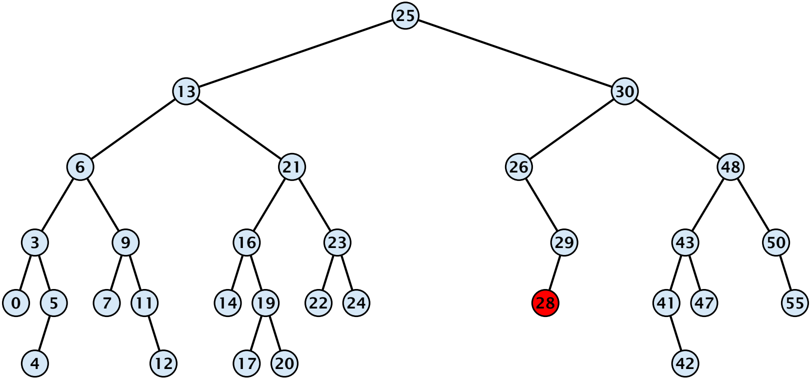

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Element does not have any children

- Simply go to the parent and set the corresponding pointer to null.

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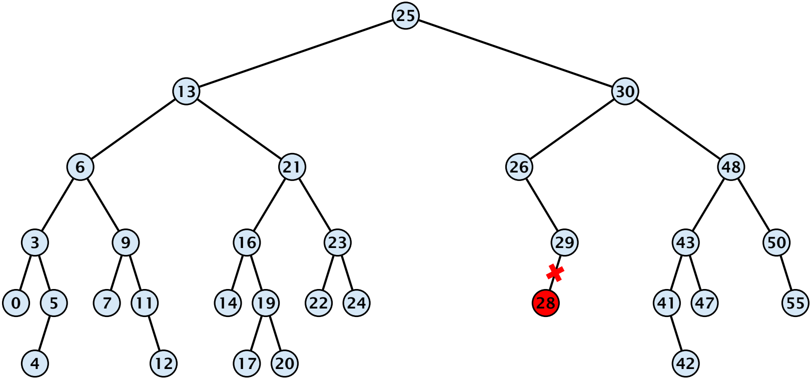

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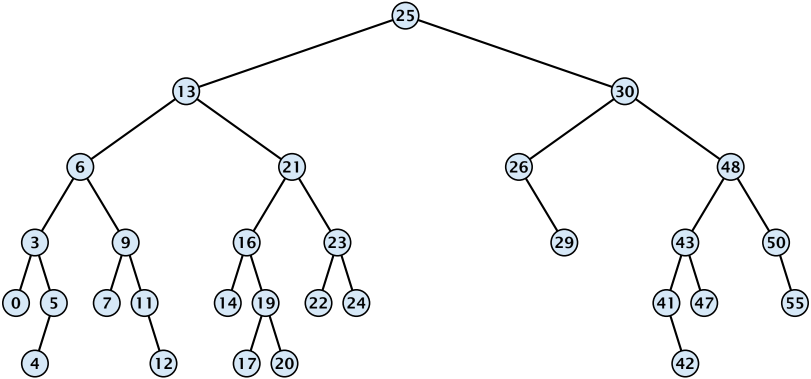

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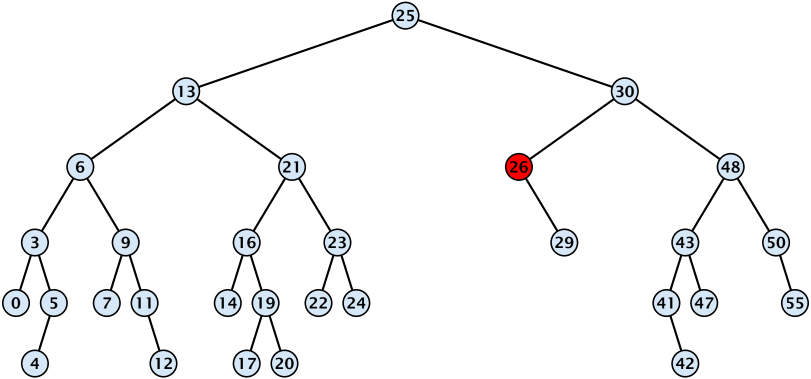

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Element has exactly one child

- Splice the element out of the tree by connecting its parent to its successor.

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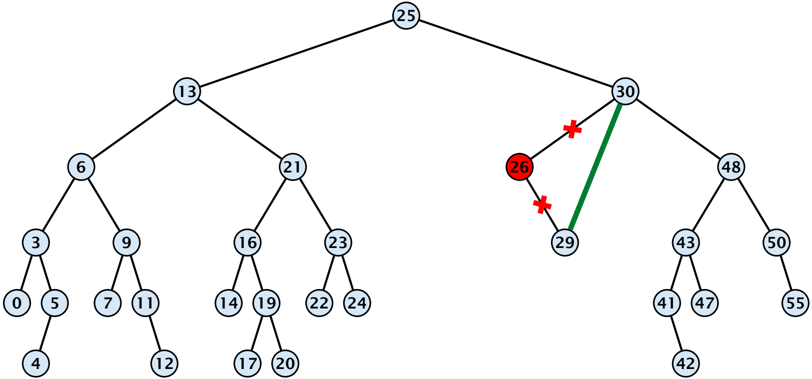

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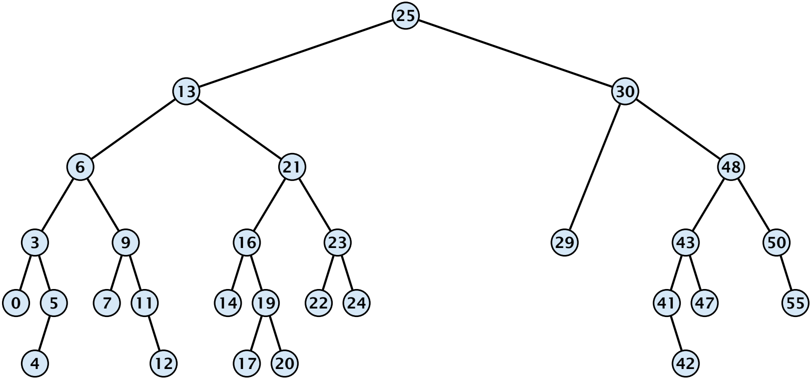

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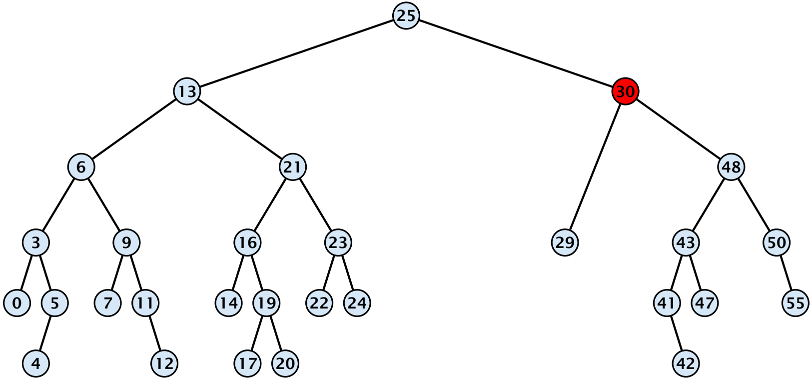

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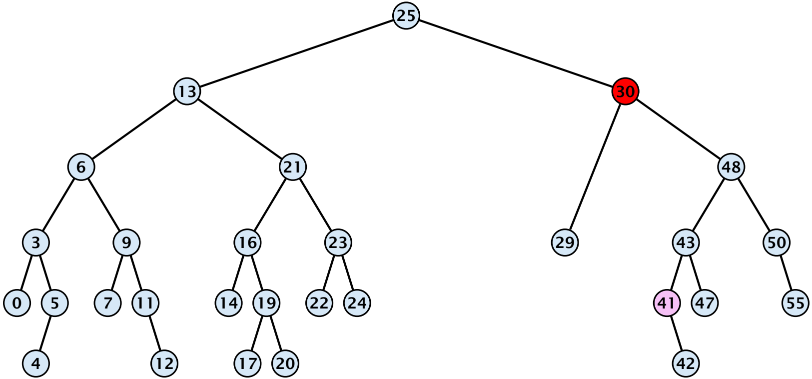

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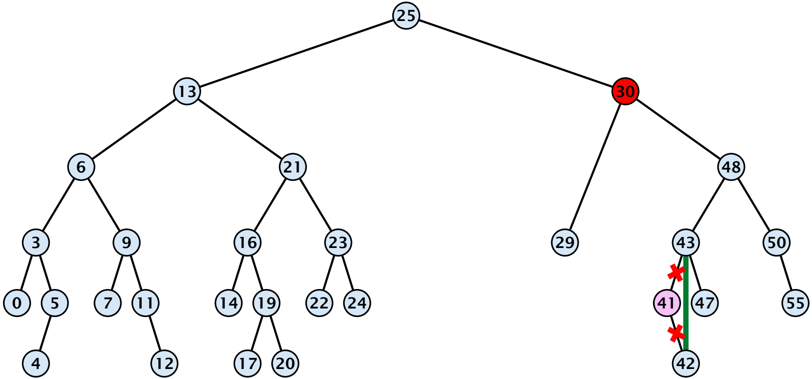

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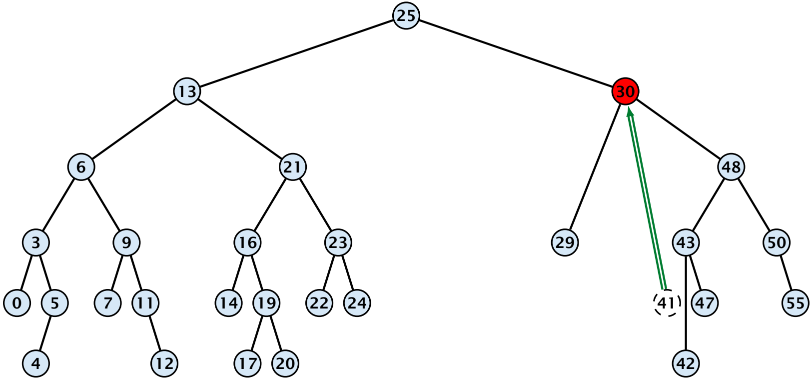

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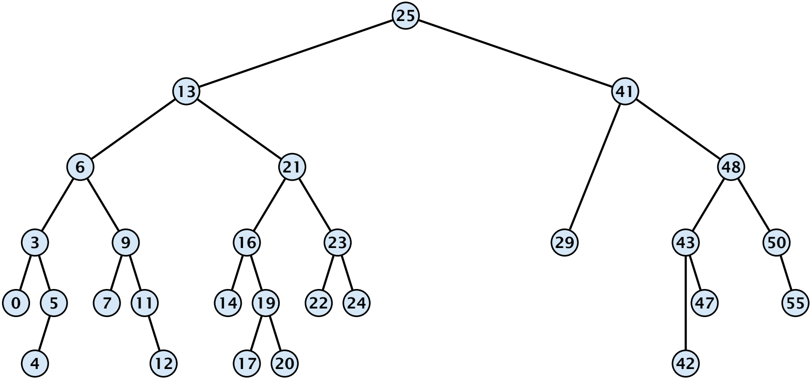

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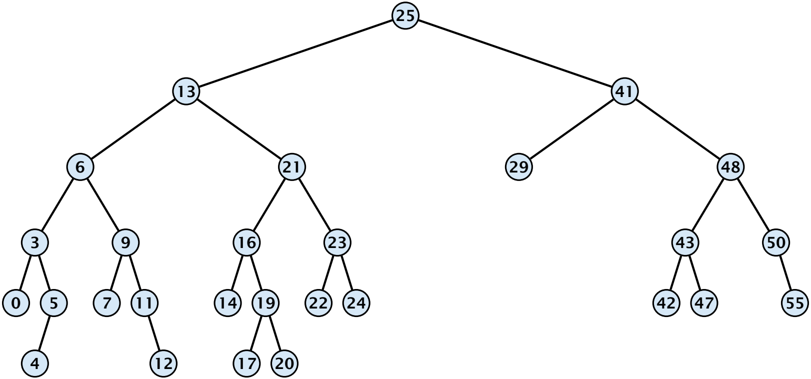

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Binary Search Trees: Delete

Algorithm 9 TreeDelete(z)

if left[z] = null **or** right[z] = null
  then $y \leftarrow z$ **else** $y \leftarrow$ TreeSucc(z); — select y to splice out
if left[y] $\neq$ null
  then $x \leftarrow$ left[y] **else** $x \leftarrow$ right[y]; — x is child of y (or null)
if $x \neq$ null **then** parent[x] $\leftarrow$ parent[y]; — parent[x] is correct
if parent[y] = null **then**
  root[T] $\leftarrow x$
else
  if y = left[parent[y]] **then** — fix pointer to x
    left[parent[y]] $\leftarrow x$
  else
    right[parent[y]] $\leftarrow x$
if $y \neq z$ **then** copy y-data to z

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AVL-trees, Red-black trees, Scapegoat trees, 2-3 trees, B-trees, AA trees, Treaps

similar: SPLAY trees.

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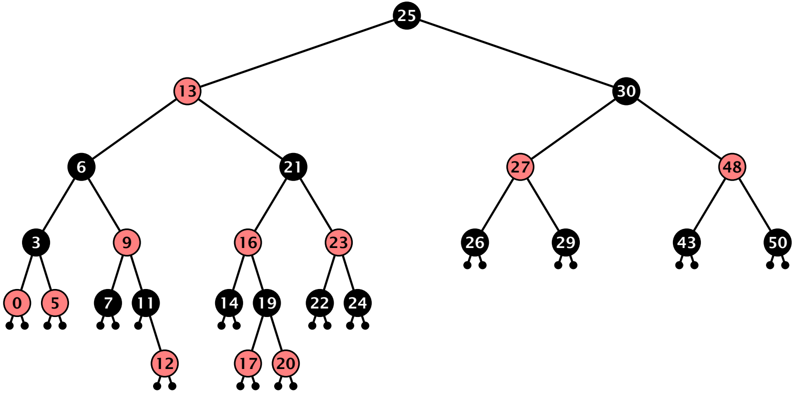
Red Black Trees: Example



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Lemma 2

A red-black tree with n internal nodes has height at most $\mathcal{O}(\log n)$.

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The black height $\mathrm{bh}(v)$ of a node v in a red black tree is the number of black nodes on a path from v to a leaf vertex (not counting v).

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We first show:

Lemma 4
A sub-tree of black height $\mathrm{bh}(v)$ in a red black tree contains at least $2^{\mathrm{bh}(v)} - 1$ internal vertices.

7.2 Red Black Trees

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- The sub-tree rooted at v contains $0 = 2^{\text{bh}(v)} - 1$ inner vertices.

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- Then T_v contains at least $2(2^{\text{bh}(v)-1} - 1) + 1 \geq 2^{\text{bh}(v)} - 1$ vertices.

$\square$

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Hence, $h \leq 2\log(n+1) = \mathcal{O}(\log n)$. $\square$

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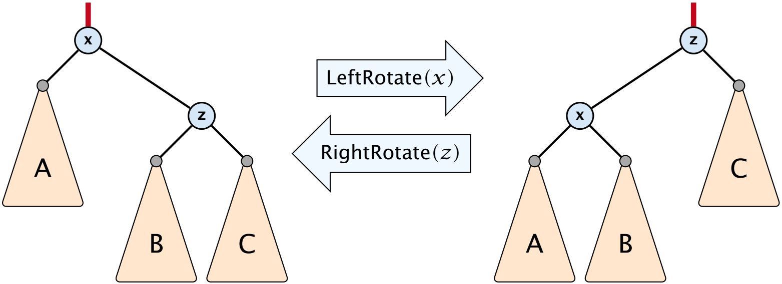
7.2 Red Black Trees

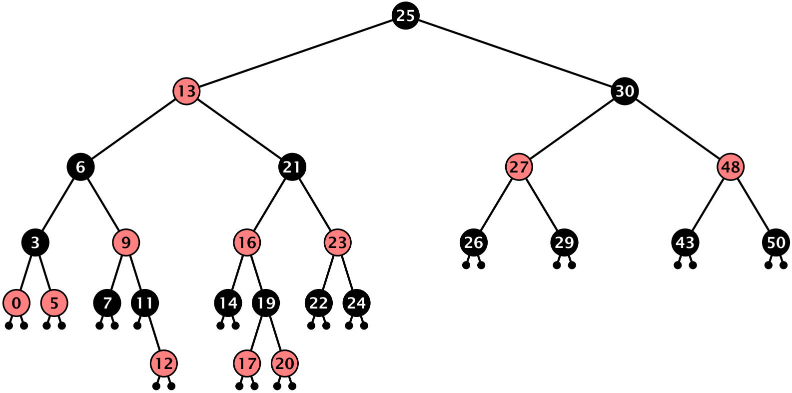
We need to adapt the insert and delete operations so that the red black properties are maintained.

Rotations

The properties will be maintained through rotations:



Red Black Trees: Insert



Insert:

- first make a normal insert into a binary search tree
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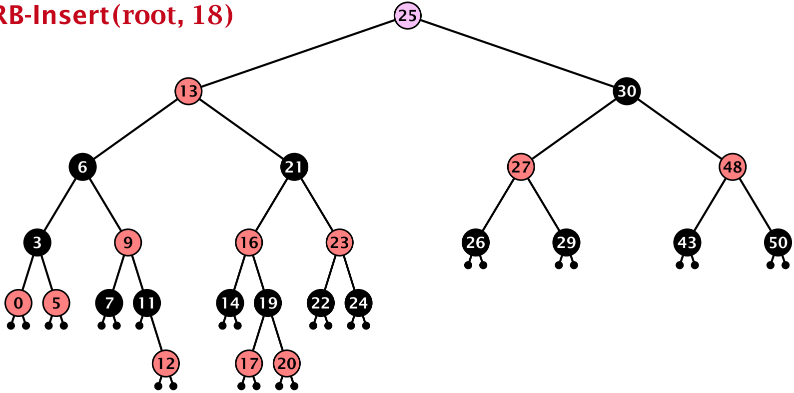
Red Black Trees: Insert

RB-Insert(root, 18)



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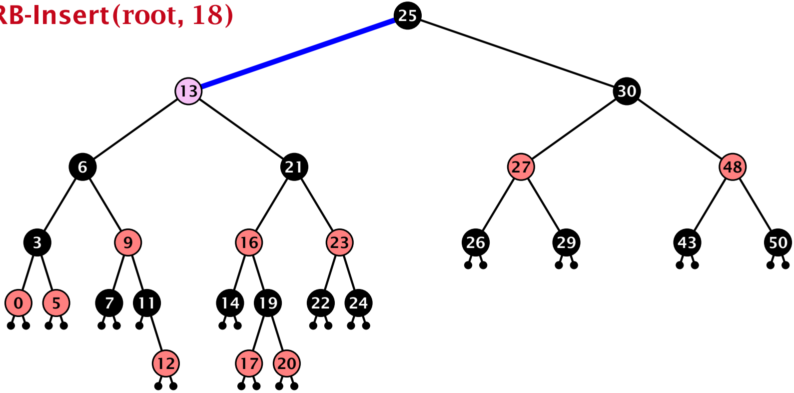

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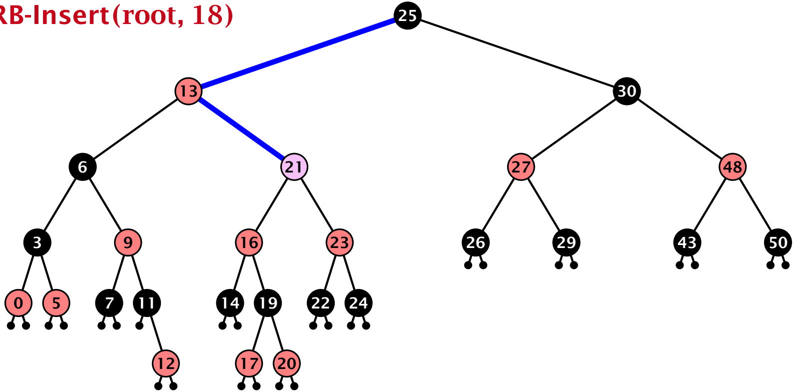

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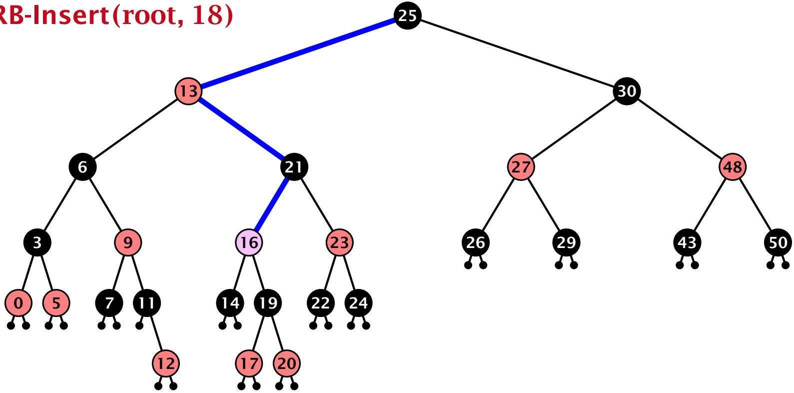

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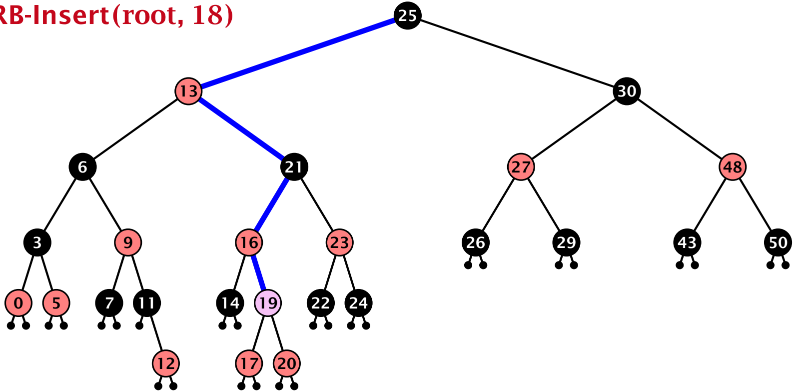

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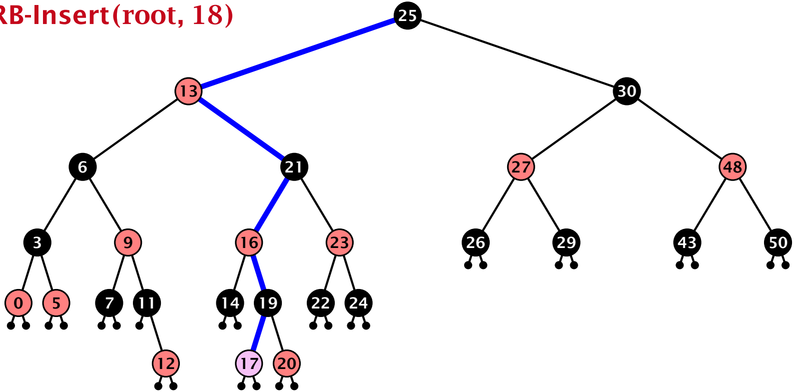

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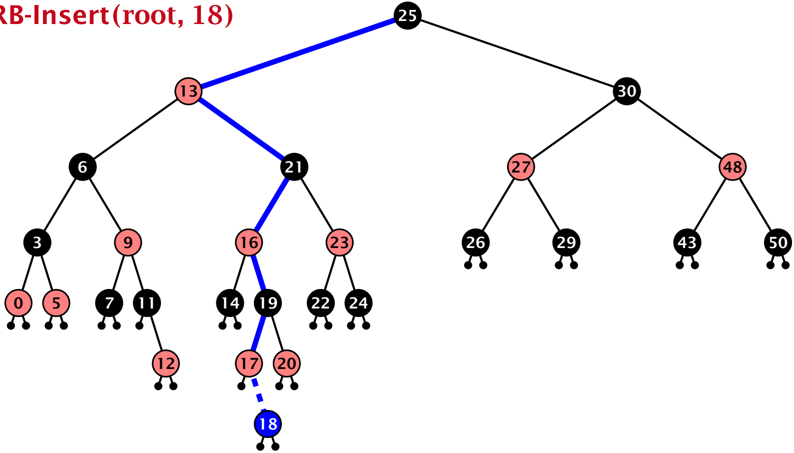

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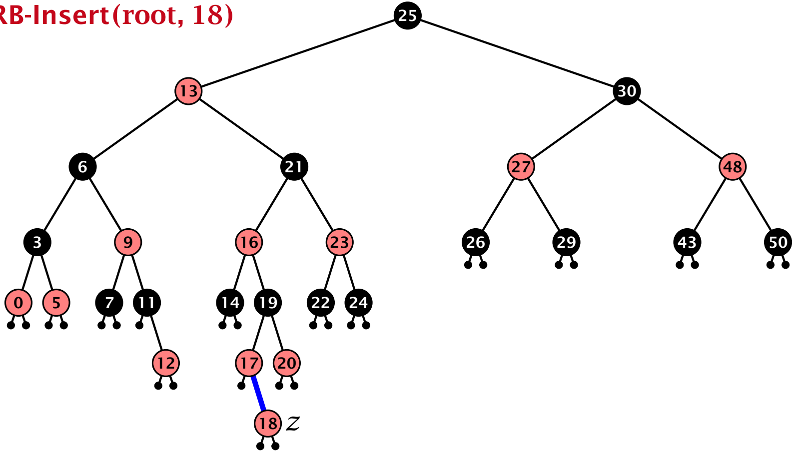

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  - either both of them are red (most important case)
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If z has a parent but no grand-parent we could simply color the parent/root black; however this case never happens.

Red Black Trees: Insert

Algorithm 10 InsertFix(z)

```
while parent[z] ≠ null and col[parent[z]] = red do
    if parent[z] = left[gp[z]] then
        uncle ← right[grandparent[z]]
        if col[uncle] = red then
            col[p[z]] ← black; col[u] ← black;
            col[gp[z]] ← red; z ← grandparent[z];
        else
            if z = right[parent[z]] then
                z ← p[z]; LeftRotate(z);
            col[p[z]] ← black; col[gp[z]] ← red;
            RightRotate(gp[z]);
    else same as then-clause but right and left exchanged
col(root[T]) ← black;
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while $\text{parent}[z] \neq \text{null}$ **and** $\text{col}[\text{parent}[z]] = \text{red}$ **do**
  if $\text{parent}[z] = \text{left}[\text{gp}[z]]$ **then** z in left subtree of grandparent
    $uncle \leftarrow \text{right}[\text{grandparent}[z]]$
    if $\text{col}[uncle] = \text{red}$ **then**
      $\text{col}[\text{p}[z]] \leftarrow \text{black}$; $\text{col}[u] \leftarrow \text{black}$;
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        if col[*uncle*] $=$ red **then**
            col[p[z]] $\leftarrow$ black; col[u] $\leftarrow$ black;
            col[gp[z]] $\leftarrow$ red; $z \leftarrow$ grandparent[z];
        else Case 2: uncle black
            if z $=$ right[parent[z]] **then**
                $z \leftarrow$ p[z]; LeftRotate(z);
            col[p[z]] $\leftarrow$ black; col[gp[z]] $\leftarrow$ red;
            RightRotate(gp[z]);
    else same as then-clause but right and left exchanged
col(root[T]) $\leftarrow$ black;

Red Black Trees: Insert

Algorithm 10 InsertFix(z)

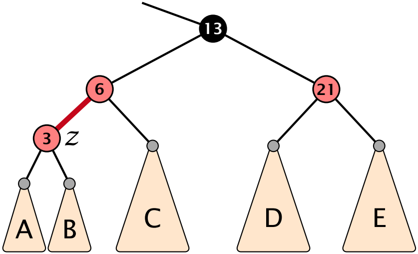
while parent[z] $\neq$ null **and** col[parent[z]] = red **do**
    if parent[z] = left[gp[z]] **then**
        uncle $\leftarrow$ right[grandparent[z]]
        if col[*uncle*] = red **then**
            col[p[z]] $\leftarrow$ black; col[u] $\leftarrow$ black;
            col[gp[z]] $\leftarrow$ red; $z \leftarrow$ grandparent[z];
        else
            if z = right[parent[z]] **then**   2a: z right child
                $z \leftarrow$ p[z]; LeftRotate(z);
            col[p[z]] $\leftarrow$ black; col[gp[z]] $\leftarrow$ red;
            RightRotate(gp[z]);
    else same as then-clause but right and left exchanged
col(root[T]) $\leftarrow$ black;

Red Black Trees: Insert

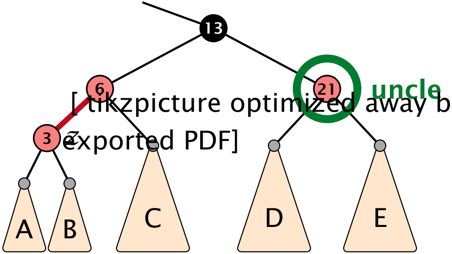
Algorithm 10 InsertFix(z)

