

Part III

Data Structures

Abstract Data Type

An abstract data type (ADT) is defined by an interface of operations or methods that can be performed and that have a defined behavior.

The data types in this lecture all operate on objects that are represented by a [key, value] pair.

- ▶ The **key** comes from a totally ordered set, and we assume that there is an efficient comparison function.
- ▶ The **value** can be anything; it usually carries satellite information important for the application that uses the ADT.

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- ▶ **S. split(k, S'):**
 $S := \{x \in S \mid \text{key}[x] \leq k\}$, $S' := \{x \in S \mid \text{key}[x] > k\}$.

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 $S := \{x \in S \mid \text{key}[x] \leq k\}$, $S' := \{x \in S \mid \text{key}[x] > k\}$.
- ▶ **S . concatenate(S'):** $S := S \cup S'$.
Requires $\text{key}[S.\text{maximum}()] \leq \text{key}[S'.\text{minimum}()]$.

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- ▶ **S.concatenate(S')**: $S := S \cup S'$.
Requires $\text{key}[S.\text{maximum}()] \leq \text{key}[S'.\text{minimum}()]$.
- ▶ **S.decrease-key(x, k)**: Replace $\text{key}[x]$ by $k \leq \text{key}[x]$.

Examples of ADTs

Stack:

- ▶ **S . push(x)**: Insert an element.
- ▶ **S . pop()**: Return the element from S that was inserted most recently; delete it from S .
- ▶ **S . empty()**: Tell if S contains any object.

Examples of ADTs

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Queue:

- ▶ **$S.$ enqueue(x)**: Insert an element.
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Priority-Queue:

- ▶ ***S.insert(x)***: Insert an element.
- ▶ ***S.delete-min()***: Return the element with lowest key-value; delete it from *S*.

7 Dictionary

Dictionary:

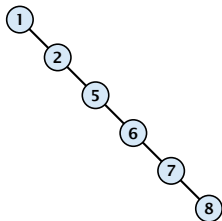
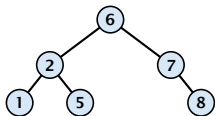
- ▶ **S .insert(x)**: Insert an element x .
- ▶ **S .delete(x)**: Delete the element pointed to by x .
- ▶ **S .search(k)**: Return a pointer to an element e with $\text{key}[e] = k$ in S if it exists; otherwise return **null**.

7.1 Binary Search Trees

An (**internal**) **binary search tree** stores the elements in a binary tree. Each tree-node corresponds to an element. All elements in the left sub-tree of a node v have a smaller key-value than $\text{key}[v]$ and elements in the right sub-tree have a larger-key value. We assume that all key-values are different.

(**External** Search Trees store objects only at leaf-vertices)

Examples:

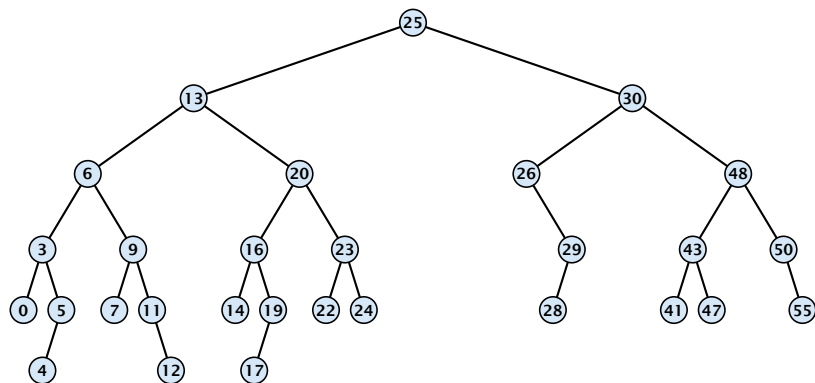


7.1 Binary Search Trees

We consider the following operations on binary search trees. Note that this is a super-set of the dictionary-operations.

- ▶ $T.\text{insert}(x)$
- ▶ $T.\text{delete}(x)$
- ▶ $T.\text{search}(k)$
- ▶ $T.\text{successor}(x)$
- ▶ $T.\text{predecessor}(x)$
- ▶ $T.\text{minimum}()$
- ▶ $T.\text{maximum}()$

Binary Search Trees: Searching

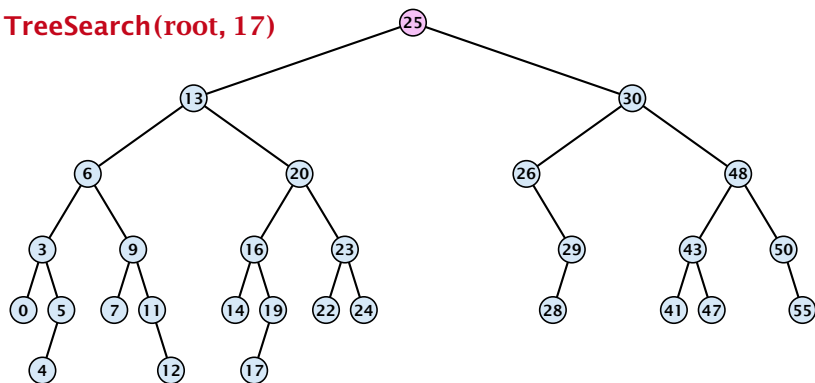


Algorithm 1 TreeSearch(x, k)

- 1: **if** $x = \text{null}$ **or** $k = \text{key}[x]$ **return** x
- 2: **if** $k < \text{key}[x]$ **return** TreeSearch(left[x], k)
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Binary Search Trees: Searching

TreeSearch(root, 17)

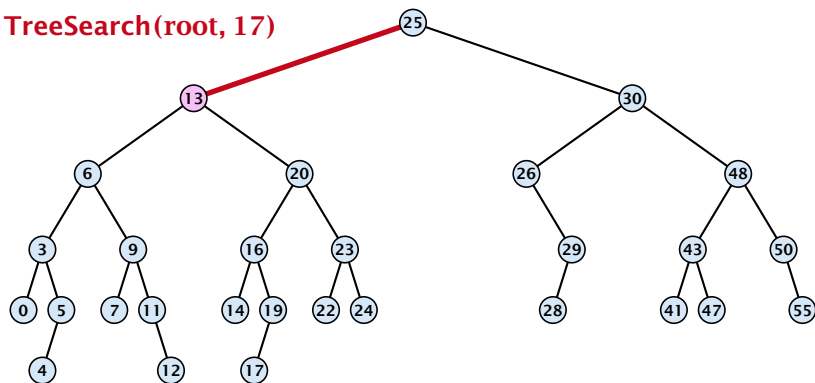


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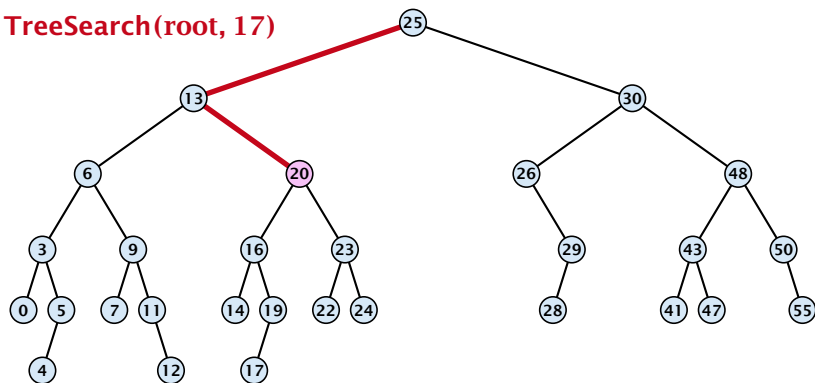


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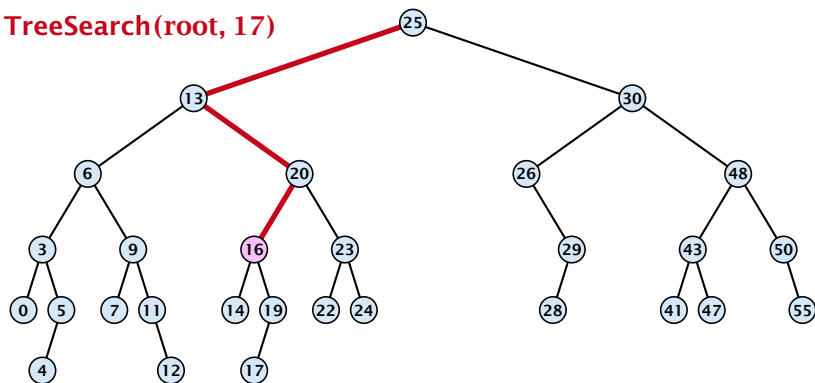


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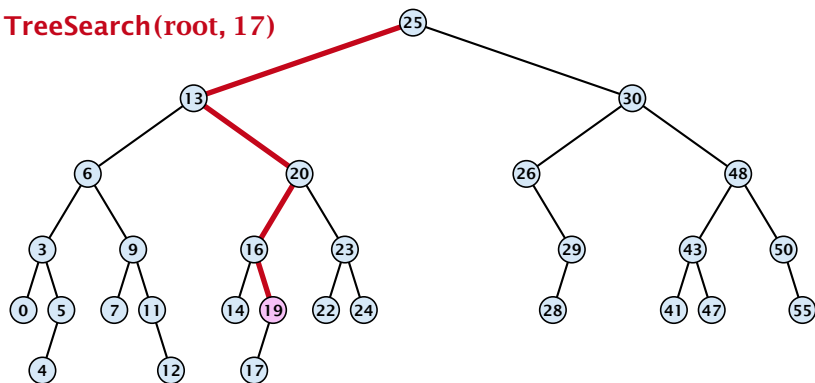


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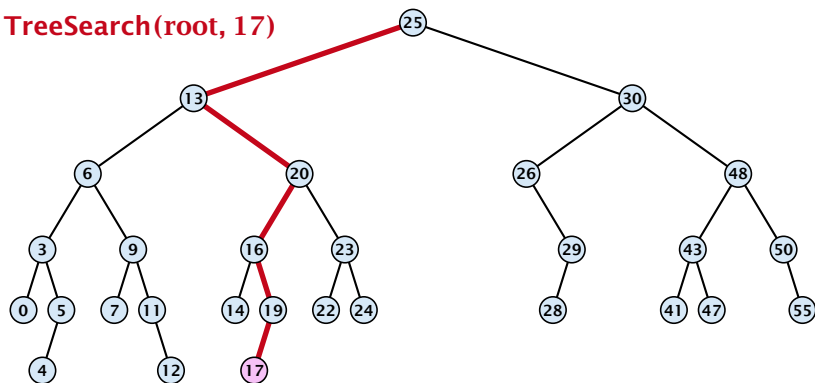


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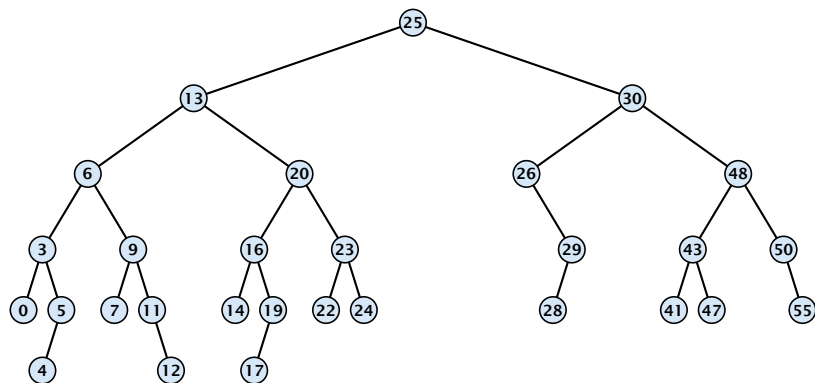
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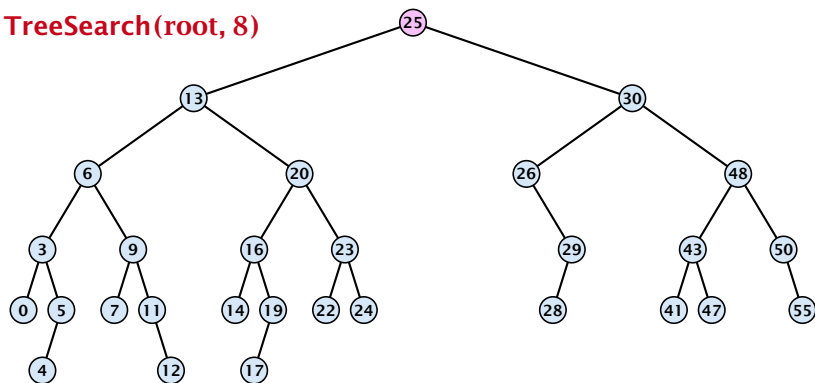


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Binary Search Trees: Searching

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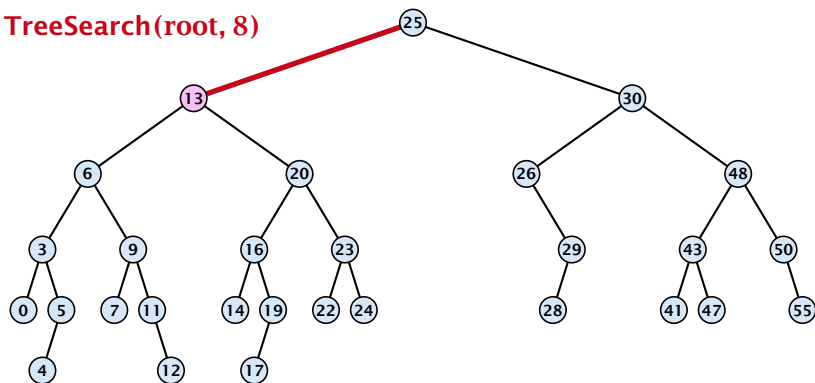


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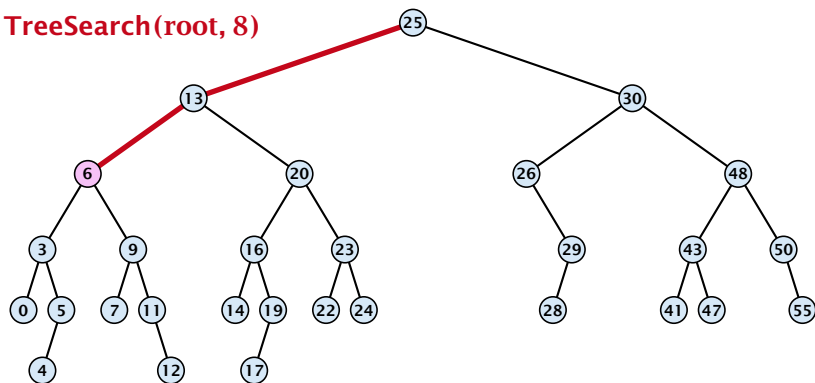


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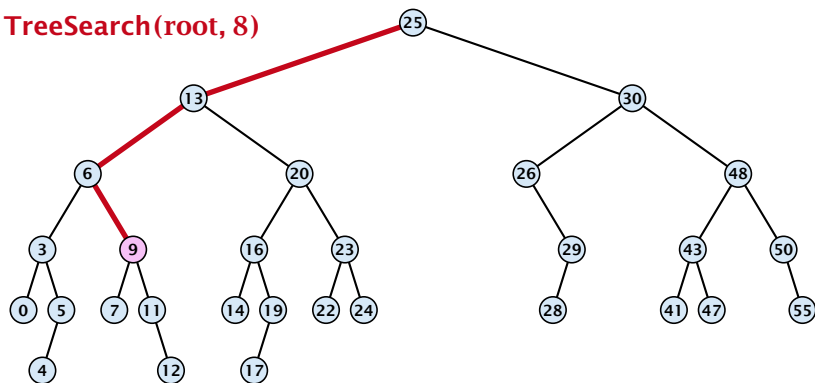


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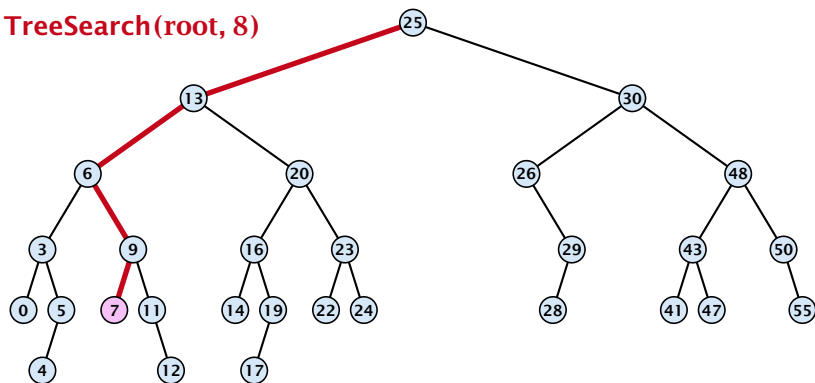


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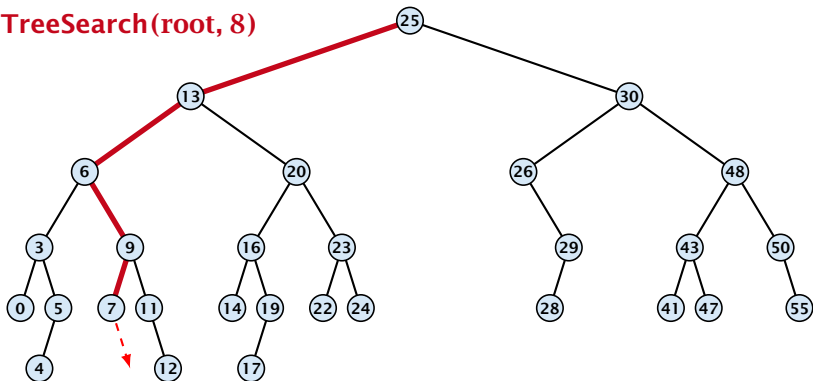


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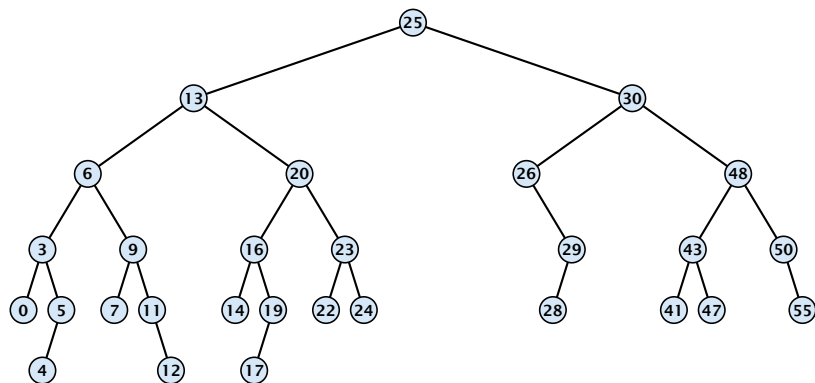
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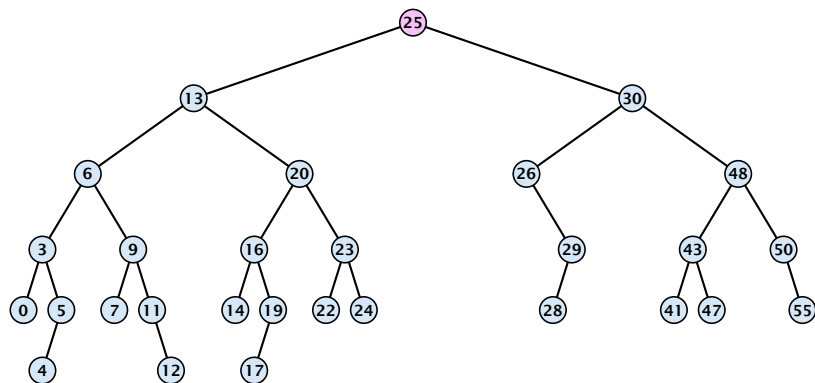
Binary Search Trees: Minimum



Algorithm 2 TreeMin(x)

- 1: **if** $x = \text{null}$ **or** $\text{left}[x] = \text{null}$ **return** x
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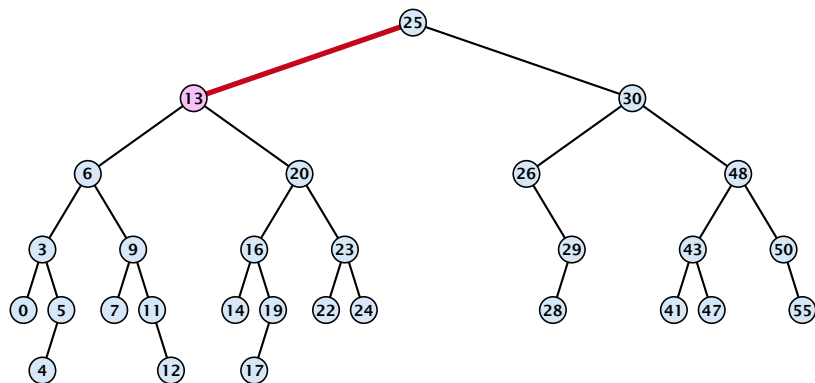
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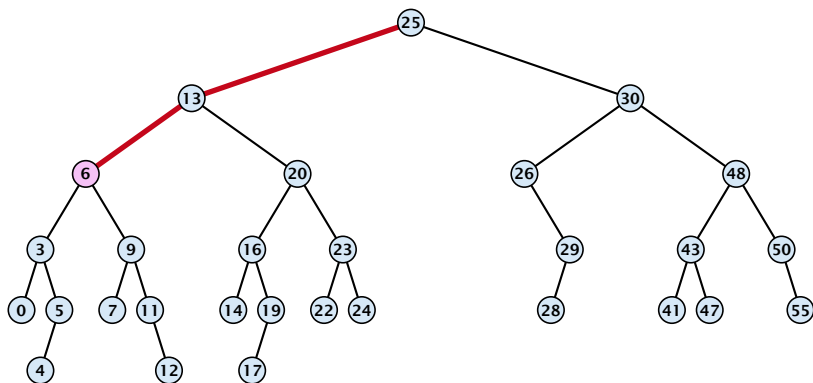
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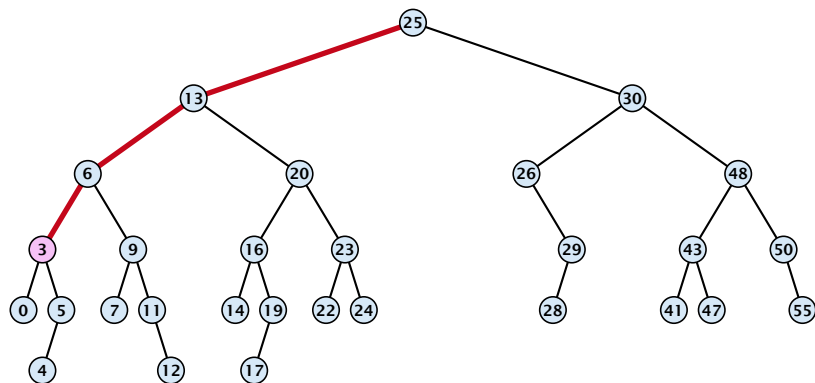
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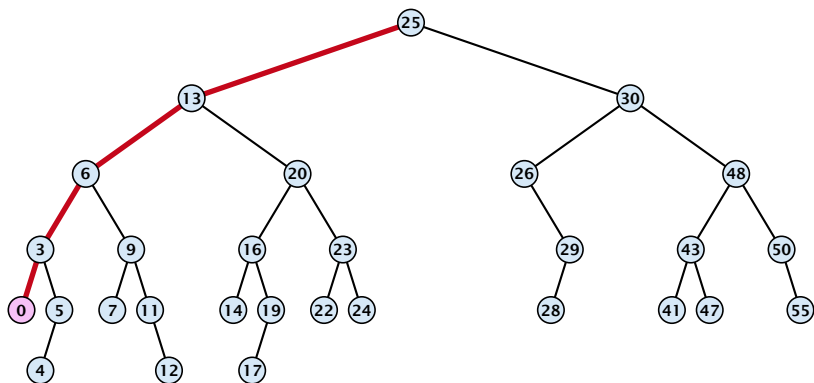
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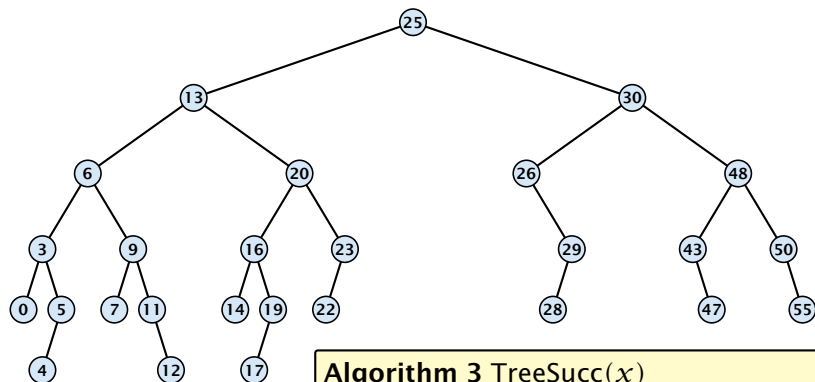
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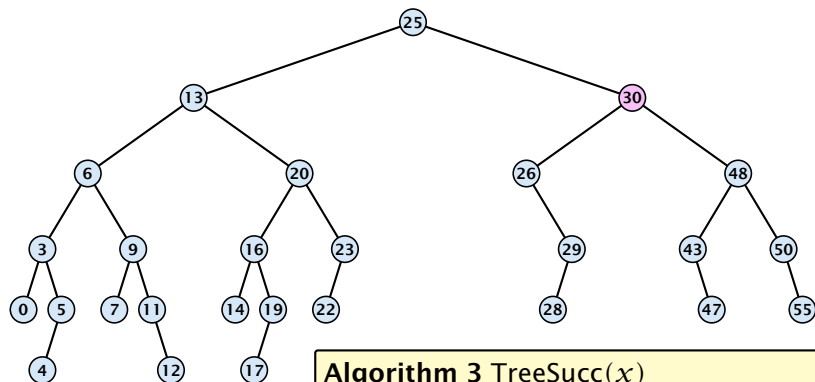
Binary Search Trees: Successor



Algorithm 3 TreeSucc(x)

- 1: **if** right[x] \neq null **return** TreeMin(right[x])
- 2: $y \leftarrow$ parent[x]
- 3: **while** $y \neq$ null **and** $x =$ right[y] **do**
- 4: $x \leftarrow y$; $y \leftarrow$ parent[x]
- 5: **return** y ;

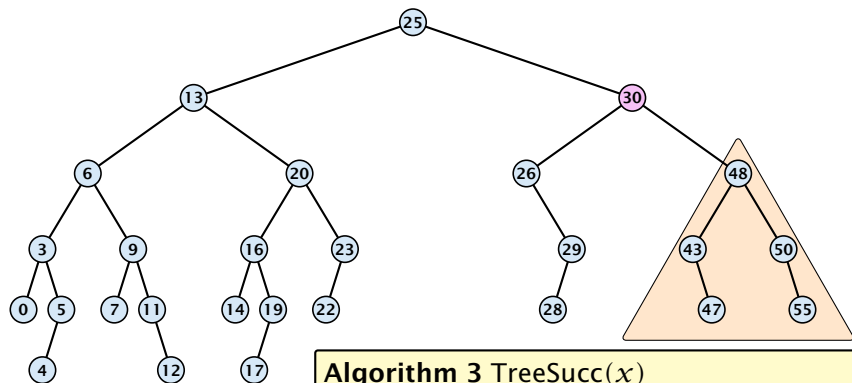
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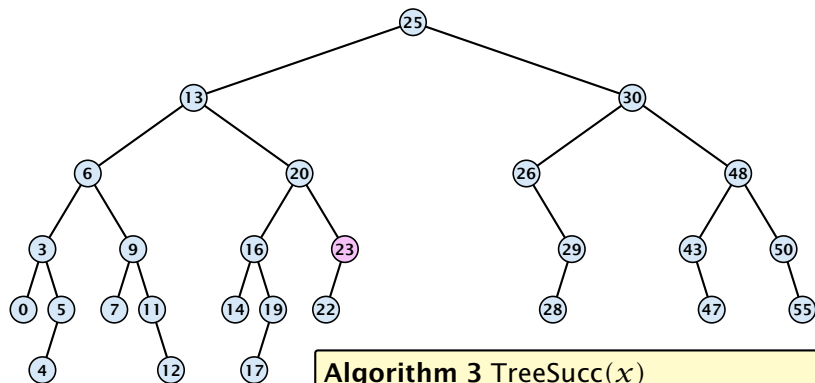
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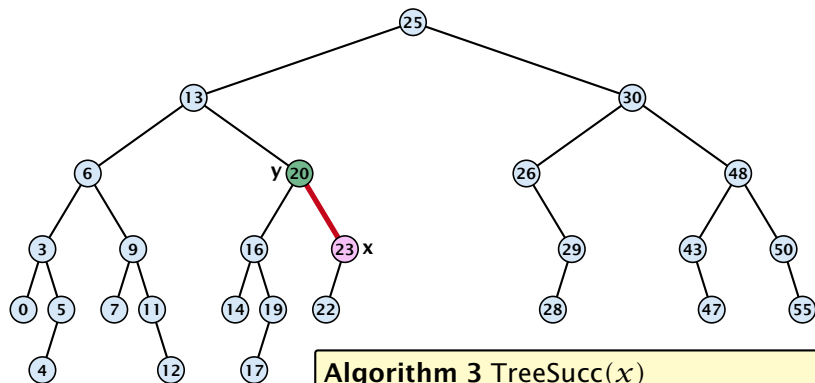
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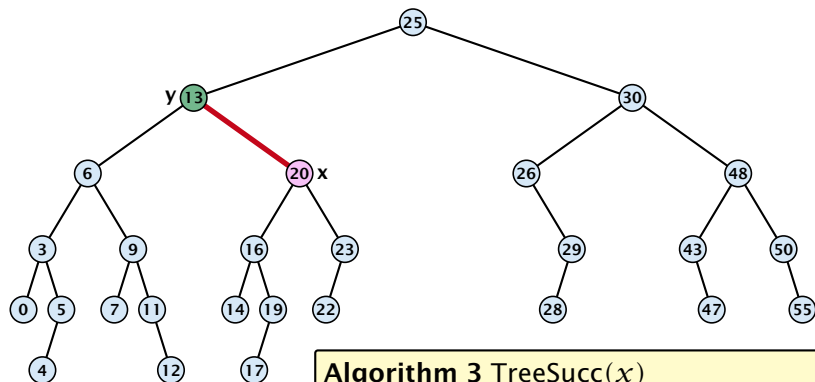
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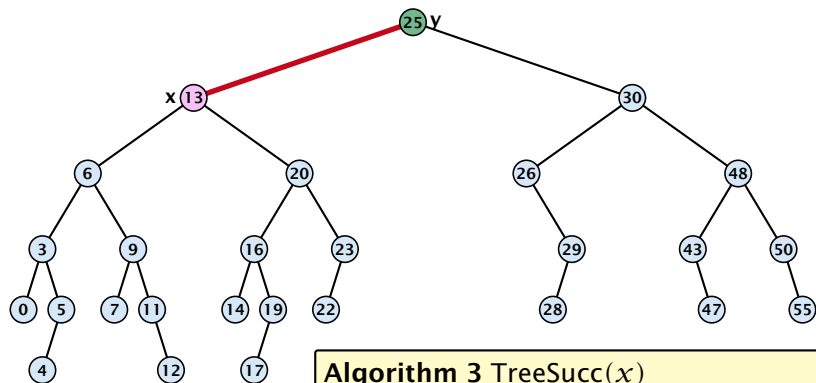
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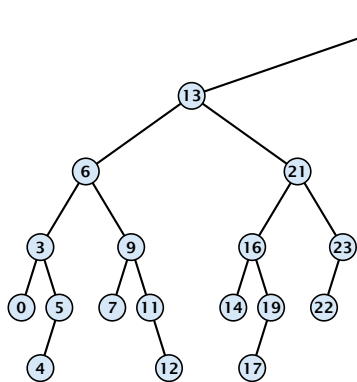
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Binary Search Trees: Insert

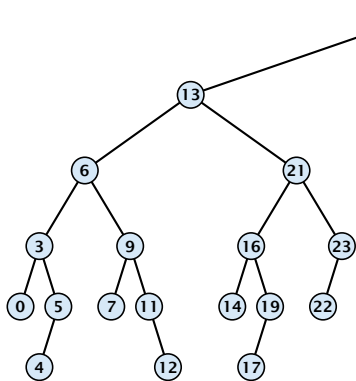


Algorithm 4 TreeInsert(x, z)

- 1: **if** $x = \text{null}$ **then**
- 2: $\text{root}[T] \leftarrow z$; $\text{parent}[z] \leftarrow \text{null}$;
- 3: **return**;
- 4: **if** $\text{key}[x] > \text{key}[z]$ **then**
- 5: **if** $\text{left}[x] = \text{null}$ **then**
- 6: $\text{left}[x] \leftarrow z$; $\text{parent}[z] \leftarrow x$;
- 7: **else** TreeInsert($\text{left}[x], z$);
- 8: **else**
- 9: **if** $\text{right}[x] = \text{null}$ **then**
- 10: $\text{right}[x] \leftarrow z$; $\text{parent}[z] \leftarrow x$;
- 11: **else** TreeInsert($\text{right}[x], z$);

Binary Search Trees: Insert

Insert element **not** in the tree.

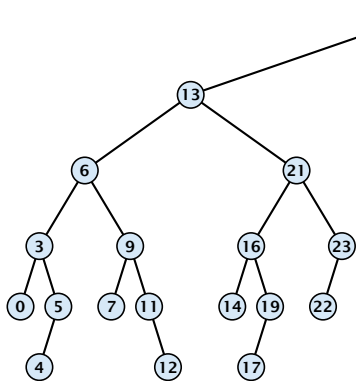


Algorithm 4 TreeInsert(x, z)

- 1: **if** $x = \text{null}$ **then**
- 2: $\text{root}[T] \leftarrow z$; $\text{parent}[z] \leftarrow \text{null}$;
- 3: **return**;
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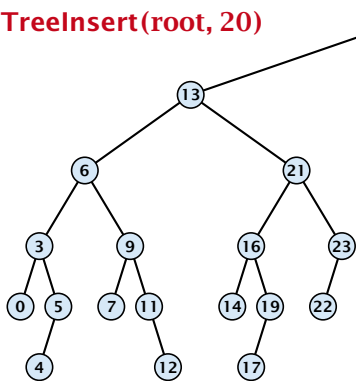
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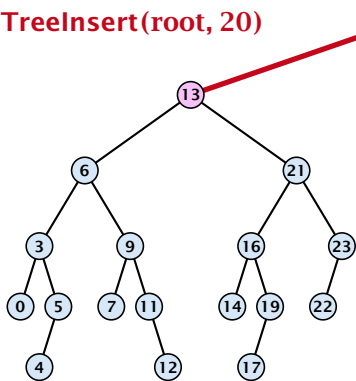
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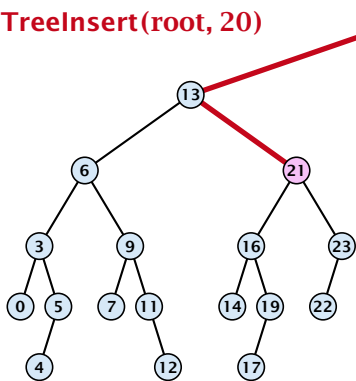
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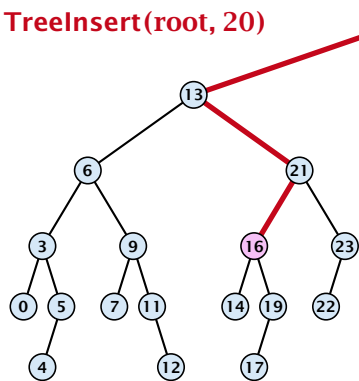
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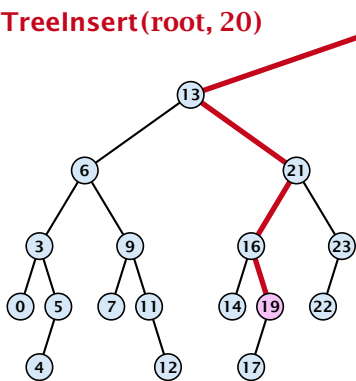
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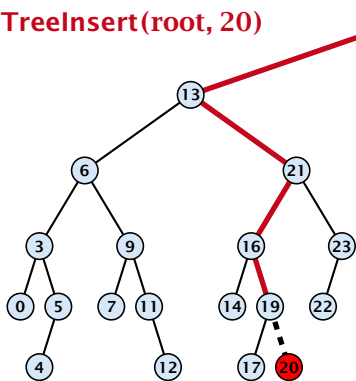
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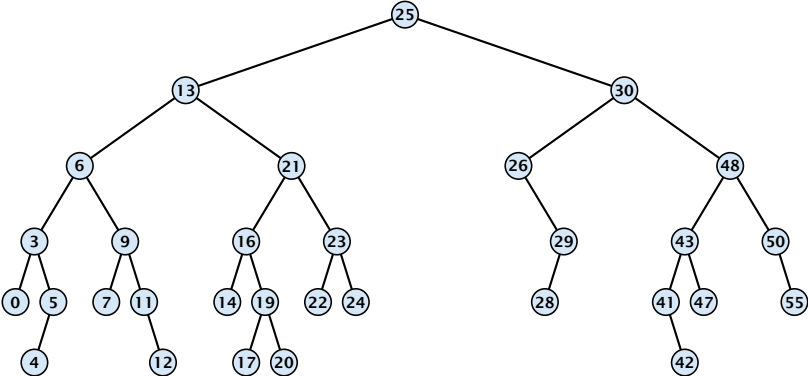


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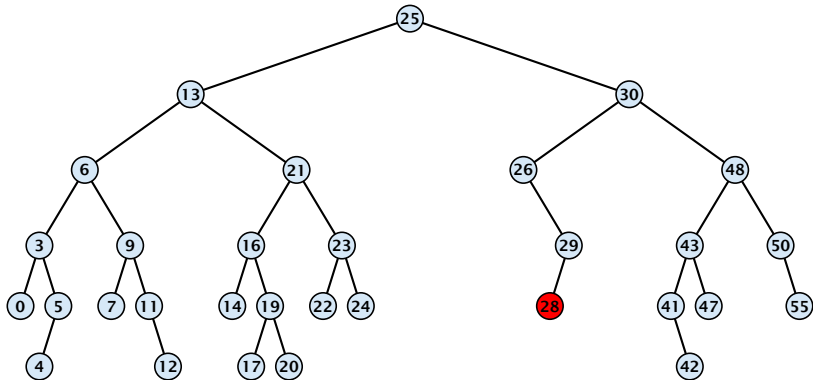
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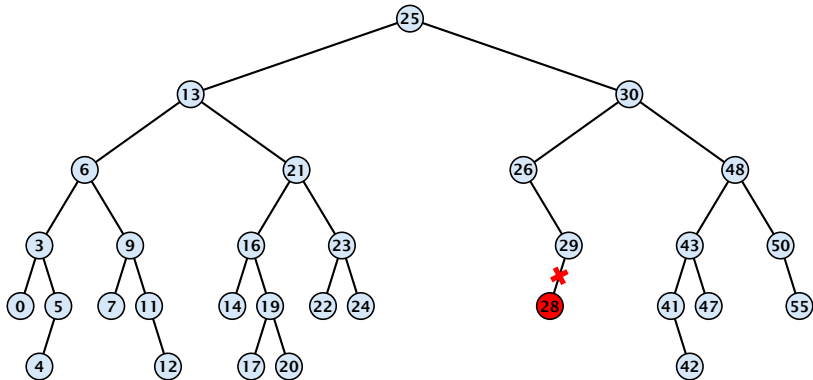


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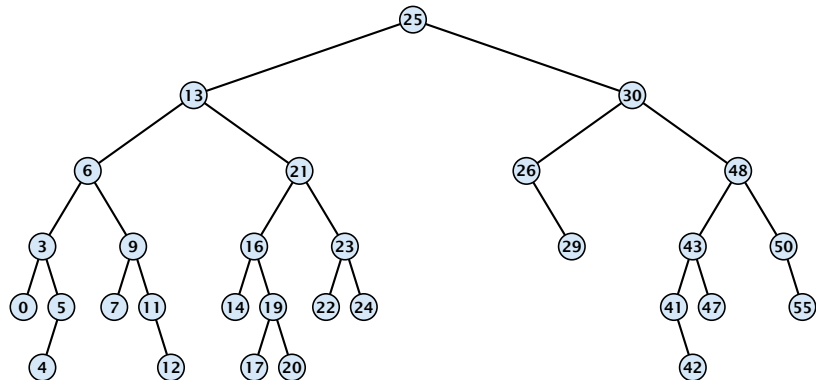


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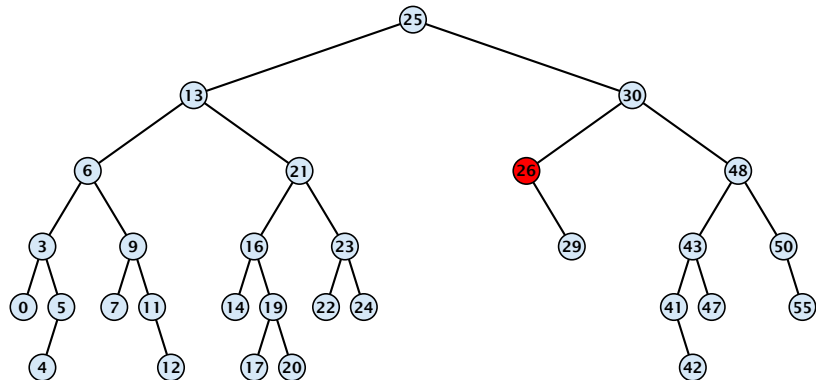


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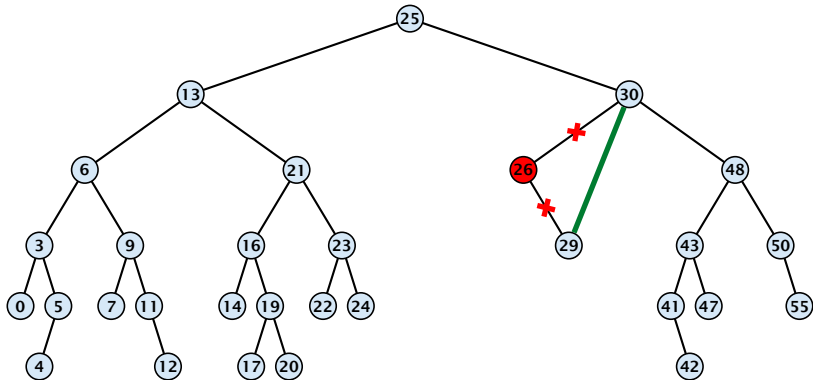


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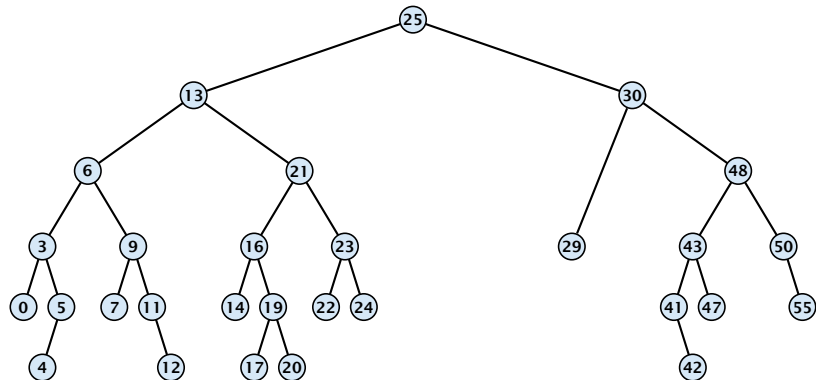


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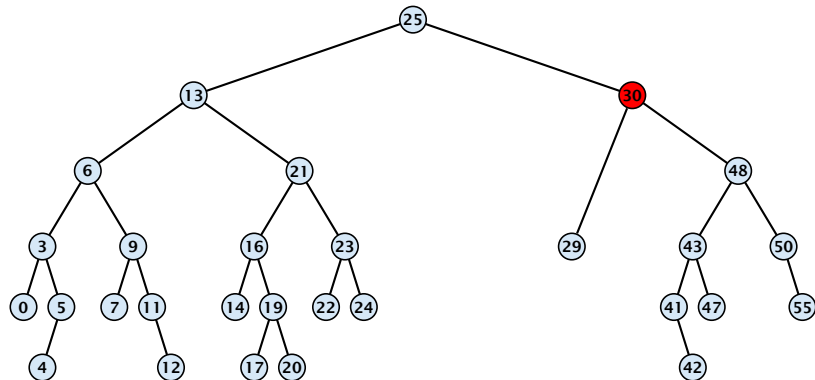


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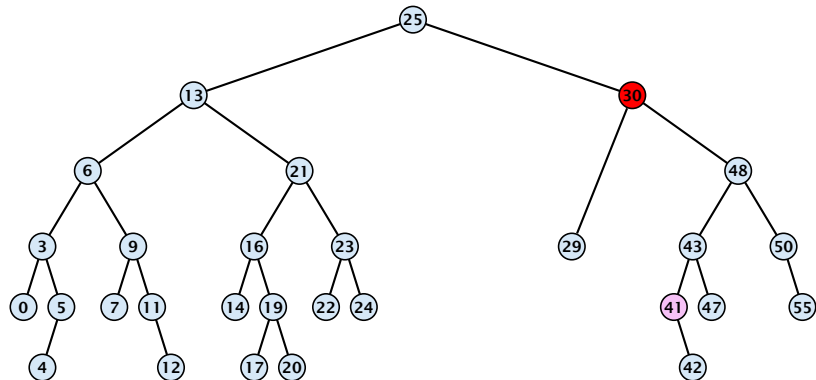


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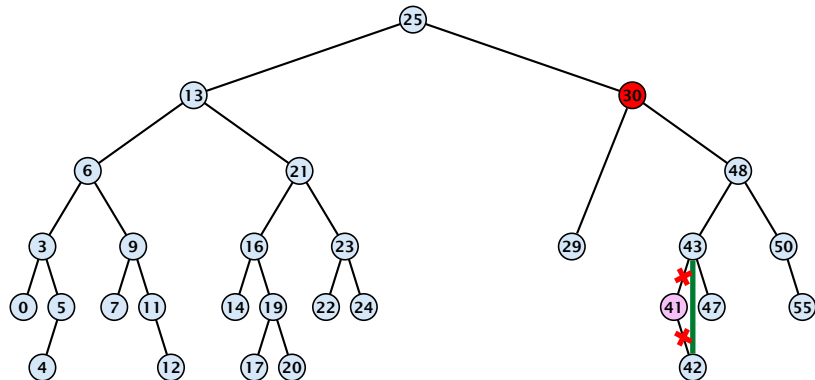


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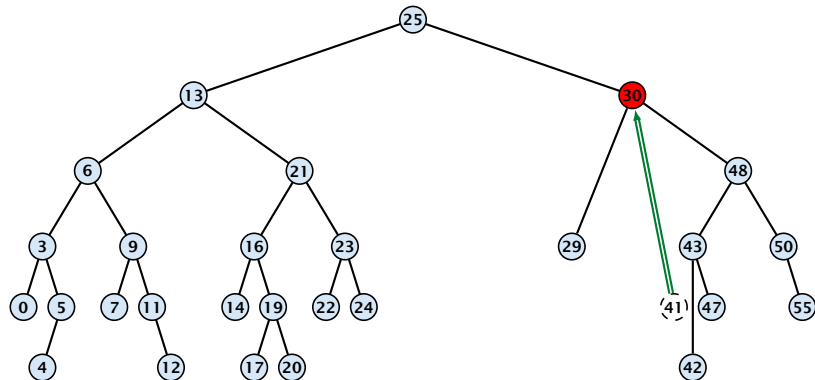


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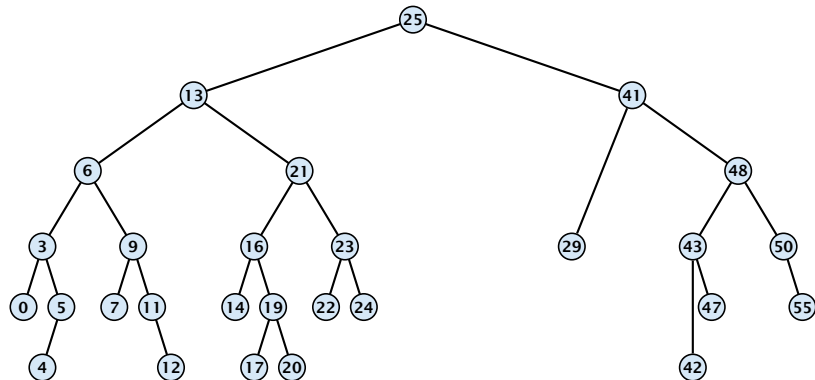


