Part III

Data Structures

Abstract Data Type

An abstract data type (ADT) is defined by an interface of operations or methods that can be performed and that have a defined behavior.

The data types in this lecture all operate on objects that are represented by a [key, value] pair.

- ▶ The key comes from a totally ordered set, and we assume that there is an efficient comparison function.
- ▶ The value can be anything; it usually carries satellite information important for the application that uses the ADT.

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- **$S.\,\mathbf{concatenate}(S')$:** $S := S \cup S'$.
  Requires $\text{key}[S.\,\text{maximum}()] \le \text{key}[S'.\,\text{minimum}()]$.

- **$S.\,\mathbf{decrease\text{-}key}(x, k)$:** Replace $\text{key}[x]$ by $k \le \text{key}[x]$.

Examples of ADTs

Stack:

- $S.\textbf{push}(x)$: Insert an element.
- $S.\textbf{pop}()$: Return the element from S that was inserted most recently; delete it from S.
- $S.\textbf{empty}()$: Tell if S contains any object.

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Queue:

- ▶ $S.\,\mathrm{enqueue}(x)$: Insert an element.
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Priority-Queue:

- ▶ $S.\,\mathrm{insert}(x)$: Insert an element.
- ▶ $S.\,\mathrm{delete\text{-}min}()$: Return the element with lowest key-value; delete it from S.

7 Dictionary

Dictionary:

▶ $S.\,\text{insert}(x)$: Insert an element x.

▶ $S.\,\text{delete}(x)$: Delete the element pointed to by x.

▶ $S.\,\text{search}(k)$: Return a pointer to an element e with
$\text{key}[e] = k$ in S if it exists; otherwise return null.

7.1 Binary Search Trees

An (internal) binary search tree stores the elements in a binary tree. Each tree-node corresponds to an element. All elements in the left sub-tree of a node v have a smaller key-value than $\text{key}[v]$ and elements in the right sub-tree have a larger-key value. We assume that all key-values are different.

(External Search Trees store objects only at leaf-vertices)

Examples:
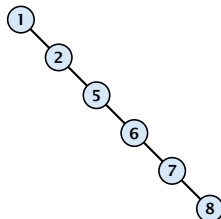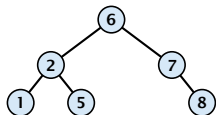
7.1 Binary Search Trees

We consider the following operations on binary search trees. Note that this is a super-set of the dictionary-operations.

- ▶ $T. \text{insert}(x)$
- ▶ $T. \text{delete}(x)$
- ▶ $T. \text{search}(k)$
- ▶ $T. \text{successor}(x)$
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Binary Search Trees: Searching

Algorithm 1 TreeSearch(x, k)

1: **if** x = null **or** k = key[x] **return** x
2: **if** $k <$ key[x] **return** TreeSearch(left[x], k)
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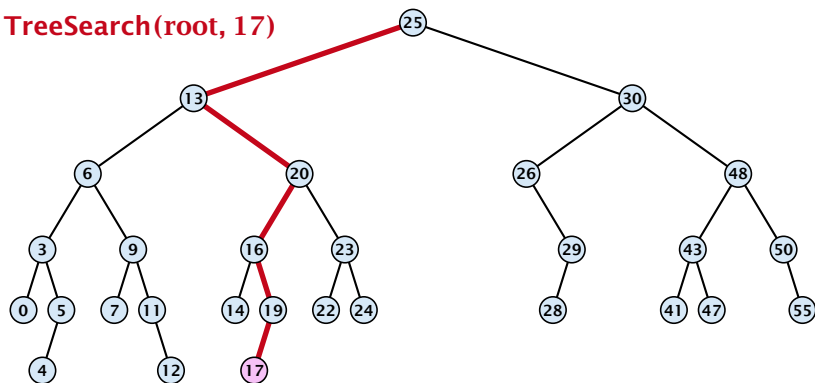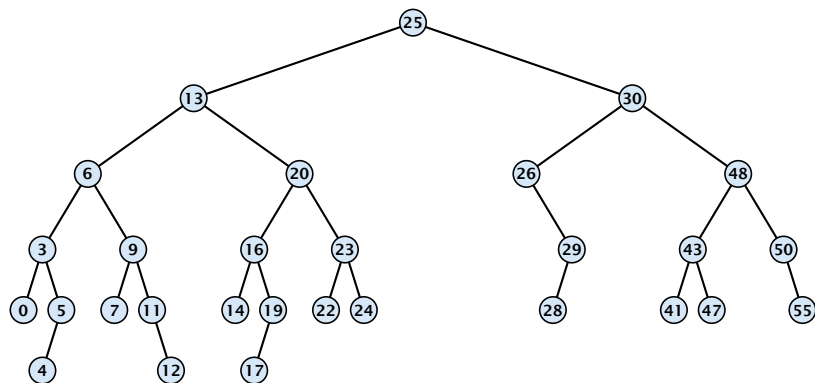# Binary Search Trees: Searching

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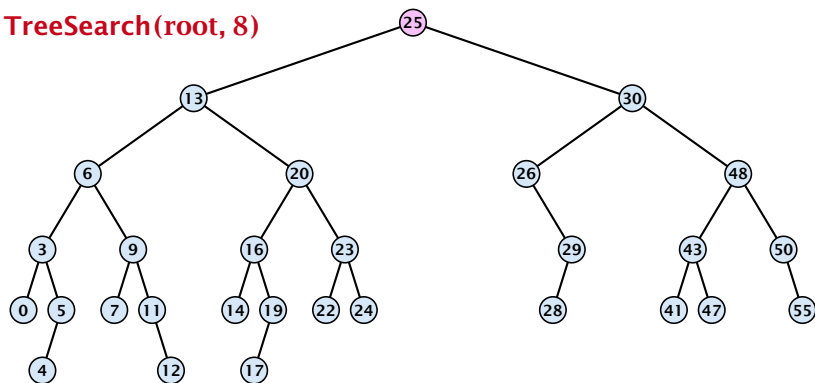3: **else return** TreeSearch(right$[x], k)$

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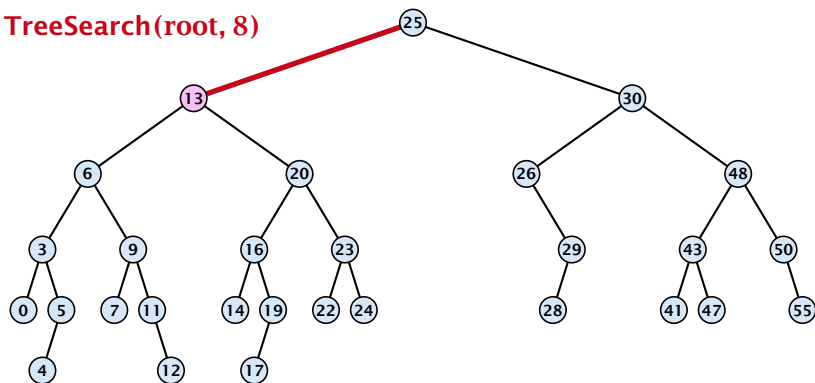
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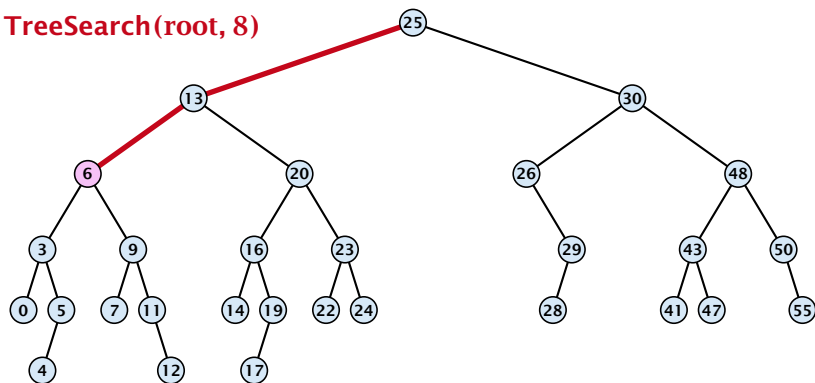

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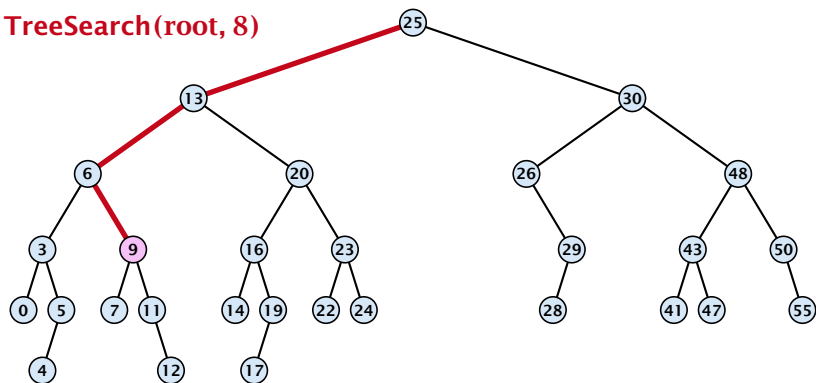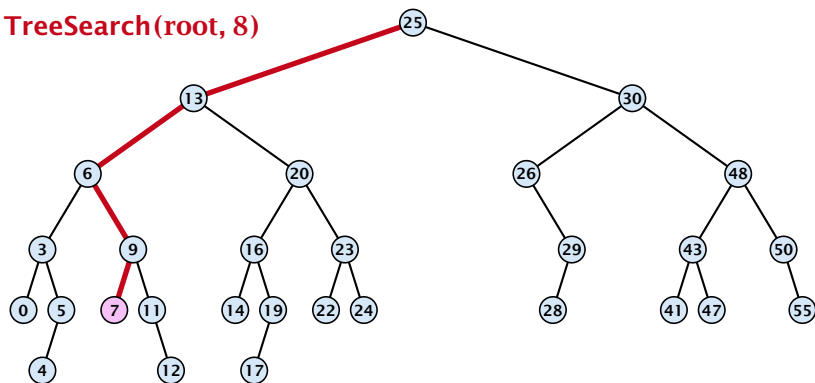
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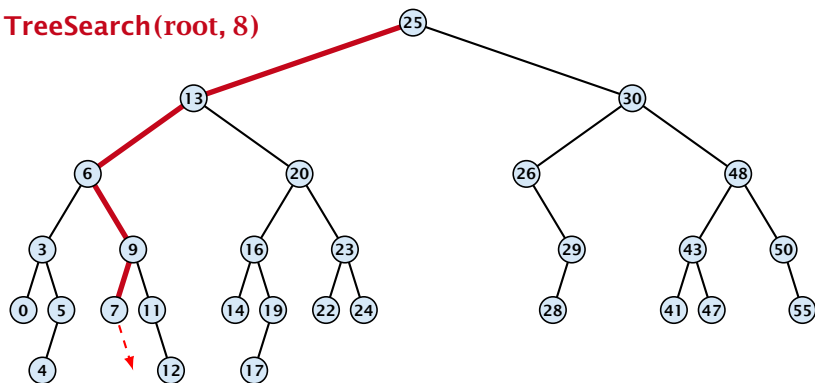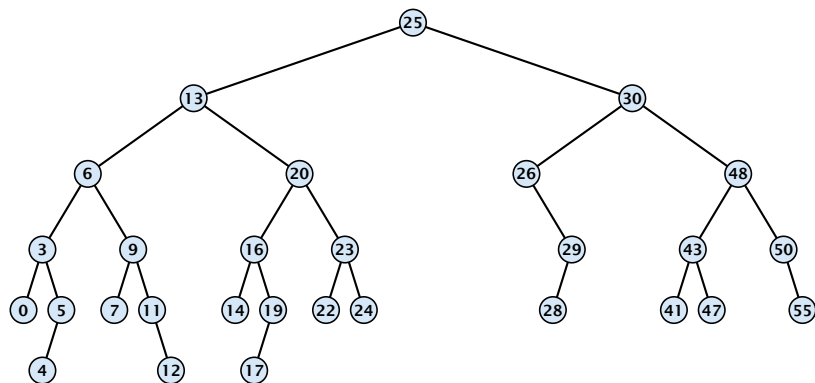
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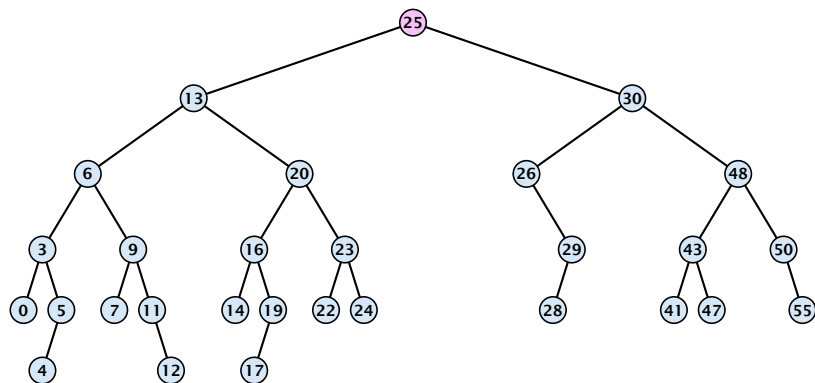3: **else return** TreeSearch(right[x], k)

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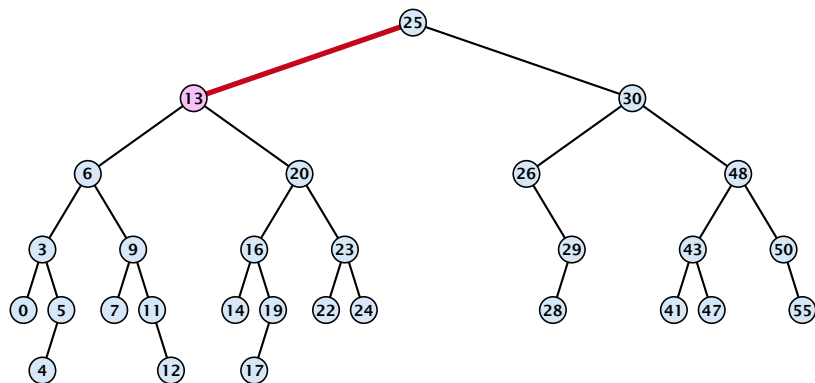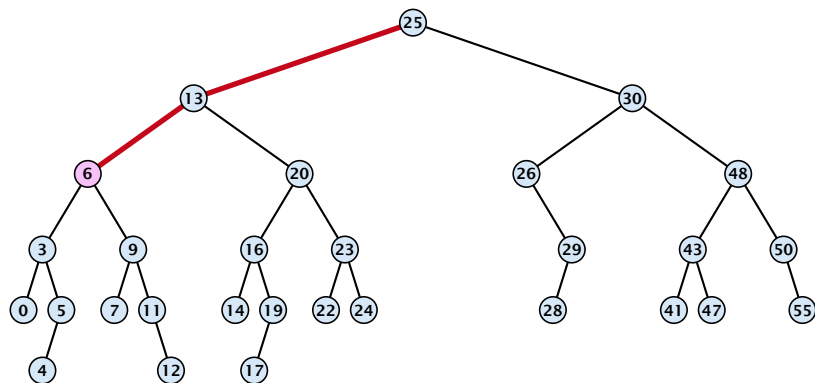
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Binary Search Trees: Minimum

Algorithm 2 TreeMin(x)

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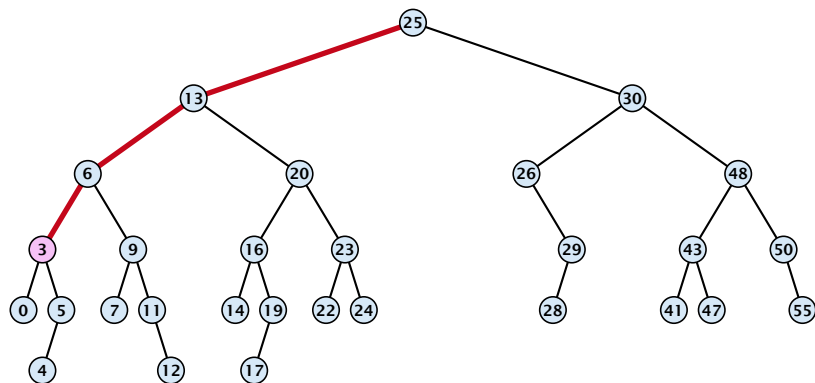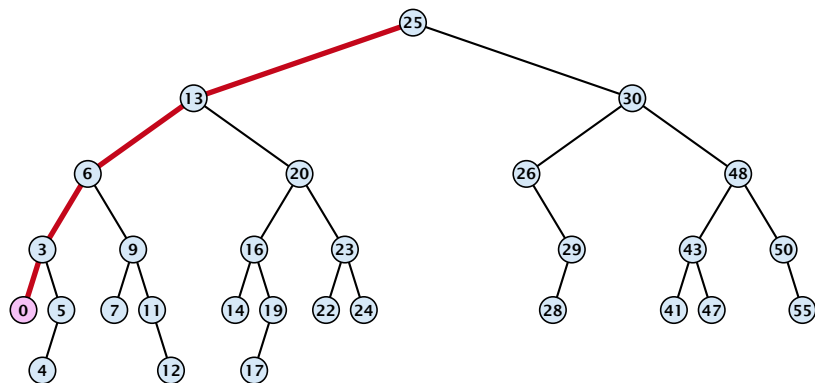# Binary Search Trees: Minimum

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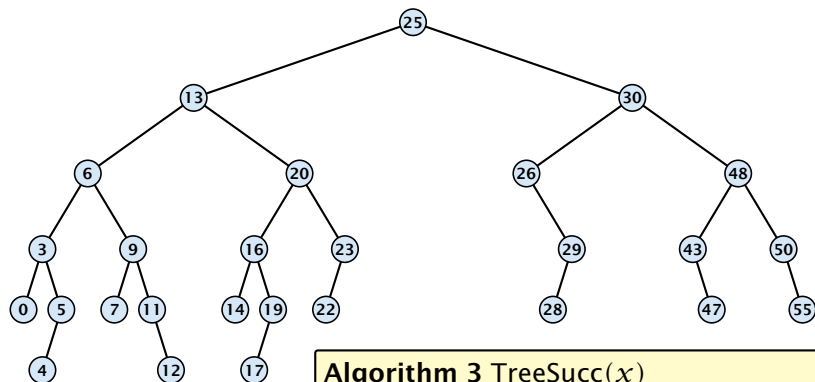1: **if** x = null **or** left[x] = null **return** x
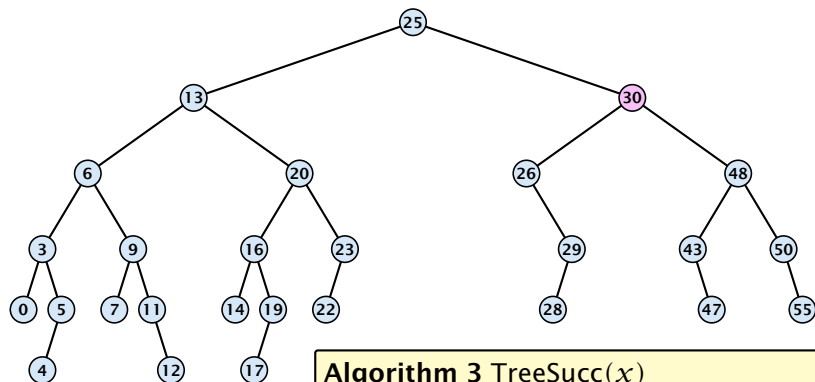2: **return** TreeMin(left[x])

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Binary Search Trees: Successor

Algorithm 3 TreeSucc(x)
1: **if** right[x] ≠ null **return** TreeMin(right[x])
2: $y \leftarrow$ parent[x]
3: **while** $y \neq$ null **and** $x =$ right[y] **do**
4:      $x \leftarrow y; y \leftarrow$ parent[x]
5: **return** y;

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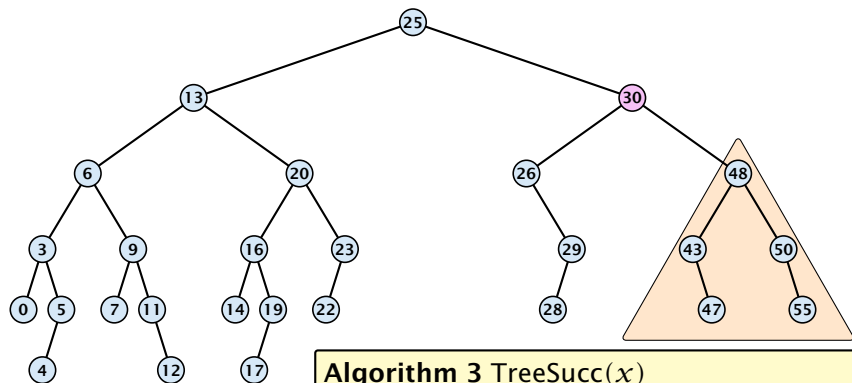5: **return** y;

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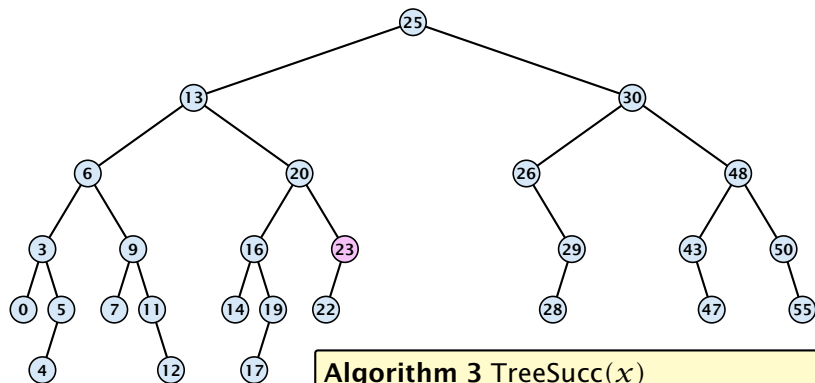4: $\quad x \leftarrow y; y \leftarrow$ parent[x]
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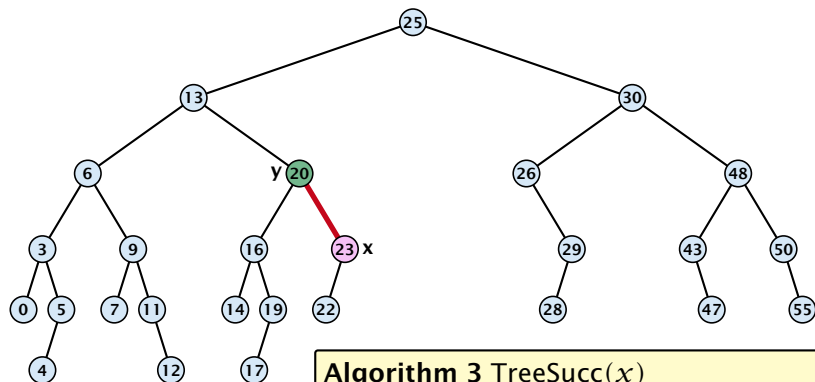4: $x \leftarrow y$; $y \leftarrow$ parent[x]
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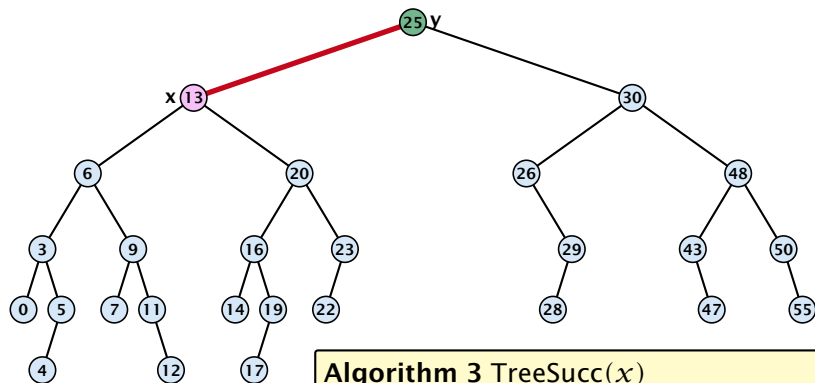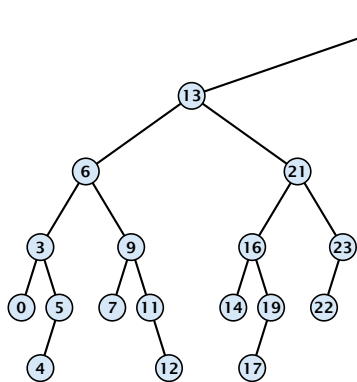4: $x \leftarrow y; y \leftarrow$ parent[x]
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Binary Search Trees: Insert

Algorithm 4 TreeInsert(x, z)

1: **if** x = null **then**
2:      root$[T] \leftarrow z$; parent$[z] \leftarrow$ null;
3:      **return**;
4: **if** key$[x]$ > key$[z]$ **then**
5:      **if** left$[x]$ = null **then**
6:          left$[x] \leftarrow z$; parent$[z] \leftarrow x$;
7:      **else** TreeInsert(left$[x], z$);
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9:      **if** right$[x]$ = null **then**
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Insert element **not** in the tree.

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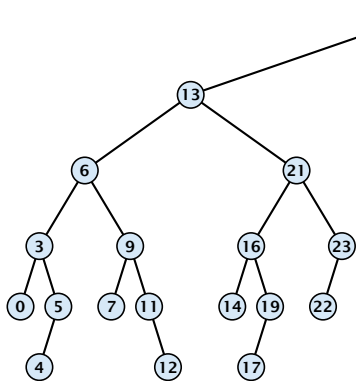**TreeInsert(root, 20)**

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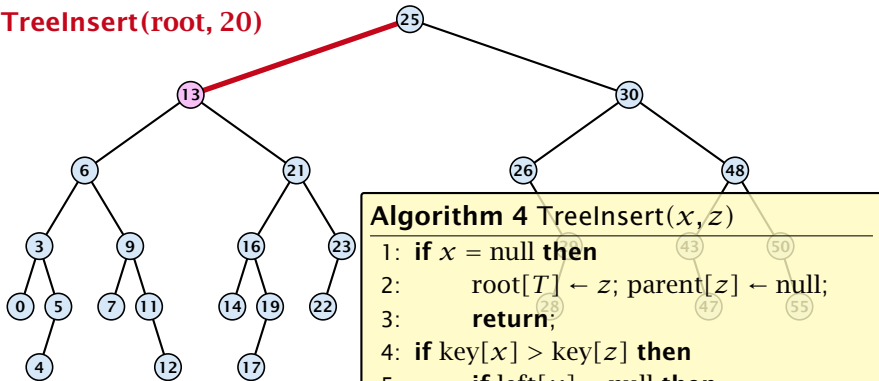
Algorithm 4 TreeInsert(x, z)

1: **if** x = null **then**
2:     root[T] ← z; parent[z] ← null;
3:     **return**;
4: **if** key[x] > key[z] **then**
5:     **if** left[x] = null **then**
6:         left[x] ← z; parent[z] ← x;
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8: **else**
9:     **if** right[x] = null **then**
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Binary Search Trees: Insert

Insert element **not** in the tree.

TreeInsert(root, 20)

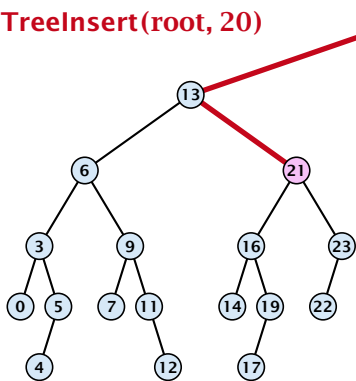Search for z. At some point the search stops at a null-pointer. This is the place to insert z.

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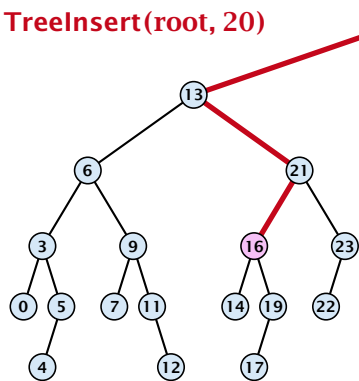
Algorithm 4 TreeInsert(x, z)

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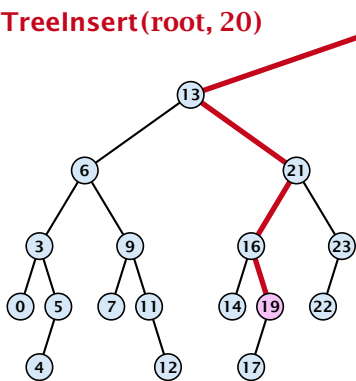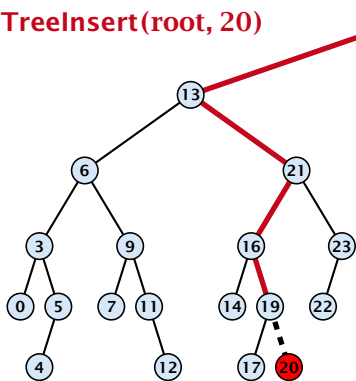Search for z. At some point the search stops at a null-pointer. This is the place to insert z.

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Case 1:

Element does not have any children

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Binary Search Trees: Delete

Case 3:

Element has two children

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Algorithm 9 TreeDelete(z)

1: **if** left[z] = null **or** right[z] = null
2:      **then** $y \leftarrow z$ **else** $y \leftarrow$ TreeSucc(z);   *select y to splice out*
3: **if** left[y] $\neq$ null
4:      **then** $x \leftarrow$ left[y] **else** $x \leftarrow$ right[y]; *x is child of y (or null)*
5: **if** $x \neq$ null **then** parent[x] $\leftarrow$ parent[y];   *parent[x] is correct*
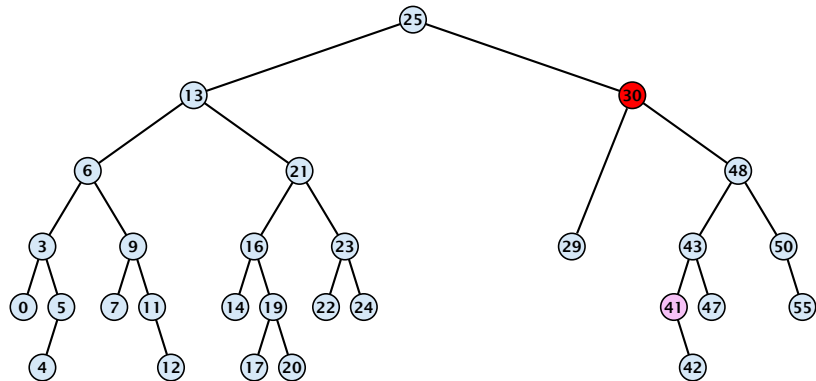6: **if** parent[y] = null **then**
7:      root[T] $\leftarrow x$
8: **else**
9:      **if** y = left[parent[y]] **then**
10:        left[parent[y]] $\leftarrow x$ *fix pointer to x*
11:      **else**
12:        right[parent[y]] $\leftarrow x$
13: **if** $y \neq z$ **then** copy y-data to z

Balanced Binary Search Trees

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All operations on a binary search tree can be performed in time $\mathcal{O}(h)$, where h denotes the height of the tree.

Ernst Mayr, Harald Räcke

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AVL-trees, Red-black trees, Scapegoat trees, 2-3 trees, B-trees, AA trees, Treaps

similar: SPLAY trees.