```
while parent[z] ≠ null and col[parent[z]] = red do
    if parent[z] = left[gp[z]] then
        uncle ← right[grandparent[z]]
        if col[uncle] = red then
            col[p[z]] ← black; col[u] ← black;
            col[gp[z]] ← red; z ← grandparent[z];
        else
            if z = right[parent[z]] then
                z ← p[z]; LeftRotate(z);
            col[p[z]] ← black; col[gp[z]] ← red; 2b: z left child
            RightRotate(gp[z]);
    else same as then-clause but right and left exchanged
col(root[T]) ← black;
```

Case 1: Red Uncle



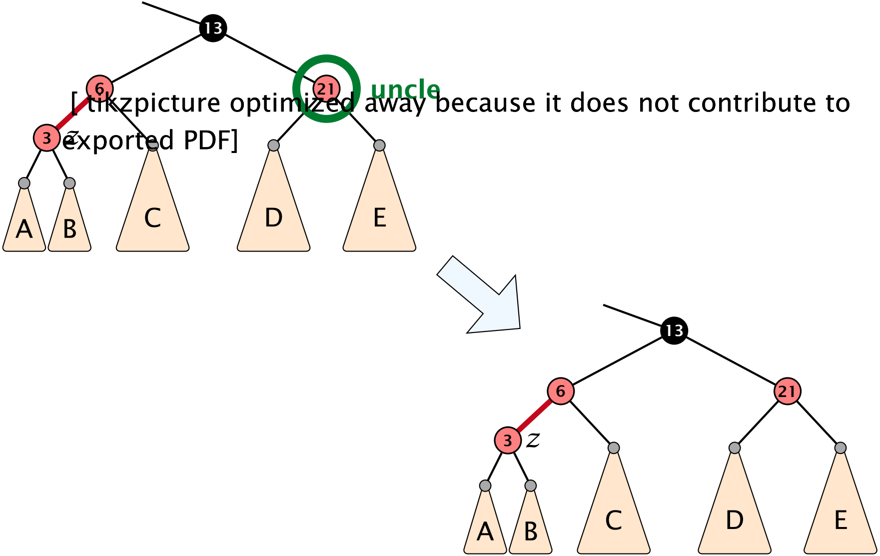
Case 1: Red Uncle



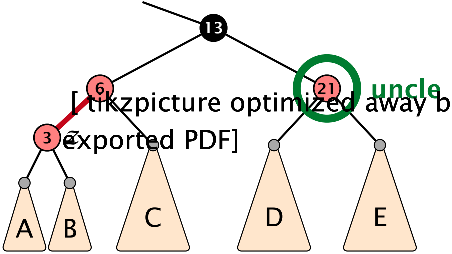
[tikzpicture optimized away because it does not contribute to exported PDF]

Case 1: Red Uncle

[tikzpicture optimized away because it does not contribute to exported PDF]

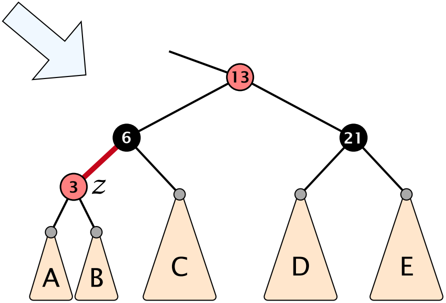


Case 1: Red Uncle

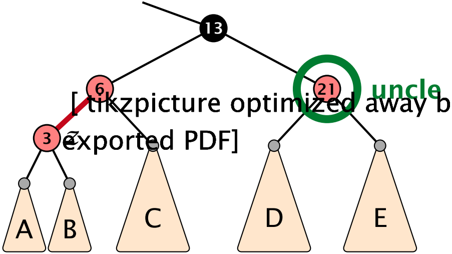


[tikzpicture optimized away because it does not contribute to exported PDF]

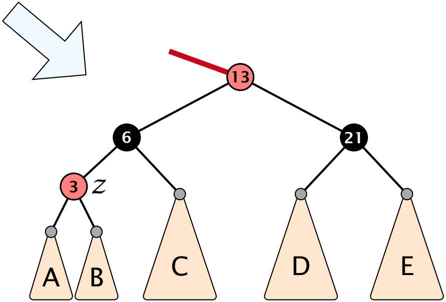
1. recolour



Case 1: Red Uncle

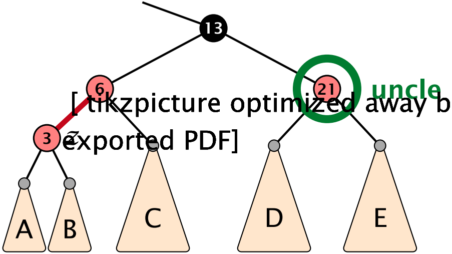


[tikzpicture optimized away because it does not contribute to exported PDF]

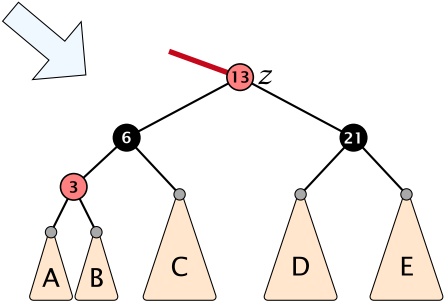


1. recolour

Case 1: Red Uncle

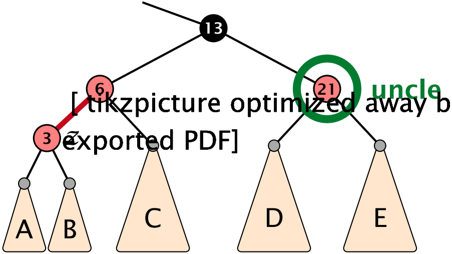


[tikzpicture optimized away because it does not contribute to exported PDF]

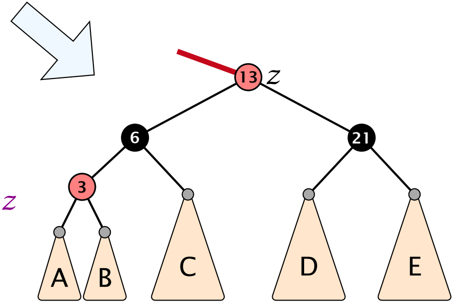


1. recolour
2. move z to grand-parent

Case 1: Red Uncle

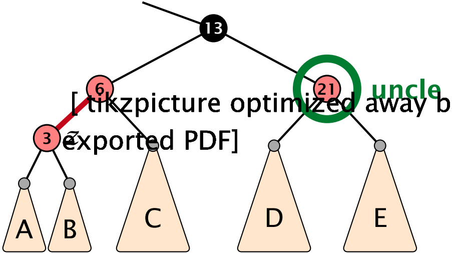


[tikzpicture optimized away because it does not contribute to exported PDF]



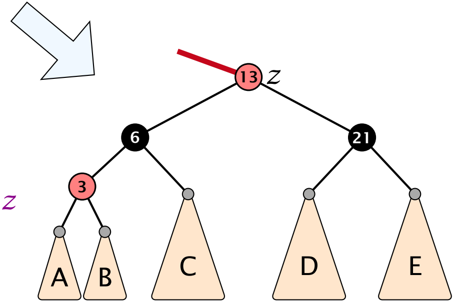
1. recolour
2. move z to grand-parent
3. invariant is fulfilled for new z

Case 1: Red Uncle

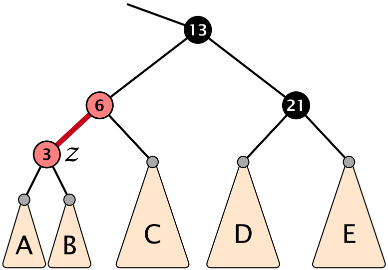


[tikzpicture optimized away because it does not contribute to exported PDF]

1. recolour
2. move z to grand-parent
3. invariant is fulfilled for new z
4. you made progress

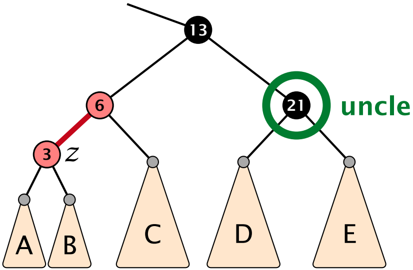


Case 2b: Black uncle and z is left child



Case 2b: Black uncle and z is left child

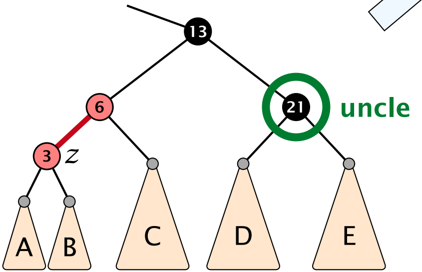
[tikzpicture optimized away because it does not contribute to exported PDF]

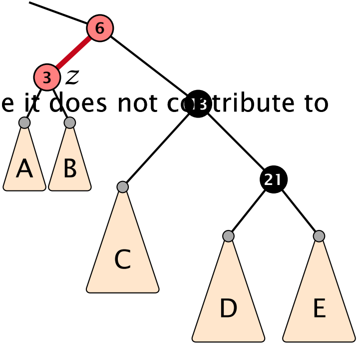


Case 2b: Black uncle and z is left child

1. rotate around grandparent

[tikzpicture optimized away because it does not contribute to exported PDF]

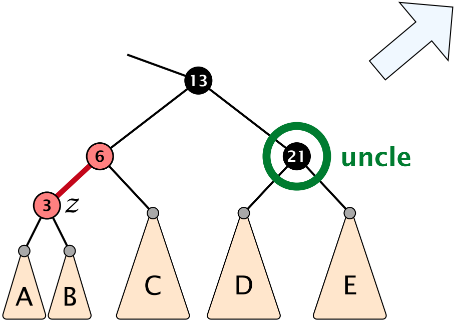


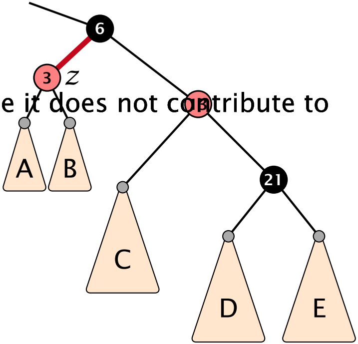


Case 2b: Black uncle and z is left child

1. rotate around grandparent
2. re-colour to ensure that black height property holds

[tikzpicture optimized away because it does not contribute to exported PDF]

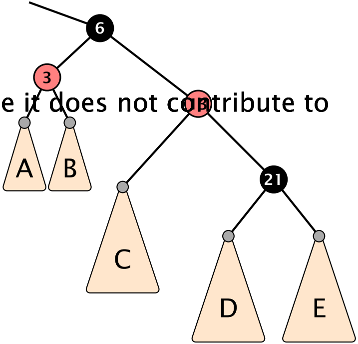


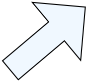


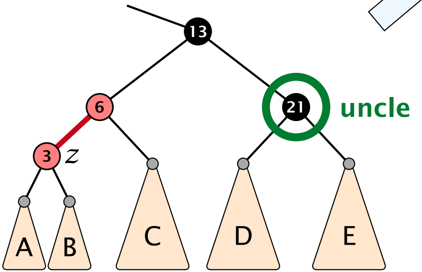
Case 2b: Black uncle and z is left child

1. rotate around grandparent
2. re-colour to ensure that black height property holds
3. you have a red black tree

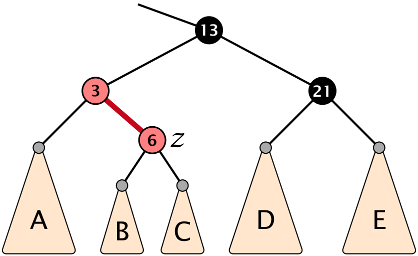
[tikzpicture optimized away because it does not contribute to exported PDF]





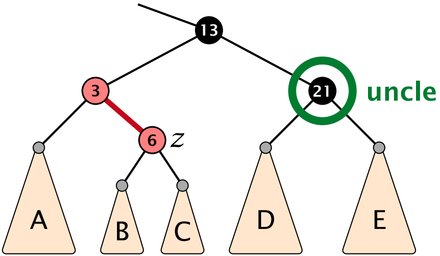


Case 2a: Black uncle and z is right child



Case 2a: Black uncle and z is right child

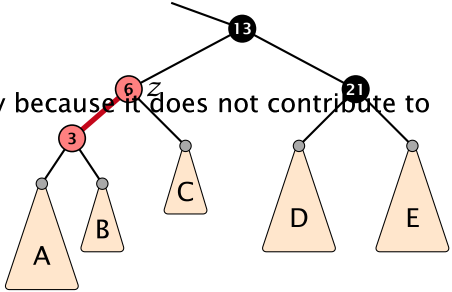
[tikzpicture optimized away because it does not contribute to exported PDF]

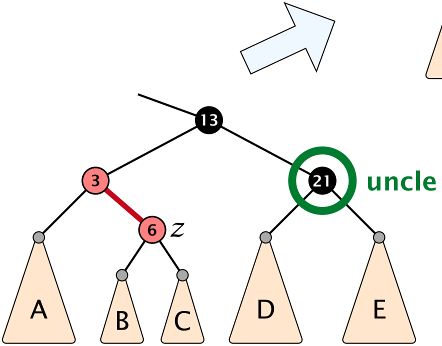


Case 2a: Black uncle and z is right child

1. rotate around parent

[tikzpicture optimized away because it does not contribute to exported PDF]

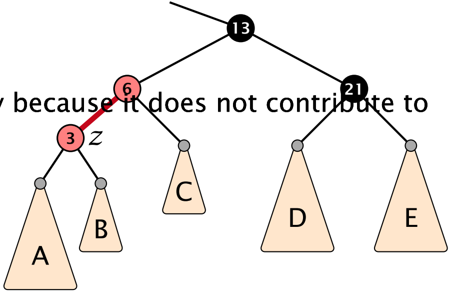


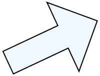


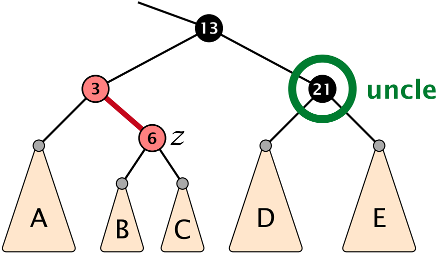
Case 2a: Black uncle and z is right child

1. rotate around parent
2. move z downwards

[tikzpicture optimized away because it does not contribute to exported PDF]



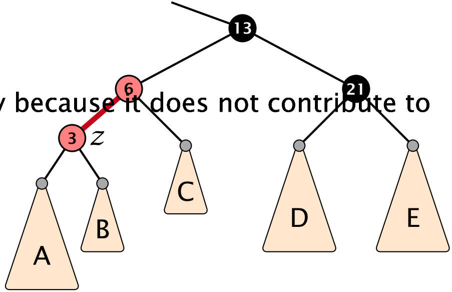


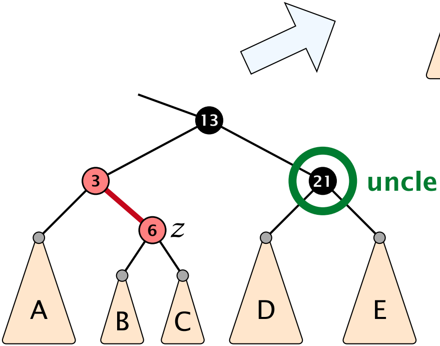


Case 2a: Black uncle and z is right child

1. rotate around parent
2. move z downwards
3. you have Case 2b.

['tikzpicture optimized away because it does not contribute to exported PDF]





Red Black Trees: Insert

Running time:

- Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.

Red Black Trees: Insert

Running time:

- Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- Case 2a $\rightarrow$ Case 2b $\rightarrow$ red-black tree

Red Black Trees: Insert

Running time:

- Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- Case 2a $\rightarrow$ Case 2b $\rightarrow$ red-black tree
- Case 2b $\rightarrow$ red-black tree

Red Black Trees: Insert

Running time:

- Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- Case 2a $\rightarrow$ Case 2b $\rightarrow$ red-black tree
- Case 2b $\rightarrow$ red-black tree

Performing Case 1 at most $\mathcal{O}(\log n)$ times and every other case at most once, we get a red-black tree. Hence $\mathcal{O}(\log n)$ re-colorings and at most 2 rotations.

Red Black Trees: Delete

Red Black Trees: Delete

First do a standard delete.

Red Black Trees: Delete

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If the spliced out node x was red everything is fine.

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If it was black there may be the following problems.

Red Black Trees: Delete

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If the spliced out node x was red everything is fine.

If it was black there may be the following problems.

- Parent and child of x were red; two adjacent red vertices.

Red Black Trees: Delete

First do a standard delete.

If the spliced out node x was red everything is fine.

If it was black there may be the following problems.

- Parent and child of x were red; two adjacent red vertices.
- If you delete the root, the root may now be red.

Red Black Trees: Delete

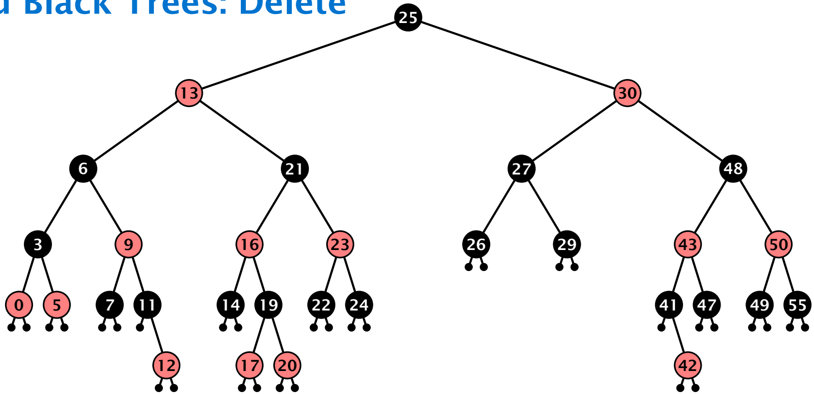
First do a standard delete.

If the spliced out node x was red everything is fine.

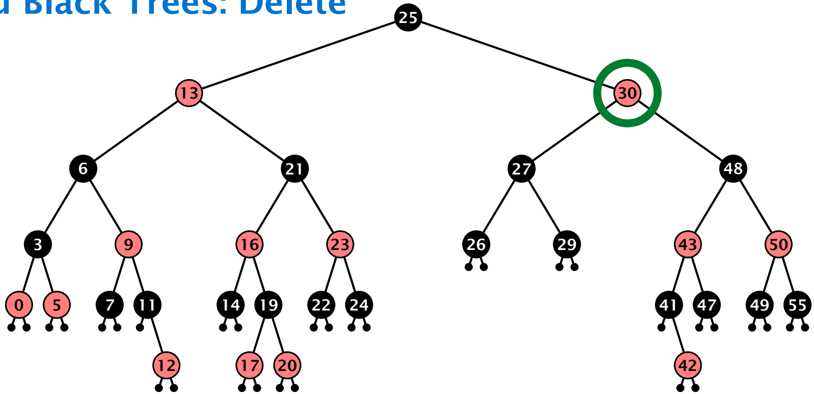
If it was black there may be the following problems.

- Parent and child of x were red; two adjacent red vertices.
- If you delete the root, the root may now be red.
- Every path from an ancestor of x to a descendant leaf of x changes the number of black nodes. Black height property might be violated.

Red Black Trees: Delete



Red Black Trees: Delete

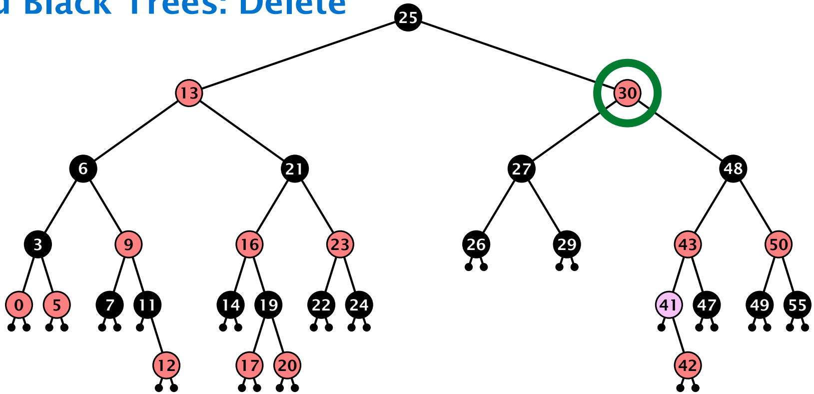


Case 3:

Element has two children

- do normal delete
- when replacing content by content of successor, don't change color of node

Red Black Trees: Delete



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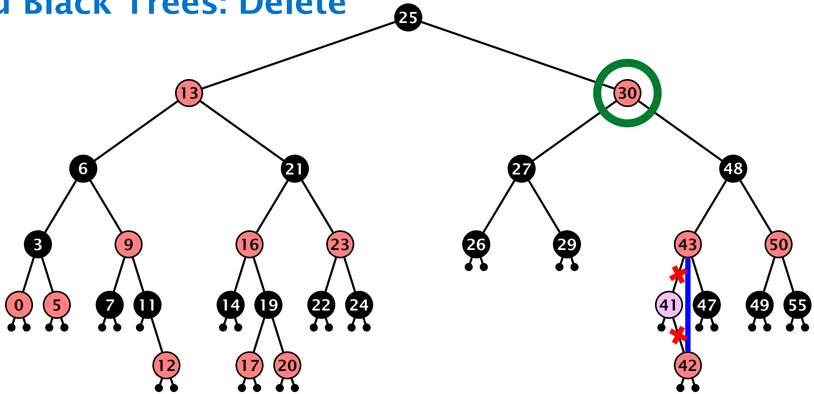
Red Black Trees: Delete



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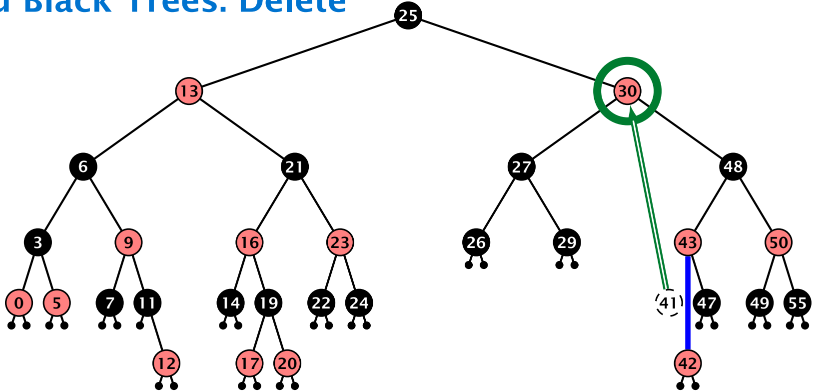
Red Black Trees: Delete



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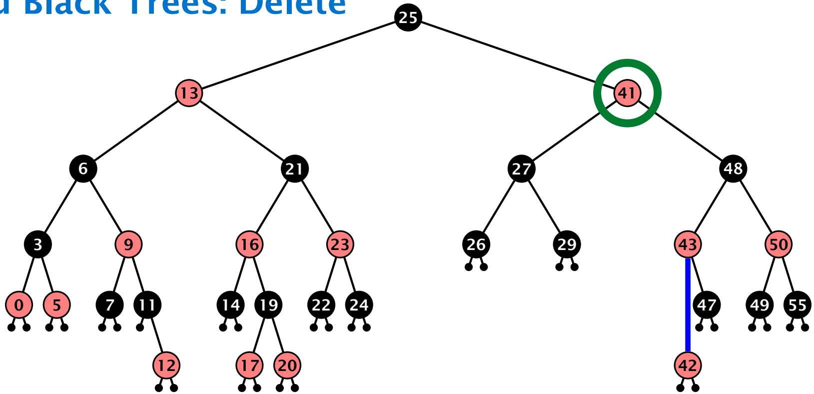
Red Black Trees: Delete



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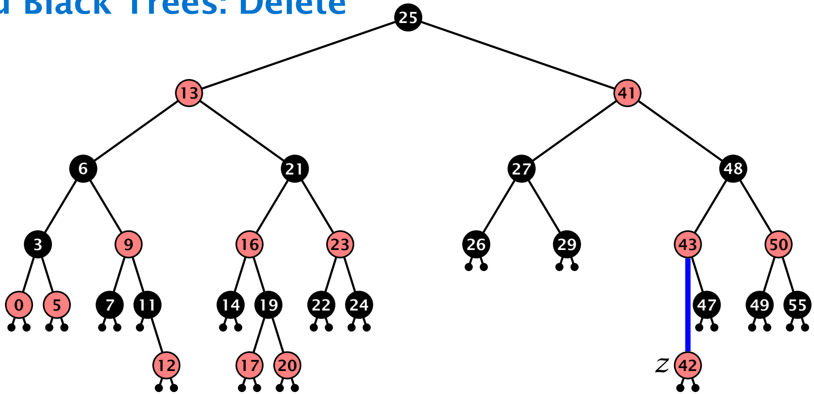
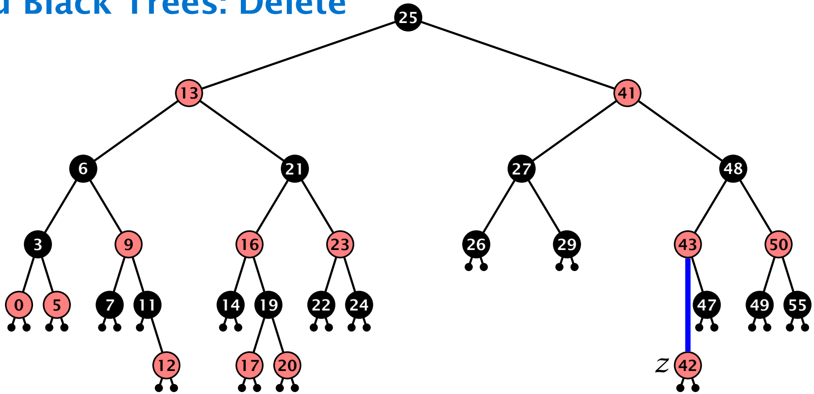


Delete:

- deleting black node messes up black-height property

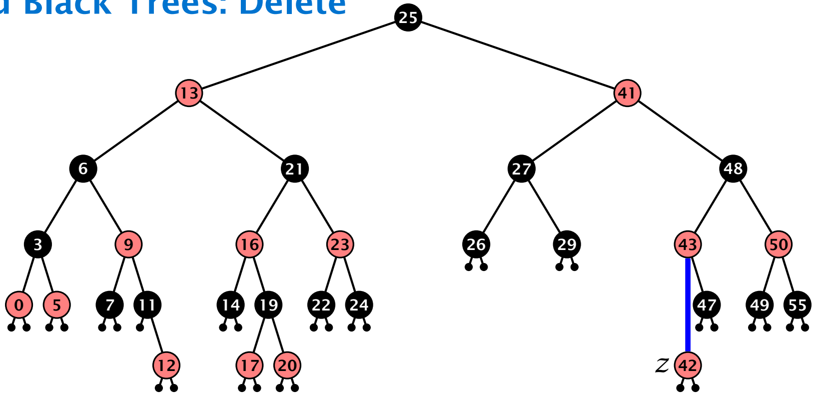
Red Black Trees: Delete



Delete:

- deleting black node messes up black-height property
- if z is red, we can simply color it black and everything is fine

Red Black Trees: Delete



Delete:

- deleting black node messes up black-height property
- if z is red, we can simply color it black and everything is fine
- the problem is if z is black (e.g. a dummy-leaf); we call a fix-up procedure to fix the problem.

Red Black Trees: Delete

Invariant of the fix-up algorithm

- the node z is black

Red Black Trees: Delete

Invariant of the fix-up algorithm

- the node z is black
- if we "assign" a fake black unit to the edge from z to its parent then the black-height property is fulfilled

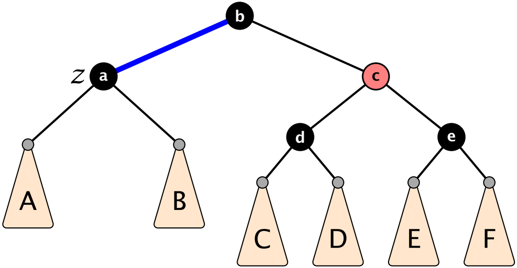
Red Black Trees: Delete

Invariant of the fix-up algorithm

- the node z is black
- if we "assign" a fake black unit to the edge from z to its parent then the black-height property is fulfilled

Goal: make rotations in such a way that you at some point can remove the fake black unit from the edge.

Case 1: Sibling of z is red



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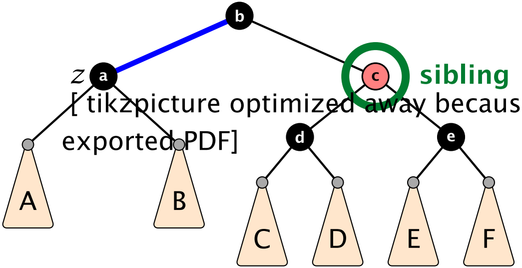

[tikzpicture optimized away because it does not contribute to exported PDF]

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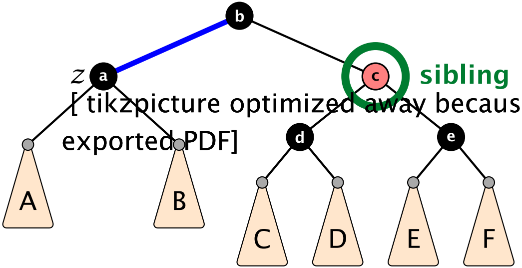
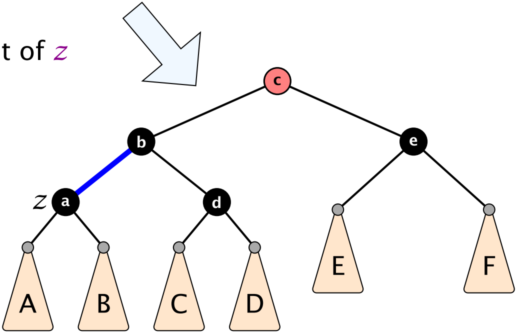


[tikzpicture optimized away because it does not contribute to exported PDF]