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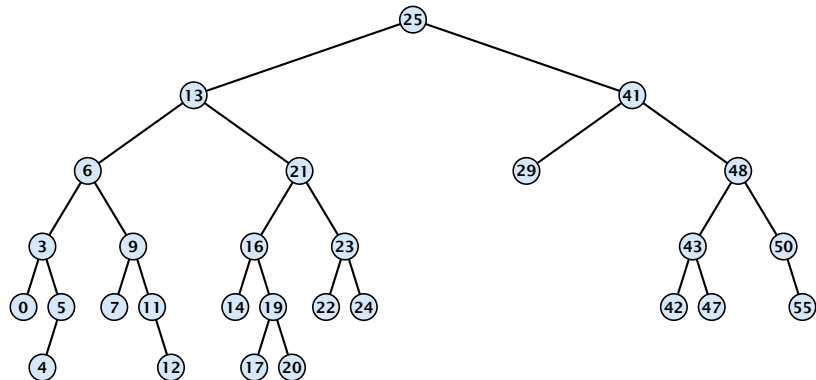


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Algorithm 9 TreeDelete(z)

```
1: if left[ $z$ ] = null or right[ $z$ ] = null
2:   then  $y \leftarrow z$  else  $y \leftarrow \text{TreeSucc}(z)$ ;   select  $y$  to splice out
3:   if left[ $y$ ]  $\neq$  null
4:     then  $x \leftarrow \text{left}[y]$  else  $x \leftarrow \text{right}[y]$ ;  $x$  is child of  $y$  (or null)
5:   if  $x \neq \text{null}$  then parent[ $x$ ]  $\leftarrow$  parent[ $y$ ];   parent[ $x$ ] is correct
6:   if parent[ $y$ ] = null then
7:     root[ $T$ ]  $\leftarrow x$ 
8:   else
9:     if  $y = \text{left}[\text{parent}[y]]$  then
10:      left[parent[ $y$ ]]  $\leftarrow x$ 
11:    else
12:      right[parent[ $y$ ]]  $\leftarrow x$ 
13:   if  $y \neq z$  then copy  $y$ -data to  $z$ 
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} fix pointer to x

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AVL-trees, Red-black trees, Scapegoat trees, 2-3 trees, B-trees, AA trees, Treaps

similar: SPLAY trees.

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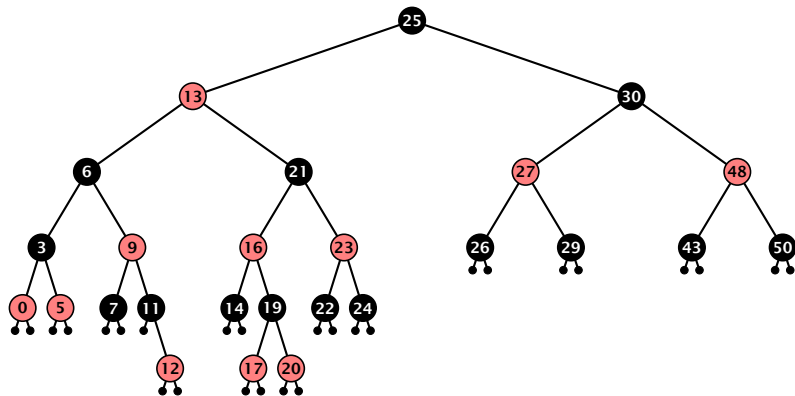
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Red Black Trees: Example



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We first show:

Lemma 4

A sub-tree of black height $\text{bh}(v)$ in a red black tree contains at least $2^{\text{bh}(v)} - 1$ internal vertices.

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- ▶ Then T_v contains at least $2(2^{\text{bh}(v)-1} - 1) + 1 \geq 2^{\text{bh}(v)} - 1$ vertices.



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Hence, $h \leq 2 \log(n + 1) = \mathcal{O}(\log n)$. □

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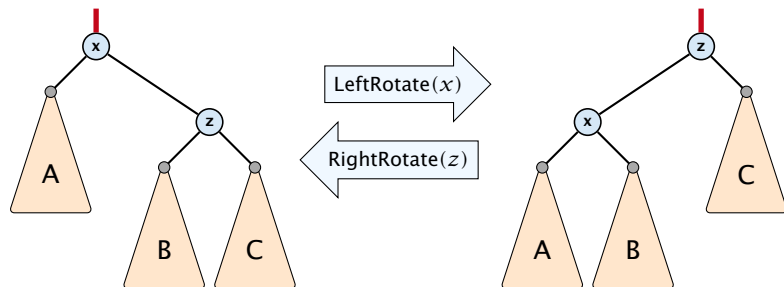
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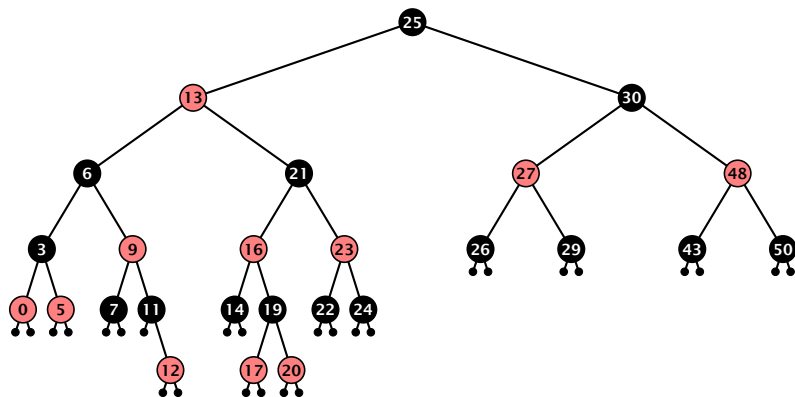
We need to adapt the insert and delete operations so that the red black properties are maintained.

Rotations

The properties will be maintained through rotations:



Red Black Trees: Insert

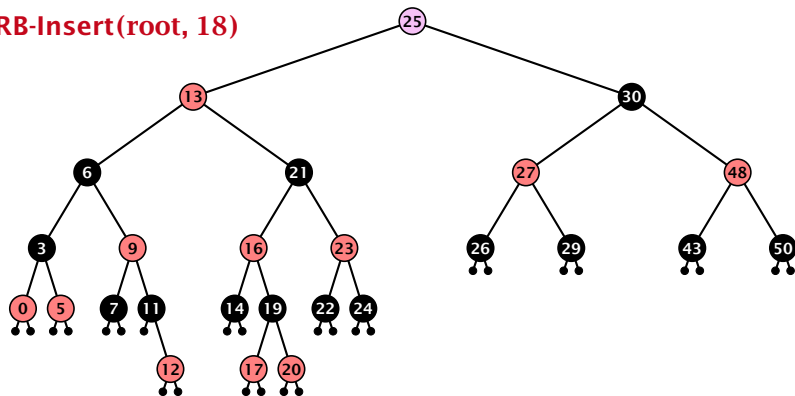


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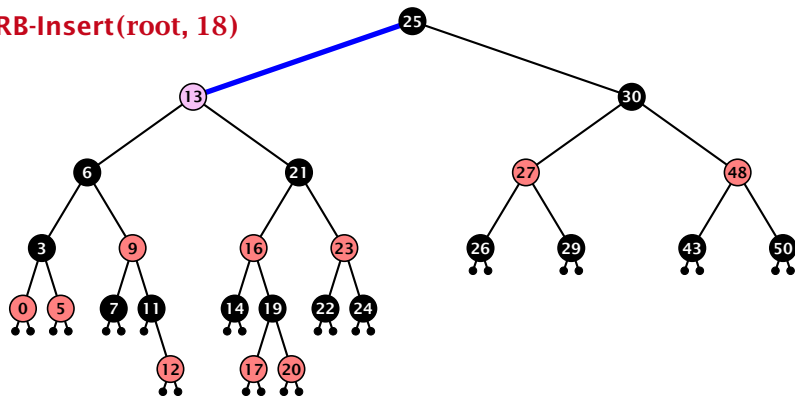


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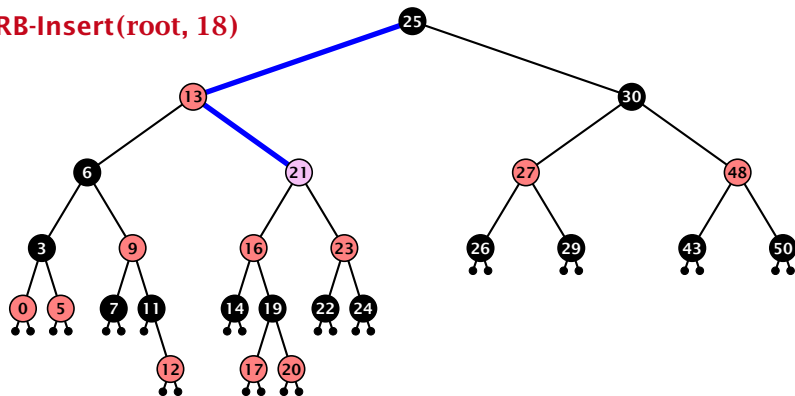


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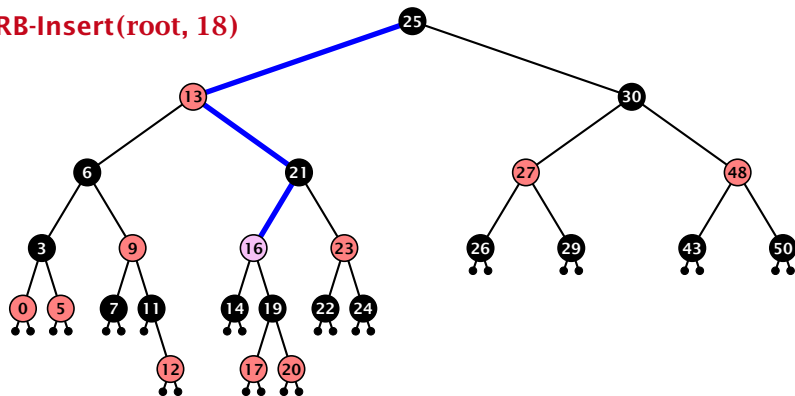


Insert:

- ▶ first make a normal insert into a binary search tree
- ▶ then fix red-black properties

Red Black Trees: Insert

RB-Insert(root, 18)

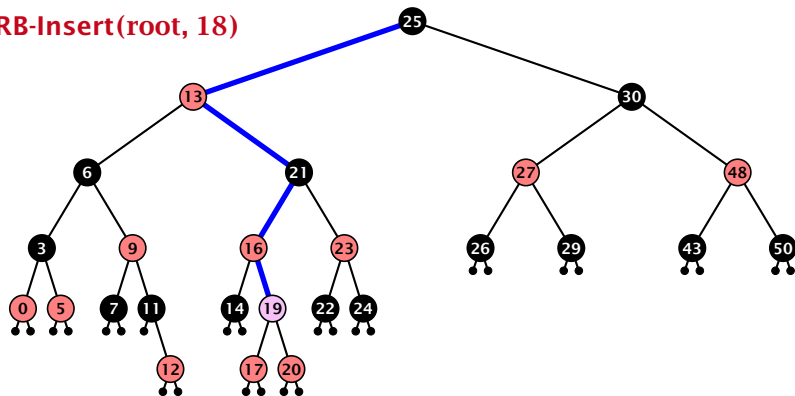


Insert:

- ▶ first make a normal insert into a binary search tree
- ▶ then fix red-black properties

Red Black Trees: Insert

RB-Insert(root, 18)

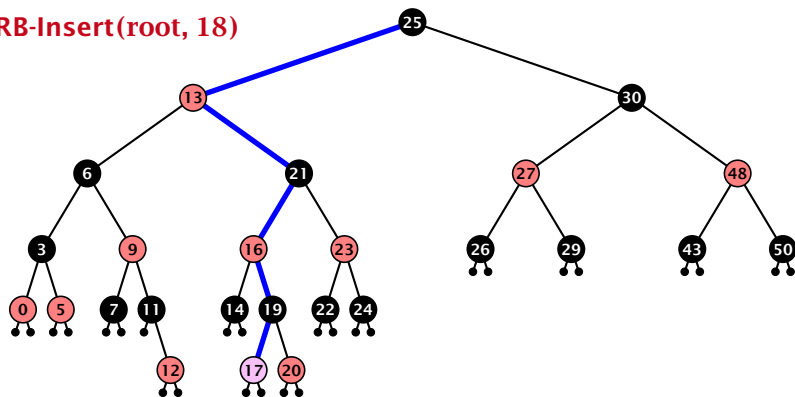


Insert:

- ▶ first make a normal insert into a binary search tree
- ▶ then fix red-black properties

Red Black Trees: Insert

RB-Insert(root, 18)

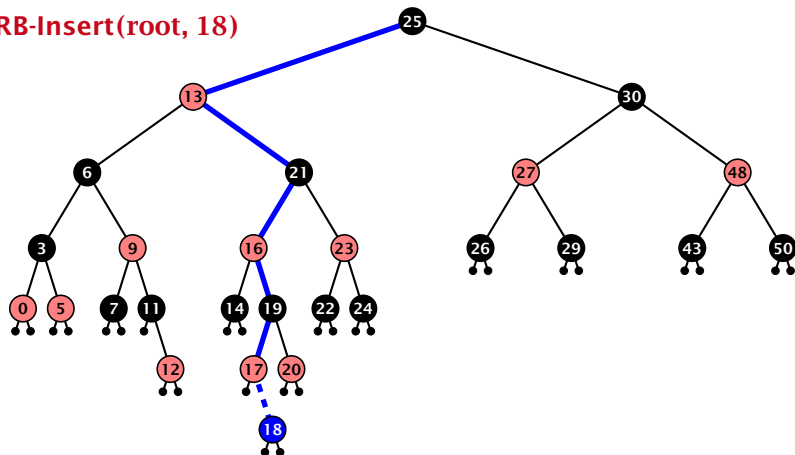


Insert:

- ▶ first make a normal insert into a binary search tree
- ▶ then fix red-black properties

Red Black Trees: Insert

RB-Insert(root, 18)

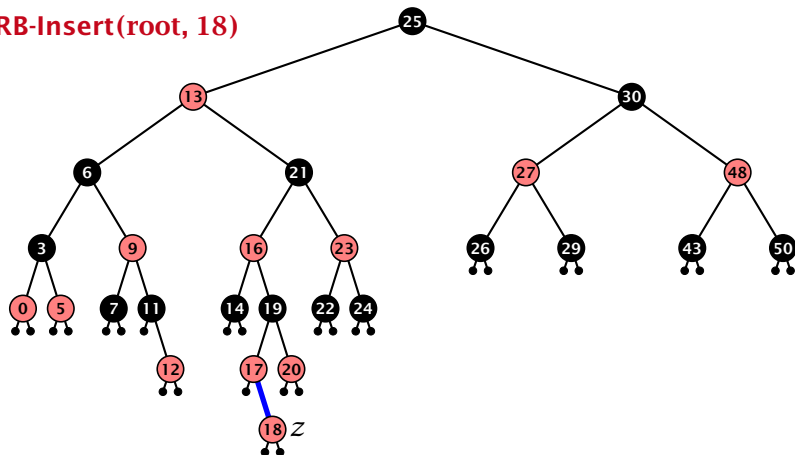


Insert:

- ▶ first make a normal insert into a binary search tree
- ▶ then fix red-black properties

Red Black Trees: Insert

RB-Insert(root, 18)



Insert:

- ▶ first make a normal insert into a binary search tree
- ▶ then fix red-black properties

Red Black Trees: Insert

Invariant of the fix-up algorithm:

- ▶ z is a red node

Red Black Trees: Insert

Invariant of the fix-up algorithm:

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- ▶ the black-height property is fulfilled at every node

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Red Black Trees: Insert

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(most important case)
 - ▶ or the parent does not exist
(violation since root must be black)

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Invariant of the fix-up algorithm:

- ▶ z is a red node
- ▶ the black-height property is fulfilled at every node
- ▶ the only violation of red-black properties occurs at z and $\text{parent}[z]$
 - ▶ either both of them are red (most important case)
 - ▶ or the parent does not exist (violation since root must be black)

If z has a parent but no grand-parent we could simply color the parent/root black; however this case never happens.

Red Black Trees: Insert

Algorithm 10 InsertFix(z)

```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
2:   if parent[ $z$ ] = left[gp[ $z$ ]] then
3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
7:     else
8:       if  $z$  = right[parent[ $z$ ]] then
9:          $z \leftarrow$  p[ $z$ ]; LeftRotate( $z$ );
10:      col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red;
11:      RightRotate(gp[ $z$ ]);
12:     else same as then-clause but right and left exchanged
13: col(root[ $T$ ])  $\leftarrow$  black;
```

Red Black Trees: Insert

Algorithm 10 InsertFix(z)

```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
2:   if parent[ $z$ ] = left[gp[ $z$ ]] then  $z$  in left subtree of grandparent
3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
7:     else
8:       if  $z$  = right[parent[ $z$ ]] then
9:          $z \leftarrow$  p[ $z$ ]; LeftRotate( $z$ );
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Red Black Trees: Insert

Algorithm 10 InsertFix(z)

```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
2:   if parent[ $z$ ] = left[gp[ $z$ ]] then
3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then Case 1: uncle red
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
7:     else
8:       if  $z$  = right[parent[ $z$ ]] then
9:          $z \leftarrow$  p[ $z$ ]; LeftRotate( $z$ );
10:      col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red;
11:      RightRotate(gp[ $z$ ]);
12:     else same as then-clause but right and left exchanged
13: col(root[ $T$ ])  $\leftarrow$  black;
```

Red Black Trees: Insert

Algorithm 10 InsertFix(z)

```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
2:   if parent[ $z$ ] = left[gp[ $z$ ]] then
3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
7:   else Case 2: uncle black
8:     if  $z$  = right[parent[ $z$ ]] then
9:        $z \leftarrow$  p[ $z$ ]; LeftRotate( $z$ );
10:    col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red;
11:    RightRotate(gp[ $z$ ]);
12:   else same as then-clause but right and left exchanged
13: col(root[ $T$ ])  $\leftarrow$  black;
```

Red Black Trees: Insert

Algorithm 10 InsertFix(z)

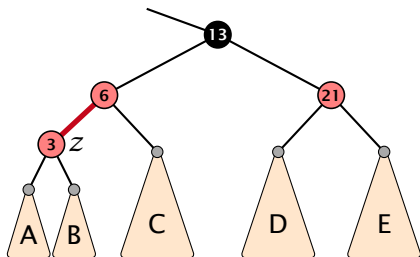
```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
2:   if parent[ $z$ ] = left[gp[ $z$ ]] then
3:     uncle  $\leftarrow$  right[grandparent[ $z$ ]]
4:     if col[uncle] = red then
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[u]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
7:     else
8:       if  $z$  = right[parent[ $z$ ]] then 2a:  $z$  right child
9:          $z \leftarrow$  p[ $z$ ]; LeftRotate( $z$ );
10:        col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red;
11:        RightRotate(gp[ $z$ ]);
12:       else same as then-clause but right and left exchanged
13: col(root[ $T$ ])  $\leftarrow$  black;
```

Red Black Trees: Insert

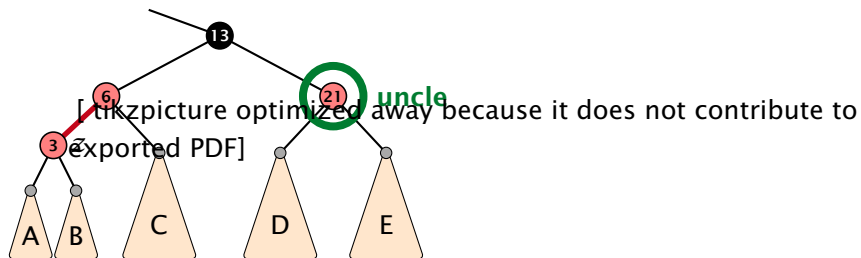
Algorithm 10 InsertFix(z)

```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
2:   if parent[ $z$ ] = left[gp[ $z$ ]] then
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4:     if col[ $uncle$ ] = red then
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6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
7:     else
8:       if  $z$  = right[parent[ $z$ ]] then
9:          $z \leftarrow$  p[ $z$ ]; LeftRotate( $z$ );
10:      col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red; 2b:  $z$  left child
11:      RightRotate(gp[ $z$ ]);
12:     else same as then-clause but right and left exchanged
13: col(root[ $T$ ])  $\leftarrow$  black;
```

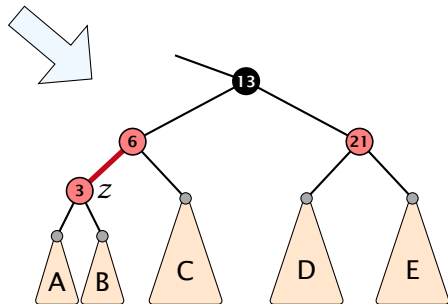
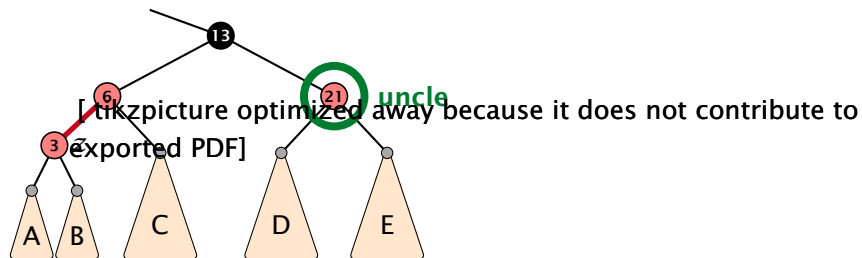
Case 1: Red Uncle



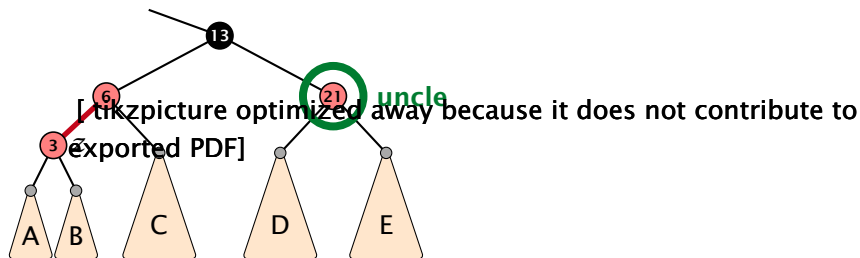
Case 1: Red Uncle



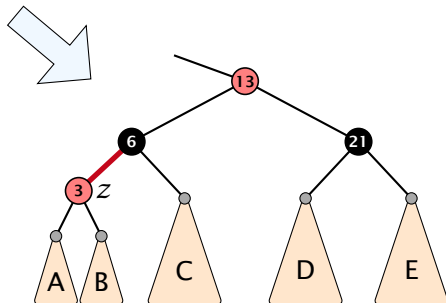
Case 1: Red Uncle



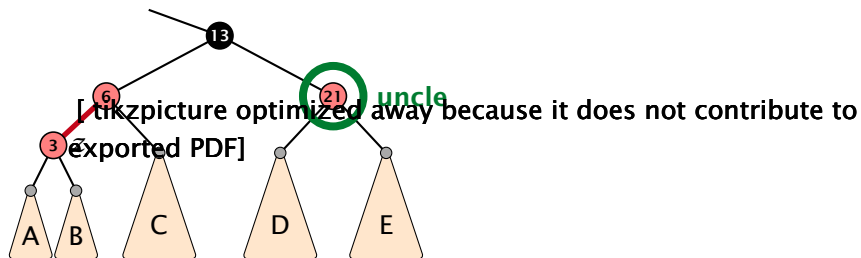
Case 1: Red Uncle



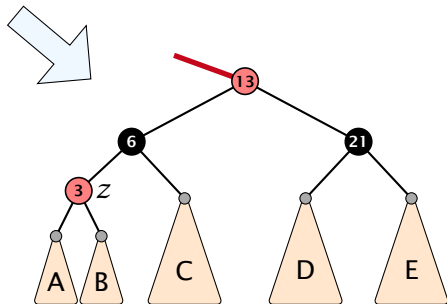
1. recolour



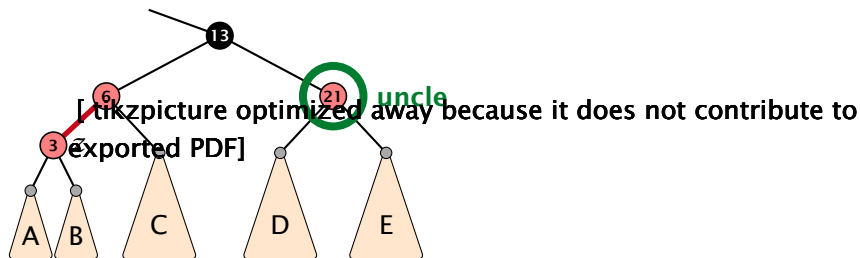
Case 1: Red Uncle



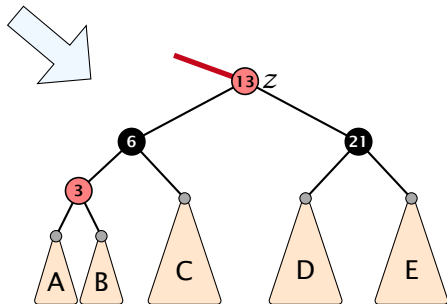
1. recolour



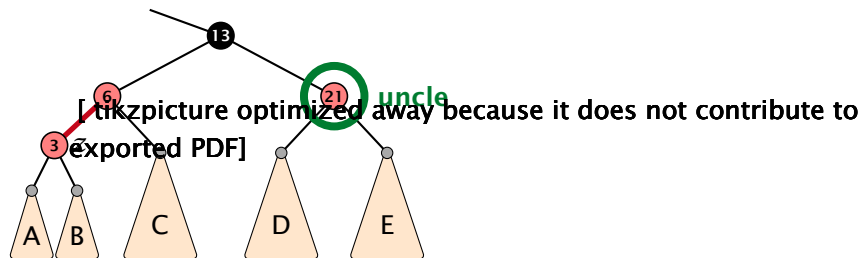
Case 1: Red Uncle



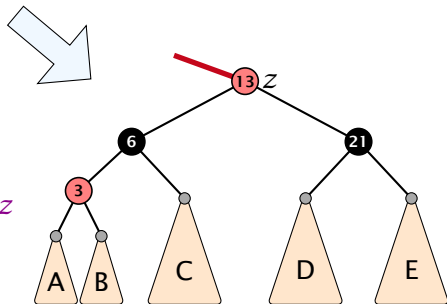
1. recolour
2. move z to grand-parent



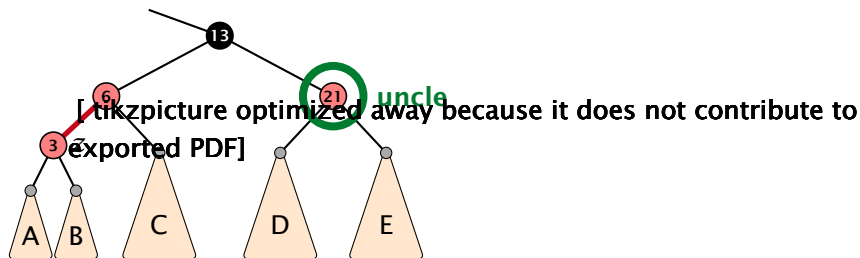
Case 1: Red Uncle



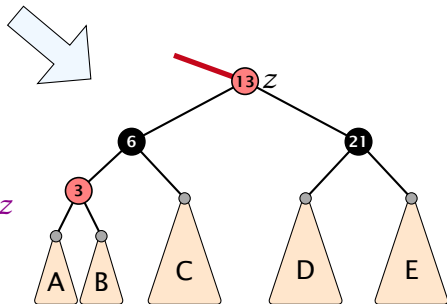
1. recolour
2. move z to grand-parent
3. invariant is fulfilled for new z



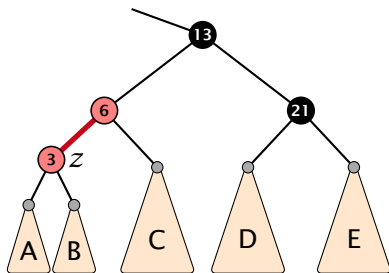
Case 1: Red Uncle



1. recolour
2. move z to grand-parent
3. invariant is fulfilled for new z
4. you made progress

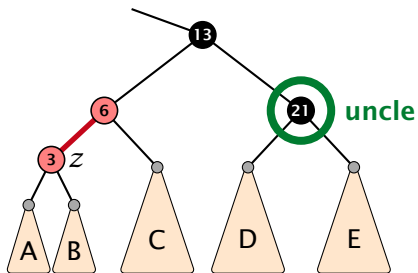


Case 2b: Black uncle and z is left child



Case 2b: Black uncle and z is left child

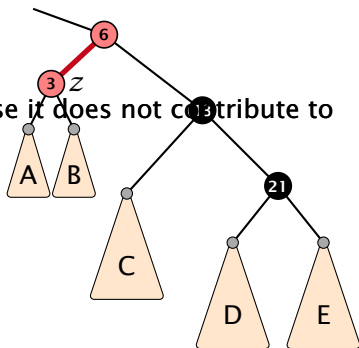
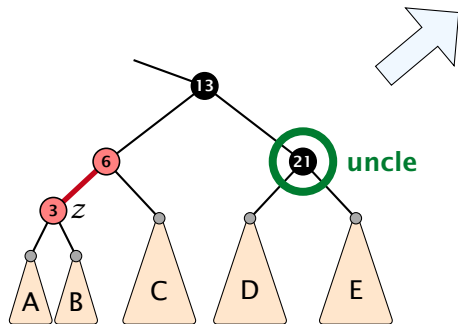
[tikzpicture optimized away because it does not contribute to exported PDF]



Case 2b: Black uncle and z is left child

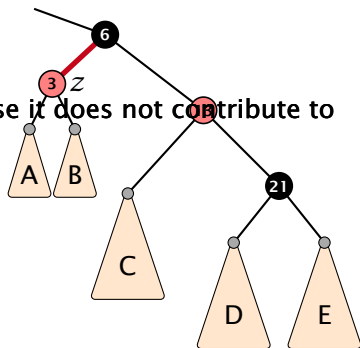
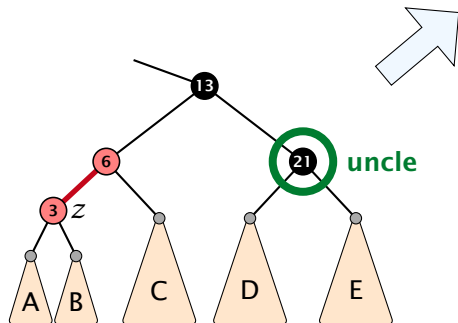
1. rotate around grandparent

[tikzpicture optimized away because it does not contribute to exported PDF]



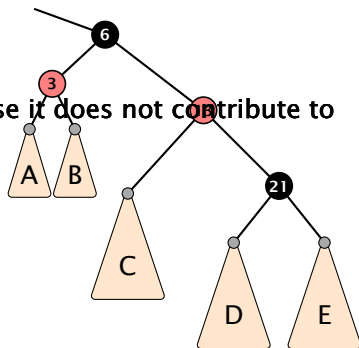
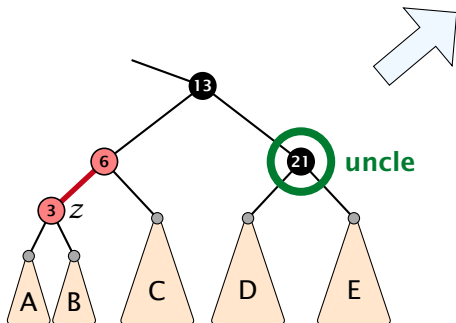
Case 2b: Black uncle and z is left child

1. rotate around grandparent
 2. re-colour to ensure that black height property holds
- [tikzpicture optimized away because it does not contribute to exported PDF]

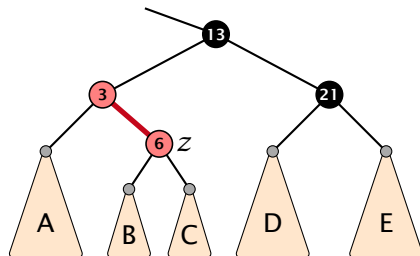


Case 2b: Black uncle and z is left child

1. rotate around grandparent
2. re-colour to ensure that black height property holds
[tikzpicture optimized away because it does not contribute to exported PDF]
3. you have a red black tree

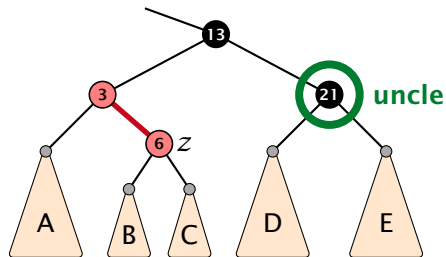


Case 2a: Black uncle and z is right child



Case 2a: Black uncle and z is right child

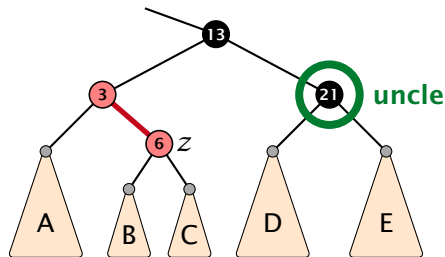
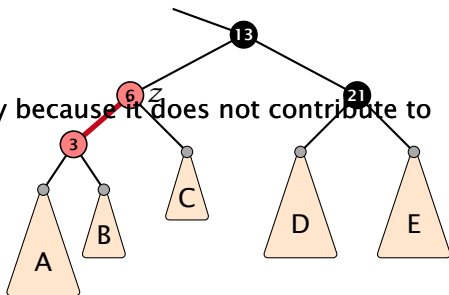
[tikzpicture optimized away because it does not contribute to exported PDF]



Case 2a: Black uncle and z is right child

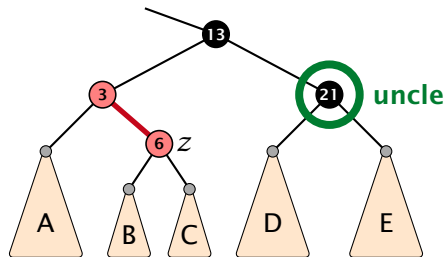
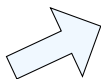
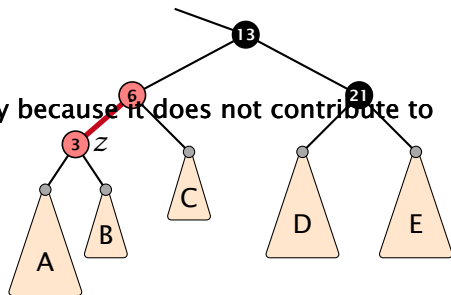
1. rotate around parent

[tikzpicture optimized away because it does not contribute to exported PDF]



Case 2a: Black uncle and z is right child

1. rotate around parent
2. move z downwards
[tikzpicture optimized away because it does not contribute to exported PDF]

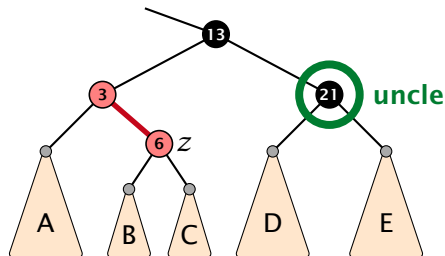
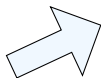
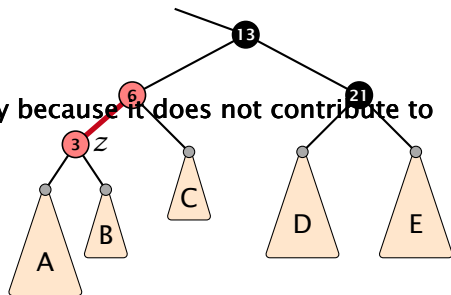


Case 2a: Black uncle and z is right child

1. rotate around parent

2. move z downwards

3. exported PDF



Red Black Trees: Insert

Running time:

- ▶ Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.

Red Black Trees: Insert

Running time:

- ▶ Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- ▶ Case 2a \rightarrow Case 2b \rightarrow red-black tree

Red Black Trees: Insert

Running time:

- ▶ Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- ▶ Case 2a → Case 2b → red-black tree
- ▶ Case 2b → red-black tree

Red Black Trees: Insert

Running time:

- ▶ Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- ▶ Case 2a \rightarrow Case 2b \rightarrow red-black tree
- ▶ Case 2b \rightarrow red-black tree

Performing Case 1 at most $\mathcal{O}(\log n)$ times and every other case at most once, we get a red-black tree. Hence $\mathcal{O}(\log n)$ re-colorings and at most 2 rotations.

Red Black Trees: Delete



Red Black Trees: Delete

First do a standard delete.

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If the spliced out node x was red everything is fine.

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If it was black there may be the following problems.

- ▶ Parent and child of x were red; two adjacent red vertices.

Red Black Trees: Delete

First do a standard delete.

If the spliced out node x was red everything is fine.

If it was black there may be the following problems.

- ▶ Parent and child of x were red; two adjacent red vertices.
- ▶ If you delete the root, the root may now be red.

Red Black Trees: Delete

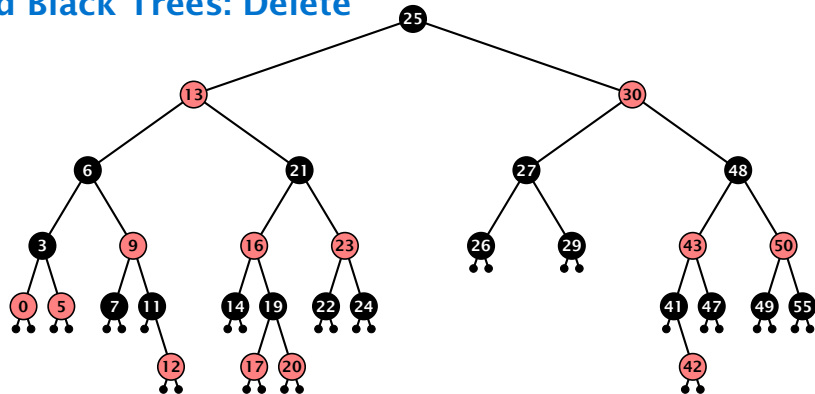
First do a standard delete.

If the spliced out node x was red everything is fine.

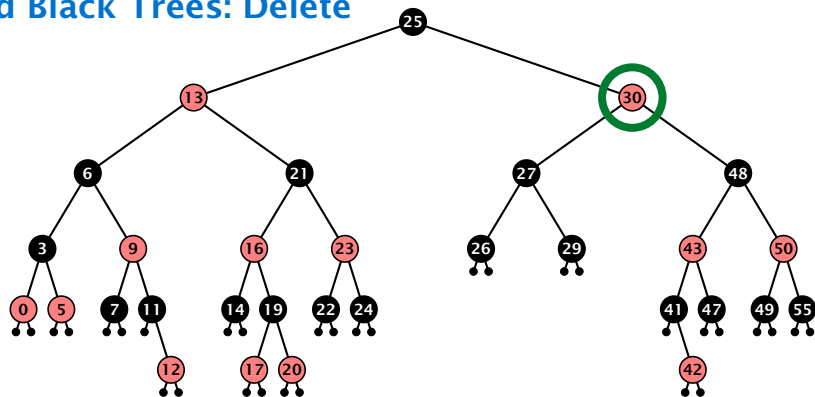
If it was black there may be the following problems.

- ▶ Parent and child of x were red; two adjacent red vertices.
- ▶ If you delete the root, the root may now be red.
- ▶ Every path from an ancestor of x to a descendant leaf of x changes the number of black nodes. Black height property might be violated.

Red Black Trees: Delete



Red Black Trees: Delete

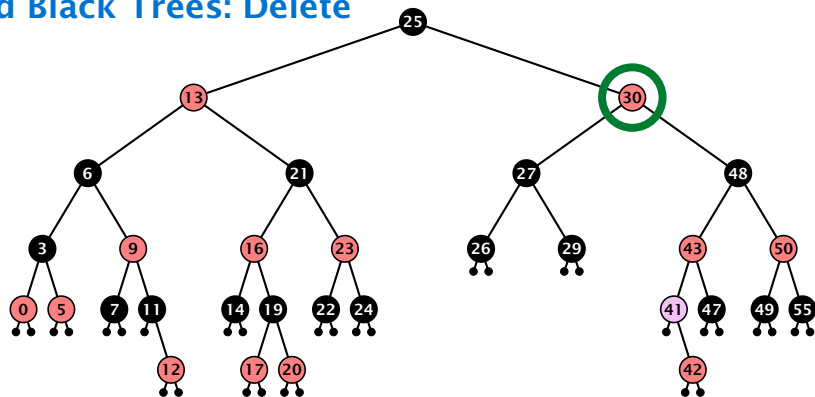


Case 3:

Element has two children

- ▶ do normal delete
- ▶ when replacing content by content of successor, don't change color of node

Red Black Trees: Delete

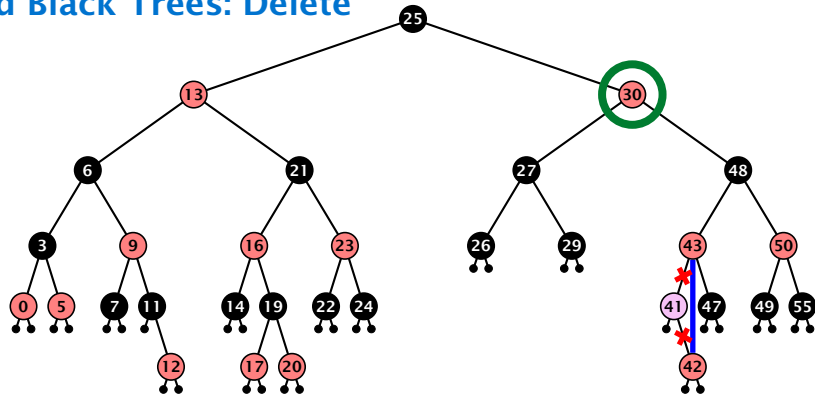


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Red Black Trees: Delete

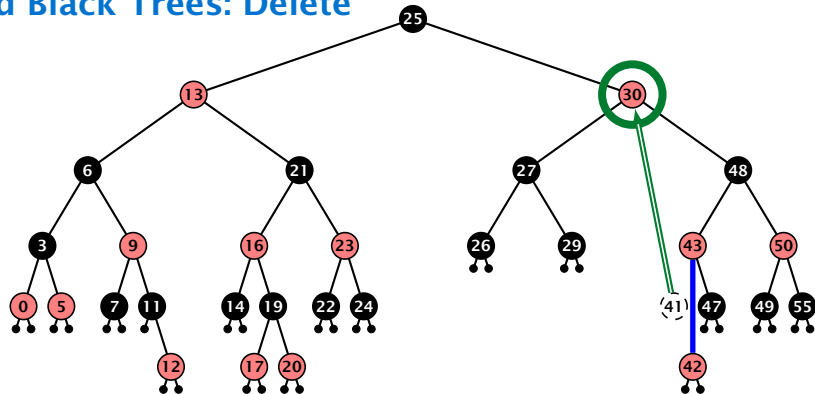


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Red Black Trees: Delete

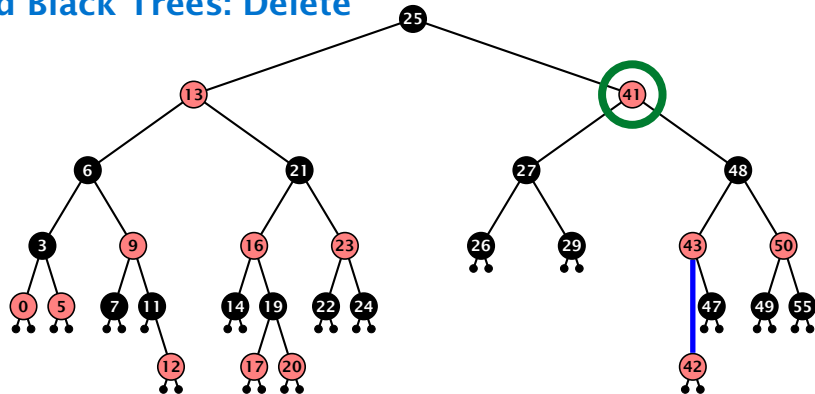


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Red Black Trees: Delete

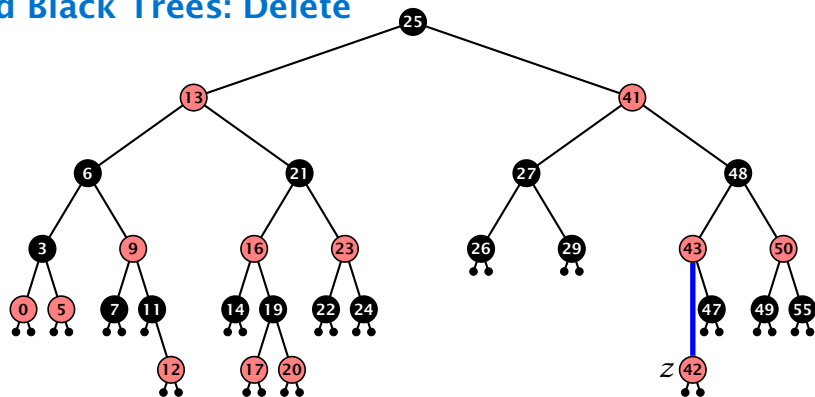


Case 3:

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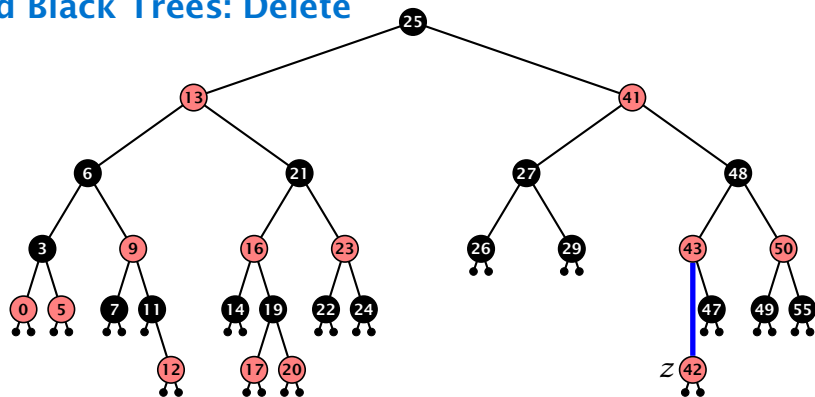
Red Black Trees: Delete



Delete:

- ▶ deleting black node messes up black-height property

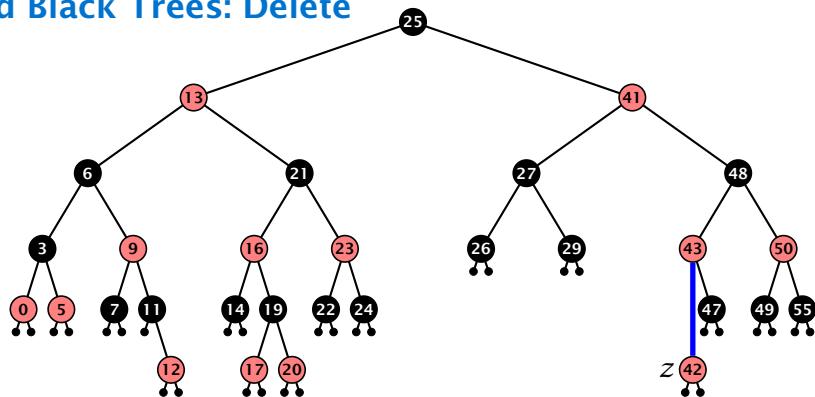
Red Black Trees: Delete



Delete:

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- ▶ if z is red, we can simply color it black and everything is fine

Red Black Trees: Delete



Delete:

- ▶ deleting black node messes up black-height property
- ▶ if z is red, we can simply color it black and everything is fine
- ▶ the problem is if z is black (e.g. a dummy-leaf); we call a fix-up procedure to fix the problem.