7.2 Red Black Trees

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Red Black Trees: Example
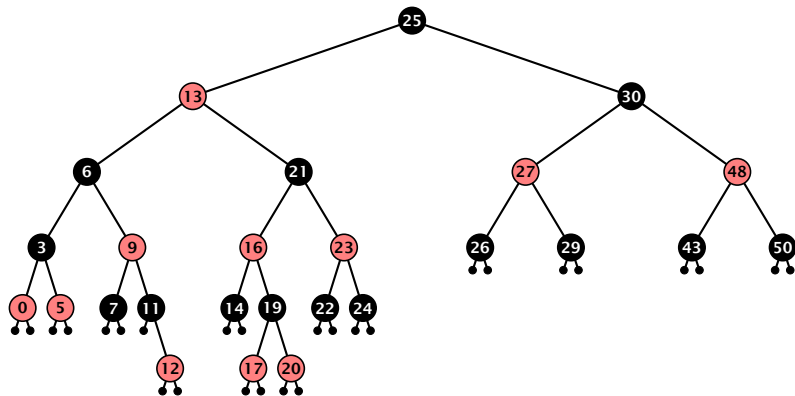
7.2 Red Black Trees

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A red-black tree with n internal nodes has height at most $\mathcal{O}(\log n)$.

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We first show:

Lemma 4

A sub-tree of black height $\mathrm{bh}(v)$ in a red black tree contains at least $2^{\mathrm{bh}(v)} - 1$ internal vertices.

Proof of Lemma 4.

7.2 Red Black Trees

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Induction on the height of v.

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- If $\text{height}(v)$ (maximum distance btw. v and a node in the sub-tree rooted at v) is 0 then v is a leaf.

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- If $\text{height}(v)$ (maximum distance btw. v and a node in the sub-tree rooted at v) is 0 then v is a leaf.
- The black height of v is 0.
- The sub-tree rooted at v contains $0 = 2^{\text{bh}(v)} - 1$ inner vertices.

Proof (cont.)

7.2 Red Black Trees

Proof (cont.)

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- ▶ Supose v is a node with $\text{height}(v) > 0$.

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▶ Then T_v contains at least $2(2^{\operatorname{bh}(v)-1} - 1) + 1 \geq 2^{\operatorname{bh}(v)} - 1$ vertices.

$\square$

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The tree contains at least $2^{h/2} - 1$ internal vertices. Hence, $2^{h/2} - 1 \leq n$.

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The tree contains at least $2^{h/2} - 1$ internal vertices. Hence, $2^{h/2} - 1 \le n$.

Hence, $h \le 2 \log(n+1) = \mathcal{O}(\log n)$. $\square$

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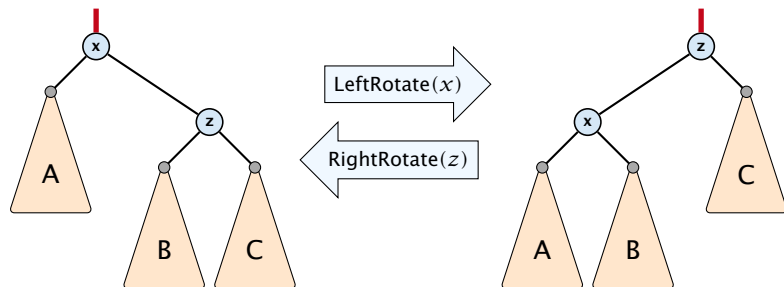4. If a node is red then both its children are black.

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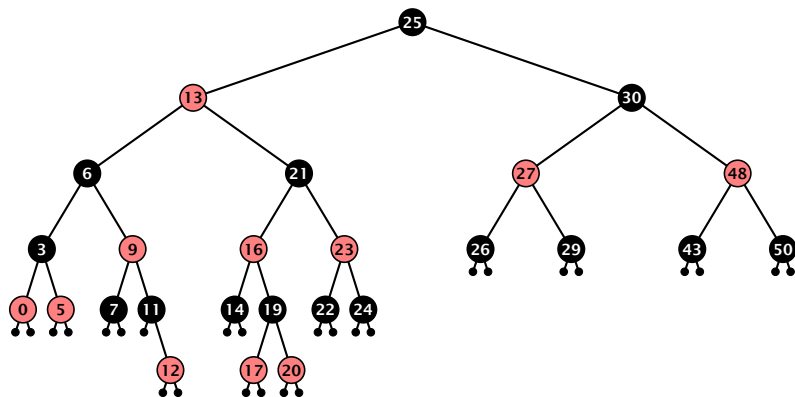
7.2 Red Black Trees

We need to adapt the insert and delete operations so that the red black properties are maintained.

Rotations

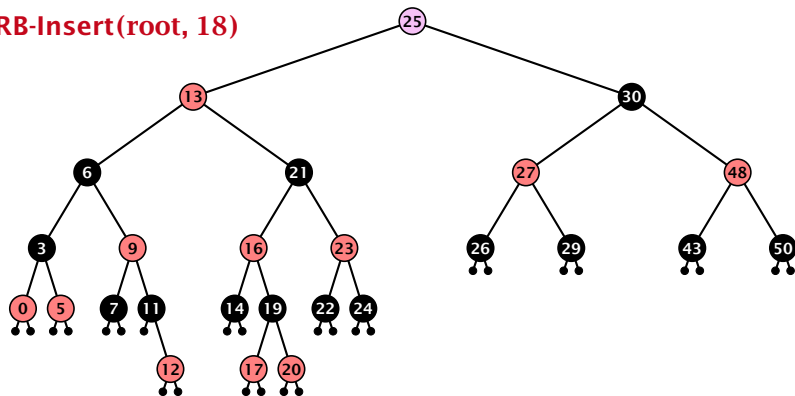The properties will be maintained through rotations:

Red Black Trees: Insert

Insert:

▶ first make a normal insert into a binary search tree

▶ then fix red-black properties

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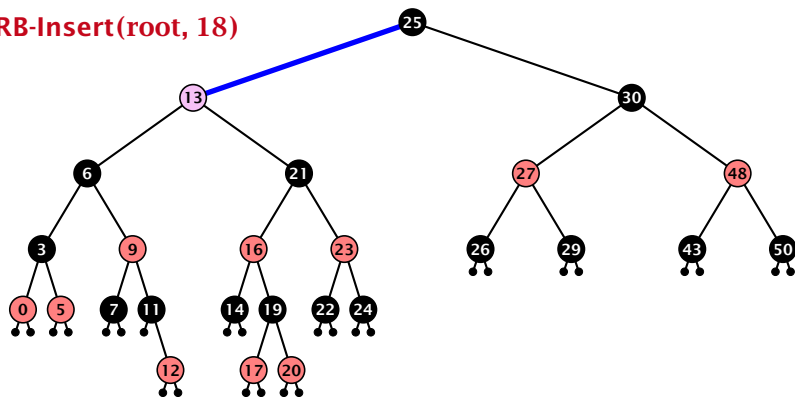

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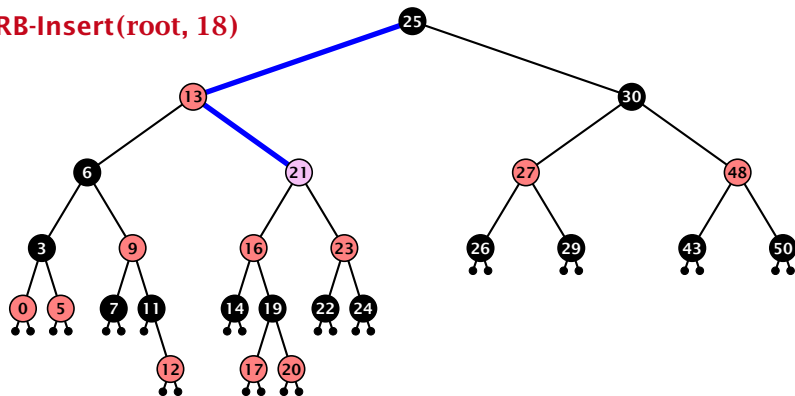
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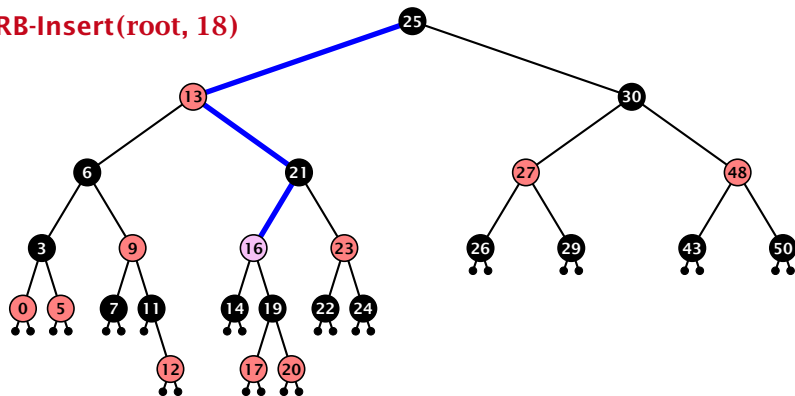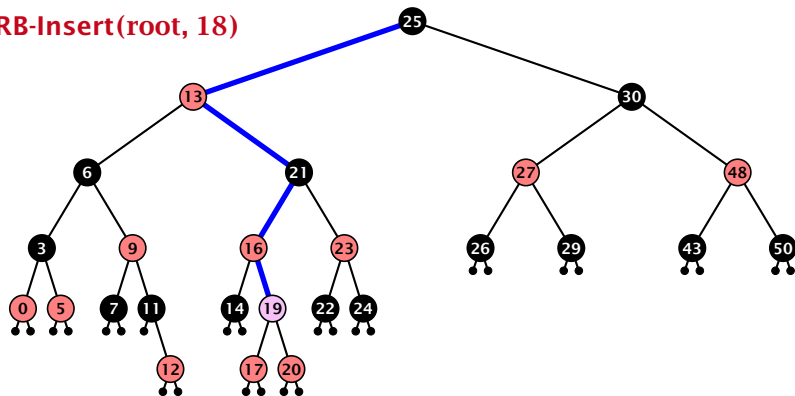### Insert:

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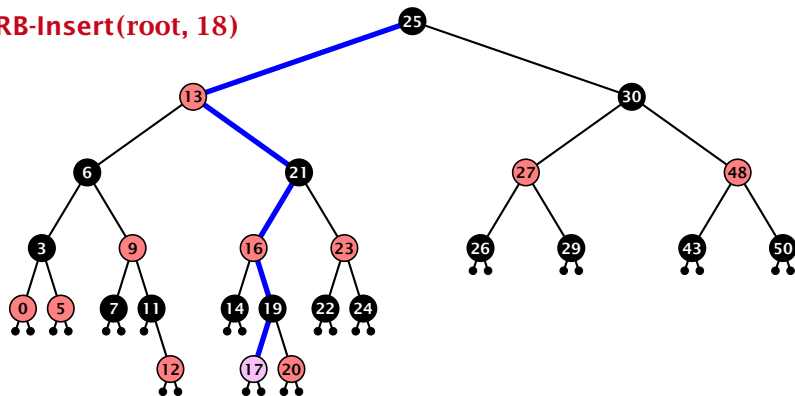**Insert:**

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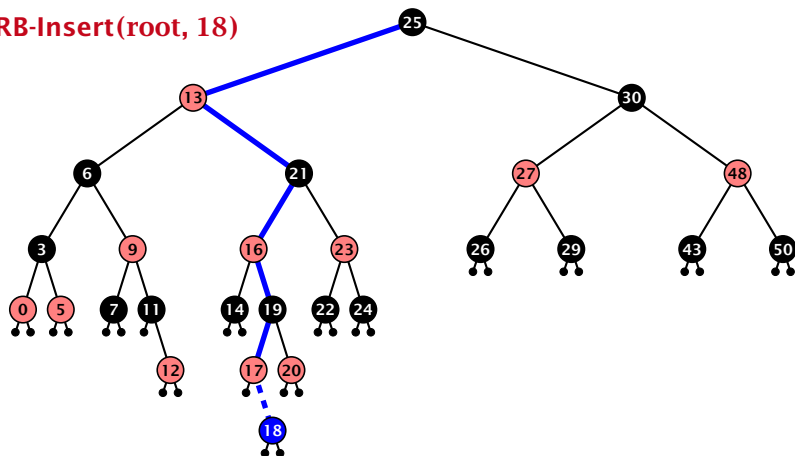## Insert:

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Invariant of the fix-up algorithm:

- ▶ z is a red node

Ernst Mayr, Harald Räcke

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If z has a parent but no grand-parent we could simply color the parent/root black; however this case never happens.

Red Black Trees: Insert

Algorithm 10 InsertFix(z)
1: **while** parent[z] ≠ null **and** col[parent[z]] = red **do**
2:     **if** parent[z] = left[gp[z]] **then**
3:         *uncle* ← right[grandparent[z]]
4:         **if** col[*uncle*] = red **then**
5:             col[p[z]] ← black; col[u] ← black;
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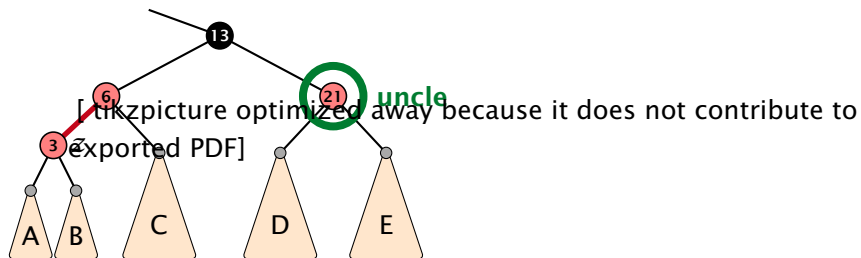| Algorithm 10 InsertFix(z) |
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| 11:　　　　　　RightRotate(gp[z]); |
| 12:　　**else** same as then-clause but right and left exchanged |
| 13: col(root[T]) ← black; |

Red Black Trees: Insert

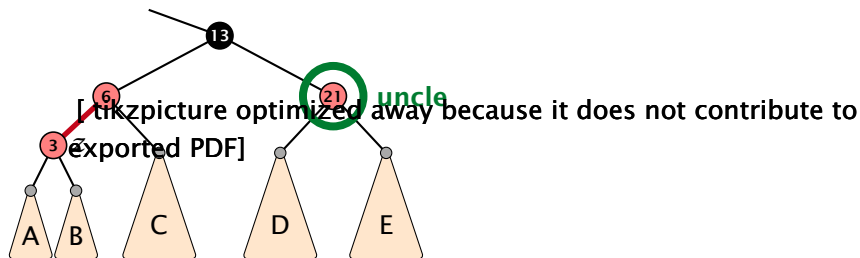|	**Algorithm 10** InsertFix(z)
1:	**while** parent[z] ≠ null **and** col[parent[z]] = red **do**
2:	**if** parent[z] = left[gp[z]] **then**
3:	*uncle* ← right[grandparent[z]]
4:	**if** col[*uncle*] = red **then**
5:	col[p[z]] ← black; col[u] ← black;
6:	col[gp[z]] ← red; z ← grandparent[z];
7:	**else**
8:	**if** z = right[parent[z]] **then**
9:	z ← p[z]; LeftRotate(z);
10:	col[p[z]] ← black; col[gp[z]] ← red; 2b: *z* left child
11:	RightRotate(gp[z]);
12:	**else** same as then-clause but right and left exchanged
13:	col(root[T]) ← black;
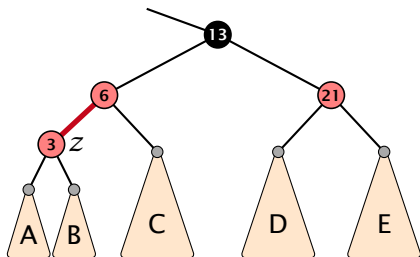
Case 1: Red Uncle

Case 1: Red Uncle

[tikzpicture optimized away because it does not contribute to exported PDF]

Case 1: Red Uncle

[tikzpicture optimized away because it does not contribute to exported PDF]

Case 1: Red Uncle

[tikzpicture optimized away because it does not contribute to exported PDF]

1. recolour

Case 1: Red Uncle

uncle

[tikzpicture optimized away because it does not contribute to exported PDF]

1. recolour

Case 1: Red Uncle

[tikzpicture optimized away because it does not contribute to exported PDF]

1. recolour
2. move z to grand-parent

Case 1: Red Uncle
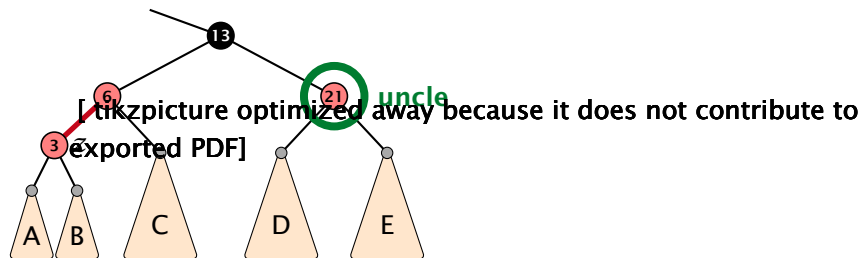
1. recolour
2. move z to grand-parent
3. invariant is fulfilled for new z

Case 1: Red Uncle

uncle

[tikzpicture optimized away because it does not contribute to exported PDF]
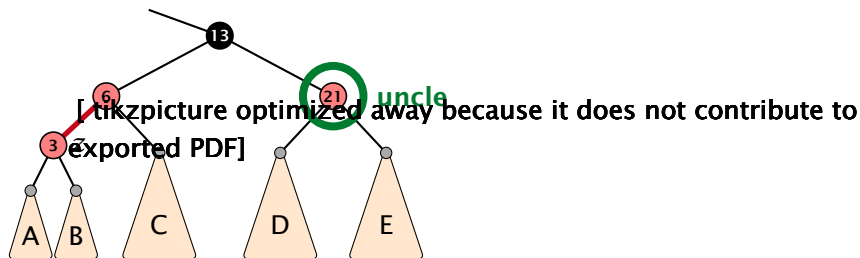
1. recolour
2. move z to grand-parent
3. invariant is fulfilled for new z
4. you made progress
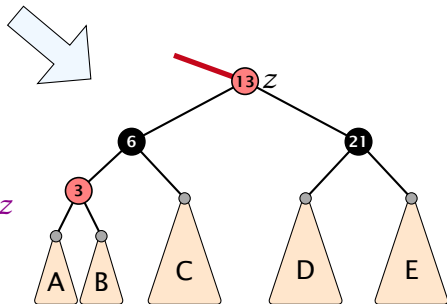
Case 2b: Black uncle and z is left child
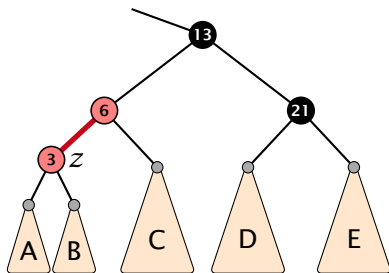
Case 2b: Black uncle and *z* is left child

[tikzpicture optimized away because it does not contribute to exported PDF]

Case 2b: Black uncle and *z* is left child

1. rotate around grandparent

   [tikzpicture optimized away because it does not contribute to exported PDF]

Case 2b: Black uncle and z is left child

1. rotate around grandparent
2. re-colour to ensure that black height property holds

[tikzpicture optimized away because it does not contribute to exported PDF]

Case 2b: Black uncle and z is left child

1. rotate around grandparent
2. re-colour to ensure that black height property holds
3. you have a red black tree

[tikzpicture optimized away because it does not contribute to exported PDF]
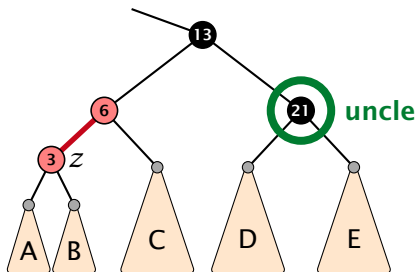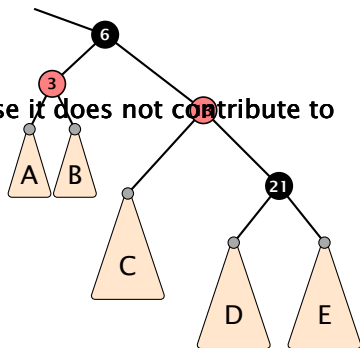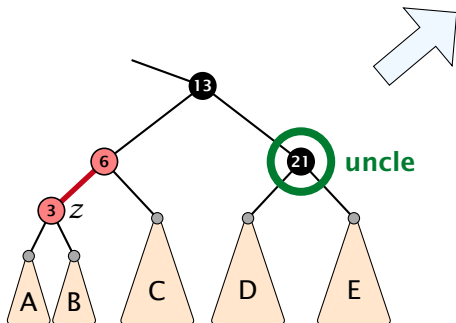
Case 2a: Black uncle and *z* is right child
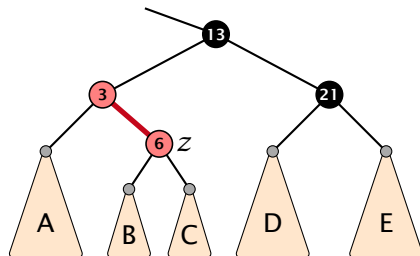
Case 2a: Black uncle and z is right child

[tikzpicture optimized away because it does not contribute to exported PDF]
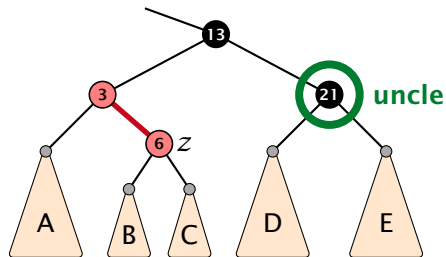
Case 2a: Black uncle and *z* is right child

1. rotate around parent

   [tikzpicture optimized away because it does not contribute to exported PDF]
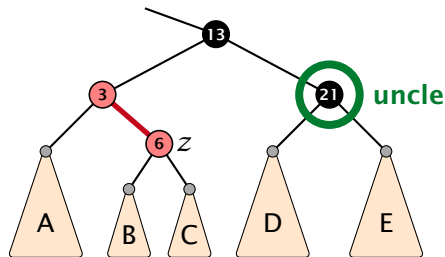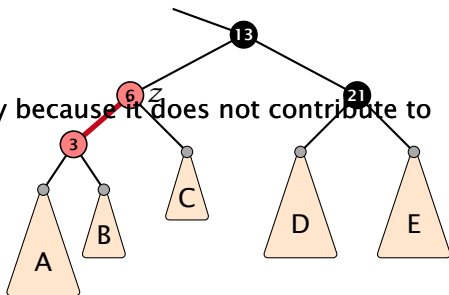
Case 2a: Black uncle and z is right child

1. rotate around parent
2. move z downwards
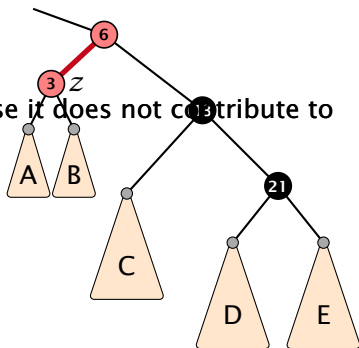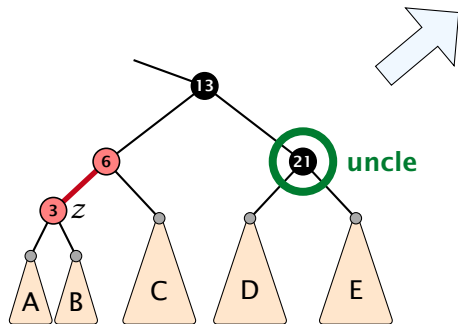
[TikZpicture optimized away because it does not contribute to exported PDF]
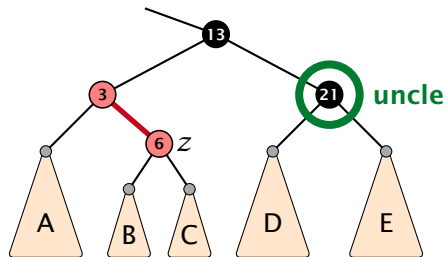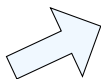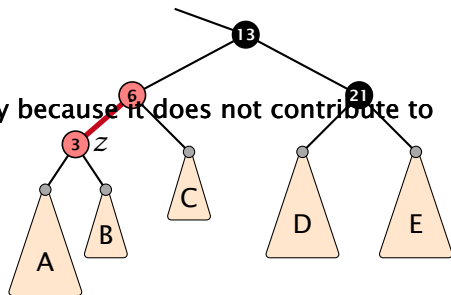
Case 2a: Black uncle and z is right child

1. rotate around parent
2. move z downwards
3. you had Case 2b.

[tikzpicture optimized away because it does not contribute to exported PDF.]
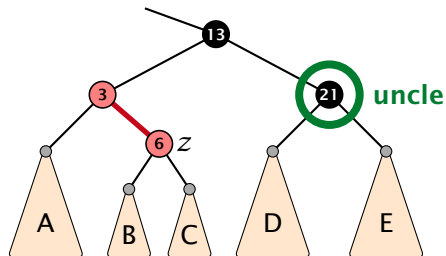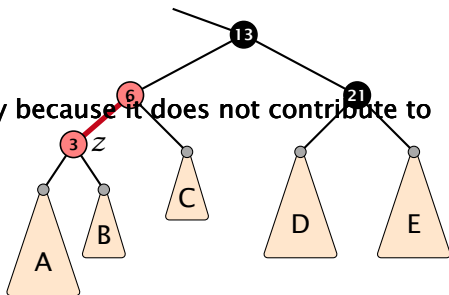
Red Black Trees: Insert

Running time:

▶ Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.

Red Black Trees: Insert

Running time:

- ▶ Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.

- ▶ Case 2a → Case 2b → red-black tree

Red Black Trees: Insert

Running time:

- ▶ Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- ▶ Case 2a → Case 2b → red-black tree
- ▶ Case 2b → red-black tree

Red Black Trees: Insert

Running time:

- ▶ Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- ▶ Case 2a → Case 2b → red-black tree
- ▶ Case 2b → red-black tree

Performing Case 1 at most $\mathcal{O}(\log n)$ times and every other case at most once, we get a red-black tree. Hence $\mathcal{O}(\log n)$ re-colorings and at most 2 rotations.

Red Black Trees: Delete

Ernst Mayr, Harald Räcke

Red Black Trees: Delete

First do a standard delete.

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If the spliced out node x was red everything is fine.

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- ▶ Parent and child of x were red; two adjacent red vertices.

Red Black Trees: Delete

First do a standard delete.

If the spliced out node x was red everything is fine.

If it was black there may be the following problems.

- ▶ Parent and child of x were red; two adjacent red vertices.
- ▶ If you delete the root, the root may now be red.

Red Black Trees: Delete

First do a standard delete.

If the spliced out node x was red everything is fine.

If it was black there may be the following problems.

- ▶ Parent and child of x were red; two adjacent red vertices.
- ▶ If you delete the root, the root may now be red.
- ▶ Every path from an ancestor of x to a descendant leaf of x changes the number of black nodes. Black height property might be violated.

Red Black Trees: Delete

Red Black Trees: Delete

Case 3:

Element has two children

- ▶ do normal delete
- ▶ when replacing content by content of successor, don't change color of node

Red Black Trees: Delete

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Red Black Trees: Delete

Delete:
▶ deleting black node messes up black-height property

Red Black Trees: Delete
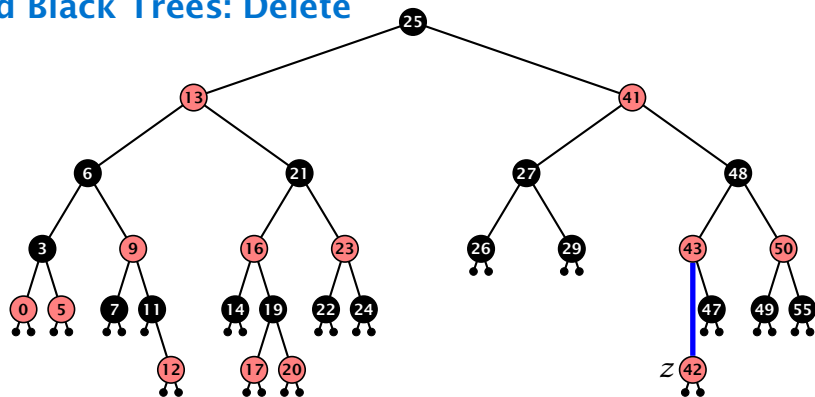
Delete:

▶ deleting black node messes up black-height property

▶ if z is red, we can simply color it black and everything is fine

Red Black Trees: Delete
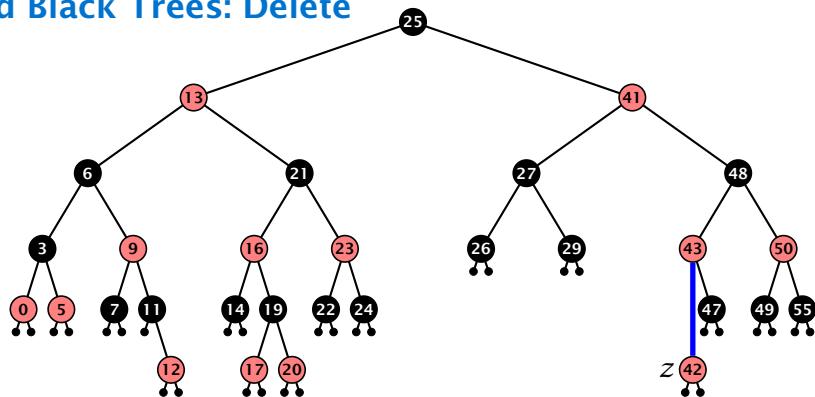
Delete:

▶ deleting black node messes up black-height property

▶ if z is red, we can simply color it black and everything is fine

▶ the problem is if z is black (e.g. a dummy-leaf); we call a
fix-up procedure to fix the problem.

Red Black Trees: Delete

Invariant of the fix-up algorithm

▶ the node z is black

Red Black Trees: Delete

Invariant of the fix-up algorithm

▶ the node z is black

▶ if we "assign" a fake black unit to the edge from z to its parent then the black-height property is fulfilled

Red Black Trees: Delete

Invariant of the fix-up algorithm

- ▶ the node z is black
- ▶ if we "assign" a fake black unit to the edge from z to its parent then the black-height property is fulfilled

Goal: make rotations in such a way that you at some point can remove the fake black unit from the edge.