1. left-rotate around parent of z



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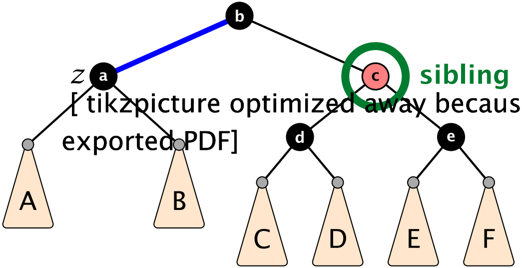
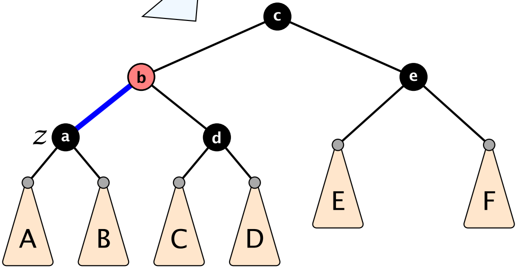


[tikzpicture optimized away because it does not contribute to exported PDF]

1. left-rotate around parent of z
2. recolor nodes b and c



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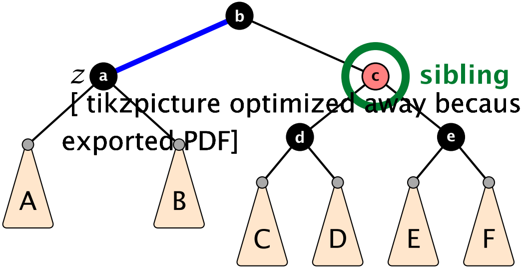
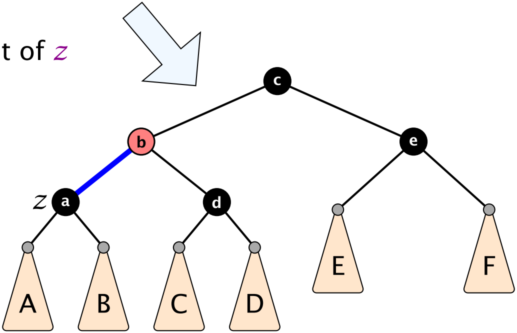
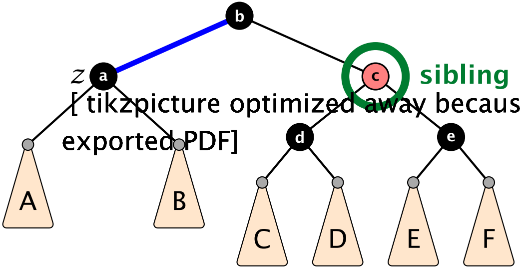


[tikzpicture optimized away because it does not contribute to exported PDF]

1. left-rotate around parent of z
2. recolor nodes b and c
3. the new sibling is black (and parent of z is red)

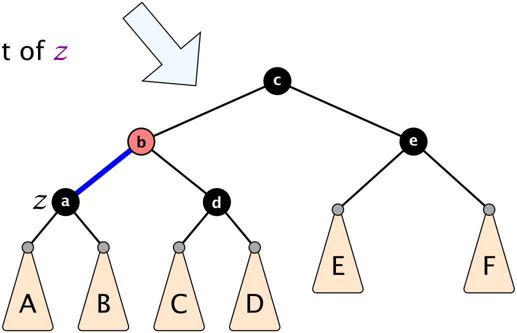


Case 1: Sibling of z is red

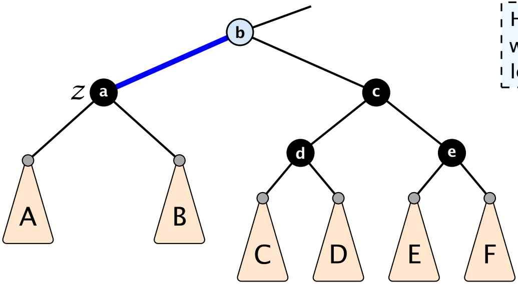


[tikzpicture optimized away because it does not contribute to exported PDF]

1. left-rotate around parent of z
2. recolor nodes b and c
3. the new sibling is black (and parent of z is red)
4. Case 2 (special), or Case 3, or Case 4

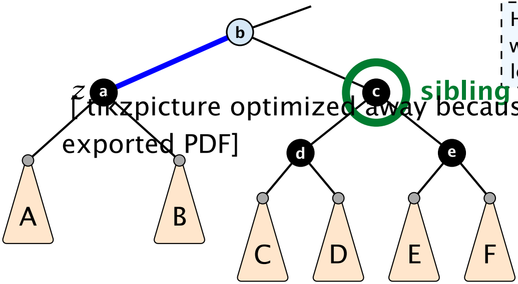


Case 2: Sibling is black with two black children



Here b is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

Case 2: Sibling is black with two black children



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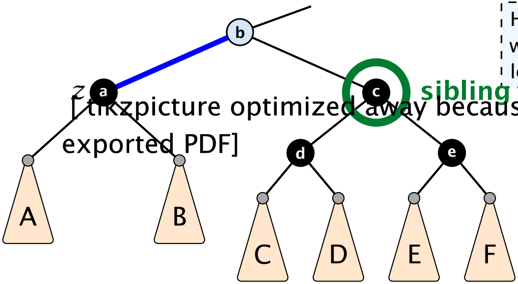
Case 2: Sibling is black with two black children



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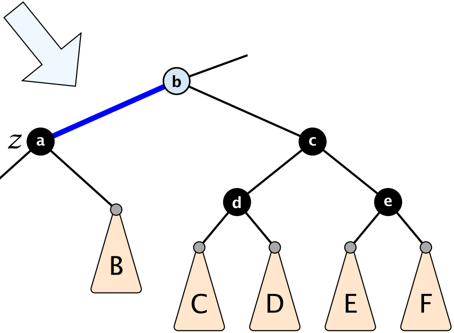
[tikzpicture optimized away because it does not contribute to exported PDF]



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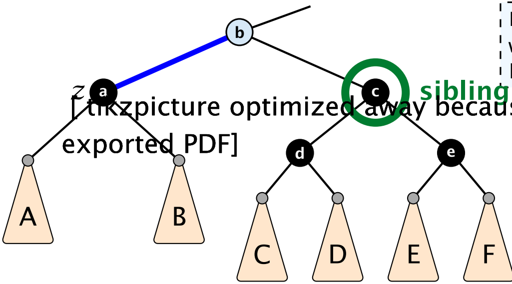

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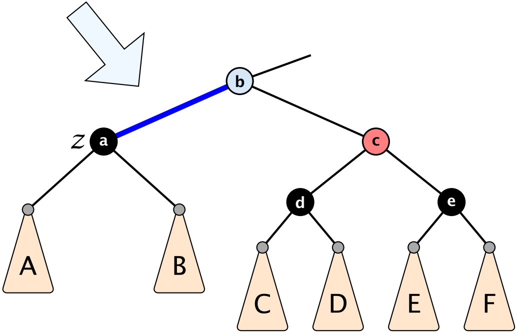

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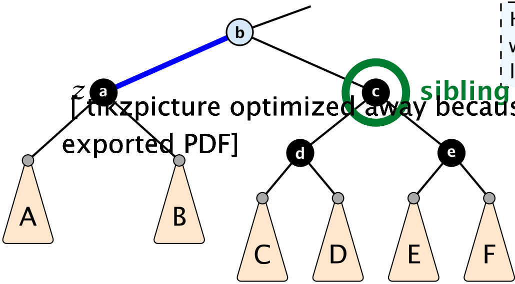
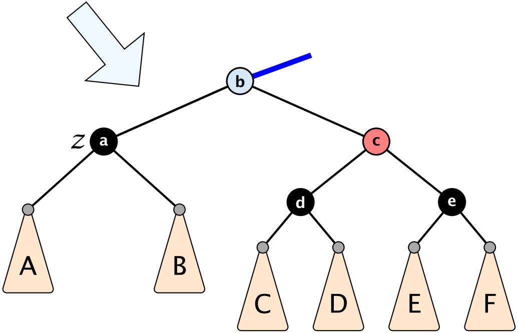


Here b is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

[tikzpicture optimized away because it does not contribute to exported PDF]

1. re-color node c
2. move fake black unit upwards



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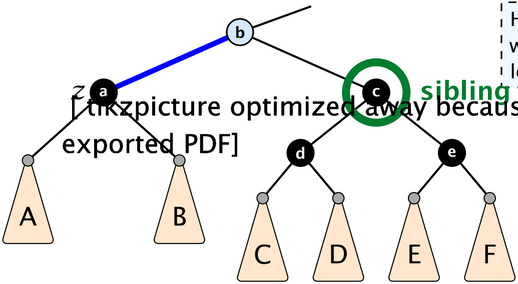
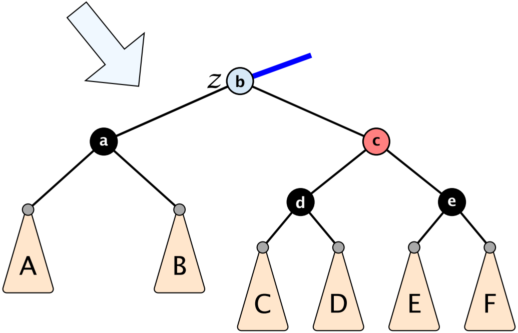


[tikzpicture optimized away because it does not contribute to exported PDF]

1. re-color node c
2. move fake black unit upwards
3. move z upwards



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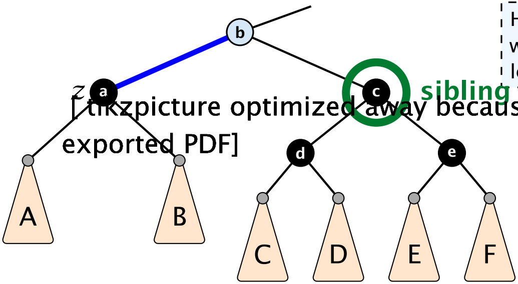

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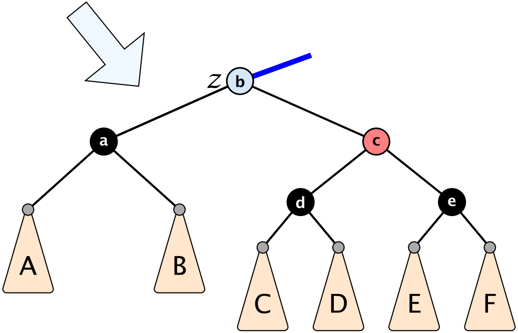
Here b is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

1. re-color node c
2. move fake black unit upwards
3. move z upwards
4. we made progress



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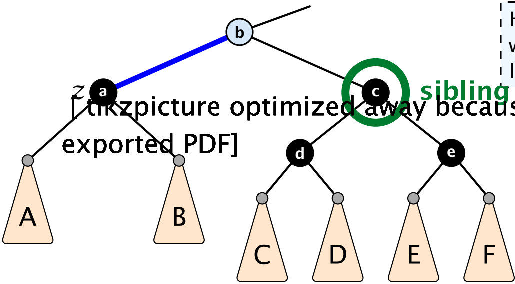
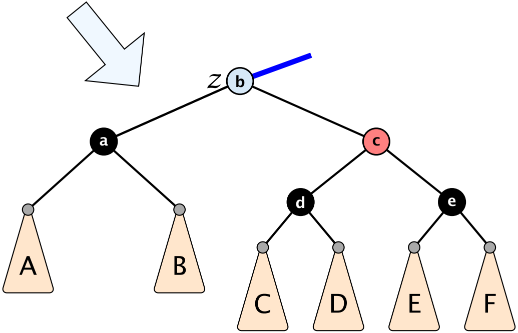
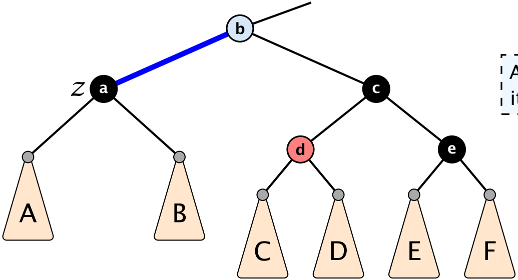


[tikzpicture optimized away because it does not contribute to exported PDF]

1. re-color node c
2. move fake black unit upwards
3. move z upwards
4. we made progress
5. if b is red we color it black and are done



Case 3: Sibling black with one black child to the right



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Case 3: Sibling black with one black child to the right

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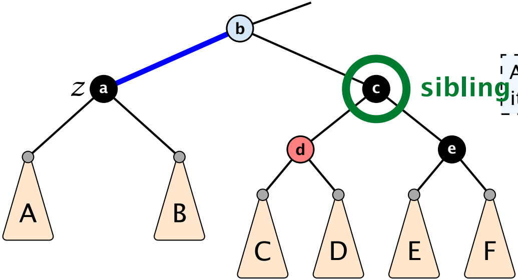

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Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling

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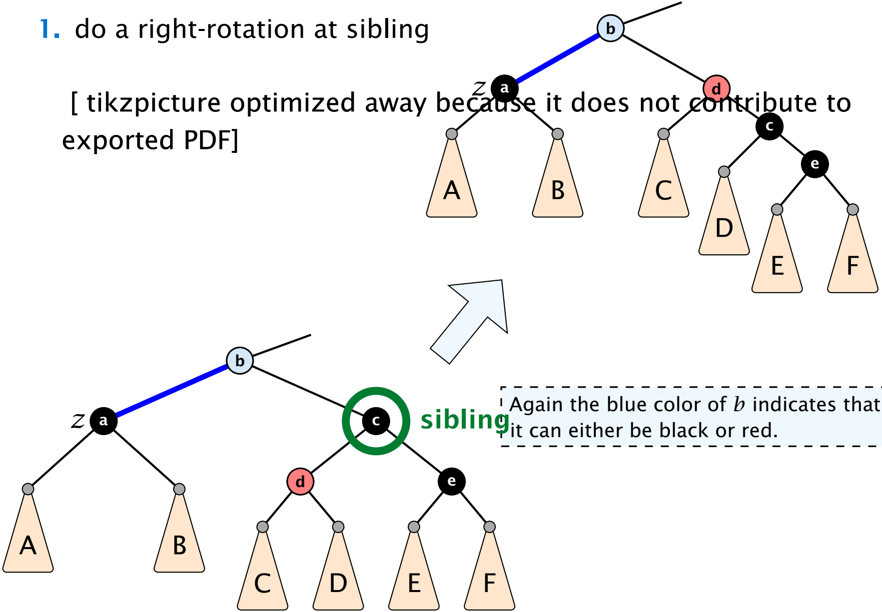

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Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling
2. recolor c and d

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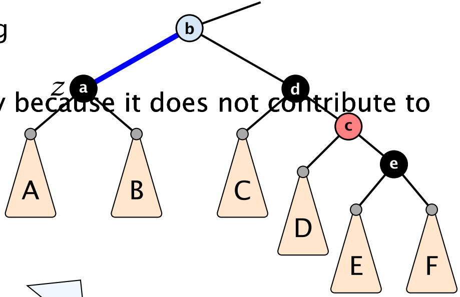

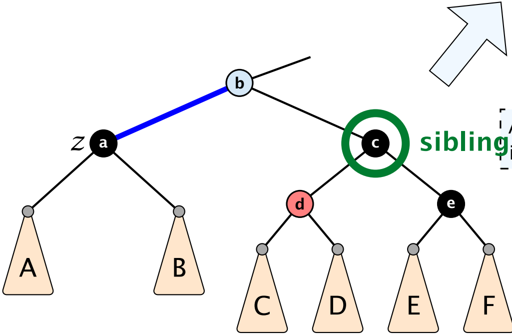

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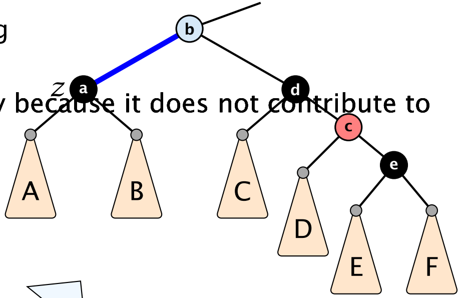
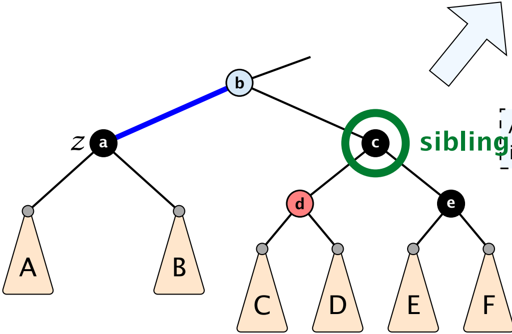
Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling
2. recolor c and d
3. new sibling is black with red right child (Case 4)

[tikzpicture optimized away because it does not contribute to exported PDF]





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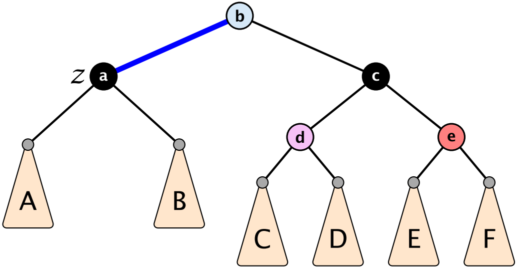
Case 4: Sibling is black with red right child



- Here b and d are either red or black but have possibly different colors.
- We recolor c by giving it the color of b.

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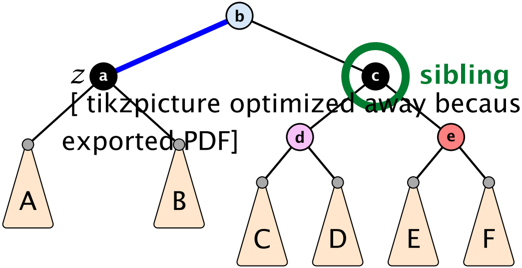

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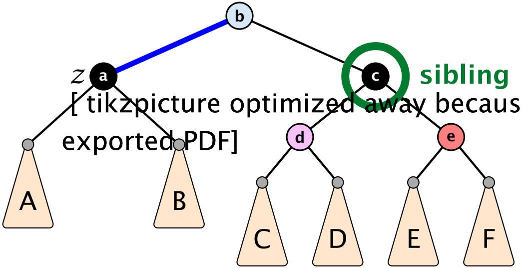

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- Here b and d are either red or black but have possibly different colors.
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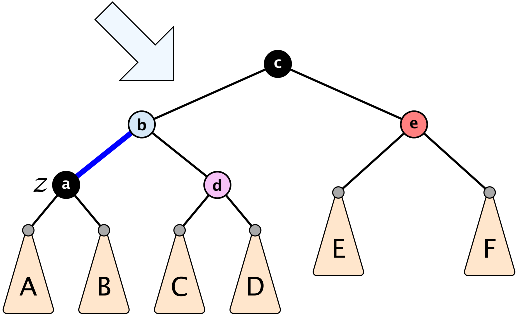
1. left-rotate around b



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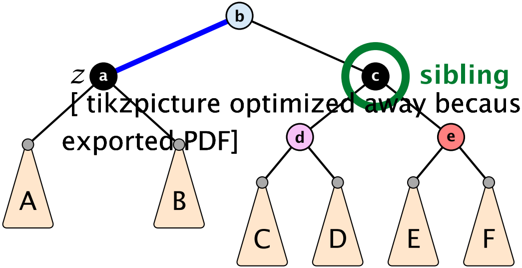
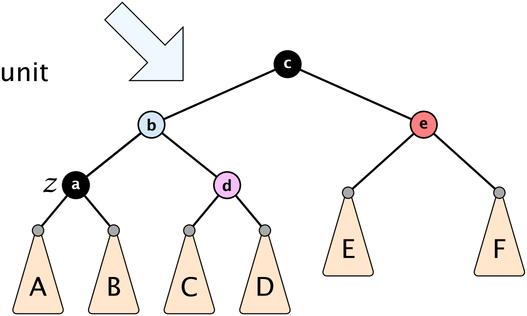


[tikzpicture optimized away because it does not contribute to exported PDF]

- Here b and d are either red or black but have possibly different colors.
- We recolor c by giving it the color of b.

1. left-rotate around b
2. remove the fake black unit



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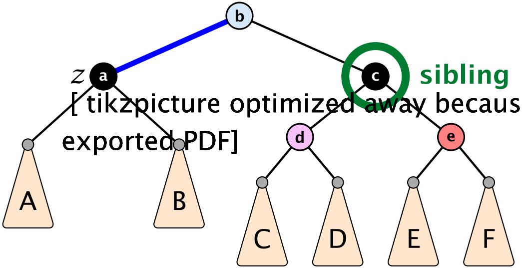

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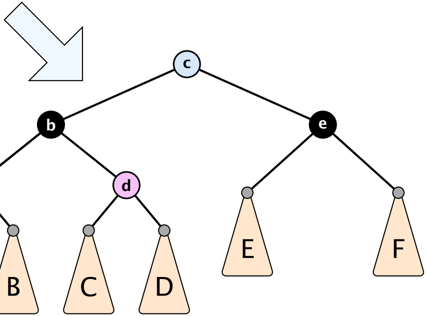
- Here b and d are either red or black but have possibly different colors.
- We recolor c by giving it the color of b.

1. left-rotate around b
2. remove the fake black unit
3. recolor nodes b, c, and e



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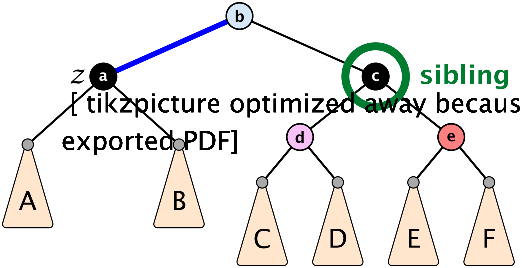

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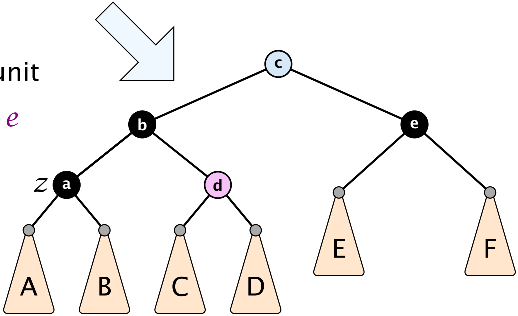
- We recolor c by giving it the color of b.

1. left-rotate around b
2. remove the fake black unit
3. recolor nodes b, c, and e
4. you have a valid red black tree



Running time:

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Performing Case 2 at most $\mathcal{O}(\log n)$ times and every other step at most once, we get a red black tree. Hence, $\mathcal{O}(\log n)$ re-colorings and at most 3 rotations.

Splay Trees

Disadvantage of balanced search trees:

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Splay Trees:

+ after access, an element is moved to the root; splay(x)
  repeated accesses are faster
- only amortized guarantee
- read-operations change the tree

Splay Trees

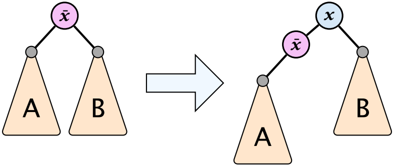
find(x)

- search for x according to a search tree
- let $\bar{x}$ be last element on search-path
- splay($\bar{x}$)

Splay Trees

insert(x)

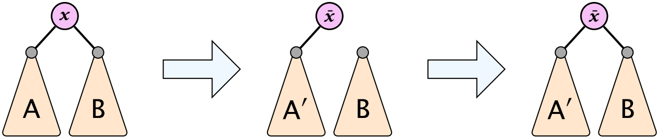
- search for x; $\bar{x}$ is last visited element during search (successer or predecessor of x)
- splay($\bar{x}$) moves $\bar{x}$ to the root
- insert x as new root



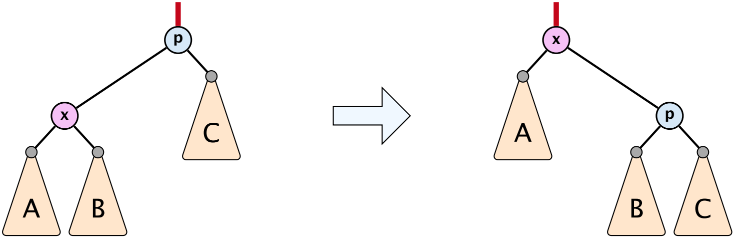
Splay Trees

delete(x)

- search for x; splay(x); remove x
- search largest element $\bar{x}$ in A
- splay($\bar{x}$) (on subtree A)
- connect root of B as right child of $\bar{x}$



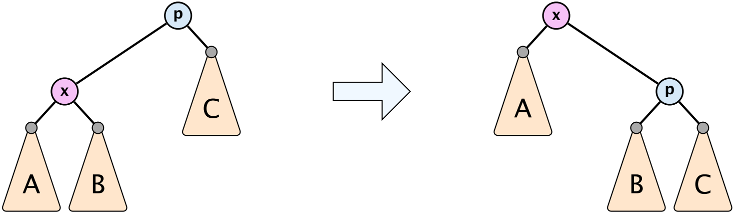
Move to Root



How to bring element to root?

- one (bad) option: moveToRoot(x)
- iteratively do rotation around parent of x until x is root
- if x is left child do right rotation otw. left rotation

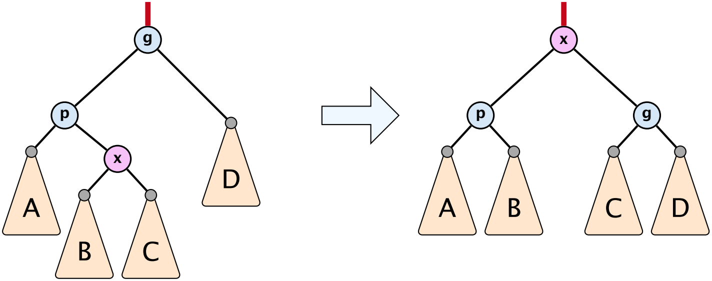
Splay: Zig Case



better option splay(x):

- zig case: if x is child of root do left rotation or right rotation around parent

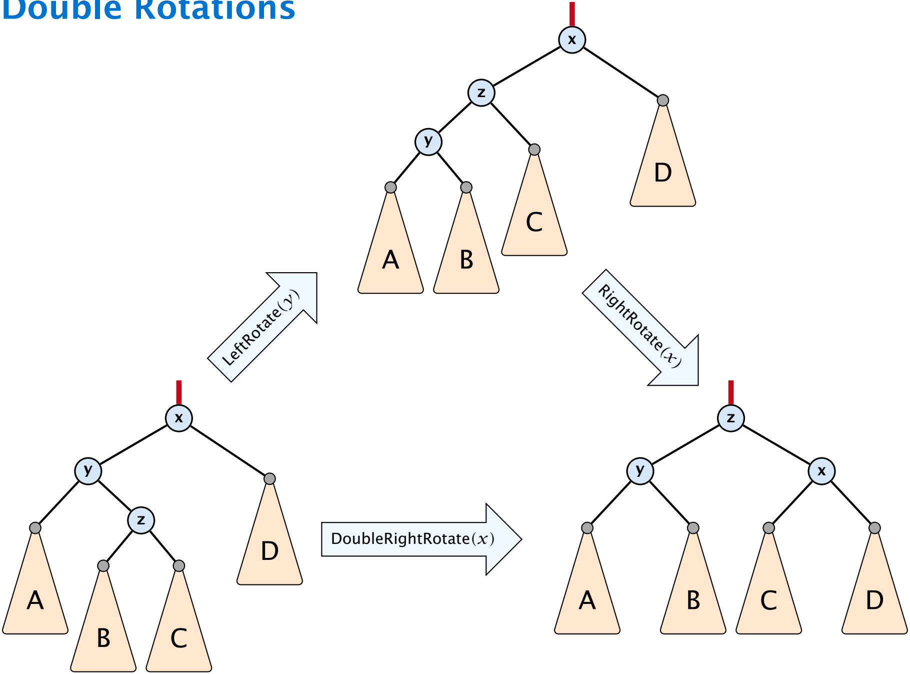
Splay: Zigzag Case



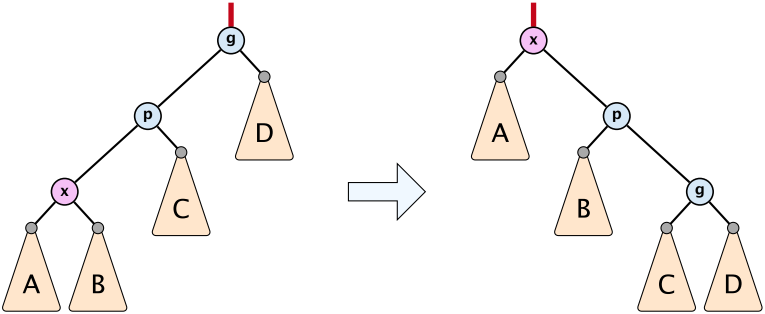
better option splay(x):

- zigzag case: if x is right child and parent of x is left child (or x left child parent of x right child)
- do double right rotation around grand-parent (resp. double left rotation)

Double Rotations



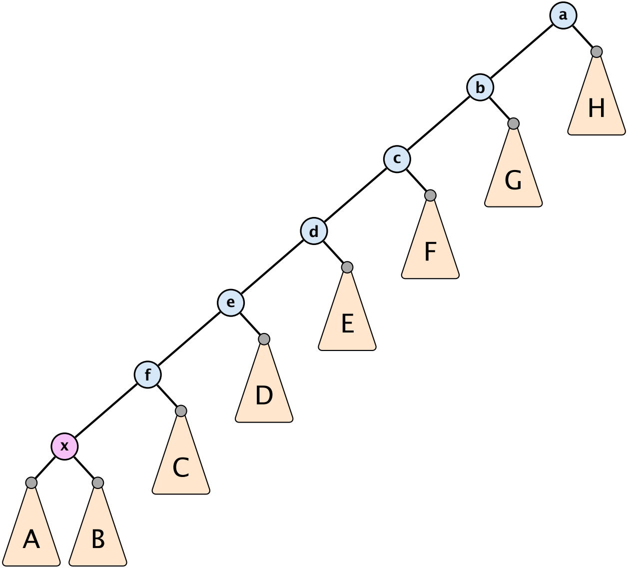
Splay: Zigzig Case



better option splay(x):

- zigzig case: if x is left child and parent of x is left child (or x right child, parent of x right child)
- do right roation around grand-parent followed by right rotation around parent (resp. left rotations)

Splay vs. Move to Root



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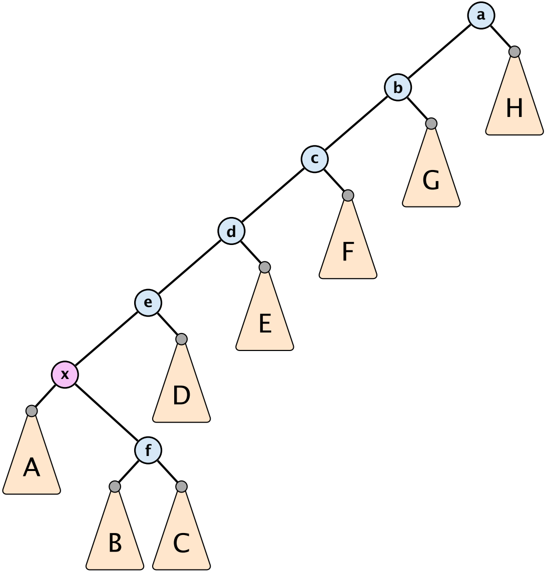

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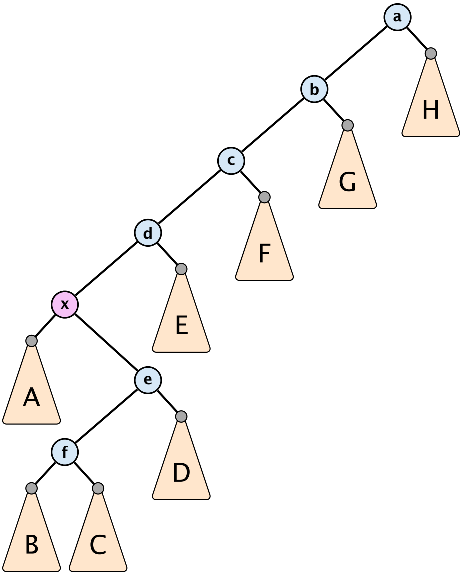

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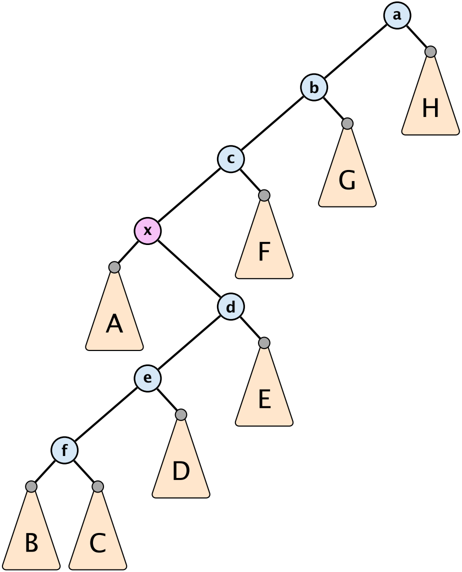

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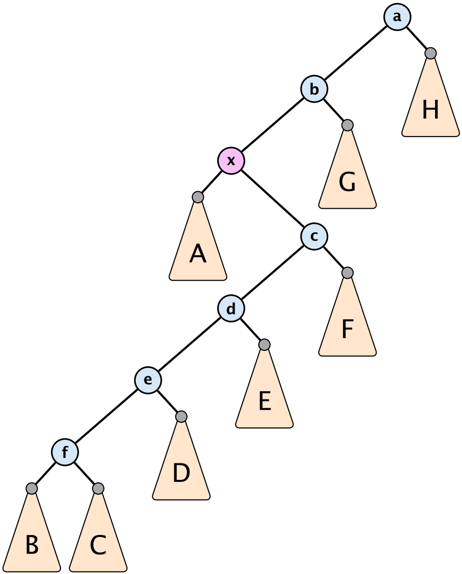