Red Black Trees: Delete

Invariant of the fix-up algorithm

- ▶ the node z is black

Red Black Trees: Delete

Invariant of the fix-up algorithm

- ▶ the node z is black
- ▶ if we “assign” a fake black unit to the edge from z to its parent then the black-height property is fulfilled

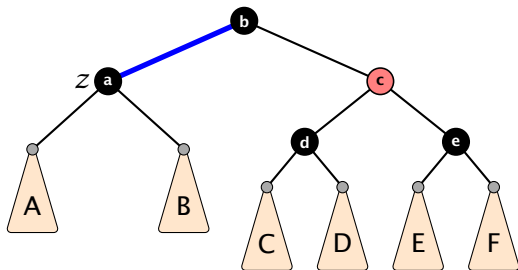
Red Black Trees: Delete

Invariant of the fix-up algorithm

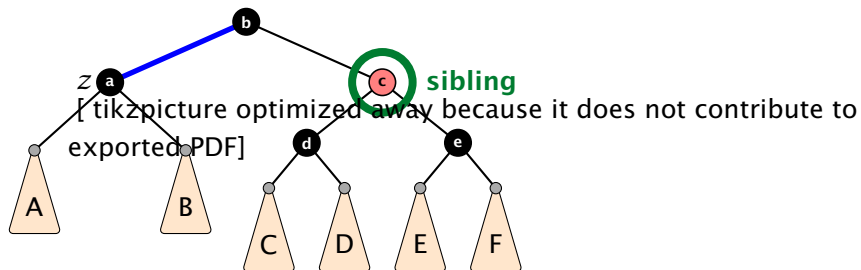
- ▶ the node z is black
- ▶ if we “assign” a fake black unit to the edge from z to its parent then the black-height property is fulfilled

Goal: make rotations in such a way that you at some point can remove the fake black unit from the edge.

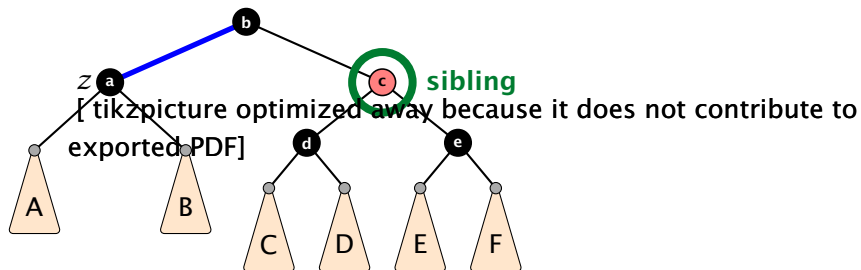
Case 1: Sibling of z is red



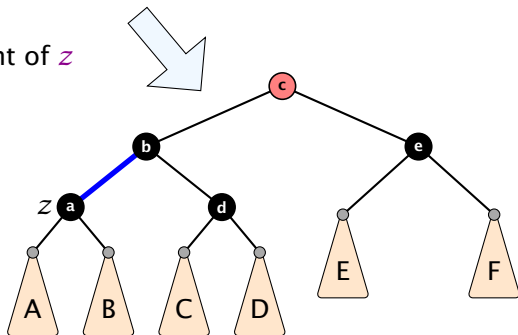
Case 1: Sibling of z is red



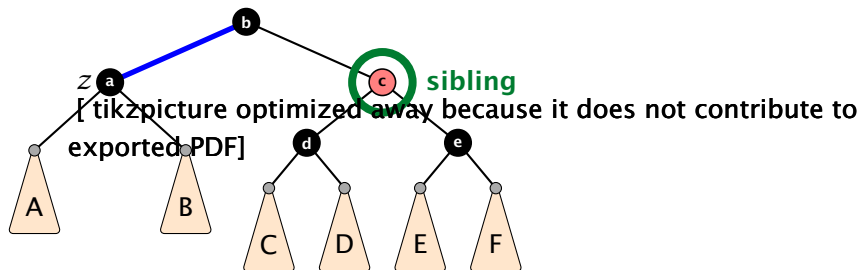
Case 1: Sibling of z is red



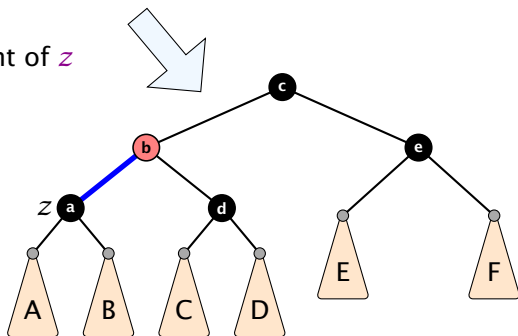
1. left-rotate around parent of z



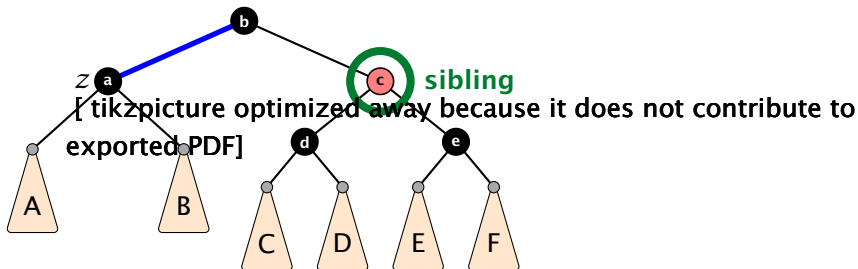
Case 1: Sibling of z is red



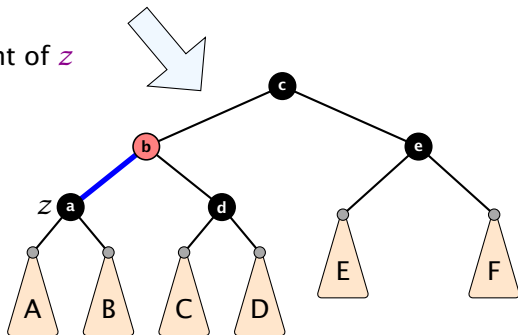
1. left-rotate around parent of z
2. recolor nodes b and c



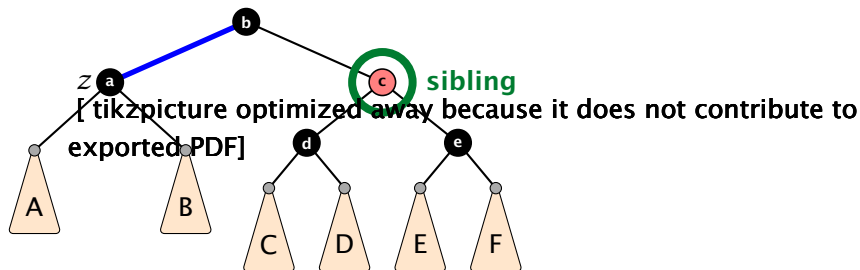
Case 1: Sibling of z is red



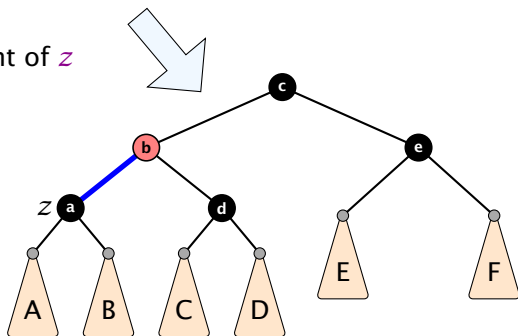
1. left-rotate around parent of z
2. recolor nodes b and c
3. the new sibling is black (and parent of z is red)



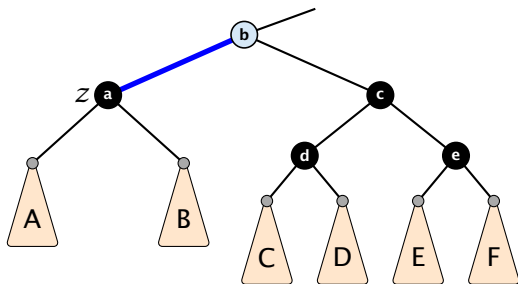
Case 1: Sibling of z is red



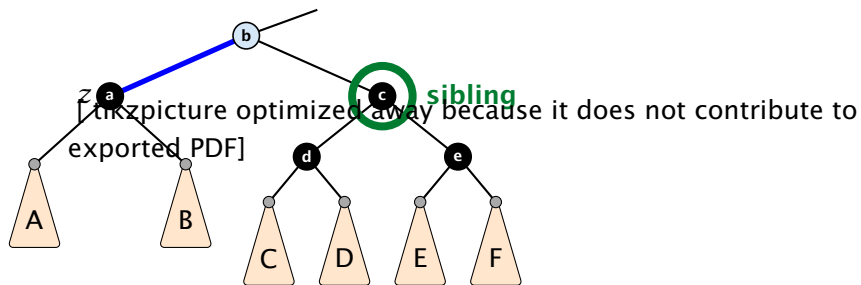
1. left-rotate around parent of z
2. recolor nodes b and c
3. the new sibling is black (and parent of z is red)
4. Case 2 (special), or Case 3, or Case 4



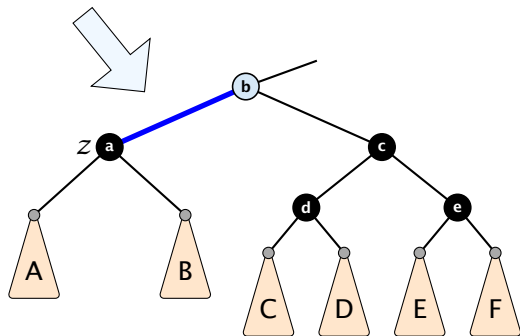
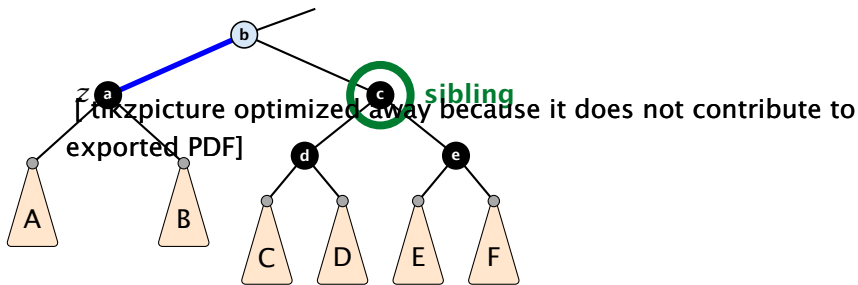
Case 2: Sibling is black with two black children



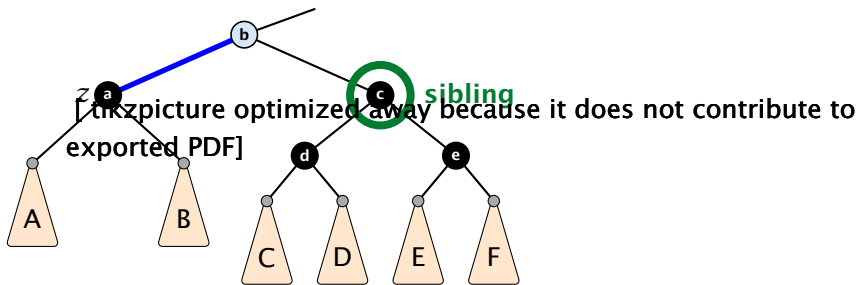
Case 2: Sibling is black with two black children



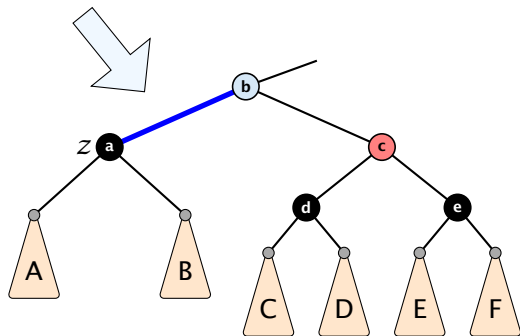
Case 2: Sibling is black with two black children



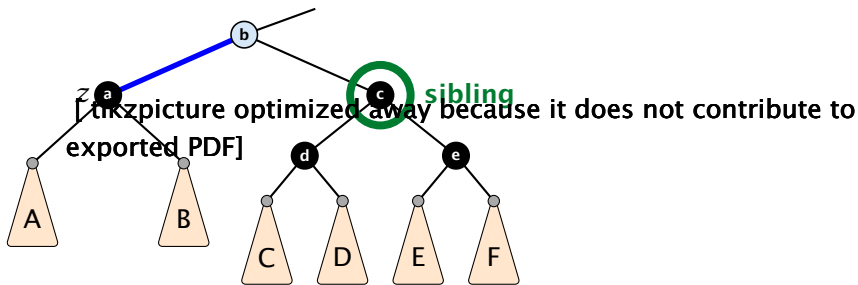
Case 2: Sibling is black with two black children



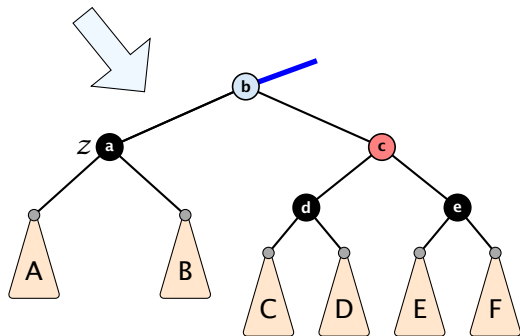
1. re-color node *c*



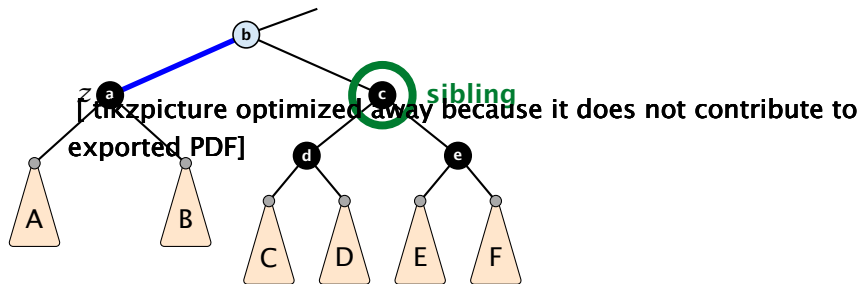
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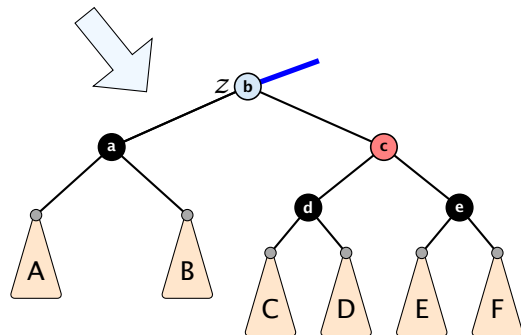
1. re-color node **c**
2. move fake black unit upwards



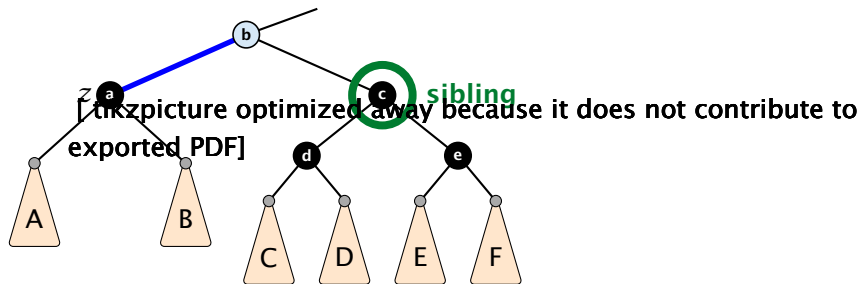
Case 2: Sibling is black with two black children



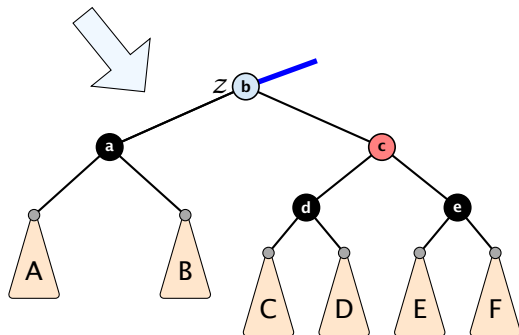
1. re-color node **c**
2. move fake black unit upwards
3. move **z** upwards



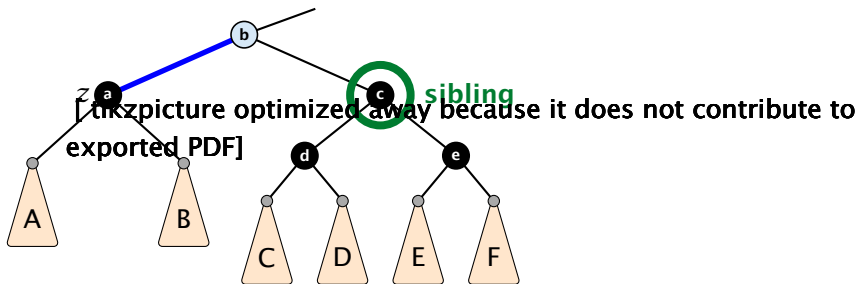
Case 2: Sibling is black with two black children



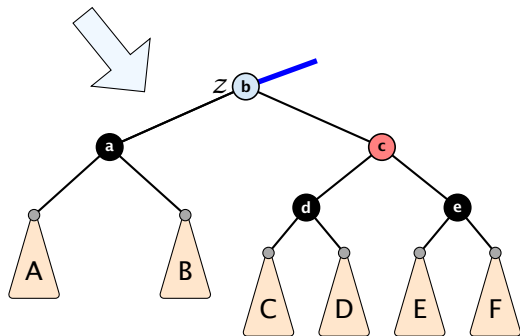
1. re-color node **c**
2. move fake black unit upwards
3. move **z** upwards
4. we made progress



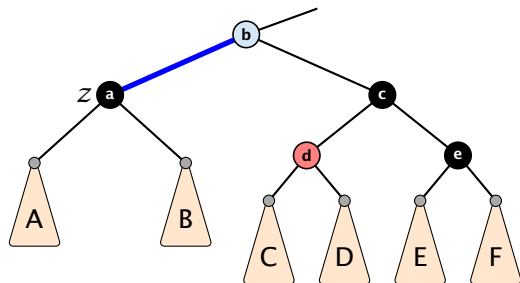
Case 2: Sibling is black with two black children



1. re-color node c
2. move fake black unit upwards
3. move z upwards
4. we made progress
5. if b is red we color it black and are done

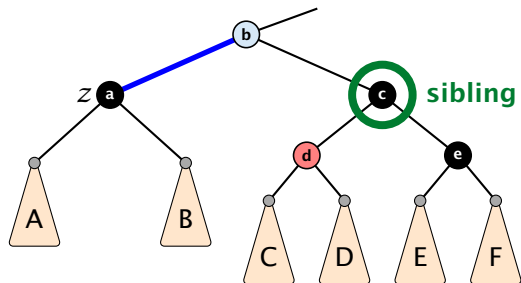


Case 3: Sibling black with one black child to the right



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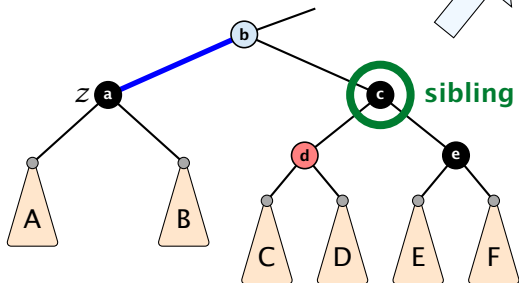
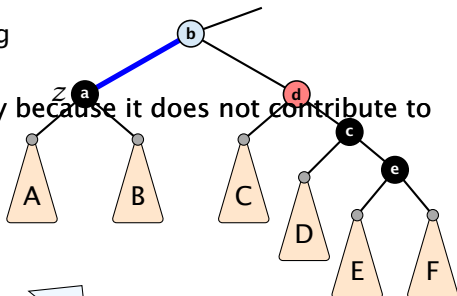
[tikzpicture optimized away because it does not contribute to exported PDF]



Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling

[tikzpicture optimized away because it does not contribute to exported PDF]

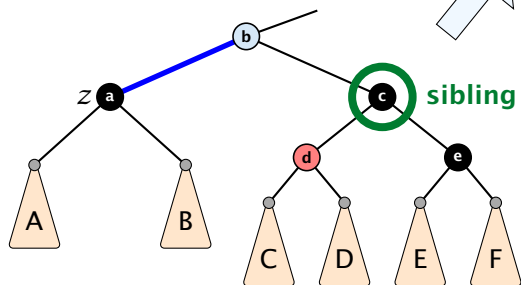
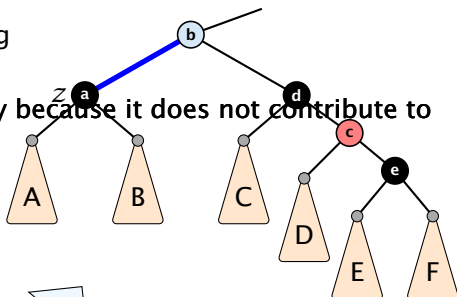


Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling

2. recolor *c* and *d*

[tikzpicture optimized away because it does not contribute to exported PDF]



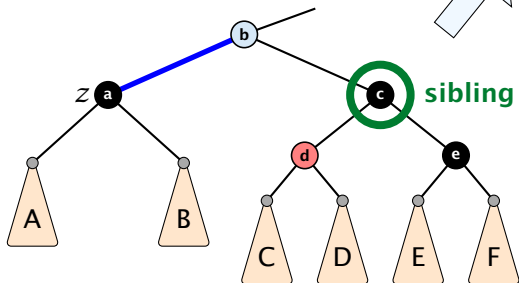
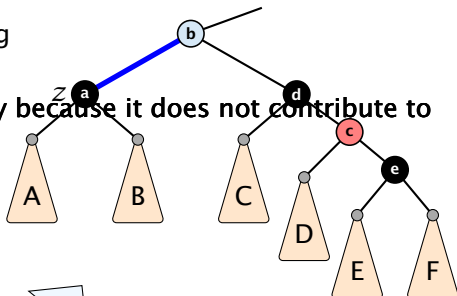
Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling

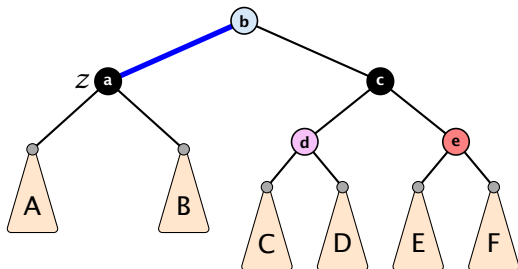
2. recolor *c* and *d*

3. new sibling is black with red right child (Case 4)

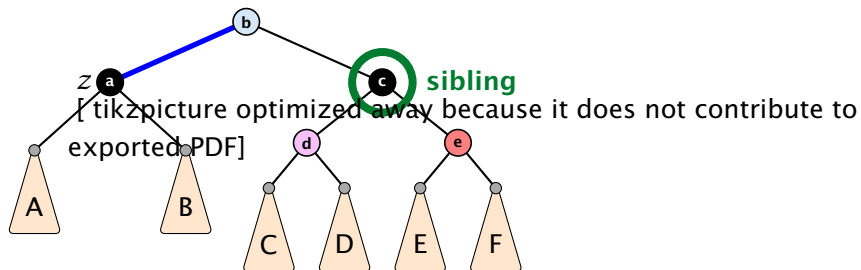
[tikzpicture optimized away because it does not contribute to exported PDF]



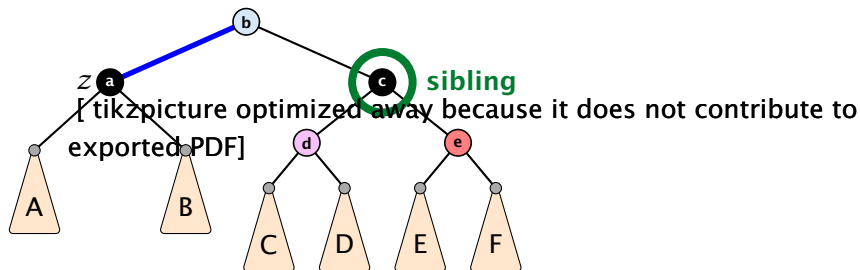
Case 4: Sibling is black with red right child



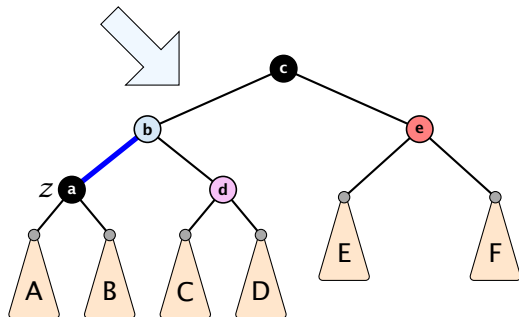
Case 4: Sibling is black with red right child



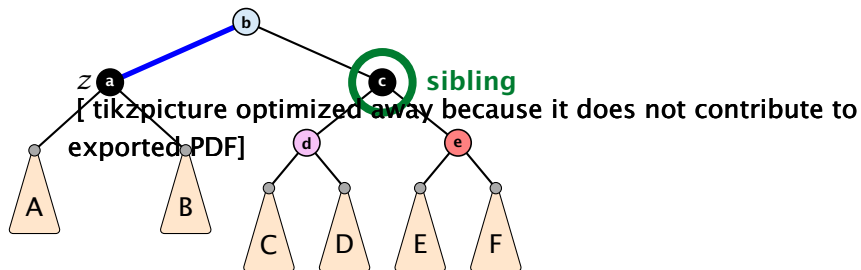
Case 4: Sibling is black with red right child



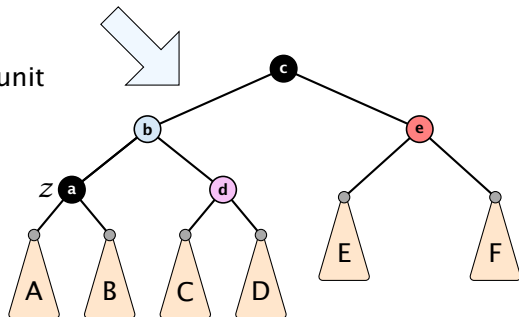
1. left-rotate around **b**



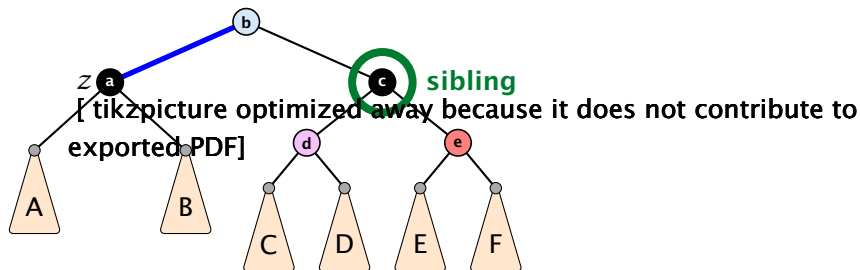
Case 4: Sibling is black with red right child



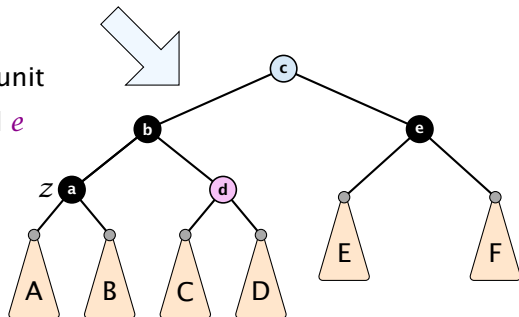
1. left-rotate around *b*
2. remove the fake black unit



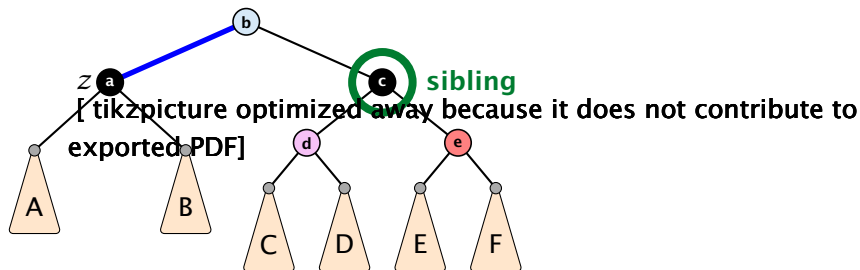
Case 4: Sibling is black with red right child



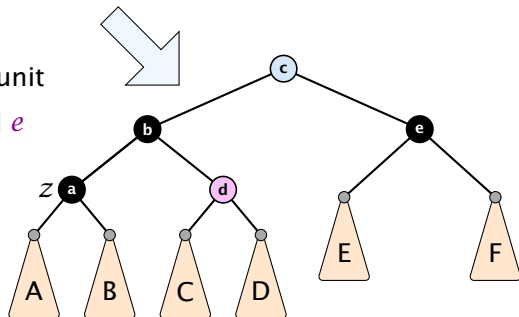
1. left-rotate around b
2. remove the fake black unit
3. recolor nodes b , c , and e



Case 4: Sibling is black with red right child



1. left-rotate around b
2. remove the fake black unit
3. recolor nodes b , c , and e
4. you have a valid red black tree



Running time:

- ▶ only Case 2 can repeat; but only h many steps, where h is the height of the tree

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Running time:

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Case 1 → Case 3 → Case 4 → red black tree
Case 1 → Case 4 → red black tree
- ▶ Case 3 → Case 4 → red black tree
- ▶ Case 4 → red black tree

Performing Case 2 at most $\mathcal{O}(\log n)$ times and every other step at most once, we get a red black tree. Hence, $\mathcal{O}(\log n)$ re-colorings and at most 3 rotations.

Disadvantage of balanced search trees:

Splay Trees

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Splay Trees:

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- only amortized guarantee
- read-operations change the tree

Splay Trees

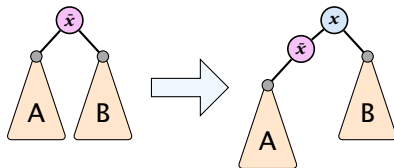
find(x)

- ▶ search for x according to a search tree
- ▶ let \tilde{x} be last element on search-path
- ▶ splay(\tilde{x})

Splay Trees

insert(x)

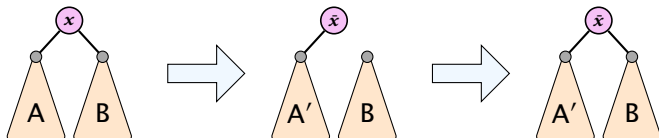
- ▶ search for x ; \bar{x} is last visited element during search (successor or predecessor of x)
- ▶ splay(\bar{x}) moves \bar{x} to the root
- ▶ insert x as new root



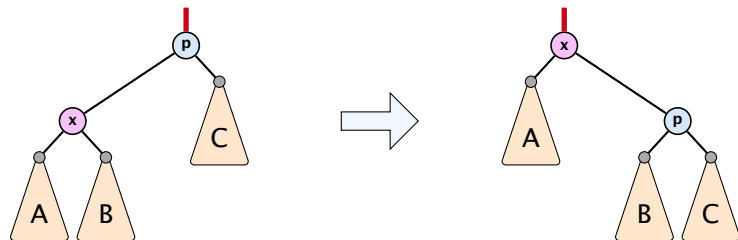
Splay Trees

delete(x)

- ▶ search for x ; splay(x); remove x
- ▶ search largest element \bar{x} in A
- ▶ splay(\bar{x}) (on subtree A)
- ▶ connect root of B as right child of \bar{x}



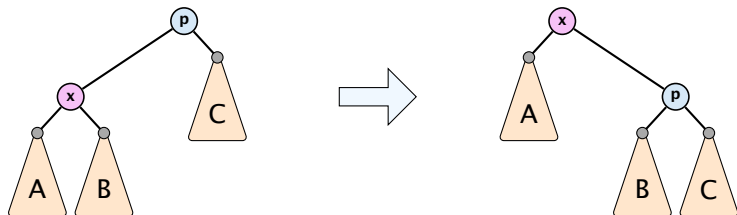
Move to Root



How to bring element to root?

- ▶ one (bad) option: `moveToRoot(x)`
- ▶ iteratively do rotation around parent of x until x is root
- ▶ if x is left child do right rotation otw. left rotation

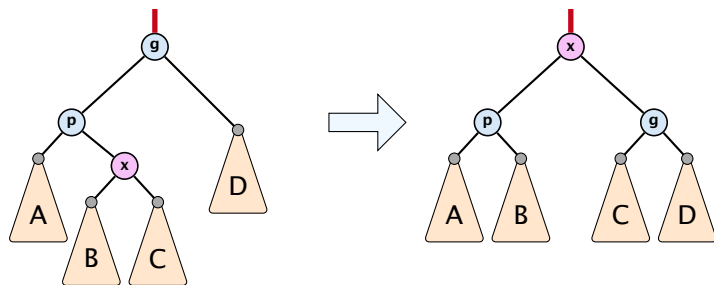
Splay: Zig Case



better option $\text{splay}(x)$:

- ▶ zig case: if x is child of root do left rotation or right rotation around parent

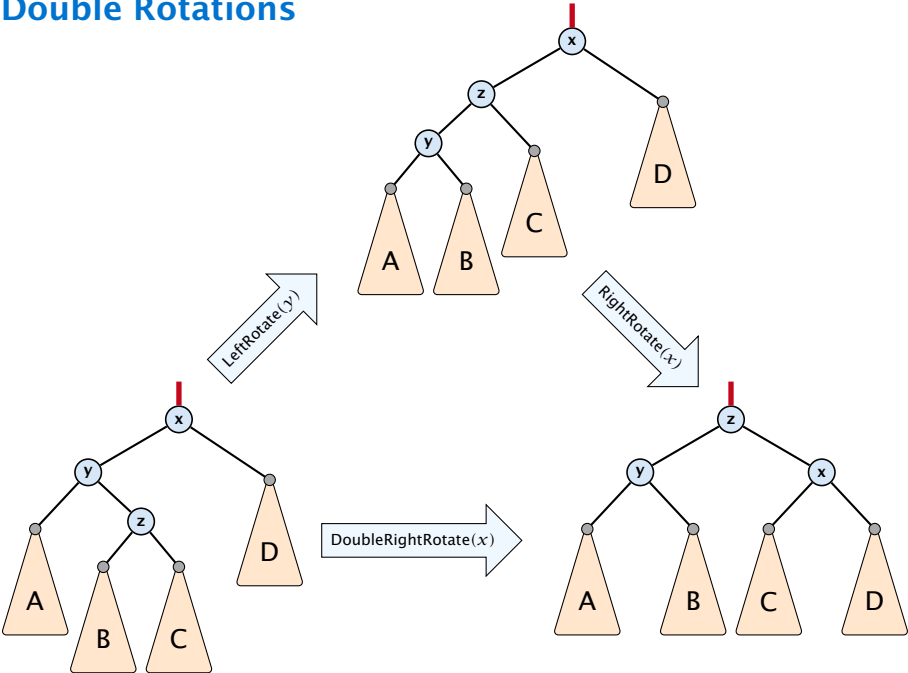
Splay: Zigzag Case



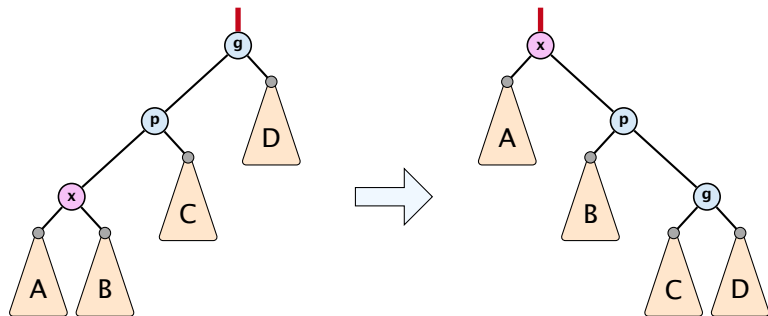
better option $\text{splay}(x)$:

- ▶ zigzag case: if x is right child and parent of x is left child (or x left child parent of x right child)
- ▶ do double right rotation around grand-parent (resp. double left rotation)

Double Rotations



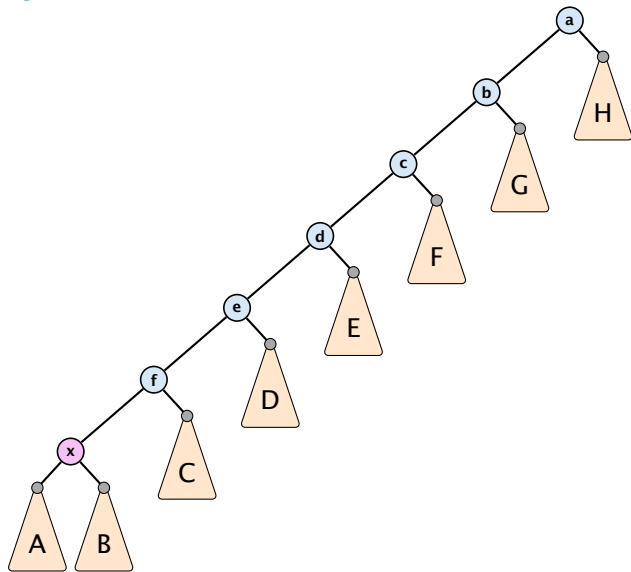
Splay: Zigzig Case



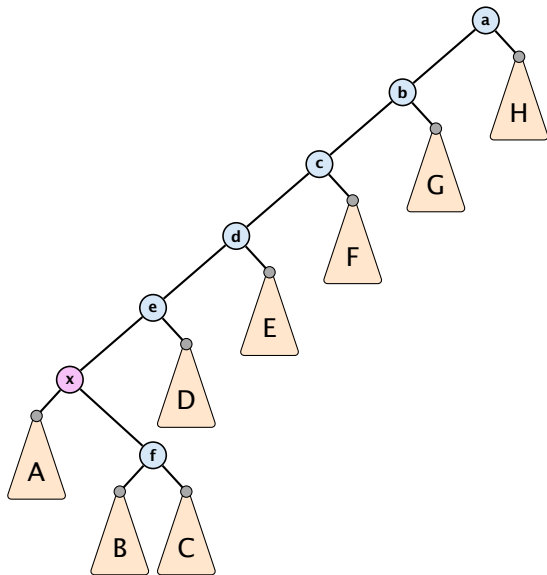
better option $\text{splay}(x)$:

- ▶ zigzig case: if x is left child and parent of x is left child (or x right child, parent of x right child)
- ▶ do right rotation around grand-parent followed by right rotation around parent (resp. left rotations)

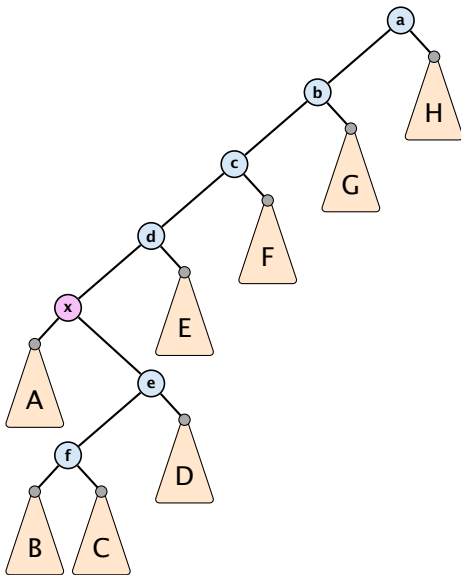
Splay vs. Move to Root



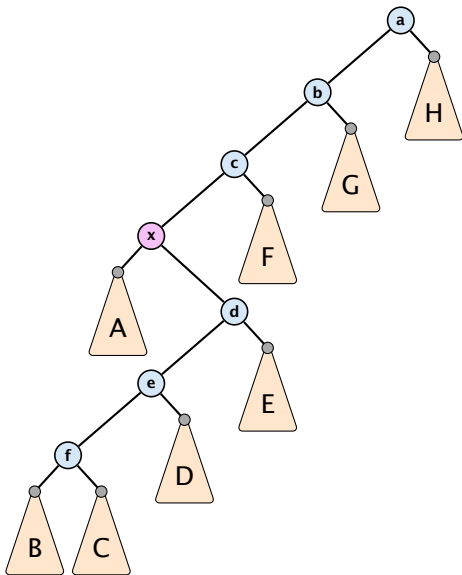
Splay vs. Move to Root



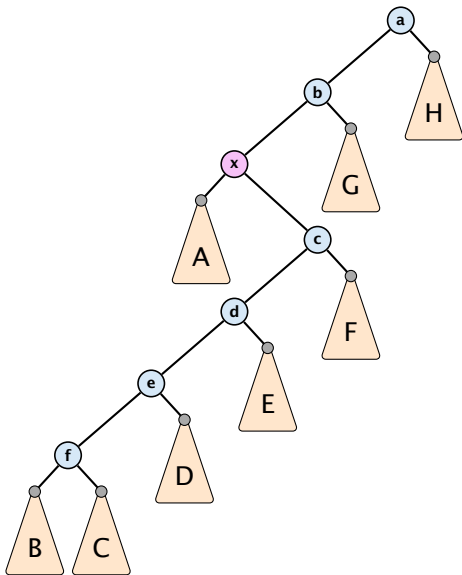
Splay vs. Move to Root



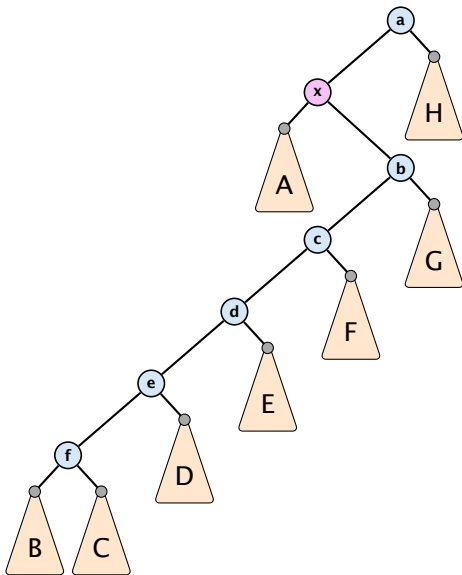
Splay vs. Move to Root



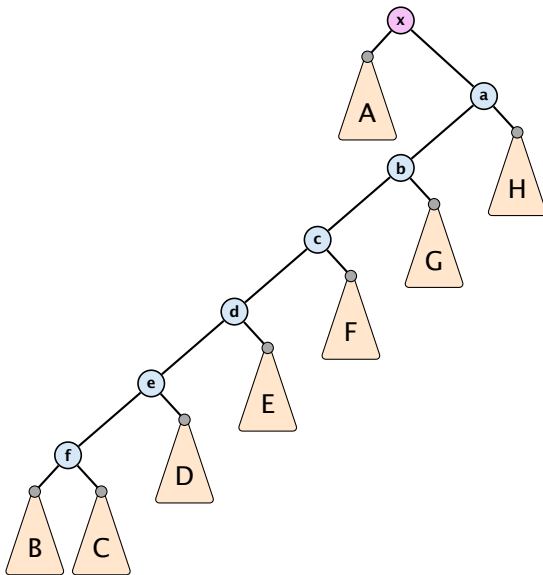
Splay vs. Move to Root



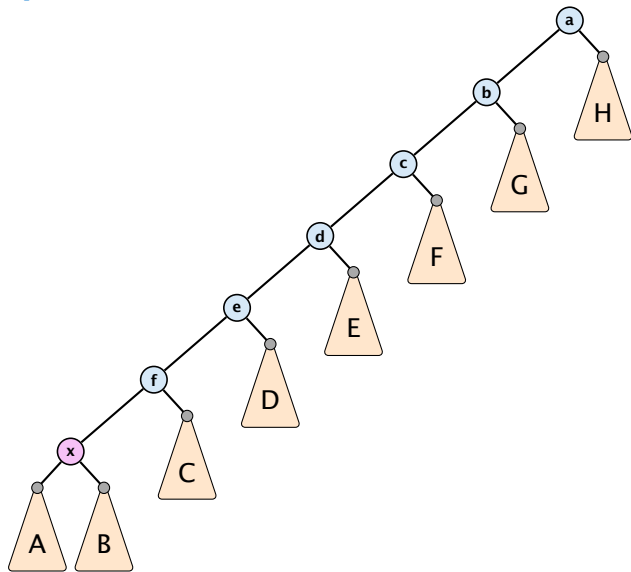
Splay vs. Move to Root



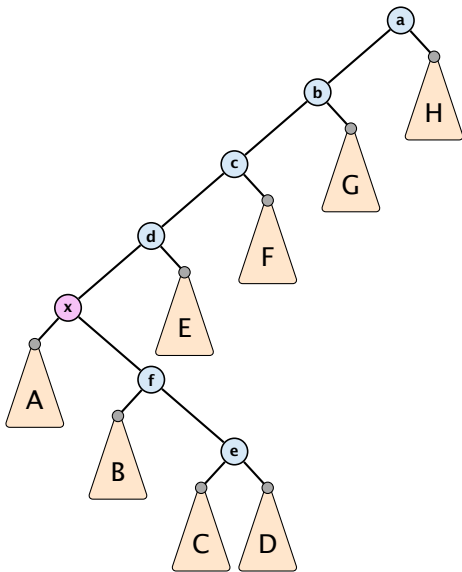
Splay vs. Move to Root



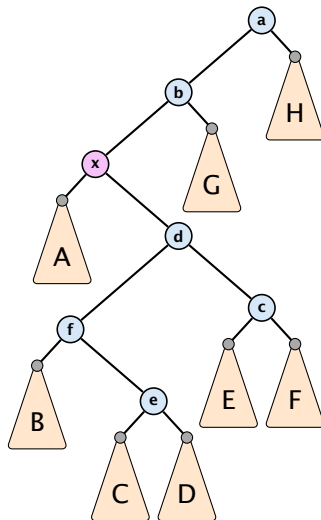
Splay vs. Move to Root



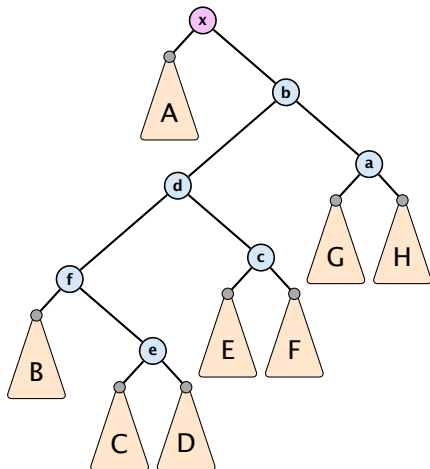
Splay vs. Move to Root



Splay vs. Move to Root



Splay vs. Move to Root



Static Optimality

Suppose we have a sequence of m find-operations. $\text{find}(x)$ appears h_x times in this sequence.

The cost of a **static** search tree T is:

$$\text{cost}(T) = m + \sum_x h_x \text{depth}_T(x)$$

The total cost for processing the sequence on a splay-tree is $\mathcal{O}(\text{cost}(T_{\min}))$, where T_{\min} is an **optimal static search tree**.

Dynamic Optimality

Let S be a sequence with m find-operations.

Let A be a data-structure based on a search tree:

- ▶ the cost for accessing element x is $1 + \text{depth}(x)$;
- ▶ after accessing x the tree may be re-arranged through rotations;

Conjecture:

A splay tree that only contains elements from S has cost $\mathcal{O}(\text{cost}(A, S))$, for processing S .

Lemma 5

*Splay Trees have an **amortized** running time of $\mathcal{O}(\log n)$ for all operations.*

Amortized Analysis

Definition 6

A data structure with operations $\text{op}_1(), \dots, \text{op}_k()$ has amortized running times t_1, \dots, t_k for these operations if the following holds.

Suppose you are given a sequence of operations (**starting with an empty data-structure**) that operate on at most n elements, and let k_i denote the number of occurrences of $\text{op}_i()$ within this sequence. Then the actual running time must be at most $\sum_i k_i \cdot t_i(n)$.

Potential Method

Introduce a potential for the data structure.

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- ▶ $\Phi(D_i)$ is the potential after the i -th operation.
- ▶ Amortized cost of the i -th operation is

$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) .$$

Potential Method

Introduce a potential for the data structure.

- ▶ $\Phi(D_i)$ is the potential after the i -th operation.
- ▶ Amortized cost of the i -th operation is

$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) .$$

- ▶ Show that $\Phi(D_i) \geq \Phi(D_0)$.

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$$\sum_{i=1}^k c_i \leq \sum_{i=1}^k c_i + \Phi(D_k) - \Phi(D_0) = \sum_{i=1}^k \hat{c}_i$$

This means the amortized costs can be used to derive a bound on the total cost.

Example: Stack

Stack

- ▶ $S.$ push()
- ▶ $S.$ pop()
- ▶ $S.$ multipop(k): removes k items from the stack. If the stack currently contains less than k items it empties the stack.
- ▶ The user has to ensure that pop and multipop do not generate an underflow.

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Actual cost:

- ▶ $S.$ push(): cost 1.
- ▶ $S.$ pop(): cost 1.
- ▶ $S.$ multipop(k): cost $\min\{\text{size}, k\} = k$.

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- ▶ **$S.\text{multipop}(k)$** : cost

$$\hat{C}_{\text{mp}} = C_{\text{mp}} + \Delta\Phi = \min\{\text{size}, k\} - \min\{\text{size}, k\} \leq 0 .$$

Example: Binary Counter

Incrementing a binary counter:

Consider a computational model where each bit-operation costs one time-unit.

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- ▶ Changing bit from 0 to 1: cost 1.
- ▶ Changing bit from 1 to 0: cost 1.
- ▶ Increment: cost is $k + 1$, where k is the number of consecutive ones in the least significant bit-positions (e.g, 001101 has $k = 1$).