Case 1: Sibling of z is red
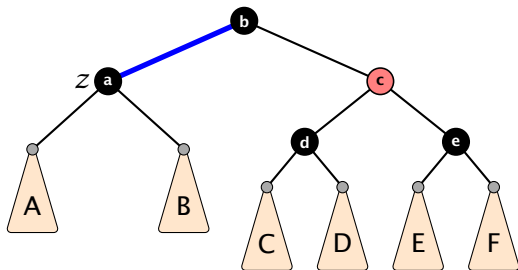
Case 1: Sibling of z is red
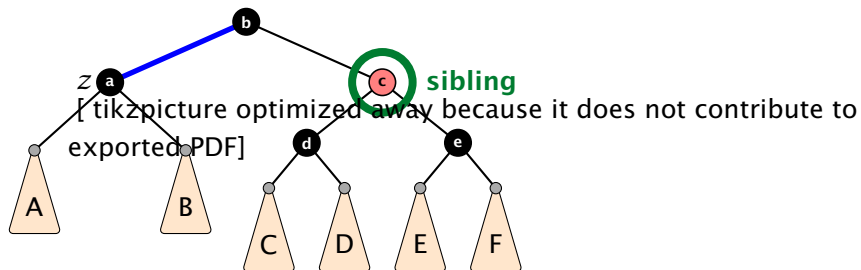
Case 1: Sibling of z is red
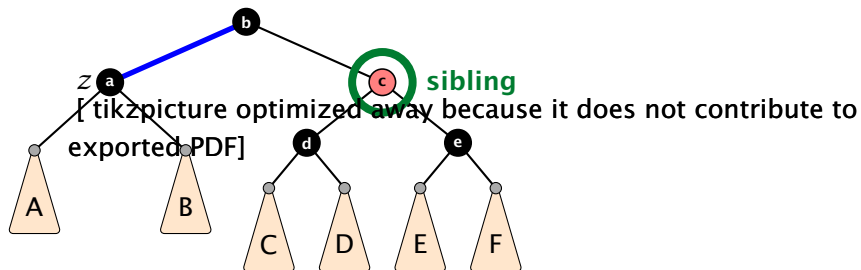
1. left-rotate around parent of z

Case 1: Sibling of z is red

[tikzpicture optimized away because it does not contribute to exported PDF]
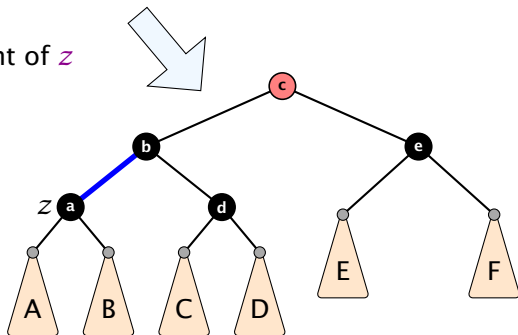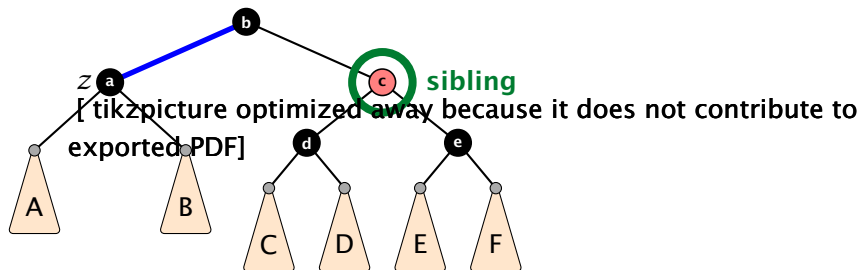
1. left-rotate around parent of z
2. recolor nodes b and c

Case 1: Sibling of z is red
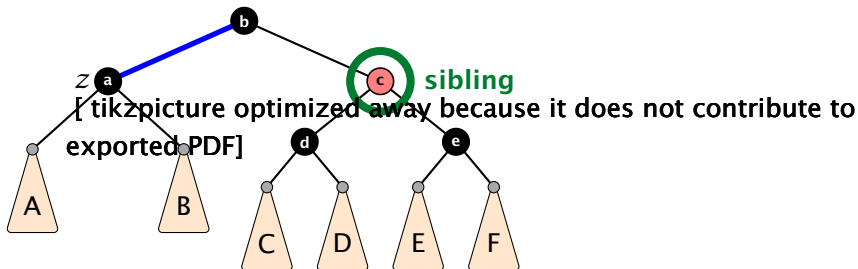
[tikzpicture optimized away because it does not contribute to exported PDF]

1. left-rotate around parent of z
2. recolor nodes b and c
3. the new sibling is black
   (and parent of z is red)
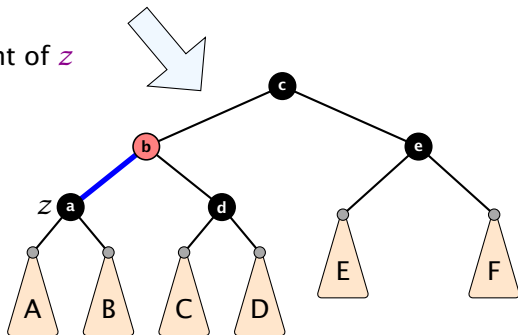
Case 1: Sibling of z is red
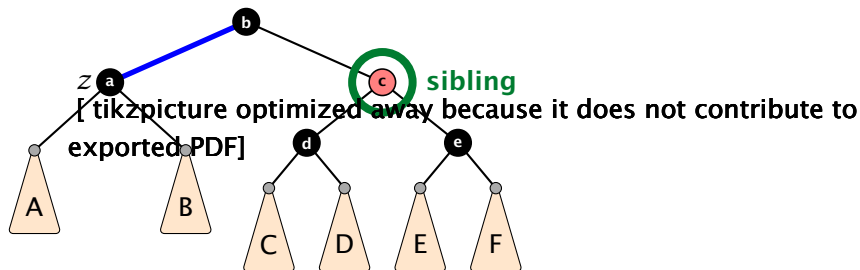
1. left-rotate around parent of z
2. recolor nodes b and c
3. the new sibling is black
   (and parent of z is red)
4. Case 2 (special),
   or Case 3, or Case 4
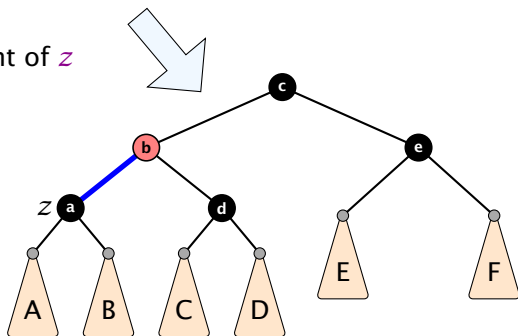
Case 2: Sibling is black with two black children
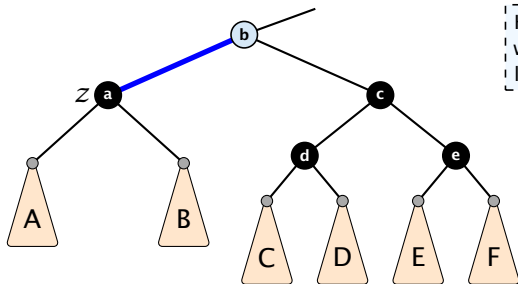
Here b is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

Case 2: Sibling is black with two black children
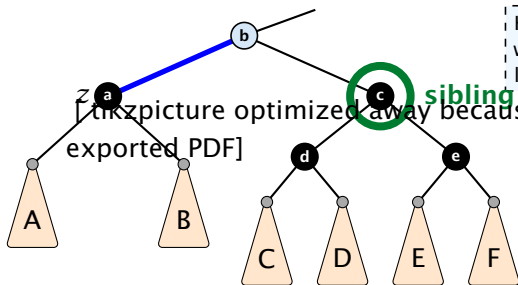
Here b is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

z [tikzpicture optimized away because it does not contribute to exported PDF]

Case 2: Sibling is black with two black children

Here b is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

sibling

Case 2: Sibling is black with two black children

Here b is either black or red. If it is red we are in a special case that directly leads to a red-black tree.
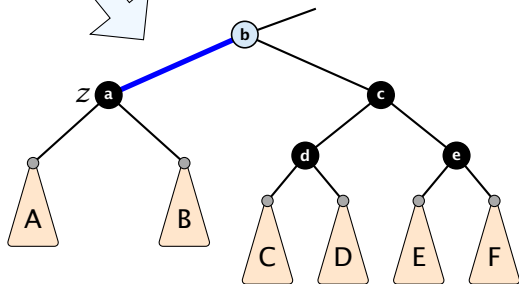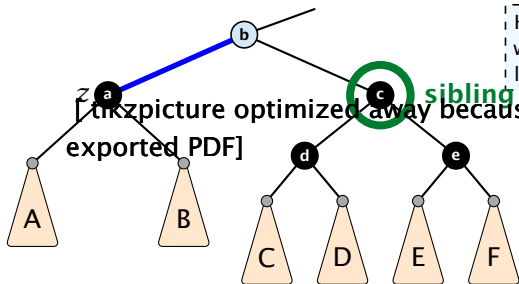
1. re-color node c

Case 2: Sibling is black with two black children
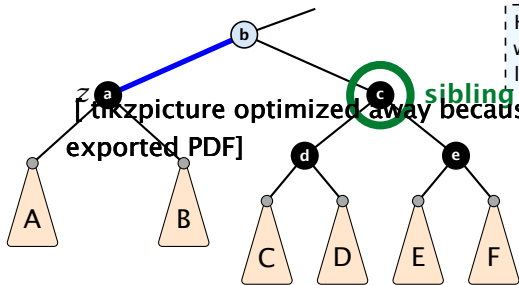
Here b is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

[tikzpicture optimized away because it does not contribute to exported PDF]

sibling

1. re-color node c
2. move fake black unit upwards

Case 2: Sibling is black with two black children
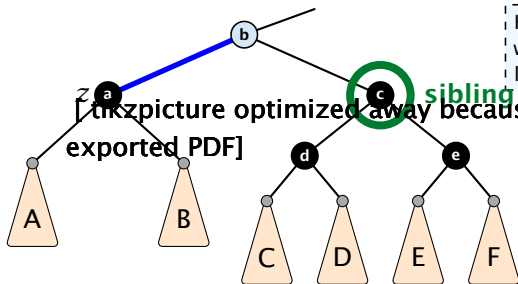
Here b is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

[tikzpicture optimized away because it does not contribute to exported PDF]

1. re-color node c
2. move fake black unit upwards
3. move z upwards
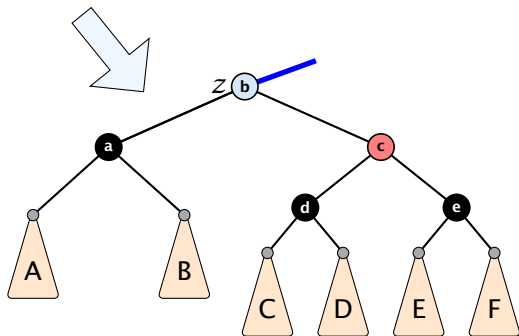
Case 2: Sibling is black with two black children

Here b is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

[tikzpicture optimized away because it does not contribute to exported PDF]

sibling

1. re-color node c
2. move fake black unit upwards
3. move z upwards
4. we made progress

Case 2: Sibling is black with two black children

Here b is either black or red. If it is red we are in a special case that directly leads to a red-black tree.
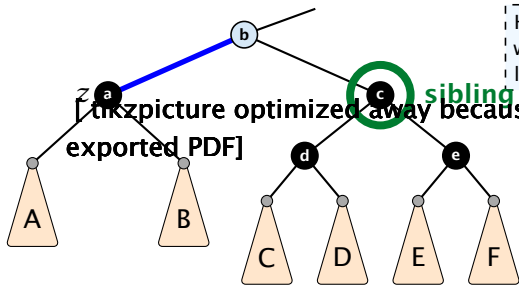
1. re-color node c
2. move fake black unit upwards
3. move z upwards
4. we made progress
5. if b is red we color it black and are done
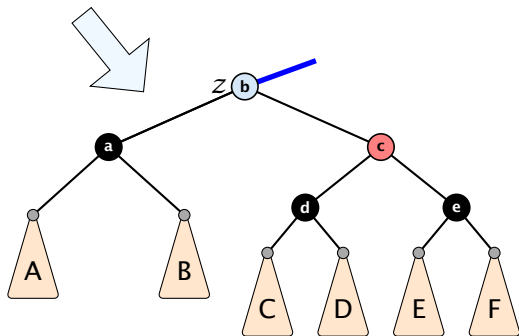
Case 3: Sibling black with one black child to the right
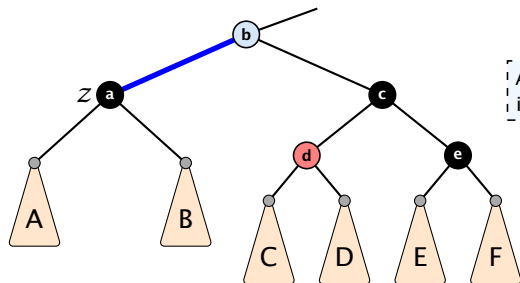
Again the blue color of b indicates that it can either be black or red.

Case 3: Sibling black with one black child to the right

[tikzpicture optimized away because it does not contribute to exported PDF]

sibling

Again the blue color of b indicates that it can either be black or red.
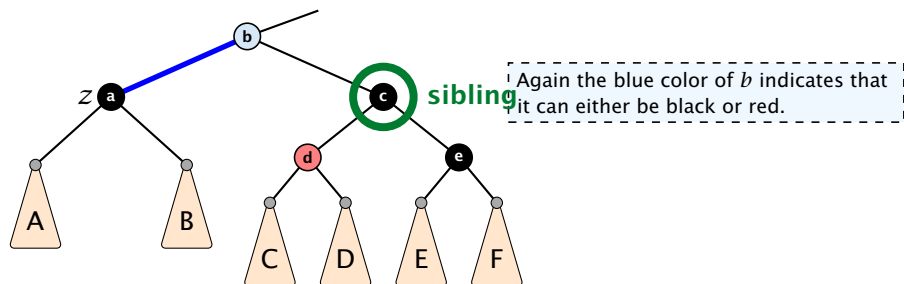
Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling

[tikzpicture optimized away because it does not contribute to exported PDF]
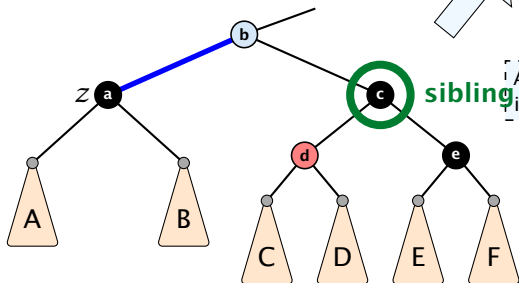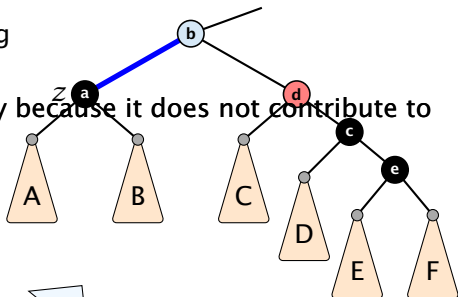
sibling

Again the blue color of b indicates that it can either be black or red.

Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling
2. recolor c and d
   [tikzpicture optimized away because it does not contribute to exported PDF]
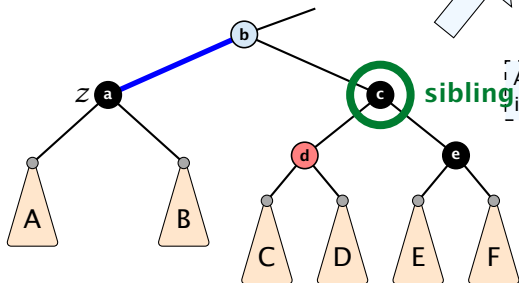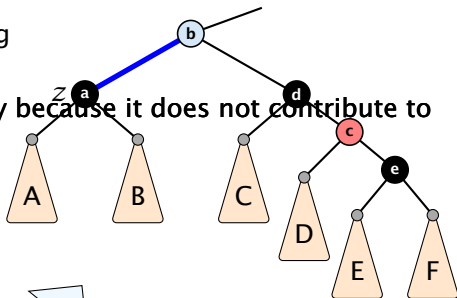
Again the blue color of b indicates that it can either be black or red.

sibling

Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling

2. recolor c and d

3. new sibling is black with red right child (Case 4)

[tikzpicture optimized away because it does not contribute to exported PDF]
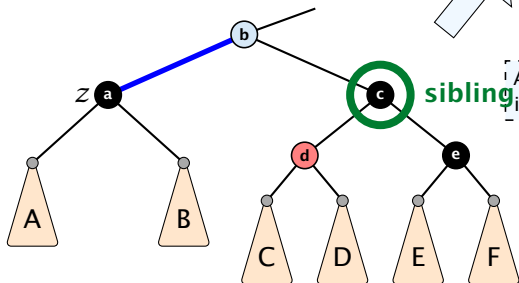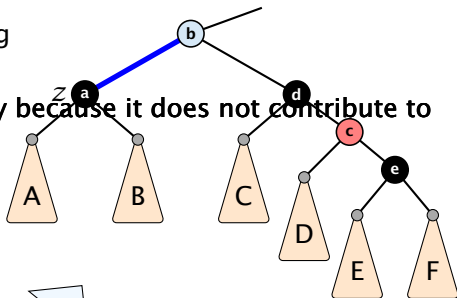
Again the blue color of b indicates that it can either be black or red.

Case 4: Sibling is black with red right child
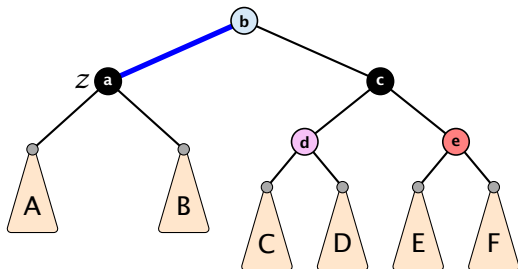
- Here b and d are either red or black but have possibly different colors.
- We recolor c by giving it the color of b.

Case 4: Sibling is black with red right child
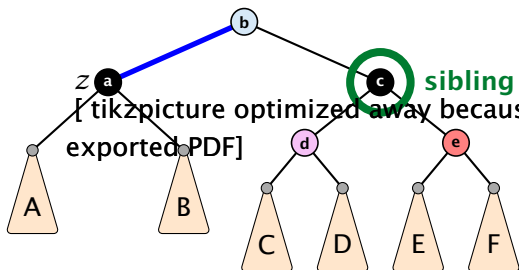
- Here b and d are either red or black but have possibly different colors.
- b does not contribute to color of b.

[tikzpicture optimized away because it does not contribute to the exported PDF]

Case 4: Sibling is black with red right child

- Here b and d are either red or black but have possibly different colors.
- Does not contribute to color of b.

1. left-rotate around b

Case 4: Sibling is black with red right child
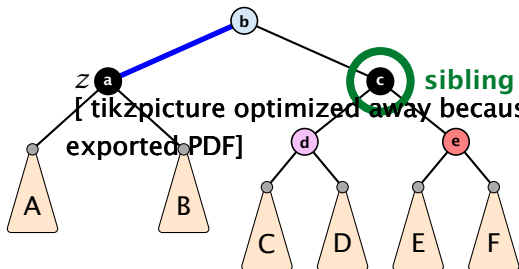
- Here b and d are either red or black but have possibly different colors.
- [overlapping text]... does not contribute to color of b.

[tikzpicture optimized away because it does not contribute to exported PDF]

sibling

1. left-rotate around b
2. remove the fake black unit

Case 4: Sibling is black with red right child

- Here b and d are either red or black but have possibly different colors.
- Does not contribute to color of b.

1. left-rotate around b
2. remove the fake black unit
3. recolor nodes b, c, and e
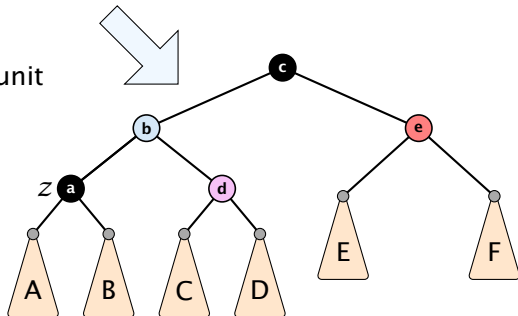
Case 4: Sibling is black with red right child
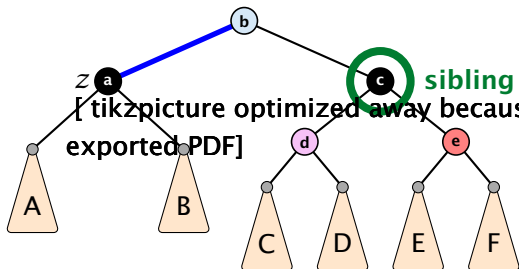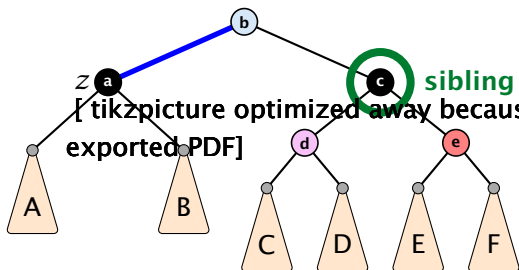
sibling

[tikzpicture optimized away because it does not contribute to exported PDF]

- Here b and d are either red or black but have possibly different colors.
- [Does not contribute to color of b.]

1. left-rotate around b
2. remove the fake black unit
3. recolor nodes b, c, and e
4. you have a valid red black tree
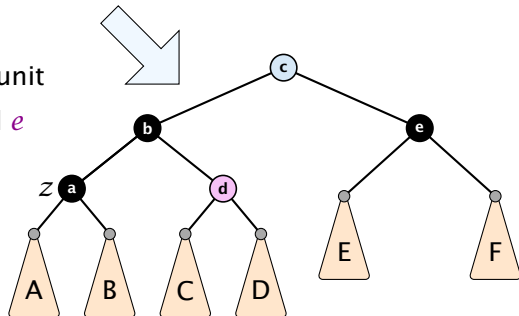
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▶ only Case 2 can repeat; but only h many steps, where h is the height of the tree

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Performing Case 2 at most $\mathcal{O}(\log n)$ times and every other step at most once, we get a red black tree. Hence, $\mathcal{O}(\log n)$ re-colorings and at most 3 rotations.

Splay Trees

Disadvantage of balanced search trees:

Splay Trees

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– worst case; no advantage for easy inputs

Ernst Mayr, Harald Räcke

Splay Trees

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- − additional memory required

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Splay Trees:

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- – only amortized guarantee
- – read-operations change the tree

Splay Trees

find(x)

- ▶ search for x according to a search tree
- ▶ let $\bar{x}$ be last element on search-path
- ▶ splay($\bar{x}$)

Splay Trees

insert(x)

- ▶ search for x; $\bar{x}$ is last visited element during search (successor or predecessor of x)
- ▶ splay($\bar{x}$) moves $\bar{x}$ to the root
- ▶ insert x as new root
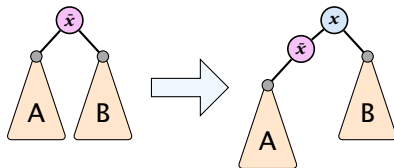
Splay Trees

delete(x)

- ▶ search for x; splay(x); remove x
- ▶ search largest element $\bar{x}$ in A
- ▶ splay($\bar{x}$) (on subtree A)
- ▶ connect root of B as right child of $\bar{x}$
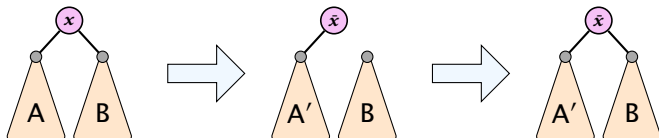
Move to Root
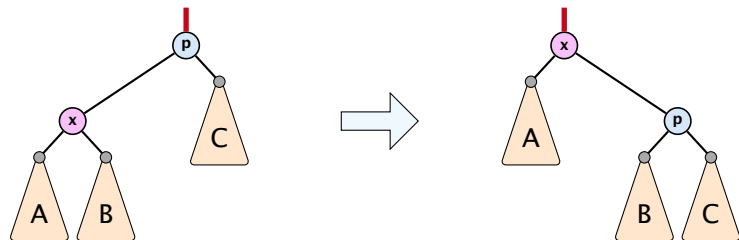
How to bring element to root?

▶ one (bad) option: moveToRoot(x)

▶ iteratively do rotation around parent of x until x is root

▶ if x is left child do right rotation otw. left rotation

Splay: Zig Case
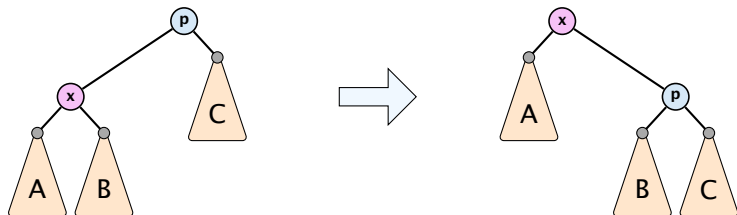
better option splay(x):

▶ zig case: if x is child of root do left rotation or right rotation around parent

Splay: Zigzag Case
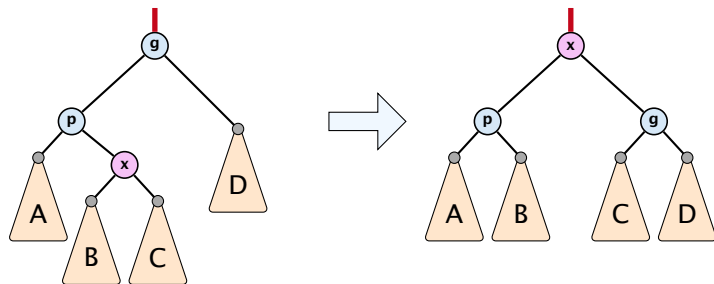
better option splay(x):

▶ zigzag case: if x is right child and parent of x is left child (or x left child parent of x right child)

▶ do double right rotation around grand-parent (resp. double left rotation)

Double Rotations
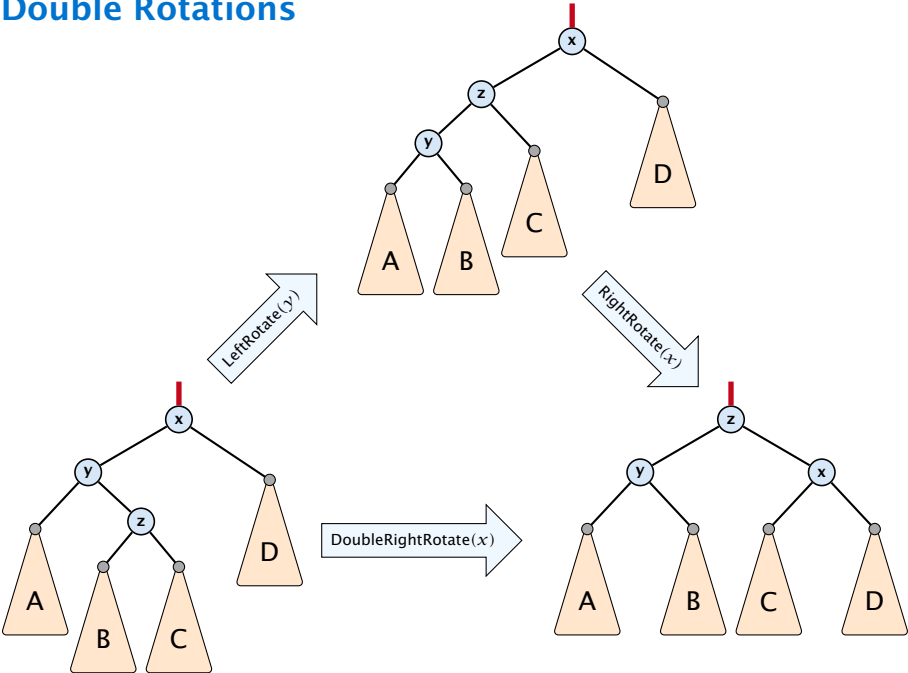
Splay: Zigzig Case
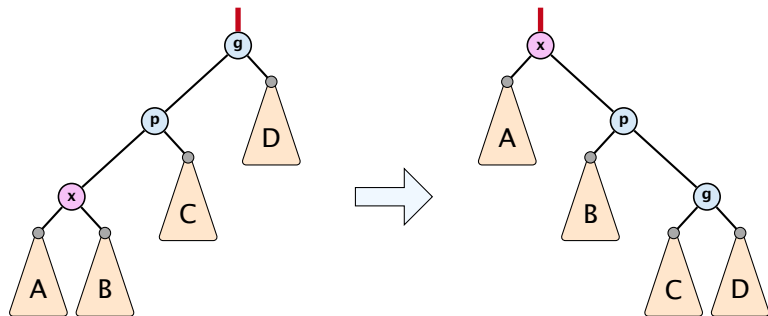
better option splay(x):

▶ zigzig case: if x is left child and parent of x is left child (or x right child, parent of x right child)

▶ do right roation around grand-parent followed by right rotation around parent (resp. left rotations)