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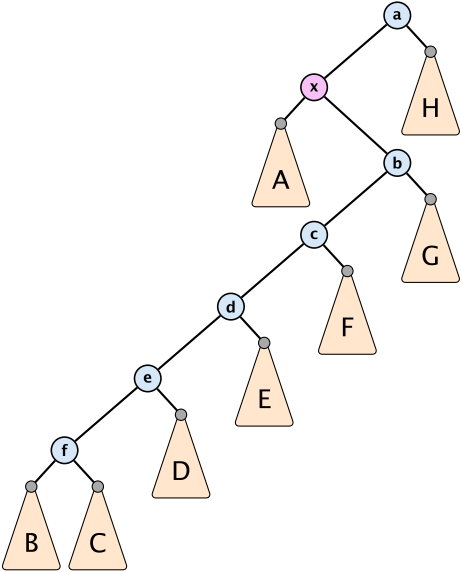

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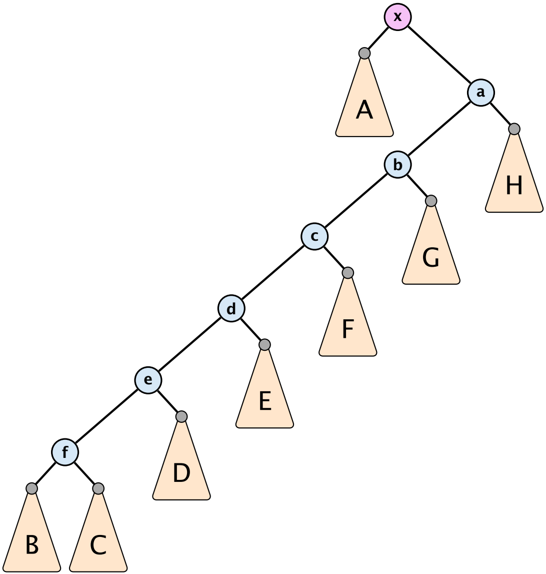

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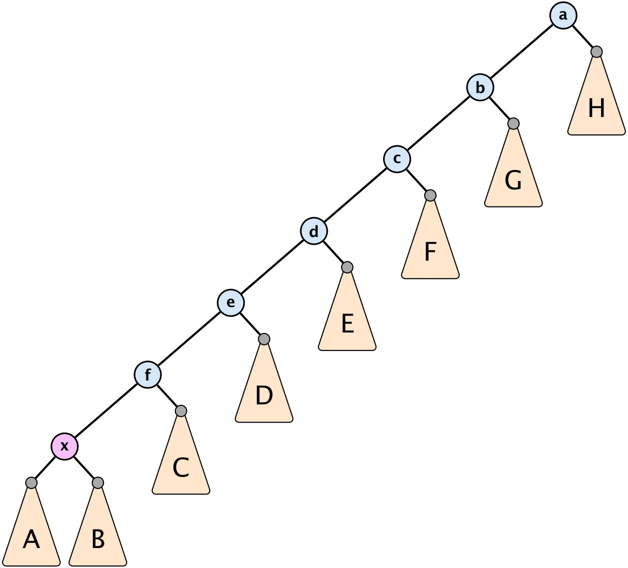

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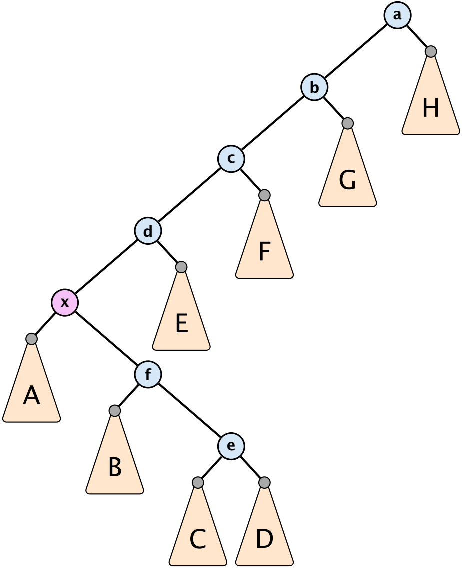

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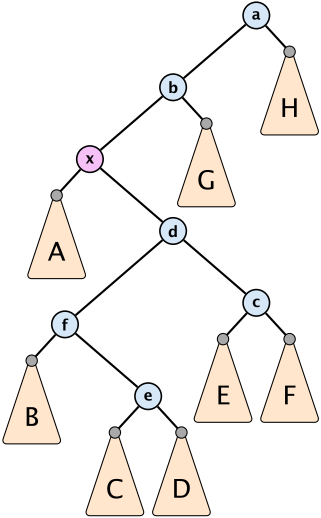

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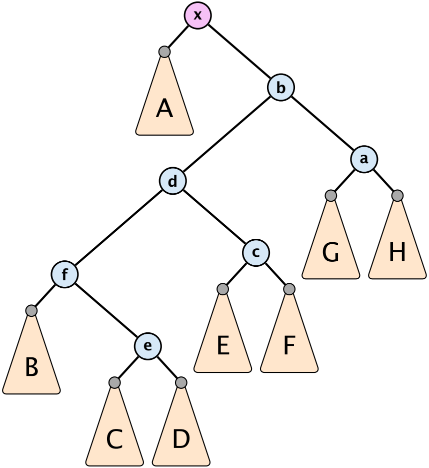

Static Optimality

Suppose we have a sequence of m find-operations. find(x) appears h_x times in this sequence.

The cost of a static search tree T is:

$$\operatorname{cost}(T) = m + \sum_x h_x \operatorname{depth}_T(x)$$

The total cost for processing the sequence on a splay-tree is $\mathcal{O}(\operatorname{cost}(T_{\text{min}}))$, where T_{min} is an optimal static search tree.

Dynamic Optimality

Let S be a sequence with m find-operations.

Let A be a data-structure based on a search tree:

- the cost for accessing element x is $1 + \text{depth}(x)$;
- after accessing x the tree may be re-arranged through rotations;

Conjecture:
A splay tree that only contains elements from S has cost $\mathcal{O}(\text{cost}(A, S))$, for processing S.

Lemma 5

Splay Trees have an amortized running time of $\mathcal{O}(\log n)$ *for all operations.*

Amortized Analysis

Definition 6

A data structure with operations $\mathrm{op}_1(), \ldots, \mathrm{op}_k()$ has amortized running times $t_1, \ldots, t_k$ for these operations if the following holds.

Suppose you are given a sequence of operations (starting with an empty data-structure) that operate on at most n elements, and let k_i denote the number of occurences of $\mathrm{op}_i()$ within this sequence. Then the actual running time must be at most $\sum_i k_i \cdot t_i(n)$.

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Introduce a potential for the data structure.

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$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) \ .$$

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Introduce a potential for the data structure.

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- Amortized cost of the i-th operation is

$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) \ .$$

- Show that $\Phi(D_i) \geq \Phi(D_0)$.

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Introduce a potential for the data structure.

- $\Phi(D_i)$ is the potential after the i-th operation.
- Amortized cost of the i-th operation is

$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) \ .$$

- Show that $\Phi(D_i) \geq \Phi(D_0)$.

Then

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$$\sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} c_i + \Phi(D_k) - \Phi(D_0) = \sum_{i=1}^{k} \hat{c}_i$$

This means the amortized costs can be used to derive a bound on the total cost.

Example: Stack

Stack

- $S.\,\mathbf{push}()$
- $S.\,\mathbf{pop}()$
- $S.\,\mathbf{multipop}(k)$: removes k items from the stack. If the stack currently contains less than k items it empties the stack.
- The user has to ensure that pop and multipop do not generate an underflow.

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- The user has to ensure that pop and multipop do not generate an underflow.

Actual cost:

- $S.\mathbf{push}()$: cost 1.
- $S.\mathbf{pop}()$: cost 1.
- $S.\mathbf{multipop}(k)$: cost $\min\{\text{size}, k\} = k$.

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Use potential function $\Phi(S) =$ number of elements on the stack.

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$$\hat{C}_{\mathrm{pop}} = C_{\mathrm{pop}} + \Delta\Phi = 1 - 1 \le 0 \enspace .$$

- $\boldsymbol{S.\,\mathrm{multipop}(k)}$: cost

$$\hat{C}_{\mathrm{mp}} = C_{\mathrm{mp}} + \Delta\Phi = \min\{\mathrm{size}, k\} - \min\{\mathrm{size}, k\} \le 0 \enspace .$$

Example: Binary Counter

Incrementing a binary counter:
Consider a computational model where each bit-operation costs one time-unit.

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Actual cost:

- Changing bit from 0 to 1: cost 1.
- Changing bit from 1 to 0: cost 1.
- Increment: cost is $k + 1$, where k is the number of consecutive ones in the least significant bit-positions (e.g, 001101 has $k = 1$).

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- Changing bit from 0 to 1:

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- Changing bit from 1 to 0:

$$\hat{C}_{1\to 0} = C_{1\to 0} + \Delta\Phi = 1 - 1 \leq 0 \ .$$

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- Changing bit from 1 to 0:

$$\hat{C}_{1\to0} = C_{1\to0} + \Delta\Phi = 1 - 1 \le 0 \ .$$

- Increment: Let k denotes the number of consecutive ones in the least significant bit-positions. An increment involves k $(1 \to 0)$-operations, and one $(0 \to 1)$-operation.

  Hence, the amortized cost is $k\hat{C}_{1\to0} + \hat{C}_{0\to1} \le 2$.

Splay Trees

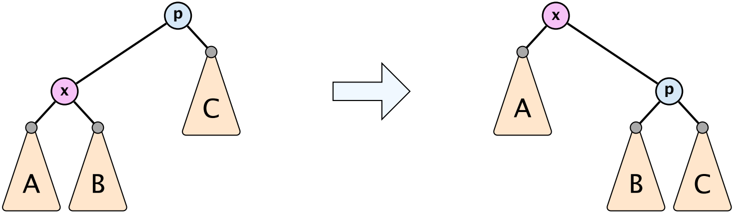
potential function for splay trees:

- size $\mathrm{s}(x) = |T_x|$
- rank $\mathrm{r}(x) = \log_2(s(x))$
- $\Phi(T) = \sum_{v \in T} r(v)$

amortized cost = real cost + potential change

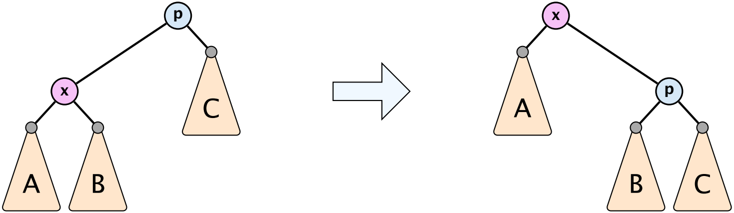
The cost is essentially the cost of the splay-operation, which is 1 plus the number of rotations.

Splay: Zig Case



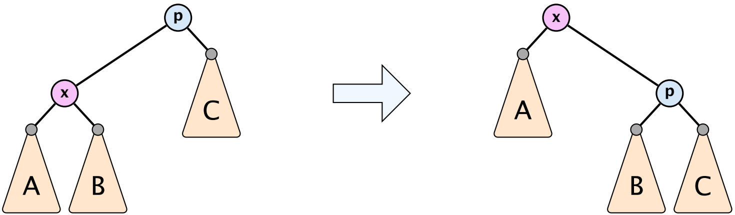
$\Delta\Phi =$

Splay: Zig Case



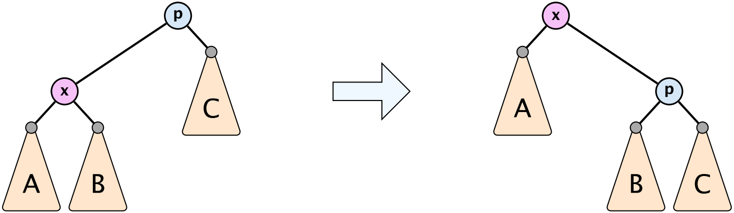
$$\Delta\Phi = r'(x) + r'(p) - r(x) - r(p)$$

Splay: Zig Case



$$\Delta\Phi = r'(x) + r'(p) - r(x) - r(p)$$
$$= r'(p) - r(x)$$

Splay: Zig Case



$$
\begin{aligned}
\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p) \\
&= r'(p) - r(x) \\
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\end{aligned}
$$

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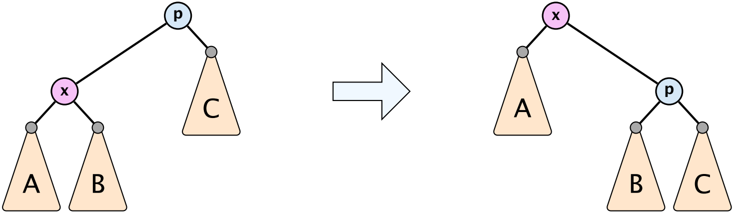
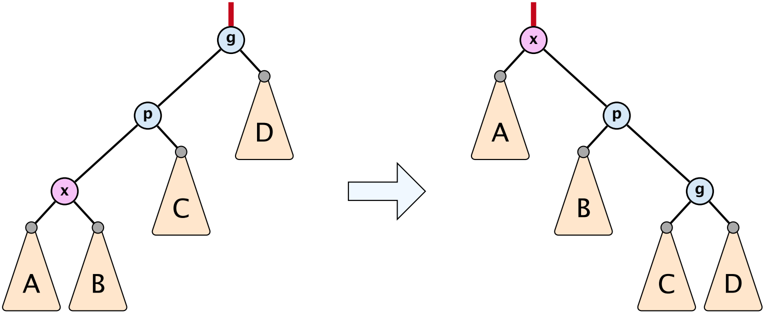


$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p) \\ &= r'(p) - r(x) \\ &\le r'(x) - r(x)\end{aligned}$$

$$\mathrm{cost}_{\mathsf{zig}} \le 1 + 3(r'(x) - r(x))$$

Splay: Zigzig Case



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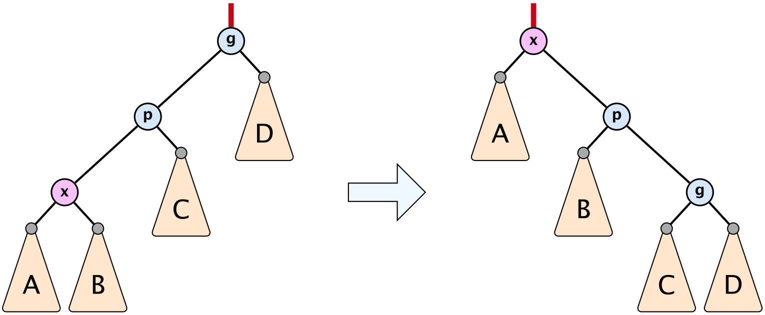

$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$

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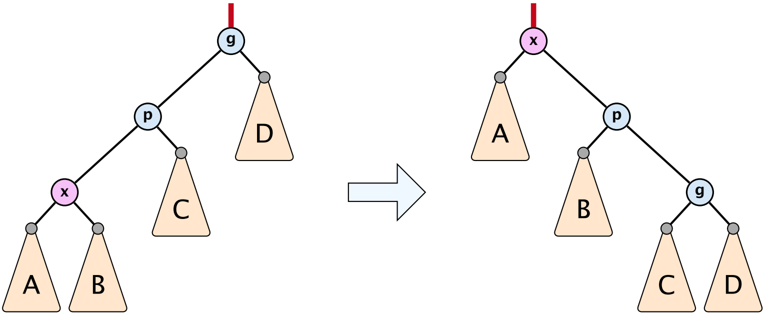

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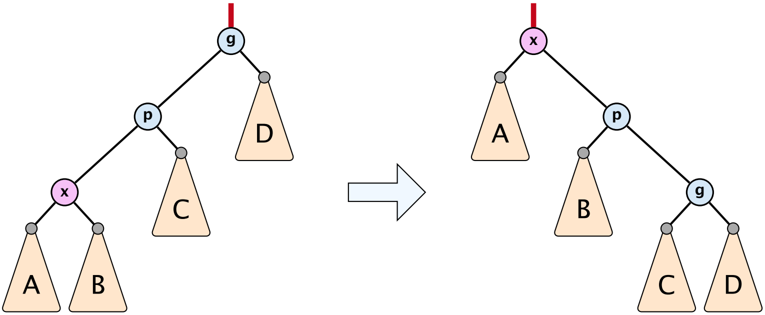

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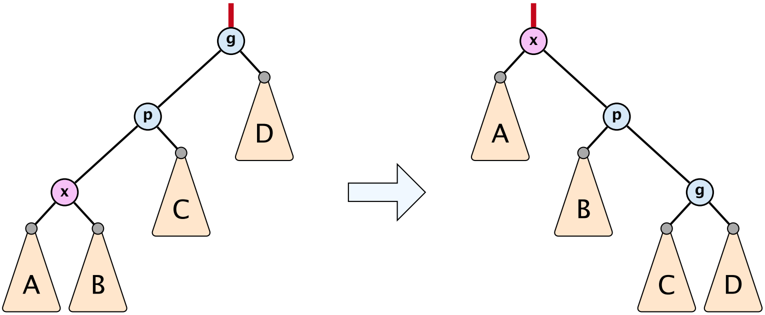

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\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\
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&= r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x)
\end{aligned}
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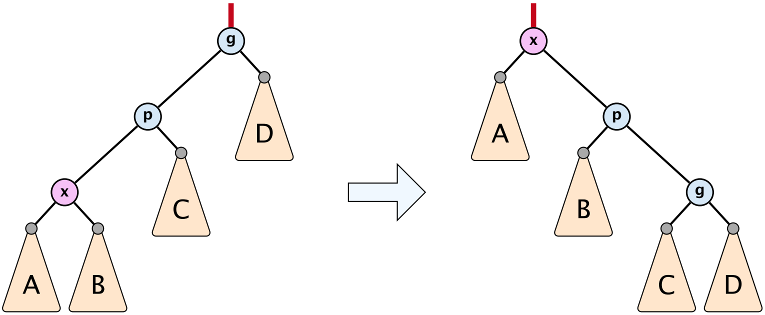

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\end{aligned}
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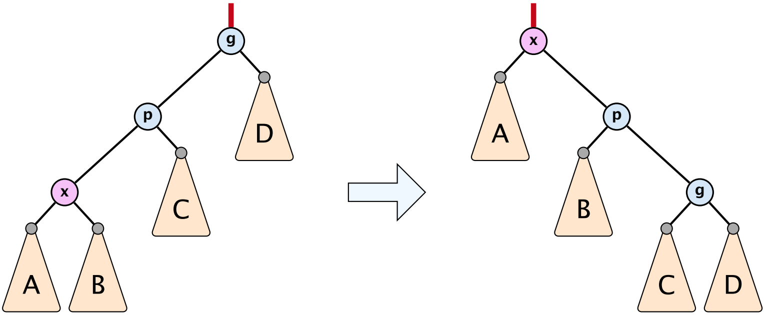

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&\le -2 + 3(r'(x) - r(x))
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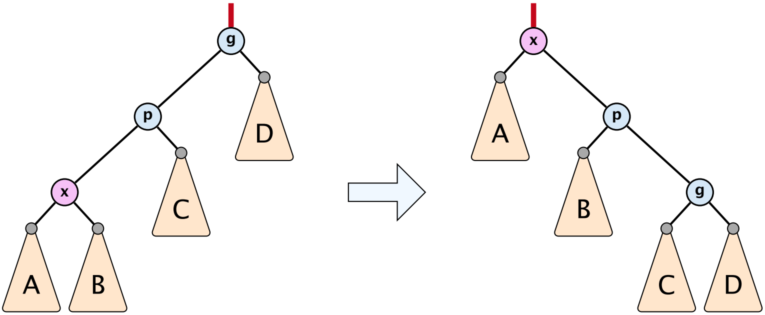

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\end{aligned}
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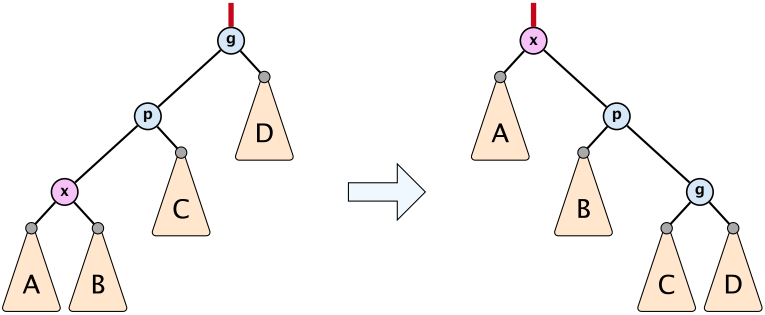

$$\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)$$

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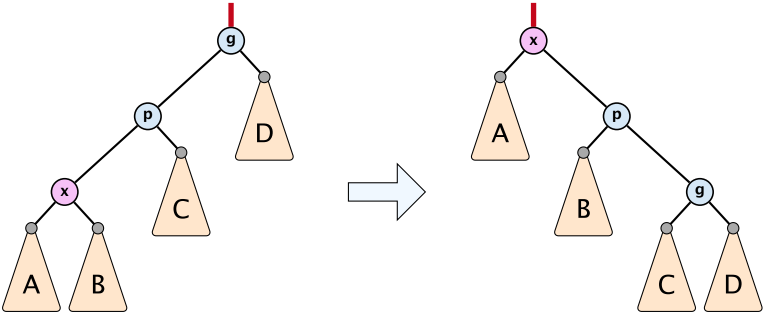

$$\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)$$

$$= \frac{1}{2}\Big(\log(s(x)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

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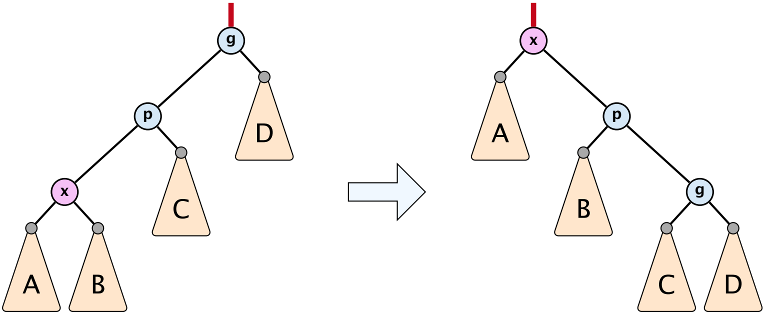
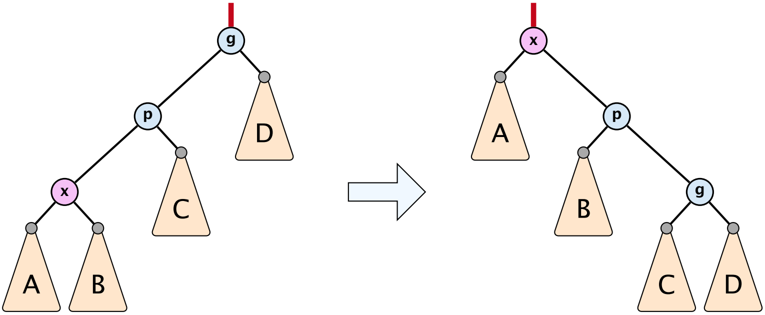


$$\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)$$
$$= \frac{1}{2}\Big(\log(s(x)) + \log(s'(g)) - 2\log(s'(x))\Big)$$
$$= \frac{1}{2}\log\Big(\frac{s(x)}{s'(x)}\Big) + \frac{1}{2}\log\Big(\frac{s'(g)}{s'(x)}\Big)$$

Splay: Zigzig Case



$$\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)$$

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$$= \frac{1}{2}\log\Big(\frac{s(x)}{s'(x)}\Big) + \frac{1}{2}\log\Big(\frac{s'(g)}{s'(x)}\Big)$$

$$\leq \log\Big(\frac{1}{2}\frac{s(x)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\Big)$$

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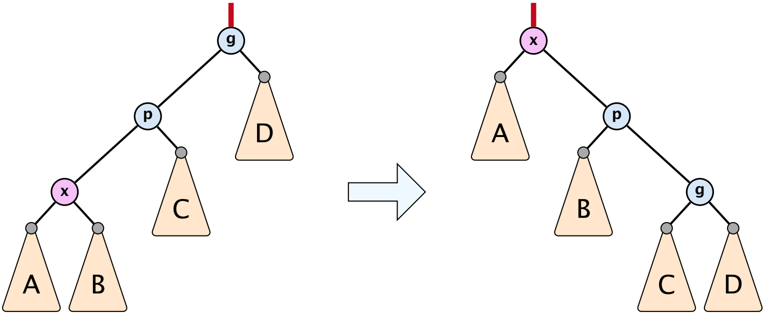

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\begin{aligned}