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- ▶ Changing bit from 1 to 0:

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- ▶ **Increment:** Let k denotes the number of consecutive ones in the least significant bit-positions. An increment involves k $(1 \rightarrow 0)$ -operations, and one $(0 \rightarrow 1)$ -operation.

Hence, the amortized cost is $k\hat{C}_{1 \rightarrow 0} + \hat{C}_{0 \rightarrow 1} \leq 2$.

Splay Trees

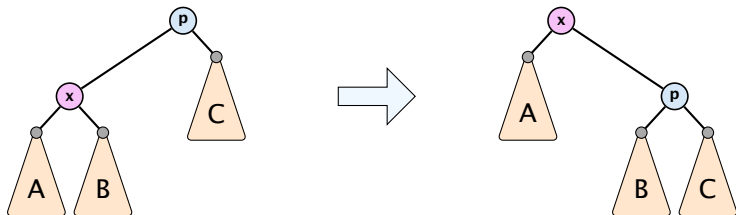
potential function for splay trees:

- ▶ size $s(x) = |T_x|$
- ▶ rank $r(x) = \log_2(s(x))$
- ▶ $\Phi(T) = \sum_{v \in T} r(v)$

amortized cost = real cost + potential change

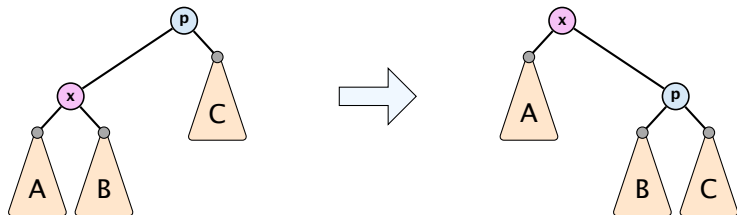
The cost is essentially the cost of the splay-operation, which is 1 plus the number of rotations.

Splay: Zig Case



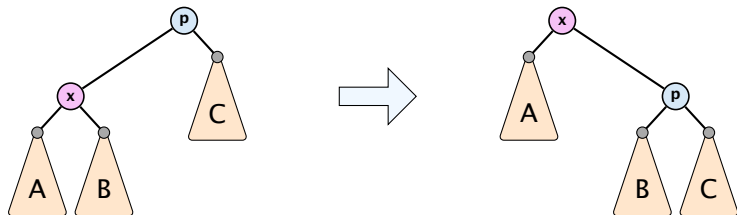
$$\Delta\Phi =$$

Splay: Zig Case



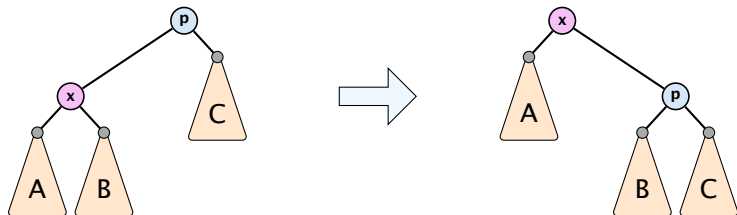
$$\Delta\Phi = r'(x) + r'(p) - r(x) - r(p)$$

Splay: Zig Case



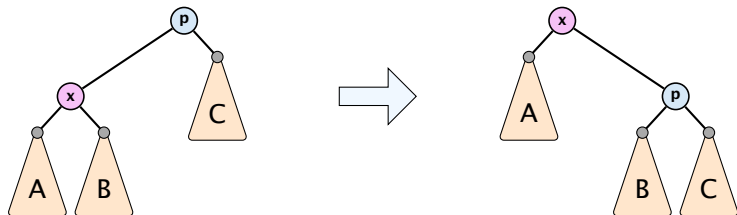
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p) \\ &= r'(p) - r(x)\end{aligned}$$

Splay: Zig Case



$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p) \\ &= r'(p) - r(x) \\ &\leq r'(x) - r(x)\end{aligned}$$

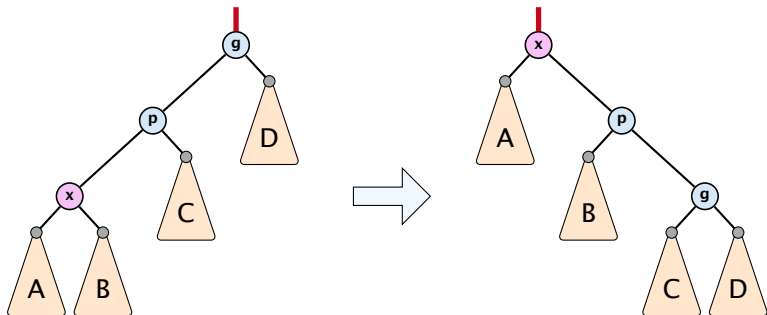
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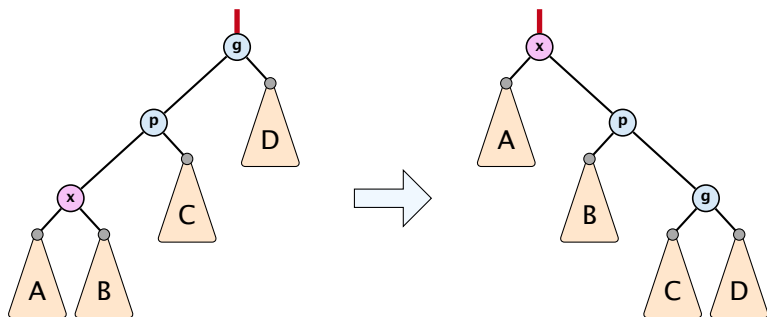
$$\text{cost}_{\text{zig}} \leq 1 + 3(r'(x) - r(x))$$

Splay: Zigzig Case



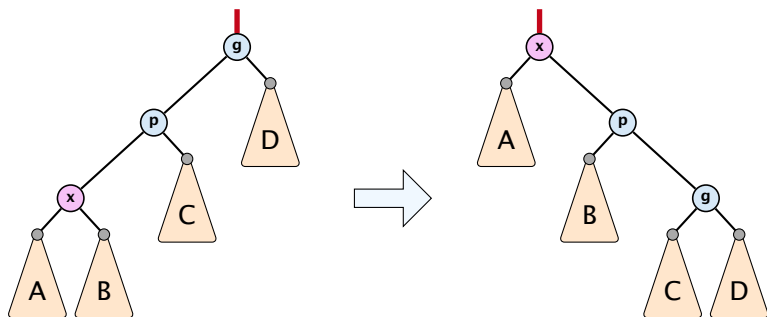
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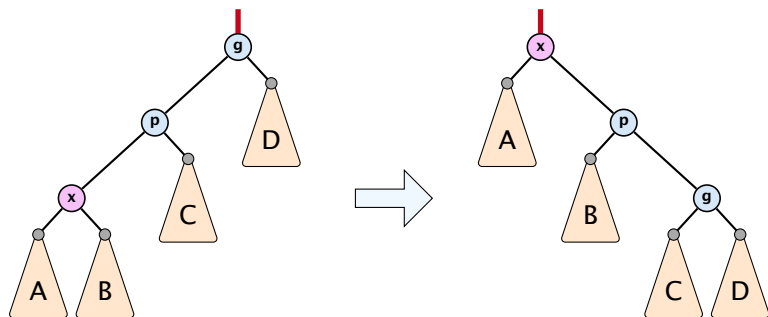
$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$

Splay: Zigzig Case



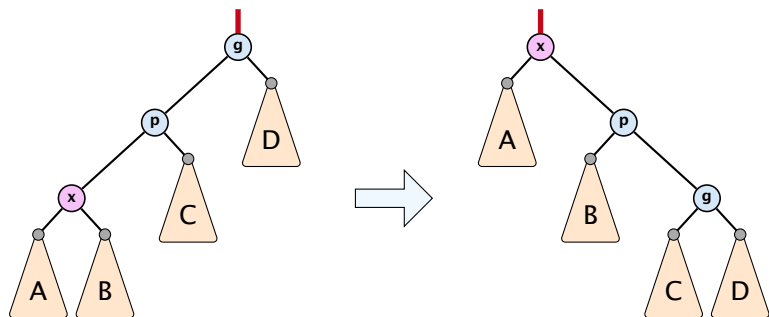
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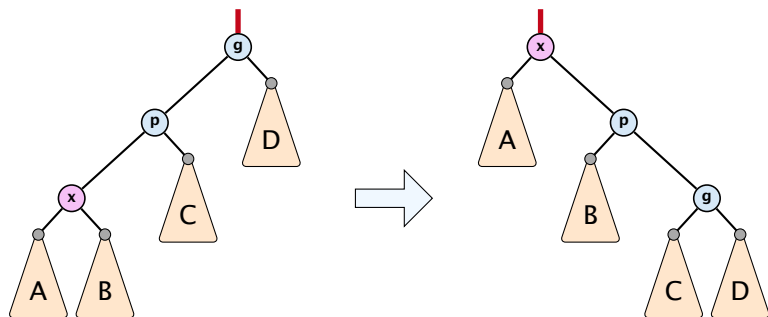
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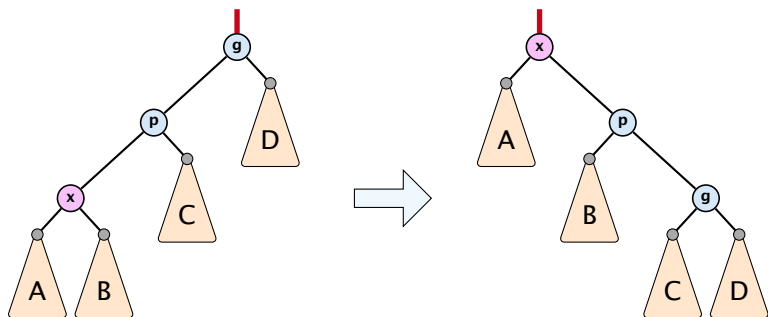
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(x) + r'(g) - r(x) - r(x) \\ &= r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x)\end{aligned}$$

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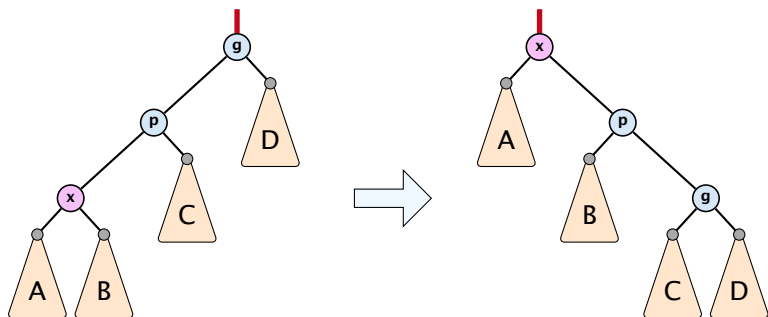
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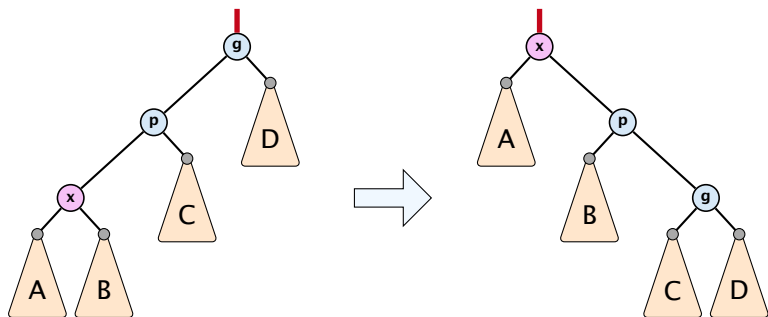
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Splay: Zigzig Case



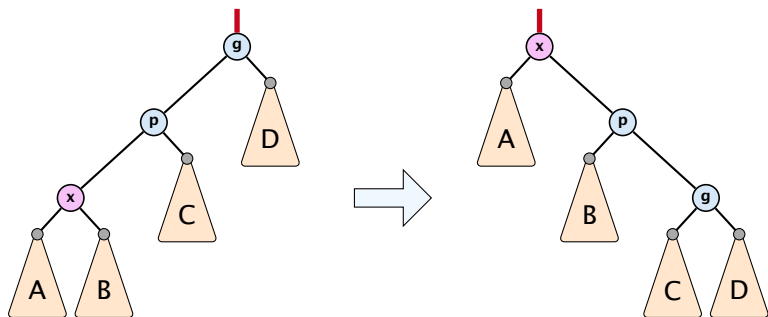
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Splay: Zigzig Case



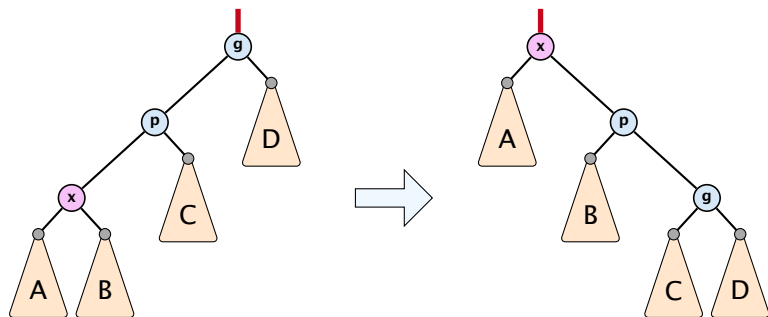
$$\frac{1}{2}(r(x) + r'(g) - 2r'(x))$$

Splay: Zigzig Case



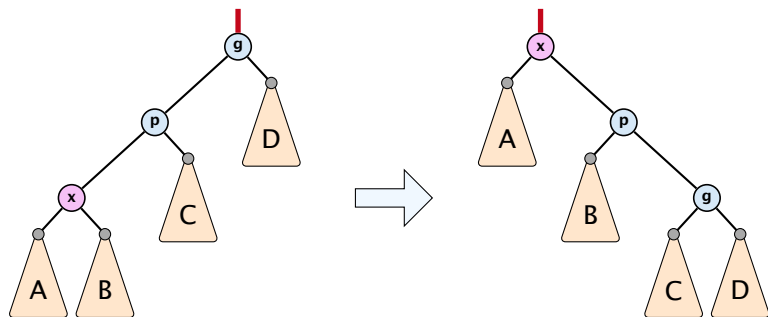
$$\begin{aligned} & \frac{1}{2} (r(x) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} (\log(s(x)) + \log(s'(g)) - 2\log(s'(x))) \end{aligned}$$

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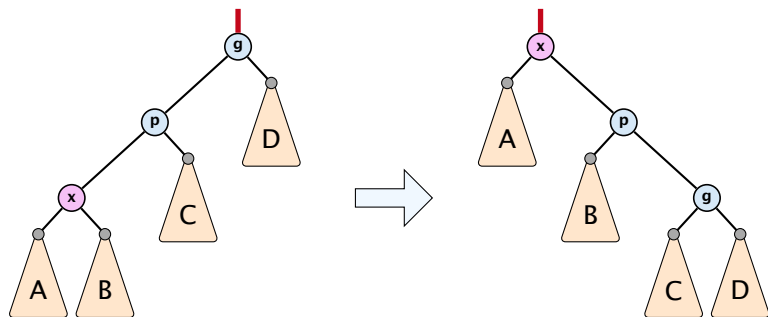
$$\begin{aligned} & \frac{1}{2} (r(x) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} (\log(s(x)) + \log(s'(g)) - 2\log(s'(x))) \\ &= \frac{1}{2} \log\left(\frac{s(x)}{s'(x)}\right) + \frac{1}{2} \log\left(\frac{s'(g)}{s'(x)}\right) \end{aligned}$$

Splay: Zigzig Case



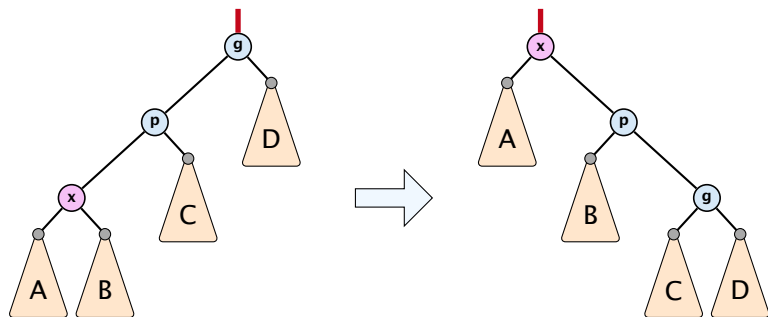
$$\begin{aligned} & \frac{1}{2} (r(x) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} (\log(s(x)) + \log(s'(g)) - 2\log(s'(x))) \\ &= \frac{1}{2} \log\left(\frac{s(x)}{s'(x)}\right) + \frac{1}{2} \log\left(\frac{s'(g)}{s'(x)}\right) \\ &\leq \log\left(\frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)}\right) \end{aligned}$$

Splay: Zigzig Case



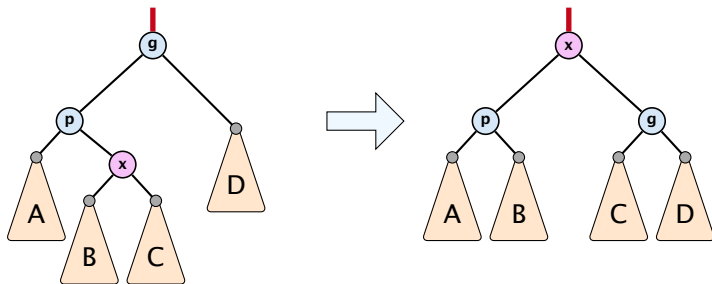
$$\begin{aligned} & \frac{1}{2} (r(x) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} \left(\log(s(x)) + \log(s'(g)) - 2 \log(s'(x)) \right) \\ &= \frac{1}{2} \log \left(\frac{s(x)}{s'(x)} \right) + \frac{1}{2} \log \left(\frac{s'(g)}{s'(x)} \right) \\ &\leq \log \left(\frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)} \right) \leq \log \left(\frac{1}{2} \right) \end{aligned}$$

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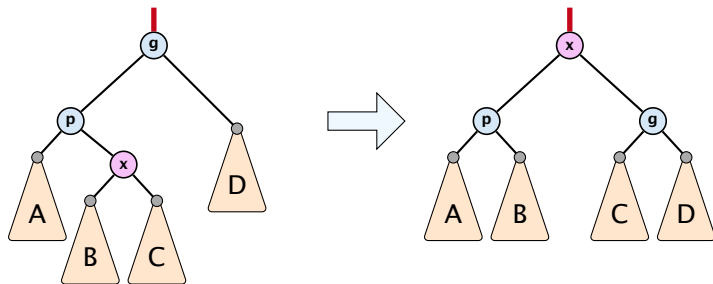
$$\begin{aligned} & \frac{1}{2} (r(x) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} \left(\log(s(x)) + \log(s'(g)) - 2 \log(s'(x)) \right) \\ &= \frac{1}{2} \log \left(\frac{s(x)}{s'(x)} \right) + \frac{1}{2} \log \left(\frac{s'(g)}{s'(x)} \right) \\ &\leq \log \left(\frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)} \right) \leq \log \left(\frac{1}{2} \right) = -1 \end{aligned}$$

Splay: Zigzag Case



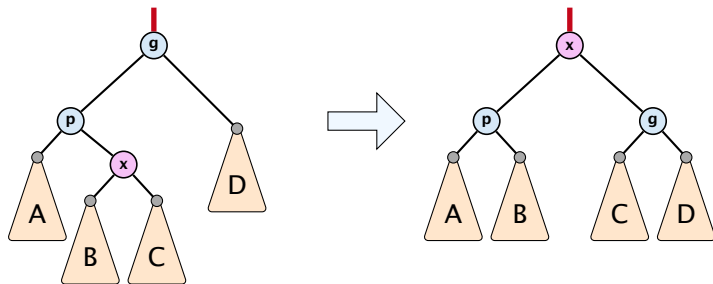
$\Delta\Phi =$

Splay: Zigzag Case



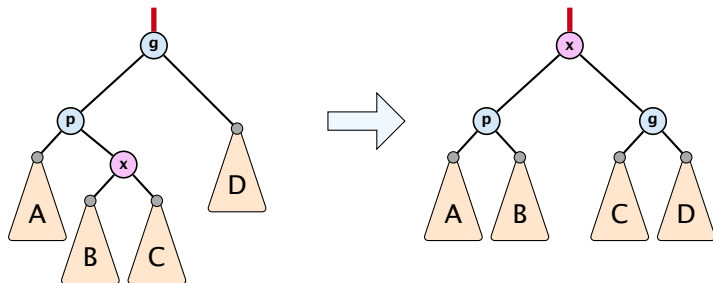
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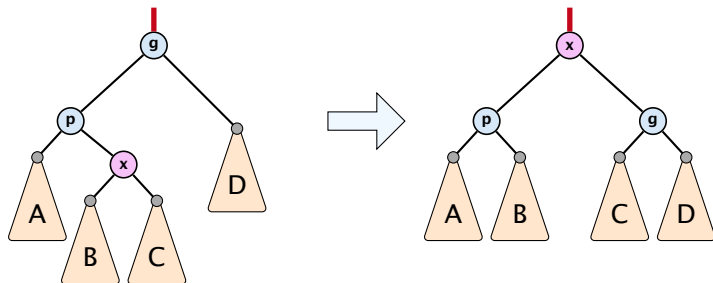
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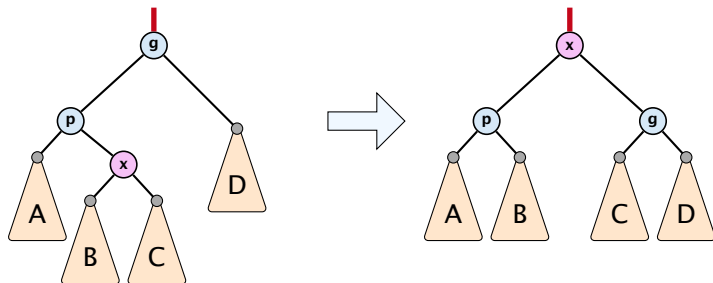
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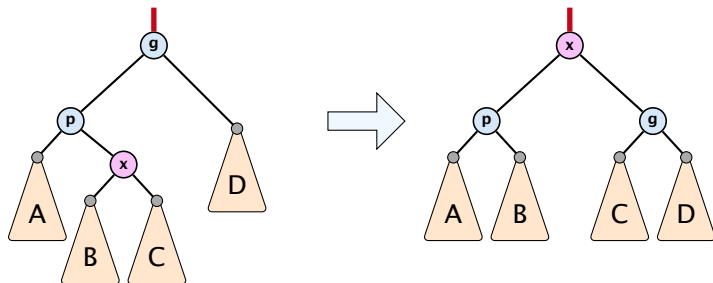
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Splay: Zigzag Case



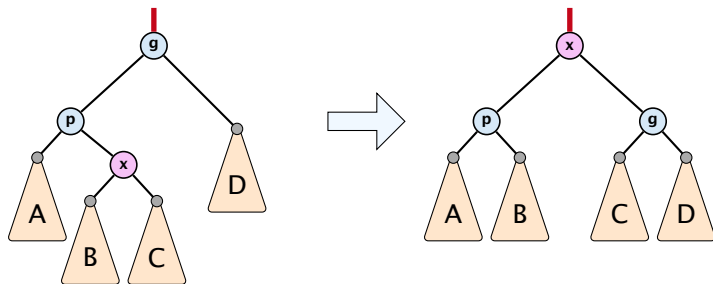
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Splay: Zigzag Case



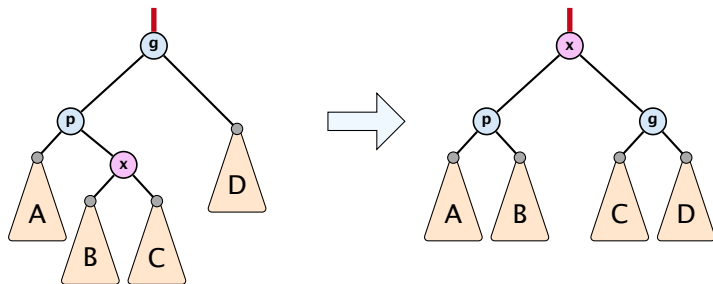
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Splay: Zigzag Case



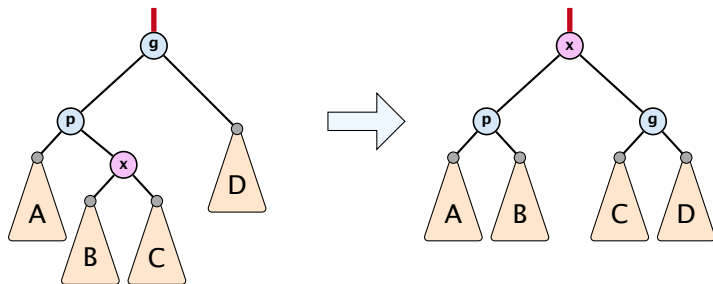
$$\frac{1}{2}(r'(p) + r'(g) - 2r'(x))$$

Splay: Zigzag Case



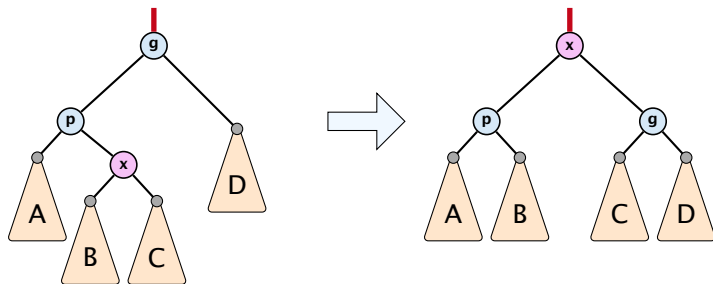
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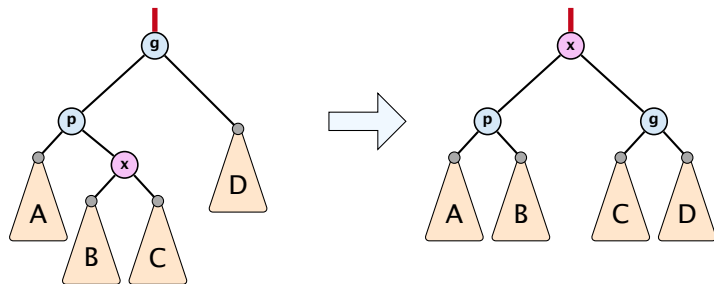
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Splay: Zigzag Case



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Amortized cost of the whole splay operation:

$$\begin{aligned} &\leq 1 + 1 + \sum_{\text{steps } t} 3(r_t(x) - r_{t-1}(x)) \\ &= 2 + 3(r(\text{root}) - r_0(x)) \\ &\leq \mathcal{O}(\log n) \end{aligned}$$

7.4 Augmenting Data Structures

Suppose you want to develop a data structure with:

- ▶ **Insert(x)**: insert element x .
- ▶ **Search(k)**: search for element with key k .
- ▶ **Delete(x)**: delete element referenced by pointer x .
- ▶ **find-by-rank(ℓ)**: return the ℓ -th element; return “error” if the data-structure contains less than ℓ elements.

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- ▶ **Search(k)**: search for element with key k .
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- ▶ **find-by-rank(ℓ)**: return the ℓ -th element; return “error” if the data-structure contains less than ℓ elements.

Augment an existing data-structure instead of developing a new one.

7.4 Augmenting Data Structures

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Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

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3. We need to be able to update the size-field in each node without asymptotically affecting the running time of insert, delete, and search. We come back to this step later...

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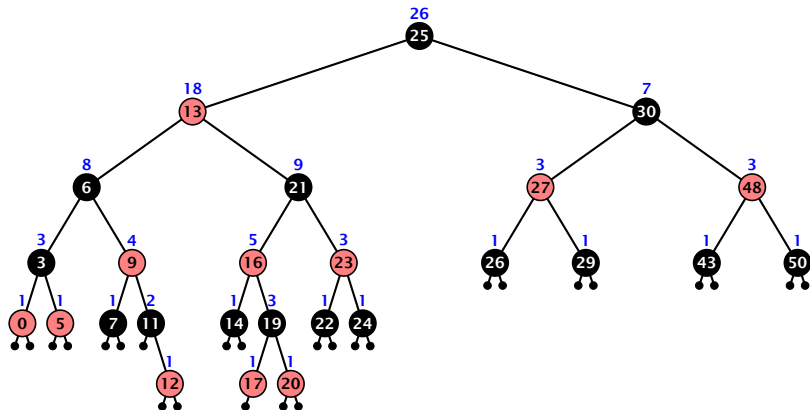
4. How does find-by-rank work?

Find-by-rank(k) := Select($root, k$) with

Algorithm 1 Select(x, i)

```
1: if  $x = \text{null}$  then return error
2: if  $\text{left}[x] \neq \text{null}$  then  $r \leftarrow \text{left}[x].\text{size} + 1$  else  $r \leftarrow 1$ 
3: if  $i = r$  then return  $x$ 
4: if  $i < r$  then
5:     return Select( $\text{left}[x], i$ )
6: else
7:     return Select( $\text{right}[x], i - r$ )
```

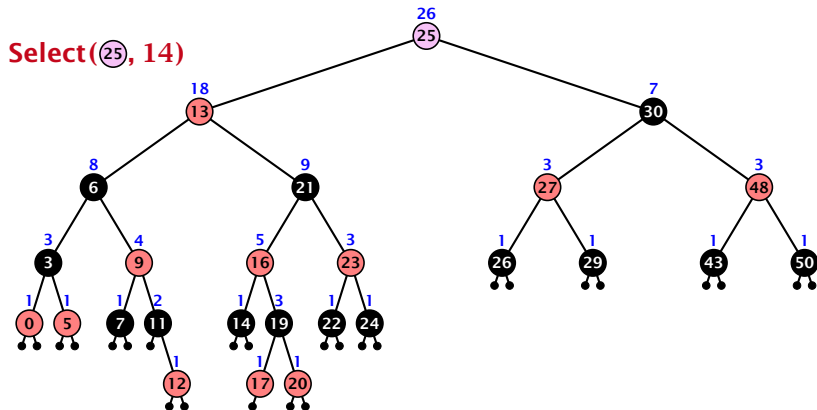
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Find-by-rank:

- ▶ decide whether you have to proceed into the left or right sub-tree
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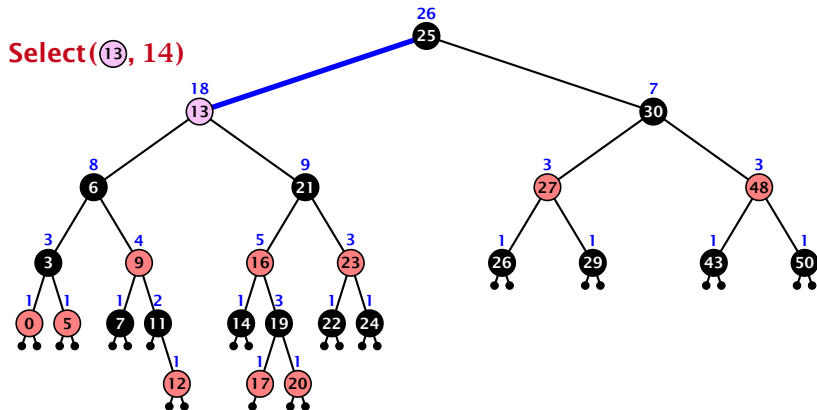
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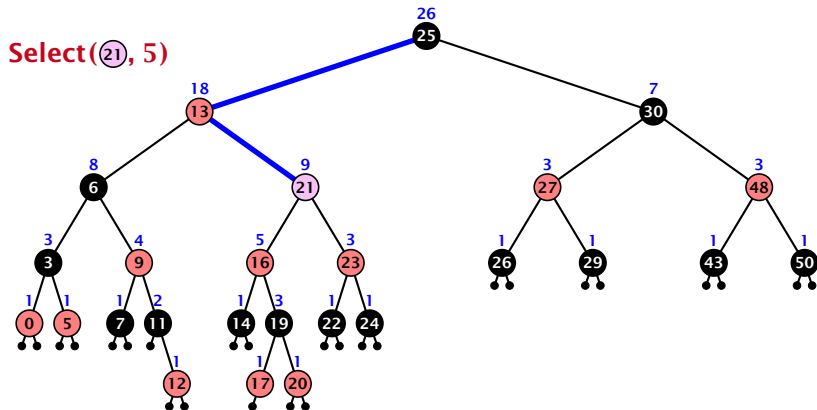
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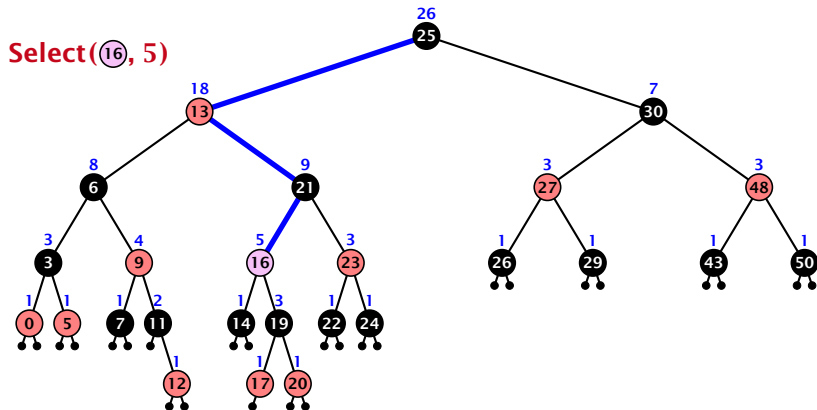
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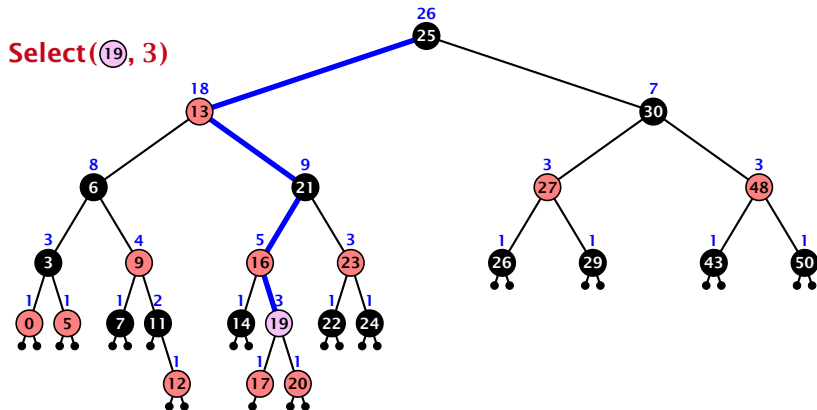
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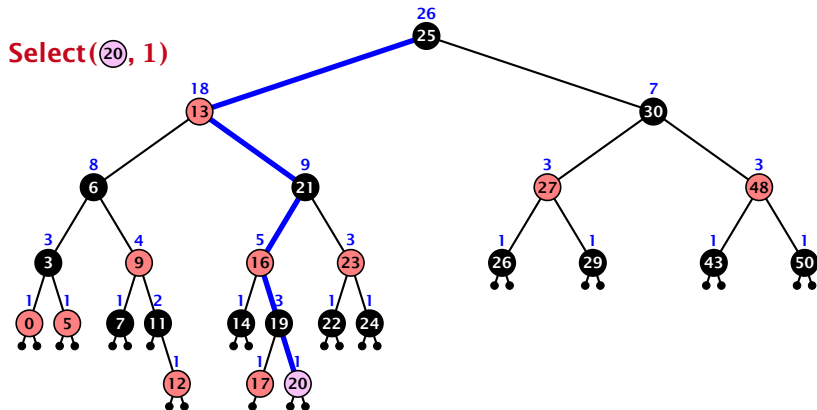
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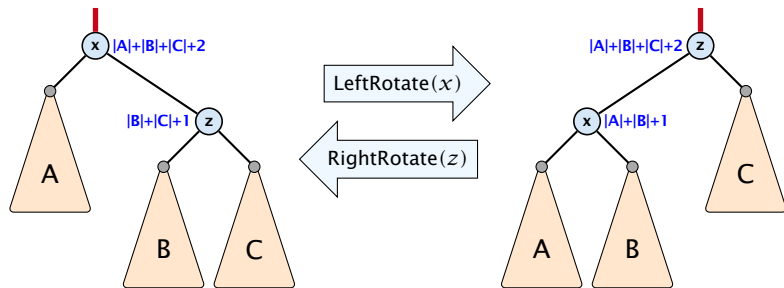
Search(k): Nothing to do.

Insert(x): When going down the search path increase the size field for each visited node. **Maintain the size field during rotations.**

Delete(x): Directly after splicing out a node traverse the path from the spliced out node upwards, and decrease the size counter on every node on this path. **Maintain the size field during rotations.**

Rotations

The only operation during the fix-up procedure that alters the tree and requires an update of the size-field:



The nodes x and z are the only nodes changing their size-fields.

The new size-fields can be computed **locally** from the size-fields of the children.

7.5 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

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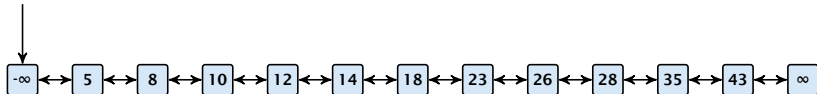
Why do we not use a list for implementing the ADT Dynamic Set?

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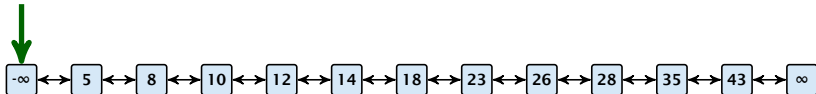
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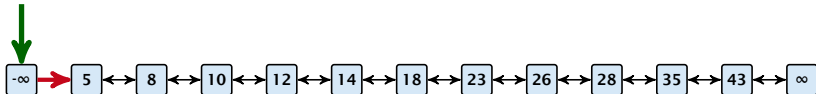
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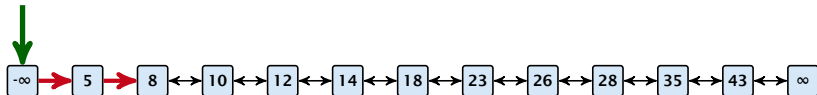
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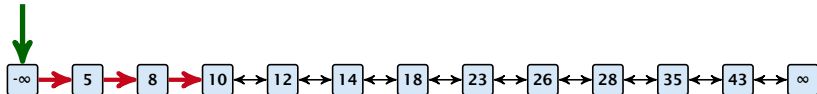
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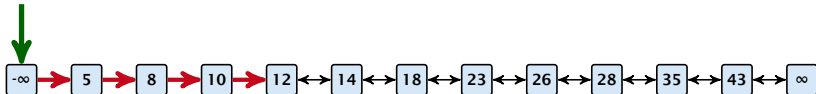
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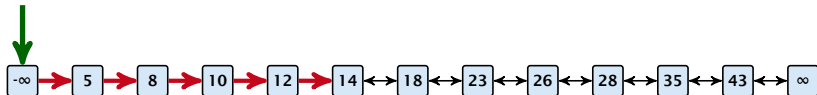
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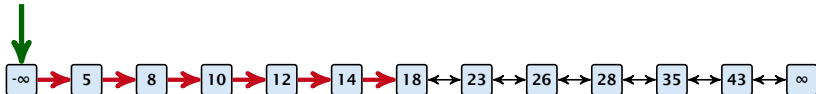
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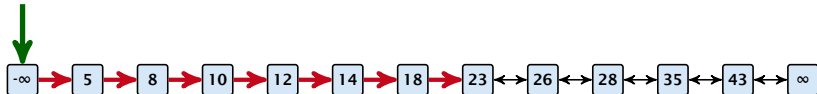
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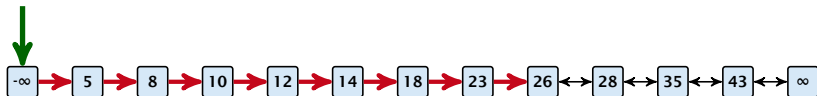
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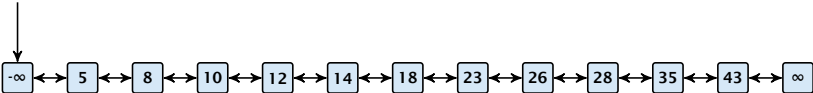
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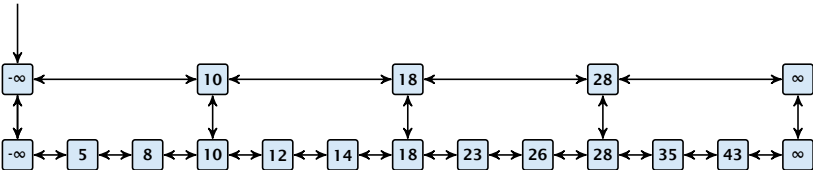
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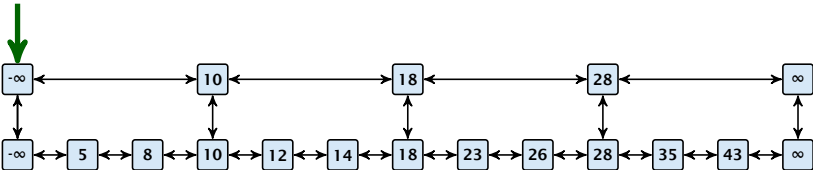
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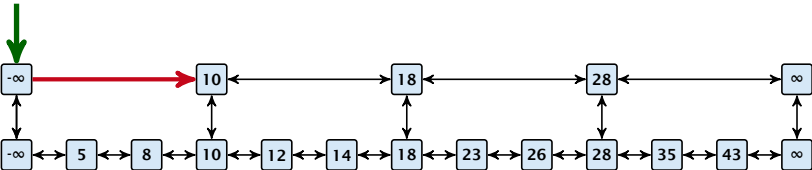
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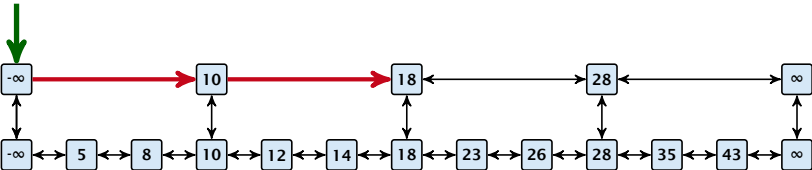
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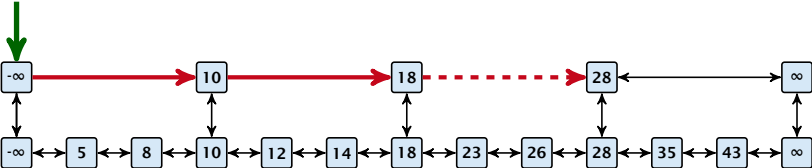
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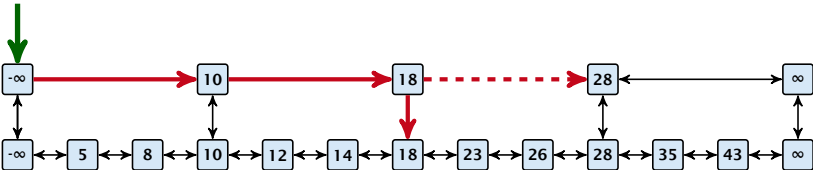
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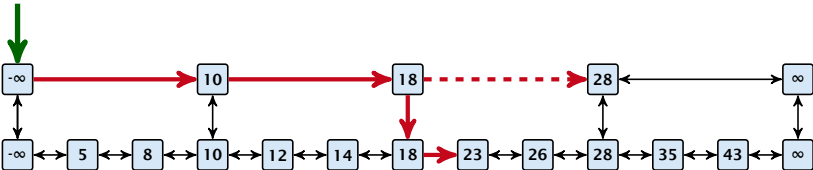
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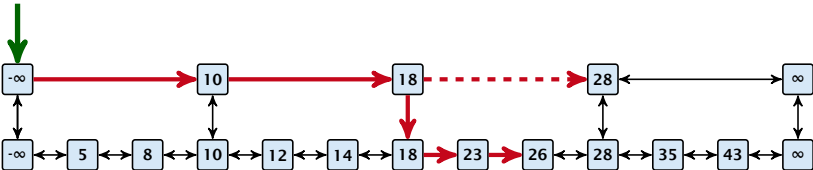
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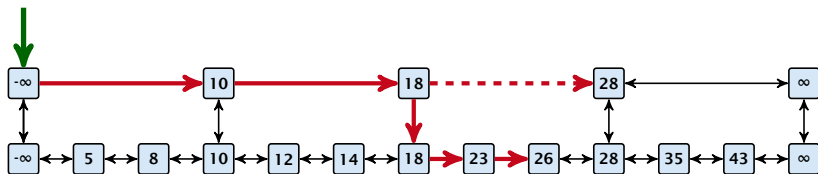
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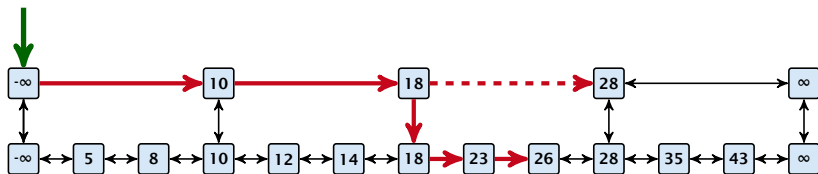


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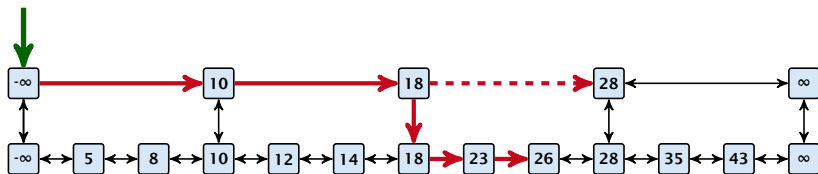
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Choose $|L_1| = \sqrt{n}$. Then search time $\Theta(\sqrt{n})$.

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- ▶ At most $|L_k| + \sum_{i=1}^k \frac{L_{i-1}}{L_i} + 3(k + 1)$ steps.

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Choosing $k = \Theta(\log n)$ gives a logarithmic running time.

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Use randomization instead!

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Insert:

- ▶ A search operation gives you the insert position for element x in every list.
- ▶ Flip a coin until it shows head, and record the number $t \in \{1, 2, \dots\}$ of trials needed.
- ▶ Insert x into lists L_0, \dots, L_{t-1} .

Delete:

- ▶ You get all predecessors via backward pointers.

7.5 Skip Lists

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- ▶ A search operation gives you the insert position for element x in every list.
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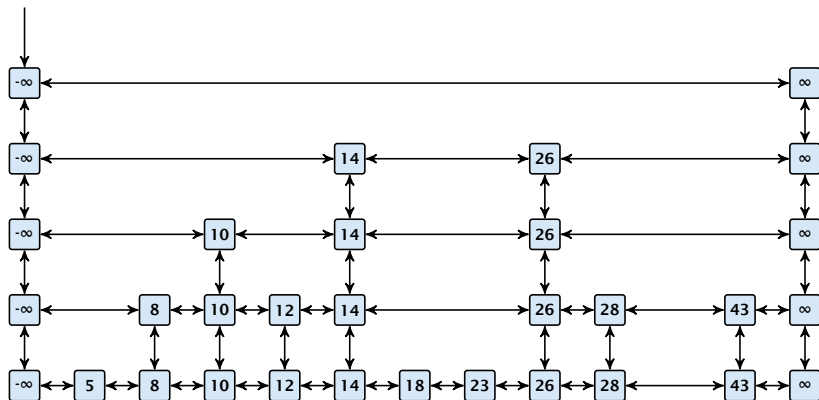
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- ▶ You get all predecessors via backward pointers.
- ▶ Delete x in all lists it actually appears in.

The time for both operations is dominated by the search time.