Splay vs. Move to Root
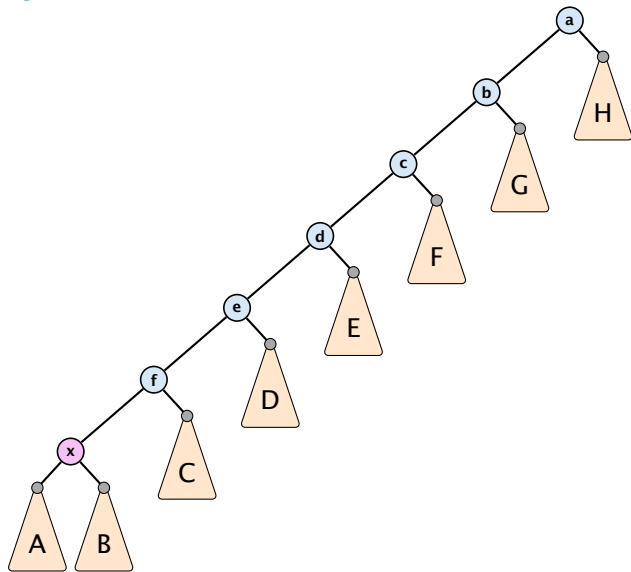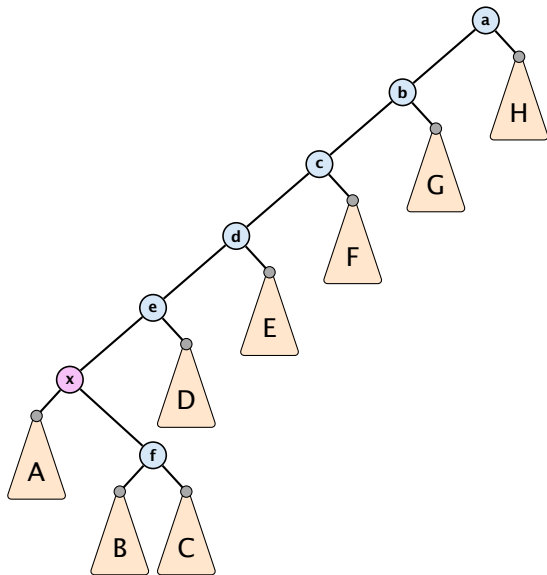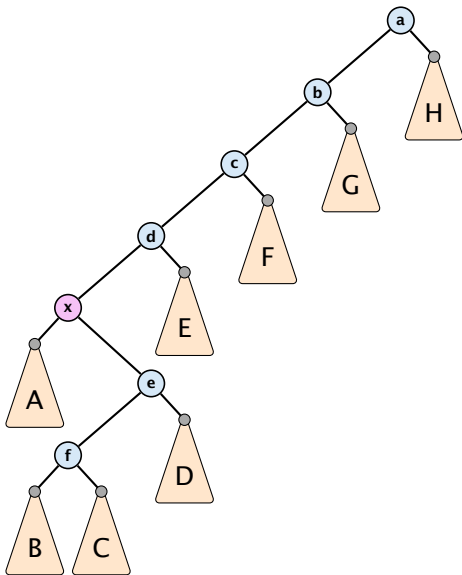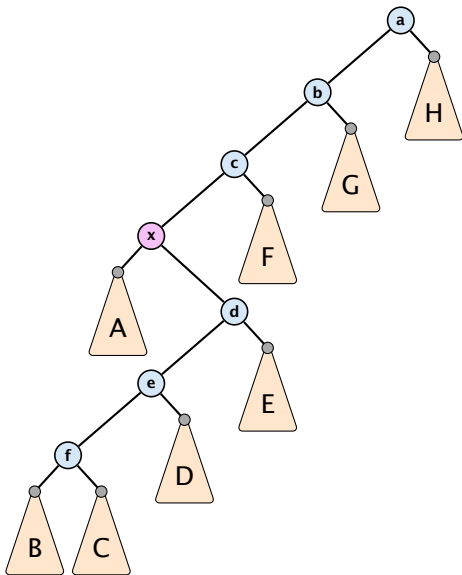
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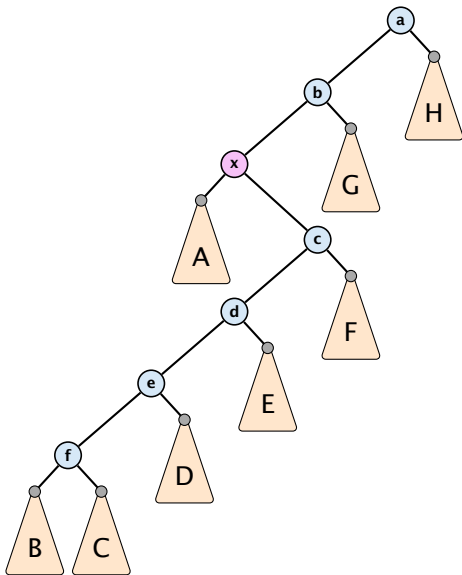
Splay vs. Move to Root
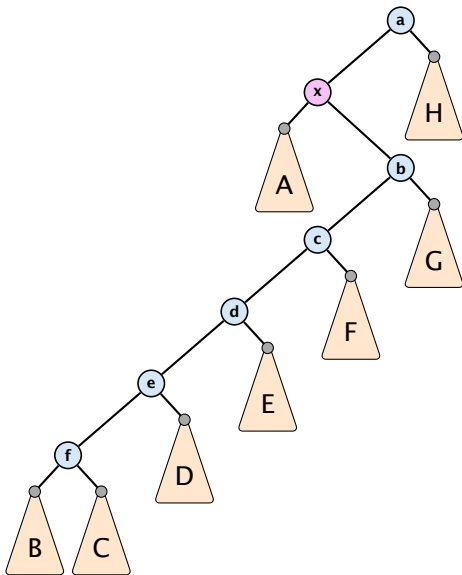
Splay vs. Move to Root
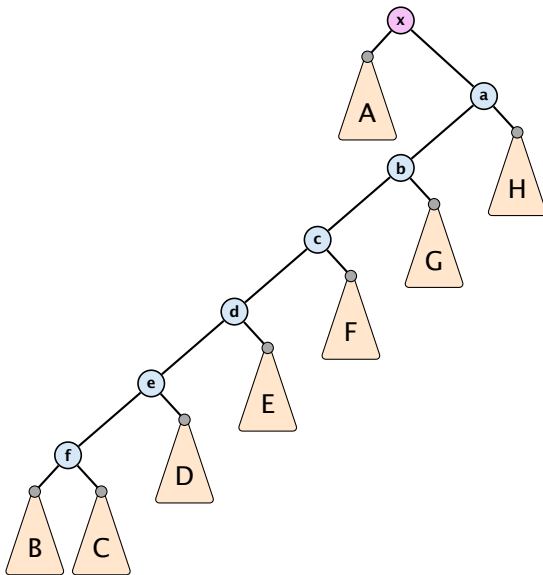
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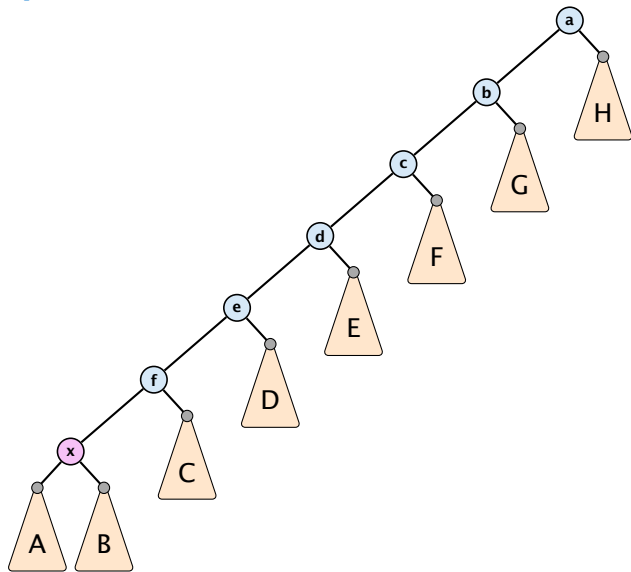
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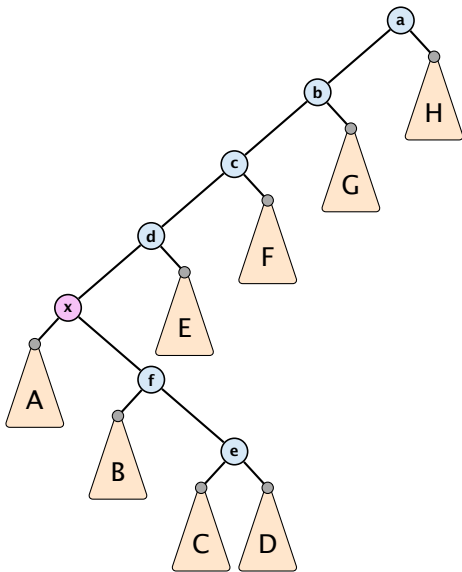
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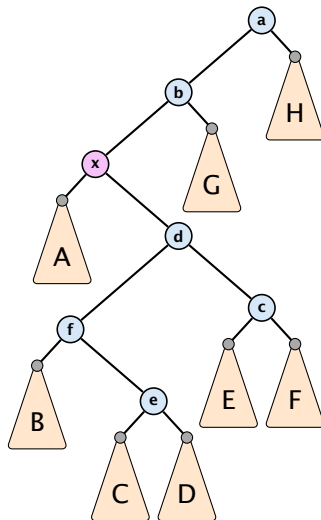
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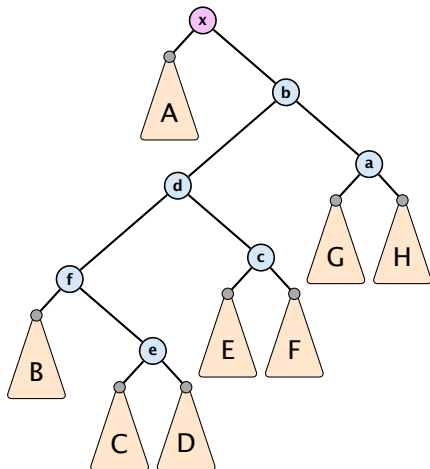
Static Optimality

Suppose we have a sequence of m find-operations. $\text{find}(x)$ appears h_x times in this sequence.

The cost of a static search tree T is:

$$\text{cost}(T) = m + \sum_x h_x \, \text{depth}_T(x)$$

The total cost for processing the sequence on a splay-tree is $\mathcal{O}(\text{cost}(T_{\min}))$, where $T_{\min}$ is an optimal static search tree.

Dynamic Optimality

Let S be a sequence with m find-operations.

Let A be a data-structure based on a search tree:

▶ the cost for accessing element x is $1 + \text{depth}(x)$;

▶ after accessing x the tree may be re-arranged through rotations;

Conjecture:
A splay tree that only contains elements from S has cost $\mathcal{O}(\text{cost}(A, S))$, for processing S.

Lemma 5

Splay Trees have an amortized running time of $\mathcal{O}(\log n)$ for all operations.

Amortized Analysis

Definition 6

A data structure with operations $\mathrm{op}_1(), \ldots, \mathrm{op}_k()$ has amortized running times $t_1, \ldots, t_k$ for these operations if the following holds.

Suppose you are given a sequence of operations (starting with an empty data-structure) that operate on at most n elements, and let k_i denote the number of occurences of $\mathrm{op}_i()$ within this sequence. Then the actual running time must be at most $\sum_i k_i \cdot t_i(n)$.

Potential Method

Introduce a potential for the data structure.

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Then

$$\sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} c_i + \Phi(D_k) - \Phi(D_0) = \sum_{i=1}^{k} \hat{c}_i$$

This means the amortized costs can be used to derive a bound on the total cost.

Example: Stack

Stack

- ▶ $S.\,\mathbf{push()}$
- ▶ $S.\,\mathbf{pop()}$
- ▶ $S.\,\mathbf{multipop(k)}$: removes k items from the stack. If the stack currently contains less than k items it empties the stack.
- ▶ The user has to ensure that pop and multipop do not generate an underflow.

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- ▶ The user has to ensure that pop and multipop do not generate an underflow.

Actual cost:

- ▶ $S.\,\mathbf{push()}$: cost 1.
- ▶ $S.\,\mathbf{pop()}$: cost 1.
- ▶ $S.\,\mathbf{multipop}(k)$: cost $\min\{\text{size}, k\} = k$.

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Use potential function $\Phi(S) =$ number of elements on the stack.

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Amortized cost:

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▶ $S. \textbf{multipop}(k)$: cost

$$\hat{C}_{\text{mp}} = C_{\text{mp}} + \Delta\Phi = \min\{\text{size}, k\} - \min\{\text{size}, k\} \le 0 \ .$$

Example: Binary Counter

Incrementing a binary counter:
Consider a computational model where each bit-operation costs
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Actual cost:

▶ Changing bit from 0 to 1: cost 1.

▶ Changing bit from 1 to 0: cost 1.

▶ Increment: cost is $k + 1$, where k is the number of
  consecutive ones in the least significant bit-positions (e.g,
  001101 has $k = 1$).

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▶ Changing bit from 1 to 0:

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▶ Changing bit from 1 to 0:

$$\hat{C}_{1 \to 0} = C_{1 \to 0} + \Delta\Phi = 1 - 1 \leq 0 \ .$$

▶ Increment: Let k denotes the number of consecutive ones in the least significant bit-positions. An increment involves k $(1 \to 0)$-operations, and one $(0 \to 1)$-operation.

Hence, the amortized cost is $k\hat{C}_{1 \to 0} + \hat{C}_{0 \to 1} \leq 2$.

Splay Trees

potential function for splay trees:

- ▶ size $s(x) = |T_x|$
- ▶ rank $r(x) = \log_2(s(x))$
- ▶ $\Phi(T) = \sum_{v \in T} r(v)$

amortized cost = real cost + potential change

The cost is essentially the cost of the splay-operation, which is 1 plus the number of rotations.

$$\Delta \Phi =$$
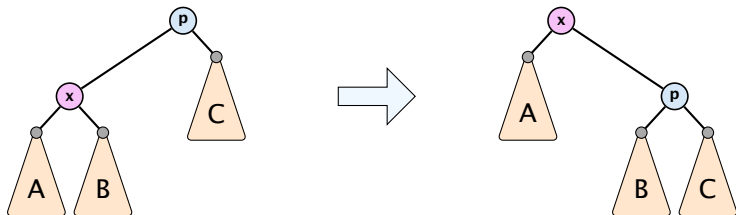
Splay: Zig Case
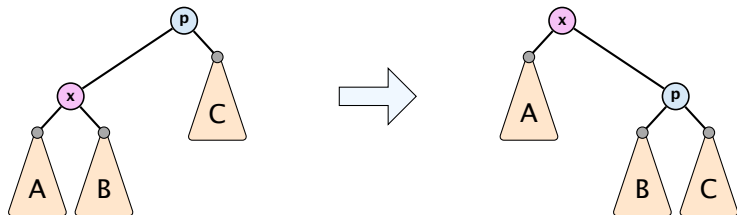
$$\Delta\Phi = r'(x) + r'(p) - r(x) - r(p)$$

Splay: Zig Case
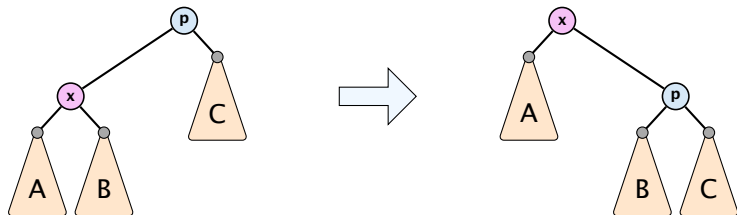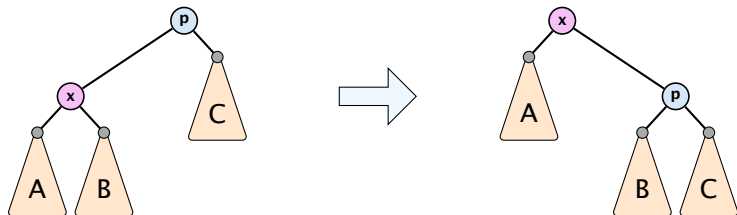
$$\Delta\Phi = r'(x) + r'(p) - r(x) - r(p)$$
$$= r'(p) - r(x)$$

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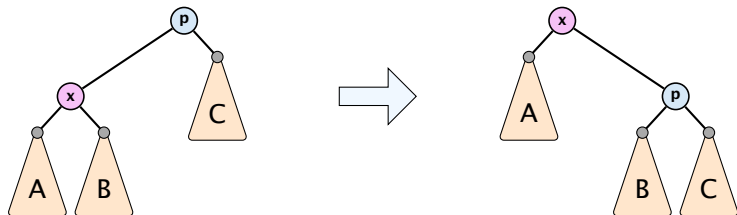
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Splay: Zig Case

$$\Delta\Phi = r'(x) + r'(p) - r(x) - r(p)$$
$$= r'(p) - r(x)$$
$$\leq r'(x) - r(x)$$

$$\text{cost}_{\text{zig}} \leq 1 + 3(r'(x) - r(x))$$

Splay: Zigzig Case

$\Delta\Phi =$
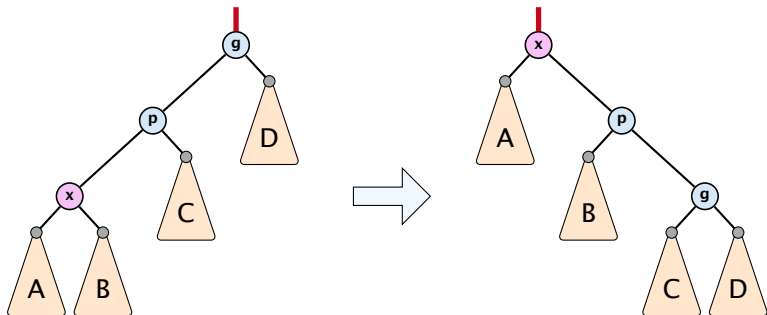
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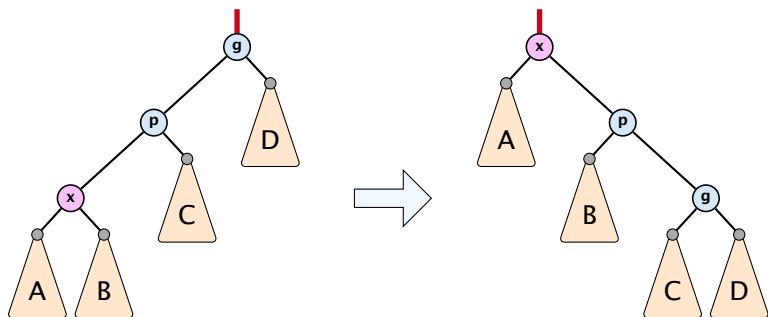
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$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$
$$= r'(p) + r'(g) - r(x) - r(p)$$

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$$= r'(p) + r'(g) - r(x) - r(p)$$
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$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$

$$= r'(p) + r'(g) - r(x) - r(p)$$

$$\le r'(x) + r'(g) - r(x) - r(x)$$

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$$= -2r'(x) + r'(g) + r(x) + 3(r'(x) - r(x))$$

Splay: Zigzig Case

$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$
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$$\leq -2 + 3(r'(x) - r(x))$$

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$$= r'(p) + r'(g) - r(x) - r(p)$$

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$$= -2r'(x) + r'(g) + r(x) + 3(r'(x) - r(x))$$

$$\leq -2 + 3(r'(x) - r(x)) \quad \Rightarrow \text{cost}_{\text{zigzig}} \leq 3(r'(x) - r(x))$$
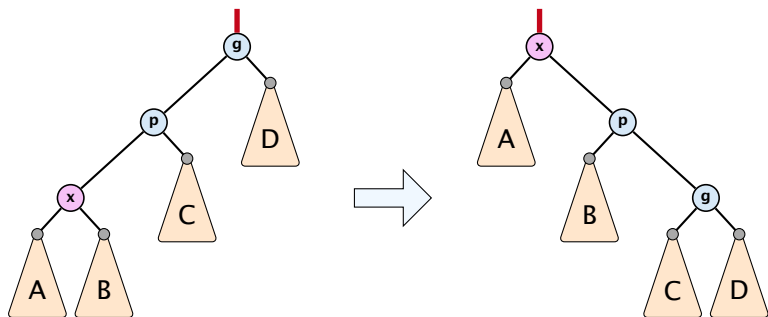
Splay: Zigzig Case

$$\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)$$

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$$\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)$$

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$$= \frac{1}{2}\Big(\log(s(x)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

$$= \frac{1}{2}\log\Big(\frac{s(x)}{s'(x)}\Big) + \frac{1}{2}\log\Big(\frac{s'(g)}{s'(x)}\Big)$$

Splay: Zigzig Case

$$\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)$$

$$= \frac{1}{2}\Big(\log(s(x)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

$$= \frac{1}{2}\log\Big(\frac{s(x)}{s'(x)}\Big) + \frac{1}{2}\log\Big(\frac{s'(g)}{s'(x)}\Big)$$

$$\leq \log\Big(\frac{1}{2}\frac{s(x)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\Big)$$

Splay: Zigzig Case

$$\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)$$

$$= \frac{1}{2}\Big(\log(s(x)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

$$= \frac{1}{2}\log\Big(\frac{s(x)}{s'(x)}\Big) + \frac{1}{2}\log\Big(\frac{s'(g)}{s'(x)}\Big)$$

$$\leq \log\Big(\frac{1}{2}\frac{s(x)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\Big) \leq \log\Big(\frac{1}{2}\Big)$$

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Splay: Zigzag Case

$\Delta\Phi =$

Splay: Zigzag Case
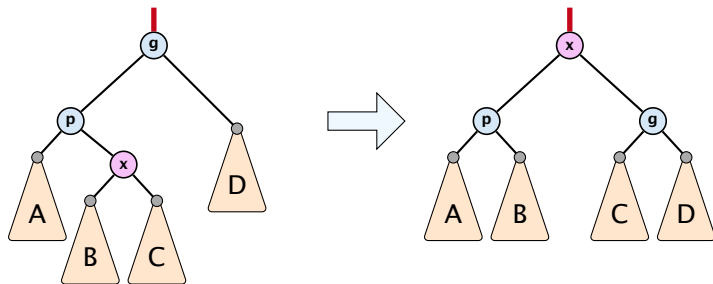
$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$

Splay: Zigzag Case

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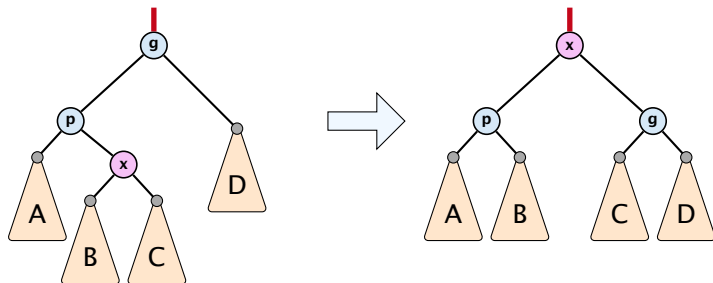
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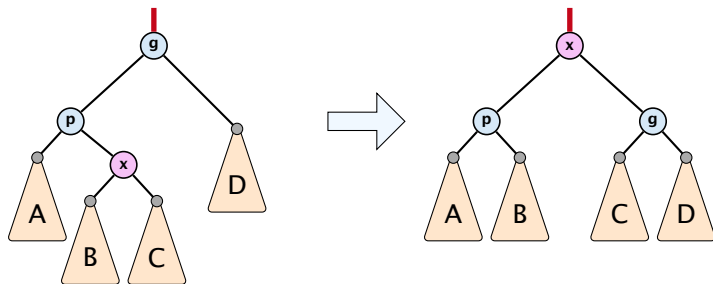
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Amortized cost of the whole splay operation:

$$\leq 1 + 1 + \sum_{\text{steps } t} 3(r_t(x) - r_{t-1}(x))$$

$$= 2 + 3(r(\text{root}) - r_0(x))$$

$$\leq \mathcal{O}(\log n)$$

7.4 Augmenting Data Structures

Suppose you want to develop a data structure with:

▶ **Insert(x):** insert element x.

▶ **Search(k):** search for element with key k.

▶ **Delete(x):** delete element referenced by pointer x.

▶ **find-by-rank(ℓ):** return the ℓ-th element; return "error" if the data-structure contains less than ℓ elements.

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Augment an existing data-structure instead of developing a new one.

7.4 Augmenting Data Structures

How to augment a data-structure

1. choose an underlying data-structure

- Of course, the above steps heavily depend on each other. For example it makes no sense to choose additional information to be stored (Step 2), and later realize that either the information cannot be maintained efficiently (Step 3) or is not sufficient to support the new operations (Step 4).

- However, the above outline is a good way to describe/document a new data-structure.

7.4 Augmenting Data Structures

How to augment a data-structure

1. choose an underlying data-structure

2. determine additional information to be stored in the underlying structure

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7.4 Augmenting Data Structures

How to augment a data-structure

1. choose an underlying data-structure

2. determine additional information to be stored in the underlying structure

3. verify/show how the additional information can be maintained for the basic modifying operations on the underlying structure.

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7.4 Augmenting Data Structures

Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

   1. We choose a red-black tree as the underlying data-structure.

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1. We choose a red-black tree as the underlying data-structure.

2. We store in each node v the size of the sub-tree rooted at v.

3. We need to be able to update the size-field in each node without asymptotically affecting the running time of insert, delete, and search. We come back to this step later...

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Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

4. How does find-by-rank work?
   Find-by-rank(k) := Select(root,k) with

Algorithm 1 Select(x, i)

1: **if** $x = $ null **then return** error
2: **if** left[x] $\neq$ null **then** $r \leftarrow$ left[x]. size $+1$ **else** $r \leftarrow 1$
3: **if** $i = r$ **then return** x
4: **if** $i < r$ **then**
5:     **return** Select(left[x],i)
6: **else**
7:     **return** Select(right[x],$i - r$)

Select(x, i)

Find-by-rank:

▶ decide whether you have to proceed into the left or right sub-tree

▶ adjust the rank that you are searching for if you go right

Select(*x*, *i*)

Select(㉕, 14)

Find-by-rank:

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Select(*x*, *i*)

Select(13, 14)

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Select(x, i)

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Select(x, i)

Select(19, 3)

Find-by-rank:

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Select(x, i)

Select(⑳, 1)

Find-by-rank:

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Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

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Search(k): Nothing to do.

Insert(x): When going down the search path increase the size field for each visited node. Maintain the size field during rotations.

Delete(x): Directly after splicing out a node traverse the path from the spliced out node upwards, and decrease the size counter on every node on this path. Maintain the size field during rotations.

Rotations

The only operation during the fix-up procedure that alters the tree and requires an update of the size-field:
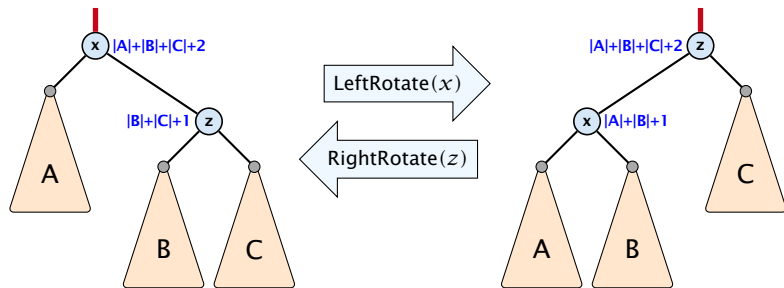
The nodes x and z are the only nodes changing their size-fields.

The new size-fields can be computed locally from the size-fields of the children.

7.5 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

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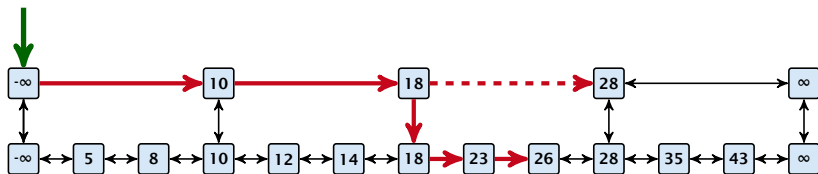
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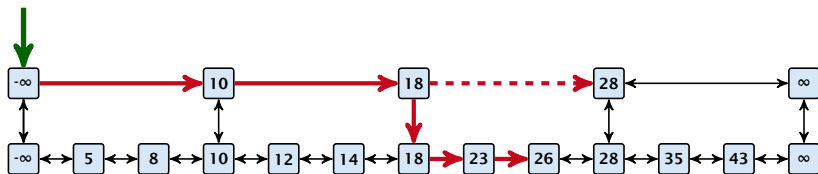
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Choose $|L_1| = \sqrt{n}$. Then search time $\Theta(\sqrt{n})$.

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▶ . . .

▶ At most $|L_k| + \sum_{i=1}^{k} \frac{L_{i-1}}{L_i} + 3(k+1)$ steps.

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Choose ratios between list-lengths evenly, i.e., $\frac{|L_{i-1}|}{|L_i|} = r$, and, hence, $L_k \approx r^{-k}n$.

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$$r^{-k}n + kr = \left(n^{\frac{1}{k+1}}\right)^{-k}n + kn^{\frac{1}{k+1}}$$
$$= n^{1-\frac{k}{k+1}} + kn^{\frac{1}{k+1}}$$

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$$
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&= n^{1 - \frac{k}{k+1}} + k n^{\frac{1}{k+1}} \\
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Choosing $k = \Theta(\log n)$ gives a logarithmic running time.

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▶ If we want that in L_i we always skip over roughly the same number of elements in L_{i-1} an insert or delete may require a lot of re-organisation.

Use randomization instead!

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Insert:

▶ A search operation gives you the insert position for element x in every list.

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▶ Insert x into lists $L_0, \dots, L_{t-1}$.

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- ▶ Flip a coin until it shows head, and record the number $t \in \{1, 2, \dots\}$ of trials needed.
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Delete:

7.5 Skip Lists

Insert:

▶ A search operation gives you the insert position for element x in every list.

▶ Flip a coin until it shows head, and record the number $t \in \{1, 2, \dots\}$ of trials needed.

▶ Insert x into lists $L_0, \dots, L_{t-1}$.

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▶ You get all predecessors via backward pointers.

7.5 Skip Lists

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7.5 Skip Lists

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▶ Delete x in all lists it actually appears in.

The time for both operations is dominated by the search time.