&\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big) \\
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&\qquad \le \log\Big(\frac{1}{2}\frac{s(x)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\Big) \le \log\Big(\frac{1}{2}\Big)
\end{aligned}
$$

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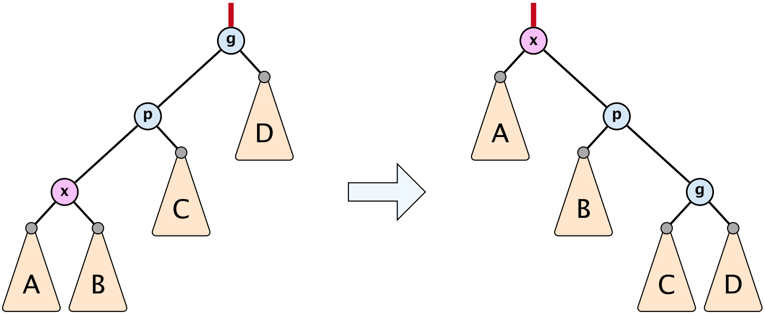

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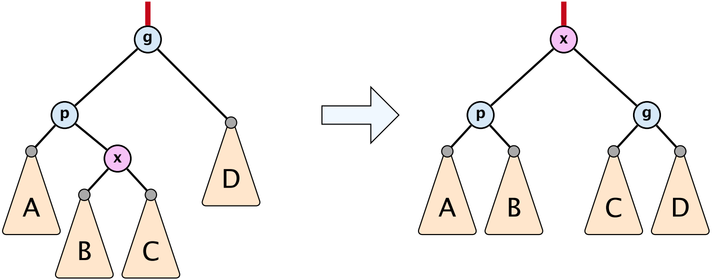
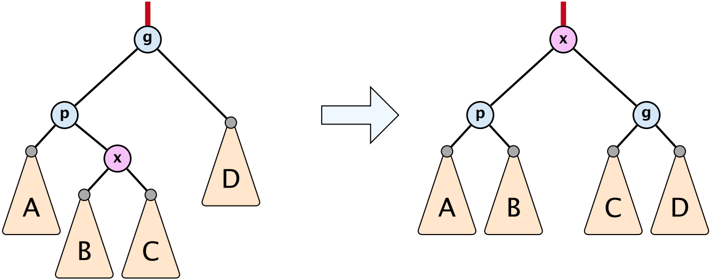
\begin{aligned}
\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big) \\
&= \frac{1}{2}\Big(\log(s(x)) + \log(s'(g)) - 2\log(s'(x))\Big) \\
&= \frac{1}{2}\log\Big(\frac{s(x)}{s'(x)}\Big) + \frac{1}{2}\log\Big(\frac{s'(g)}{s'(x)}\Big) \\
&\leq \log\Big(\frac{1}{2}\frac{s(x)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\Big) \leq \log\Big(\frac{1}{2}\Big) = -1
\end{aligned}
$$

Splay: Zigzag Case



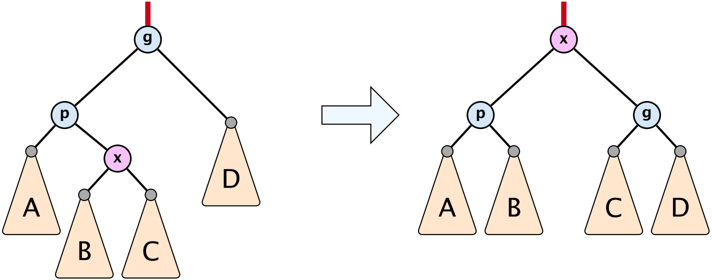
$\Delta\Phi =$

Splay: Zigzag Case



$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$

Splay: Zigzag Case



$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$
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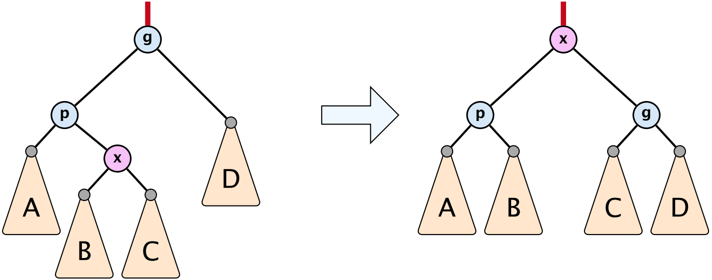
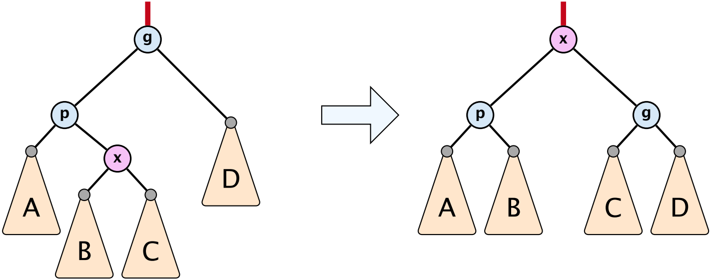
Splay: Zigzag Case



$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)\\ &= r'(p) + r'(g) - r(x) - r(p)\\ &\leq r'(p) + r'(g) - r(x) - r(x)\end{aligned}$$

Splay: Zigzag Case



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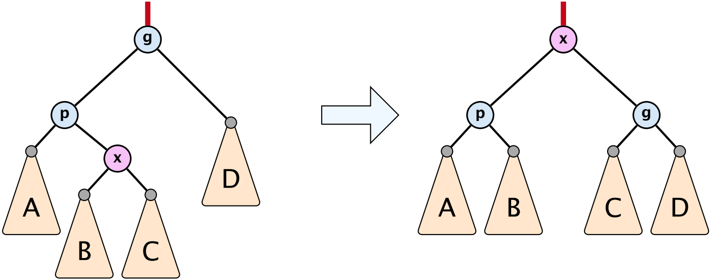
Splay: Zigzag Case



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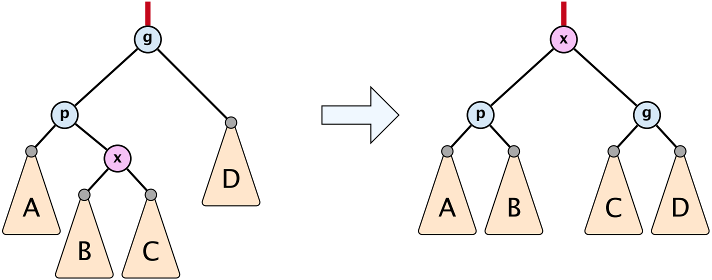
\begin{aligned}
\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\
&= r'(p) + r'(g) - r(x) - r(p) \\
&\leq r'(p) + r'(g) - r(x) - r(x) \\
&= r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x) \\
&\leq -2 + 2(r'(x) - r(x))
\end{aligned}
$$

Splay: Zigzag Case



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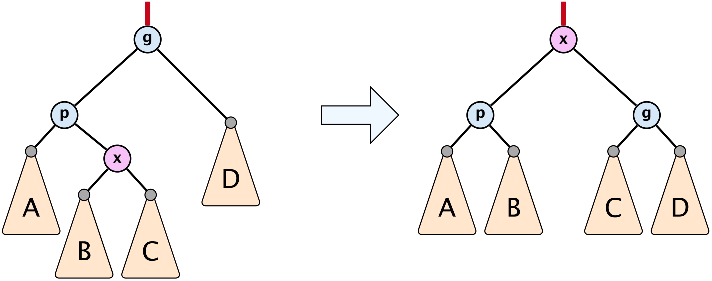
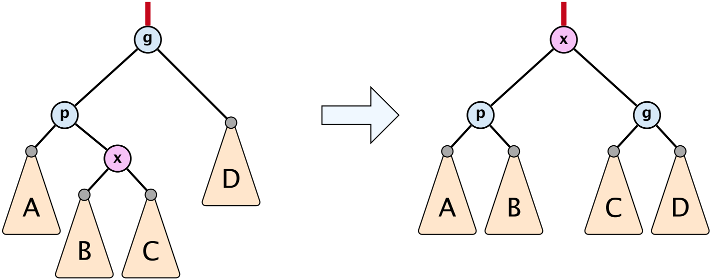
&\le -2 + 2(r'(x) - r(x)) \quad \Rightarrow \mathrm{cost}_{\mathrm{zigzag}} \le 3(r'(x) - r(x))
\end{aligned}
$$

Splay: Zigzag Case



$$\frac{1}{2}\Big(r'(p) + r'(g) - 2r'(x)\Big)$$

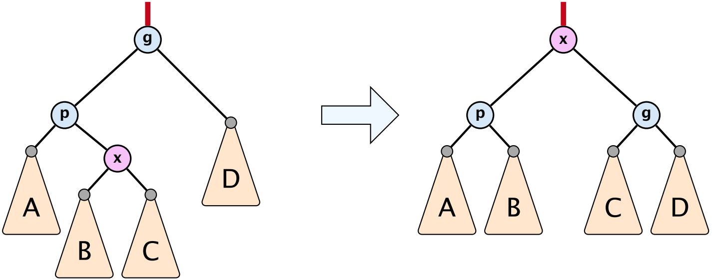
Splay: Zigzag Case



$$\frac{1}{2}\Big(r'(p) + r'(g) - 2r'(x)\Big)$$

$$= \frac{1}{2}\Big(\log(s'(p)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

Splay: Zigzag Case

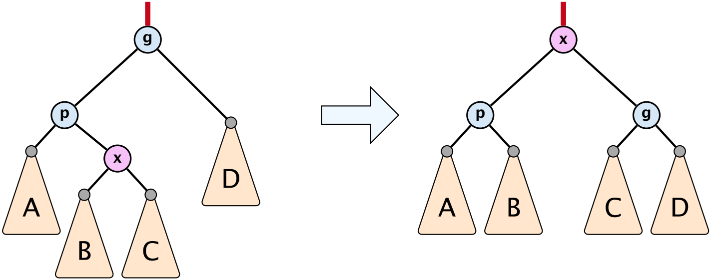


$$\frac{1}{2}\Big(r'(p) + r'(g) - 2r'(x)\Big)$$

$$= \frac{1}{2}\Big(\log(s'(p)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

$$\leq \log\Big(\frac{1}{2}\frac{s'(p)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\Big)$$

Splay: Zigzag Case

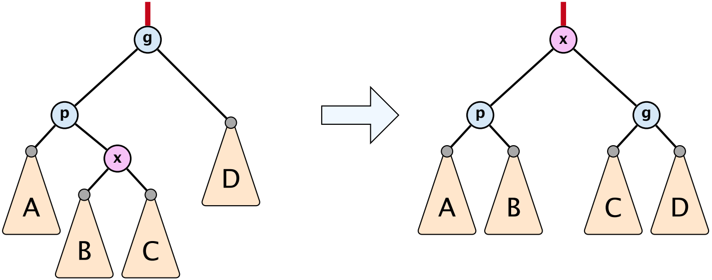


$$\frac{1}{2}\Big(r'(p) + r'(g) - 2r'(x)\Big)$$

$$= \frac{1}{2}\Big(\log(s'(p)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

$$\leq \log\left(\frac{1}{2}\frac{s'(p)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\right) \leq \log\left(\frac{1}{2}\right)$$

Splay: Zigzag Case



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$$\leq \log\left(\frac{1}{2}\frac{s'(p)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\right) \leq \log\left(\frac{1}{2}\right) = -1$$

Amortized cost of the whole splay operation:

$$
\begin{aligned}
&\le 1 + 1 + \sum_{\text{steps } t} 3(r_t(x) - r_{t-1}(x)) \\
&= 2 + 3(r(\text{root}) - r_0(x)) \\
&\le \mathcal{O}(\log n)
\end{aligned}
$$

7.4 Augmenting Data Structures

Suppose you want to develop a data structure with:

- **Insert(x)**: insert element x.
- **Search(k)**: search for element with key k.
- **Delete(x)**: delete element referenced by pointer x.
- **find-by-rank(ℓ)**: return the ℓ-th element; return "error" if the data-structure contains less than ℓ elements.

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Augment an existing data-structure instead of developing a new one.

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How to augment a data-structure

1. choose an underlying data-structure

- Of course, the above steps heavily depend on each other. For example it makes no sense to choose additional information to be stored (Step 2), and later realize that either the information cannot be maintained efficiently (Step 3) or is not sufficient to support the new operations (Step 4).
- However, the above outline is a good way to describe/document a new data-structure.

7.4 Augmenting Data Structures

How to augment a data-structure

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How to augment a data-structure

1. choose an underlying data-structure
2. determine additional information to be stored in the underlying structure
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Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

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3. We need to be able to update the size-field in each node without asymptotically affecting the running time of insert, delete, and search. We come back to this step later...

7.4 Augmenting Data Structures

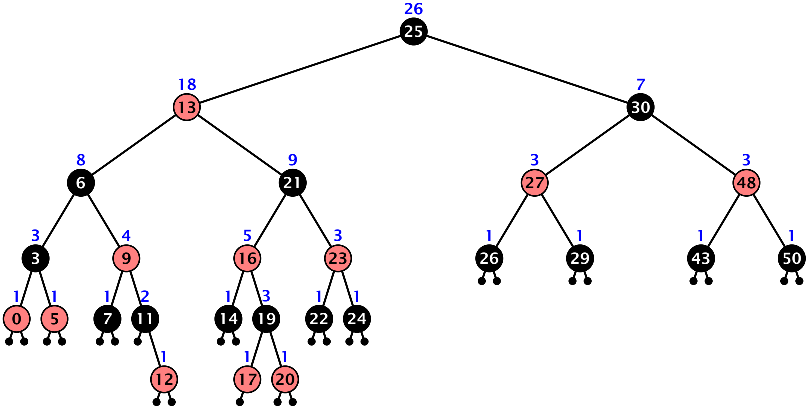
Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

4. How does find-by-rank work?
   Find-by-rank(k) := Select(root,k) with

Algorithm 1 Select(x, i)

if $x = \mathrm{null}$ **then return** error
if $\mathrm{left}[x] \neq \mathrm{null}$ **then** $r \leftarrow \mathrm{left}[x].\,\mathrm{size} + 1$ **else** $r \leftarrow 1$
if $i = r$ **then return** x
if $i < r$ **then**
    return Select($\mathrm{left}[x], i$)
else
    return Select($\mathrm{right}[x], i - r$)

Select(x, i)



Find-by-rank:

- decide whether you have to proceed into the left or right sub-tree
- adjust the rank that you are searching for if you go right

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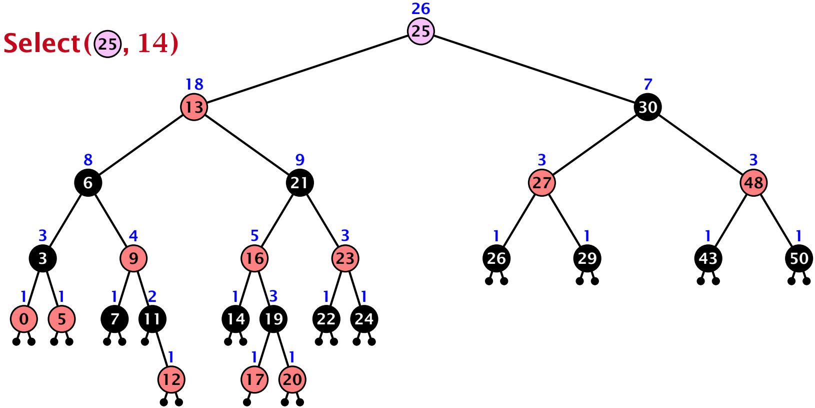

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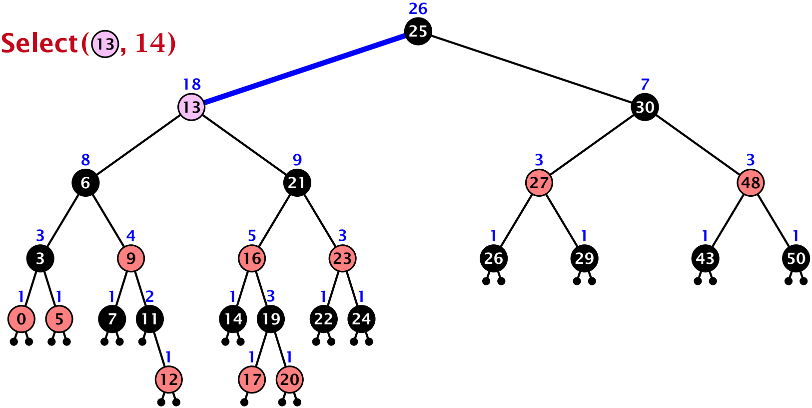

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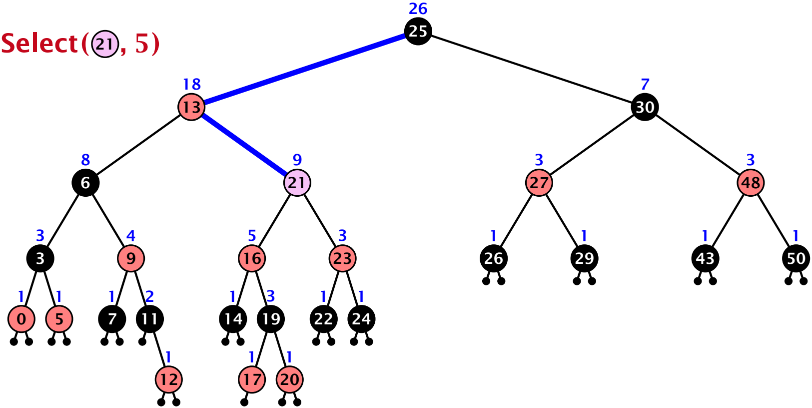
Select(x, i)

Select(21, 5)



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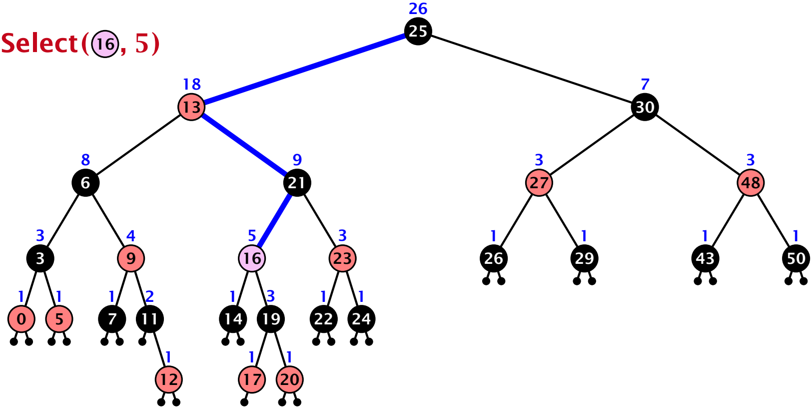
Select(x, i)

Select(16, 5)



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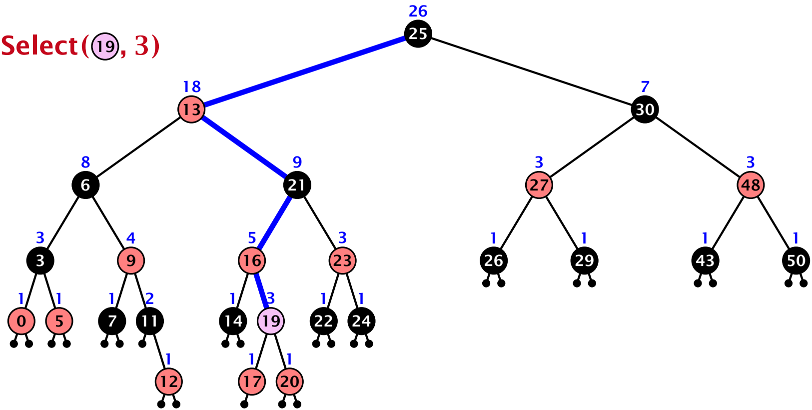

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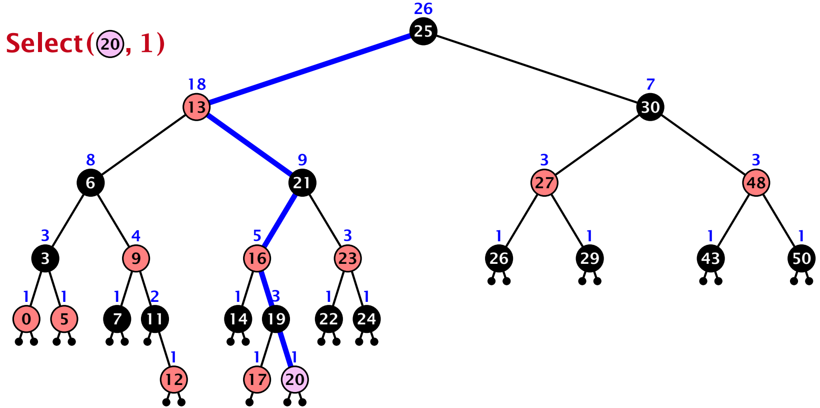

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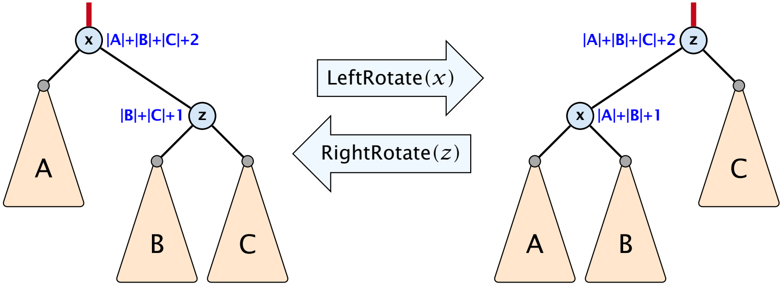
Search(k): Nothing to do.

Insert(x): When going down the search path increase the size field for each visited node. Maintain the size field during rotations.

Delete(x): Directly after splicing out a node traverse the path from the spliced out node upwards, and decrease the size counter on every node on this path. Maintain the size field during rotations.

Rotations

The only operation during the fix-up procedure that alters the tree and requires an update of the size-field:



The nodes x and z are the only nodes changing their size-fields.

The new size-fields can be computed locally from the size-fields of the children.

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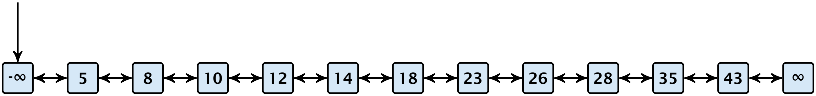

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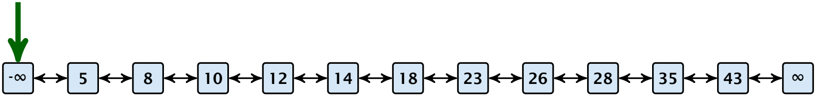

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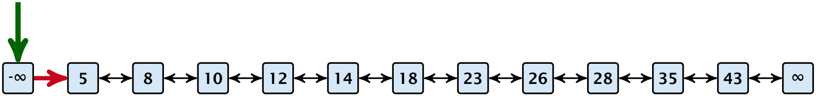

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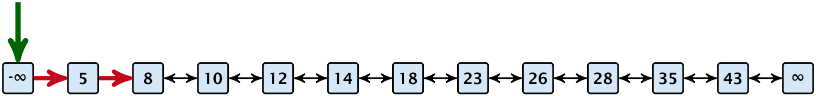

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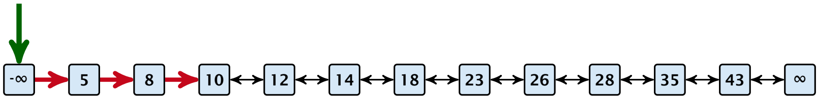

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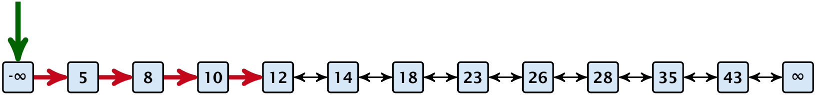

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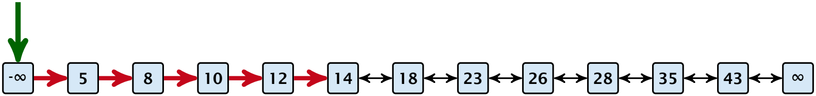

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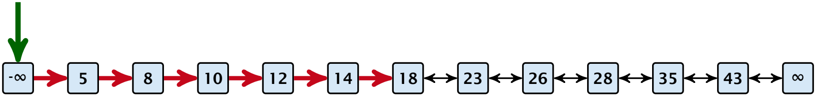

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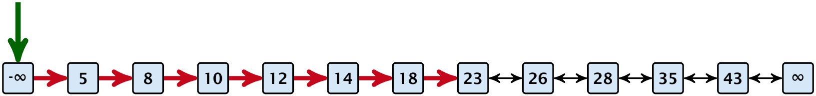

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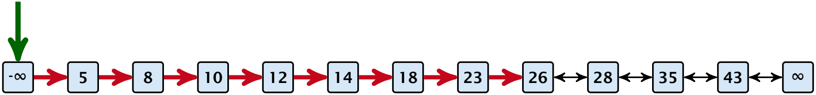

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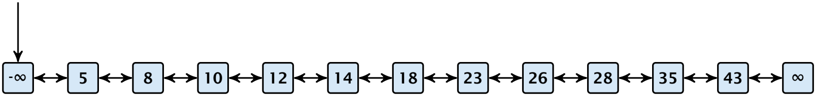

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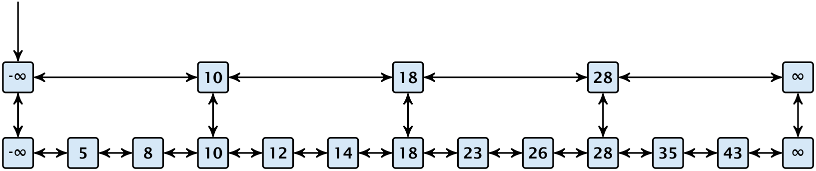

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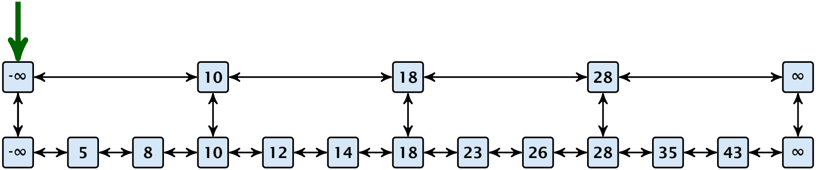

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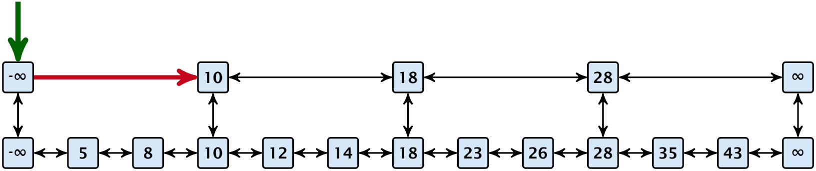

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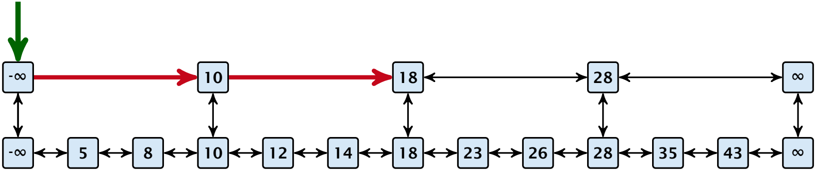