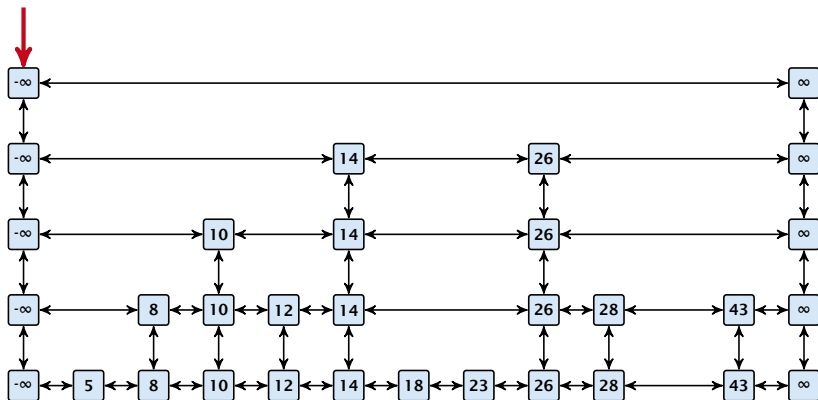
7.5 Skip Lists

Insert (35):



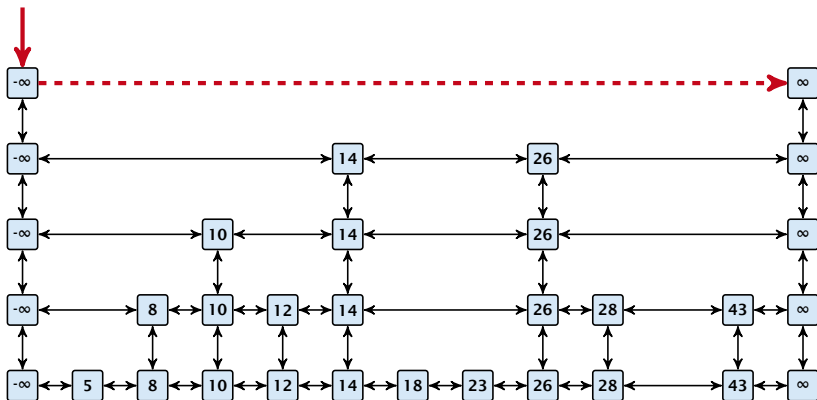
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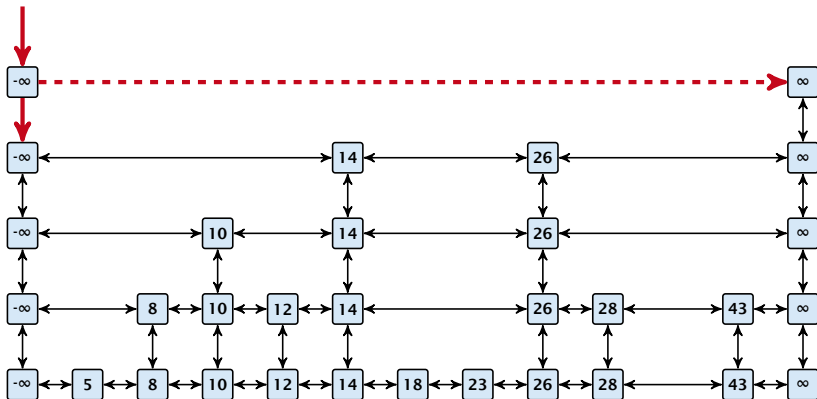
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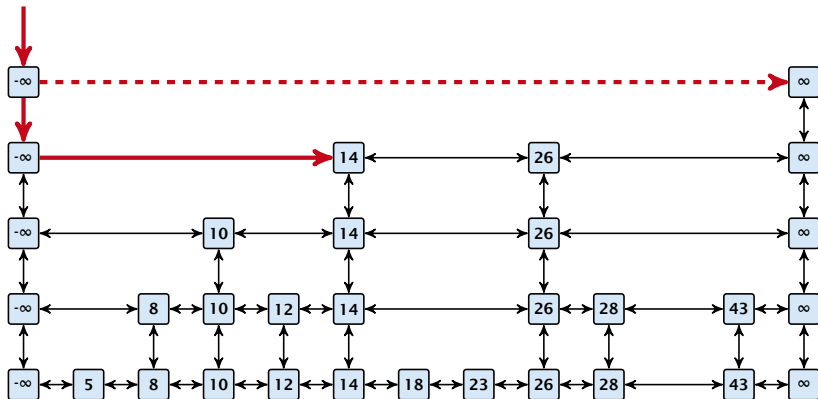
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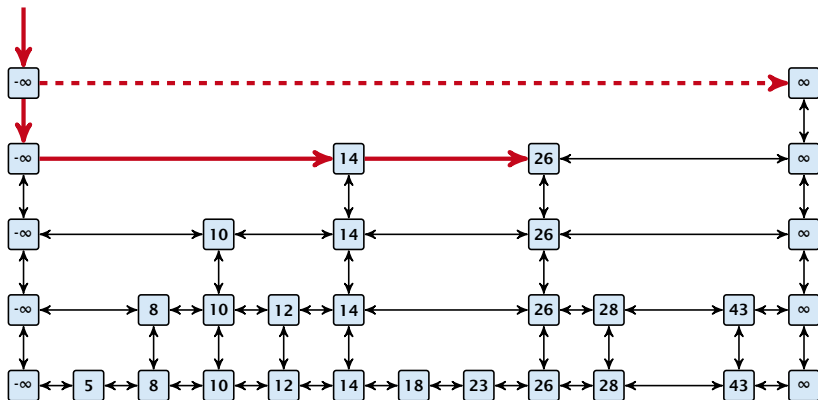
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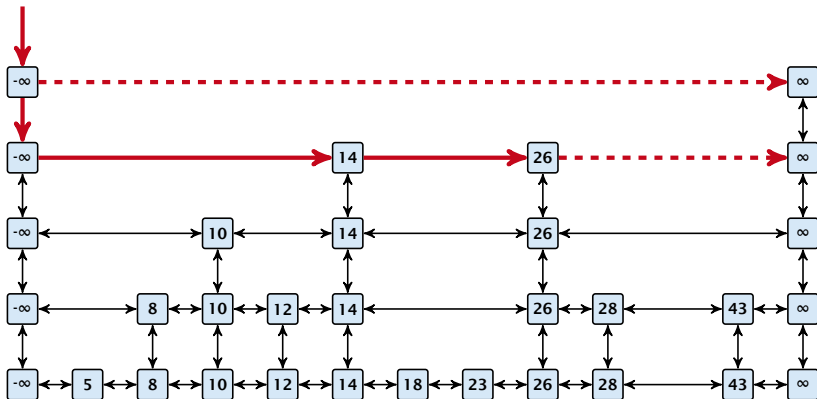
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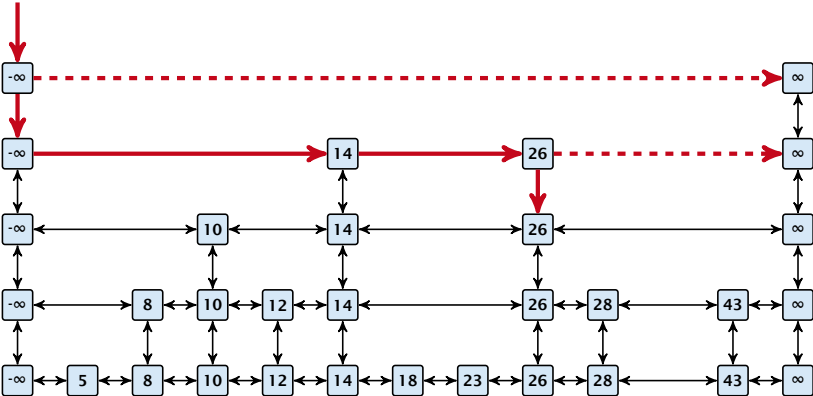
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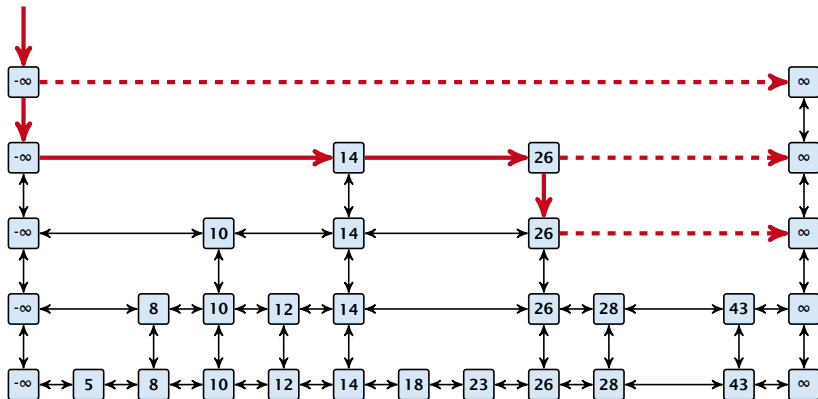
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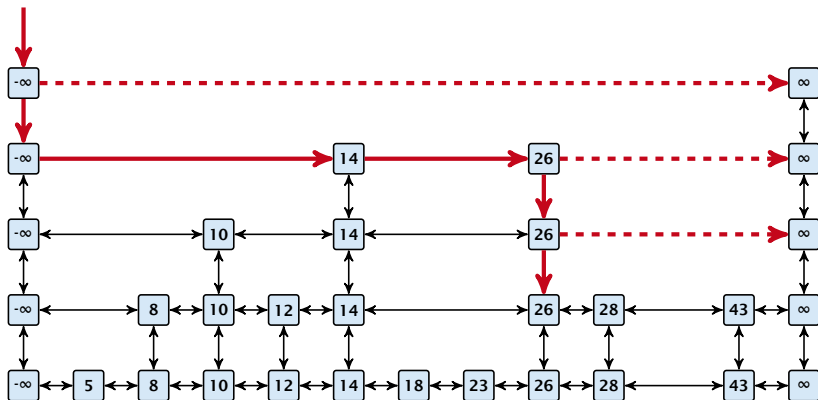
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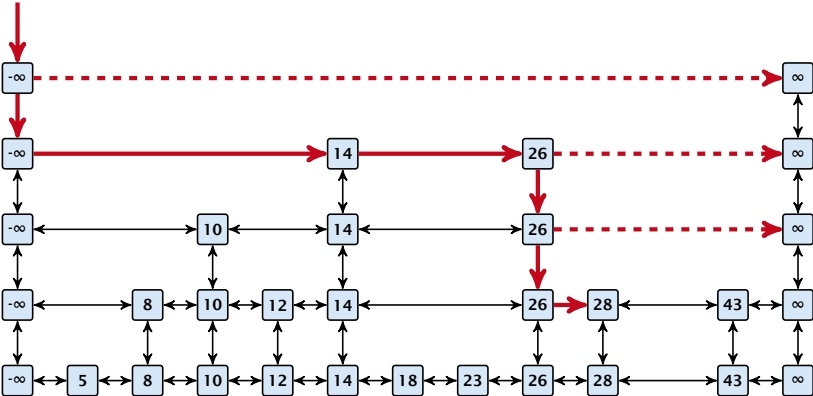
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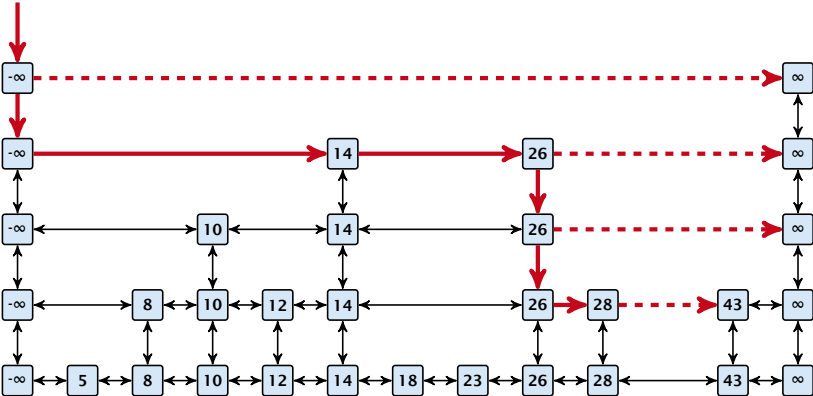
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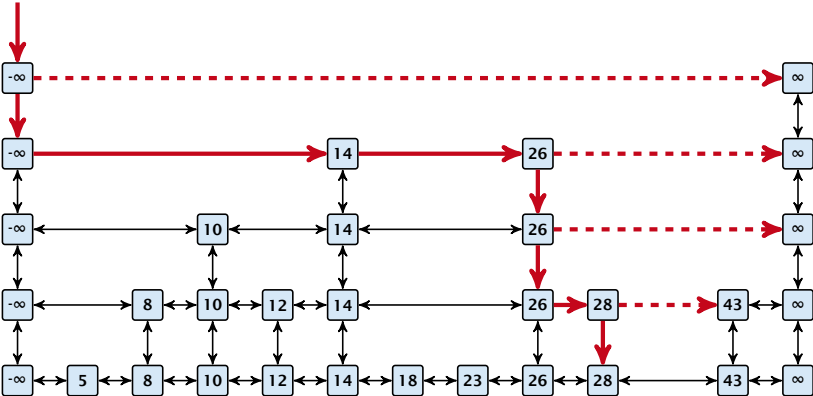
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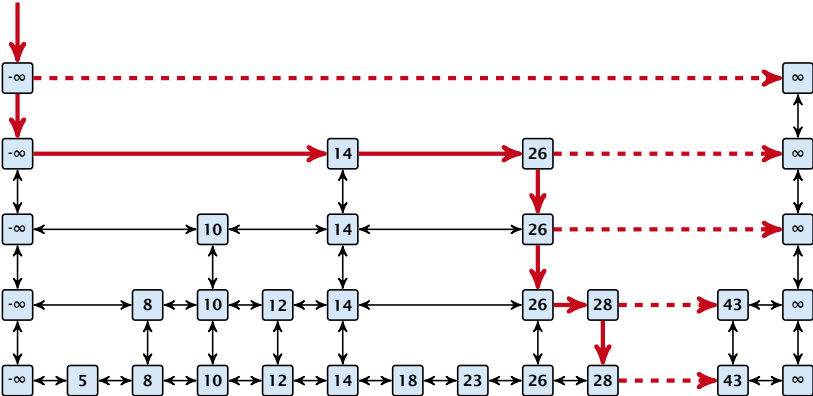
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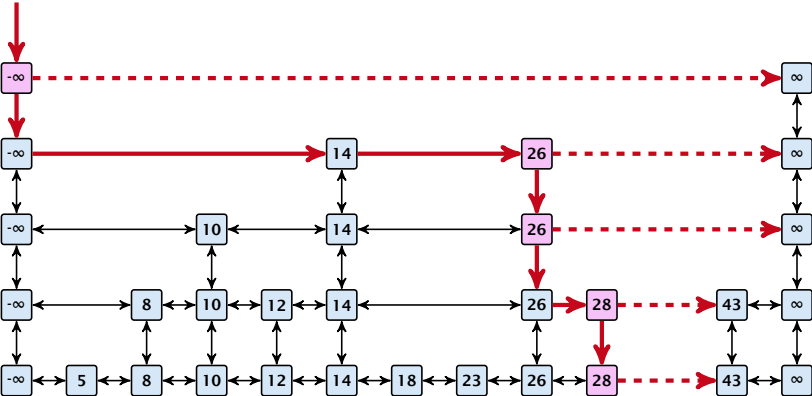
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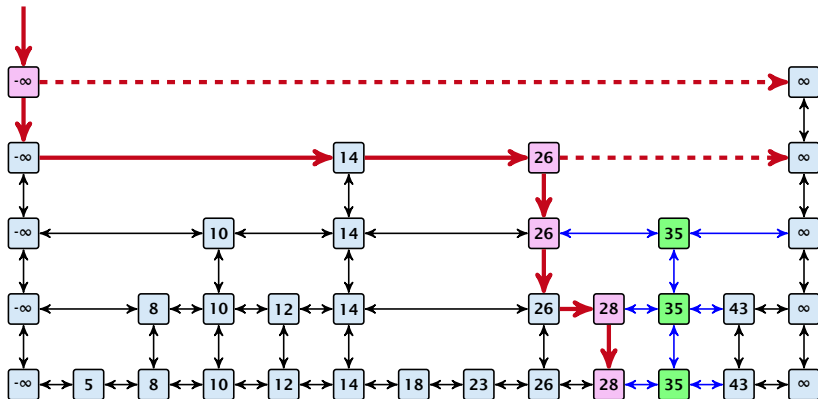
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High Probability

Definition 7 (High Probability)

We say a **randomized** algorithm has running time $\mathcal{O}(\log n)$ with **high probability** if for any constant α the running time is at most $\mathcal{O}(\log n)$ with probability at least $1 - \frac{1}{n^\alpha}$.

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Here the \mathcal{O} -notation hides a constant that may depend on α .

High Probability

Suppose there are **polynomially** many events E_1, E_2, \dots, E_ℓ , $\ell = n^c$ each holding with high probability (e.g. E_i may be the event that the i -th search in a skip list takes time at most $\mathcal{O}(\log n)$).

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This means $\Pr[E_1 \wedge \dots \wedge E_\ell]$ holds with high probability.

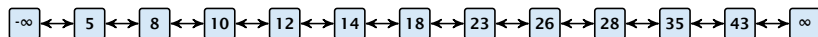
7.5 Skip Lists

Lemma 8

A search (and, hence, also insert and delete) in a skip list with n elements takes time $\mathcal{O}(\log n)$ with high probability (w. h. p.).

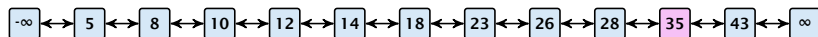
7.5 Skip Lists

Backward analysis:



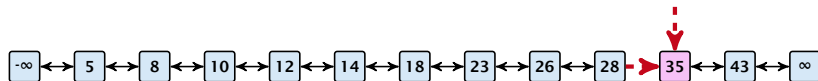
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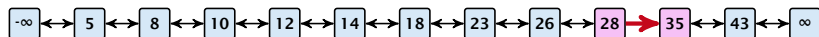
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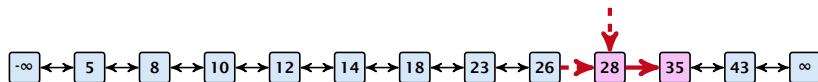
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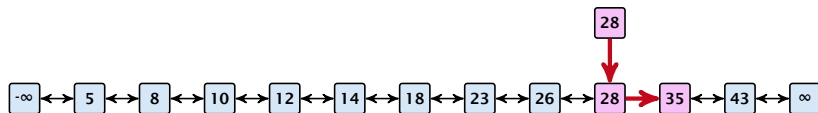
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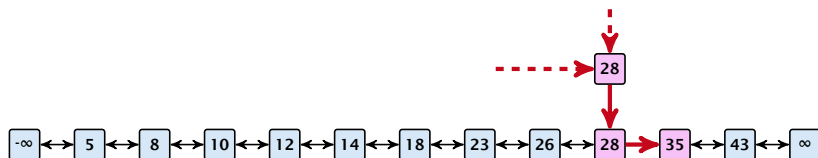
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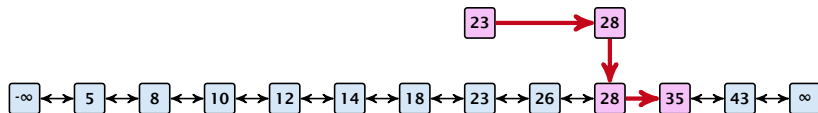
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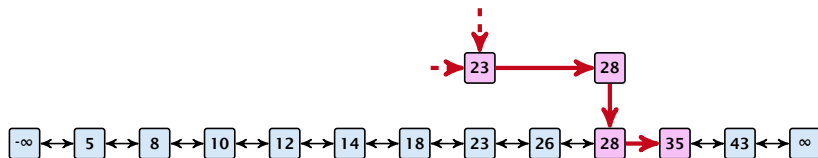
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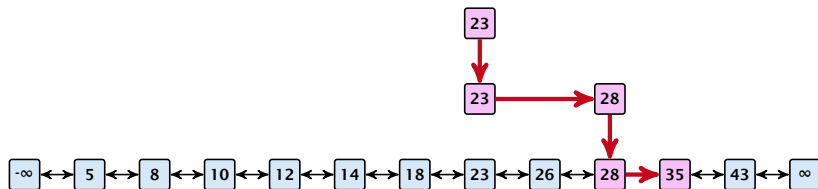
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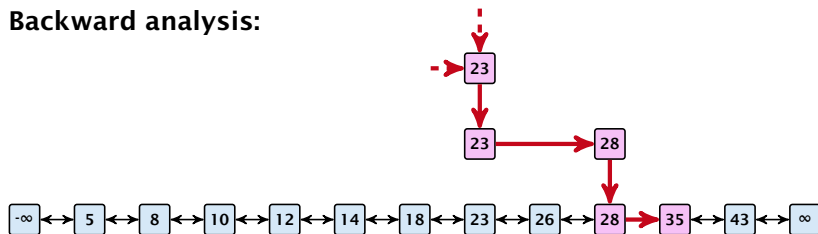
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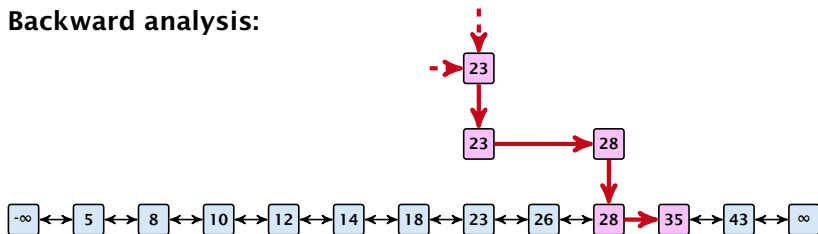
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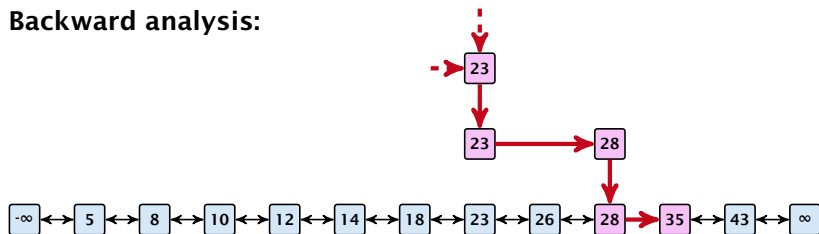
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At each point the path goes up with probability $1/2$ and left with probability $1/2$.

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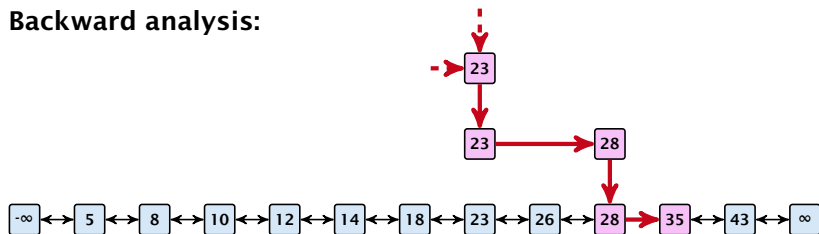
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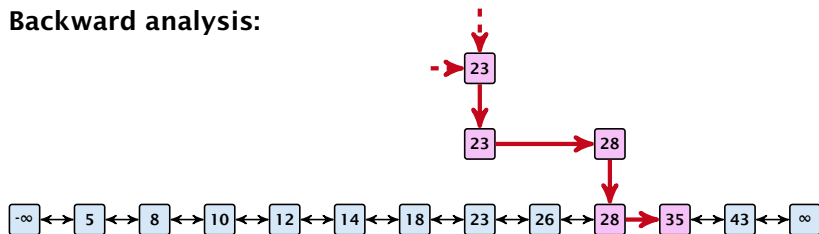
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From this it follows that w.h.p. there are no long paths.

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Estimation for Binomial Coefficients

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In particular, this means that during the construction in the backward analysis we see at most k heads (i.e., coin flips that tell you to go up) in z trials.

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choosing $k = \gamma \log n$ with $\gamma \geq 1$ and $z = (\beta + \alpha)\gamma \log n$

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now choosing $\beta = 6\alpha$ gives

7.5 Skip Lists

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now choosing $\beta = 6\alpha$ gives

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This means, the search requires at most z steps, w. h. p.

7.6 Hashing

Dictionary:

- ▶ **S . insert(x)**: Insert an element x .
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Hashing tries to **directly** compute the memory location from the given key. The goal is to have constant search time.

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Definitions:

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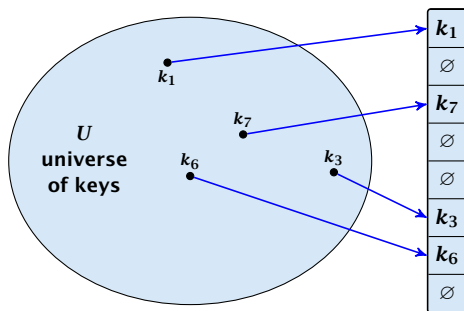
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The hash-function h should fulfill:

- ▶ Fast to evaluate.
- ▶ Small storage requirement.
- ▶ Good distribution of elements over the whole table.

Direct Addressing

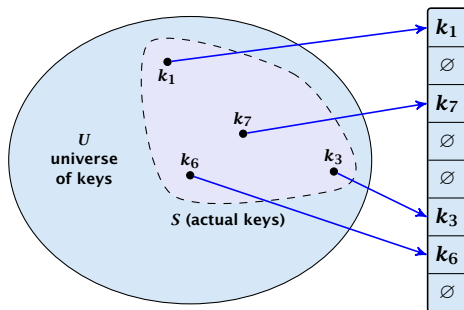
Ideally the hash function maps **all** keys to different memory locations.



This special case is known as **Direct Addressing**. It is usually very unrealistic as the universe of keys typically is quite large, and in particular larger than the available memory.

Perfect Hashing

Suppose that we **know** the set S of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



Such a hash function h is called a **perfect hash function** for set S .

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Hence, there may be two elements k_1, k_2 from the set S that map to the same memory location (i.e., $h(k_1) = h(k_2)$). This is called a **collision**.

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Lemma 9

The probability of having a collision when hashing m elements into a table of size n under uniform hashing is at least

$$1 - e^{-\frac{m(m-1)}{2n}} \approx 1 - e^{-\frac{m^2}{2n}} .$$

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Uniform hashing:

Choose a hash function uniformly at random from all functions $f : U \rightarrow [0, \dots, n-1]$.

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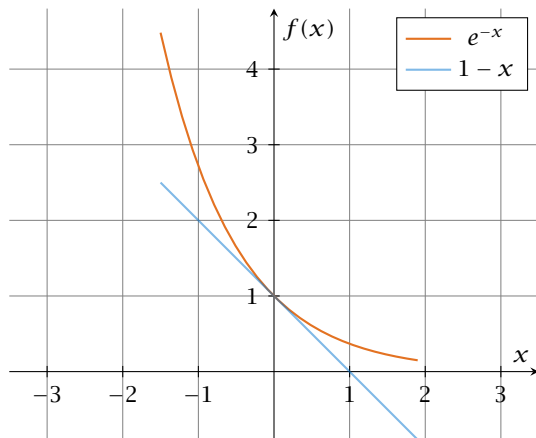
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Here the first equality follows since the ℓ -th element that is hashed has a probability of $\frac{n-\ell+1}{n}$ to not generate a collision under the condition that the previous elements did not induce collisions. □

Collisions



The inequality $1 - x \leq e^{-x}$ is derived by stopping the Taylor-expansion of e^{-x} after the second term.

Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

- ▶ **open addressing**, aka. closed hashing
- ▶ **hashing with chaining**, aka. closed addressing, open hashing.

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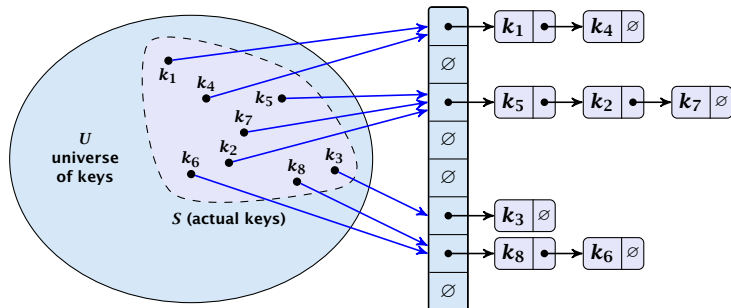
- ▶ **open addressing**, aka. closed hashing
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There are applications e.g. computer chess where you do not resolve collisions at all.

Hashing with Chaining

Arrange elements that map to the same position in a linear list.

- ▶ Access: compute $h(x)$ and search list for $\text{key}[x]$.
- ▶ Insert: insert at the front of the list.



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We assume **uniform hashing** for the following analysis.

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The time required for an unsuccessful search is 1 plus the length of the list that is examined. The average length of a list is $\alpha = \frac{m}{n}$. Hence, if A is the collision resolving strategy “Hashing with Chaining” we have

$$A^- = 1 + \alpha .$$

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Hashing with Chaining

$$\begin{aligned} E \left[\frac{1}{m} \sum_{i=1}^m \left(1 + \sum_{j=i+1}^m X_{ij} \right) \right] &= \frac{1}{m} \sum_{i=1}^m \left(1 + \sum_{j=i+1}^m E[X_{ij}] \right) \\ &= \frac{1}{m} \sum_{i=1}^m \left(1 + \sum_{j=i+1}^m \frac{1}{n} \right) \\ &= 1 + \frac{1}{mn} \sum_{i=1}^m (m - i) \\ &= 1 + \frac{1}{mn} \left(m^2 - \frac{m(m+1)}{2} \right) \\ &= 1 + \frac{m-1}{2n} = 1 + \frac{\alpha}{2} - \frac{\alpha}{2m} . \end{aligned}$$

Hence, the expected cost for a successful search is $A^+ \leq 1 + \frac{\alpha}{2}$.

Hashing with Chaining

Disadvantages:

- ▶ pointers increase memory requirements
- ▶ pointers may lead to bad cache efficiency

Advantages:

- ▶ no à priori limit on the number of elements
- ▶ deletion can be implemented efficiently
- ▶ by using balanced trees instead of linked list one can also obtain worst-case guarantees.

Open Addressing

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All objects are stored in the table itself.

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Define a function $h(k, j)$ that determines the table-position to be examined in the j -th step. The values $h(k, 0), \dots, h(k, n - 1)$ must form a permutation of $0, \dots, n - 1$.

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Search(k): Try position $h(k, 0)$; if it is empty your search fails; otw. continue with $h(k, 1), h(k, 2), \dots$

Insert(x): Search until you find an empty slot; insert your element there. If your search reaches $h(k, n - 1)$, and this slot is non-empty then your table is full.

Open Addressing

Choices for $h(k, j)$:

- ▶ **Linear probing:**

$$h(k, i) = h(k) + i \bmod n$$

(sometimes: $h(k, i) = h(k) + ci \bmod n$).

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For quadratic probing and double hashing one has to ensure that the search covers all positions in the table (i.e., for double hashing $h_2(k)$ must be relatively prime to n (**teilerfremd**); for quadratic probing c_1 and c_2 have to be chosen carefully).

Linear Probing

- ▶ Advantage: **Cache-efficiency**. The new probe position is very likely to be in the cache.

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Lemma 10

Let L be the method of linear probing for resolving collisions:

$$L^+ \approx \frac{1}{2} \left(1 + \frac{1}{1 - \alpha} \right)$$

$$L^- \approx \frac{1}{2} \left(1 + \frac{1}{(1 - \alpha)^2} \right)$$

Quadratic Probing

- ▶ Not as cache-efficient as Linear Probing.
- ▶ **Secondary clustering**: caused by the fact that all keys mapped to the same position have the same probe sequence.

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Lemma 11

Let Q be the method of quadratic probing for resolving collisions:

$$Q^+ \approx 1 + \ln\left(\frac{1}{1-\alpha}\right) - \frac{\alpha}{2}$$

$$Q^- \approx \frac{1}{1-\alpha} + \ln\left(\frac{1}{1-\alpha}\right) - \alpha$$

Double Hashing

- ▶ Any probe into the hash-table usually creates a cache-miss.

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Lemma 12

Let D be the method of double hashing for resolving collisions:

$$D^+ \approx \frac{1}{\alpha} \ln \left(\frac{1}{1 - \alpha} \right)$$

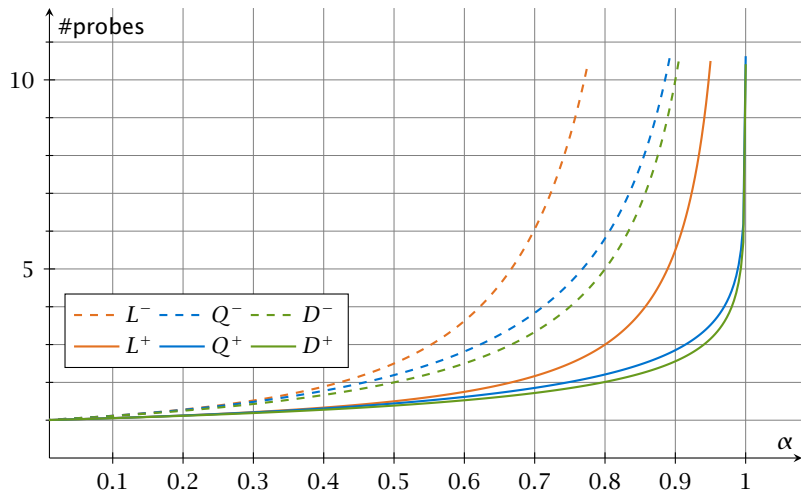
$$D^- \approx \frac{1}{1 - \alpha}$$

Open Addressing

Some values:

| α | <i>Linear Probing</i> | | <i>Quadratic Probing</i> | | <i>Double Hashing</i> | |
|----------|-----------------------|-------|--------------------------|-------|-----------------------|-------|
| | L^+ | L^- | Q^+ | Q^- | D^+ | D^- |
| 0.5 | 1.5 | 2.5 | 1.44 | 2.19 | 1.39 | 2 |
| 0.9 | 5.5 | 50.5 | 2.85 | 11.40 | 2.55 | 10 |
| 0.95 | 10.5 | 200.5 | 3.52 | 22.05 | 3.15 | 20 |

Open Addressing



Analysis of Idealized Open Address Hashing

We analyze the time for a search in a very idealized Open Addressing scheme.

- ▶ The probe sequence $h(k, 0), h(k, 1), h(k, 2), \dots$ is equally likely to be any permutation of $\langle 0, 1, \dots, n - 1 \rangle$.

Analysis of Idealized Open Address Hashing



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$$\Pr[X \geq i] = \frac{m}{n} \cdot \frac{m-1}{n-1} \cdot \frac{m-2}{n-2} \cdot \dots \cdot \frac{m-i+2}{n-i+2}$$

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$E[X]$

Analysis of Idealized Open Address Hashing

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Analysis of Idealized Open Address Hashing

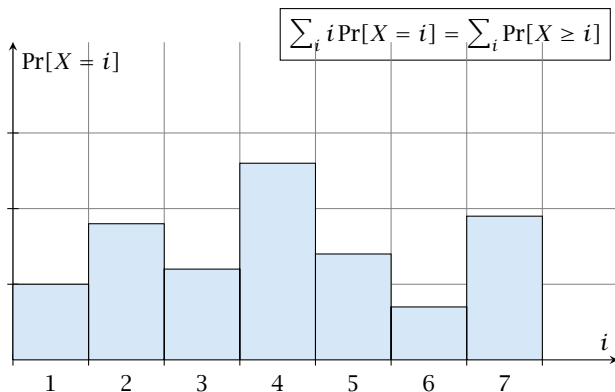
$$E[X] = \sum_{i=1}^{\infty} \Pr[X \geq i] \leq \sum_{i=1}^{\infty} \alpha^{i-1} = \sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha} .$$

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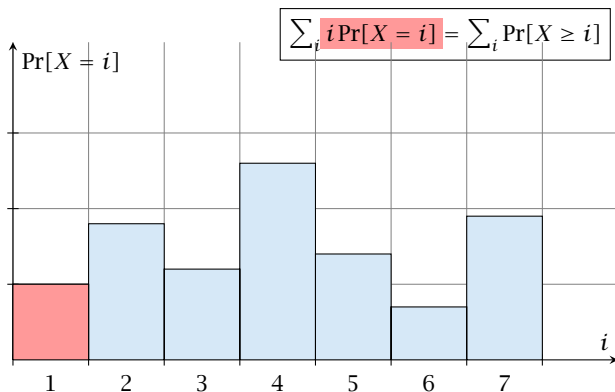
$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \dots$$

Analysis of Idealized Open Address Hashing



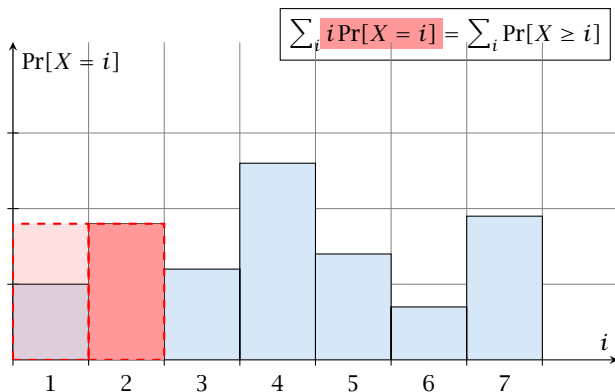
Analysis of Idealized Open Address Hashing

$i = 1$



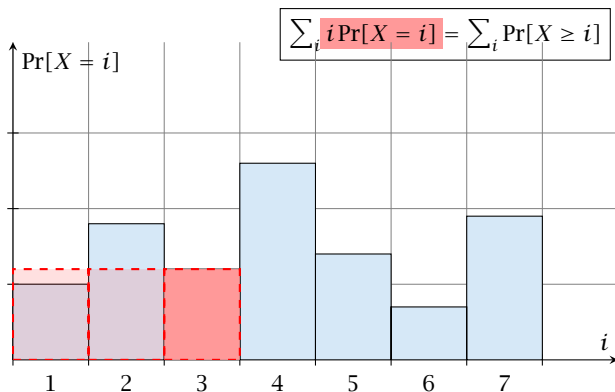
Analysis of Idealized Open Address Hashing

$i = 2$



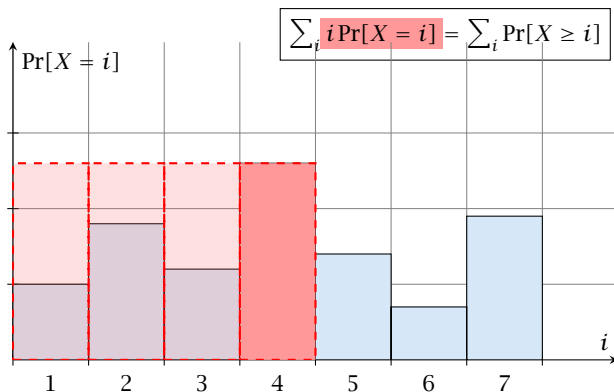
Analysis of Idealized Open Address Hashing

$i = 3$



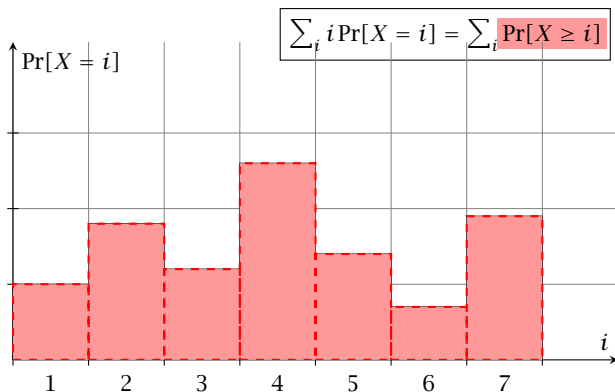
Analysis of Idealized Open Address Hashing

$i = 4$



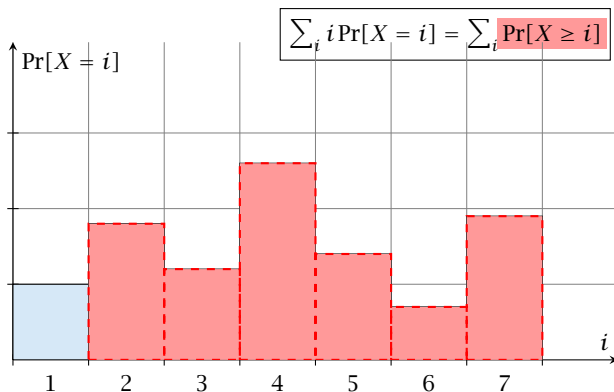
Analysis of Idealized Open Address Hashing

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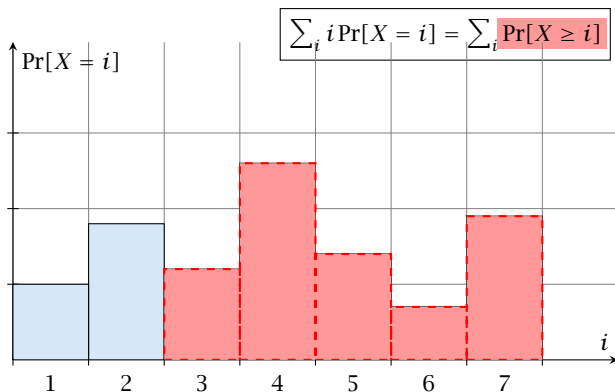
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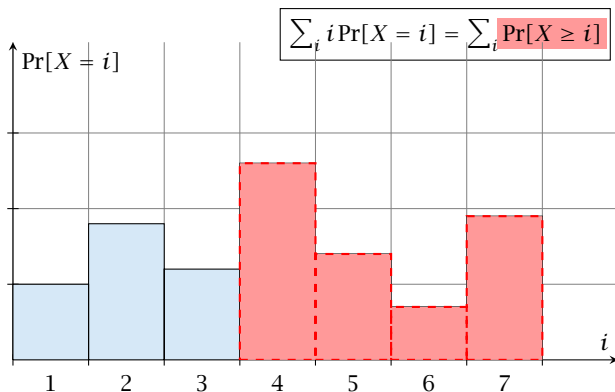
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$i = 3$

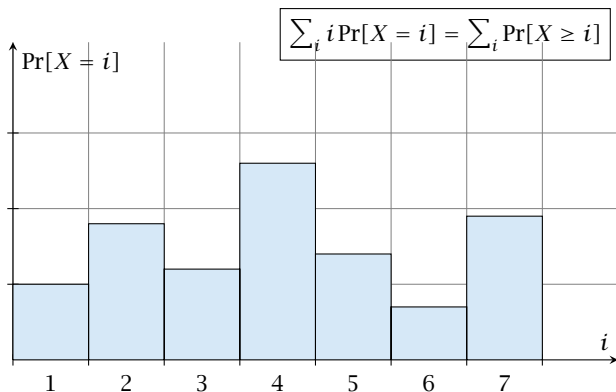


Analysis of Idealized Open Address Hashing

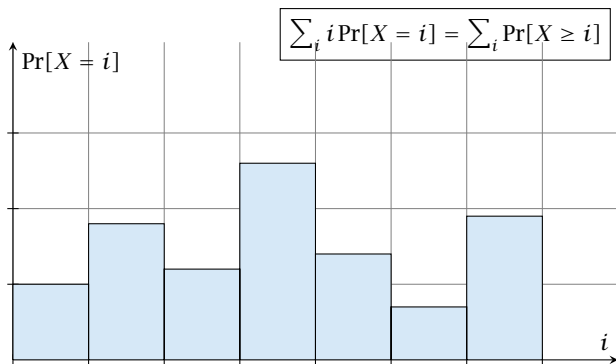
$i = 4$



Analysis of Idealized Open Address Hashing



Analysis of Idealized Open Address Hashing



The j -th rectangle appears in both sums j times. (j times in the first due to multiplication with j ; and j times in the second for summands $i = 1, 2, \dots, j$)

Analysis of Idealized Open Address Hashing



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$$\frac{1}{m} \sum_{i=0}^{m-1} \frac{n}{n-i} = \frac{n}{m} \sum_{i=0}^{m-1} \frac{1}{n-i} = \frac{1}{\alpha} \sum_{k=n-m+1}^n \frac{1}{k}$$

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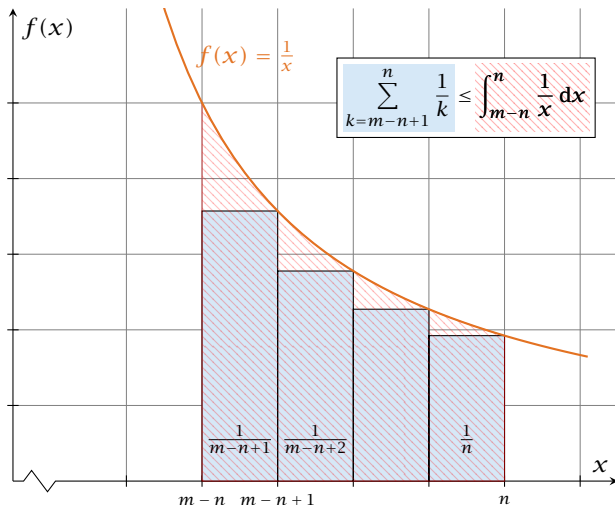
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- ▶ For open addressing this is difficult.

Deletions in Hashtables

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- ▶ The table could fill up with **deleted**-markers leading to bad performance.
- ▶ If a table contains many deleted-markers (linear fraction of the keys) one can rehash the whole table and amortize the cost for this rehash against the cost for the deletions.

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- ▶ For Linear Probing one can delete elements without using **deletion**-markers.

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- ▶ Upon a deletion elements that are further down in the probe-sequence may be moved to guarantee that they are still found during a search.