7.5 Skip Lists

Insert (35):
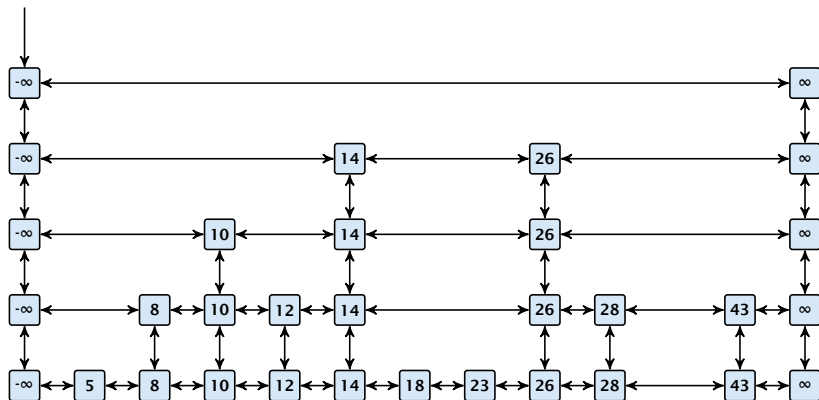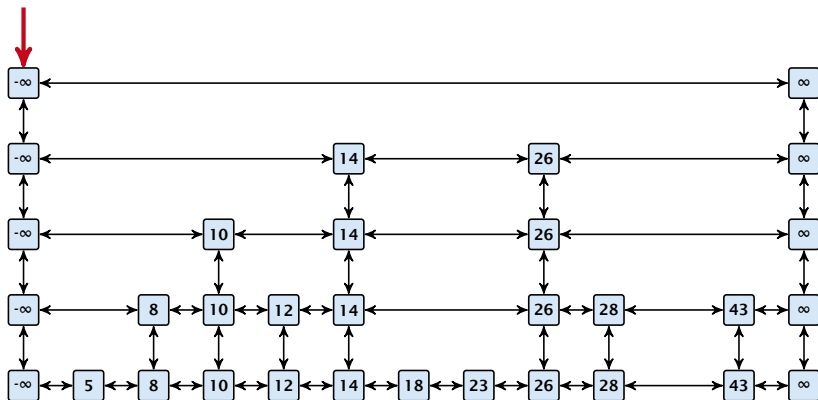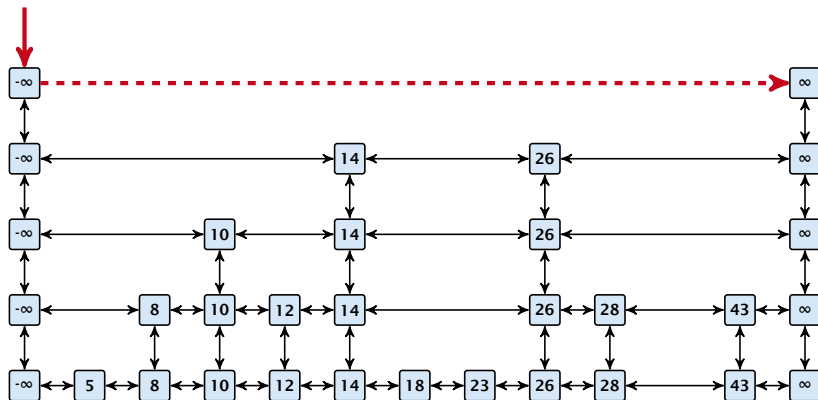
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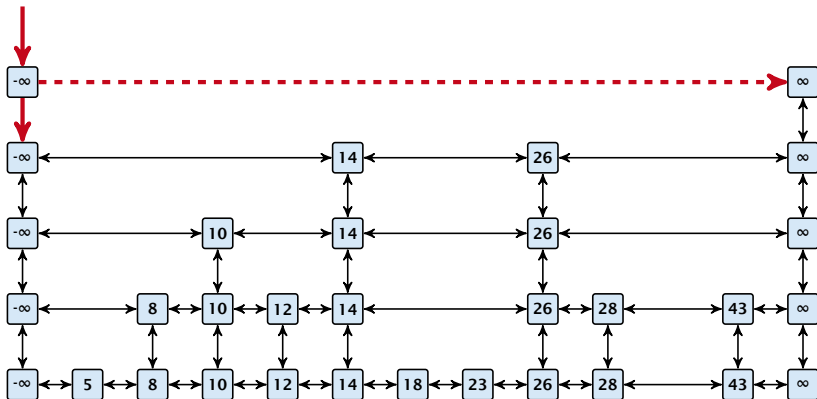
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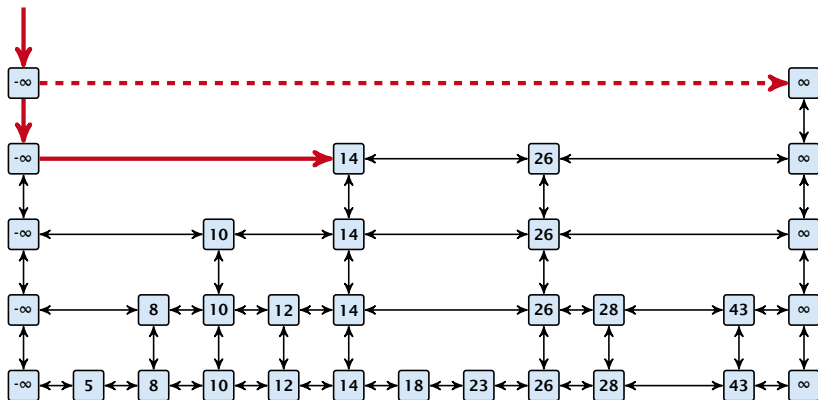
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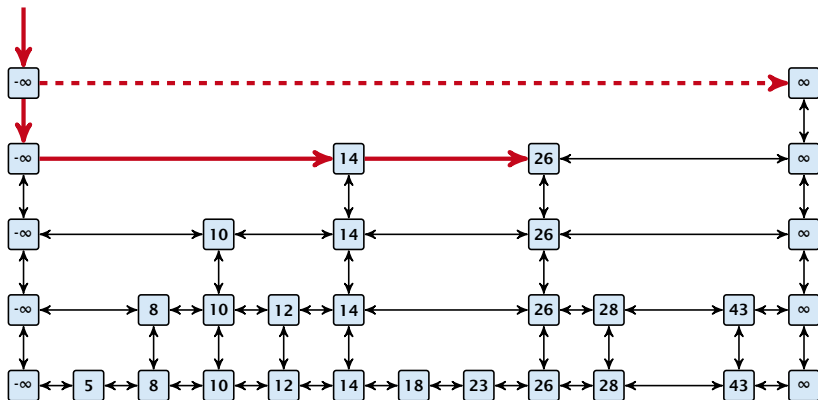
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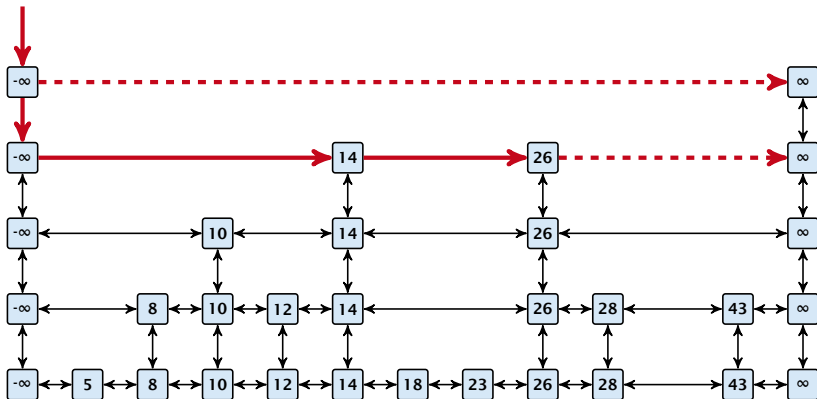
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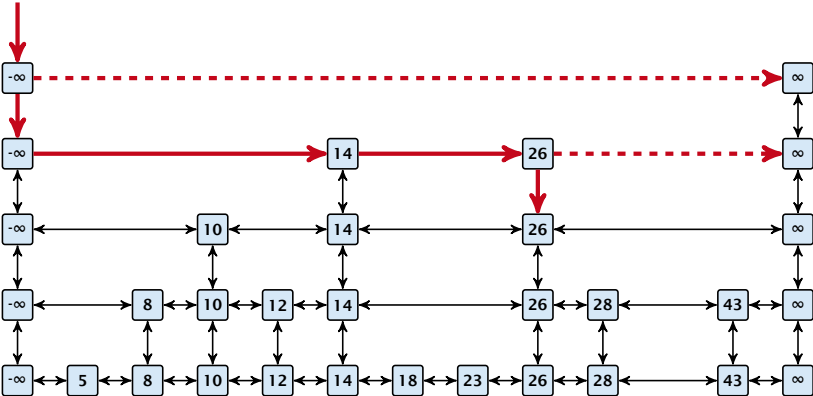
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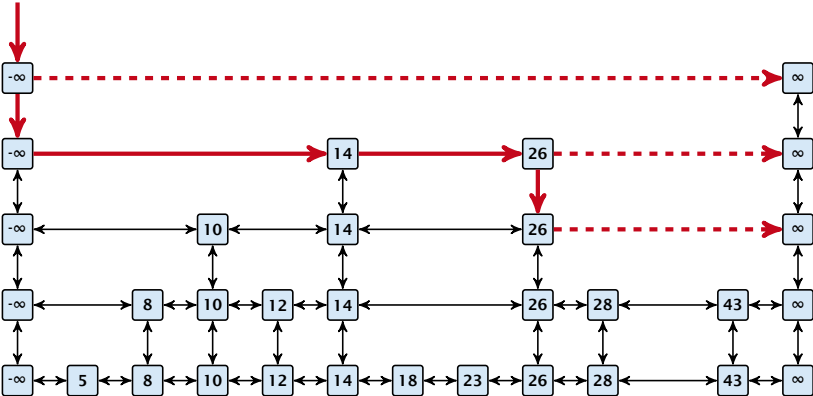
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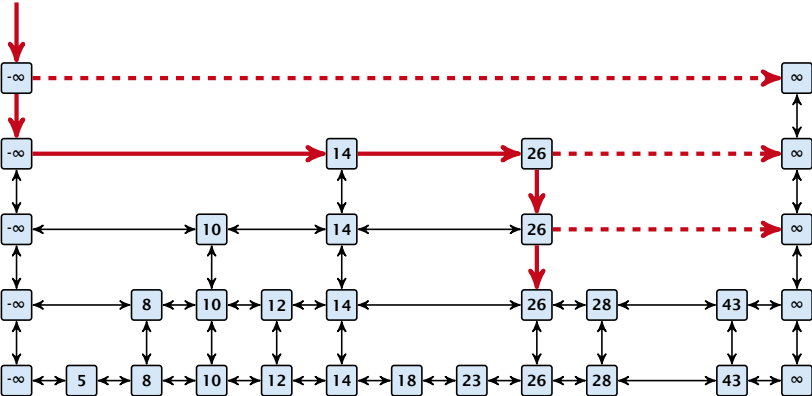
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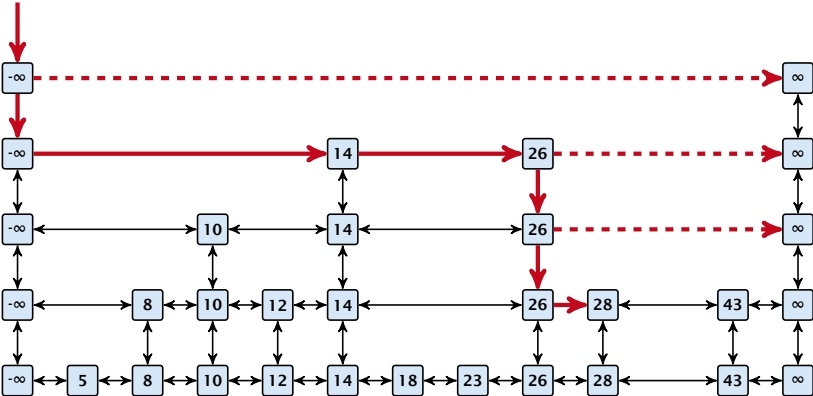
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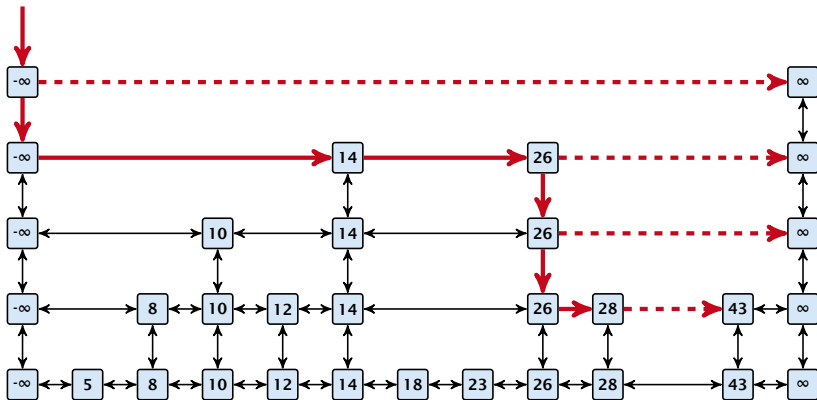
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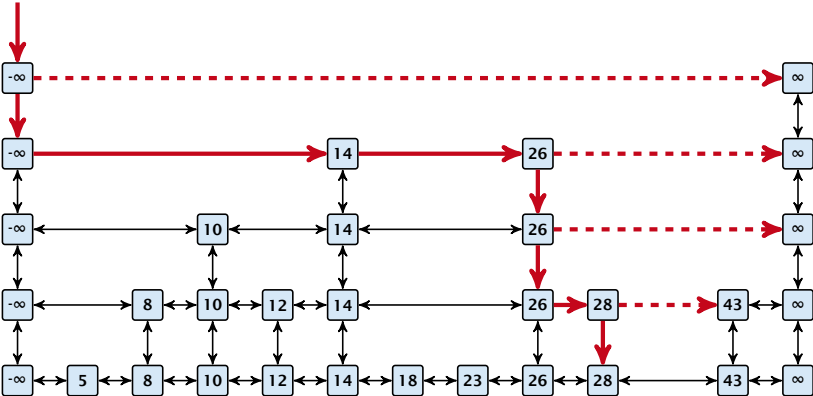
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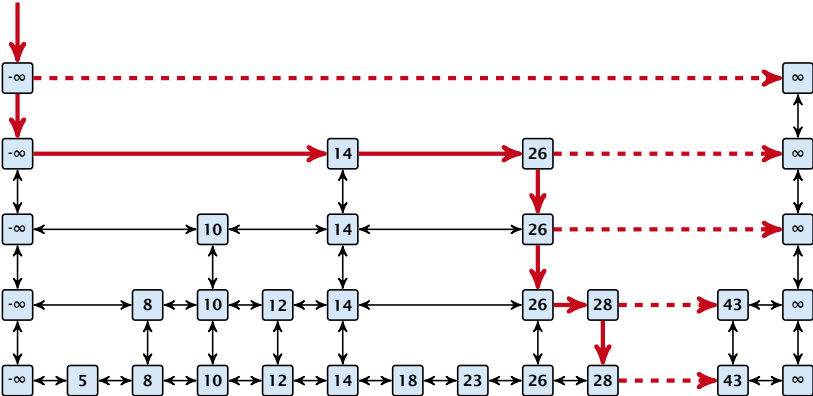
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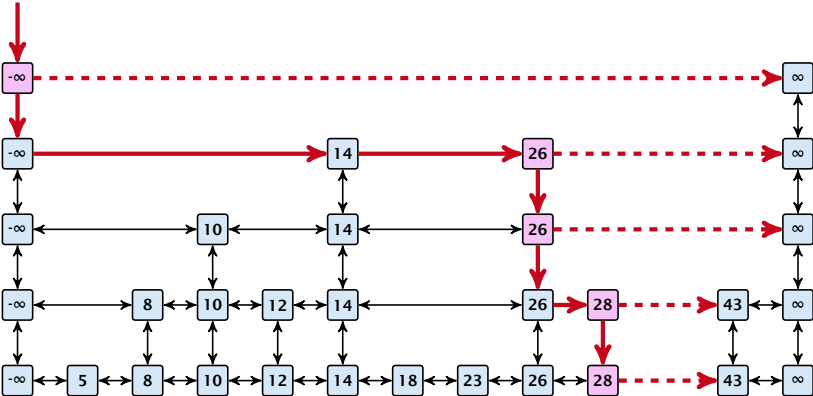
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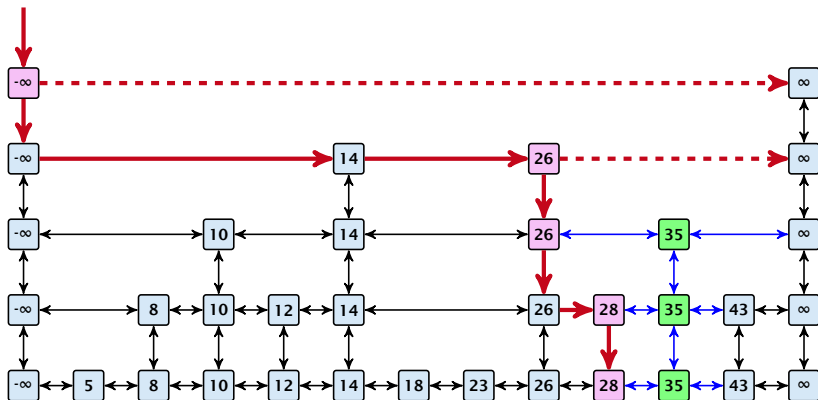
High Probability

Definition 7 (High Probability)

We say a **randomized** algorithm has running time $\mathcal{O}(\log n)$ with high probability if for any constant α the running time is at most $\mathcal{O}(\log n)$ with probability at least $1 - \frac{1}{n^\alpha}$.

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Here the $\mathcal{O}$-notation hides a constant that may depend on α.

High Probability

Suppose there are polynomially many events $E_1, E_2, \ldots, E_\ell$, $\ell = n^c$ each holding with high probability (e.g. E_i may be the event that the i-th search in a skip list takes time at most $\mathcal{O}(\log n)$).

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This means $\Pr[E_1 \wedge \cdots \wedge E_\ell]$ holds with high probability.

7.5 Skip Lists

Lemma 8

A search (and, hence, also insert and delete) in a skip list with n elements takes time $\mathcal{O}(\log n)$ with high probability (w. h. p.).

7.5 Skip Lists

Backward analysis:
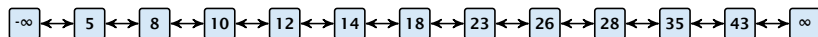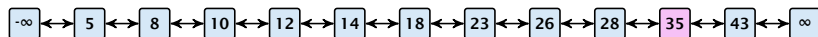
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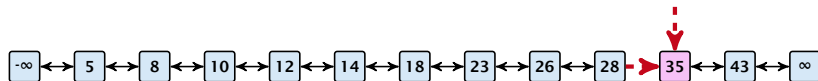
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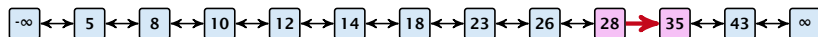
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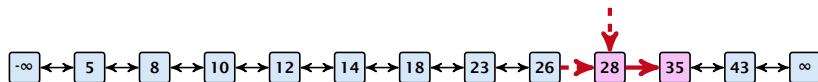
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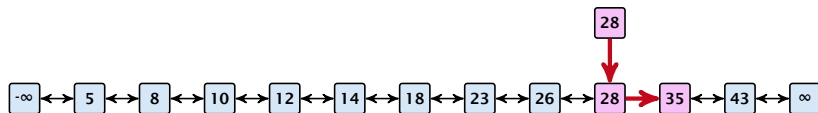
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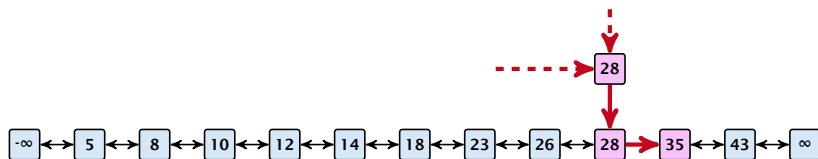
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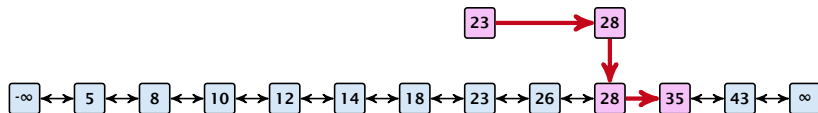
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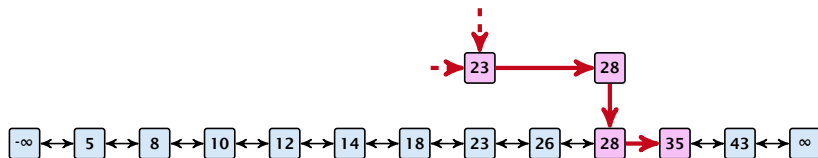
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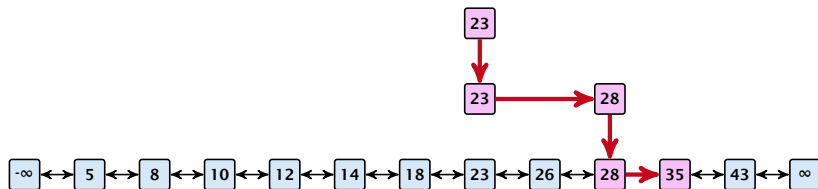
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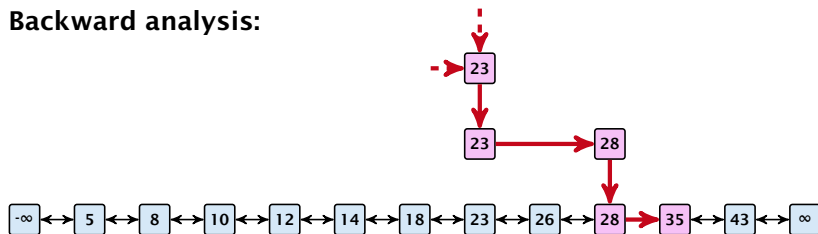
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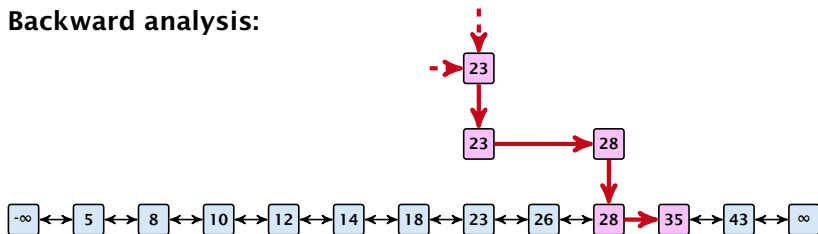
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At each point the path goes up with probability $1/2$ and left with probability $1/2$.

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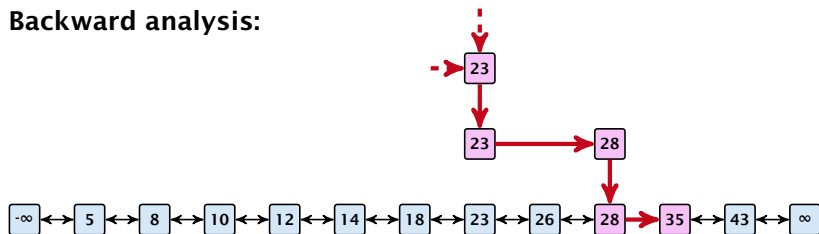
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We show that w.h.p:

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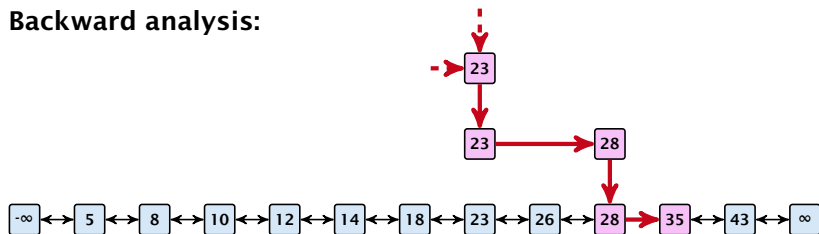
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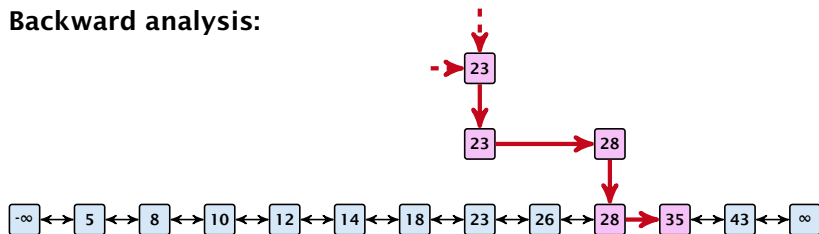
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We show that w.h.p:

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From this it follows that w.h.p. there are no long paths.

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Estimation for Binomial Coefficients

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

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In particular, this means that during the construction in the backward analysis we see at most k heads (i.e., coin flips that tell you to go up) in z trials.

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$\Pr[E_{z,k}]$

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now choosing $\beta = 6\alpha$ gives

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This means, the search requires at most z steps, w. h. p.

7.6 Hashing

Dictionary:

- ▶ $S.\,\mathbf{insert}(x)$: Insert an element x.
- ▶ $S.\,\mathbf{delete}(x)$: Delete the element pointed to by x.
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Hashing tries to directly compute the memory location from the given key. The goal is to have constant search time.

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Definitions:

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- ▶ Fast to evaluate.
- ▶ Small storage requirement.
- ▶ Good distribution of elements over the whole table.

Direct Addressing

Ideally the hash function maps all keys to different memory locations.
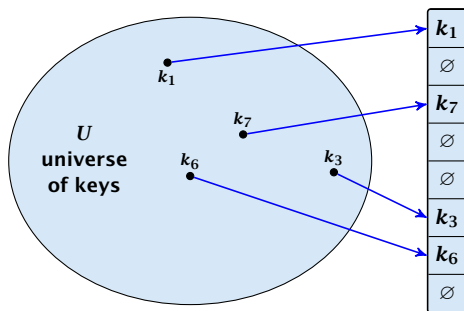
This special case is known as Direct Addressing. It is usually very unrealistic as the universe of keys typically is quite large, and in particular larger than the available memory.

Perfect Hashing

Suppose that we know the set S of actual keys (no insert/no delete). Then we may want to design a simple hash-function that maps all these keys to different memory locations.
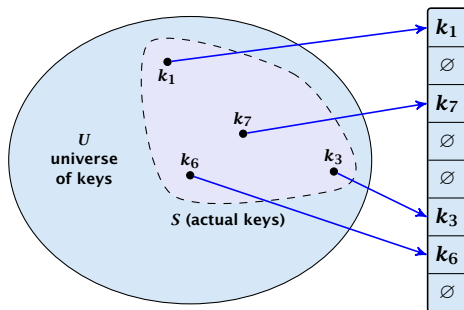
Such a hash function h is called a perfect hash function for set S.

Collisions

If we do not know the keys in advance, the best we can hope for is that the hash function distributes keys evenly across the table.

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Hence, there may be two elements k_1, k_2 from the set S that map to the same memory location (i.e., $h(k_1) = h(k_2)$). This is called a collision.

Collisions

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Lemma 9

The probability of having a collision when hashing m elements into a table of size n under *uniform hashing* *is at least*

$$1 - e^{-\frac{m(m-1)}{2n}} \approx 1 - e^{-\frac{m^2}{2n}} \ .$$

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Uniform hashing:
Choose a hash function uniformly at random from all functions $f : U \to [0, \dots, n-1]$.

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Proof.
Let $A_{m,n}$ denote the event that inserting m keys into a table of size n does **not** generate a collision. Then

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Here the first equality follows since the ℓ-th element that is hashed has a probability of $\frac{n-\ell+1}{n}$ to not generate a collision under the condition that the previous elements did not induce collisions. □

Collisions
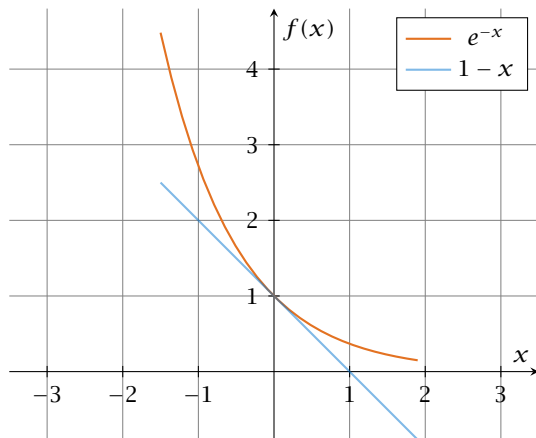
The inequality $1 - x \le e^{-x}$ is derived by stopping the Taylor-expansion of e^{-x} after the second term.

7.6 Hashing

Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

- ▶ **open addressing**, aka. closed hashing
- ▶ **hashing with chaining**, aka. closed addressing, open hashing.

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There are applications e.g. computer chess where you do not resolve collisions at all.

Hashing with Chaining

Arrange elements that map to the same position in a linear list.

▶ Access: compute $h(x)$ and search list for $\text{key}[x]$.

▶ Insert: insert at the front of the list.
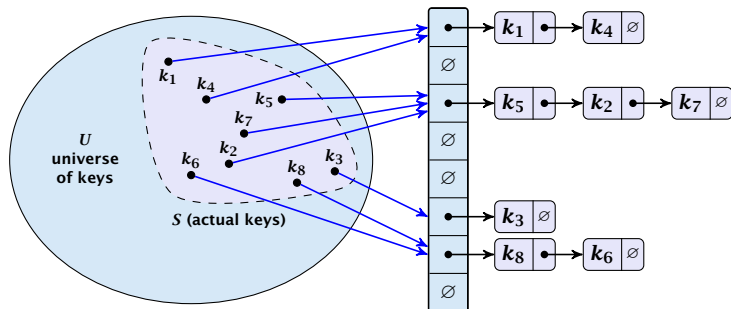
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We assume uniform hashing for the following analysis.

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Hashing with Chaining

The time required for an unsuccessful search is 1 plus the length of the list that is examined. The average length of a list is $\alpha = \frac{m}{n}$. Hence, if A is the collision resolving strategy "Hashing with Chaining" we have

$$A^- = 1 + \alpha \ .$$

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\mathrm{E}\Big[\frac{1}{m}\sum_{i=1}^{m}\Big(1+\sum_{j=i+1}^{m}X_{ij}\Big)\Big] &= \frac{1}{m}\sum_{i=1}^{m}\Big(1+\sum_{j=i+1}^{m}\mathrm{E}\,[X_{ij}]\Big) \\
&= \frac{1}{m}\sum_{i=1}^{m}\Big(1+\sum_{j=i+1}^{m}\frac{1}{n}\Big) \\
&= 1+\frac{1}{mn}\sum_{i=1}^{m}(m-i) \\
&= 1+\frac{1}{mn}\Big(m^2-\frac{m(m+1)}{2}\Big) \\
&= 1+\frac{m-1}{2n} = 1+\frac{\alpha}{2}-\frac{\alpha}{2m} \ .
\end{aligned}
$$

Hence, the expected cost for a successful search is $A^{+}\le 1+\frac{\alpha}{2}$.

Hashing with Chaining

Disadvantages:

▶ pointers increase memory requirements

▶ pointers may lead to bad cache efficiency

Advantages:

▶ no à priori limit on the number of elements

▶ deletion can be implemented efficiently

▶ by using balanced trees instead of linked list one can also obtain worst-case guarantees.

Open Addressing

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All objects are stored in the table itself.

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Define a function $h(k, j)$ that determines the table-position to be examined in the j-th step. The values $h(k, 0), \ldots, h(k, n-1)$ must form a permutation of $0, \ldots, n-1$.

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Insert(x): Search until you find an empty slot; insert your element there. If your search reaches $h(k, n - 1)$, and this slot is non-empty then your table is full.

Open Addressing

Choices for $h(k, j)$:

▶ Linear probing:
$h(k, i) = h(k) + i \mod n$
(sometimes: $h(k, i) = h(k) + ci \mod n$).

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For quadratic probing and double hashing one has to ensure that the search covers all positions in the table (i.e., for double hashing $h_2(k)$ must be relatively prime to n (teilerfremd); for quadratic probing c_1 and c_2 have to be chosen carefully).

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▶ Advantage: Cache-efficiency. The new probe position is very likely to be in the cache.

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Lemma 10

Let L be the method of linear probing for resolving collisions:

$$L^+ \approx \frac{1}{2}\left(1 + \frac{1}{1 - \alpha}\right)$$

$$L^- \approx \frac{1}{2}\left(1 + \frac{1}{(1 - \alpha)^2}\right)$$

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Let Q be the method of quadratic probing for resolving collisions:

$$Q^+ \approx 1 + \ln\left(\frac{1}{1-\alpha}\right) - \frac{\alpha}{2}$$

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Double Hashing

- ▶ Any probe into the hash-table usually creates a cache-miss.

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Lemma 12

Let D be the method of double hashing for resolving collisions:

$$D^+ \approx \frac{1}{\alpha} \ln\left(\frac{1}{1-\alpha}\right)$$

$$D^- \approx \frac{1}{1-\alpha}$$

Open Addressing

Some values:

α	Linear Probing		Quadratic Probing		Double Hashing	
	L^+	L^-	Q^+	Q^-	D^+	D^-
0.5	1.5	2.5	1.44	2.19	1.39	2
0.9	5.5	50.5	2.85	11.40	2.55	10
0.95	10.5	200.5	3.52	22.05	3.15	20

Open Addressing
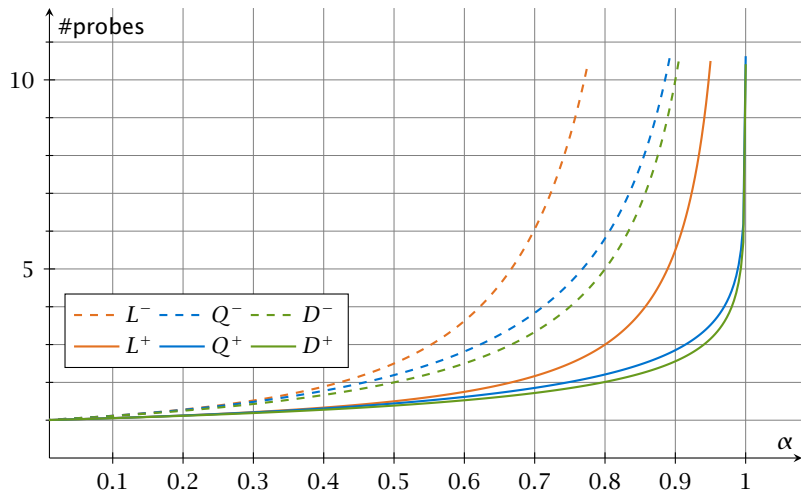
Analysis of Idealized Open Address Hashing

We analyze the time for a search in a very idealized Open Addressing scheme.

▶ The probe sequence $h(k,0), h(k,1), h(k,2), \ldots$ is equally likely to be any permutation of $\langle 0, 1, \ldots, n-1 \rangle$.