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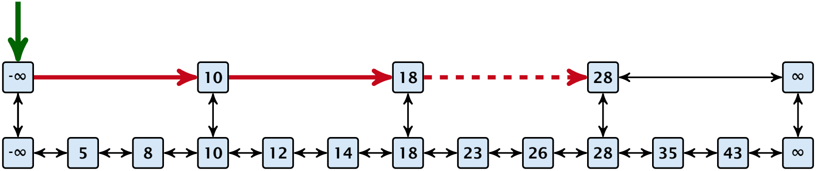

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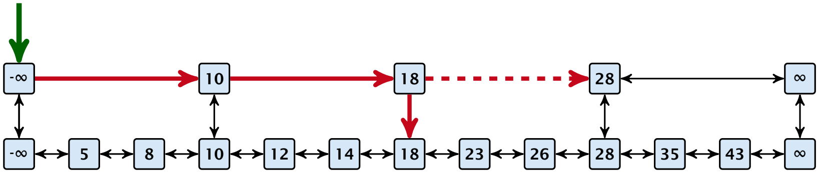

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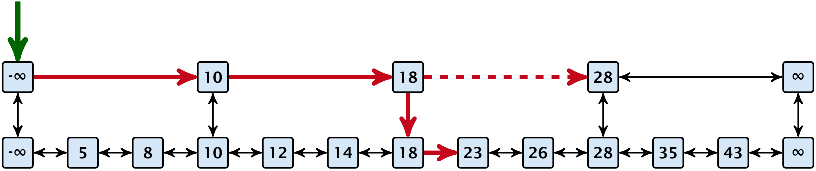

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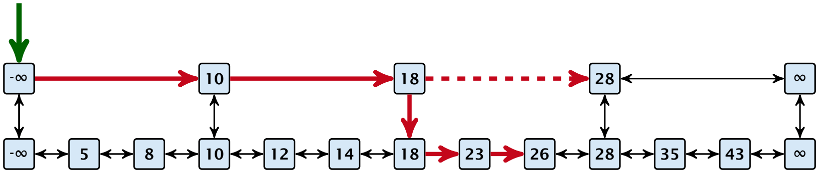

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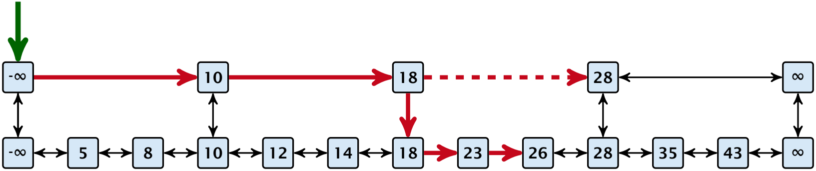

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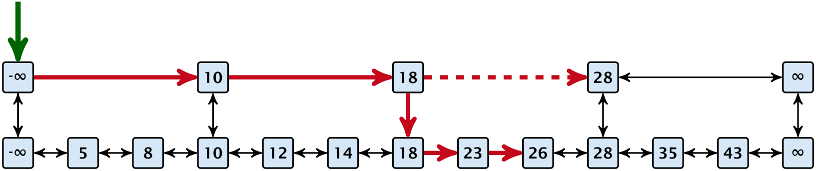

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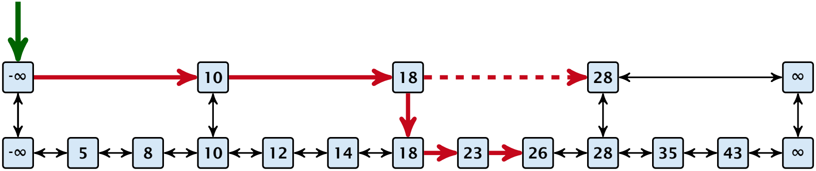

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Choose $|L_1| = \sqrt{n}$. Then search time $\Theta(\sqrt{n})$.

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- ...
- At most $|L_k| + \sum_{i=1}^{k} \frac{L_{i-1}}{L_i} + 3(k+1)$ steps.

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Choose ratios between list-lengths evenly, i.e., $\frac{|L_{i-1}|}{|L_i|} = r$, and, hence, $L_k \approx r^{-k}n$.

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Choosing $k = \Theta(\log n)$ gives a logarithmic running time.

7.5 Skip Lists

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Use randomization instead!

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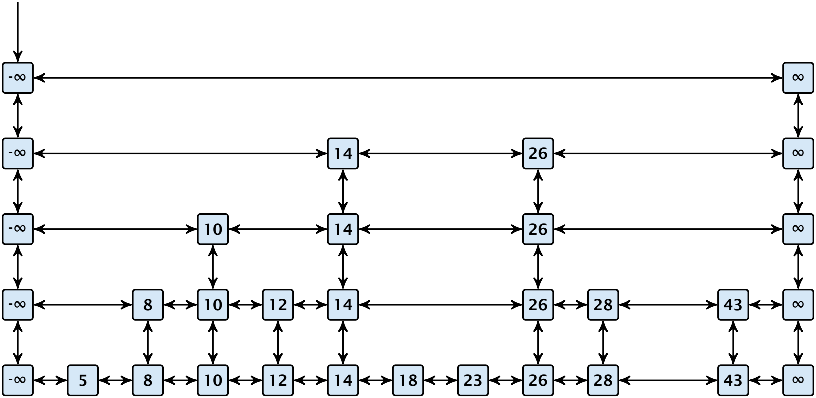
The time for both operations is dominated by the search time.

Insert (35):



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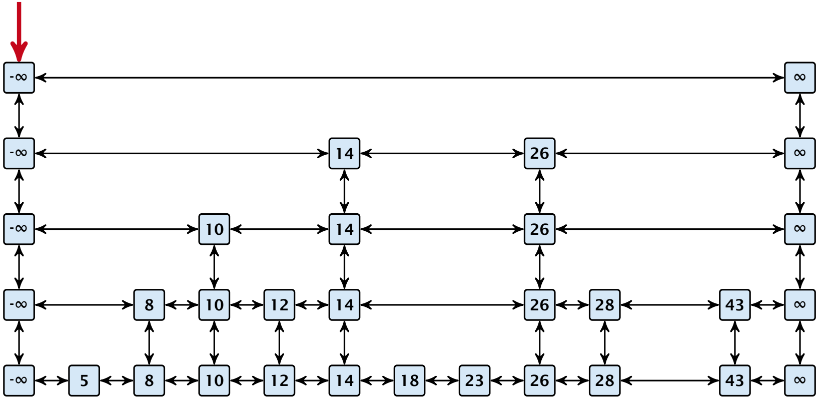

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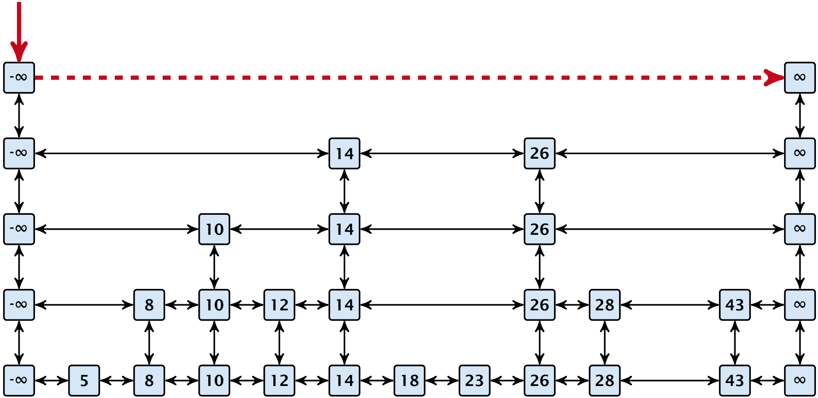

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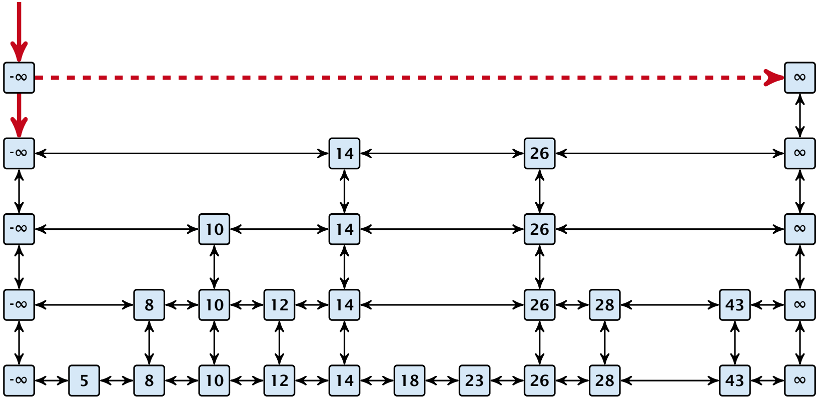

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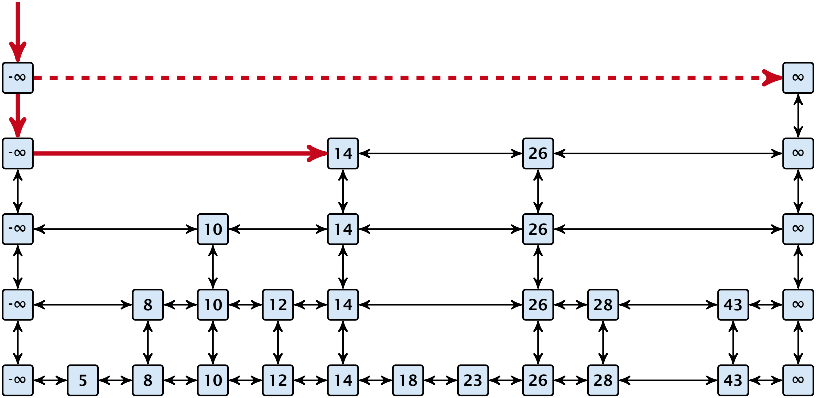

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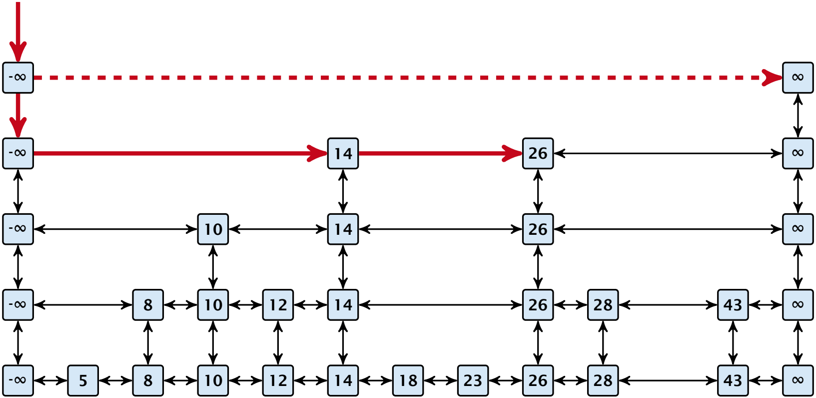

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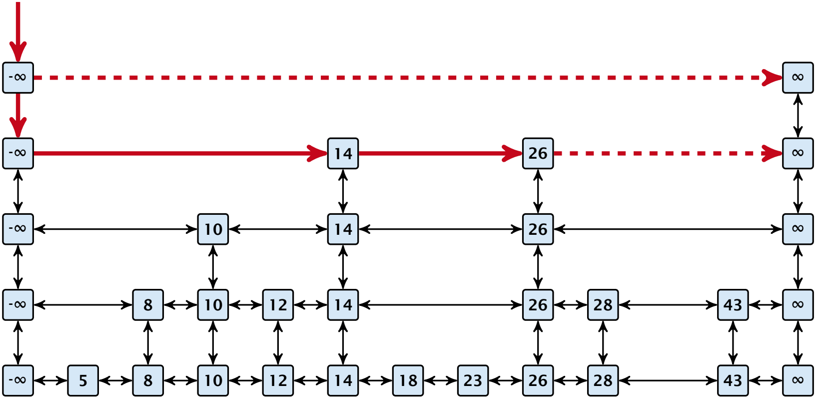

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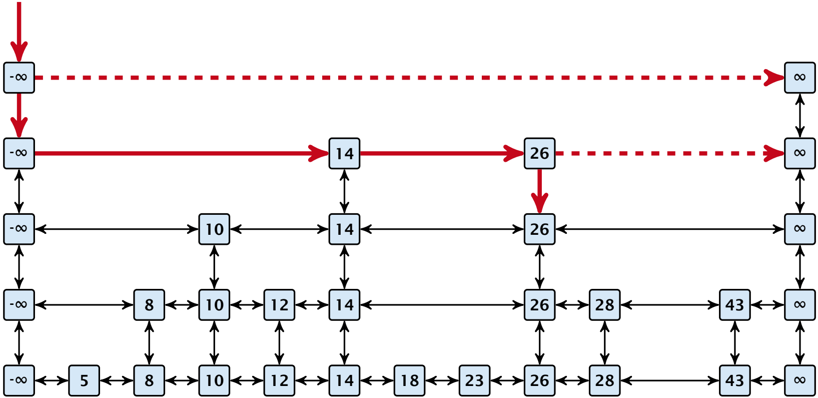

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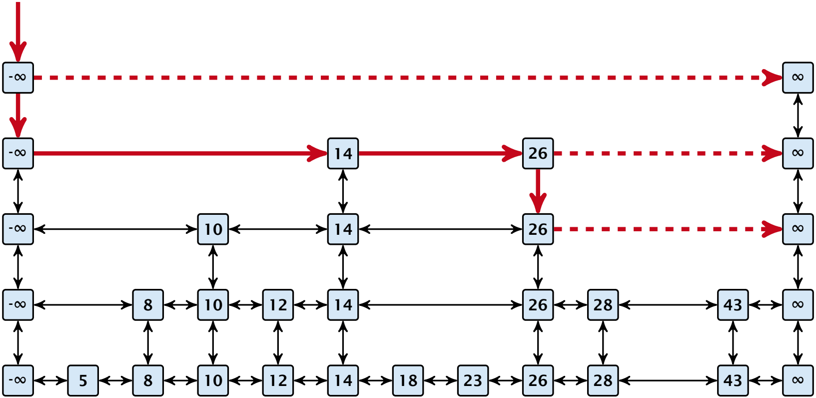

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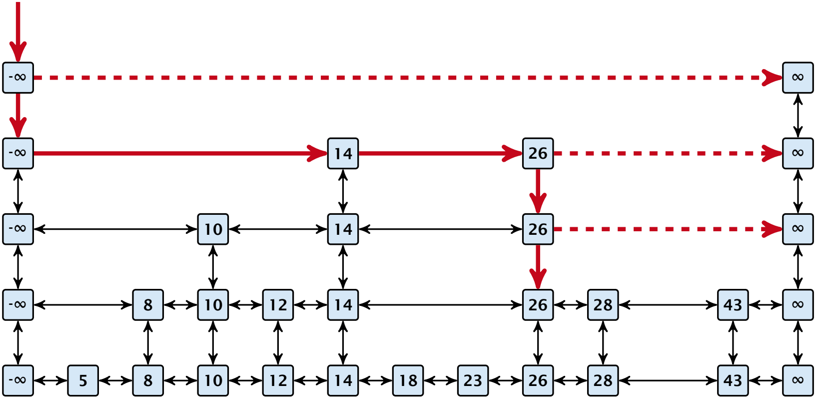

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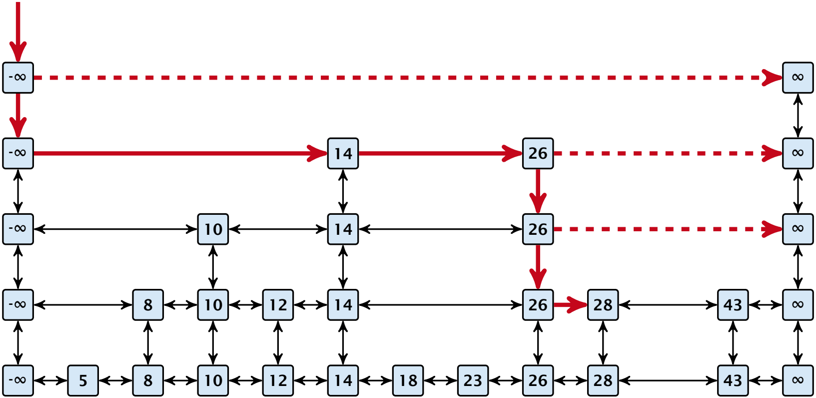

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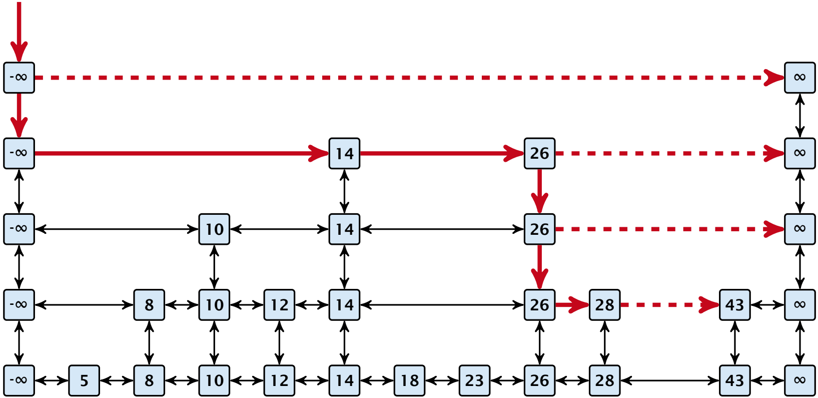

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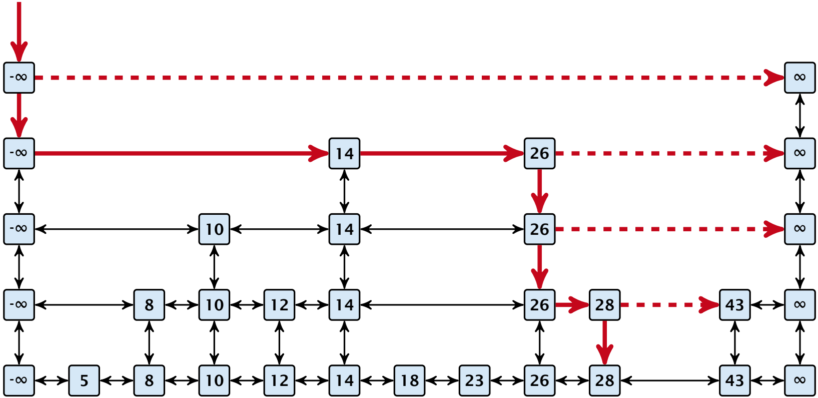

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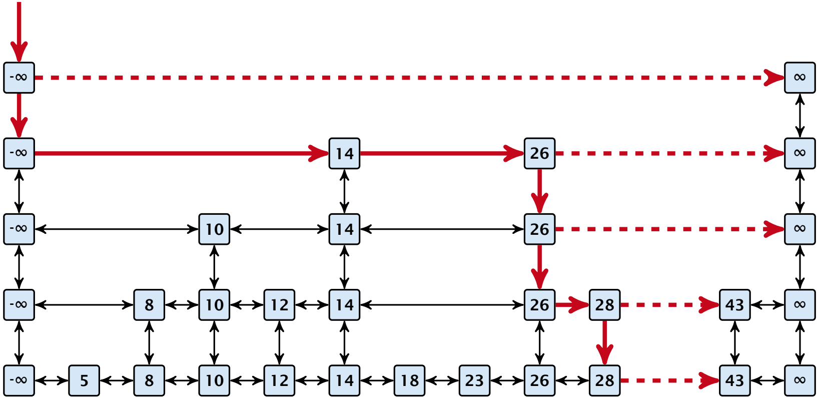

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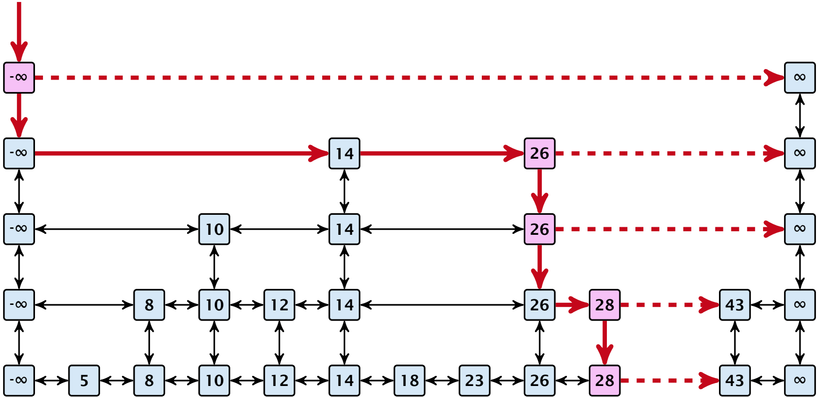

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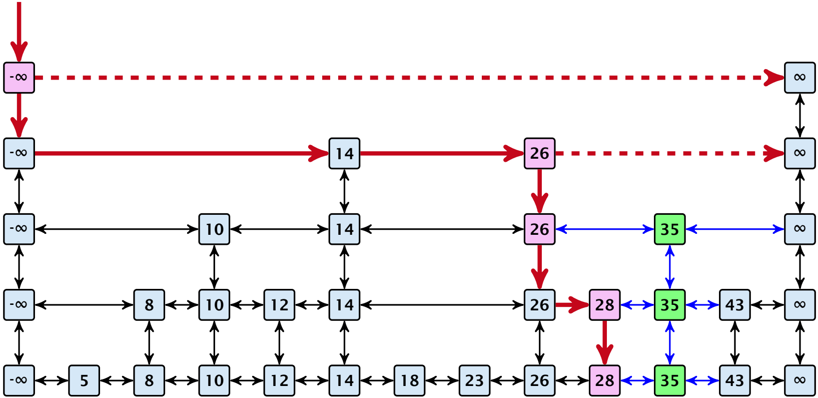

High Probability

Definition 7 (High Probability)

We say a **randomized** algorithm has running time $\mathcal{O}(\log n)$ with high probability if for any constant α the running time is at most $\mathcal{O}(\log n)$ with probability at least $1 - \frac{1}{n^\alpha}$.

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Here the $\mathcal{O}$-notation hides a constant that may depend on α.

High Probability

Suppose there are polynomially many events $E_1, E_2, \ldots, E_\ell$, $\ell = n^c$ each holding with high probability (e.g. E_i may be the event that the i-th search in a skip list takes time at most $\mathcal{O}(\log n)$).

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Then the probability that all E_i hold is at least

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This means $\Pr[E_1 \wedge \cdots \wedge E_\ell]$ holds with high probability.

Lemma 8

A search (and, hence, also insert and delete) in a skip list with n *elements takes time* $\mathcal{O}(\log n)$ *with high probability (w. h. p.).*

7.5 Skip Lists

Backward analysis:

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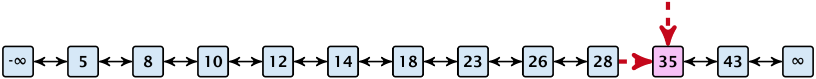

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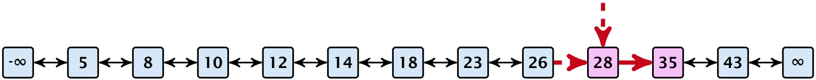

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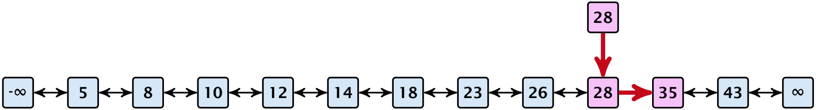

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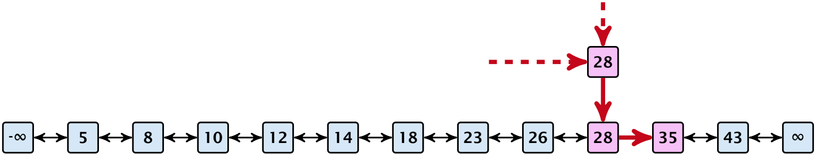

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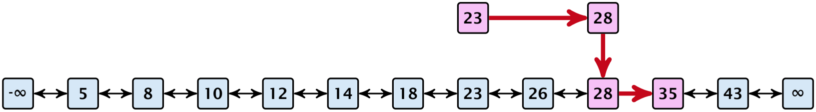

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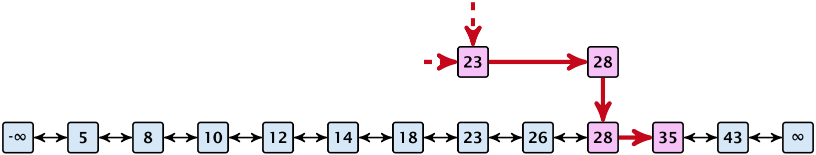

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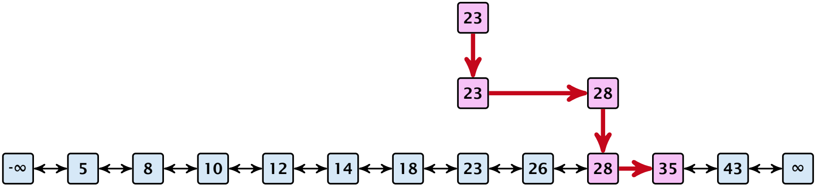

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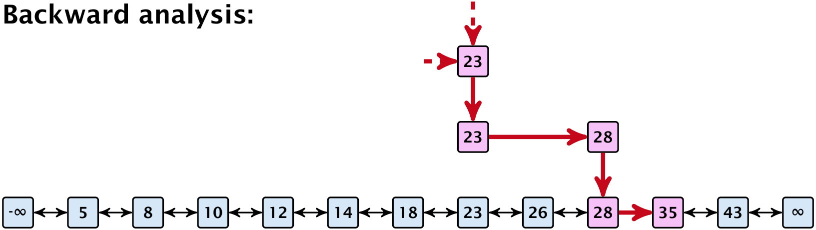

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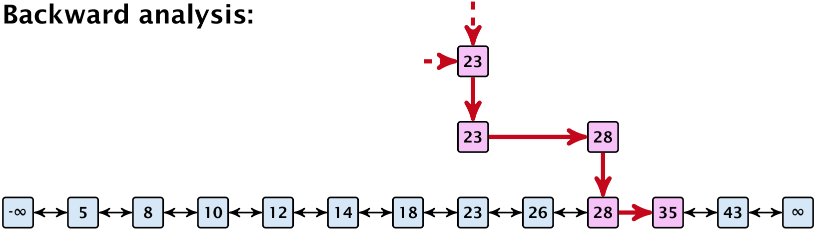

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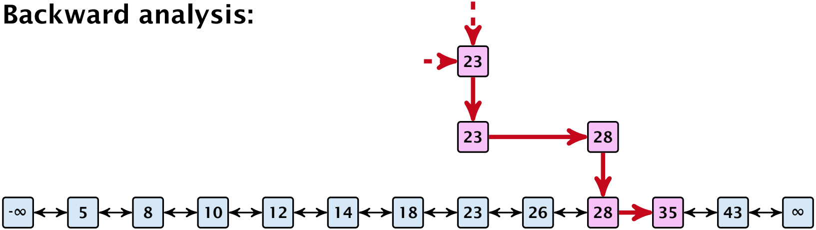

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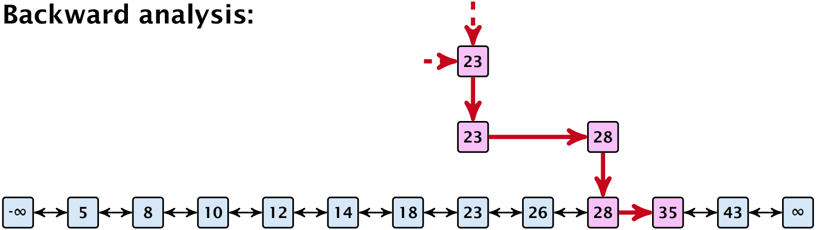

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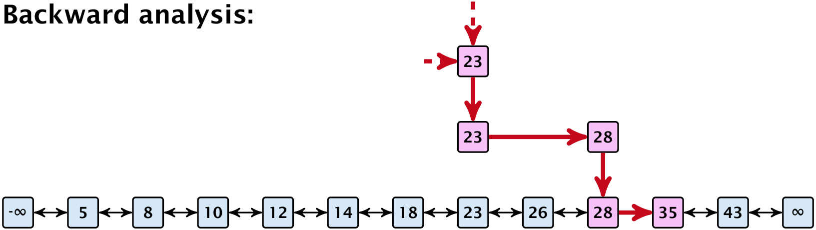

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From this it follows that w.h.p. there are no long paths.

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$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

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In particular, this means that during the construction in the backward analysis we see at most k heads (i.e., coin flips that tell you to go up) in z trials.

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This means, the search requires at most z steps, w. h. p.

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Dictionary:

- $S.\mathbf{insert}(\boldsymbol{x})$: Insert an element x.
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Hashing tries to directly compute the memory location from the given key. The goal is to have constant search time.

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Definitions:

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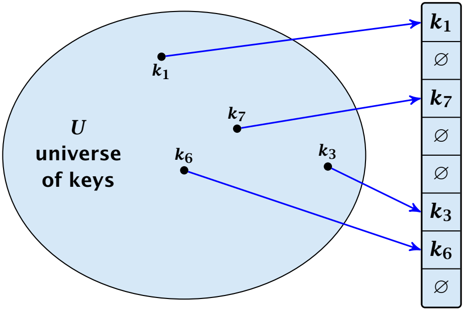
- Good distribution of elements over the whole table.

Direct Addressing

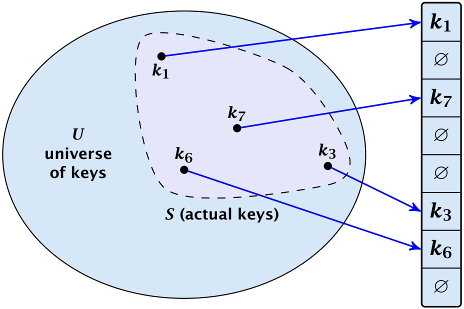
Ideally the hash function maps all keys to different memory locations.



This special case is known as Direct Addressing. It is usually very unrealistic as the universe of keys typically is quite large, and in particular larger than the available memory.

Perfect Hashing

Suppose that we know the set S of actual keys (no insert/no delete). Then we may want to design a simple hash-function that maps all these keys to different memory locations.



Such a hash function h is called a perfect hash function for set S.

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Hence, there may be two elements k_1, k_2 from the set S that map to the same memory location (i.e., $h(k_1) = h(k_2)$). This is called a collision.

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Lemma 9

The probability of having a collision when hashing m elements into a table of size n under uniform hashing is at least

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Uniform hashing:

Choose a hash function uniformly at random from all functions $f : U \to [0, \ldots, n-1]$.

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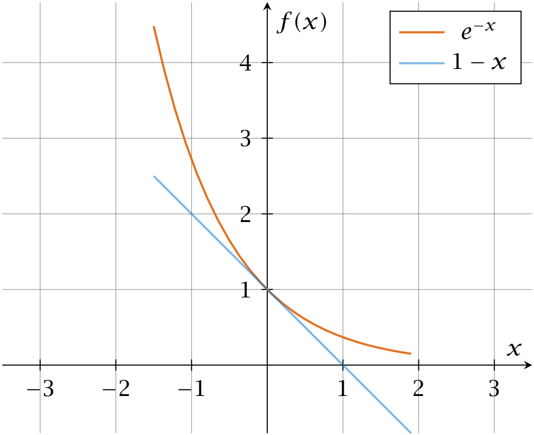
Here the first equality follows since the ℓ-th element that is hashed has a probability of $\frac{n-\ell+1}{n}$ to not generate a collision under the condition that the previous elements did not induce collisions. □

Collisions



The inequality $1 - x \leq e^{-x}$ is derived by stopping the Taylor-expansion of e^{-x} after the second term.

Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

- **open addressing**, aka. closed hashing
- **hashing with chaining**, aka. closed addressing, open hashing.

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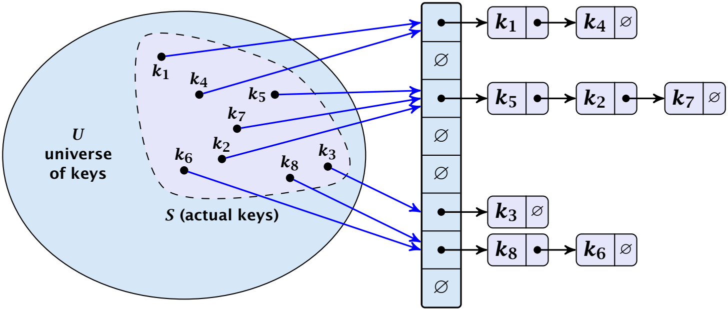
There are applications e.g. computer chess where you do not resolve collisions at all.

Hashing with Chaining

Arrange elements that map to the same position in a linear list.

- Access: compute $h(x)$ and search list for $\text{key}[x]$.
- Insert: insert at the front of the list.



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We assume uniform hashing for the following analysis.

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$$A^{-} = 1 + \alpha \ .$$

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Hashing with Chaining

For a successful search observe that we do **not** choose a list at random, but we consider a random key k in the hash-table and ask for the search-time for k.