Deletions for Linear Probing

Algorithm 12 delete(p)

```
1:  $T[p] \leftarrow \text{null}$ 
2:  $p \leftarrow \text{succ}(p)$ 
3: while  $T[p] \neq \text{null}$  do
4:    $y \leftarrow T[p]$ 
5:    $T[p] \leftarrow \text{null}$ 
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p is the index into the table-cell that contains the object to be deleted.

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Pointers into the hash-table become invalid.

Universal Hashing



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Universal hashing tries to define a set \mathcal{H} of functions that is much smaller but still leads to good average case behaviour when selecting a hash-function uniformly at random from \mathcal{H} .

Universal Hashing

Definition 13

A class \mathcal{H} of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called **universal** if for all $u_1, u_2 \in U$ with $u_1 \neq u_2$

$$\Pr[h(u_1) = h(u_2)] \leq \frac{1}{n} ,$$

where the probability is w. r. t. the choice of a random hash-function from set \mathcal{H} .

Universal Hashing

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Note that this means that the probability of a collision between two arbitrary elements is at most $\frac{1}{n}$.

Universal Hashing

Definition 14

A class \mathcal{H} of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called **2-independent** (pairwise independent) if the following two conditions hold

- ▶ For any key $u \in U$, and $t \in \{0, \dots, n-1\}$ $\Pr[h(u) = t] = \frac{1}{n}$, i.e., a key is distributed uniformly within the hash-table.
- ▶ For all $u_1, u_2 \in U$ with $u_1 \neq u_2$, and for any two hash-positions t_1, t_2 :

$$\Pr[h(u_1) = t_1 \wedge h(u_2) = t_2] \leq \frac{1}{n^2} .$$

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This requirement clearly implies a universal hash-function.

Definition 15

A class \mathcal{H} of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called **k -independent** if for any choice of $\ell \leq k$ distinct keys $u_1, \dots, u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions t_1, \dots, t_ℓ :

$$\Pr[h(u_1) = t_1 \wedge \dots \wedge h(u_\ell) = t_\ell] \leq \frac{1}{n^\ell} ,$$

where the probability is w. r. t. the choice of a random hash-function from set \mathcal{H} .

Universal Hashing

Definition 16

A class \mathcal{H} of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called (μ, k) -independent if for any choice of $\ell \leq k$ distinct keys $u_1, \dots, u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions t_1, \dots, t_ℓ :

$$\Pr[h(u_1) = t_1 \wedge \dots \wedge h(u_\ell) = t_\ell] \leq \frac{\mu}{n^\ell},$$

where the probability is w. r. t. the choice of a random hash-function from set \mathcal{H} .

Universal Hashing



Universal Hashing

Let $U := \{0, \dots, p - 1\}$ for a prime p . Let $\mathbb{Z}_p := \{0, \dots, p - 1\}$, and let $\mathbb{Z}_p^* := \{1, \dots, p - 1\}$ denote the set of invertible elements in \mathbb{Z}_p .

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Lemma 17

The class

$$\mathcal{H} = \{h_{a,b} \mid a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$$

is a universal class of hash-functions from U to $\{0, \dots, n-1\}$.

Universal Hashing



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Proof.

Let $x, y \in U$ be two distinct keys. We have to show that the probability of a collision is only $1/n$.

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where we use that \mathbb{Z}_p is a field (Körper) and, hence, has no zero divisors (nullteilerfrei).

Universal Hashing

- ▶ The hash-function does not generate collisions before the $(\text{mod } n)$ -operation. Furthermore, every choice (a, b) is mapped to a different pair (t_x, t_y) with $t_x := ax + b$ and $t_y := ay + b$.

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$$t_y \equiv ay + b \pmod{p}$$

$$a \equiv (t_x - t_y)(x - y)^{-1} \pmod{p}$$

$$b \equiv t_y - ay \pmod{p}$$

Universal Hashing



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There is a one-to-one correspondence between hash-functions (pairs (a, b) , $a \neq 0$) and pairs (t_x, t_y) , $t_x \neq t_y$.

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What happens when we do the $\text{mod } n$ operation?

Fix a value t_x . There are $p - 1$ possible values for choosing t_y .

From the range $0, \dots, p - 1$ the values $t_x, t_x + n, t_x + 2n, \dots$ map to t_x after the modulo-operation. These are at most $\lceil p/n \rceil$ values.

Universal Hashing



Universal Hashing

As $t_y \neq t_x$ there are

$$\left[\frac{p}{n} \right] - 1$$

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This happens with probability at most $\frac{1}{n}$.

Universal Hashing



Universal Hashing

It is also possible to show that \mathcal{H} is an (almost) pairwise independent class of hash-functions.

$$\Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[\begin{array}{l} t_x \bmod n = h_1 \\ t_y \bmod n = h_2 \end{array} \right]$$

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$$\frac{\lfloor \frac{p}{n} \rfloor^2}{p(p-1)} \leq \Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[\begin{array}{l} t_x \bmod n = h_1 \\ t_y \bmod n = h_2 \end{array} \right] \leq \frac{\lfloor \frac{p}{n} \rfloor^2}{p(p-1)}$$

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Note that the middle is the probability that $h(x) = h_1$ and $h(y) = h_2$. The total number of choices for (t_x, t_y) is $p(p-1)$. The number of choices for t_x (t_y) such that $t_x \bmod n = h_1$ ($t_y \bmod n = h_2$) lies between $\lfloor \frac{p}{n} \rfloor$ and $\lceil \frac{p}{n} \rceil$.

Universal Hashing

Definition 18

Let $d \in \mathbb{N}$; $q \geq (d + 1)n$ be a prime; and let $\bar{a} \in \{0, \dots, q - 1\}^{d+1}$. Define for $x \in \{0, \dots, q - 1\}$

$$h_{\bar{a}}(x) := \left(\sum_{i=0}^d a_i x^i \bmod q \right) \bmod n .$$

Let $\mathcal{H}_n^d := \{h_{\bar{a}} \mid \bar{a} \in \{0, \dots, q - 1\}^{d+1}\}$. The class \mathcal{H}_n^d is $(e, d + 1)$ -independent.

Note that in the previous case we had $d = 1$ and chose $a_d \neq 0$.

Universal Hashing



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For the coefficients $\bar{a} \in \{0, \dots, q-1\}^{d+1}$ let $f_{\bar{a}}$ denote the polynomial

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The polynomial is defined by $d+1$ distinct points.

Universal Hashing

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Fix $\ell \leq d + 1$; let $x_1, \dots, x_\ell \in \{0, \dots, q - 1\}$ be keys, and let t_1, \dots, t_ℓ denote the corresponding hash-function values.

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Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

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We have

$$|B_1| \cdot \dots \cdot |B_\ell|$$

possibilities to do this (so that $h_{\bar{a}}(x_i) = t_i$).

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Now, we choose $d - \ell + 1$ other inputs and choose their value arbitrarily. We have $q^{d-\ell+1}$ possibilities to do this.

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Therefore we have

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possibilities to choose \bar{a} such that $h_{\bar{a}} \in A_\ell$.

Universal Hashing

Therefore the probability of choosing $h_{\bar{a}}$ from A_ℓ is only

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$$\begin{aligned}\frac{\lceil \frac{q}{n} \rceil^\ell \cdot q^{d-\ell+1}}{q^{d+1}} &\leq \frac{(\frac{q+n}{n})^\ell}{q^\ell} \leq \left(\frac{q+n}{q}\right)^\ell \cdot \frac{1}{n^\ell} \\ &\leq \left(1 + \frac{1}{\ell}\right)^\ell \cdot \frac{1}{n^\ell}\end{aligned}$$

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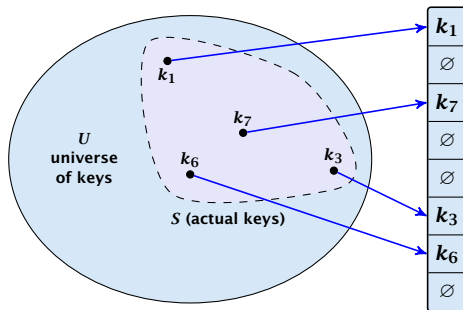
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This shows that the \mathcal{H} is $(e, d+1)$ -universal.

The last step followed from $q \geq (d+1)n$, and $\ell \leq d+1$.

Perfect Hashing

Suppose that we **know** the set S of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



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The probability of having **1** or more collisions can be at most $\frac{1}{2}$ as otherwise the expectation would be larger than $\frac{1}{2}$.

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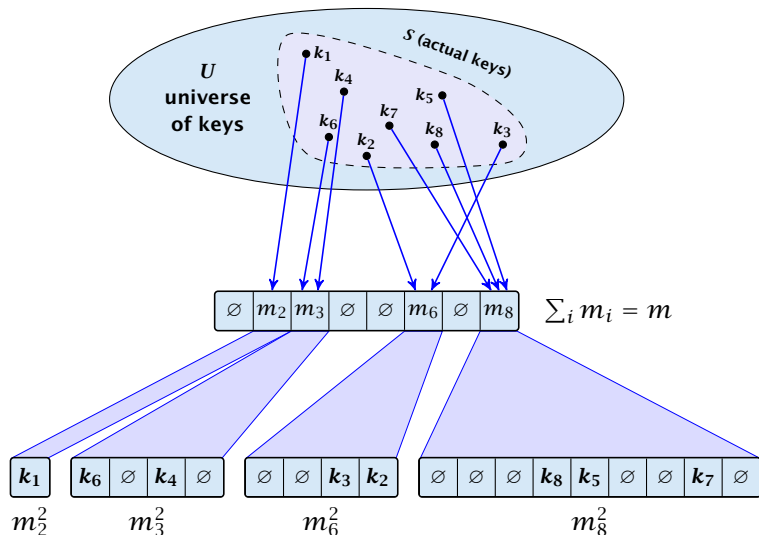
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However, a hash-table size of $n = m^2$ is very very high.

We construct a two-level scheme. We first use a hash-function that maps elements from S to m buckets.

Let m_j denote the number of items that are hashed to the j -th bucket. For each bucket we choose a second hash-function that maps the elements of the bucket into a table of size m_j^2 . The second function can be chosen such that all elements are mapped to different locations.

Perfect Hashing



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The total memory that is required by all hash-tables is $\mathcal{O}(\sum_j m_j^2)$.
Note that m_j is a random variable.

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The first expectation is simply the expected number of collisions, for the first level. Since we use universal hashing we have

Perfect Hashing

The total memory that is required by all hash-tables is $\mathcal{O}(\sum_j m_j^2)$. Note that m_j is a random variable.

$$\begin{aligned} \mathbb{E} \left[\sum_j m_j^2 \right] &= \mathbb{E} \left[2 \sum_j \binom{m_j}{2} + \sum_j m_j \right] \\ &= 2 \mathbb{E} \left[\sum_j \binom{m_j}{2} \right] + \mathbb{E} \left[\sum_j m_j \right] \end{aligned}$$

The first expectation is simply the expected number of collisions, for the first level. Since we use universal hashing we have

$$= 2 \binom{m}{2} \frac{1}{m} + m = 2m - 1 .$$

Perfect Hashing

We need only $\mathcal{O}(m)$ time to construct a hash-function h with $\sum_j m_j^2 = \mathcal{O}(4m)$, because with probability at least $1/2$ a random function from a universal family will have this property.

Then we construct a hash-table h_j for every bucket. This takes expected time $\mathcal{O}(m_j)$ for every bucket. A random function h_j is collision-free with probability at least $1/2$. We need $\mathcal{O}(m_j)$ to test this.

We only need that the hash-functions are chosen from a universal family!!!

Cuckoo Hashing



Cuckoo Hashing

Goal:

Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

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Cuckoo Hashing

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Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

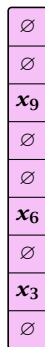
- ▶ Two hash-tables $T_1[0, \dots, n - 1]$ and $T_2[0, \dots, n - 1]$, with hash-functions h_1 , and h_2 .
- ▶ An object x is either stored at location $T_1[h_1(x)]$ or $T_2[h_2(x)]$.
- ▶ A search clearly takes constant time if the above constraint is met.

Cuckoo Hashing

Insert:



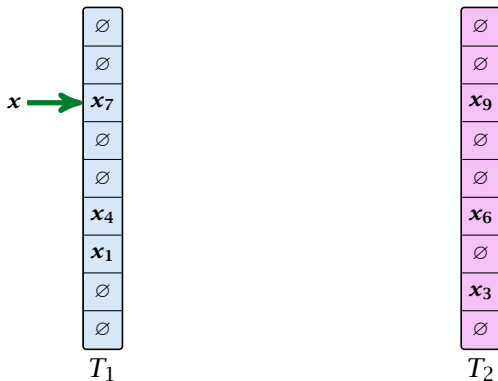
T_1



T_2

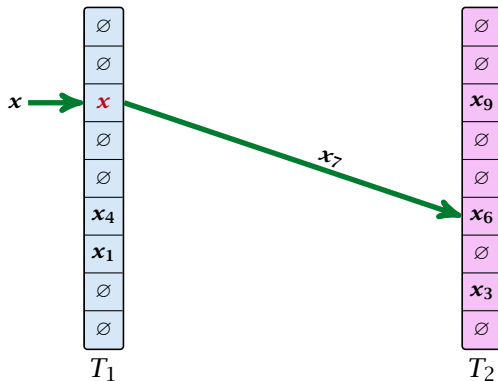
Cuckoo Hashing

Insert:



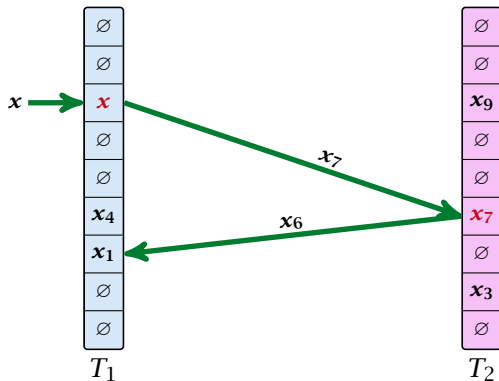
Cuckoo Hashing

Insert:



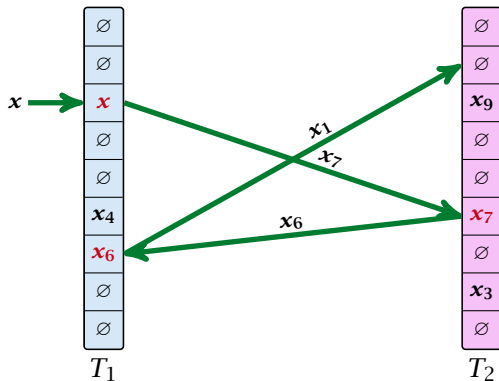
Cuckoo Hashing

Insert:



Cuckoo Hashing

Insert:



Algorithm 13 Cuckoo-Insert(x)

```
1: if  $T_1[h_1(x)] = x \vee T_2[h_2(x)] = x$  then return  
2: steps  $\leftarrow$  1  
3: while steps  $\leq$  maxsteps do  
4:     exchange  $x$  and  $T_1[h_1(x)]$   
5:     if  $x = \text{null}$  then return  
6:     exchange  $x$  and  $T_2[h_2(x)]$   
7:     if  $x = \text{null}$  then return  
8:     steps  $\leftarrow$  steps + 1  
9: rehash() // change hash-functions; rehash everything  
10: Cuckoo-Insert( $x$ )
```

Cuckoo Hashing

- ▶ We call one iteration through the while-loop a **step** of the algorithm.

Cuckoo Hashing

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- ▶ We call a sequence of iterations through the while-loop without the termination condition becoming true a **phase** of the algorithm.

Cuckoo Hashing

- ▶ We call one iteration through the while-loop a **step** of the algorithm.
- ▶ We call a sequence of iterations through the while-loop without the termination condition becoming true a **phase** of the algorithm.
- ▶ We say a phase is **successful** if it is not terminated by the **maxstep**-condition, but the while loop is left because $x = \text{null}$.

Cuckoo Hashing



What is the expected time for an insert-operation?

What is the expected time for an insert-operation?

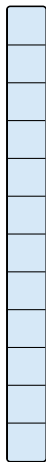
We first analyze the probability that we end-up in an infinite loop (that is then terminated after **maxsteps** steps).

What is the expected time for an insert-operation?

We first analyze the probability that we end-up in an infinite loop (that is then terminated after **maxsteps** steps).

Formally what is the probability to enter an infinite loop that touches s different keys?

Cuckoo Hashing: Insert

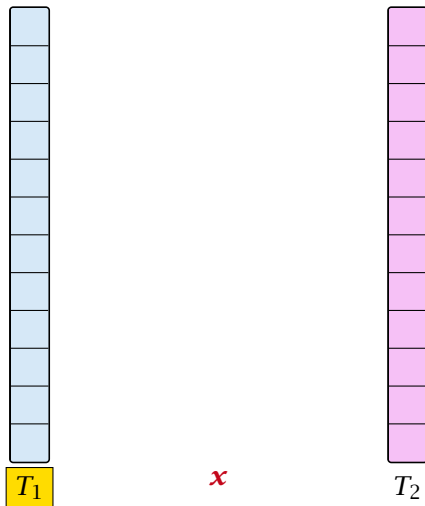


T_1

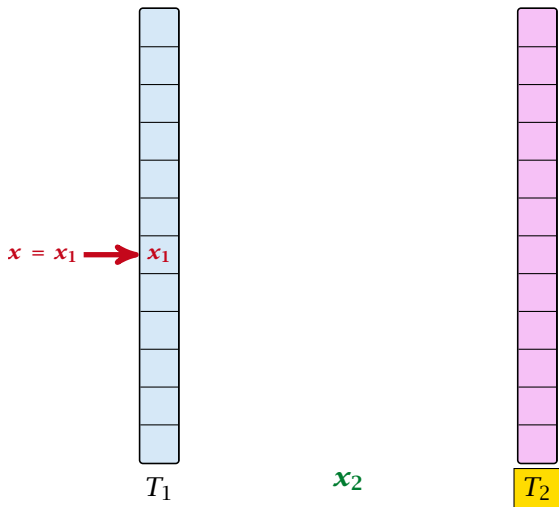


T_2

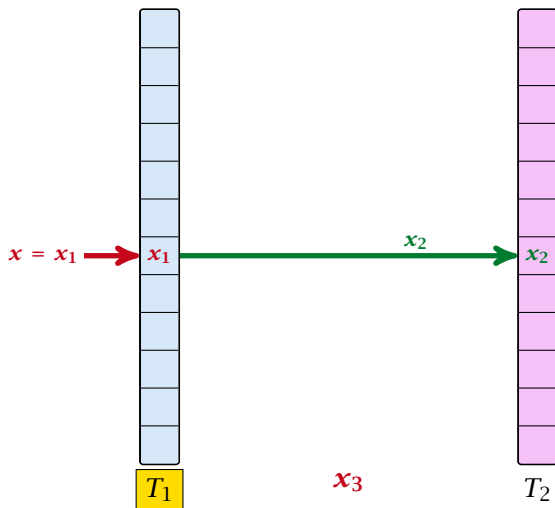
Cuckoo Hashing: Insert



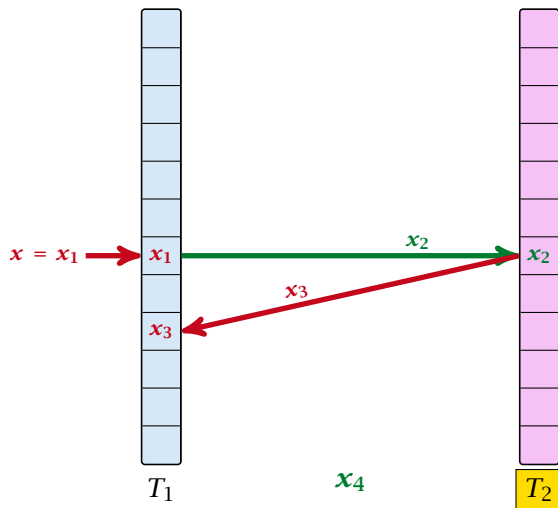
Cuckoo Hashing: Insert



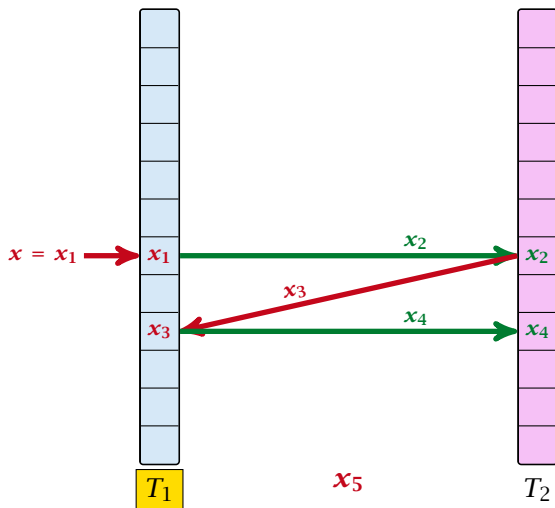
Cuckoo Hashing: Insert



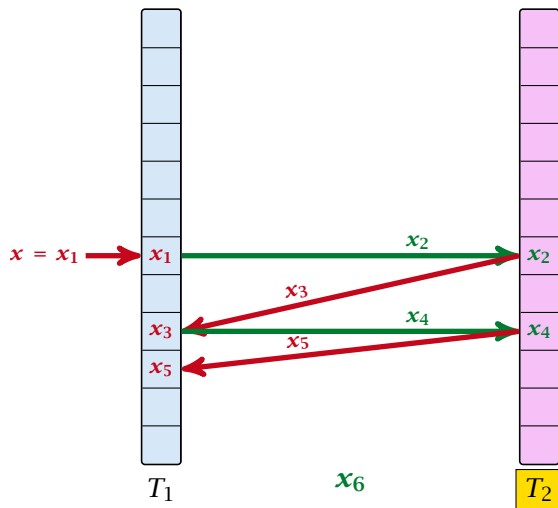
Cuckoo Hashing: Insert



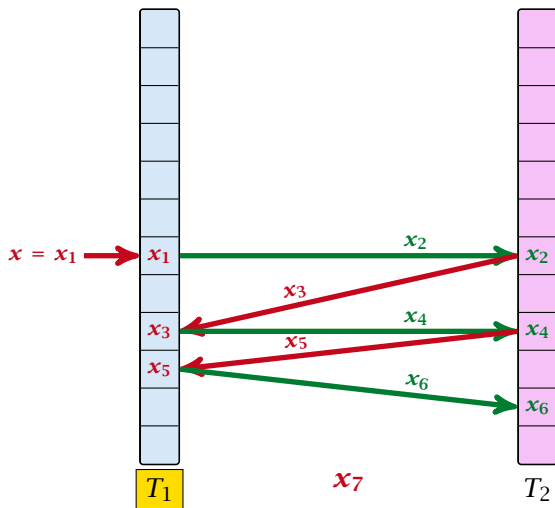
Cuckoo Hashing: Insert



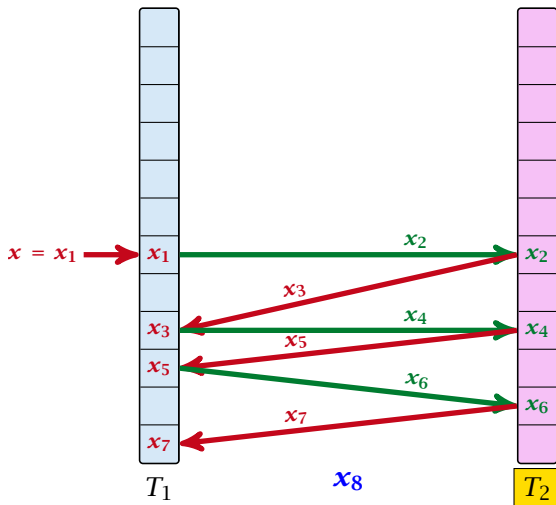
Cuckoo Hashing: Insert



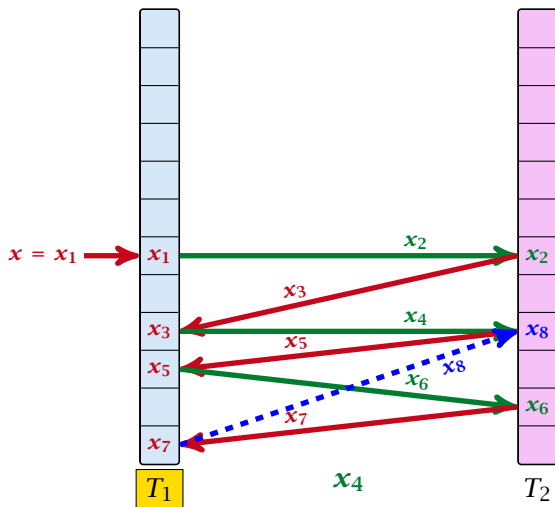
Cuckoo Hashing: Insert



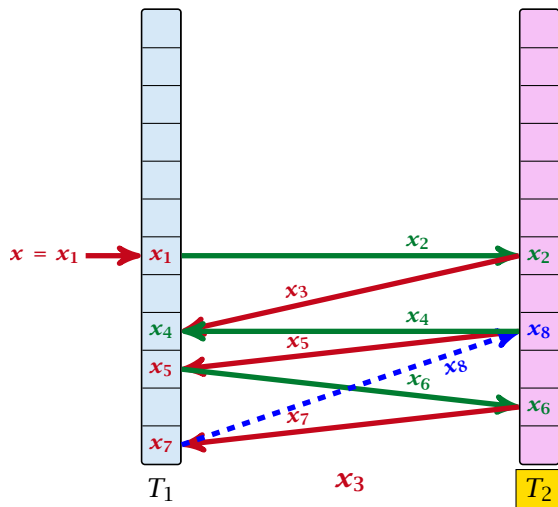
Cuckoo Hashing: Insert



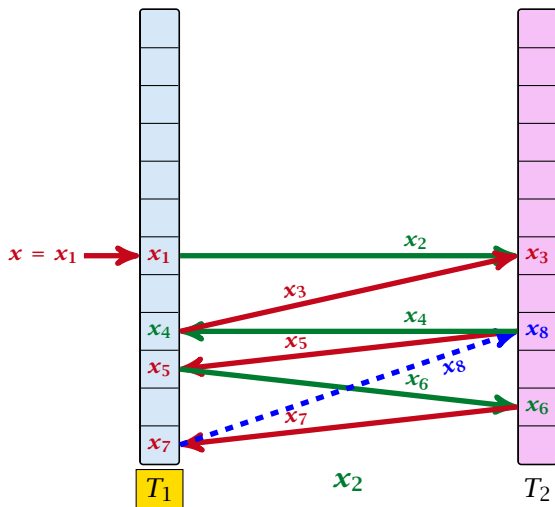
Cuckoo Hashing: Insert



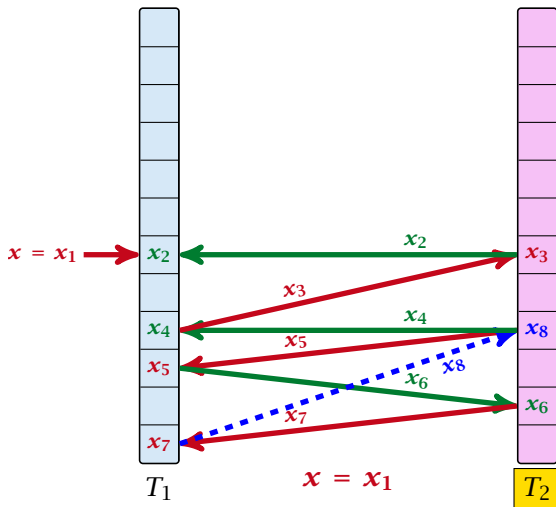
Cuckoo Hashing: Insert



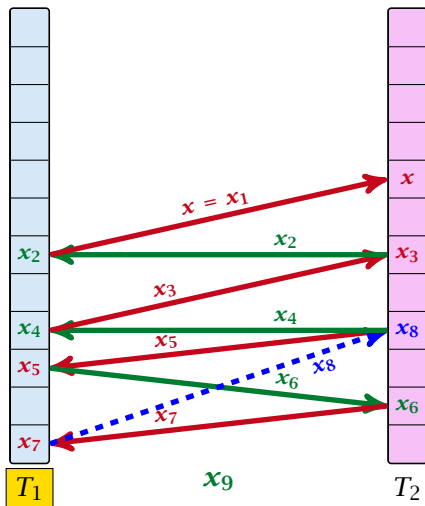
Cuckoo Hashing: Insert



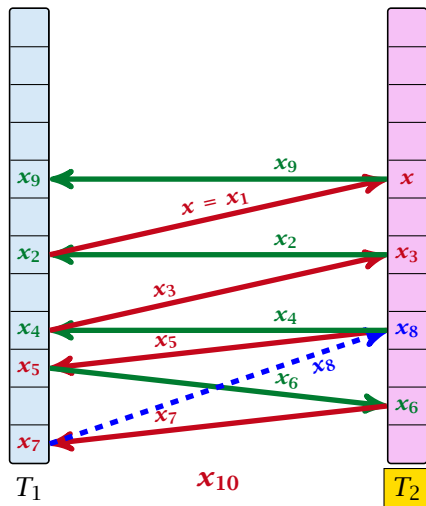
Cuckoo Hashing: Insert



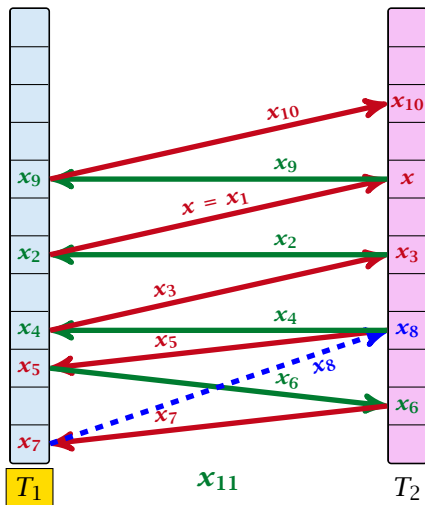
Cuckoo Hashing: Insert



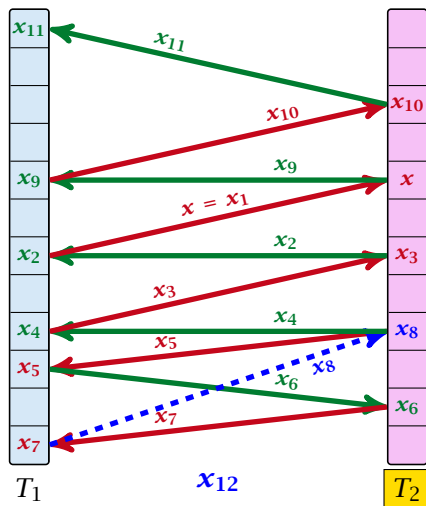
Cuckoo Hashing: Insert



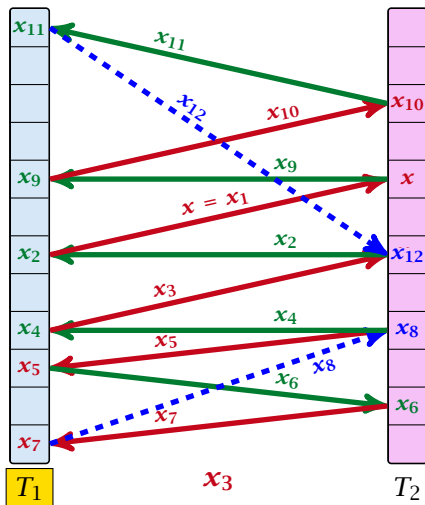
Cuckoo Hashing: Insert



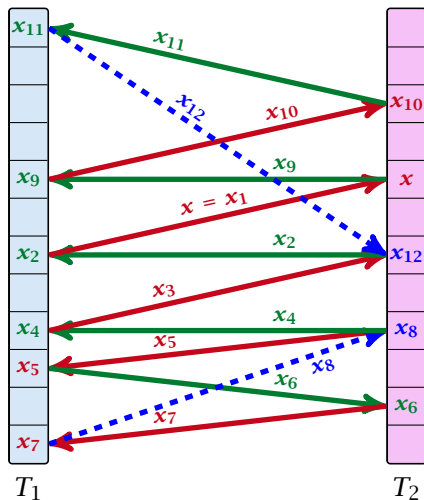
Cuckoo Hashing: Insert



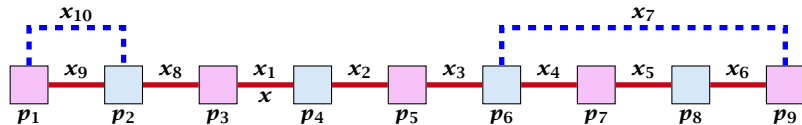
Cuckoo Hashing: Insert



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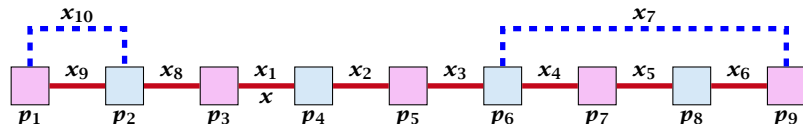


Cuckoo Hashing



A cycle-structure of size s is defined by

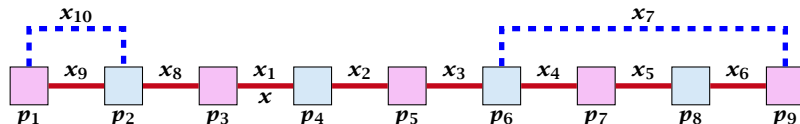
Cuckoo Hashing



A cycle-structure of size s is defined by

- ▶ $s - 1$ different cells (alternating btw. cells from T_1 and T_2).

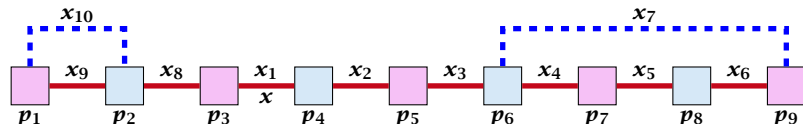
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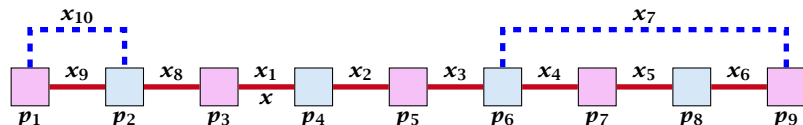
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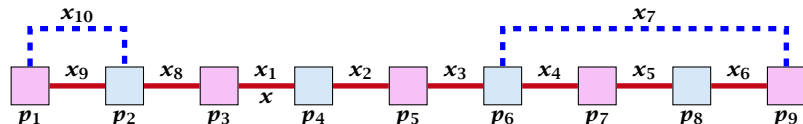
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- ▶ The leftmost cell is “linked forward” to some cell on the right.
- ▶ The rightmost cell is “linked backward” to a cell on the left.
- ▶ One link represents key x ; this is where the counting starts.

Cuckoo Hashing

A cycle-structure is **active** if for every key x_ℓ (linking a cell p_i from T_1 and a cell p_j from T_2) we have

$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

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Observation:

If during a phase the insert-procedure runs into a cycle there must exist an active cycle structure of size $s \geq 3$.

Cuckoo Hashing

What is the probability that all keys in a cycle-structure of size s correctly map into their T_1 -cell?

Cuckoo Hashing

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These events are independent.

Cuckoo Hashing

The probability that a given cycle-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

Cuckoo Hashing

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What is the probability that **there exists** an active cycle structure of size s ?

Cuckoo Hashing

The number of cycle-structures of size s is at most

$$s^3 \cdot n^{s-1} \cdot m^{s-1} .$$

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- ▶ There are at most s possibilities to choose where to place key x .
- ▶ There are m^{s-1} possibilities to choose the keys apart from x .
- ▶ There are n^{s-1} possibilities to choose the cells.

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} = \frac{\mu^2}{nm} \sum_{s=3}^{\infty} s^3 \left(\frac{m}{n}\right)^s$$

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

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Here we used the fact that $(1 + \epsilon)m \leq n$.

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The probability that there exists an active cycle-structure is therefore at most

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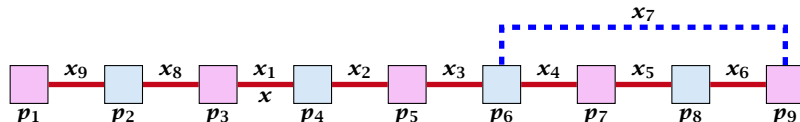
Hence,

$$\Pr[\text{cycle}] = \mathcal{O}\left(\frac{1}{m^2}\right).$$

Cuckoo Hashing

Now, we analyze the probability that a phase is not successful without running into a closed cycle.

Cuckoo Hashing



Sequence of visited keys:

$x = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_3, x_2, x_1 = x, x_8, x_9, \dots$

Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with x in the order that they are visited during the phase.

Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with x in the order that they are visited during the phase.

Lemma 19

*If the sequence is of length p then there exists a sub-sequence of at least $\frac{p+2}{3}$ keys starting with x of *distinct* keys.*

Cuckoo Hashing

Proof.

Let i be the number of keys (including x) that we see before the first repeated key. Let j denote the total number of distinct keys.

The sequence is of the form:

$$x = x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i \rightarrow x_r \rightarrow x_{r-1} \rightarrow \dots \rightarrow x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$$

As $r \leq i - 1$ the length p of the sequence is

$$p = i + r + (j - i) \leq i + j - 1 .$$

Cuckoo Hashing

Proof.

Let i be the number of keys (including x) that we see before the first repeated key. Let j denote the total number of distinct keys.

The sequence is of the form:

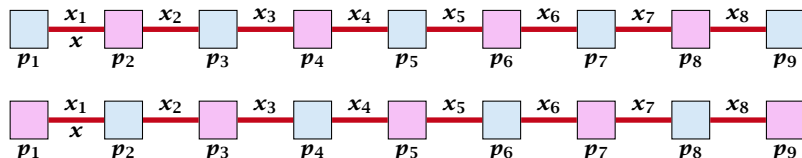
$$x = x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i \rightarrow x_r \rightarrow x_{r-1} \rightarrow \dots \rightarrow x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$$

As $r \leq i - 1$ the length p of the sequence is

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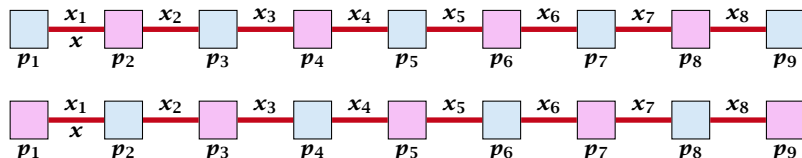
Either sub-sequence $x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i$ or sub-sequence $x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$ has at least $\frac{p+2}{3}$ elements. □

Cuckoo Hashing



A path-structure of size s is defined by

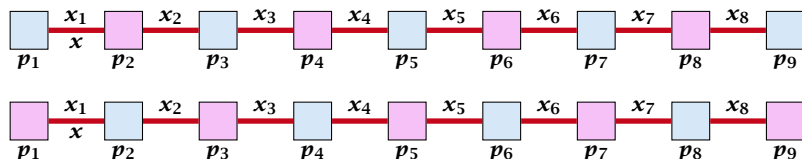
Cuckoo Hashing



A path-structure of size s is defined by

- ▶ $s + 1$ different cells (alternating btw. cells from T_1 and T_2).

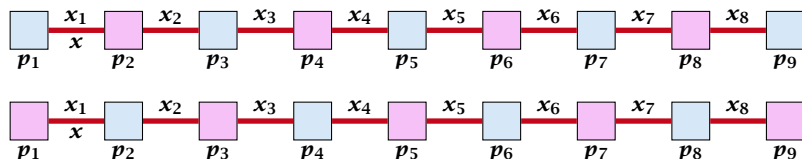
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A **path-structure of size s** is defined by

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Cuckoo Hashing



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- ▶ $s + 1$ different cells (alternating btw. cells from T_1 and T_2).
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Cuckoo Hashing

A path-structure is **active** if for every key x_ℓ (linking a cell p_i from T_1 and a cell p_j from T_2) we have

$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

Observation:

If a phase takes at least t steps without running into a cycle there must exist an active path-structure of size $(2t + 2)/3$.

Cuckoo Hashing

The probability that a given path-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

Cuckoo Hashing

The probability that a given path-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

The probability that there exists an active path-structure of size s is at most

$$2 \cdot n^{s+1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$

Cuckoo Hashing

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So far we estimated

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This means the expected cost for a successful phase is constant (even after accounting for the cost of the incomplete step that finishes the phase).

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Therefore the expected cost for re-hashes is $\mathcal{O}(m) \cdot \mathcal{O}(p) = \mathcal{O}(1)$.

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Therefore, it is sufficient to have $(\mu, \Theta(\log m))$ -independent hash-functions.

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- ▶ Whenever m drops below $\alpha n/4$ we divide n by 2 and do a rehash (**table-shrink**).
- ▶ Note that right after a change in table-size we have $m = \alpha n/2$. In order for a table-expand to occur at least $\alpha n/2$ insertions are required. Similar, for a table-shrink at least $\alpha n/4$ deletions must occur.

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How do we make sure that $n \geq (1 + \epsilon)m$?

- ▶ Let $\alpha := 1/(1 + \epsilon)$.
- ▶ Keep track of the number of elements in the table. When $m \geq \alpha n$ we double n and do a complete re-hash (**table-expand**).
- ▶ Whenever m drops below $\alpha n/4$ we divide n by 2 and do a rehash (**table-shrink**).
- ▶ Note that right after a change in table-size we have $m = \alpha n/2$. In order for a table-expand to occur at least $\alpha n/2$ insertions are required. Similar, for a table-shrink at least $\alpha n/4$ deletions must occur.
- ▶ Therefore we can amortize the rehash cost after a change in table-size against the cost for insertions and deletions.

Cuckoo Hashing

Lemma 20

Cuckoo Hashing has an expected constant insert-time and a worst-case constant search-time.

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Note that the above lemma only holds if the fill-factor (number of keys/total number of hash-table slots) is at most $\frac{1}{2(1+\epsilon)}$.

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Sometimes we also have

- ▶ **S . merge(S')**: $S := S \cup S'$; $S' := \emptyset$.

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- ▶ **S . decrease-key(h, k)**: Decreases the key of the element specified by handle h to k . Assumes that the key is at least k before the operation.

Dijkstra's Shortest Path Algorithm

Algorithm 1 Shortest-Path($G = (V, E, d), s \in V$)

```
1: Input: weighted graph  $G = (V, E, d)$ ; start vertex  $s$ ;  
2: Output: key-field of every node contains distance from  $s$ ;  
3:  $S.build()$ ; // build empty priority queue  
4: for all  $v \in V \setminus \{s\}$  do  
5:      $v.key \leftarrow \infty$ ;  
6:      $h_v \leftarrow S.insert(v)$ ;  
7:  $s.key \leftarrow 0$ ;  $S.insert(s)$ ;  
8: while  $S.is-empty() = false$  do  
9:      $v \leftarrow S.delete-min()$ ;  
10:    for all  $x \in V$  s.t.  $(v, x) \in E$  do  
11:        if  $x.key > v.key + d(v, x)$  then  
12:             $S.decrease-key(h_x, v.key + d(v, x))$ ;  
13:             $x.key \leftarrow v.key + d(v, x)$ ;
```

Prim's Minimum Spanning Tree Algorithm

Algorithm 2 Prim-MST($G = (V, E, d), s \in V$)

```
1: Input: weighted graph  $G = (V, E, d)$ ; start vertex  $s$ ;  
2: Output: pred-fields encode MST;  
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14:             $x.pred \leftarrow v$ ;
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Analysis of Dijkstra and Prim

Both algorithms require:

- ▶ 1 build() operation
- ▶ $|V|$ insert() operations
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How good a running time can we obtain?

8 Priority Queues

| <i>Operation</i> | <i>Binary Heap</i> | <i>BST</i> | <i>Binomial Heap</i> | <i>Fibonacci Heap*</i> |
|------------------|--------------------|------------|----------------------|------------------------|
| build | n | $n \log n$ | $n \log n$ | n |
| minimum | 1 | $\log n$ | $\log n$ | 1 |
| is-empty | 1 | 1 | 1 | 1 |
| insert | $\log n$ | $\log n$ | $\log n$ | 1 |
| delete | $\log n^{**}$ | $\log n$ | $\log n$ | $\log n$ |
| delete-min | $\log n$ | $\log n$ | $\log n$ | $\log n$ |
| decrease-key | $\log n$ | $\log n$ | $\log n$ | 1 |
| merge | n | $n \log n$ | $\log n$ | 1 |

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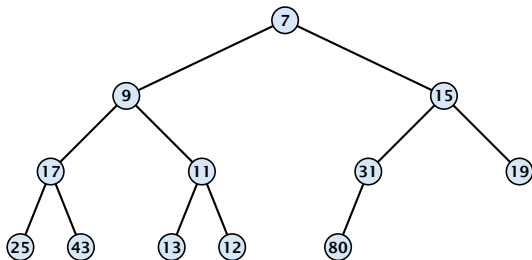
Fibonacci heaps only give an **amortized** guarantee.

8 Priority Queues

Using Binary Heaps, Prim and Dijkstra run in time $\mathcal{O}((|V| + |E|) \log |V|)$.

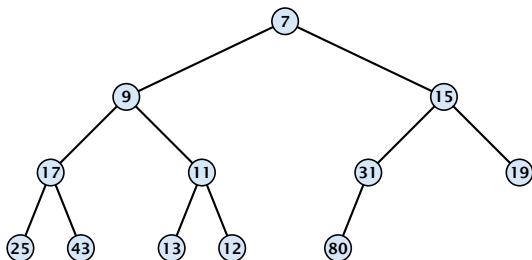
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8.1 Binary Heaps



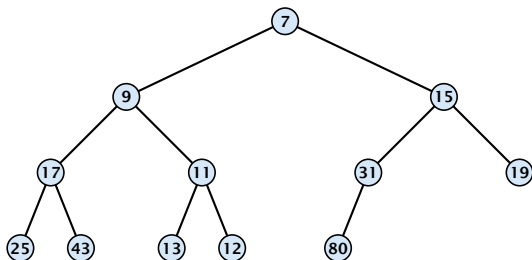
8.1 Binary Heaps

- ▶ Nearly complete binary tree; only the last level is not full, and this one is filled from left to right.



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- ▶ Nearly complete binary tree; only the last level is not full, and this one is filled from left to right.
- ▶ **Heap property:** A node's key is not larger than the key of one of its children.



Binary Heaps

Operations:

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Operations:

- ▶ **minimum()**: return the root-element. Time $\mathcal{O}(1)$.

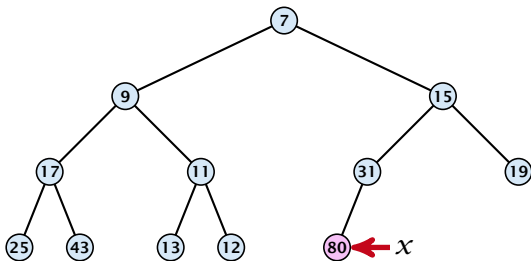
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8.1 Binary Heaps

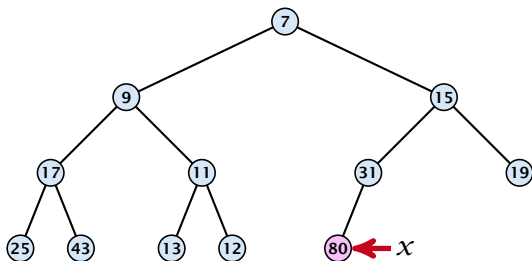
Maintain a pointer to the **last element** x .



8.1 Binary Heaps

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- ▶ We can compute the predecessor of x (last element when x is deleted) in time $\mathcal{O}(\log n)$.



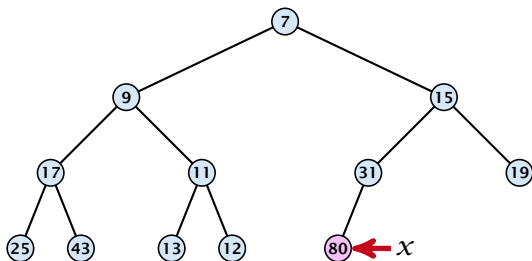
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go up until the last edge used was a right edge.

go left; go right until you reach a leaf



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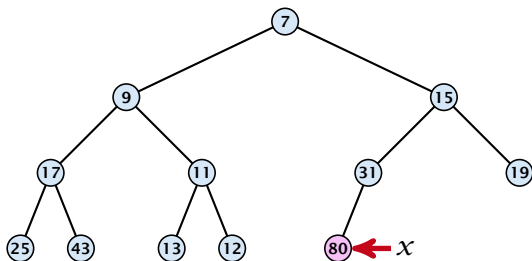
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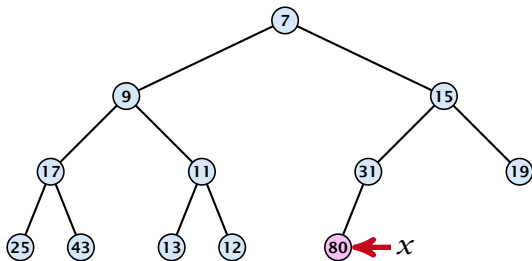
go left; go right until you reach a leaf

if you hit the root on the way up, go to the rightmost element



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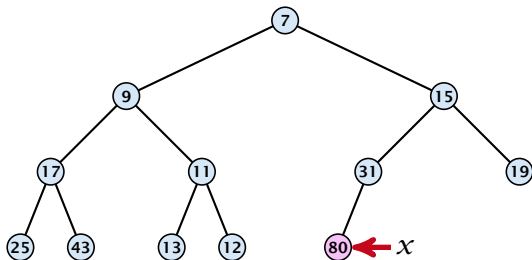
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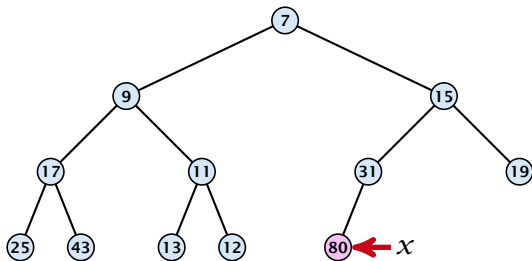
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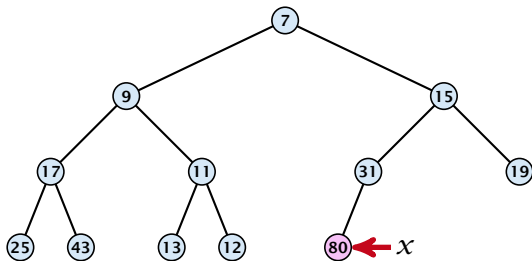
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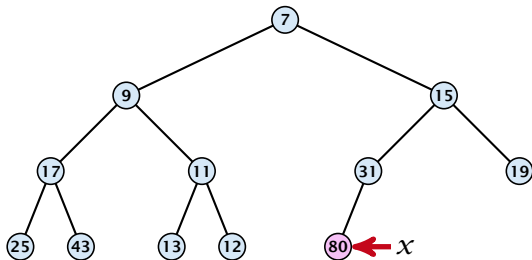
if you hit the root on the way up, go to the leftmost element;

insert a new element as a left child;



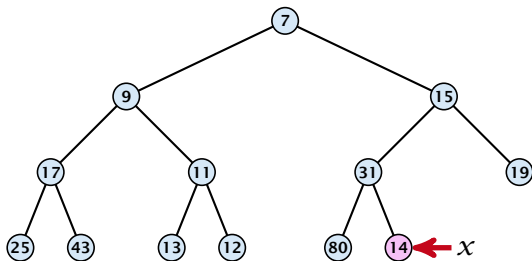
Insert

1. Insert element at successor of x .



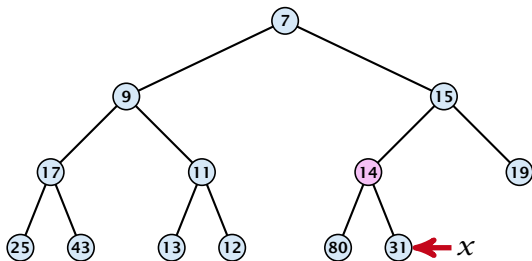
Insert

1. Insert element at successor of x .
2. Exchange with parent until heap property is fulfilled.



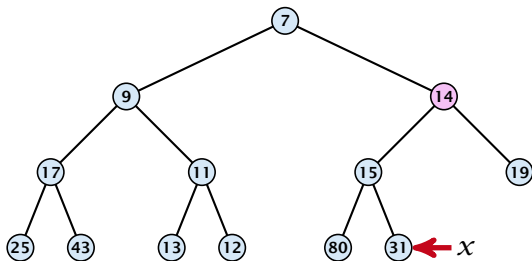
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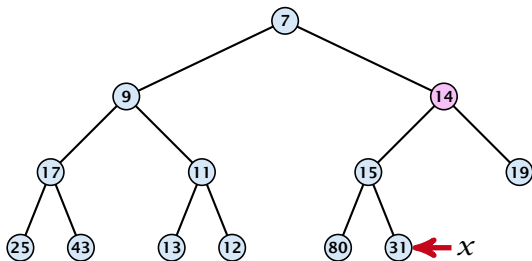
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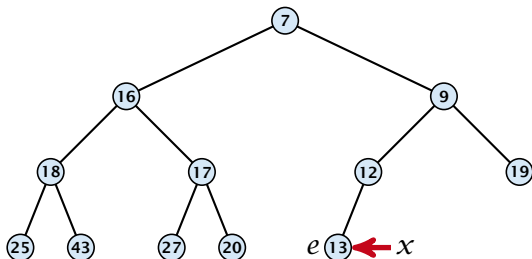
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Note that an exchange can either be done by moving the data or by changing pointers. The latter method leads to an addressable priority queue.