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$$\leq \left(\frac{m}{n}\right)^{i-1} = \alpha^{i-1} \ .$$

Analysis of Idealized Open Address Hashing

$E[X]$

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Ernst Mayr, Harald Räcke

Analysis of Idealized Open Address Hashing

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$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \dots$$

Analysis of Idealized Open Address Hashing

$$\sum_i i \Pr[X = i] = \sum_i \Pr[X \geq i]$$

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Analysis of Idealized Open Address Hashing

$i = 1$

$$\sum_i i \Pr[X = i] = \sum_i \Pr[X \geq i]$$

Analysis of Idealized Open Address Hashing

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$i = 4$

$$\sum_i i \Pr[X = i] = \sum_i \Pr[X \geq i]$$
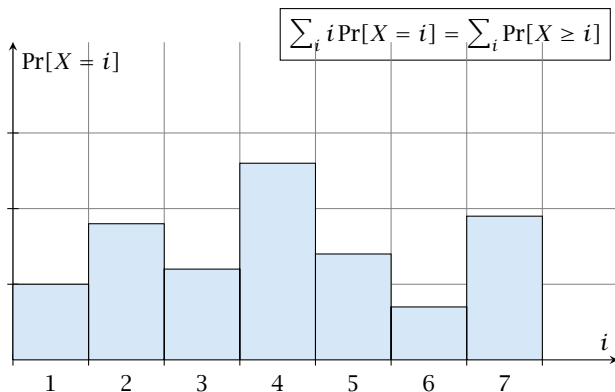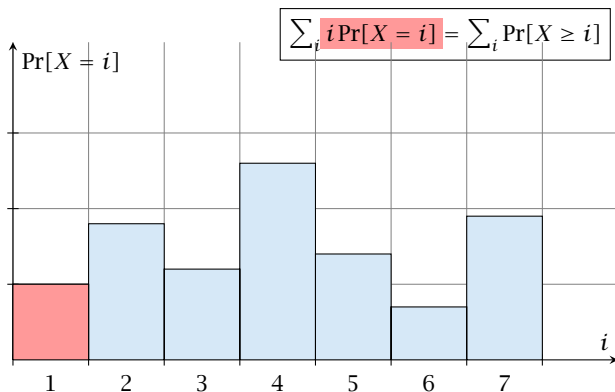
Analysis of Idealized Open Address Hashing

$i = 1$

Analysis of Idealized Open Address Hashing

$i = 2$
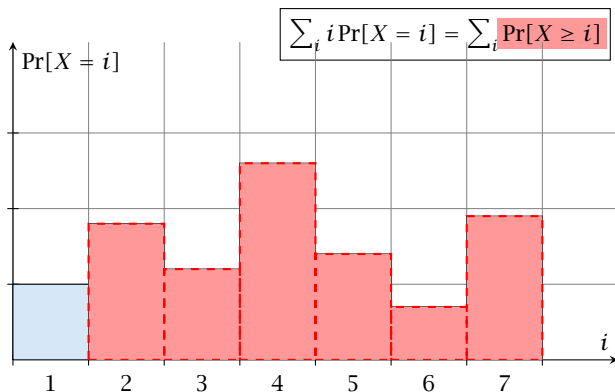
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Analysis of Idealized Open Address Hashing

$i = 3$
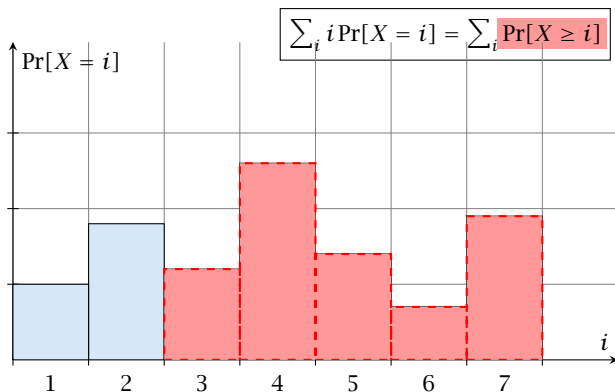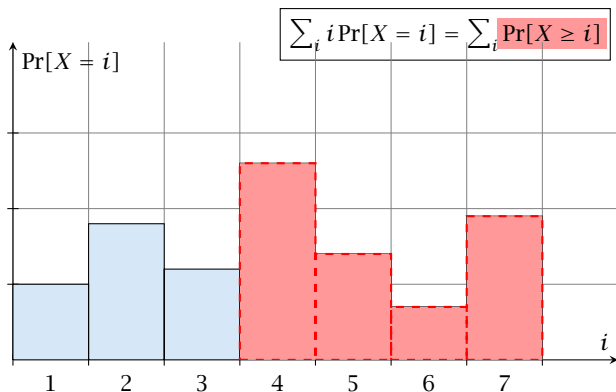
$$\sum_i i \Pr[X = i] = \sum_i \Pr[X \geq i]$$

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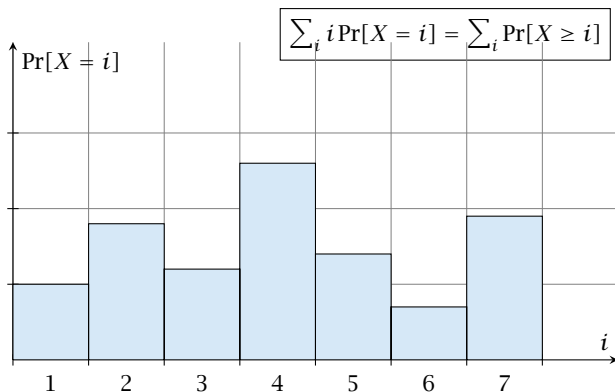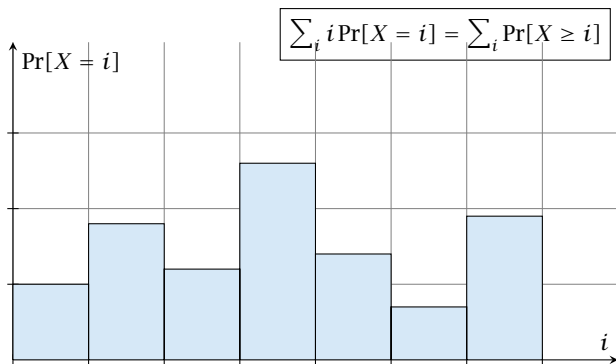
The j-th rectangle appears in both sums j times. (j times in the first due to multiplication with j; and j times in the second for summands $i = 1, 2, \ldots, j$)

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Deletions in Hashtables

How do we delete in a hash-table?

▶ For hashing with chaining this is not a problem. Simply search for the key, and delete the item in the corresponding list.

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▶ For open addressing this is difficult.

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▶ The table could fill up with deleted-markers leading to bad performance.

▶ If a table contains many deleted-markers (linear fraction of the keys) one can rehash the whole table and amortize the cost for this rehash against the cost for the deletions.

Deletions for Linear Probing

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- ▶ For Linear Probing one can delete elements without using deletion-markers.
- ▶ Upon a deletion elements that are further down in the probe-sequence may be moved to guarantee that they are still found during a search.

Deletions for Linear Probing

Algorithm 12 delete(p)
1: $T[p] \leftarrow$ null
2: $p \leftarrow \text{succ}(p)$
3: **while** $T[p] \neq$ null **do**
4: $\quad y \leftarrow T[p]$
5: $\quad T[p] \leftarrow$ null
6: $\quad p \leftarrow \text{succ}(p)$
7: $\quad \text{insert}(y)$

p is the index into the table-cell that contains the object to be deleted.

Deletions for Linear Probing

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Pointers into the hash-table become invalid.

Universal Hashing

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However, the assumption of uniform hashing that h is chosen randomly from all functions $f : U \to [0, \ldots, n-1]$ is clearly unrealistic as there are $n^{|U|}$ such functions. Even writing down such a function would take $|U| \log n$ bits.

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Universal hashing tries to define a set $\mathcal{H}$ of functions that is much smaller but still leads to good average case behaviour when selecting a hash-function uniformly at random from $\mathcal{H}$.

Universal Hashing

Definition 13

A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0, \ldots, n-1\}$ is called universal if for all $u_1, u_2 \in U$ with $u_1 \neq u_2$

$$\Pr[h(u_1) = h(u_2)] \leq \frac{1}{n} \ ,$$

where the probability is w. r. t. the choice of a random hash-function from set $\mathcal{H}$.

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Note that this means that the probability of a collision between two arbitrary elements is at most $\frac{1}{n}$.

Universal Hashing

Definition 14

A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0, \ldots, n-1\}$ is called 2-independent (pairwise independent) if the following two conditions hold

▶ For any key $u \in U$, and $t \in \{0, \ldots, n-1\}$ $\Pr[h(u) = t] = \frac{1}{n}$, i.e., a key is distributed uniformly within the hash-table.

▶ For all $u_1, u_2 \in U$ with $u_1 \neq u_2$, and for any two hash-positions t_1, t_2:

$$\Pr[h(u_1) = t_1 \wedge h(u_2) = t_2] \leq \frac{1}{n^2} \ .$$

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This requirement clearly implies a universal hash-function.

Universal Hashing

Definition 15

A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0, \ldots, n-1\}$ is called k-independent if for any choice of $\ell \le k$ distinct keys $u_1, \ldots, u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions $t_1, \ldots, t_\ell$:

$$\Pr[h(u_1) = t_1 \wedge \cdots \wedge h(u_\ell) = t_\ell] \le \frac{1}{n^\ell} \ ,$$

where the probability is w.r.t. the choice of a random hash-function from set $\mathcal{H}$.

Universal Hashing

Definition 16

A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0, \ldots, n-1\}$ is called (μ, k)-independent if for any choice of $\ell \le k$ distinct keys $u_1, \ldots, u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions $t_1, \ldots, t_\ell$:

$$\Pr[h(u_1) = t_1 \wedge \cdots \wedge h(u_\ell) = t_\ell] \le \frac{\mu}{n^\ell} \ ,$$

where the probability is w.r.t. the choice of a random hash-function from set $\mathcal{H}$.

Universal Hashing

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Let $U := \{0, \ldots, p-1\}$ for a prime p. Let $\mathbb{Z}_p := \{0, \ldots, p-1\}$, and let $\mathbb{Z}_p^* := \{1, \ldots, p-1\}$ denote the set of invertible elements in $\mathbb{Z}_p$.

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Lemma 17

The class
$$\mathcal{H} = \{h_{a,b} \mid a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$$

is a universal class of hash-functions from U to $\{0, \dots, n-1\}$.

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Let $x, y \in U$ be two distinct keys. We have to show that the probability of a collision is only $1/n$.

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  where we use that $\mathbb{Z}_p$ is a field (Körper) and, hence, has no zero divisors (nullteilerfrei).

Universal Hashing

▶ The hash-function does not generate collisions before the $(\bmod\ n)$-operation. Furthermore, every choice (a, b) is mapped to a different pair (t_x, t_y) with $t_x := ax + b$ and $t_y := ay + b$.

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$$t_y \equiv ay + b \qquad (\text{mod } p)$$

$$a \equiv (t_x - t_y)(x - y)^{-1} \qquad (\text{mod } p)$$
$$b \equiv t_y - ay \qquad (\text{mod } p)$$

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What happens when we do the mod n operation?

Fix a value t_x. There are $p - 1$ possible values for choosing t_y.

From the range $0, \ldots, p - 1$ the values $t_x, t_x + n, t_x + 2n, \ldots$ map to t_x after the modulo-operation. These are at most $\lceil p/n \rceil$ values.

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This happens with probability at most $\frac{1}{n}$.

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It is also possible to show that $\mathcal{H}$ is an (almost) pairwise independent class of hash-functions.

$$\Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[\begin{array}{c} t_x \bmod n = h_1 \\ \wedge \\ t_y \bmod n = h_2 \end{array} \right]$$

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$$\frac{\left\lfloor \frac{p}{n} \right\rfloor^2}{p(p-1)} \leq \Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[\begin{array}{c} t_x \bmod n = h_1 \\ \overset{\wedge}{t_y \bmod n = h_2} \end{array} \right] \leq \frac{\left\lceil \frac{p}{n} \right\rceil^2}{p(p-1)}$$

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Note that the middle is the probability that $h(x) = h_1$ and $h(y) = h_2$. The total number of choices for (t_x, t_y) is $p(p-1)$. The number of choices for t_x (t_y) such that $t_x \bmod n = h_1$ ($t_y \bmod n = h_2$) lies between $\lfloor \frac{p}{n} \rfloor$ and $\lceil \frac{p}{n} \rceil$.

Universal Hashing

Definition 18

Let $d \in \mathbb{N}$; $q \geq (d+1)n$ be a prime; and let $\bar{a} \in \{0, \ldots, q-1\}^{d+1}$. Define for $x \in \{0, \ldots, q-1\}$

$$h_{\bar{a}}(x) := \Big(\sum_{i=0}^{d} a_i x^i \bmod q \Big) \bmod n \ .$$

Let $\mathcal{H}_n^d := \{h_{\bar{a}} \mid \bar{a} \in \{0, \ldots, q-1\}^{d+1}\}$. The class $\mathcal{H}_n^d$ is $(e, d+1)$-independent.

Note that in the previous case we had $d = 1$ and chose $a_d \neq 0$.

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For the coefficients $\bar{a} \in \{0, \ldots, q-1\}^{d+1}$ let $f_{\bar{a}}$ denote the polynomial

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The polynomial is defined by $d + 1$ distinct points.

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Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

$$f_{\bar{a}}(x_i) \in \underbrace{\{t_i + \alpha \cdot n \mid \alpha \in \{0, \ldots, \lceil \frac{q}{n} \rceil - 1\}\}}_{=:B_i}$$

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We first fix the values for inputs $x_1, \ldots, x_\ell$.
We have

$$|B_1| \cdot \ldots \cdot |B_\ell|$$

possibilities to do this (so that $h_{\bar{a}}(x_i) = t_i$).

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Therefore we have

$$|B_1| \cdot \ldots \cdot |B_\ell| \cdot q^{d-\ell+1} \leq \lceil \frac{q}{n} \rceil^\ell \cdot q^{d-\ell+1}$$

possibilities to choose $\bar{a}$ such that $h_{\bar{a}} \in A_\ell$.

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Therefore the probability of choosing $h_{\bar{a}}$ from A_ℓ is only

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$$\leq \left(1 + \frac{1}{\ell} \right)^\ell \cdot \frac{1}{n^\ell} \leq \frac{e}{n^\ell} \ .$$

This shows that the $\mathcal{H}$ is $(e, d+1)$-universal.

The last step followed from $q \geq (d+1)n$, and $\ell \leq d+1$.

Perfect Hashing

Suppose that we **know** the set S of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.
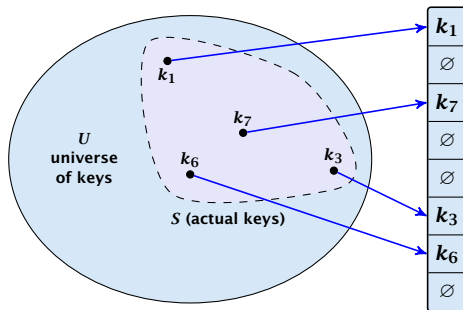
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The probability of having 1 or more collisions can be at most $\frac{1}{2}$ as otherwise the expectation would be larger than $\frac{1}{2}$.

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Ernst Mayr, Harald Räcke

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Let m_j denote the number of items that are hashed to the j-th bucket. For each bucket we choose a second hash-function that maps the elements of the bucket into a table of size m_j^2. The second function can be chosen such that all elements are mapped to different locations.

Perfect Hashing

$$\sum_i m_i = m$$
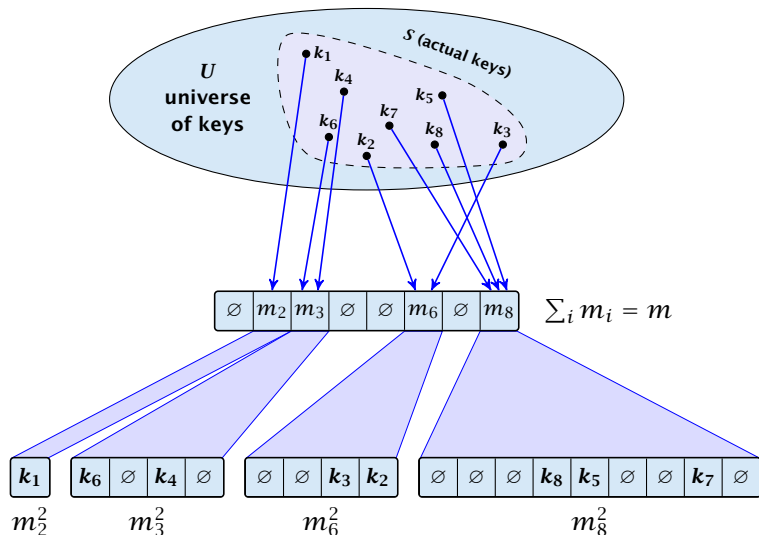
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$$= 2\binom{m}{2}\frac{1}{m} + m = 2m - 1 \ .$$

Perfect Hashing

We need only $\mathcal{O}(m)$ time to construct a hash-function h with $\sum_j m_j^2 = \mathcal{O}(4m)$, because with probability at least $1/2$ a random function from a universal family will have this property.

Then we construct a hash-table h_j for every bucket. This takes expected time $\mathcal{O}(m_j)$ for every bucket. A random function h_j is collision-free with probability at least $1/2$. We need $\mathcal{O}(m_j)$ to test this.

We only need that the hash-functions are chosen from a universal family!!!

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- ▶ An object x is either stored at location $T_1[h_1(x)]$ or $T_2[h_2(x)]$.

- ▶ A search clearly takes constant time if the above constraint is met.

Cuckoo Hashing

Insert:

Cuckoo Hashing

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Algorithm 13 Cuckoo-Insert(x)

1: **if** $T_1[h_1(x)] = x \lor T_2[h_2(x)] = x$ **then return**
2: steps $\leftarrow 1$
3: **while** steps $\leq$ maxsteps **do**
4:     exchange x and $T_1[h_1(x)]$
5:     **if** $x =$ null **then return**
6:     exchange x and $T_2[h_2(x)]$
7:     **if** $x =$ null **then return**
8:     steps $\leftarrow$ steps $+1$
9: rehash() // change hash-functions; rehash everything
10: Cuckoo-Insert(x)

Cuckoo Hashing

- ► We call one iteration through the while-loop a step of the algorithm.

Cuckoo Hashing

- We call one iteration through the while-loop a step of the algorithm.
- We call a sequence of iterations through the while-loop without the termination condition becoming true a phase of the algorithm.

Cuckoo Hashing

- ▶ We call one iteration through the while-loop a step of the algorithm.
- ▶ We call a sequence of iterations through the while-loop without the termination condition becoming true a phase of the algorithm.
- ▶ We say a phase is successful if it is not terminated by the maxstep-condition, but the while loop is left because $x = \text{null}$.

Cuckoo Hashing

Cuckoo Hashing

What is the expected time for an insert-operation?

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We first analyze the probability that we end-up in an infinite loop (that is then terminated after maxsteps steps).

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We first analyze the probability that we end-up in an infinite loop (that is then terminated after maxsteps steps).

Formally what is the probability to enter an infinite loop that touches s different keys?

Cuckoo Hashing: Insert

T_1 T_2

Cuckoo Hashing: Insert
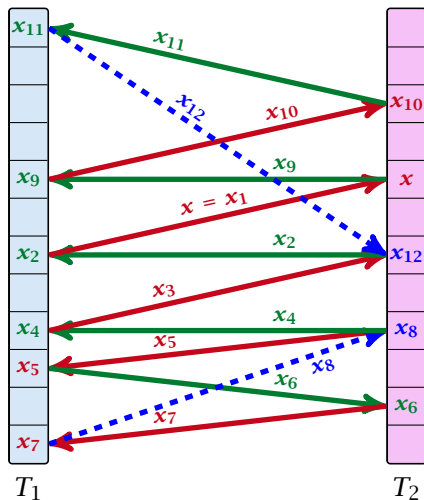
Cuckoo Hashing: Insert

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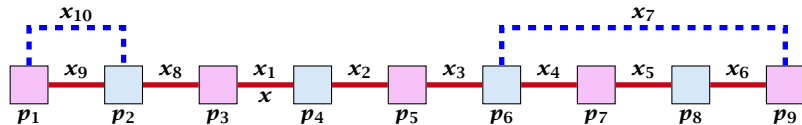
Cuckoo Hashing: Insert

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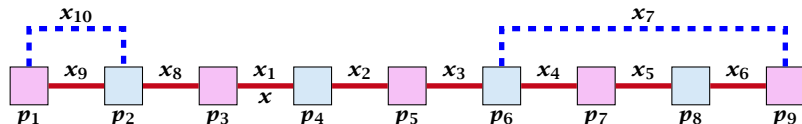# Cuckoo Hashing

A cycle-structure of size s is defined by

Cuckoo Hashing

A cycle-structure of size s is defined by

- $s - 1$ different cells (alternating btw. cells from T_1 and T_2).
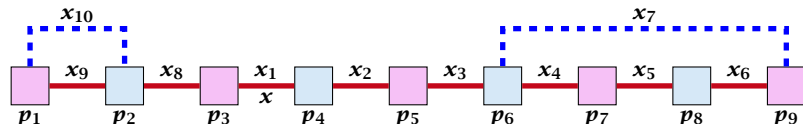
Cuckoo Hashing

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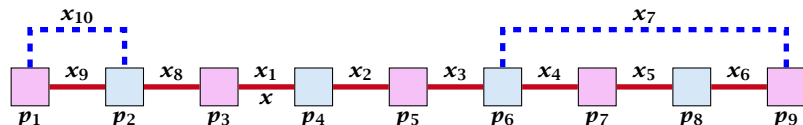▶ s distinct keys $x = x_1, x_2, \ldots, x_s$, linking the cells.

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- ▶ $s - 1$ different cells (alternating btw. cells from T_1 and T_2).

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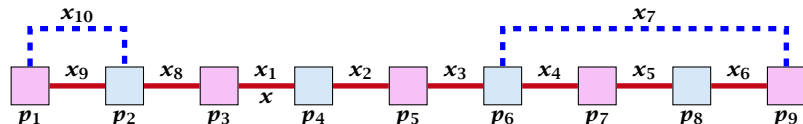- ▶ The leftmost cell is "linked forward" to some cell on the right.

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Cuckoo Hashing

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- ▶ $s - 1$ different cells (alternating btw. cells from T_1 and T_2).
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- ▶ The leftmost cell is "linked forward" to some cell on the right.
- ▶ The rightmost cell is "linked backward" to a cell on the left.
- ▶ One link represents key x; this is where the counting starts.

Cuckoo Hashing

A cycle-structure is active if for every key x_ℓ (linking a cell p_i from T_1 and a cell p_j from T_2) we have

$$h_1(x_\ell) = p_i \qquad \text{and} \qquad h_2(x_\ell) = p_j$$

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Observation:
If during a phase the insert-procedure runs into a cycle there must exist an active cycle structure of size $s \geq 3$.

Cuckoo Hashing

What is the probability that all keys in a cycle-structure of size s correctly map into their T_1-cell?

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This probability is at most $\frac{\mu}{n^s}$ since h_2 is a (μ, s)-independent hash-function.

These events are independent.

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What is the probability that there exists an active cycle structure of size s?

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The number of cycle-structures of size s is at most

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- ▶ There are at most s possibilities to choose where to place key x.
- ▶ There are m^{s-1} possibilities to choose the keys apart from x.
- ▶ There are n^{s-1} possibilities to choose the cells.

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} = \frac{\mu^2}{nm} \sum_{s=3}^{\infty} s^3 \left(\frac{m}{n} \right)^s$$

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Here we used the fact that $(1+\epsilon)m \leq n$.

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

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$$

$$
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$$

Here we used the fact that $(1 + \epsilon)m \leq n$.

Hence,

$$
\Pr[\text{cycle}] = \mathcal{O} \left(\frac{1}{m^2} \right) \ .
$$

Cuckoo Hashing

Now, we analyze the probability that a phase is not successful without running into a closed cycle.

Cuckoo Hashing
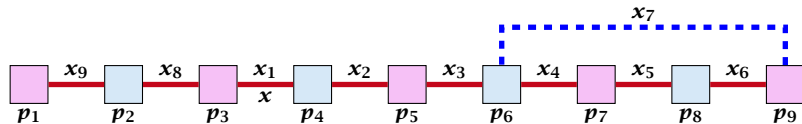
Sequence of visited keys:

$x = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_3, x_2, x_1 = x, x_8, x_9, \ldots$

Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with x in the order that they are visited during the phase.

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Lemma 19
If the sequence is of length p then there exists a sub-sequence of at least $\frac{p+2}{3}$ keys starting with x of *distinct* keys.

Cuckoo Hashing

Proof.
Let i be the number of keys (including x) that we see before the first repeated key. Let j denote the total number of distinct keys.

The sequence is of the form:

$$x = x_1 \to x_2 \to \cdots \to x_i \to x_r \to x_{r-1} \to \cdots \to x_1 \to x_{i+1} \to \cdots \to x_j$$

As $r \le i - 1$ the length p of the sequence is

$$p = i + r + (j - i) \le i + j - 1 \ .$$

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Either sub-sequence $x_1 \to x_2 \to \cdots \to x_i$ or sub-sequence $x_1 \to x_{i+1} \to \cdots \to x_j$ has at least $\frac{p+2}{3}$ elements. $\square$
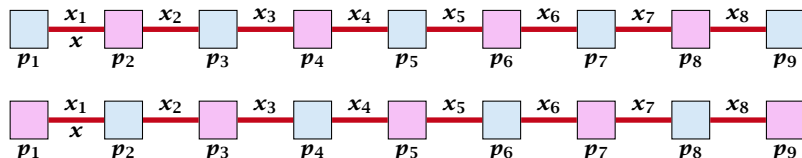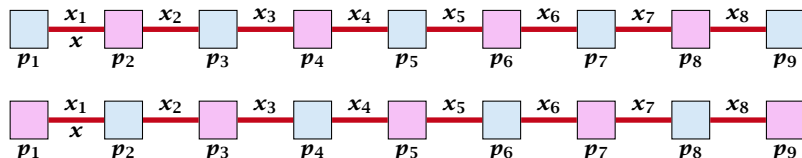
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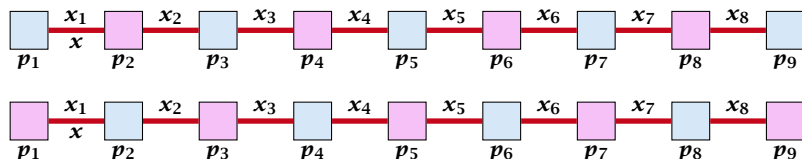
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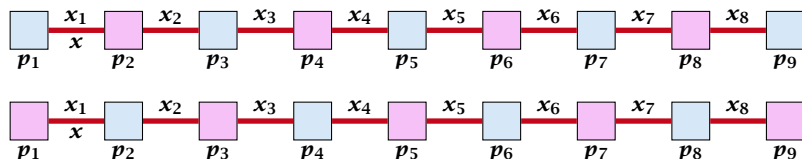
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$$h_1(x_\ell) = p_i \qquad \text{and} \qquad h_2(x_\ell) = p_j$$

Observation:
If a phase takes at least t steps without running into a cycle there must exist an active path-structure of size $(2t + 2)/3$.

Cuckoo Hashing

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Cuckoo Hashing

We choose maxsteps $\geq 3\ell/2 + 1/2$.

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We choose $\text{maxsteps} \geq 3\ell/2 + 1/2$. Then the probability that a phase terminates unsuccessfully without running into a cycle is at most

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$\quad \leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{\ell} \leq \frac{1}{m^2}$

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by choosing $\ell \geq \log\big(\frac{1}{2\mu^2 m^2}\big)/\log\big(\frac{1}{1+\epsilon}\big) = \log\big(2\mu^2 m^2\big)/\log\big(1 + \epsilon\big)$

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This gives $\text{maxsteps} = \Theta(\log m)$.

Cuckoo Hashing

So far we estimated

$$\Pr[\text{cycle}] \le \mathcal{O}\left(\frac{1}{m^2}\right)$$

and

$$\Pr[\text{unsuccessful} \mid \text{no cycle}] \le \mathcal{O}\left(\frac{1}{m^2}\right)$$

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Observe that

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for a suitable constant $c > 0$.

Cuckoo Hashing

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This means the expected cost for a successful phase is constant (even after accounting for the cost of the incomplete step that finishes the phase).

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Therefore the expected cost for re-hashes is $\mathcal{O}(m) \cdot \mathcal{O}(p) = \mathcal{O}(1)$.

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The 0-th (re)hash is the initial configuration when doing the insert.

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7.6 Hashing

19. Oct. 2021

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Therefore, it is sufficient to have $(\mu, \Theta(\log m))$-independent hash-functions.

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▶ Therefore we can amortize the rehash cost after a change in table-size against the cost for insertions and deletions.

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Cuckoo Hashing has an expected constant insert-time and a worst-case constant search-time.

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Note that the above lemma only holds if the fill-factor (number of keys/total number of hash-table slots) is at most $\frac{1}{2(1+\epsilon)}$.

> The $1/(2(1+\epsilon))$ fill-factor comes from the fact that the total hash-table is of size $2n$ (because we have two tables of size n); moreover $m \le (1+\epsilon)n$.

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Sometimes we also have

- ▶ $S.\text{merge}(S')$: $S := S \cup S'$; $S' := \varnothing$.

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- **handle $S.$ insert(x):** Adds element x to the data-structure, and returns a handle to the object for future reference.

- **$S.$ delete(h):** Deletes element specified through handle h.

- **$S.$ decrease-key(h, k):** Decreases the key of the element specified by handle h to k. Assumes that the key is at least k before the operation.

Dijkstra's Shortest Path Algorithm

Algorithm 1 Shortest-Path($G = (V, E, d), s \in V$)

1: **Input:** weighted graph $G = (V, E, d)$; start vertex s;
2: **Output:** key-field of every node contains distance from s;
3: S.build(); // build empty priority queue
4: **for all** $v \in V \setminus \{s\}$ **do**
5:      $v.\text{key} \leftarrow \infty$;
6:      $h_v \leftarrow S.\text{insert}(v)$;
7: $s.\text{key} \leftarrow 0$; $S.\text{insert}(s)$;
8: **while** S.is-empty() = false **do**
9:      $v \leftarrow S.\text{delete-min}()$;
10:      **for all** $x \in V$ s.t. $(v, x) \in E$ **do**
11:          **if** $x.\text{key} > v.\text{key} + d(v, x)$ **then**
12:              $S.\text{decrease-key}(h_x, v.\text{key} + d(v, x))$;
13:              $x.\text{key} \leftarrow v.\text{key} + d(v, x)$;

Prim's Minimum Spanning Tree Algorithm

Algorithm 2 Prim-MST($G = (V, E, d), s \in V$)

1: **Input:** weighted graph $G = (V, E, d)$; start vertex s;
2: **Output:** pred-fields encode MST;
3: S.build(); // build empty priority queue
4: **for all** $v \in V \setminus \{s\}$ **do**
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6:      $h_v \leftarrow S$.insert(v);
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10:      **for all** $x \in V$ s.t. $\{v, x\} \in E$ **do**
11:           **if** x.key $> d(v, x)$ **then**
12:                S.decrease-key($h_x, d(v, x)$);
13:                x.key $\leftarrow d(v, x)$;
14:                x.pred $\leftarrow v$;

Analysis of Dijkstra and Prim

Both algorithms require:

- ▶ 1 build() operation
- ▶ $|V|$ insert() operations
- ▶ $|V|$ delete-min() operations
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How good a running time can we obtain?

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Operation	Binary Heap	BST	Binomial Heap	Fibonacci Heap[*]
build	n	$n \log n$	$n \log n$	n
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n$[**]	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	n	$n \log n$	$\log n$	1

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Note that most applications use **build()** only to create an empty heap which then costs time 1.

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The standard version of binary heaps is not addressable, and hence does not support a delete operation.

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delete	$\log n$[**]	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	n	$n \log n$	$\log n$	1

Note that most applications use **build()** only to create an empty heap which then costs time 1.

The standard version of binary heaps is not addressable, and hence does not support a delete operation.

Fibonacci heaps only give an amortized guarantee.

8 Priority Queues

Using Binary Heaps, Prim and Dijkstra run in time
$\mathcal{O}((|V| + |E|) \log |V|)$.

Using Fibonacci Heaps, Prim and Dijkstra run in time
$\mathcal{O}(|V| \log |V| + |E|)$.

8.1 Binary Heaps
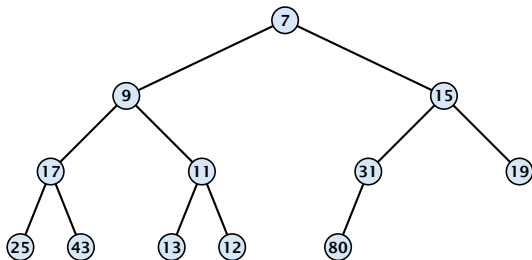
8.1 Binary Heaps

▶ Nearly complete binary tree; only the last level is not full, and this one is filled from left to right.
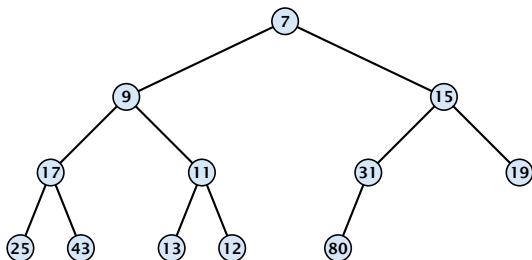
8.1 Binary Heaps

▶ Nearly complete binary tree; only the last level is not full, and this one is filled from left to right.