This is 1 plus the number of elements that lie before k in k's list.

Let k_ℓ denote the ℓ-th key inserted into the table.

Let for two keys k_i and k_j, X_{ij} denote the indicator variable for the event that k_i and k_j hash to the same position. Clearly, $\Pr[X_{ij} = 1] = 1/n$ for uniform hashing.

The expected successful search cost is

$$\mathrm{E}\Big[\frac{1}{m}\sum_{i=1}^{m}\underbrace{\Big(1 + \sum_{j=i+1}^{m} X_{ij}\Big)}_{\text{cost for key } k_i}\Big]$$

Hashing with Chaining

$$\mathrm{E}\left[\frac{1}{m}\sum_{i=1}^{m}\left(1+\sum_{j=i+1}^{m}X_{ij}\right)\right]$$

Hashing with Chaining

$$\mathrm{E}\left[\frac{1}{m}\sum_{i=1}^{m}\left(1+\sum_{j=i+1}^{m}X_{ij}\right)\right]=\frac{1}{m}\sum_{i=1}^{m}\left(1+\sum_{j=i+1}^{m}\mathrm{E}\left[X_{ij}\right]\right)$$

Hashing with Chaining

$$
\begin{aligned}
\mathrm{E}\left[\frac{1}{m}\sum_{i=1}^{m}\left(1+\sum_{j=i+1}^{m}X_{ij}\right)\right] &= \frac{1}{m}\sum_{i=1}^{m}\left(1+\sum_{j=i+1}^{m}\mathrm{E}\left[X_{ij}\right]\right) \\
&= \frac{1}{m}\sum_{i=1}^{m}\left(1+\sum_{j=i+1}^{m}\frac{1}{n}\right)
\end{aligned}
$$

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&= \frac{1}{m}\sum_{i=1}^{m}\left(1+\sum_{j=i+1}^{m}\frac{1}{n}\right)\\
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\end{aligned}
$$

Hence, the expected cost for a successful search is $A^{+} \leq 1 + \frac{\alpha}{2}$.

Hashing with Chaining

Disadvantages:

- pointers increase memory requirements
- pointers may lead to bad cache efficiency

Advantages:

- no à priori limit on the number of elements
- deletion can be implemented efficiently
- by using balanced trees instead of linked list one can also obtain worst-case guarantees.

Open Addressing

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All objects are stored in the table itself.

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Define a function $h(k,j)$ that determines the table-position to be examined in the j-th step. The values $h(k,0)$,. . . ,$h(k,n-1)$ must form a permutation of 0, . . . , $n-1$.

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Search(k): Try position $h(k, 0)$; if it is empty your search fails; otw. continue with $h(k, 1)$, $h(k, 2)$, ...

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Search(k): Try position $h(k,0)$; if it is empty your search fails; otw. continue with $h(k,1)$, $h(k,2)$, ...

Insert(x): Search until you find an empty slot; insert your element there. If your search reaches $h(k,n-1)$, and this slot is non-empty then your table is full.

Open Addressing

Choices for $h(k,j)$:

- Linear probing:
  $h(k,i) = h(k) + i \mod n$
  (sometimes: $h(k,i) = h(k) + ci \mod n$).

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For quadratic probing and double hashing one has to ensure that the search covers all positions in the table (i.e., for double hashing $h_2(k)$ must be relatively prime to n (teilerfremd); for quadratic probing c_1 and c_2 have to be chosen carefully).

Linear Probing

- Advantage: Cache-efficiency. The new probe position is very likely to be in the cache.

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Lemma 10

Let L be the method of linear probing for resolving collisions:

$$L^+ \approx \frac{1}{2}\left(1 + \frac{1}{1-\alpha}\right)$$

$$L^- \approx \frac{1}{2}\left(1 + \frac{1}{(1-\alpha)^2}\right)$$

Quadratic Probing

- Not as cache-efficient as Linear Probing.
- Secondary clustering: caused by the fact that all keys mapped to the same position have the same probe sequence.

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Lemma 11

Let Q be the method of quadratic probing for resolving collisions:

$$Q^+ \approx 1 + \ln\left(\frac{1}{1-\alpha}\right) - \frac{\alpha}{2}$$

$$Q^- \approx \frac{1}{1-\alpha} + \ln\left(\frac{1}{1-\alpha}\right) - \alpha$$

Double Hashing

- Any probe into the hash-table usually creates a cache-miss.

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Lemma 12

Let D be the method of double hashing for resolving collisions:

$$D^+ \approx \frac{1}{\alpha}\ln\left(\frac{1}{1-\alpha}\right)$$

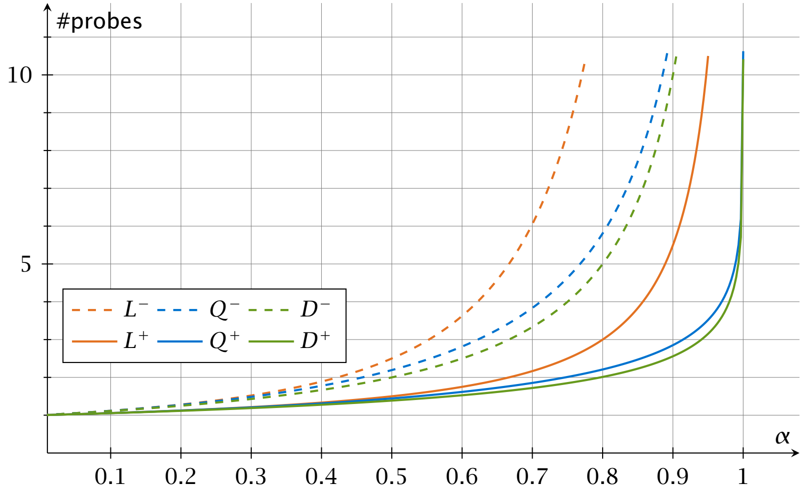
$$D^- \approx \frac{1}{1-\alpha}$$

Open Addressing

Some values:

α	*Linear Probing*		*Quadratic Probing*		*Double Hashing*	
	L^+	L^-	Q^+	Q^-	D^+	D^-
0.5	1.5	2.5	1.44	2.19	1.39	2
0.9	5.5	50.5	2.85	11.40	2.55	10
0.95	10.5	200.5	3.52	22.05	3.15	20

Open Addressing



Analysis of Idealized Open Address Hashing

We analyze the time for a search in a very idealized Open Addressing scheme.

- The probe sequence $h(k,0), h(k,1), h(k,2), \ldots$ is equally likely to be any permutation of $\langle 0, 1, \ldots, n-1 \rangle$.

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$$\Pr[A_1 \cap A_2 \cap \cdots \cap A_{i-1}]$$

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$$\Pr[X \geq i]$$

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$$\Pr[X \geq i] = \frac{m}{n} \cdot \frac{m-1}{n-1} \cdot \frac{m-2}{n-2} \cdot \ldots \cdot \frac{m-i+2}{n-i+2}$$

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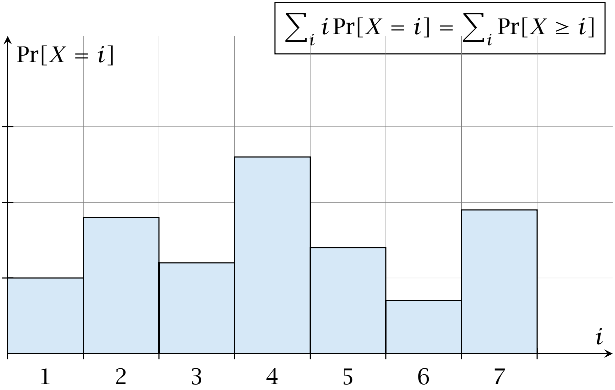
$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \ldots$$

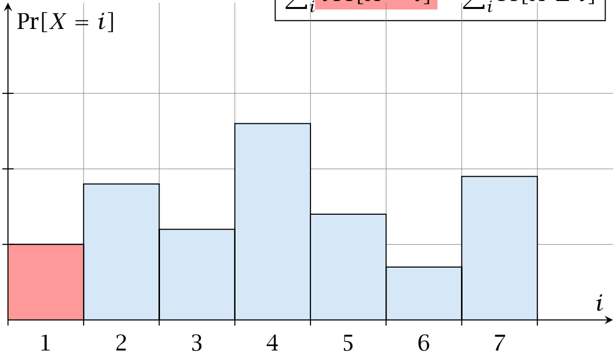
Analysis of Idealized Open Address Hashing



Analysis of Idealized Open Address Hashing

$\boldsymbol{i = 1}$

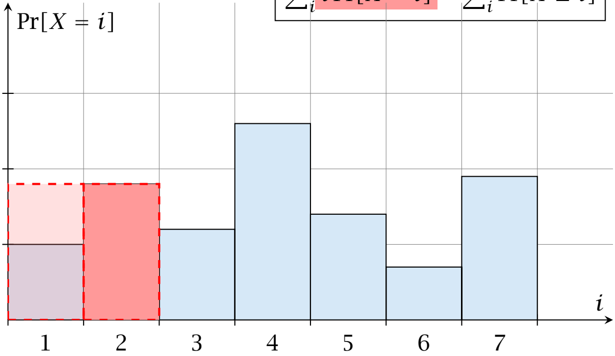
$$\sum_i i \Pr[X = i] = \sum_i \Pr[X \geq i]$$



Analysis of Idealized Open Address Hashing

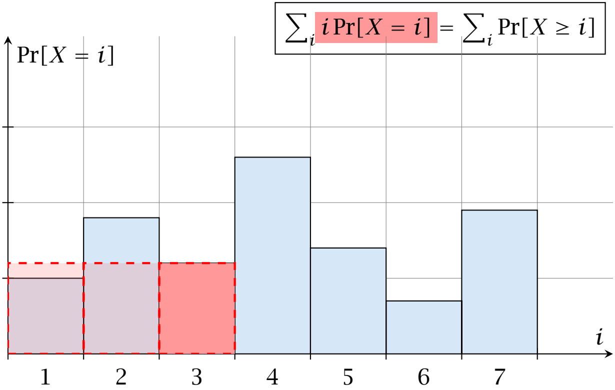
$i = 2$

$$\sum_i i \Pr[X = i] = \sum_i \Pr[X \geq i]$$



Analysis of Idealized Open Address Hashing

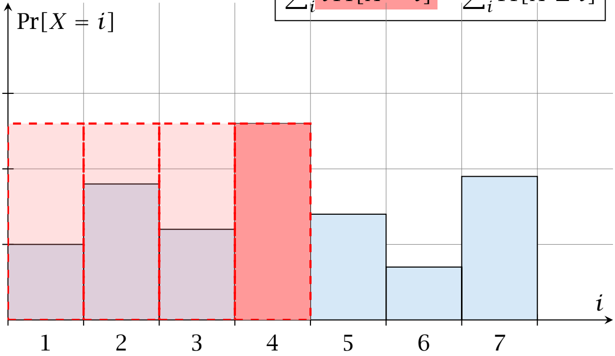
$i = 3$



Analysis of Idealized Open Address Hashing

$i = 4$

$$\sum_i i \Pr[X = i] = \sum_i \Pr[X \ge i]$$



Analysis of Idealized Open Address Hashing

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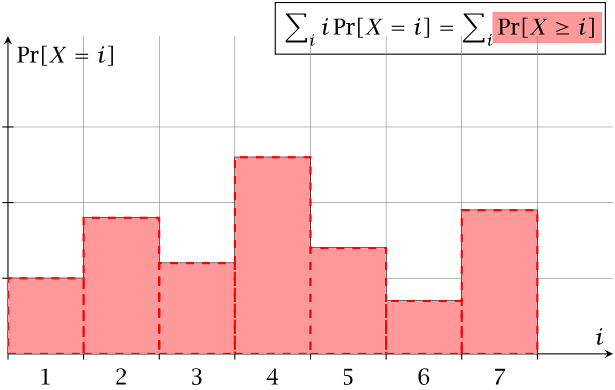

Analysis of Idealized Open Address Hashing

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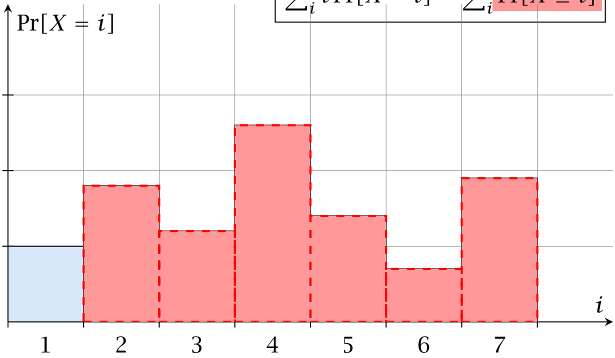
$\sum_i i \Pr[X = i] = \sum_i \Pr[X \geq i]$



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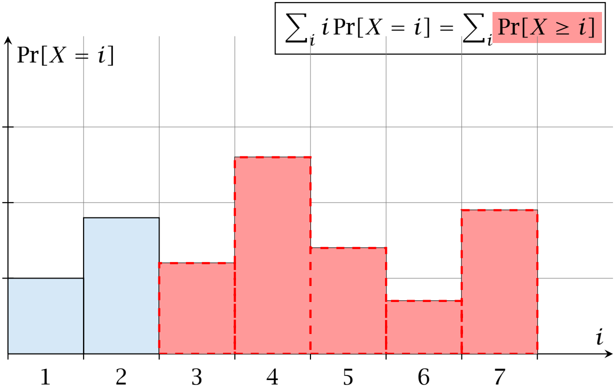

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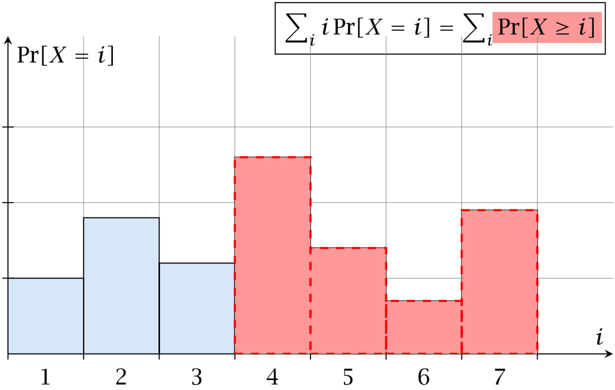

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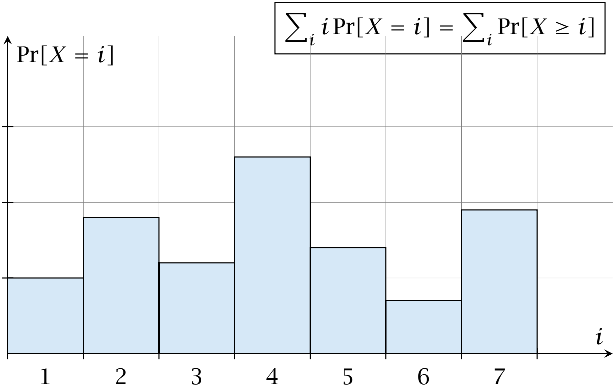

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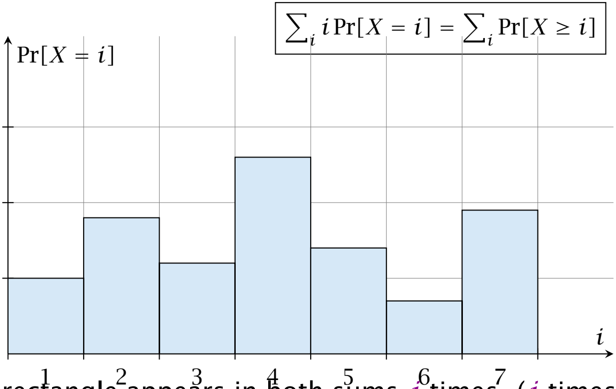

The j-th rectangle appears in both sums j times. (j times in the first due to multiplication with j; and j times in the second for summands $i = 1, 2, \ldots, j$)

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$$\frac{1}{m}\sum_{i=0}^{m-1}\frac{n}{n-i} = \frac{n}{m}\sum_{i=0}^{m-1}\frac{1}{n-i} = \frac{1}{\alpha}\sum_{k=n-m+1}^{n}\frac{1}{k}$$

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$$\le \frac{1}{\alpha}\int_{n-m}^{n}\frac{1}{x}\mathrm{d}x = \frac{1}{\alpha}\ln\frac{n}{n-m}$$

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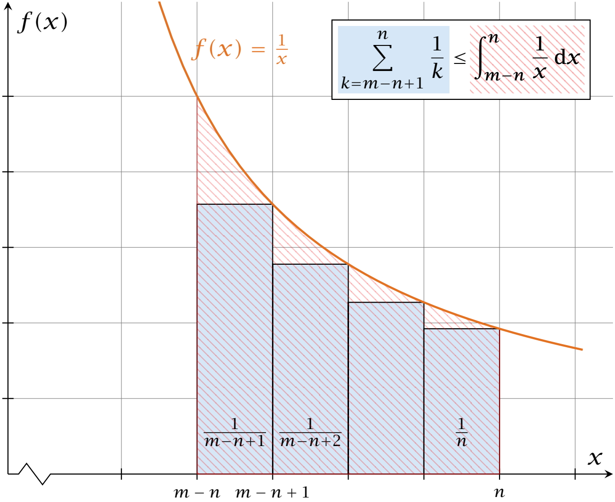
Analysis of Idealized Open Address Hashing



Deletions in Hashtables

How do we delete in a hash-table?

- For hashing with chaining this is not a problem. Simply search for the key, and delete the item in the corresponding list.

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- For open addressing this is difficult.

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- Simply removing a key might interrupt the probe sequence of other keys which then cannot be found anymore.

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  - During an insertion if a deleted-marker is encountered an element can be inserted there.

Deletions in Hashtables

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  - During a search a deleted-marker must not be used to terminate the probe sequence.
- The table could fill up with deleted-markers leading to bad performance.
- If a table contains many deleted-markers (linear fraction of the keys) one can rehash the whole table and amortize the cost for this rehash against the cost for the deletions.

Deletions for Linear Probing

- For Linear Probing one can delete elements without using deletion-markers.

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- For Linear Probing one can delete elements without using deletion-markers.
- Upon a deletion elements that are further down in the probe-sequence may be moved to guarantee that they are still found during a search.

Deletions for Linear Probing

Algorithm 12 delete(p)