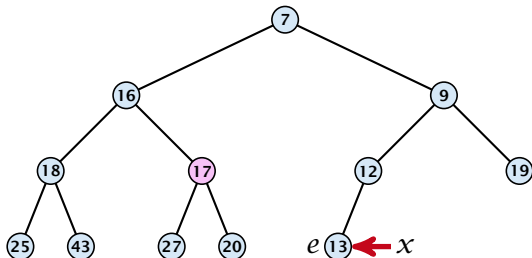
Delete

1. Exchange the element to be deleted with the element e pointed to by x .



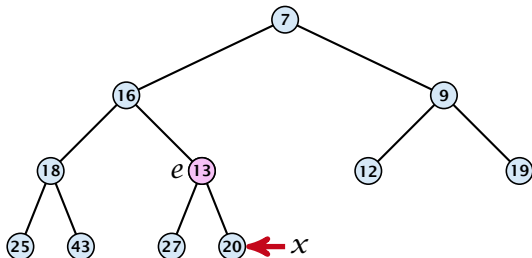
Delete

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2. Restore the heap-property for the element e .



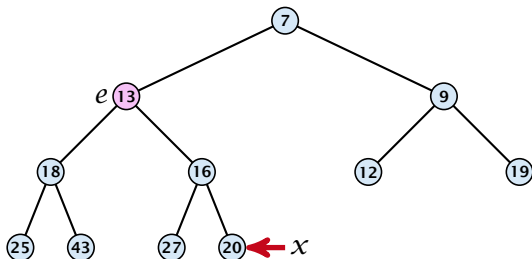
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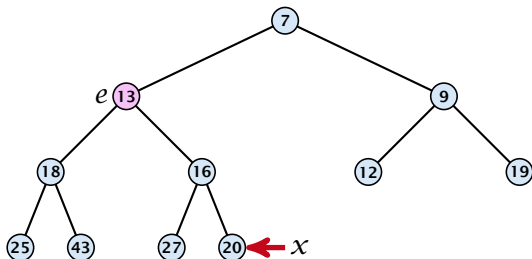
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At its new position e may either travel up or down in the tree (but not both directions).

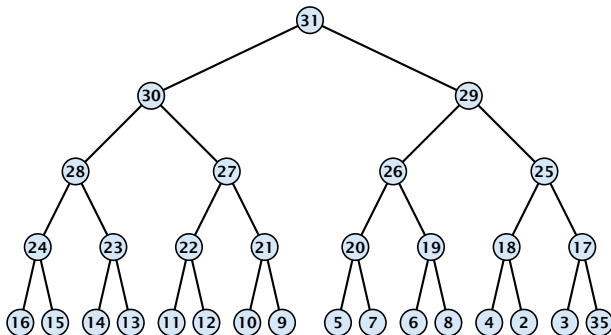
Binary Heaps

Operations:

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- ▶ **insert(k)**: insert at successor of x and bubble up. Time $\mathcal{O}(\log n)$.
- ▶ **delete(h)**: swap with x and bubble up or sift-down. Time $\mathcal{O}(\log n)$.

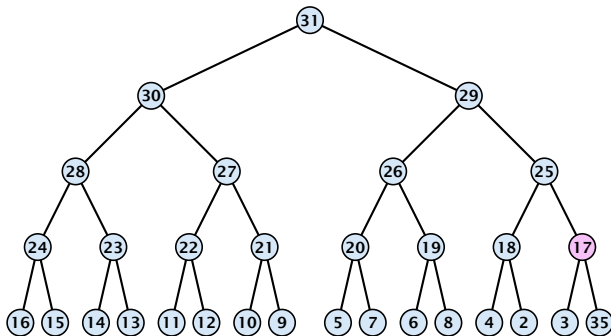
Build Heap

We can build a heap in linear time:



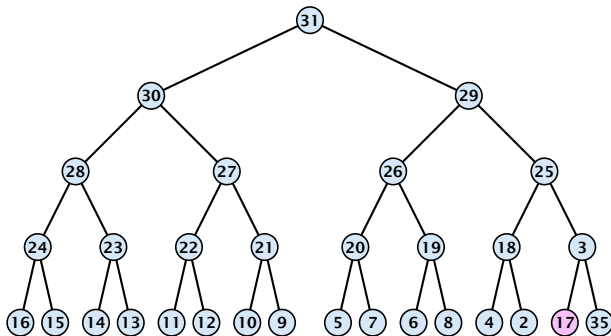
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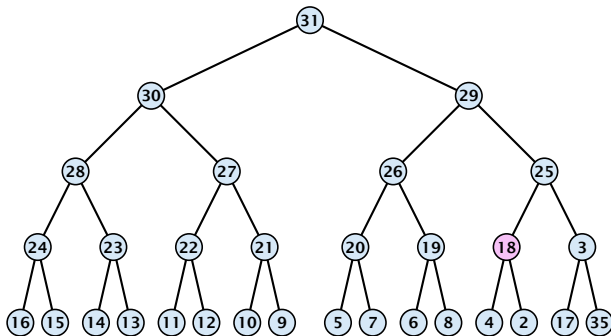
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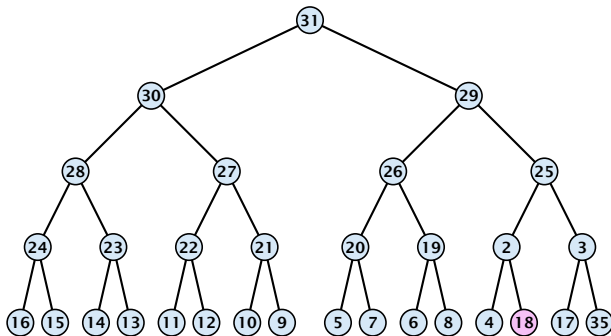
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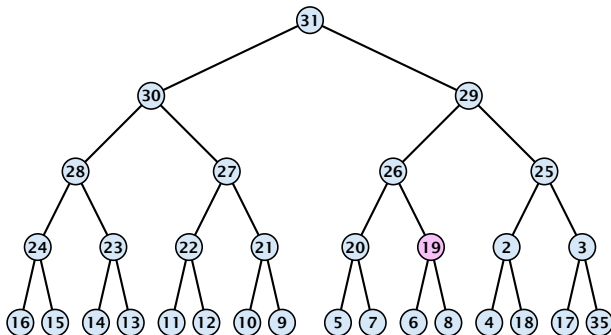
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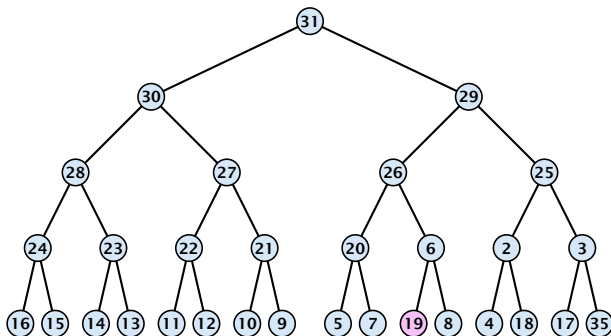
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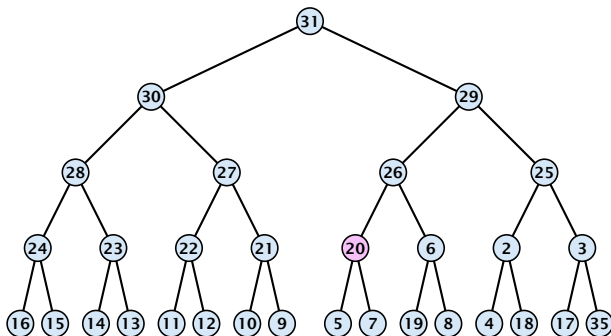
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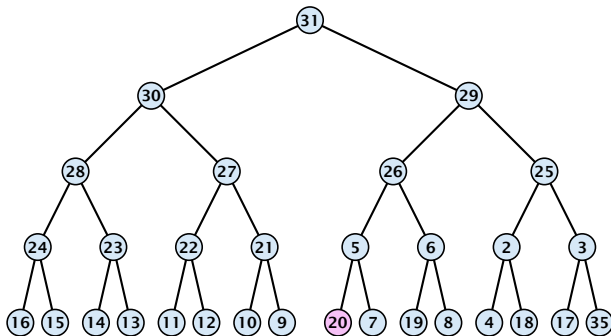
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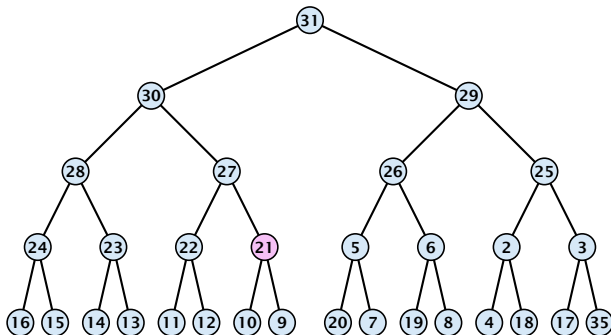
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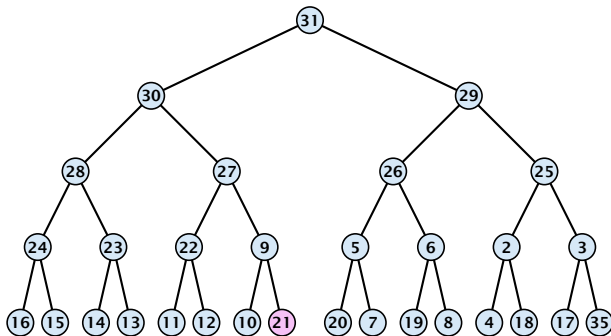
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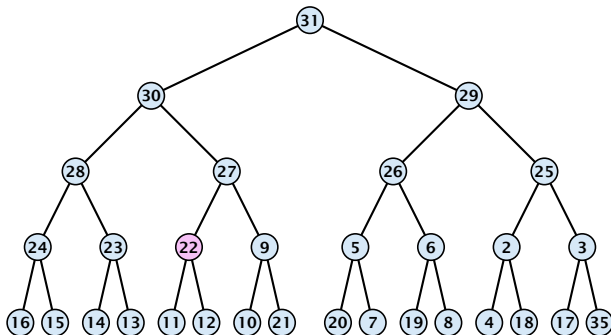
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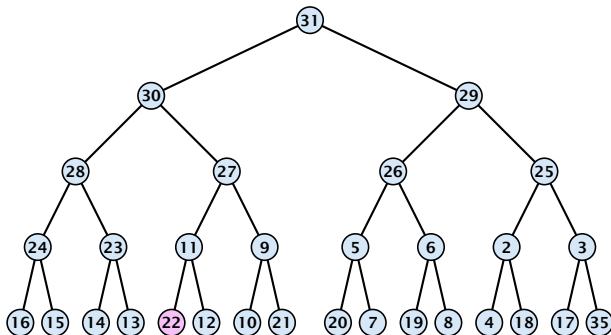
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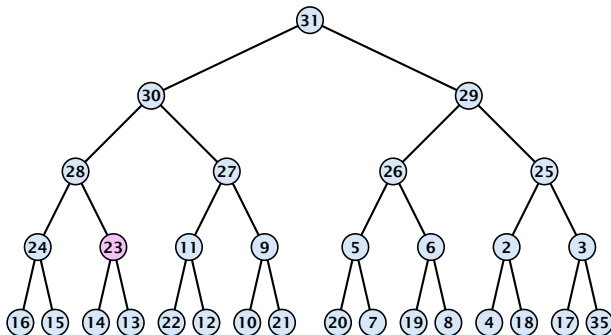
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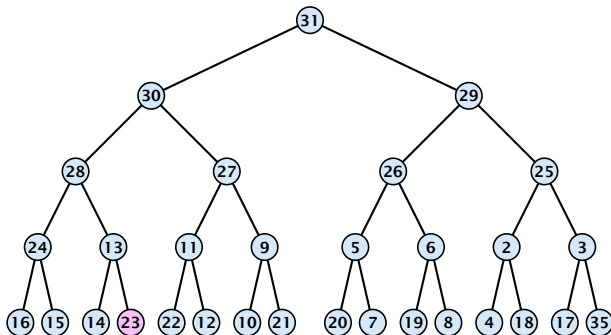
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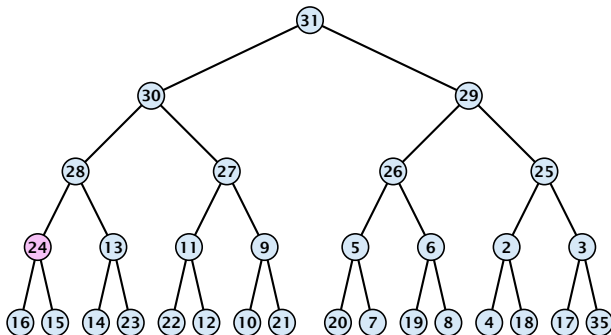
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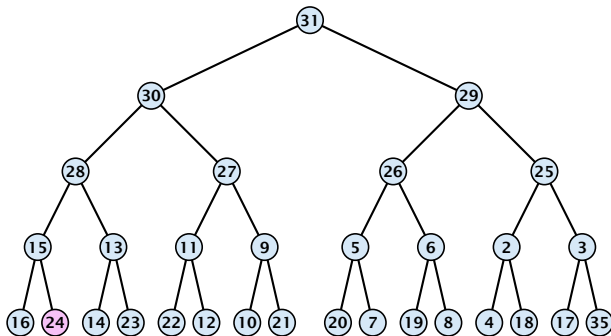
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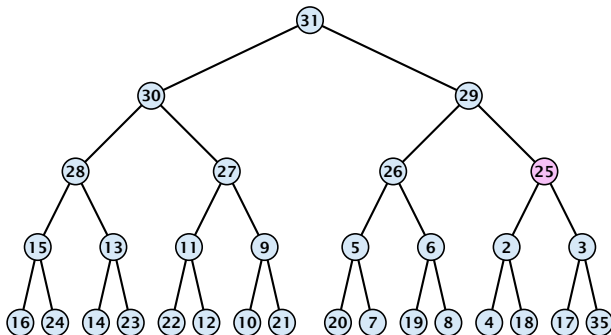
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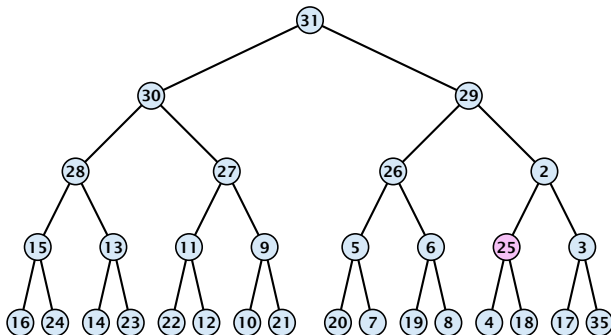
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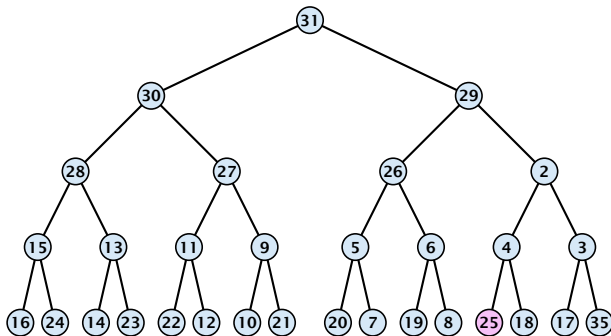
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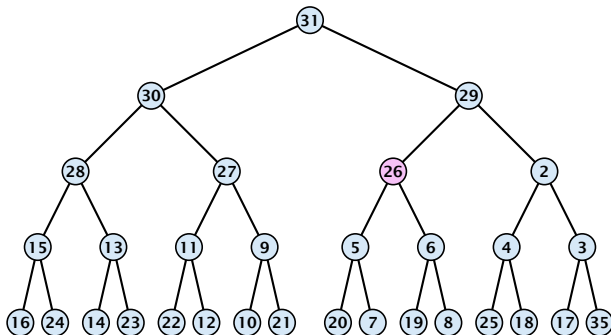
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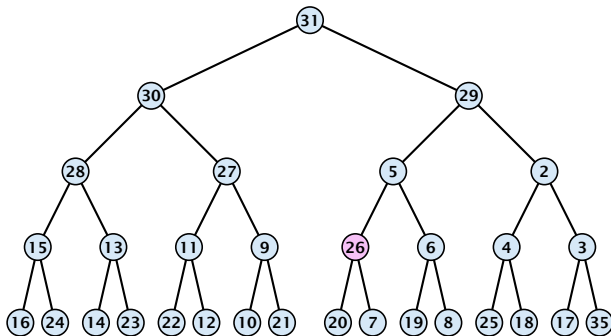
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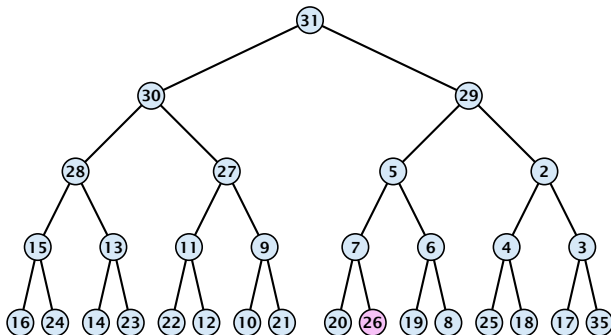
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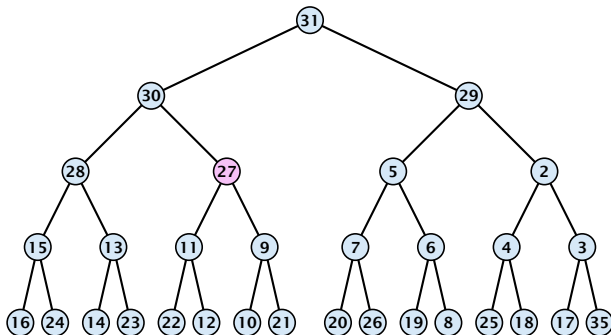
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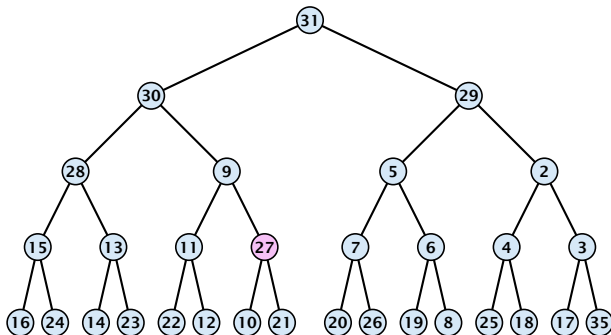
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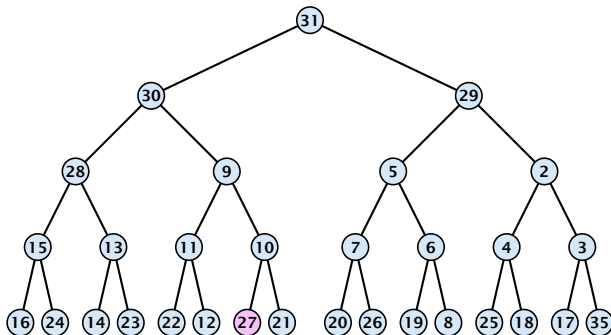
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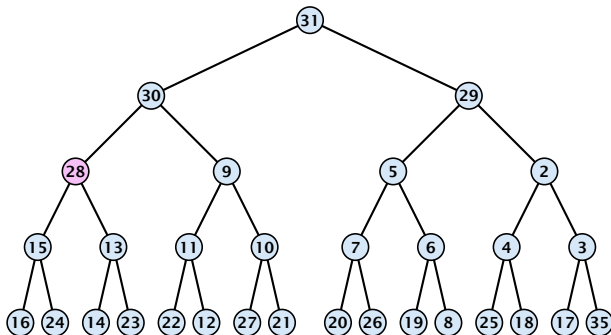
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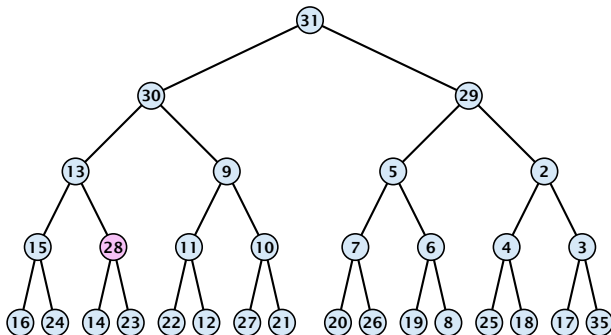
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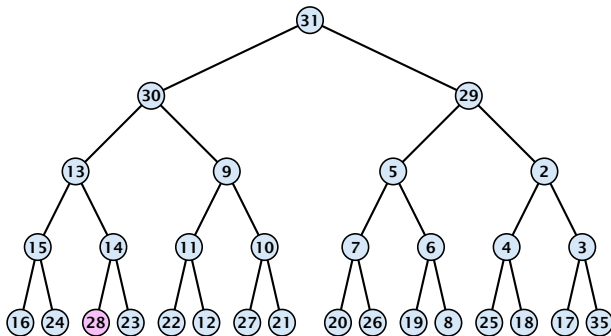
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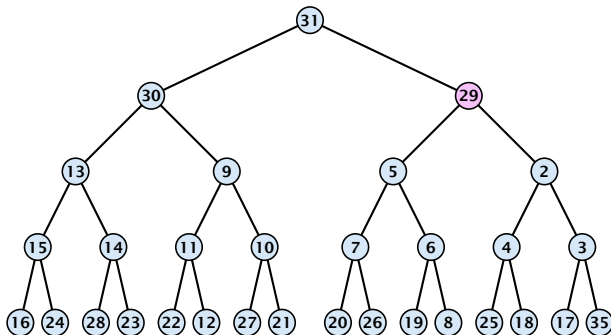
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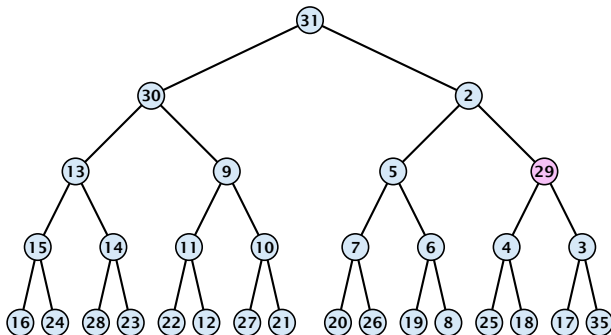
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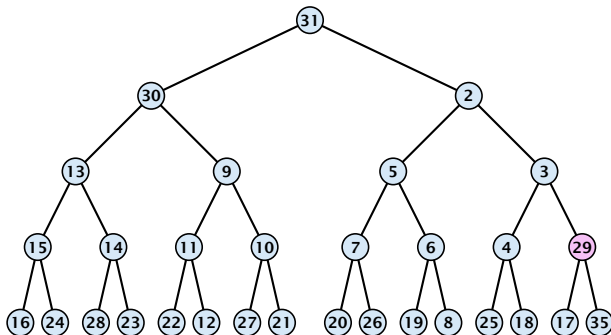
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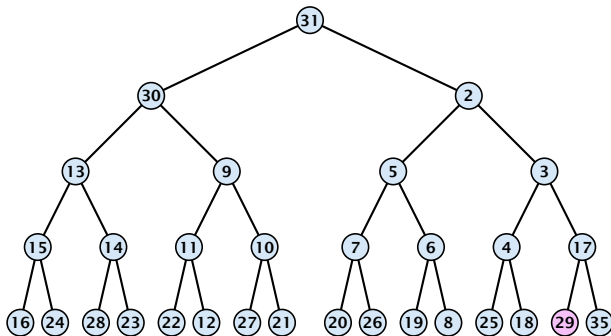
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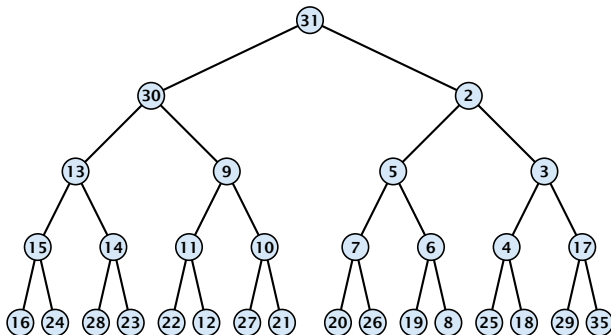
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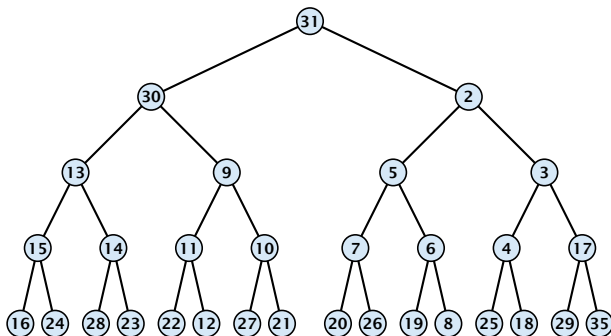
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We can build a heap in linear time:



$$\sum_{\text{levels } \ell} 2^\ell \cdot (h - \ell) = \sum_i i 2^{h-i} = \mathcal{O}(2^h) = \mathcal{O}(n)$$

Binary Heaps

Operations:

- ▶ **minimum()**: Return the root-element. Time $\mathcal{O}(1)$.
- ▶ **is-empty()**: Check whether root-pointer is **null**. Time $\mathcal{O}(1)$.
- ▶ **insert(k)**: Insert at x and bubble up. Time $\mathcal{O}(\log n)$.
- ▶ **delete(h)**: Swap with x and bubble up or sift-down. Time $\mathcal{O}(\log n)$.
- ▶ **build(x_1, \dots, x_n)**: Insert elements arbitrarily; then do sift-down operations starting with the lowest layer in the tree. Time $\mathcal{O}(n)$.

Binary Heaps



Binary Heaps

The standard implementation of binary heaps is via arrays. Let $A[0, \dots, n - 1]$ be an array

- ▶ The parent of i -th element is at position $\lfloor \frac{i-1}{2} \rfloor$.
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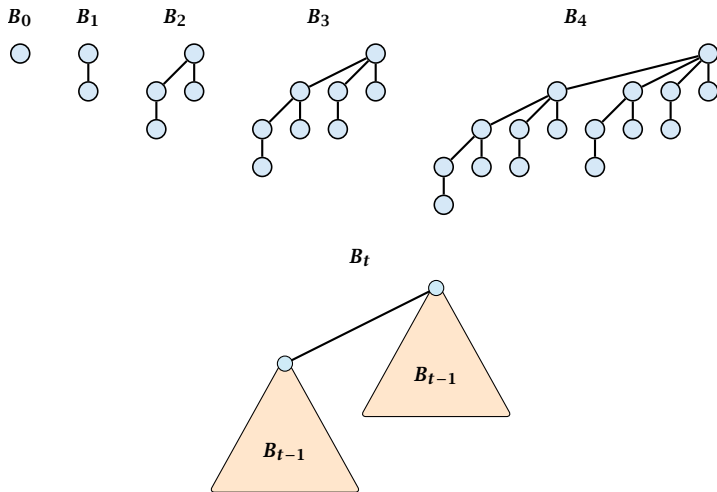
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The resulting binary heap is not addressable. The elements don't maintain their positions and therefore there are no stable handles.

8.2 Binomial Heaps

| <i>Operation</i> | <i>Binary Heap</i> | <i>BST</i> | <i>Binomial Heap</i> | <i>Fibonacci Heap*</i> |
|------------------|--------------------|------------|----------------------------|------------------------|
| build | n | $n \log n$ | $n \log n$ | n |
| minimum | 1 | $\log n$ | $\log n$ | 1 |
| is-empty | 1 | 1 | 1 | 1 |
| insert | $\log n$ | $\log n$ | $\log n$ | 1 |
| delete | $\log n^{**}$ | $\log n$ | $\log n$ | $\log n$ |
| delete-min | $\log n$ | $\log n$ | $\log n$ | $\log n$ |
| decrease-key | $\log n$ | $\log n$ | $\log n$ | 1 |
| merge | n | $n \log n$ | $\log n$ | 1 |

Binomial Trees



Properties of Binomial Trees

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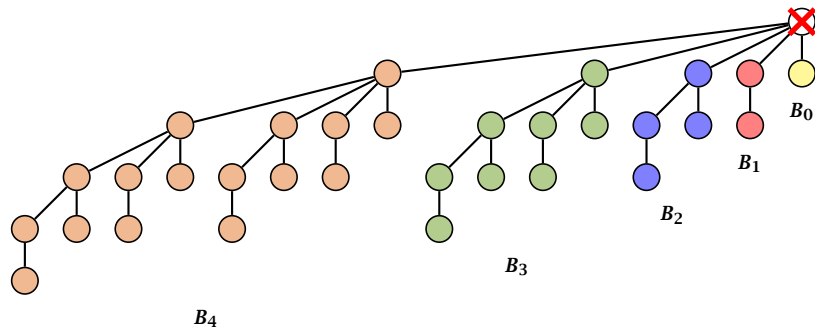
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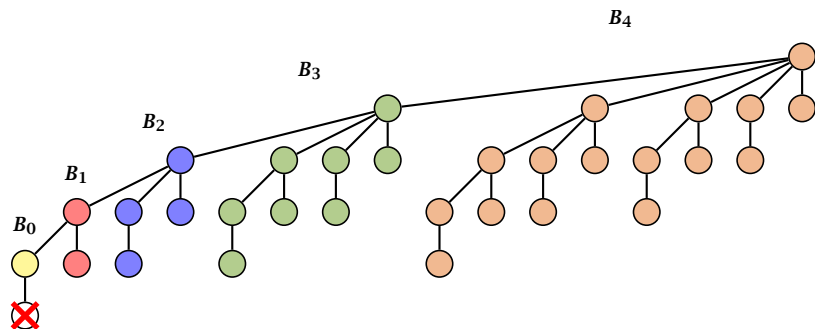
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- ▶ The root of B_k has degree k .
- ▶ B_k has $\binom{k}{\ell}$ nodes on level ℓ .
- ▶ Deleting the root of B_k gives trees B_0, B_1, \dots, B_{k-1} .

Binomial Trees



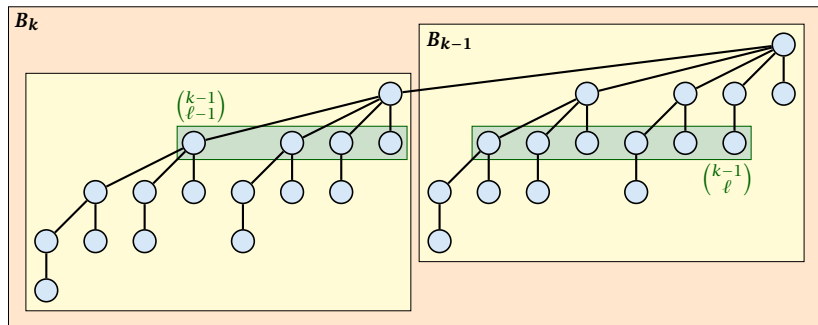
Deleting the root of B_5 leaves sub-trees B_4 , B_3 , B_2 , B_1 , and B_0 .

Binomial Trees



Deleting the leaf furthest from the root (in B_5) leaves a path that connects the roots of sub-trees B_4 , B_3 , B_2 , B_1 , and B_0 .

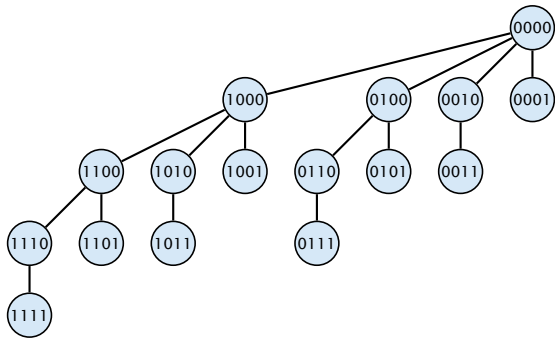
Binomial Trees



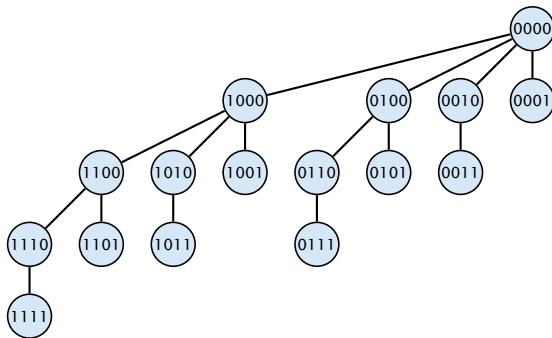
The number of nodes on level ℓ in tree B_k is therefore

$$\binom{k-1}{\ell-1} + \binom{k-1}{\ell} = \binom{k}{\ell}$$

Binomial Trees

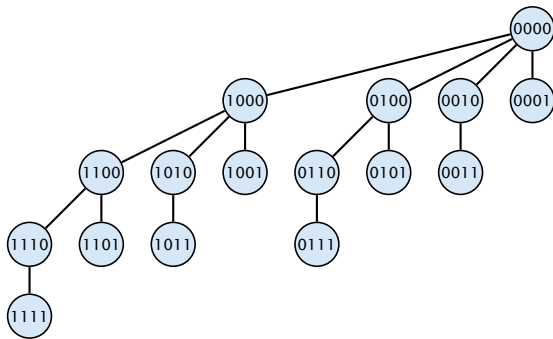


Binomial Trees



The binomial tree B_k is a sub-graph of the hypercube H_k .

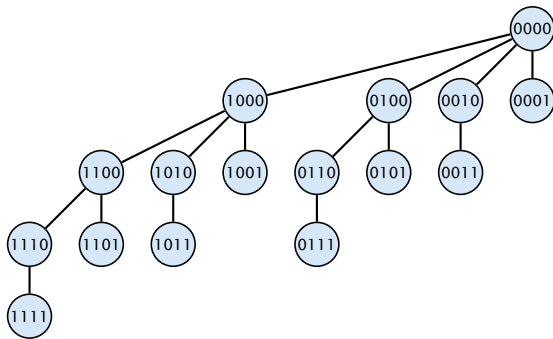
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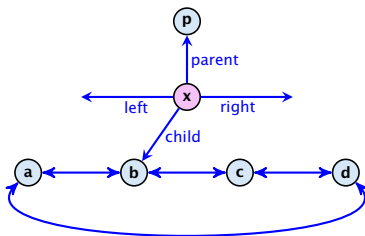
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The ℓ -th level contains nodes that have ℓ 1's in their label.

8.2 Binomial Heaps

How do we implement trees with non-constant degree?

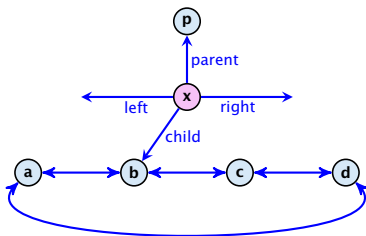
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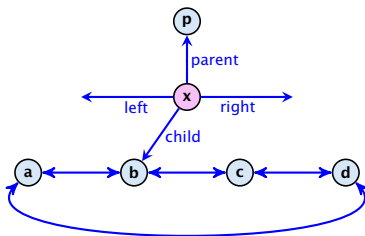
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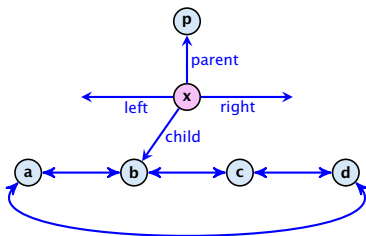
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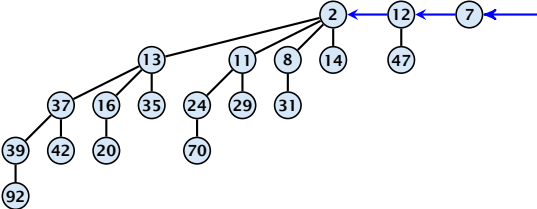
- ▶ The children of a node are arranged in a **circular linked list**.
- ▶ A child-pointer points to an arbitrary node within the list.
- ▶ A parent-pointer points to the parent node.
- ▶ Pointers $x.\text{left}$ and $x.\text{right}$ point to the left and right sibling of x (if x does not have siblings then $x.\text{left} = x.\text{right} = x$).



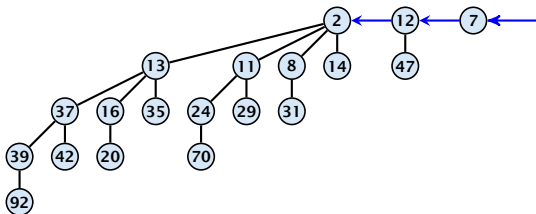
8.2 Binomial Heaps

- ▶ Given a pointer to a node x we can splice out the sub-tree rooted at x in constant time.
- ▶ We can add a child-tree T to a node x in constant time if we are given a pointer to x and a pointer to the root of T .

Binomial Heap

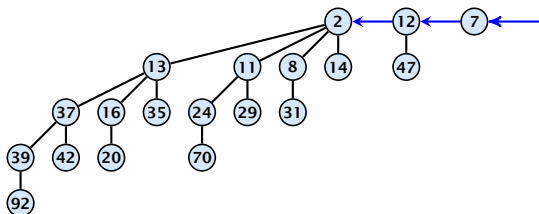


Binomial Heap



In a binomial heap the keys are arranged in a collection of binomial trees.

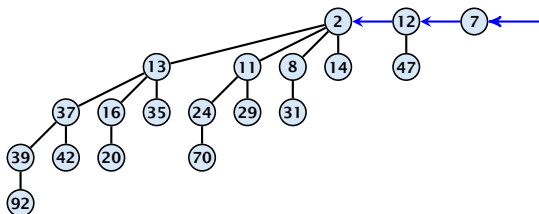
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There is at most one tree for every dimension/order. For example the above heap contains trees B_0 , B_1 , and B_4 .

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Let $B_{k_1}, B_{k_2}, B_{k_3}, k_i < k_{i+1}$ denote the binomial trees in the collection and recall that every tree may be contained at most once.

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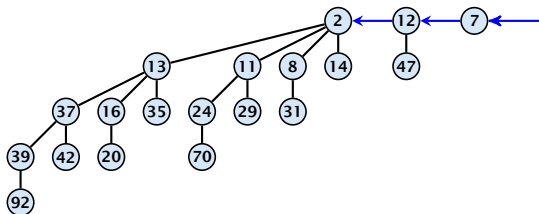
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Then $n = \sum_i 2^{k_i}$ must hold. But since the k_i are all distinct this means that the k_i define the non-zero bit-positions in the binary representation of n .

Binomial Heap

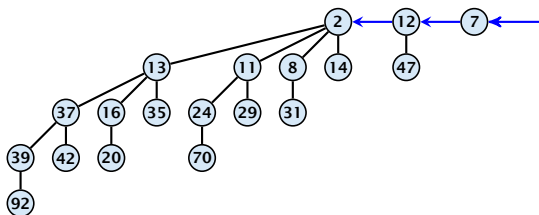
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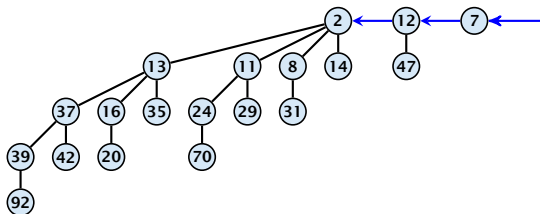
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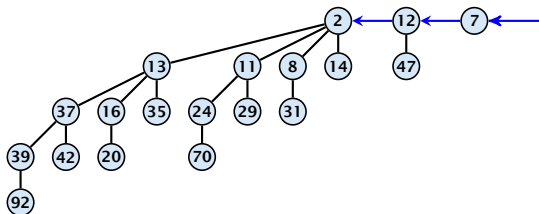
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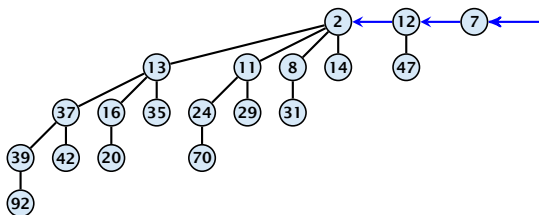
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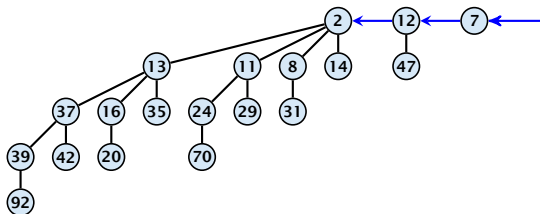
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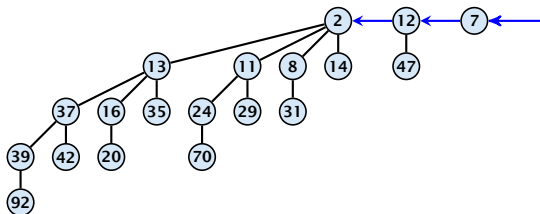
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- ▶ The trees are stored in a single-linked list; ordered by dimension/size.



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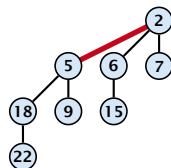
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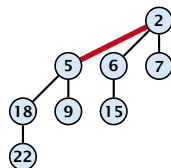
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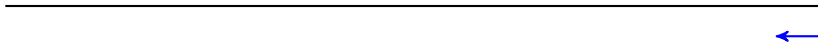
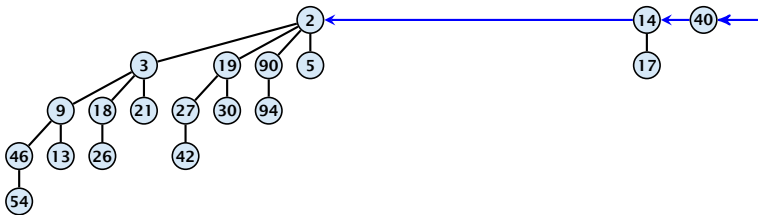
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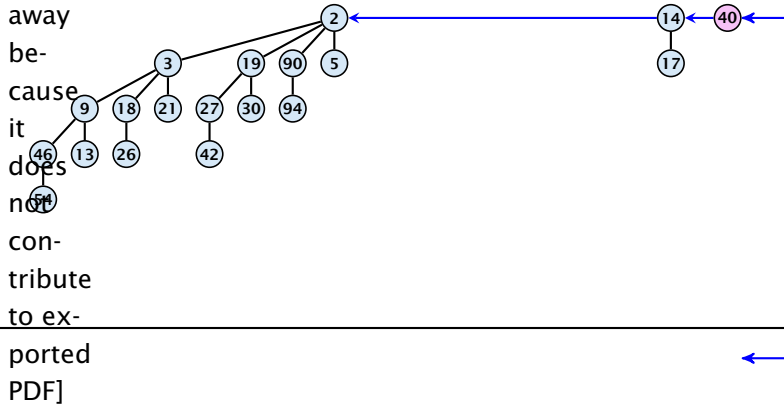
For more trees the technique is analogous to binary addition.