▶ Heap property: A node's key is not larger than the key of one of its children.
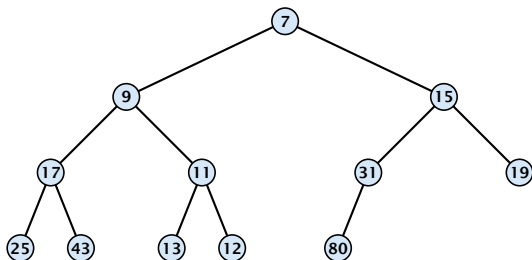
Binary Heaps

Operations:

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- **minimum()**: return the root-element. Time $\mathcal{O}(1)$.
- **is-empty()**: check whether root-pointer is null. Time $\mathcal{O}(1)$.

8.1 Binary Heaps

Maintain a pointer to the last element x.
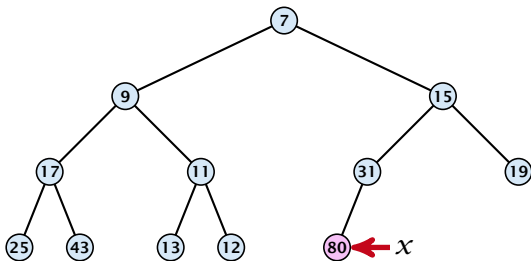
8.1 Binary Heaps

Maintain a pointer to the last element x.

▶ We can compute the predecessor of x
(last element when x is deleted) in time $\mathcal{O}(\log n)$.
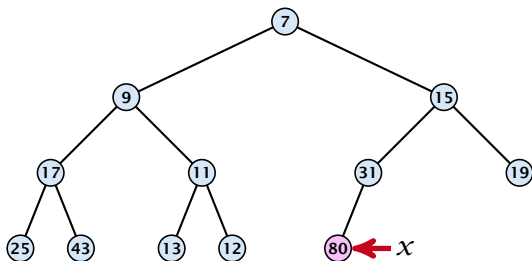
8.1 Binary Heaps

Maintain a pointer to the last element x.

▶ We can compute the predecessor of x
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  go up until the last edge used was a right edge.
  go left; go right until you reach a leaf
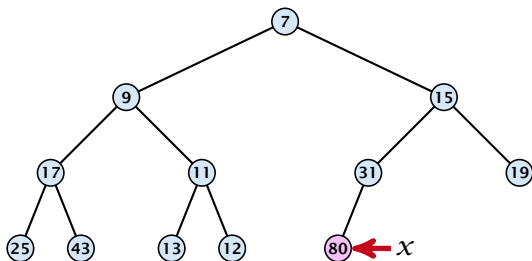
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if you hit the root on the way up, go to the rightmost element
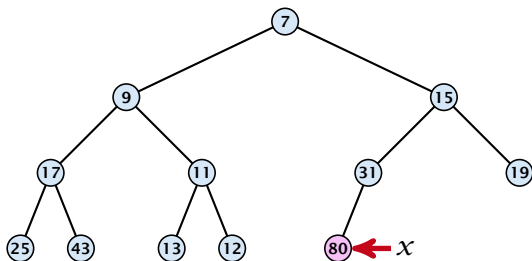
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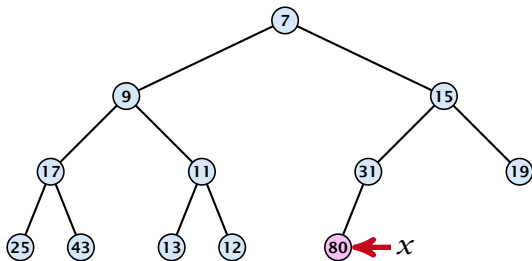
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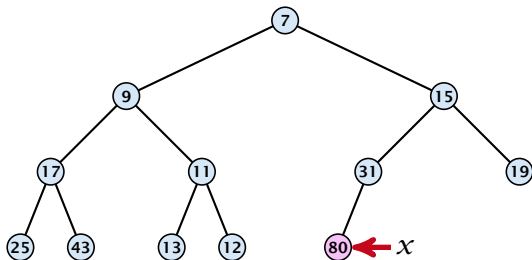
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Maintain a pointer to the last element x.

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go up until the last edge used was a left edge.
go right; go left until you reach a null-pointer.
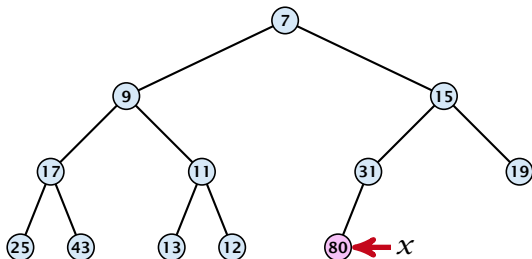
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Maintain a pointer to the last element x.

- ▶ We can compute the successor of x
  (last element when an element is inserted) in time $\mathcal{O}(\log n)$.

  go up until the last edge used was a left edge.
  go right; go left until you reach a null-pointer.

  if you hit the root on the way up, go to the leftmost element;
  insert a new element as a left child;
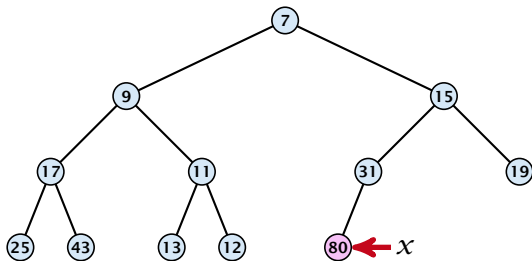
Insert

1. Insert element at successor of x.
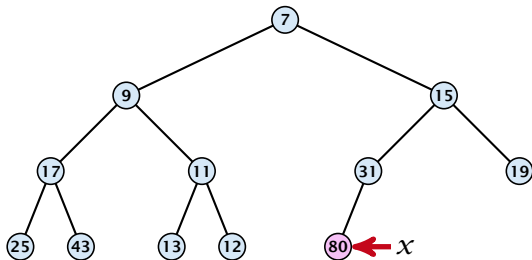
Insert

1. Insert element at successor of x.
2. Exchange with parent until heap property is fulfilled.
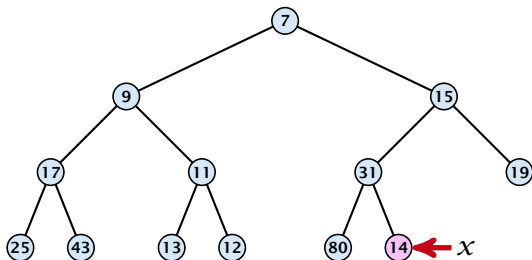
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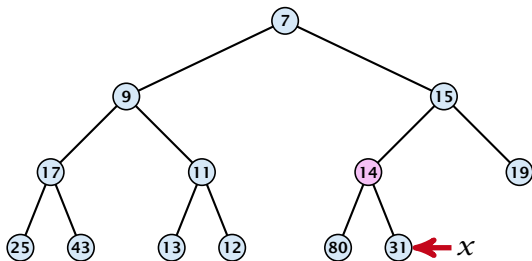
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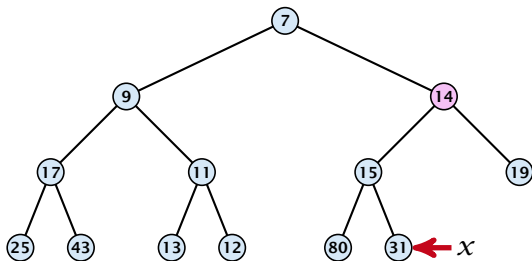
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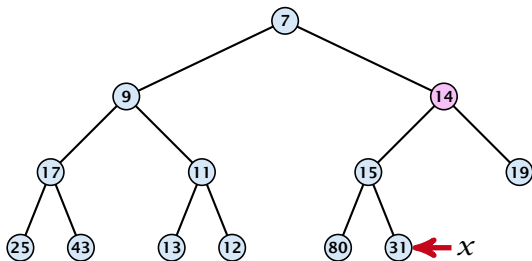
Note that an exchange can either be done by moving the data or by changing pointers. The latter method leads to an addressable priority queue.

Delete

1. Exchange the element to be deleted with the element *e*
   pointed to by *x*.
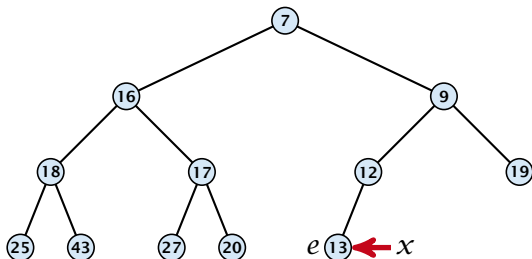
Delete

1. Exchange the element to be deleted with the element e pointed to by x.

2. Restore the heap-property for the element e.
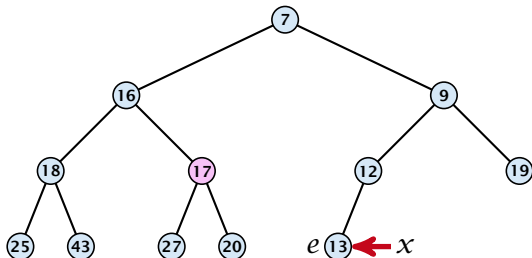
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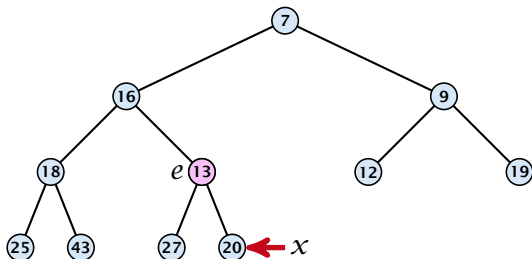
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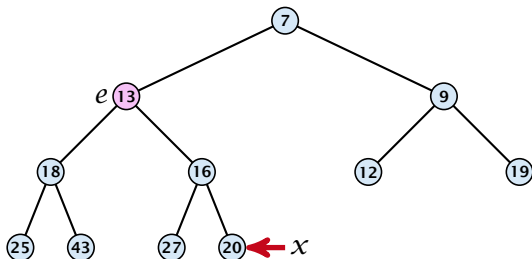
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1. Exchange the element to be deleted with the element e pointed to by x.

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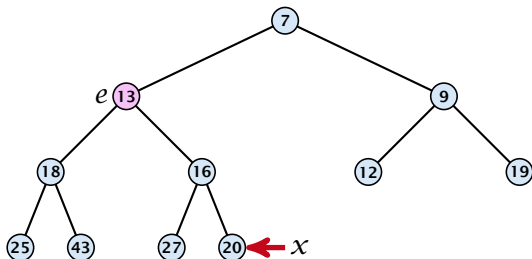
At its new position e may either travel up or down in the tree (but not both directions).

Binary Heaps

Operations:

- ▶ **minimum()**: return the root-element. Time $\mathcal{O}(1)$.

- ▶ **is-empty()**: check whether root-pointer is null. Time $\mathcal{O}(1)$.

- ▶ **insert(k)**: insert at successor of x and bubble up. Time $\mathcal{O}(\log n)$.

- ▶ **delete(h)**: swap with x and bubble up or sift-down. Time $\mathcal{O}(\log n)$.

Build Heap

We can build a heap in linear time:

Build Heap

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Ernst Mayr, Harald Räcke

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We can build a heap in linear time:

$$\sum_{\text{levels } \ell} 2^\ell \cdot (h - \ell) = \sum_i i 2^{h-i} = \mathcal{O}(2^h) = \mathcal{O}(n)$$

Binary Heaps

Operations:

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- ▶ **is-empty()**: Check whether root-pointer is null. Time $\mathcal{O}(1)$.
- ▶ **insert(k)**: Insert at x and bubble up. Time $\mathcal{O}(\log n)$.
- ▶ **delete(h)**: Swap with x and bubble up or sift-down. Time $\mathcal{O}(\log n)$.
- ▶ **build($x_1, \ldots, x_n$)**: Insert elements arbitrarily; then do sift-down operations starting with the lowest layer in the tree. Time $\mathcal{O}(n)$.

Binary Heaps

Binary Heaps

The standard implementation of binary heaps is via arrays. Let $A[0, \ldots, n-1]$ be an array

- ▶ The parent of i-th element is at position $\lfloor \frac{i-1}{2} \rfloor$.
- ▶ The left child of i-th element is at position $2i + 1$.
- ▶ The right child of i-th element is at position $2i + 2$.

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Finding the successor of x is much easier than in the description on the previous slide. Simply increase or decrease x.

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Finding the successor of x is much easier than in the description on the previous slide. Simply increase or decrease x.

The resulting binary heap is not addressable. The elements don't maintain their positions and therefore there are no stable handles.

8.2 Binomial Heaps

Operation	Binary Heap	BST	Binomial Heap	Fibonacci Heap[*]
build	n	$n \log n$	$n \log n$	n
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n$[**]	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	n	$n \log n$	**$\log n$**	1

Binomial Trees
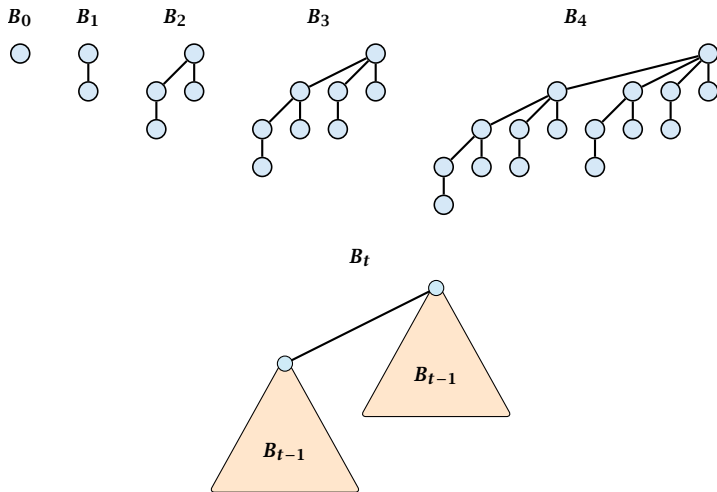
Binomial Trees

Properties of Binomial Trees

▶ B_k has 2^k nodes.

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Binomial Trees

Properties of Binomial Trees

▶ B_k has 2^k nodes.

▶ B_k has height k.

▶ The root of B_k has degree k.

▶ B_k has $\binom{k}{\ell}$ nodes on level ℓ.

▶ Deleting the root of B_k gives trees $B_0, B_1, \ldots, B_{k-1}$.

Binomial Trees
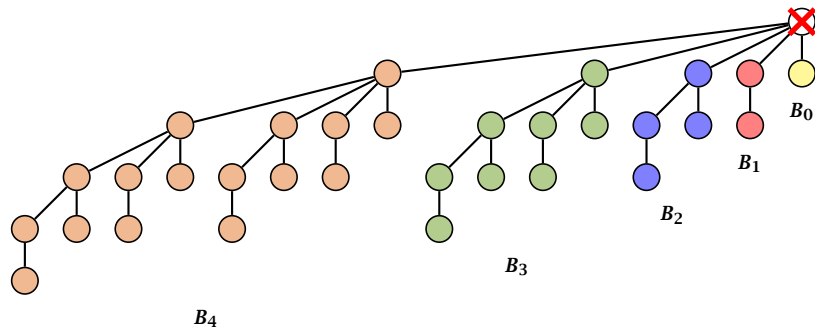
Deleting the root of B_5 leaves sub-trees B_4, B_3, B_2, B_1, and B_0.
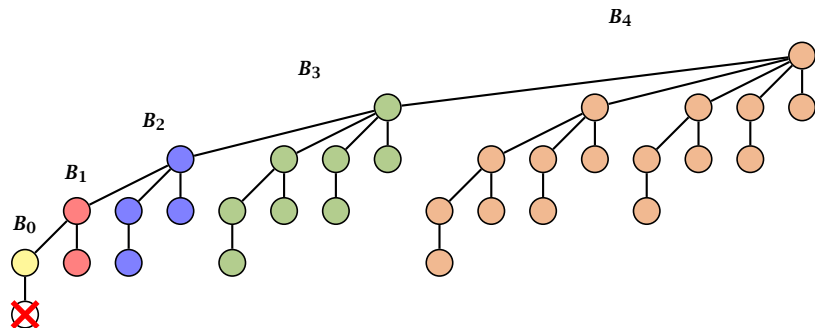
Binomial Trees

Deleting the leaf furthest from the root (in B_5) leaves a path that connects the roots of sub-trees B_4, B_3, B_2, B_1, and B_0.
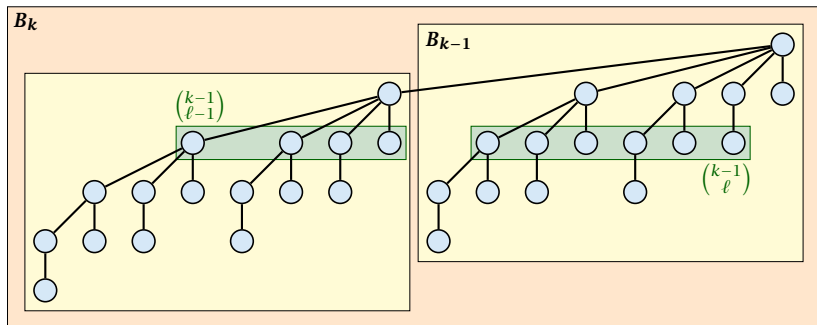
Binomial Trees

The number of nodes on level ℓ in tree B_k is therefore

$$\binom{k-1}{\ell-1} + \binom{k-1}{\ell} = \binom{k}{\ell}$$

Binomial Trees
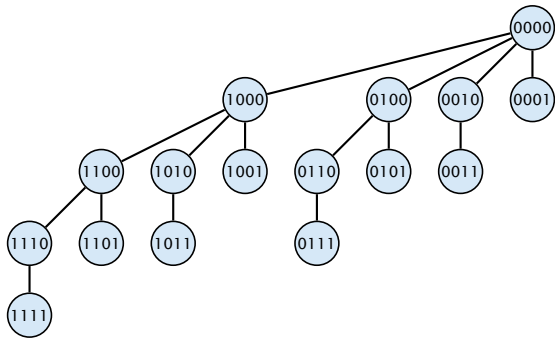
Binomial Trees
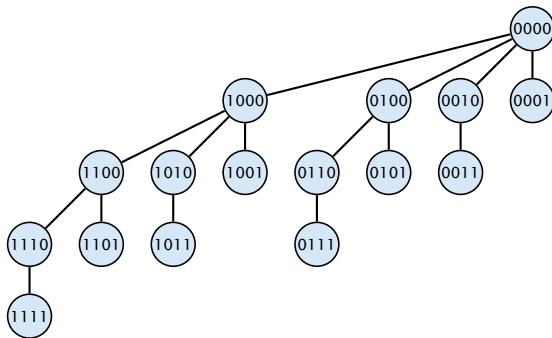
The binomial tree B_k is a sub-graph of the hypercube H_k.

Binomial Trees
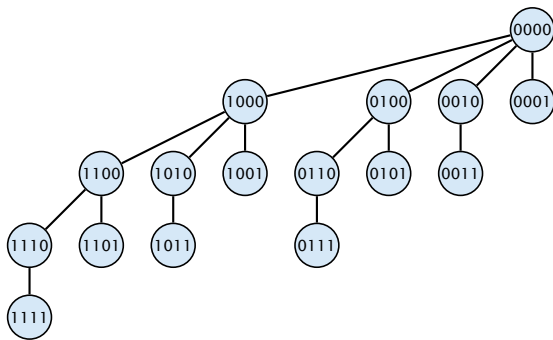
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The parent of a node with label $b_k, \ldots, b_1$ is obtained by setting the least significant 1-bit to 0.

Binomial Trees
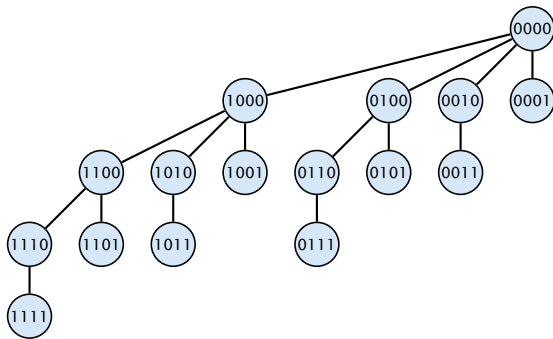
The binomial tree B_k is a sub-graph of the hypercube H_k.

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The ℓ-th level contains nodes that have ℓ 1's in their label.

8.2 Binomial Heaps

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How do we implement trees with non-constant degree?

▶ The children of a node are arranged in a circular linked list.
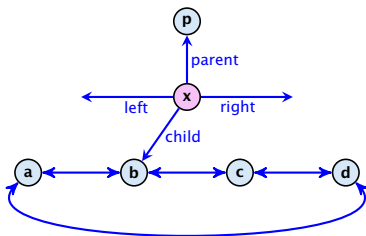
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▶ A child-pointer points to an arbitrary node within the list.
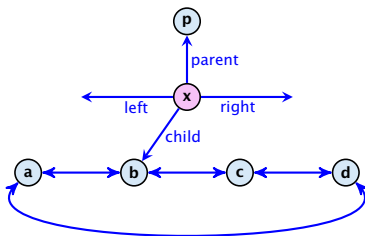
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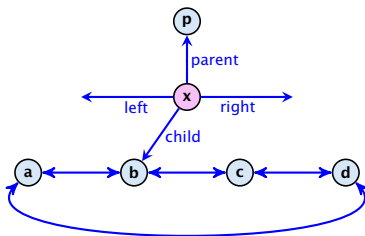
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How do we implement trees with non-constant degree?

▶ The children of a node are arranged in a circular linked list.

▶ A child-pointer points to an arbitrary node within the list.

▶ A parent-pointer points to the parent node.

▶ Pointers $x.\text{left}$ and $x.\text{right}$ point to the left and right sibling of x (if x does not have siblings then $x.\text{left} = x.\text{right} = x$).
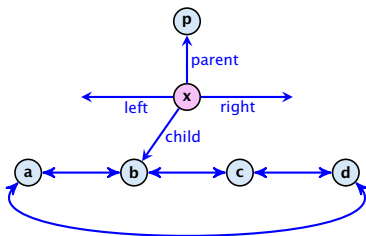
8.2 Binomial Heaps

- Given a pointer to a node x we can splice out the sub-tree rooted at x in constant time.
- We can add a child-tree T to a node x in constant time if we are given a pointer to x and a pointer to the root of T.

Binomial Heap
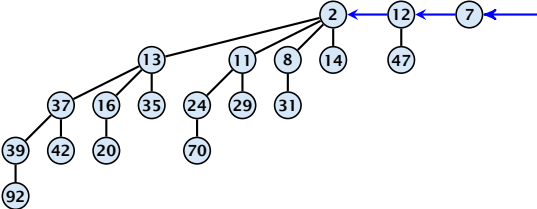
Binomial Heap
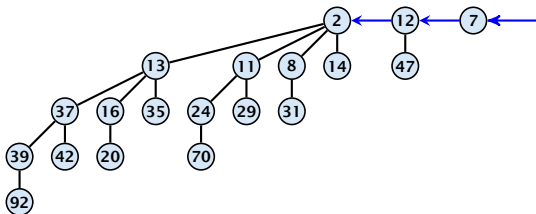
In a binomial heap the keys are arranged in a collection of binomial trees.

Binomial Heap
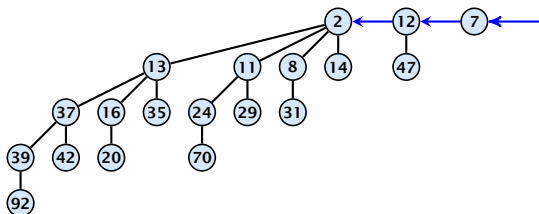
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Every tree fulfills the heap-property

Binomial Heap
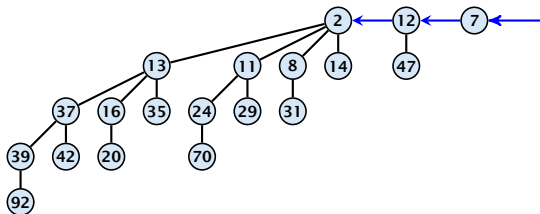
In a binomial heap the keys are arranged in a collection of binomial trees.

Every tree fulfills the heap-property

There is at most one tree for every dimension/order. For example the above heap contains trees B_0, B_1, and B_4.

Binomial Heap: Merge

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Given the number n of keys to be stored in a binomial heap we can deduce the binomial trees that will be contained in the collection.

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Let B_{k_1}, B_{k_2}, B_{k_3}, $k_i < k_{i+1}$ denote the binomial trees in the collection and recall that every tree may be contained at most once.

Binomial Heap: Merge

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Let B_{k_1}, B_{k_2}, B_{k_3}, $k_i < k_{i+1}$ denote the binomial trees in the collection and recall that every tree may be contained at most once.

Then $n = \sum_i 2^{k_i}$ must hold. But since the k_i are all distinct this means that the k_i define the non-zero bit-positions in the binary representation of n.
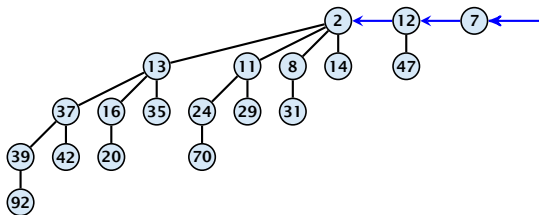
Binomial Heap

Properties of a heap with *n* keys:

Binomial Heap

Properties of a heap with n keys:

▶ Let $n = b_d b_{d-1}, \ldots, b_0$ denote binary representation of n.
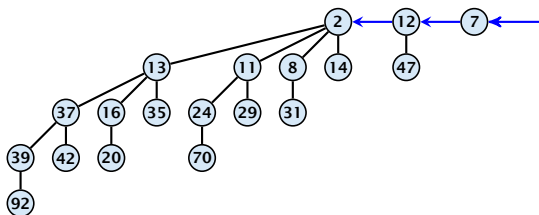
Binomial Heap

Properties of a heap with n keys:

▶ Let $n = b_d b_{d-1}, \ldots, b_0$ denote binary representation of n.

▶ The heap contains tree B_i iff $b_i = 1$.
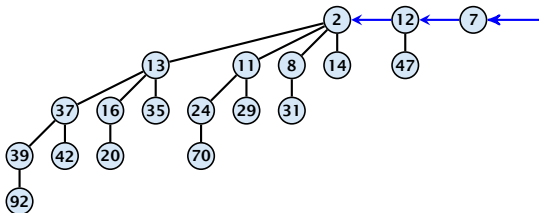
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▶ Hence, at most $\lfloor \log n \rfloor + 1$ trees.
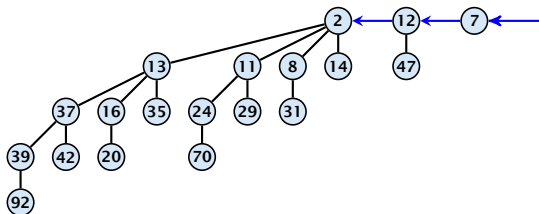
Binomial Heap

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▶ Let $n = b_d b_{d-1}, \ldots, b_0$ denote binary representation of n.

▶ The heap contains tree B_i iff $b_i = 1$.

▶ Hence, at most $\lfloor \log n \rfloor + 1$ trees.

▶ The minimum must be contained in one of the roots.
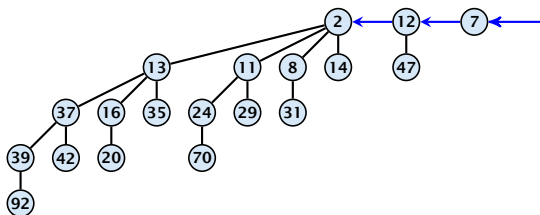
Binomial Heap

Properties of a heap with n keys:

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▶ The heap contains tree B_i iff $b_i = 1$.

▶ Hence, at most $\lfloor \log n \rfloor + 1$ trees.

▶ The minimum must be contained in one of the roots.

▶ The height of the largest tree is at most $\lfloor \log n \rfloor$.
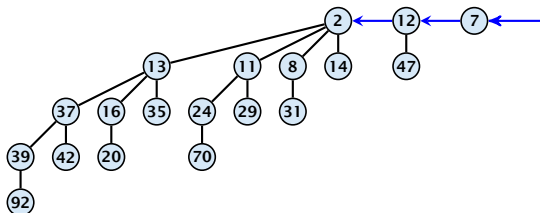
Binomial Heap

Properties of a heap with n keys:

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▶ Hence, at most $\lfloor \log n \rfloor + 1$ trees.

▶ The minimum must be contained in one of the roots.

▶ The height of the largest tree is at most $\lfloor \log n \rfloor$.

▶ The trees are stored in a single-linked list; ordered by dimension/size.
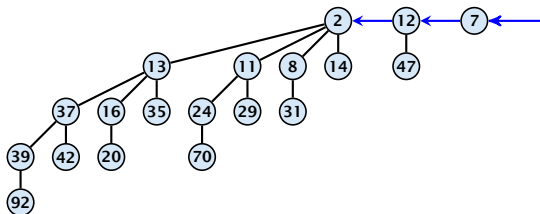
Binomial Heap: Merge

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Merging two trees of the same size: Add the tree with larger root-value as a child to the other tree.
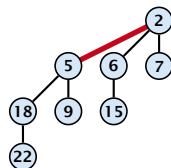
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Merging two trees of the same size: Add the tree with larger root-value as a child to the other tree.

For more trees the technique is analogous to binary addition.

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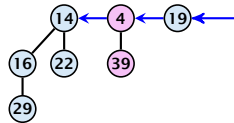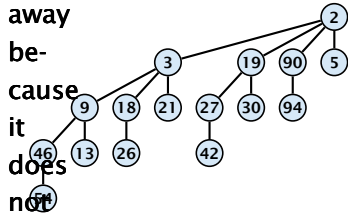cause
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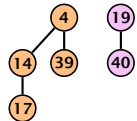[tikzpicture optimized away because it does not contribute to exported PDF]

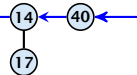[tikzpicture optimized away because it does not contribute to exported PDF]

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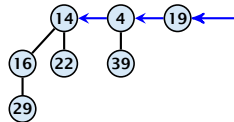mized
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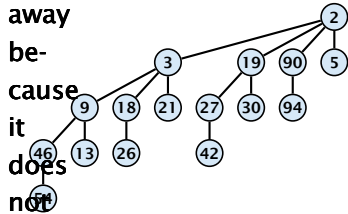[tikzpicture optimized away because it does not contribute to exported PDF]

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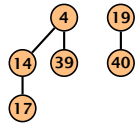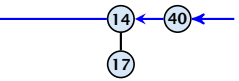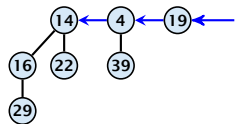opti-
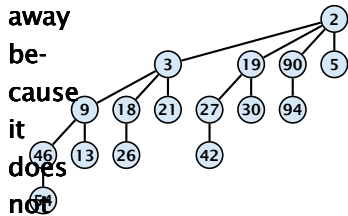mized
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8.2 Binomial Heaps

$S_1.\,\mathrm{merge}(S_2)$:

- ▶ Analogous to binary addition.

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8.2 Binomial Heaps

All other operations can be reduced to merge().