```
T[p] ← null
p ← succ(p)
while T[p] ≠ null do
    y ← T[p]
    T[p] ← null
    p ← succ(p)
    insert(y)
```

p is the index into the table-cell that contains the object to be deleted.

Deletions for Linear Probing

Algorithm 12 delete(p)

$T[p] \leftarrow \text{null}$
$p \leftarrow \text{succ}(p)$
while $T[p] \neq \text{null}$ **do**
    $y \leftarrow T[p]$
    $T[p] \leftarrow \text{null}$
    $p \leftarrow \text{succ}(p)$
    $\text{insert}(y)$

p is the index into the table-cell that contains the object to be deleted.

Pointers into the hash-table become invalid.

Universal Hashing

Universal Hashing

Regardless, of the choice of hash-function there is always an input (a set of keys) that has a very poor worst-case behaviour.

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However, the assumption of uniform hashing that h is chosen randomly from all functions $f : U \to [0, \ldots, n-1]$ is clearly unrealistic as there are $n^{|U|}$ such functions. Even writing down such a function would take $|U| \log n$ bits.

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Universal hashing tries to define a set $\mathcal{H}$ of functions that is much smaller but still leads to good average case behaviour when selecting a hash-function uniformly at random from $\mathcal{H}$.

Universal Hashing

Definition 13
A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0,\ldots,n-1\}$ is called universal if for all $u_1, u_2 \in U$ with $u_1 \neq u_2$

$$\Pr[h(u_1) = h(u_2)] \leq \frac{1}{n} \ ,$$

where the probability is w. r. t. the choice of a random hash-function from set $\mathcal{H}$.

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where the probability is w. r. t. the choice of a random hash-function from set $\mathcal{H}$.

Note that this means that the probability of a collision between two arbitrary elements is at most $\frac{1}{n}$.

Universal Hashing

Definition 14

A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0, \ldots, n-1\}$ is called 2-independent (pairwise independent) if the following two conditions hold

- For any key $u \in U$, and $t \in \{0, \ldots, n-1\}$ $\Pr[h(u) = t] = \frac{1}{n}$, i.e., a key is distributed uniformly within the hash-table.
- For all $u_1, u_2 \in U$ with $u_1 \neq u_2$, and for any two hash-positions t_1, t_2:

$$\Pr[h(u_1) = t_1 \wedge h(u_2) = t_2] \leq \frac{1}{n^2} \ .$$

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$$\Pr[h(u_1) = t_1 \wedge h(u_2) = t_2] \leq \frac{1}{n^2} \ .$$

This requirement clearly implies a universal hash-function.

Universal Hashing

Definition 15

A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0,\ldots,n-1\}$ is called k-independent if for any choice of $\ell \le k$ distinct keys $u_1,\ldots,u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions $t_1,\ldots,t_\ell$:

$$\Pr[h(u_1) = t_1 \wedge \cdots \wedge h(u_\ell) = t_\ell] \le \frac{1}{n^\ell} \ ,$$

where the probability is w.r.t. the choice of a random hash-function from set $\mathcal{H}$.

Universal Hashing

Definition 16
A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0, \ldots, n-1\}$ is called (μ, k)-independent if for any choice of $\ell \leq k$ distinct keys $u_1, \ldots, u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions $t_1, \ldots, t_\ell$:

$$\Pr[h(u_1) = t_1 \wedge \cdots \wedge h(u_\ell) = t_\ell] \leq \frac{\mu}{n^\ell} ,$$

where the probability is w. r. t. the choice of a random hash-function from set $\mathcal{H}$.

Universal Hashing

Universal Hashing

Let $U := \{0, \ldots, p-1\}$ for a prime p. Let $\mathbb{Z}_p := \{0, \ldots, p-1\}$, and let $\mathbb{Z}_p^* := \{1, \ldots, p-1\}$ denote the set of invertible elements in $\mathbb{Z}_p$.

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Define

$$h_{a,b}(x) := (ax + b \bmod p) \bmod n$$

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Define

$$h_{a,b}(x) := (ax + b \bmod p) \bmod n$$

Lemma 17

The class

$$\mathcal{H} = \{h_{a,b} \mid a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$$

is a universal class of hash-functions from U to $\{0, \dots, n-1\}$.

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Proof.

Let $x, y \in U$ be two distinct keys. We have to show that the probability of a collision is only $1/n$.

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  Multiplying with $a \not\equiv 0 \pmod{p}$ gives

$$a(x - y) \not\equiv 0 \pmod{p}$$

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  Multiplying with $a \not\equiv 0 \pmod{p}$ gives

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  where we use that $\mathbb{Z}_p$ is a field (Körper) and, hence, has no zero divisors (nullteilerfrei).

Universal Hashing

- The hash-function does not generate collisions before the $\pmod{n}$-operation. Furthermore, every choice (a, b) is mapped to a different pair (t_x, t_y) with $t_x := ax + b$ and $t_y := ay + b$.

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t_x &\equiv ax + b \pmod{p} \\
t_y &\equiv ay + b \pmod{p}
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  This holds because we can compute a and b when given t_x and t_y:

$$\begin{aligned} t_x &\equiv ax+b && \pmod{p} \\ t_y &\equiv ay+b && \pmod{p} \end{aligned}$$

$$\begin{aligned} t_x - t_y &\equiv a(x-y) && \pmod{p} \\ t_y &\equiv ay+b && \pmod{p} \end{aligned}$$

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t_x - t_y &\equiv a(x - y) &&\pmod{p}\\
t_y &\equiv ay + b &&\pmod{p}
\end{aligned}
$$

$$
\begin{aligned}
a &\equiv (t_x - t_y)(x - y)^{-1} &&\pmod{p}\\
b &\equiv t_y - ay &&\pmod{p}
\end{aligned}
$$

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There is a one-to-one correspondence between hash-functions (pairs (a,b), $a \neq 0$) and pairs (t_x, t_y), $t_x \neq t_y$.

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Fix a value t_x. There are $p - 1$ possible values for choosing t_y.

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What happens when we do the $\bmod n$ operation?

Fix a value t_x. There are $p-1$ possible values for choosing t_y.

From the range $0, \dots, p-1$ the values $t_x, t_x + n, t_x + 2n, \dots$ map to t_x after the modulo-operation. These are at most $\lceil p/n \rceil$ values.

Universal Hashing

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As $t_y \neq t_x$ there are

$$\left\lceil \frac{p}{n} \right\rceil - 1$$

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possibilities for choosing t_y such that the final hash-value creates a collision.

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possibilities for choosing t_y such that the final hash-value creates a collision.

This happens with probability at most $\frac{1}{n}$.

Universal Hashing

Universal Hashing

It is also possible to show that $\mathcal{H}$ is an (almost) pairwise independent class of hash-functions.

$$\Pr_{t_x \neq t_y \in \mathbb{Z}_p^2}\left[\begin{array}{c} t_x \bmod n = h_1 \\ \wedge \\ t_y \bmod n = h_2 \end{array}\right]$$

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It is also possible to show that $\mathcal{H}$ is an (almost) pairwise independent class of hash-functions.

$$\frac{\left\lfloor\frac{p}{n}\right\rfloor^2}{p(p-1)} \leq \Pr_{t_x \neq t_y \in \mathbb{Z}_p^2}\left[\begin{array}{c} t_x \bmod n = h_1 \\ \wedge \\ t_y \bmod n = h_2 \end{array}\right] \leq \frac{\left\lceil\frac{p}{n}\right\rceil^2}{p(p-1)}$$

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Note that the middle is the probability that $h(x) = h_1$ and $h(y) = h_2$. The total number of choices for (t_x, t_y) is $p(p-1)$. The number of choices for t_x (t_y) such that $t_x \bmod n = h_1$ ($t_y \bmod n = h_2$) lies between $\lfloor \frac{p}{n} \rfloor$ and $\lceil \frac{p}{n} \rceil$.

Universal Hashing

Definition 18
Let $d \in \mathbb{N}$; $q \geq (d+1)n$ be a prime; and let $\bar{a} \in \{0, \ldots, q-1\}^{d+1}$. Define for $x \in \{0, \ldots, q-1\}$

$$h_{\bar{a}}(x) := \Big(\sum_{i=0}^{d} a_i x^i \bmod q \Big) \bmod n \ .$$

Let $\mathcal{H}_n^d := \{h_{\bar{a}} \mid \bar{a} \in \{0, \ldots, q-1\}^{d+1}\}$. The class $\mathcal{H}_n^d$ is $(e, d+1)$-independent.

Note that in the previous case we had $d = 1$ and chose $a_d \neq 0$.

Universal Hashing

Universal Hashing

For the coefficients $\bar{a} \in \{0, \ldots, q-1\}^{d+1}$ let $f_{\bar{a}}$ denote the polynomial

$$f_{\bar{a}}(x) = \Big(\sum_{i=0}^{d} a_i x^i\Big) \bmod q$$

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The polynomial is defined by $d+1$ distinct points.

Universal Hashing

Universal Hashing

Fix $\ell \leq d+1$; let $x_1, \ldots, x_\ell \in \{0, \ldots, q-1\}$ be keys, and let $t_1, \ldots, t_\ell$ denote the corresponding hash-function values.

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Let $A^\ell = \{h_{\bar{a}} \in \mathcal{H} \mid h_{\bar{a}}(x_i) = t_i \text{ for all } i \in \{1,\ldots,\ell\}\}$

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Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

$$f_{\bar{a}}(x_i) \in \underbrace{\{t_i + \alpha \cdot n \mid \alpha \in \{0, \ldots, \lceil \tfrac{q}{n} \rceil - 1\}\}}_{=: B_i}$$

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In order to obtain the cardinality of A^ℓ we choose our polynomial by fixing $d+1$ points.

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In order to obtain the cardinality of A^ℓ we choose our polynomial by fixing $d+1$ points.

We first fix the values for inputs $x_1, \dots, x_\ell$.
We have

$$|B_1| \cdot \ldots \cdot |B_\ell|$$

possibilities to do this (so that $h_{\bar{a}}(x_i) = t_i$).

Universal Hashing

Now, we choose $d - \ell + 1$ other inputs and choose their value arbitrarily. We have $q^{d-\ell+1}$ possibilities to do this.

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Therefore we have

$$|B_1| \cdot \ldots \cdot |B_\ell| \cdot q^{d-\ell+1} \leq \lceil \frac{q}{n} \rceil^\ell \cdot q^{d-\ell+1}$$

possibilities to choose $\bar{a}$ such that $h_{\bar{a}} \in A_\ell$.

Universal Hashing

Therefore the probability of choosing $h_{\bar{a}}$ from A_ℓ is only

$$\frac{\lceil \frac{q}{n} \rceil^\ell \cdot q^{d-\ell+1}}{q^{d+1}}$$

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$$\frac{\lceil \frac{q}{n} \rceil^\ell \cdot q^{d-\ell+1}}{q^{d+1}} \le \frac{(\frac{q+n}{n})^\ell}{q^\ell} \le \Big(\frac{q+n}{q}\Big)^\ell \cdot \frac{1}{n^\ell}$$

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Therefore the probability of choosing $h_{\bar{a}}$ from A_ℓ is only

$$\frac{\lceil \frac{q}{n} \rceil^\ell \cdot q^{d-\ell+1}}{q^{d+1}} \le \frac{(\frac{q+n}{n})^\ell}{q^\ell} \le \Big(\frac{q+n}{q}\Big)^\ell \cdot \frac{1}{n^\ell}$$
$$\le \Big(1 + \frac{1}{\ell}\Big)^\ell \cdot \frac{1}{n^\ell}$$

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Therefore the probability of choosing $h_{\bar{a}}$ from A_ℓ is only

$$\begin{aligned}\frac{\lceil\frac{q}{n}\rceil^{\ell}\cdot q^{d-\ell+1}}{q^{d+1}} &\le \frac{(\frac{q+n}{n})^{\ell}}{q^{\ell}} \le \Big(\frac{q+n}{q}\Big)^{\ell}\cdot\frac{1}{n^{\ell}}\\ &\le \Big(1+\frac{1}{\ell}\Big)^{\ell}\cdot\frac{1}{n^{\ell}} \le \frac{e}{n^{\ell}}\ .\end{aligned}$$

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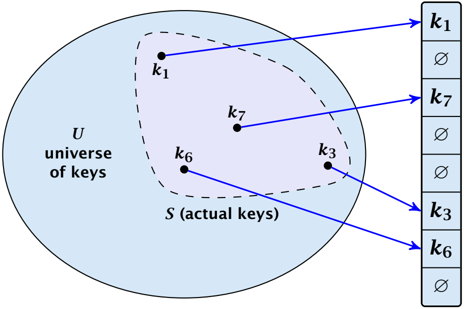
$$\le \Big(1 + \frac{1}{\ell}\Big)^\ell \cdot \frac{1}{n^\ell} \le \frac{e}{n^\ell} \ .$$

This shows that the $\mathcal{H}$ is $(e, d+1)$-universal.

The last step followed from $q \ge (d+1)n$, and $\ell \le d+1$.

Perfect Hashing

Suppose that we **know** the set S of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



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Using a universal hash-function the expected number of collisions is

$$\mathrm{E}[\#\text{Collisions}] = \binom{m}{2} \cdot \frac{1}{n} \ .$$

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The probability of having 1 or more collisions can be at most $\frac{1}{2}$ as otherwise the expectation would be larger than $\frac{1}{2}$.

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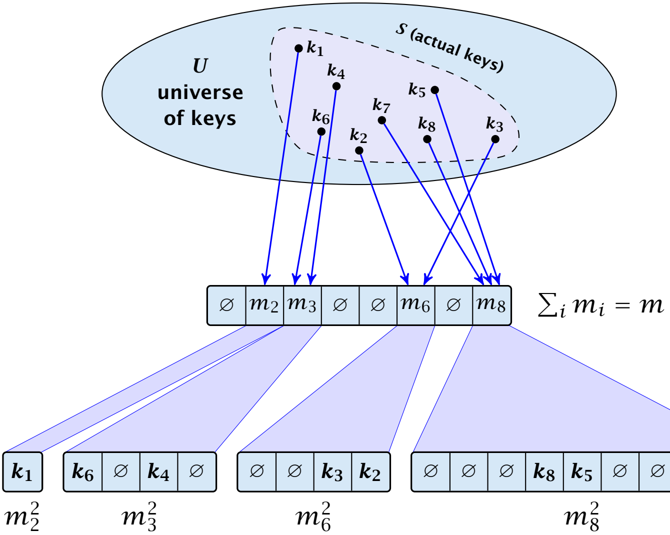
Let m_j denote the number of items that are hashed to the j-th bucket. For each bucket we choose a second hash-function that maps the elements of the bucket into a table of size m_j^2. The second function can be chosen such that all elements are mapped to different locations.

Perfect Hashing



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$$= 2\binom{m}{2}\frac{1}{m} + m = 2m - 1 \;.$$

Perfect Hashing

We need only $\mathcal{O}(m)$ time to construct a hash-function h with $\sum_j m_j^2 = \mathcal{O}(4m)$, because with probability at least $1/2$ a random function from a universal family will have this property.

Then we construct a hash-table h_j for every bucket. This takes expected time $\mathcal{O}(m_j)$ for every bucket. A random function h_j is collision-free with probability at least $1/2$. We need $\mathcal{O}(m_j)$ to test this.

We only need that the hash-functions are chosen from a universal family!!!

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Goal:
Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

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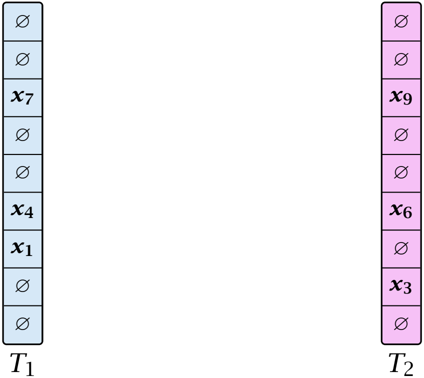
- A search clearly takes constant time if the above constraint is met.

Cuckoo Hashing

Insert:



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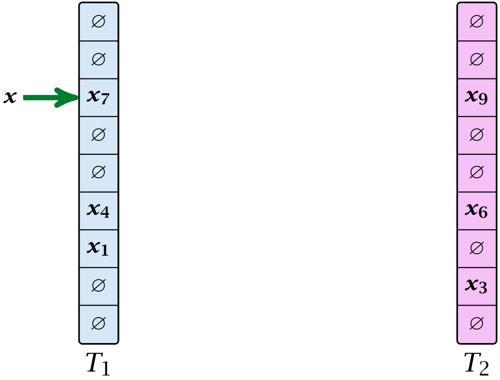

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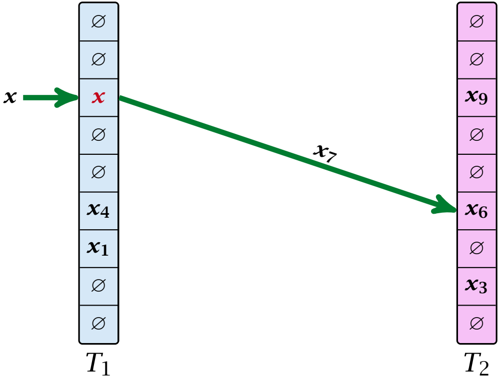

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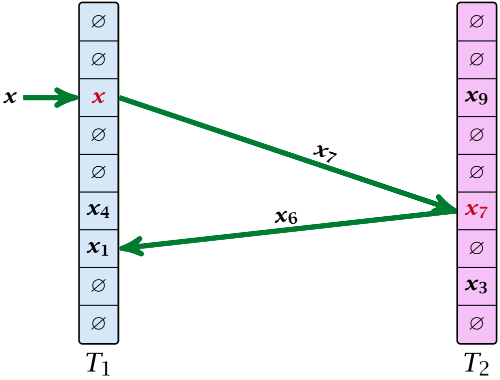

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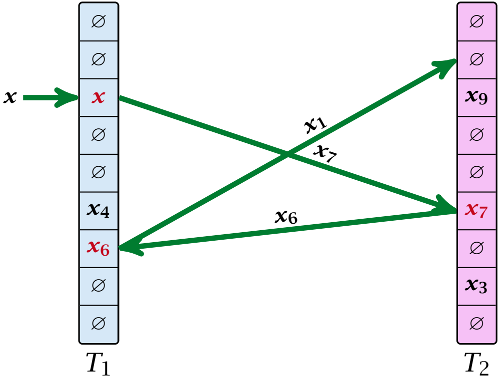

Cuckoo Hashing

Algorithm 13 Cuckoo-Insert(x)

if $T_1[h_1(x)] = x \vee T_2[h_2(x)] = x$ **then return**
steps $\leftarrow 1$
while steps $\leq$ maxsteps **do**
    exchange x and $T_1[h_1(x)]$
    if $x =$ null **then return**
    exchange x and $T_2[h_2(x)]$
    if $x =$ null **then return**
    steps $\leftarrow$ steps $+1$
rehash() // change hash-functions; rehash everything
Cuckoo-Insert(x)

Cuckoo Hashing

- We call one iteration through the while-loop a step of the algorithm.

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- We call a sequence of iterations through the while-loop without the termination condition becoming true a phase of the algorithm.
- We say a phase is successful if it is not terminated by the $\mathrm{maxstep}$-condition, but the while loop is left because $x = \mathrm{null}$.

Cuckoo Hashing

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We first analyze the probability that we end-up in an infinite loop (that is then terminated after $\mathrm{maxsteps}$ steps).

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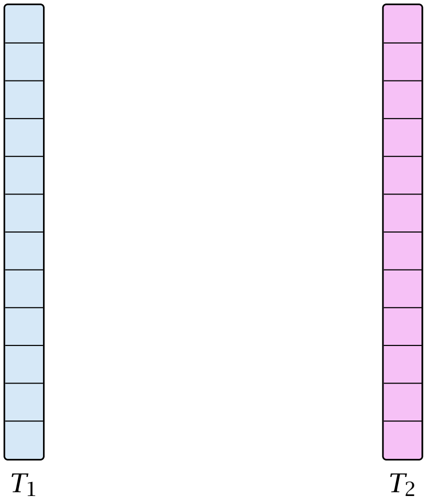
Formally what is the probability to enter an infinite loop that touches s different keys?

Cuckoo Hashing: Insert



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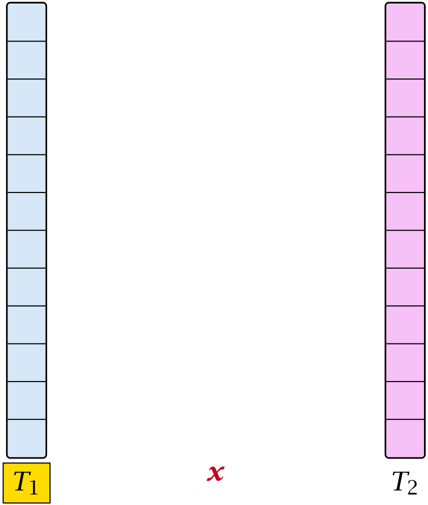

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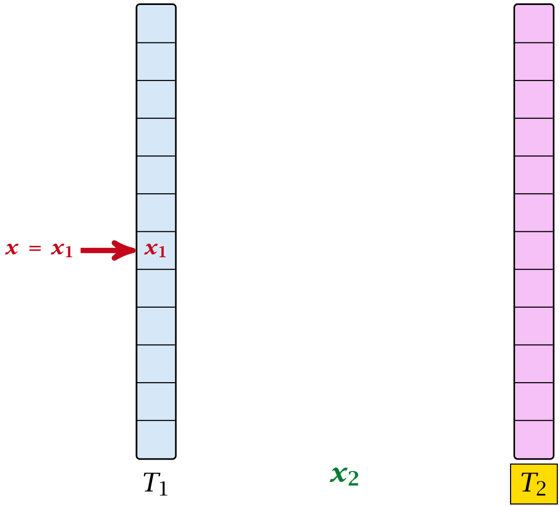

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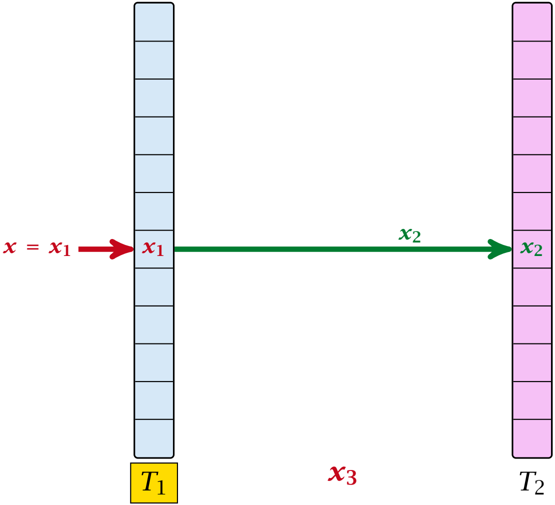

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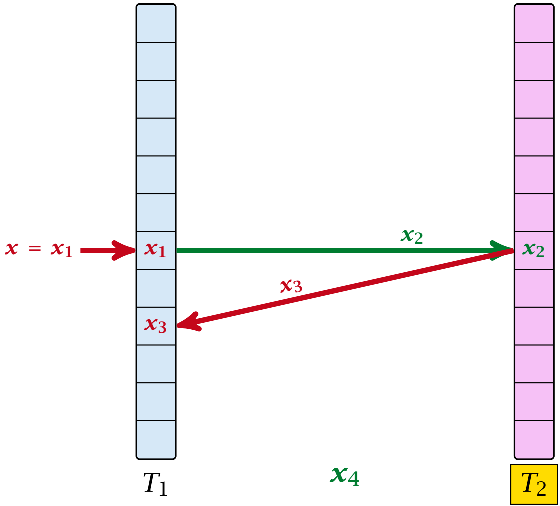

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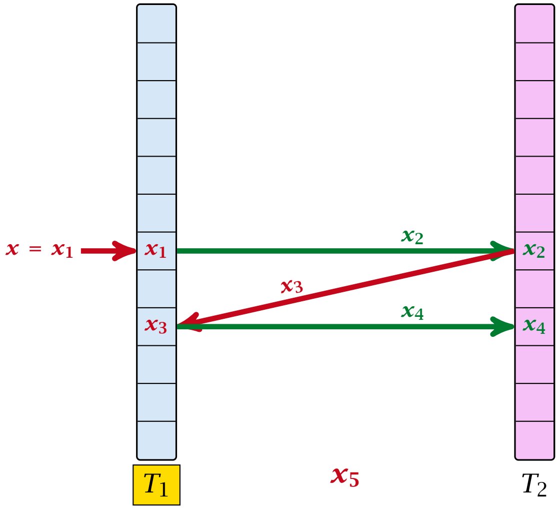

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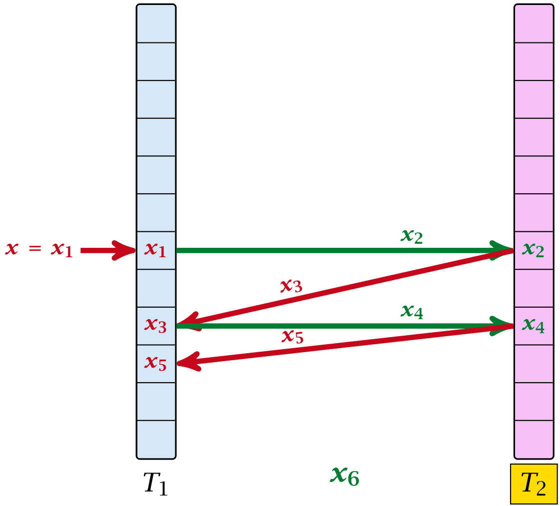

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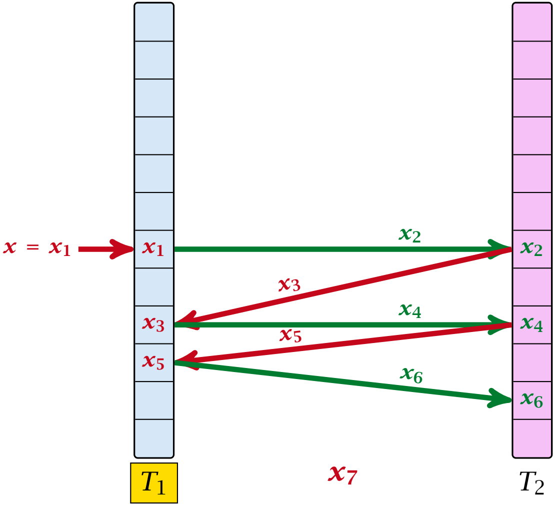

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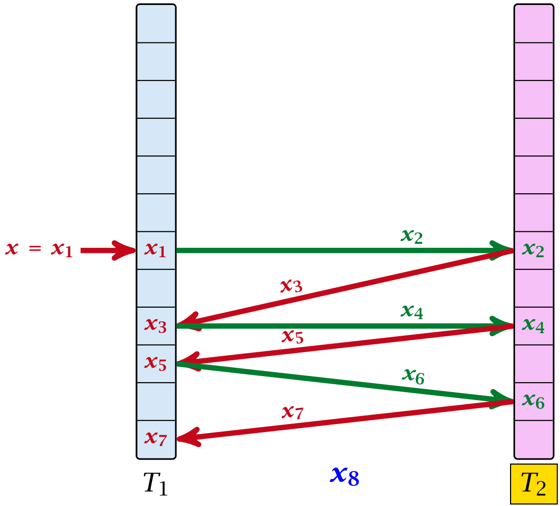

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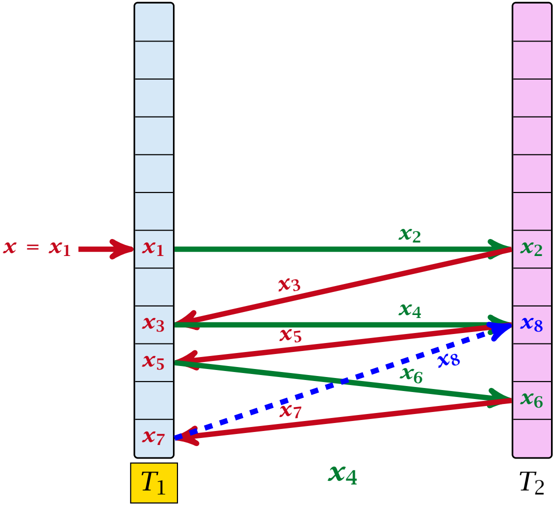

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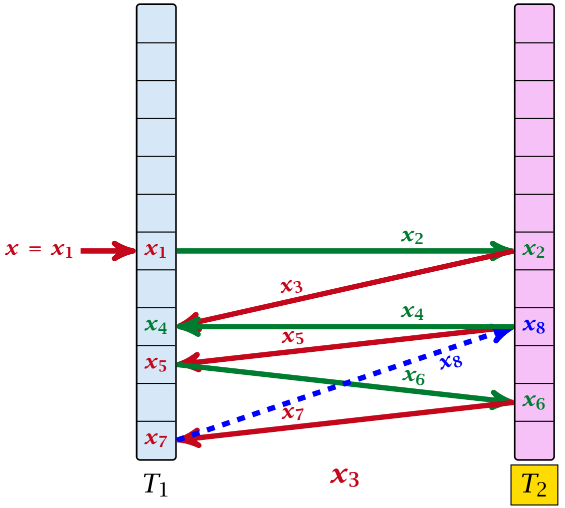

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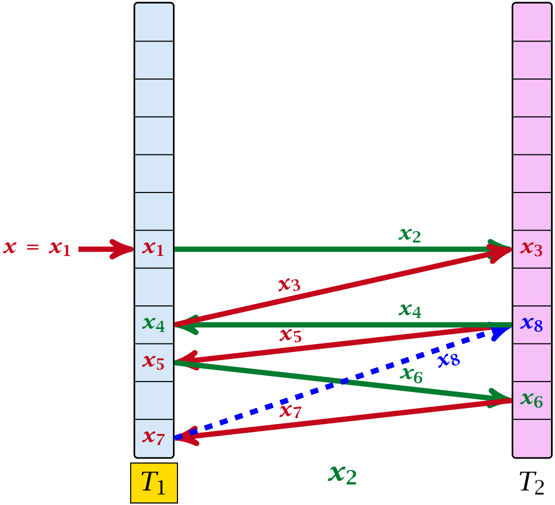

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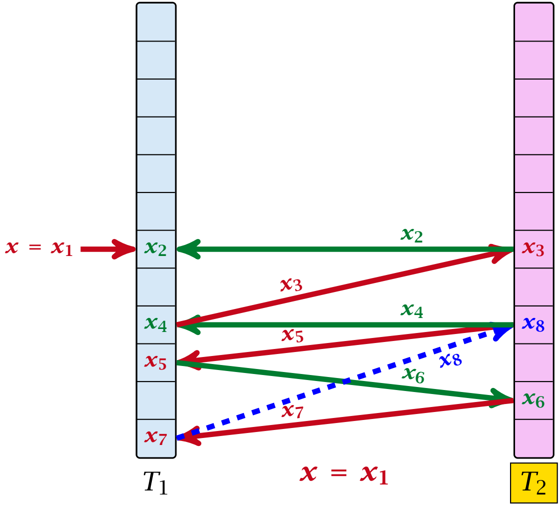
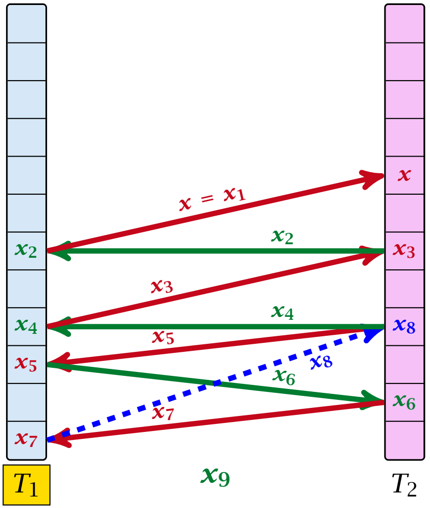


$x = x_1$

Cuckoo Hashing: Insert



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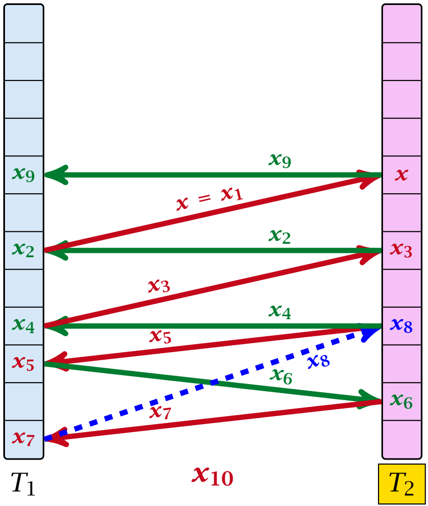

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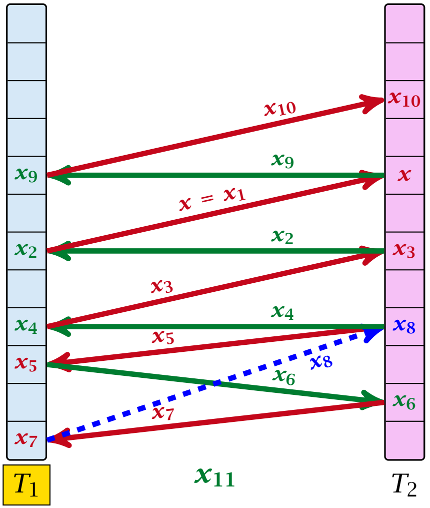

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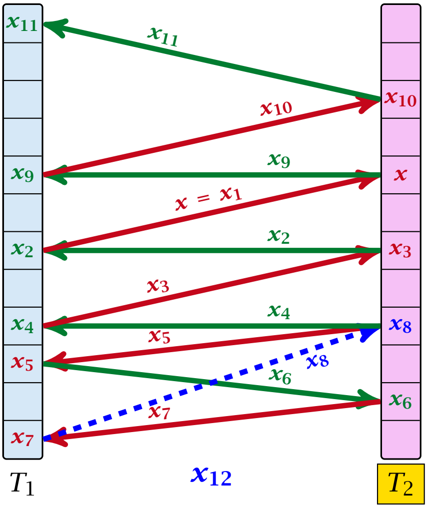

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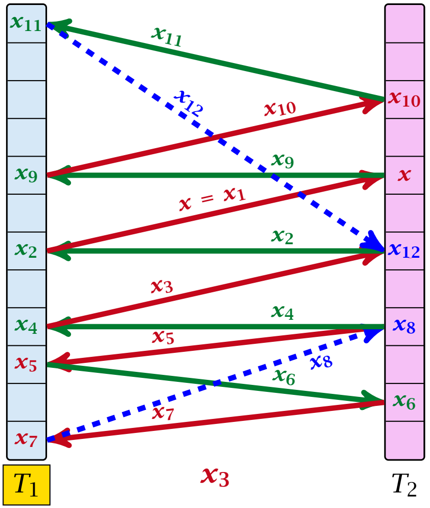

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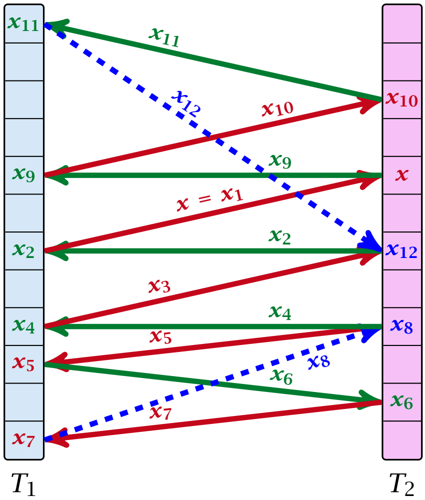

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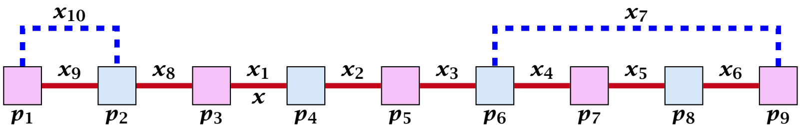

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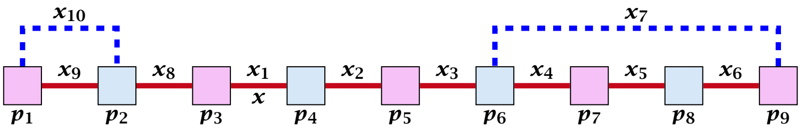

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- $s-1$ different cells (alternating btw. cells from T_1 and T_2).

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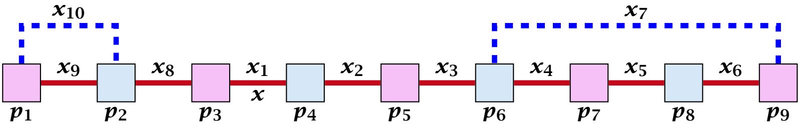

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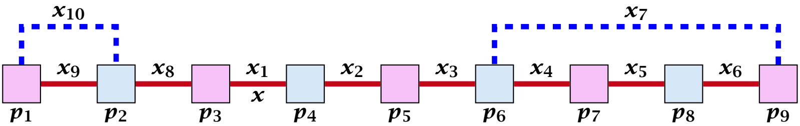

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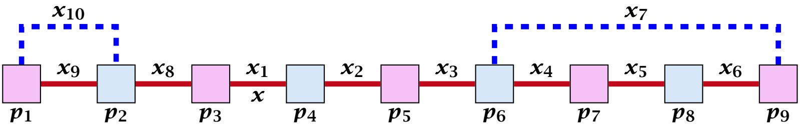

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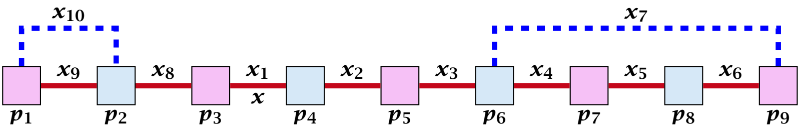

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- The leftmost cell is "linked forward" to some cell on the right.
- The rightmost cell is "linked backward" to a cell on the left.
- One link represents key x; this is where the counting starts.

Cuckoo Hashing

A cycle-structure is active if for every key x_ℓ (linking a cell p_i from T_1 and a cell p_j from T_2) we have

$$h_1(x_\ell) = p_i \qquad \text{and} \qquad h_2(x_\ell) = p_j$$

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Observation:
If during a phase the insert-procedure runs into a cycle there must exist an active cycle structure of size $s \geq 3$.

Cuckoo Hashing

What is the probability that all keys in a cycle-structure of size s correctly map into their T_1-cell?

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This probability is at most $\frac{\mu}{n^s}$ since h_1 is a (μ, s)-independent hash-function.

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These events are independent.

Cuckoo Hashing

The probability that a given cycle-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

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What is the probability that there exists an active cycle structure of size s?

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- There are at most s possibilities to choose where to place key x.
- There are m^{s-1} possibilities to choose the keys apart from x.
- There are n^{s-1} possibilities to choose the cells.

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} = \frac{\mu^2}{nm} \sum_{s=3}^{\infty} s^3 \left(\frac{m}{n}\right)^s$$

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$$\leq \frac{\mu^2}{m^2} \sum_{s=3}^{\infty} s^3 \left(\frac{1}{1+\epsilon}\right)^s$$

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$$\leq \frac{\mu^2}{m^2} \sum_{s=3}^{\infty} s^3 \left(\frac{1}{1+\epsilon}\right)^s \leq \mathcal{O}\left(\frac{1}{m^2}\right) \; .$$

Here we used the fact that $(1+\epsilon)m \leq n$.

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} = \frac{\mu^2}{nm} \sum_{s=3}^{\infty} s^3 \left(\frac{m}{n}\right)^s$$

$$\leq \frac{\mu^2}{m^2} \sum_{s=3}^{\infty} s^3 \left(\frac{1}{1+\epsilon}\right)^s \leq \mathcal{O}\left(\frac{1}{m^2}\right) \quad .$$

Here we used the fact that $(1+\epsilon)m \leq n$.

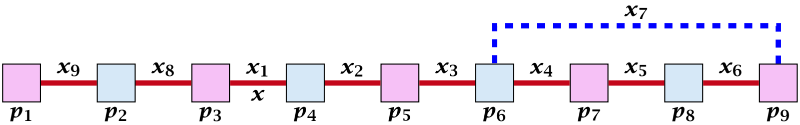
Hence,

$$\Pr[\text{cycle}] = \mathcal{O}\left(\frac{1}{m^2}\right) \quad .$$

Cuckoo Hashing

Now, we analyze the probability that a phase is not successful without running into a closed cycle.

Cuckoo Hashing



Sequence of visited keys:

$x = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_3, x_2, x_1 = x, x_8, x_9, \ldots$

Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with x in the order that they are visited during the phase.

Cuckoo Hashing

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Lemma 19

If the sequence is of length p then there exists a sub-sequence of at least $\frac{p+2}{3}$ keys starting with x of distinct keys.

Cuckoo Hashing

Proof.

Let i be the number of keys (including x) that we see before the first repeated key. Let j denote the total number of distinct keys.

The sequence is of the form:

$x = x_1 \to x_2 \to \cdots \to x_i \to x_r \to x_{r-1} \to \cdots \to x_1 \to x_{i+1} \to \cdots \to x_j$

As $r \le i - 1$ the length p of the sequence is

$$p = i + r + (j - i) \le i + j - 1 \ .$$

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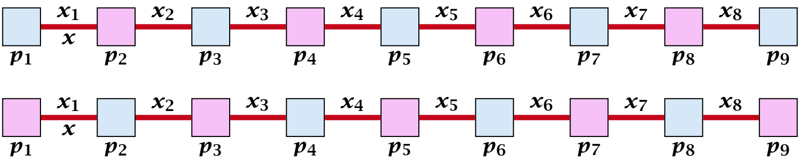
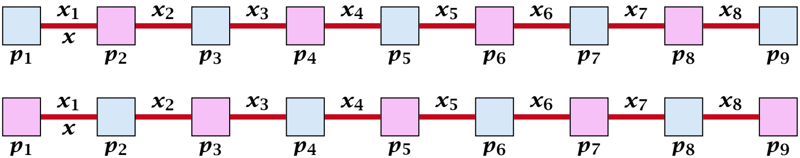
Either sub-sequence $x_1 \to x_2 \to \cdots \to x_i$ or sub-sequence $x_1 \to x_{i+1} \to \cdots \to x_j$ has at least $\frac{p+2}{3}$ elements. $\square$

Cuckoo Hashing



A path-structure of size s is defined by

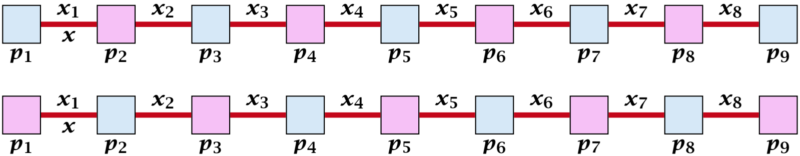
Cuckoo Hashing



A path-structure of size s is defined by

- $s + 1$ different cells (alternating btw. cells from T_1 and T_2).

Cuckoo Hashing



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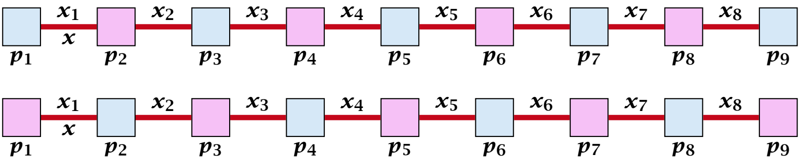

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- The leftmost cell is either from T_1 or T_2.

Cuckoo Hashing

A path-structure is active if for every key x_ℓ (linking a cell p_i from T_1 and a cell p_j from T_2) we have

$$h_1(x_\ell) = p_i \qquad \text{and} \qquad h_2(x_\ell) = p_j$$

Observation:
If a phase takes at least t steps without running into a cycle there must exist an active path-structure of size $(2t+2)/3$.

Cuckoo Hashing

The probability that a given path-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

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This gives $\text{maxsteps} = \Theta(\log m)$.

Cuckoo Hashing

So far we estimated

$$\Pr[\text{cycle}] \leq \mathcal{O}\Big(\frac{1}{m^2}\Big)$$

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for a suitable constant $c > 0$.

Cuckoo Hashing

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This means the expected cost for a successful phase is constant (even after accounting for the cost of the incomplete step that finishes the phase).

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Therefore the expected cost for re-hashes is $\mathcal{O}(m) \cdot \mathcal{O}(p) = \mathcal{O}(1)$.

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$$\Pr[Y_i | Z_i] \le (m+1) \cdot \mathcal{O}(1/m^2) \le \mathcal{O}(1/m) =: p \ .$$

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The 0-th (re)hash is the initial configuration when doing the insert.

$$\Pr[Z_i] \le \Pr[\wedge_{j=0}^{i-1} Y_j] \le p^i$$

Let X_i^s, $s \in \{1, \ldots, m+1\}$ denote the cost for inserting the s-th element during the i-th rehash (assuming i-th rehash occurs):

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Therefore, it is sufficient to have $(\mu, \Theta(\log m))$-independent hash-functions.

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- Therefore we can amortize the rehash cost after a change in table-size against the cost for insertions and deletions.

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Note that the above lemma only holds if the fill-factor (number of keys/total number of hash-table slots) is at most $\frac{1}{2(1+\epsilon)}$.

The $1/(2(1+\epsilon))$ fill-factor comes from the fact that the total hash-table is of size $2n$ (because we have two tables of size n); moreover $m \leq (1+\epsilon)n$.