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$S.\,\textbf{insert}(x)$:

▶ Create a new heap S' that contains just the element x.

▶ Execute $S.\,\mathrm{merge}(S')$.

▶ Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

$S.\,\mathbf{minimum}()$:

- ▶ Find the minimum key-value among all roots.
- ▶ Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

S. delete-min():

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S. **delete-min()**:

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▶ Remove the corresponding tree $T_{\min}$ from the heap.

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- ▶ Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

$S.$ **decrease-key(handle h):**

8.2 Binomial Heaps

S. decrease-key(handle *h*):

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S. **decrease-key(handle h):**

- ▶ Decrease the key of the element pointed to by h.
- ▶ Bubble the element up in the tree until the heap property is fulfilled.
- ▶ Time: $\mathcal{O}(\log n)$ since the trees have height $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

$S.$ **delete(handle h):**

8.2 Binomial Heaps

S. delete(handle h):

▶ Execute $S.\,\text{decrease-key}(h, -\infty)$.

8.2 Binomial Heaps

S. delete(handle h):

- ▶ Execute S. decrease-key$(h, -\infty)$.
- ▶ Execute S. delete-min$()$.

8.2 Binomial Heaps

S. delete(handle h):

- ▶ Execute S. decrease-key($h, -\infty$).
- ▶ Execute S. delete-min().
- ▶ Time: $\mathcal{O}(\log n)$.

8.3 Fibonacci Heaps

Collection of trees that fulfill the heap property.

Structure is much more relaxed than binomial heaps.
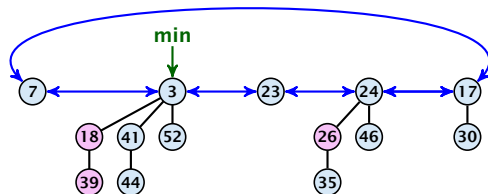
8.3 Fibonacci Heaps

Additional implementation details:

▶ Every node x stores its degree in a field $x.\,\text{degree}$. Note that this can be updated in constant time when adding a child to x.

▶ Every node stores a boolean value $x.\,\text{marked}$ that specifies whether x is marked or not.

8.3 Fibonacci Heaps

The potential function:

- ▶ $t(S)$ denotes the number of trees in the heap.
- ▶ $m(S)$ denotes the number of marked nodes.
- ▶ We use the potential function $\Phi(S) = t(S) + 2m(S)$.
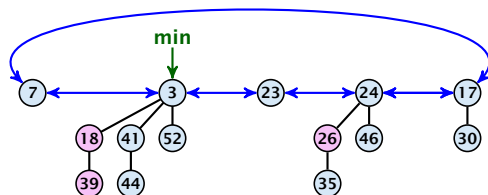
The potential is $\Phi(S) = 5 + 2 \cdot 3 = 11$.

8.3 Fibonacci Heaps

We assume that one unit of potential can pay for a constant amount of work, where the constant is chosen "big enough" (to take care of the constants that occur).

To make this more explicit we use c to denote the amount of work that a unit of potential can pay for.

8.3 Fibonacci Heaps

S. minimum()

- ▶ Access through the min-pointer.
- ▶ Actual cost $\mathcal{O}(1)$.
- ▶ No change in potential.
- ▶ Amortized cost $\mathcal{O}(1)$.

8.3 Fibonacci Heaps
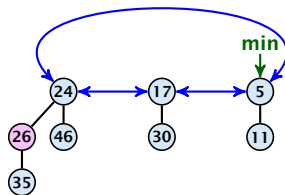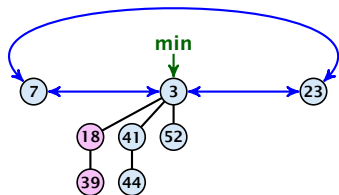
$S.\,\textbf{merge}(S')$

▶ Merge the root lists.

▶ Adjust the min-pointer

8.3 Fibonacci Heaps

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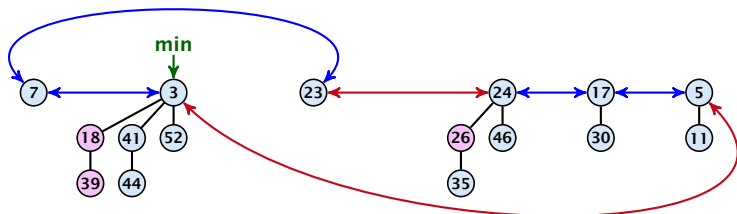
Running time:

- ▶ Actual cost $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

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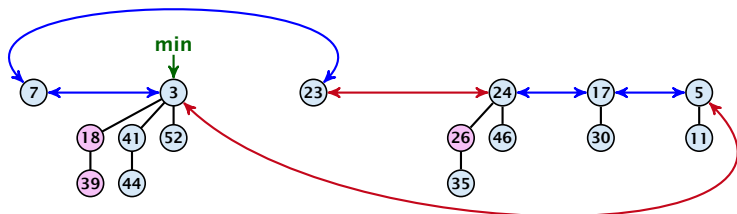
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8.3 Fibonacci Heaps

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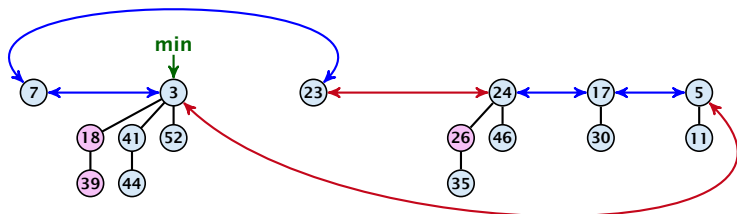
Running time:

- ▶ Actual cost $\mathcal{O}(1)$.
- ▶ No change in potential.
- ▶ Hence, amortized cost is $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

$S.\,\textbf{insert}(x)$

- ▶ Create a new tree containing x.
- ▶ Insert x into the root-list.
- ▶ Update min-pointer, if necessary.
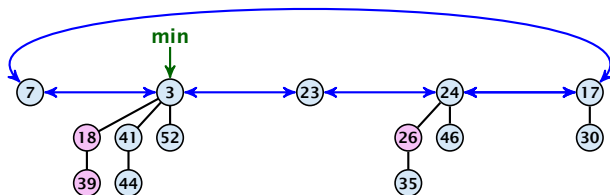
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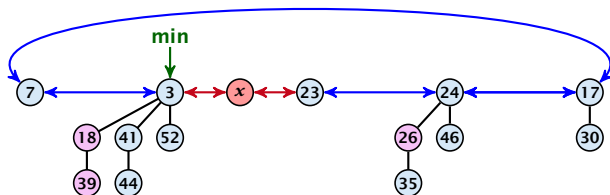
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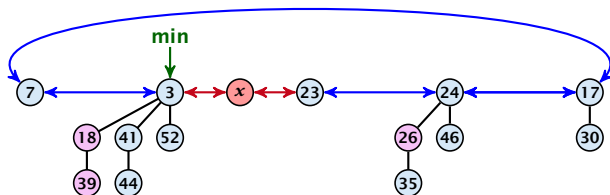
Running time:

▶ Actual cost $\mathcal{O}(1)$.

▶ Change in potential is $+1$.

▶ Amortized cost is $c + \mathcal{O}(1) = \mathcal{O}(1)$.

8.3 Fibonacci Heaps
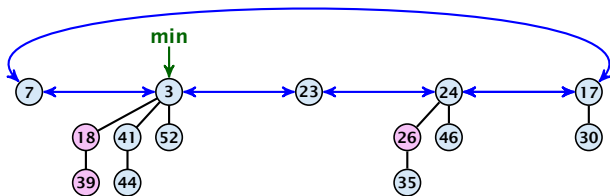
$S.$ delete-min(x)

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▶ Delete minimum; add child-trees to heap;
  time: $D(\min) \cdot \mathcal{O}(1)$.

8.3 Fibonacci Heaps

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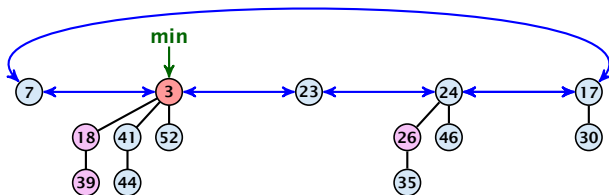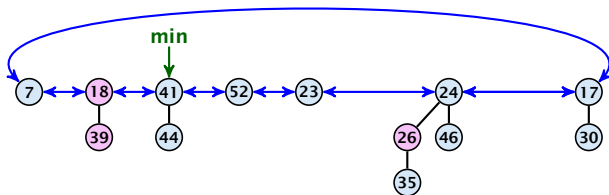
▶ Update min-pointer; time: $(t + D(\min)) \cdot \mathcal{O}(1)$.

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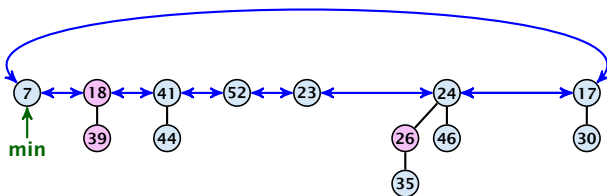
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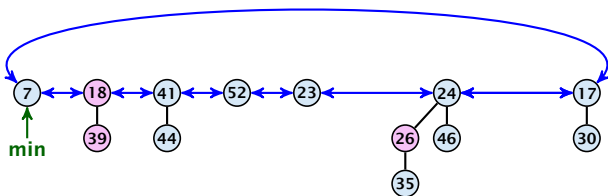
▶ Consolidate root-list so that no roots have the same degree.
   Time $t \cdot \mathcal{O}(1)$ (see next slide).

8.3 Fibonacci Heaps

Consolidate:

Consolidate:

8.3 Fibonacci Heaps

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8.3 Fibonacci Heaps

Actual cost for delete-min()

▶ At most $D_n + t$ elements in root-list before consolidate.

8.3 Fibonacci Heaps

Actual cost for delete-min()

- At most $D_n + t$ elements in root-list before consolidate.

- Actual cost for a delete-min is at most $\mathcal{O}(1) \cdot (D_n + t)$.
  Hence, there exists c_1 s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

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▶ $t' \leq D_n + 1$ as degrees are different after consolidating.

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$$\leq (c_1 + c)D_n + (c_1 - c)t + c$$

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- ▶ Therefore $\Delta\Phi \leq D_n + 1 - t$;

- ▶ We can pay $c \cdot (t - D_n - 1)$ from the potential decrease.

- ▶ The amortized cost is

  $c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1)$

  $$\leq (c_1 + c)D_n + (c_1 - c)t + c \leq 2c(D_n + 1) \leq \mathcal{O}(D_n)$$

  for $c \geq c_1$.

8.3 Fibonacci Heaps

If the input trees of the consolidation procedure are binomial trees (for example only singleton vertices) then the output will be a set of distinct binomial trees, and, hence, the Fibonacci heap will be (more or less) a Binomial heap right after the consolidation.

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If we do not have delete or decrease-key operations then $D_n \leq \log n$.
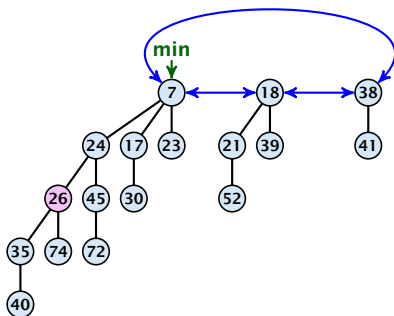
Ernst Mayr, Harald Räcke

Case 1: decrease-key does not violate heap-property

▶ Just decrease the key-value of element referenced by h. Nothing else to do.

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Case 2: heap-property is violated, but parent is not marked

▶ Decrease key-value of element x reference by h.

▶ If the heap-property is violated, cut the parent edge of x, and make x into a root.

▶ Adjust min-pointers, if necessary.

▶ Mark the (previous) parent of x (unless it's a root).

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- Decrease key-value of element x reference by h.
- Cut the parent edge of x, and make x into a root.
- Adjust min-pointers, if necessary.
- Execute the following:

  $p \leftarrow \text{parent}[x]$;
  while (p is marked)
    $pp \leftarrow \text{parent}[p]$;
    cut of p; make it into a root; unmark it;
    $p \leftarrow pp$;
  if p is unmarked and not a root mark it;

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▶ $t' = t + \ell$, as every cut creates one new root.

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  $$c_2(\ell + 1) + c(4 - \ell) \leq (c_2 - c)\ell + 4c + c_2 = \mathcal{O}(1),$$

  if $c \geq c_2$.

Delete node

$H.\,\mathbf{delete}(x)$:

- ▶ decrease value of x to $-\infty$.
- ▶ delete-min.

Amortized cost: $\mathcal{O}(D_n)$

- ▶ $\mathcal{O}(1)$ for decrease-key.
- ▶ $\mathcal{O}(D_n)$ for delete-min.

8.3 Fibonacci Heaps

Lemma 21

Let x be a node with degree k and let $y_1, \ldots, y_k$ denote the children of x in the order that they were linked to x. Then

$$\text{degree}(y_i) \geq \begin{cases} 0 & \text{if } i = 1 \\ i - 2 & \text{if } i > 1 \end{cases}$$

8.3 Fibonacci Heaps

Proof

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▶ Since, then y_i has lost at most one child.

▶ Therefore, $\mathrm{degree}(y_i) \geq i - 2$.

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Let x be a degree k node of size s_k and let $y_1, \ldots, y_k$ be its children.

$$s_k = 2 + \sum_{i=2}^{k} \text{size}(y_i)$$

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$$
\begin{aligned}
s_k &= 2 + \sum_{i=2}^{k} \text{size}(y_i) \\
&\geq 2 + \sum_{i=2}^{k} s_{i-2} \\
&= 2 + \sum_{i=0}^{k-2} s_i
\end{aligned}
$$

8.3 Fibonacci Heaps

Definition 22

Consider the following non-standard Fibonacci type sequence:

$$F_k = \begin{cases} 1 & \text{if } k = 0 \\ 2 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

Facts:

1. $F_k \geq \phi^k$.
2. For $k \geq 2$: $F_k = 2 + \sum_{i=0}^{k-2} F_i$.

The above facts can be easily proved by induction. From this it follows that $s_k \geq F_k \geq \phi^k$, which gives that the maximum degree in a Fibonacci heap is logarithmic.

k=0:	$1 = F_0 \geq \Phi^0 = 1$
k=1:	$2 = F_1 \geq \Phi^1 \approx 1.61$
k-2,k-1→ k:	$F_k = F_{k-1} + F_{k-2} \geq \Phi^{k-1} + \Phi^{k-2} = \Phi^{k-2}\overbrace{(\Phi+1)}^{\Phi^2} = \Phi^k$

| k=2: | $3 = F_2 = 2 + 1 = 2 + F_0$ |
| k-1→ k: | $F_k = F_{k-1} + F_{k-2} = 2 + \sum_{i=0}^{k-3} F_i + F_{k-2} = 2 + \sum_{i=0}^{k-2} F_i$ |

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Union Find Data Structure $\mathcal{P}$**:** Maintains a partition of disjoint sets over elements.

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- ▶ **$\mathcal{P}.\operatorname{union}(x, y)$:** Given two elements x, and y that are currently in sets S_x and S_y, respectively, the function replaces S_x and S_y by $S_x \cup S_y$ and returns an identifier for the new set.

9 Union Find

Applications:

▶ Keep track of the connected components of a dynamic graph that changes due to insertion of nodes and edges.

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- ▶ Kruskals Minimum Spanning Tree Algorithm

9 Union Find

Algorithm 1 Kruskal-MST($G = (V, E), w$)

1: $A \leftarrow \emptyset$;
2: **for all** $v \in V$ **do**
3:      $v.\text{set} \leftarrow \mathcal{P}.\text{makeset}(v.\text{label})$
4: sort edges in non-decreasing order of weight w
5: **for all** $(u, v) \in E$ in non-decreasing order **do**
6:      **if** $\mathcal{P}.\text{find}(u.\text{set}) \neq \mathcal{P}.\text{find}(v.\text{set})$ **then**
7:          $A \leftarrow A \cup \{(u, v)\}$
8:          $\mathcal{P}.\text{union}(u.\text{set}, v.\text{set})$

List Implementation

▶ The elements of a set are stored in a list; each node has a backward pointer to the head.

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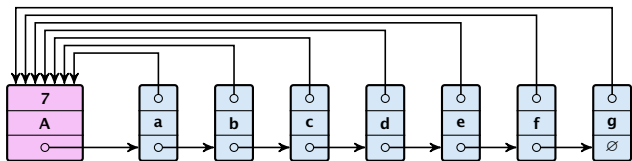
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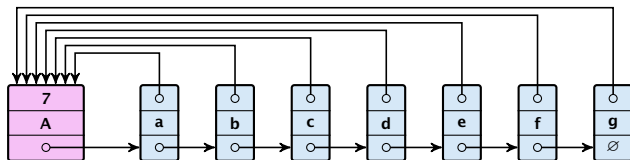
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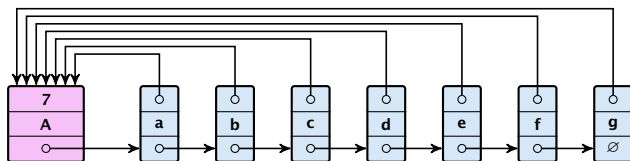
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union(*x*, *y*)

▶ Determine sets S_x and S_y.

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- ▶ Time: $\min\{|S_x|, |S_y|\}$.

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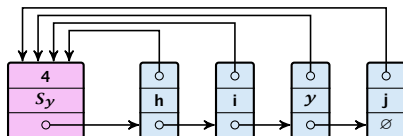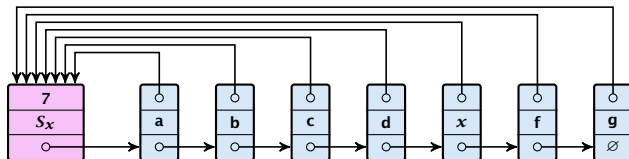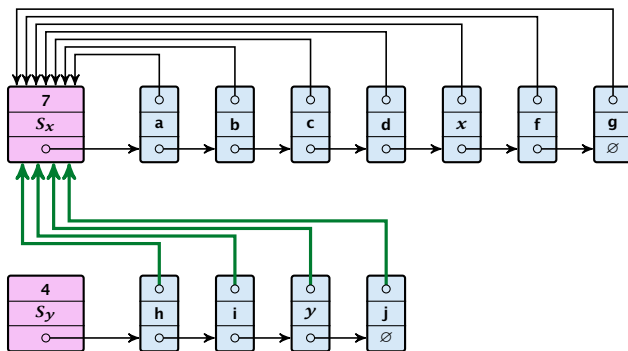
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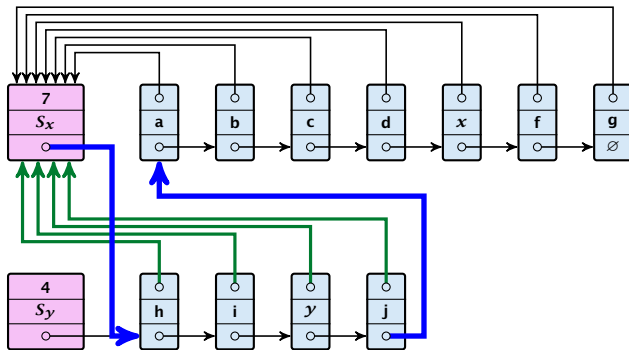
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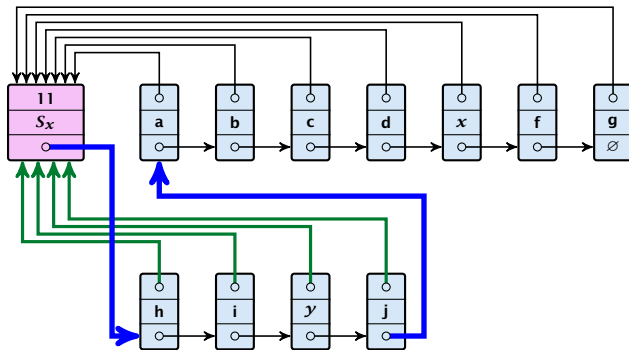
List Implementation

Running times:

- ▶ $\text{find}(x)$: constant
- ▶ $\text{makeset}(x)$: constant
- ▶ $\text{union}(x, y)$: $\mathcal{O}(n)$, where n denotes the number of elements contained in the set system.

List Implementation

Lemma 23

*The list implementation for the ADT union find fulfills the
following amortized time bounds:*

- find(x)*: $\mathcal{O}(1)$.*
- makeset(x)*: $\mathcal{O}(\log n)$.*
- union(x, y)*: $\mathcal{O}(1)$.*

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▶ If we can find a charging scheme that guarantees that balances always stay positive the amortized time bounds are proven.

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▶ We inflate the amortized cost of the makeset-operation to $\Theta(\log n)$, i.e., at this point we fill the bank account of the element to $\Theta(\log n)$.

List Implementation

▶ For an operation whose actual cost exceeds the amortized cost we charge the excess to the elements involved.

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▶ Later operations charge the account but the balance never drops below zero.

List Implementation

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▶ Assume wlog. that S_x is the smaller set; let c denote the hidden constant, i.e., the actual cost is at most $c \cdot |S_x|$.

▶ Charge c to every element in set S_x.

List Implementation

Lemma 24

An element is charged at most $\lfloor \log_2 n \rfloor$ times, where n is the total number of elements in the set system.

List Implementation

Lemma 24
An element is charged at most $\lfloor \log_2 n \rfloor$ times, where n is the total number of elements in the set system.

Proof.
Whenever an element x is charged the number of elements in x's set doubles. This can happen at most $\lfloor \log n \rfloor$ times. □

Implementation via Trees

- ▶ Maintain nodes of a set in a tree.
- ▶ The root of the tree is the label of the set.
- ▶ Only pointer to parent exists; we cannot list all elements of a given set.

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▶ Example:
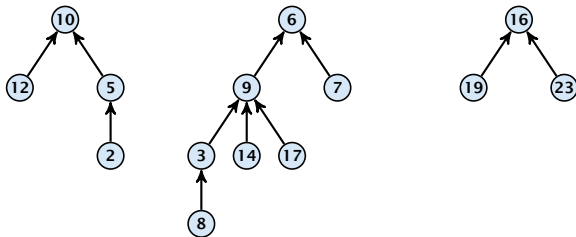
Set system $\{2, 5, 10, 12\}$, $\{3, 6, 7, 8, 9, 14, 17\}$, $\{16, 19, 23\}$.

makeset(x)

▶ Create a singleton tree. Return pointer to the root.

Implementation via Trees

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- ▶ Start at element x in the tree. Go upwards until you reach the root.
- ▶ Time: $\mathcal{O}(\text{level}(x))$, where $\text{level}(x)$ is the distance of element x to the root in its tree. Not constant.

Implementation via Trees

To support union we store the size of a tree in its root.

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▶ Perform $a \leftarrow \text{find}(x)$; $b \leftarrow \text{find}(y)$. Then: $\text{link}(a, b)$.

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- ▶ Perform $a \leftarrow \text{find}(x)$; $b \leftarrow \text{find}(y)$. Then: $\text{link}(a, b)$.
- ▶ $\text{link}(a, b)$ attaches the smaller tree as the child of the larger.
- ▶ In addition it updates the size-field of the new root.

- ▶ Time: constant for $\text{link}(a, b)$ plus two find-operations.

Implementation via Trees

Lemma 25

The running time (non-amortized!!!) for $\mathrm{find}(x)$ *is* $\mathcal{O}(\log n)$.

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☐

Path Compression

find(x):

- ▶ Go upward until you find the root.

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- ▶ Note that the size-fields now only give an upper bound on the size of a sub-tree.

Path Compression

Asymptotically the cost for a find-operation does not increase due to the path compression heuristic.

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However, for a worst-case analysis there is no improvement on the running time. It can still happen that a find-operation takes time $\mathcal{O}(\log n)$.

Amortized Analysis

Definitions:

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▶ $\text{size}(v) \coloneqq$ the number of nodes that were in the sub-tree rooted at v when v became the child of another node (or the number of nodes if v is the root).

Note that this is the same as the size of v's subtree in the case that there are no find-operations.

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Lemma 26
The rank of a parent must be strictly larger than the rank of a child.

Amortized Analysis

Lemma 27

There are at most $n/2^s$ nodes of rank s.

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▶ Let's say a node v sees node x if v is in x's sub-tree at the time that x becomes a child.

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▶ Let's say a node v sees node x if v is in x's sub-tree at the time that x becomes a child.

▶ A node v sees at most one node of rank s during the running time of the algorithm.

□

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▶ This holds because the rank-sequence of the roots of the different trees that contain v during the running time of the algorithm is a strictly increasing sequence.

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▶ This holds because the rank-sequence of the roots of the different trees that contain v during the running time of the algorithm is a strictly increasing sequence.

▶ Hence, every node *sees* at most one rank s node, but every rank s node is seen by at least 2^s different nodes. □

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We define

$$\text{tow}(i) := \begin{cases} 1 & \text{if } i = 0 \\ 2^{\text{tow}(i-1)} & \text{otw.} \end{cases}$$

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Theorem 28

Union find with path compression fulfills the following amortized running times:

- ▶ makeset$(x) : \mathcal{O}(\log^*(n))$
- ▶ find$(x) : \mathcal{O}(\log^*(n))$
- ▶ union$(x, y) : \mathcal{O}(\log^*(n))$

Amortized Analysis

In the following we assume $n \geq 2$.

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- ▶ A rank group $g \geq 1$ contains ranks $\mathrm{tow}(g-1)+1, \ldots, \mathrm{tow}(g)$.

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- ▶ The rank-group $g = 0$ contains only nodes with rank 0 or rank 1.
- ▶ A rank group $g \geq 1$ contains ranks $\mathrm{tow}(g-1)+1, \ldots, \mathrm{tow}(g)$.
- ▶ The maximum non-empty rank group is $\log^*(\lfloor \log n \rfloor) \leq \log^*(n) - 1$ (which holds for $n \geq 2$).

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In the following we assume $n \geq 2$.

rank-group:

- ▶ A node with rank $\text{rank}(v)$ is in rank group $\log^*(\text{rank}(v))$.

- ▶ The rank-group $g = 0$ contains only nodes with rank 0 or rank 1.

- ▶ A rank group $g \geq 1$ contains ranks $\text{tow}(g-1) + 1, \ldots, \text{tow}(g)$.

- ▶ The maximum non-empty rank group is $\log^*(\lfloor \log n \rfloor) \leq \log^*(n) - 1$ (which holds for $n \geq 2$).

- ▶ Hence, the total number of rank-groups is at most $\log^* n$.

Amortized Analysis

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Accounting Scheme:

▶ create an account for every find-operation

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The cost for a find-operation is equal to the length of the path traversed. We charge the cost for going from v to parent$[v]$ as follows:

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The cost for a find-operation is equal to the length of the path traversed. We charge the cost for going from v to $\text{parent}[v]$ as follows:

- ▶ If $\text{parent}[v]$ is the root we charge the cost to the find-account.

Amortized Analysis

Accounting Scheme:

▶ create an account for every find-operation

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The cost for a find-operation is equal to the length of the path traversed. We charge the cost for going from v to parent$[v]$ as follows:

▶ If parent$[v]$ is the root we charge the cost to the find-account.

▶ If the group-number of rank(v) is the same as that of rank(parent$[v]$) (before starting path compression) we charge the cost to the node-account of v.

Amortized Analysis

Accounting Scheme:

- ▶ create an account for every find-operation
- ▶ create an account for every node v

The cost for a find-operation is equal to the length of the path traversed. We charge the cost for going from v to $\mathrm{parent}[v]$ as follows:

- ▶ If $\mathrm{parent}[v]$ is the root we charge the cost to the find-account.

- ▶ If the group-number of $\mathrm{rank}(v)$ is the same as that of $\mathrm{rank}(\mathrm{parent}[v])$ (before starting path compression) we charge the cost to the node-account of v.

- ▶ Otherwise we charge the cost to the find-account.

Amortized Analysis

Observations:

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▶ A find-account is charged at most $\log^*(n)$ times (once for the root and at most $\log^*(n) - 1$ times when increasing the rank-group).

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- A find-account is charged at most $\log^*(n)$ times (once for the root and at most $\log^*(n) - 1$ times when increasing the rank-group).

- After a node v is charged its parent-edge is re-assigned. The rank of the parent strictly increases.

- After some charges to v the parent will be in a larger rank-group. $\implies v$ will never be charged again.

- The total charge made to a node in rank-group g is at most $\text{tow}(g) - \text{tow}(g - 1) - 1 \le \text{tow}(g)$.

Amortized Analysis

What is the total charge made to nodes?

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What is the total charge made to nodes?

▶ The total charge is at most

$$\sum_g n(g) \cdot \text{tow}(g) \ ,$$

where $n(g)$ is the number of nodes in group g.

Amortized Analysis

For $g \geq 1$ we have

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For $g \geq 1$ we have

$$n(g) \leq \sum_{s=\text{tow}(g-1)+1}^{\text{tow}(g)} \frac{n}{2^s} \leq \sum_{s=\text{tow}(g-1)+1}^{\infty} \frac{n}{2^s}$$

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$$= \frac{n}{2^{\text{tow}(g-1)+1}} \sum_{s=0}^{\infty} \frac{1}{2^s}$$

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$$= \frac{n}{2^{\text{tow}(g-1)+1}} \sum_{s=0}^{\infty} \frac{1}{2^s} = \frac{n}{2^{\text{tow}(g-1)+1}} \cdot 2$$

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$$
\begin{aligned}
n(g) &\leq \sum_{s=\text{tow}(g-1)+1}^{\text{tow}(g)} \frac{n}{2^s} \leq \sum_{s=\text{tow}(g-1)+1}^{\infty} \frac{n}{2^s} \\
&= \frac{n}{2^{\text{tow}(g-1)+1}} \sum_{s=0}^{\infty} \frac{1}{2^s} = \frac{n}{2^{\text{tow}(g-1)+1}} \cdot 2 \\
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\end{aligned}
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Hence,

$$
\sum_g n(g) \, \text{tow}(g)
$$

Amortized Analysis

For $g \geq 1$ we have

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$$= \frac{n}{2^{\text{tow}(g-1)}} = \frac{n}{\text{tow}(g)} \; .$$

Hence,

$$\sum_g n(g) \, \text{tow}(g) \leq n(0) \, \text{tow}(0) + \sum_{g \geq 1} n(g) \, \text{tow}(g)$$

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Hence,

$$\sum_{g} n(g) \, \text{tow}(g) \leq n(0) \, \text{tow}(0) + \sum_{g \geq 1} n(g) \, \text{tow}(g) \leq n \log^*(n)$$

Amortized Analysis

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This means if we inflate the cost of makeset to $\log^* n$ and add this to the node account of v then the balances of all node accounts will sum up to a positive value (this is sufficient to obtain an amortized bound).

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The analysis is not tight. In fact it has been shown that the amortized time for the union-find data structure with path compression is $\mathcal{O}(\alpha(m, n))$, where $\alpha(m, n)$ is the inverse Ackermann function which grows a lot lot slower than $\log^* n$. (Here, we consider the average running time of m operations on at most n elements).

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There is also a lower bound of $\Omega(\alpha(m, n))$.

Amortized Analysis

$$A(x, y) = \begin{cases} y + 1 & \text{if } x = 0 \\ A(x - 1, 1) & \text{if } y = 0 \\ A(x - 1, A(x, y - 1)) & \text{otw.} \end{cases}$$

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▶ $A(0, y) = y + 1$
▶ $A(1, y) = y + 2$
▶ $A(2, y) = 2y + 3$
▶ $A(3, y) = 2^{y+3} - 3$
▶ $A(4, y) = \underbrace{2^{2^{2^{2}}}}_{y+3 \text{ times}} - 3$