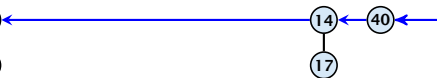
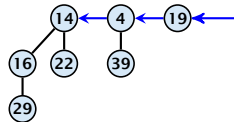
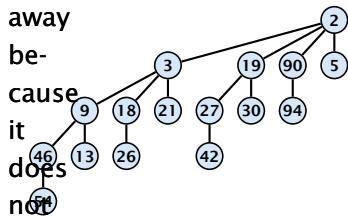
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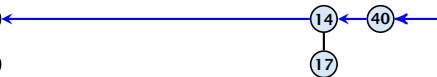
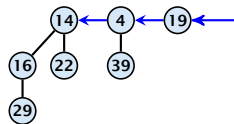
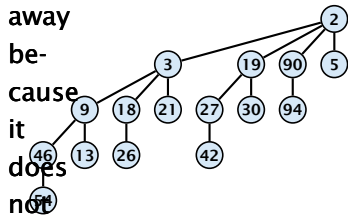
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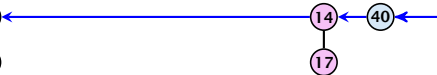
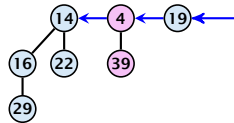
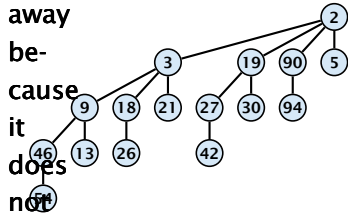
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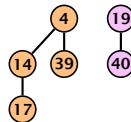
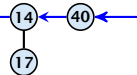
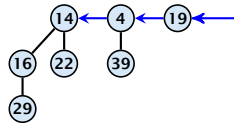
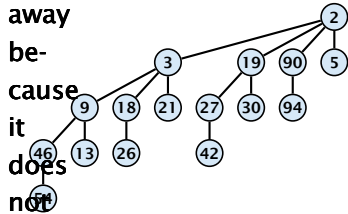
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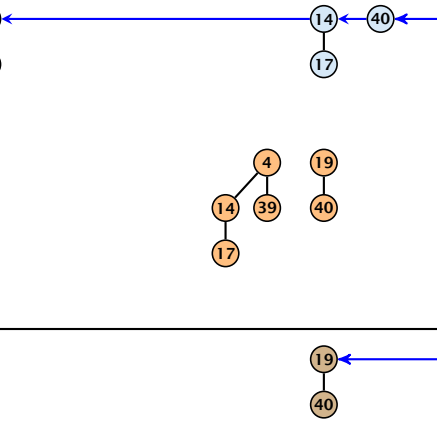
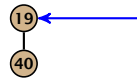
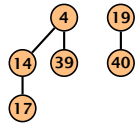
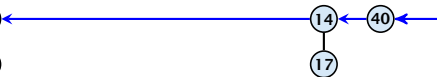
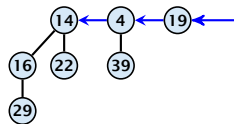
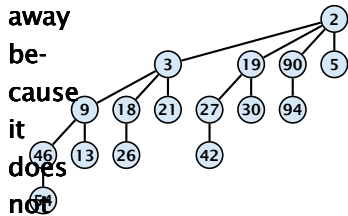
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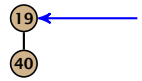
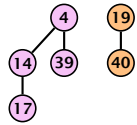
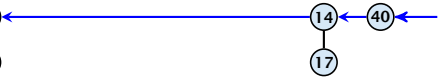
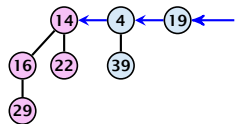
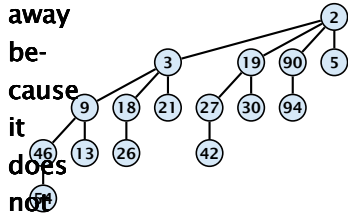
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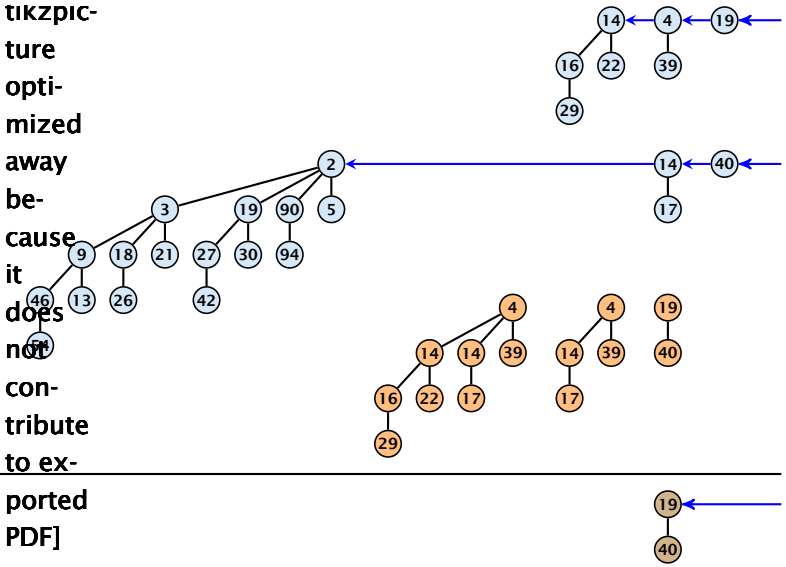
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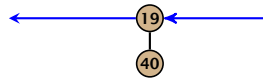
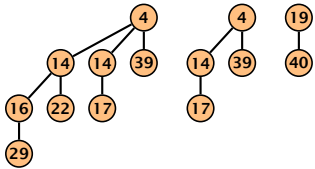
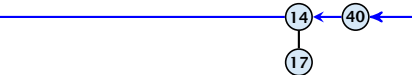
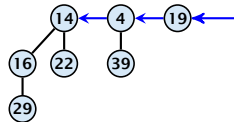
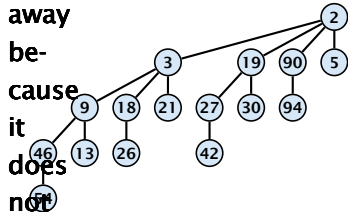
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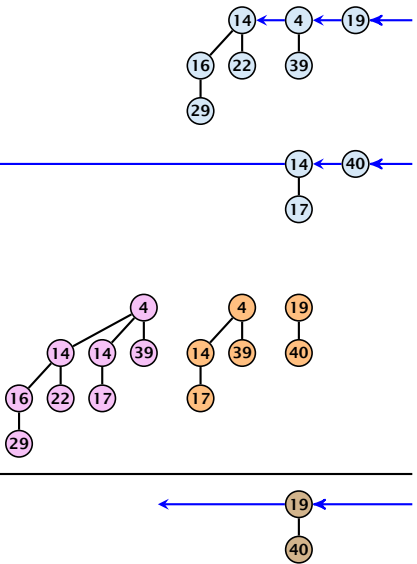
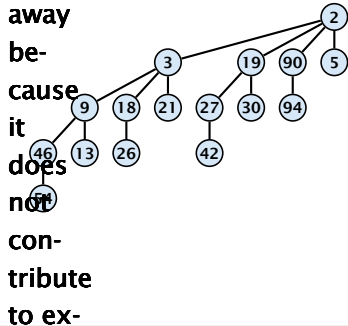
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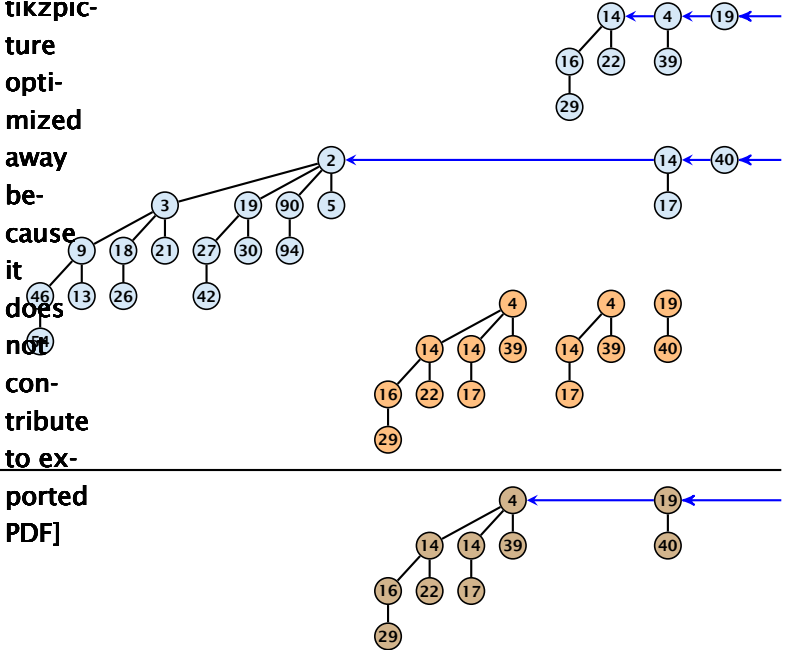
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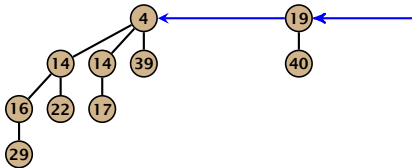
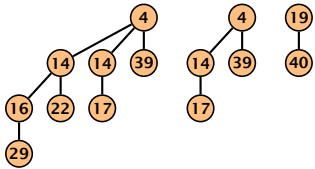
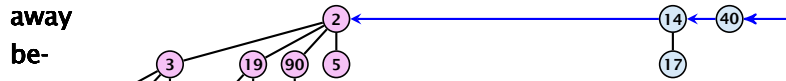
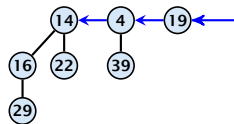
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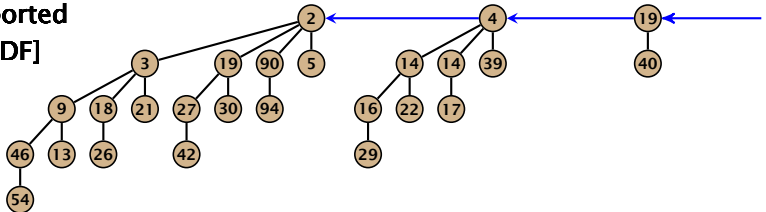
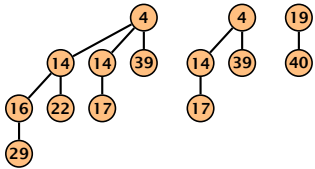
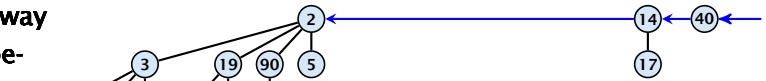
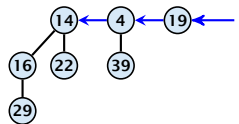
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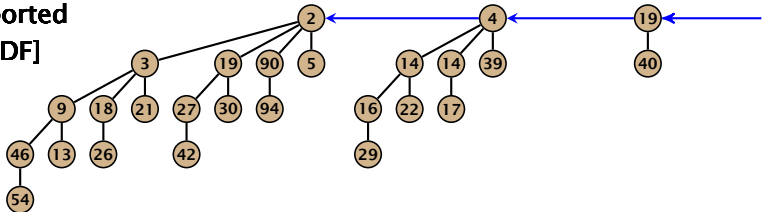
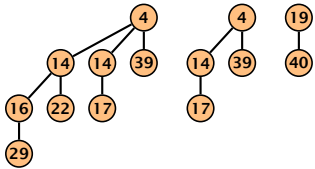
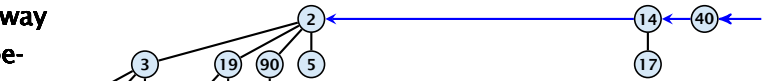
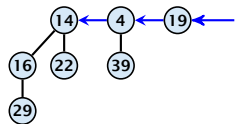
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8.2 Binomial Heaps

S_1 . merge(S_2):

- ▶ Analogous to binary addition.

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8.2 Binomial Heaps

All other operations can be reduced to `merge()`.

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- ▶ Create a new heap S' that contains just the element x .
- ▶ Execute `S.merge(S')`.
- ▶ Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

S. minimum():

- ▶ Find the minimum key-value among all roots.
- ▶ Time: $\mathcal{O}(\log n)$.

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- ▶ Create a new heap S' that contains the trees obtained from T_{\min} after deleting the root (note that these are just $\mathcal{O}(\log n)$ trees).
- ▶ Compute $S.\text{merge}(S')$.
- ▶ Time: $\mathcal{O}(\log n)$.

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S. decrease-key(handle h):

- ▶ Decrease the key of the element pointed to by h .
- ▶ Bubble the element up in the tree until the heap property is fulfilled.
- ▶ Time: $\mathcal{O}(\log n)$ since the trees have height $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

***S.* delete(handle *h*):**

8.2 Binomial Heaps

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8.2 Binomial Heaps

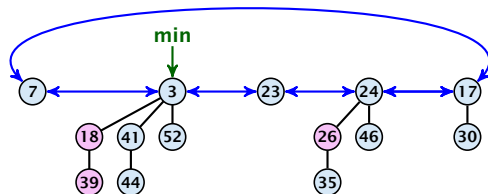
S . delete(handle h):

- ▶ Execute S . decrease-key($h, -\infty$).
- ▶ Execute S . delete-min().
- ▶ Time: $\mathcal{O}(\log n)$.

8.3 Fibonacci Heaps

Collection of trees that fulfill the heap property.

Structure is much more relaxed than binomial heaps.



8.3 Fibonacci Heaps

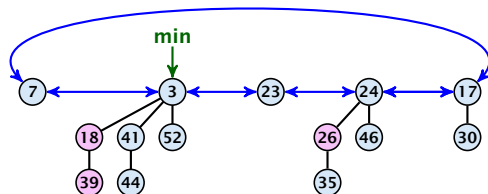
Additional implementation details:

- ▶ Every node x stores its degree in a field $x.degree$. Note that this can be updated in constant time when adding a child to x .
- ▶ Every node stores a boolean value $x.marked$ that specifies whether x is **marked** or not.

8.3 Fibonacci Heaps

The potential function:

- ▶ $t(S)$ denotes the number of trees in the heap.
- ▶ $m(S)$ denotes the number of marked nodes.
- ▶ We use the potential function $\Phi(S) = t(S) + 2m(S)$.



The potential is $\Phi(S) = 5 + 2 \cdot 3 = 11$.

8.3 Fibonacci Heaps

We assume that one unit of potential can pay for a constant amount of work, where the constant is chosen “big enough” (to take care of the constants that occur).

To make this more explicit we use c to denote the amount of work that a unit of potential can pay for.

8.3 Fibonacci Heaps

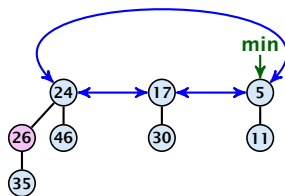
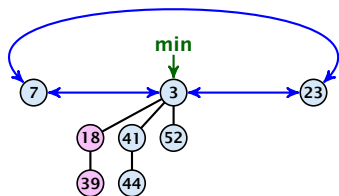
S. minimum()

- ▶ Access through the min-pointer.
- ▶ Actual cost $\mathcal{O}(1)$.
- ▶ No change in potential.
- ▶ Amortized cost $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

S . merge(S')

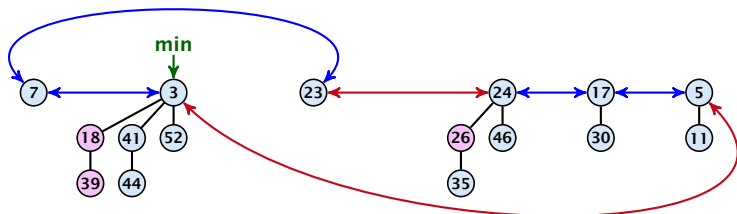
- ▶ Merge the root lists.
- ▶ Adjust the min-pointer



8.3 Fibonacci Heaps

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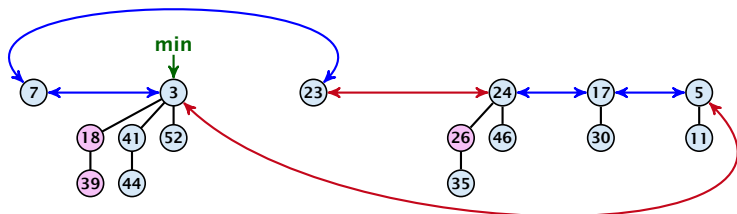
Running time:

- ▶ Actual cost $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

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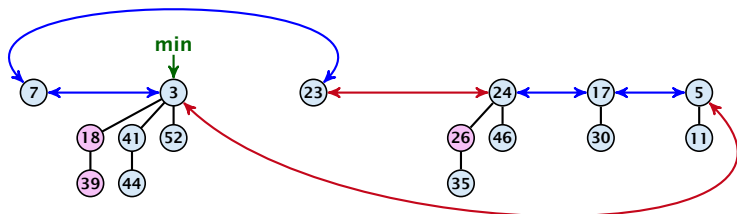
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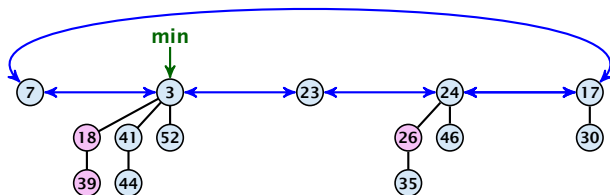
Running time:

- ▶ Actual cost $\mathcal{O}(1)$.
- ▶ No change in potential.
- ▶ Hence, amortized cost is $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

S. insert(x)

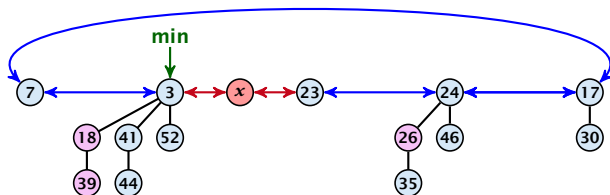
- ▶ Create a new tree containing x .
- ▶ Insert x into the root-list.
- ▶ Update min-pointer, if necessary.



8.3 Fibonacci Heaps

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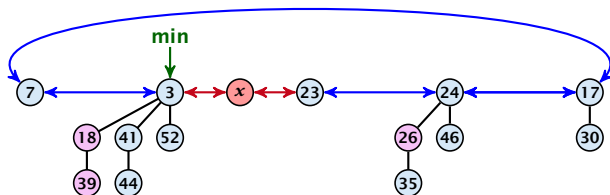
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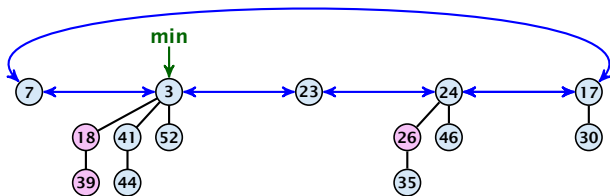


Running time:

- ▶ Actual cost $\mathcal{O}(1)$.
- ▶ Change in potential is $+1$.
- ▶ Amortized cost is $c + \mathcal{O}(1) = \mathcal{O}(1)$.

8.3 Fibonacci Heaps

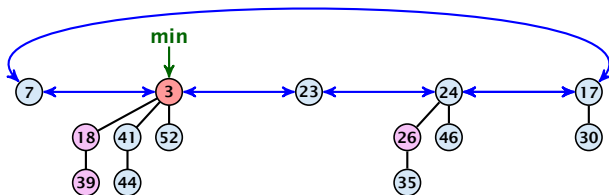
S. delete-min(x)



8.3 Fibonacci Heaps

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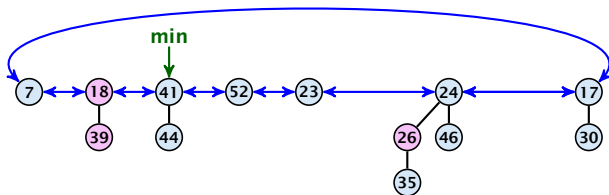
- ▶ Delete minimum; add child-trees to heap;
time: $D(\min) \cdot \mathcal{O}(1)$.



8.3 Fibonacci Heaps

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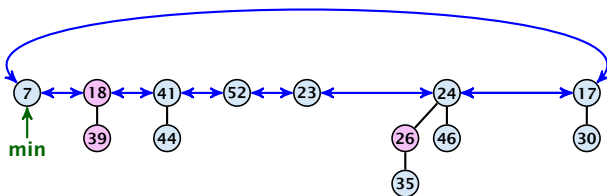
- ▶ Delete minimum; add child-trees to heap; time: $D(\min) \cdot \mathcal{O}(1)$.
- ▶ Update min-pointer; time: $(t + D(\min)) \cdot \mathcal{O}(1)$.



8.3 Fibonacci Heaps

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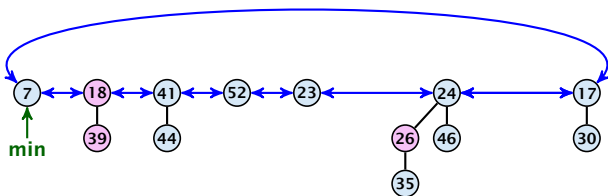
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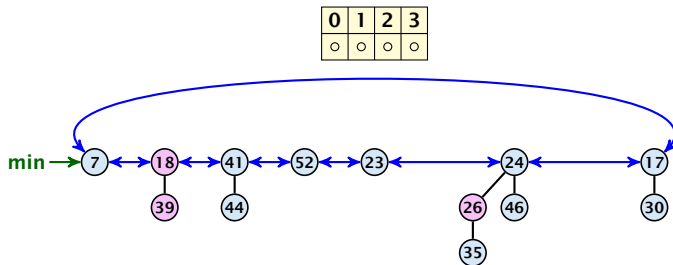
- ▶ Delete minimum; add child-trees to heap; time: $D(\min) \cdot \mathcal{O}(1)$.
- ▶ Update min-pointer; time: $(t + D(\min)) \cdot \mathcal{O}(1)$.



- ▶ Consolidate root-list so that no roots have the same degree. Time $t \cdot \mathcal{O}(1)$ (see next slide).

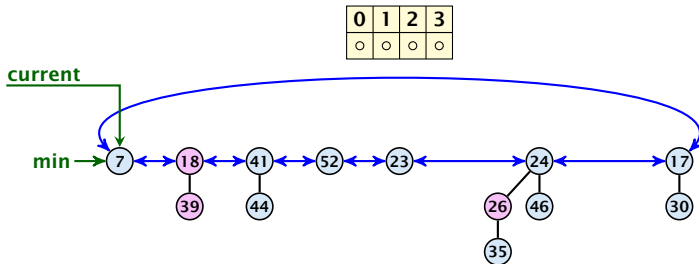
8.3 Fibonacci Heaps

Consolidate:



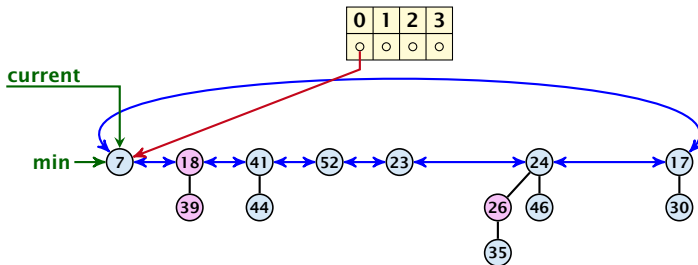
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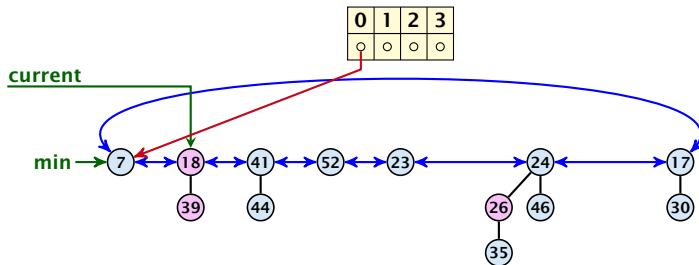
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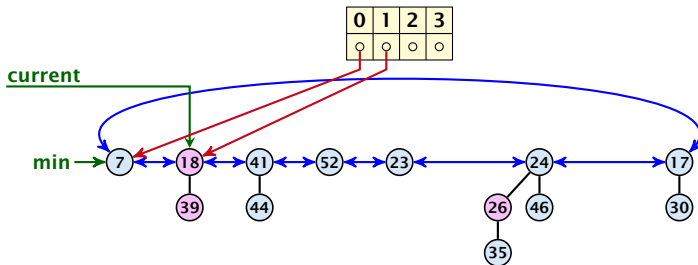
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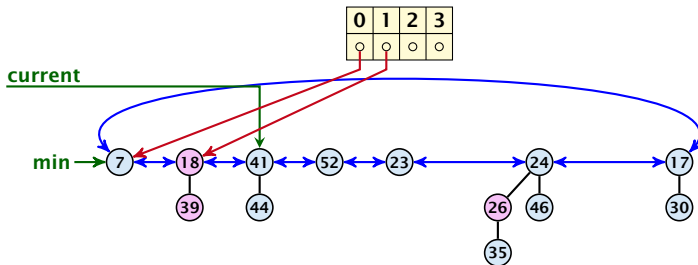
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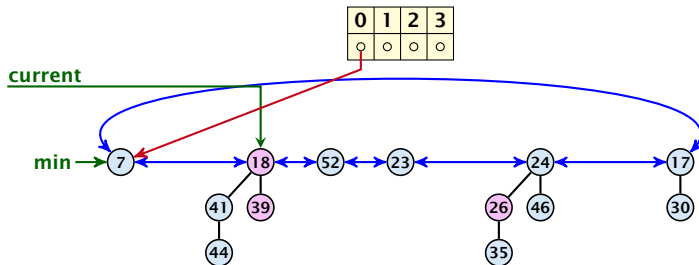
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Consolidate:



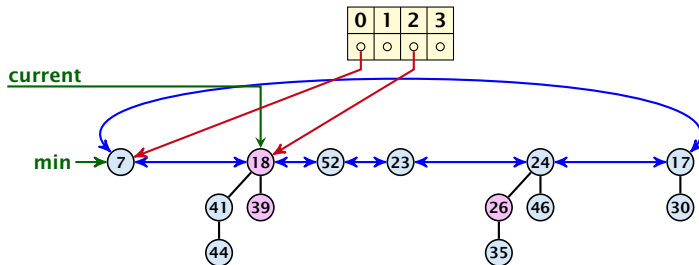
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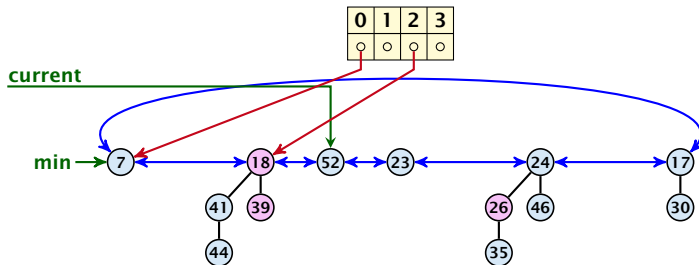
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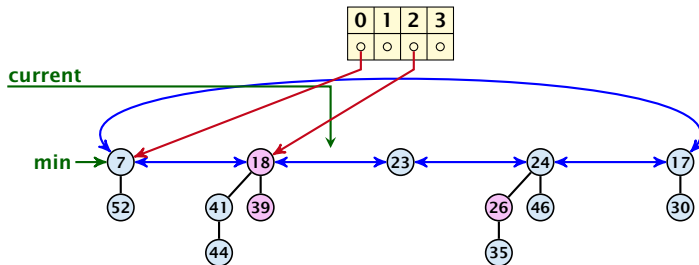
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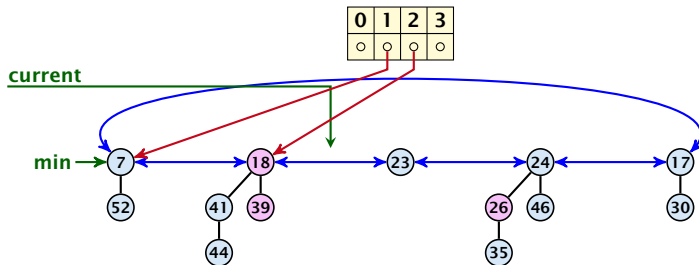
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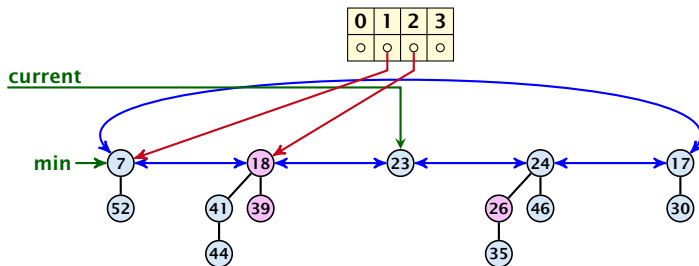
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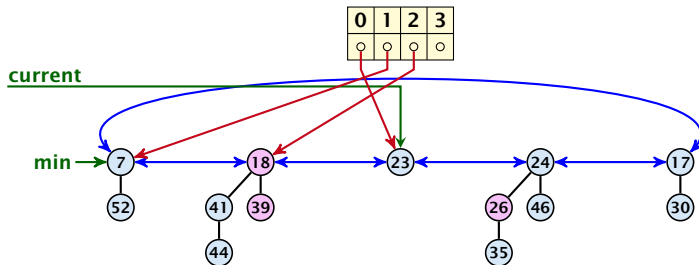
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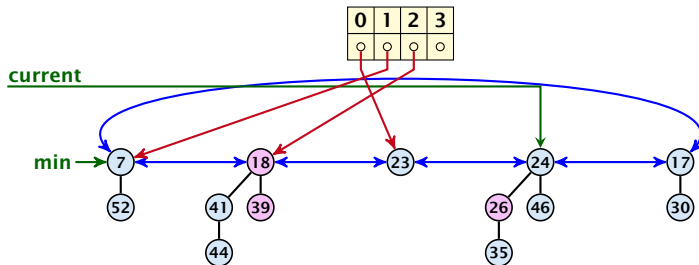
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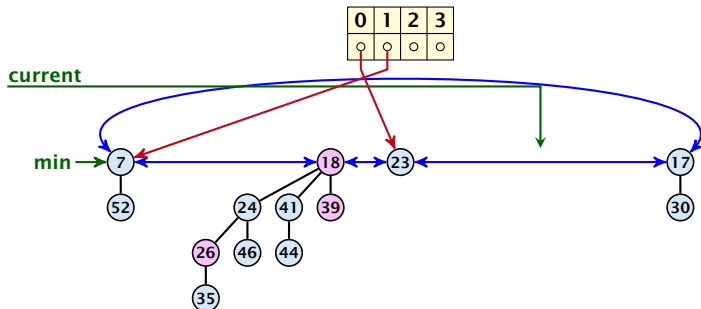
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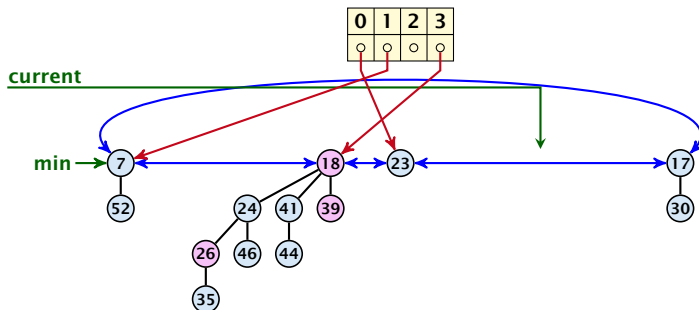
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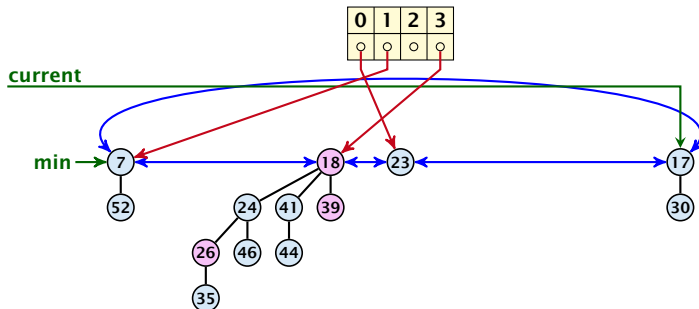
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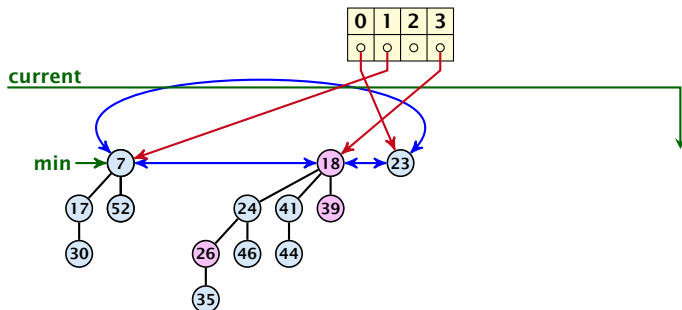
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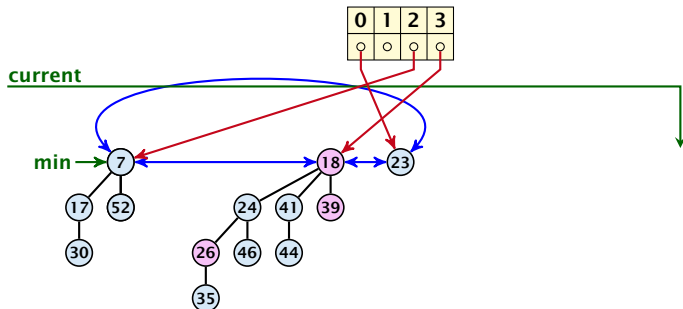
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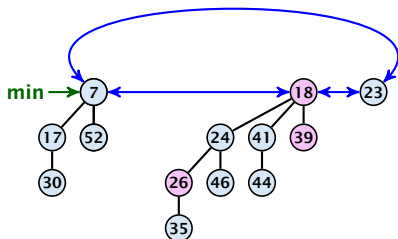
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for $c \geq c_1$.

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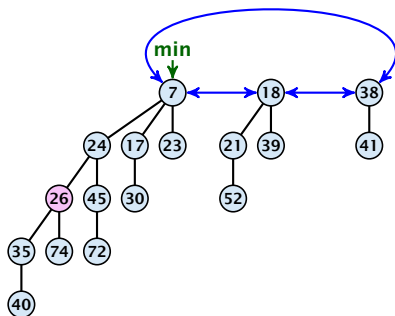
If the input trees of the consolidation procedure are binomial trees (for example only singleton vertices) then the output will be a set of distinct binomial trees, and, hence, the Fibonacci heap will be (more or less) a Binomial heap right after the consolidation.

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If we do not have delete or decrease-key operations then
 $D_n \leq \log n$.

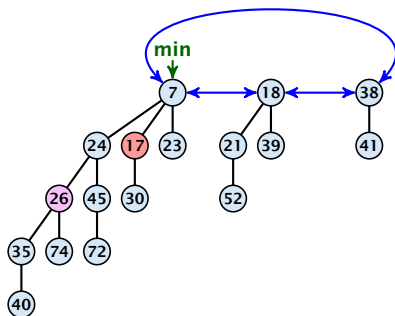
Fibonacci Heaps: decrease-key(handle h, v)



Case 1: decrease-key does not violate heap-property

- ▶ Just decrease the key-value of element referenced by h . Nothing else to do.

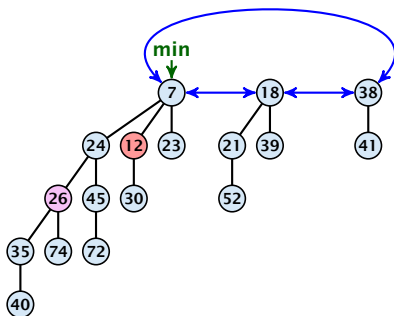
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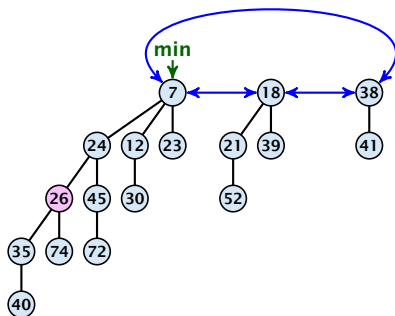
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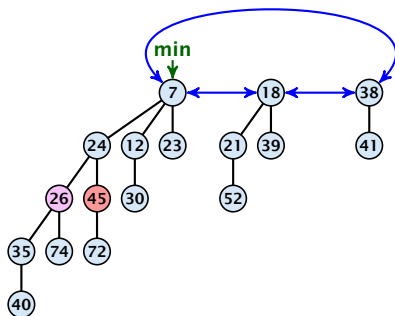
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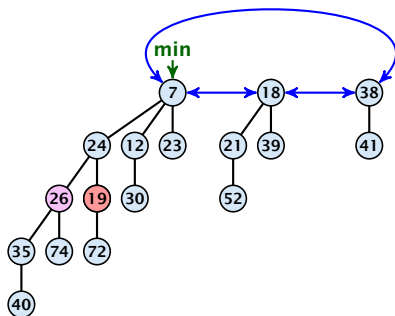
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Case 2: heap-property is violated, but parent is not marked

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- ▶ Adjust min-pointers, if necessary.
- ▶ Mark the (previous) parent of x (unless it's a root).

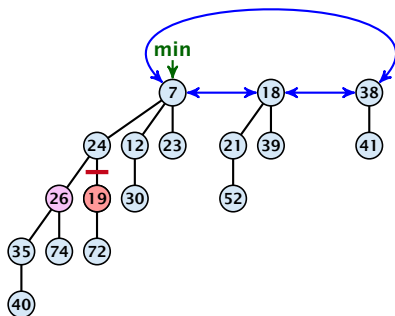
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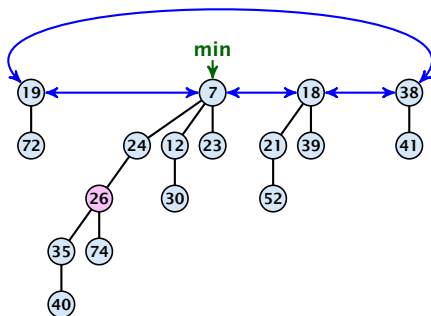
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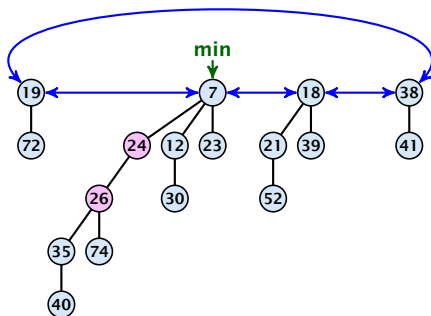
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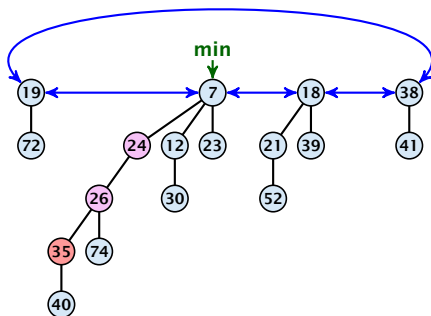
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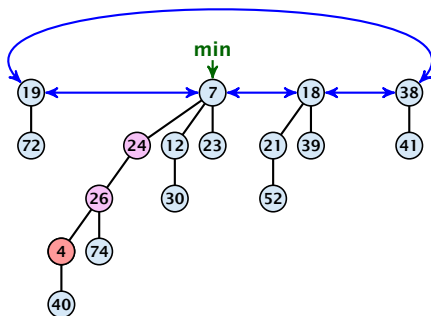
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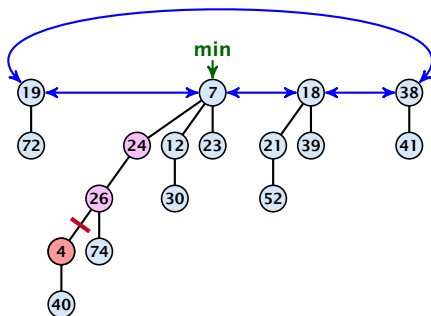
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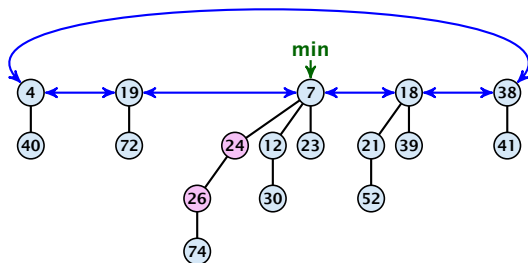
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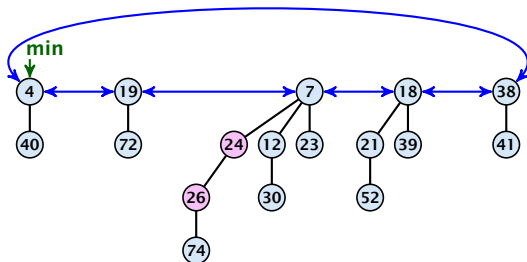
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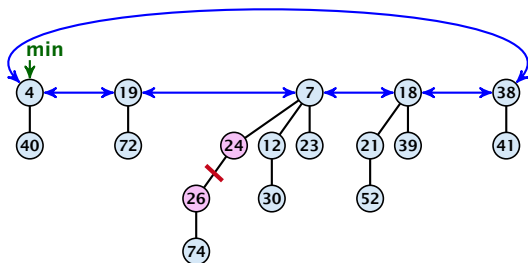
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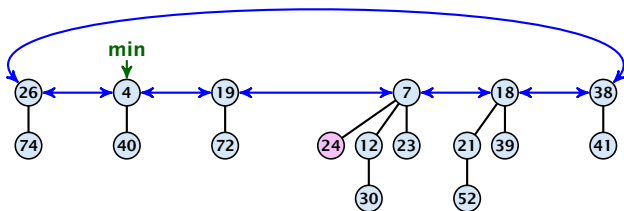
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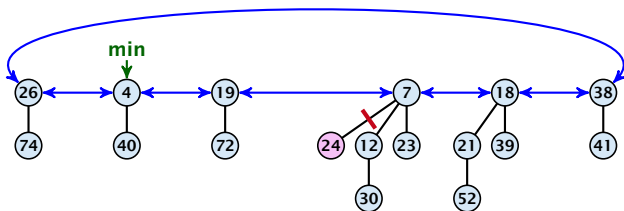
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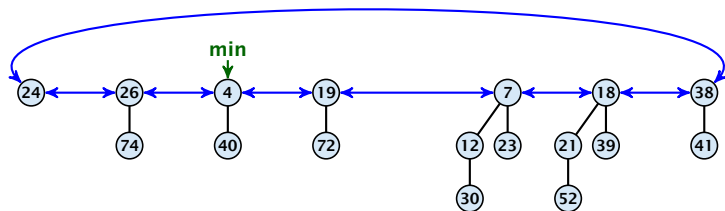
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- ▶ Cut the parent edge of x , and make x into a root.
- ▶ Adjust min-pointers, if necessary.
- ▶ Execute the following:

```
 $p \leftarrow \text{parent}[x];$   
while ( $p$  is marked)  
     $pp \leftarrow \text{parent}[p];$   
    cut of  $p$ ; make it into a root; unmark it;  
     $p \leftarrow pp;$   
if  $p$  is unmarked and not a root mark it;
```

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

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- ▶ $t' = t + \ell$, as every cut creates one new root.

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- ▶ $t' = t + \ell$, as every cut creates one new root.
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Amortized cost:

- ▶ $t' = t + \ell$, as every cut creates one new root.
- ▶ $m' \leq m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.
- ▶ $\Delta\Phi \leq \ell + 2(-\ell + 2) = 4 - \ell$
- ▶ Amortized cost is at most

$$c_2(\ell + 1) + c(4 - \ell) \leq (c_2 - c)\ell + 4c + c_2$$

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

- ▶ Constant cost for decreasing the value.
- ▶ Constant cost for each of ℓ cuts.
- ▶ Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2 .

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- ▶ $t' = t + \ell$, as every cut creates one new root.
- ▶ $m' \leq m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.
- ▶ $\Delta\Phi \leq \ell + 2(-\ell + 2) = 4 - \ell$
- ▶ Amortized cost is at most
$$c_2(\ell + 1) + c(4 - \ell) \leq (c_2 - c)\ell + 4c + c_2 = \mathcal{O}(1),$$
if $c \geq c_2$.

Delete node

H. delete(x):

- ▶ decrease value of x to $-\infty$.
- ▶ delete-min.

Amortized cost: $\mathcal{O}(D_n)$

- ▶ $\mathcal{O}(1)$ for decrease-key.
- ▶ $\mathcal{O}(D_n)$ for delete-min.

8.3 Fibonacci Heaps

Lemma 21

Let x be a node with degree k and let y_1, \dots, y_k denote the children of x in the order that they were linked to x . Then

$$\text{degree}(y_i) \geq \begin{cases} 0 & \text{if } i = 1 \\ i - 2 & \text{if } i > 1 \end{cases}$$

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- ▶ Since, then y_i has lost at most one child.
- ▶ Therefore, $\text{degree}(y_i) \geq i - 2$.

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$$\begin{aligned} s_k &= 2 + \sum_{i=2}^k \text{size}(y_i) \\ &\geq 2 + \sum_{i=2}^k s_{i-2} \\ &= 2 + \sum_{i=0}^{k-2} s_i \end{aligned}$$

8.3 Fibonacci Heaps

Definition 22

Consider the following non-standard Fibonacci type sequence:

$$F_k = \begin{cases} 1 & \text{if } k = 0 \\ 2 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

Facts:

1. $F_k \geq \phi^k$.
2. For $k \geq 2$: $F_k = 2 + \sum_{i=0}^{k-2} F_i$.

The above facts can be easily proved by induction. From this it follows that $s_k \geq F_k \geq \phi^k$, which gives that the maximum degree in a Fibonacci heap is logarithmic.

$$k=0: \quad 1 = F_0 \geq \Phi^0 = 1$$

$$k=1: \quad 2 = F_1 \geq \Phi^1 \approx 1.61$$

$$k-2, k-1 \rightarrow k: \quad F_k = F_{k-1} + F_{k-2} \geq \Phi^{k-1} + \Phi^{k-2} = \Phi^{k-2} \underbrace{(\Phi + 1)}_{\Phi^2} = \Phi^k$$

$$k=2: \quad 3 = F_2 = 2 + 1 = 2 + F_0$$

$$k-1 \rightarrow k: \quad F_k = F_{k-1} + F_{k-2} = 2 + \sum_{i=0}^{k-3} F_i + F_{k-2} = 2 + \sum_{i=0}^{k-2} F_i$$

9 Union Find

Union Find Data Structure \mathcal{P} : Maintains a partition of **disjoint** sets over elements.

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- ▶ **\mathcal{P} . find(x):** Given a handle for an element x ; find the set that contains x . Returns a representative/identifier for this set.
- ▶ **\mathcal{P} . union(x, y):** Given two elements x , and y that are currently in sets S_x and S_y , respectively, the function replaces S_x and S_y by $S_x \cup S_y$ and returns an identifier for the new set.

9 Union Find

Applications:

- ▶ Keep track of the connected components of a dynamic graph that changes due to insertion of nodes and edges.

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- ▶ Keep track of the connected components of a dynamic graph that changes due to insertion of nodes and edges.
- ▶ Kruskals Minimum Spanning Tree Algorithm

9 Union Find

Algorithm 1 Kruskal-MST($G = (V, E), w$)

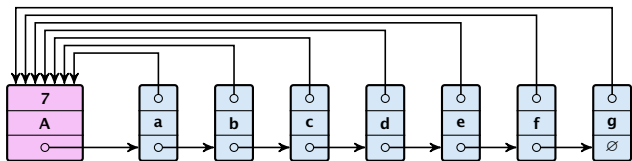
```
1:  $A \leftarrow \emptyset$ ;  
2: for all  $v \in V$  do  
3:    $v.\text{set} \leftarrow \mathcal{P}.\text{makeset}(v.\text{label})$   
4: sort edges in non-decreasing order of weight  $w$   
5: for all  $(u, v) \in E$  in non-decreasing order do  
6:   if  $\mathcal{P}.\text{find}(u.\text{set}) \neq \mathcal{P}.\text{find}(v.\text{set})$  then  
7:      $A \leftarrow A \cup \{(u, v)\}$   
8:      $\mathcal{P}.\text{union}(u.\text{set}, v.\text{set})$ 
```

List Implementation

- ▶ The elements of a set are stored in a list; each node has a backward pointer to the head.

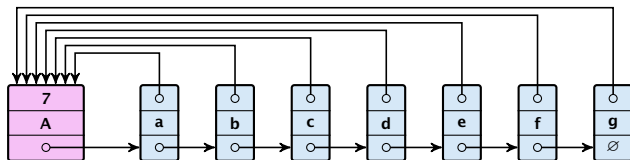
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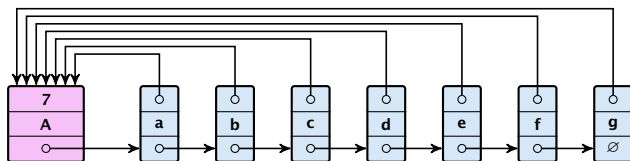
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- ▶ **makeset(x)** can be performed in constant time.
- ▶ **find(x)** can be performed in constant time.

List Implementation

union(x, y)

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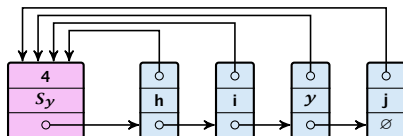
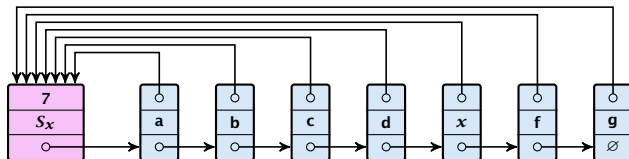
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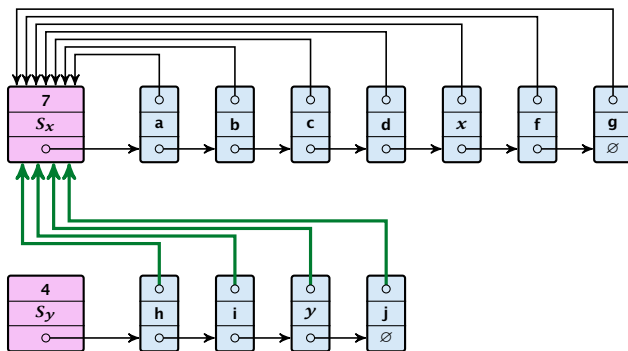
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- ▶ Time: $\min\{|S_x|, |S_y|\}$.

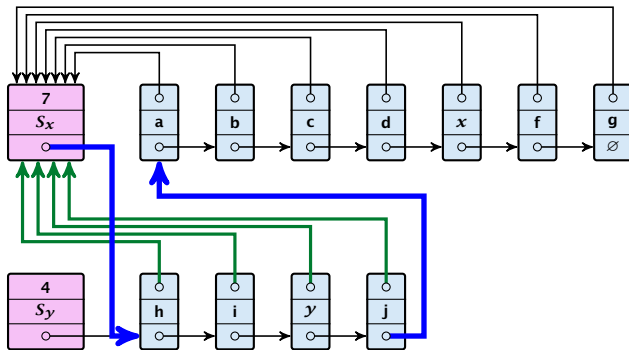
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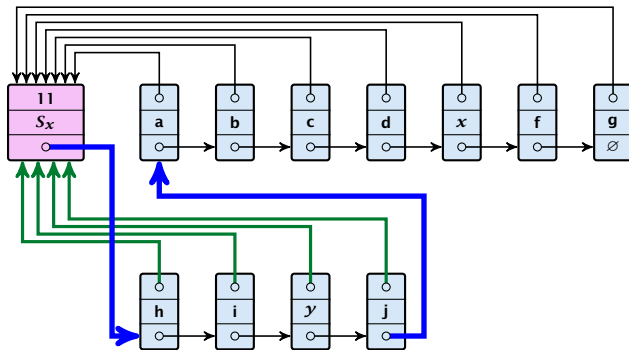
List Implementation



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Running times:

- ▶ $\text{find}(x)$: constant
- ▶ $\text{makeset}(x)$: constant
- ▶ $\text{union}(x, y)$: $\mathcal{O}(n)$, where n denotes the number of elements contained in the set system.

List Implementation

Lemma 23

The list implementation for the ADT union find fulfills the following amortized time bounds:

- ▶ $\text{find}(x): \mathcal{O}(1)$.
- ▶ $\text{makeset}(x): \mathcal{O}(\log n)$.
- ▶ $\text{union}(x, y): \mathcal{O}(1)$.

The Accounting Method for Amortized Time Bounds

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- ▶ If we can find a charging scheme that guarantees that balances always stay positive the amortized time bounds are proven.

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- ▶ Later operations charge the account but the balance never drops below zero.

List Implementation

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- ▶ Charge c to every element in set S_x .

List Implementation

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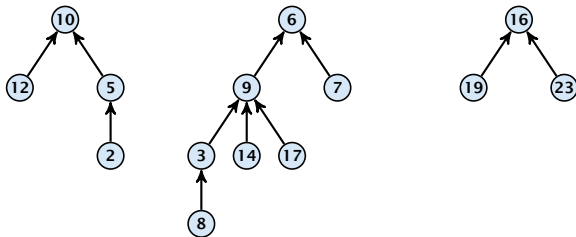
Whenever an element x is charged the number of elements in x 's set doubles. This can happen at most $\lfloor \log n \rfloor$ times. \square

Implementation via Trees

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- ▶ Example:



Set system $\{2, 5, 10, 12\}$, $\{3, 6, 7, 8, 9, 14, 17\}$, $\{16, 19, 23\}$.

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- ▶ Time: $\mathcal{O}(\text{level}(x))$, where $\text{level}(x)$ is the distance of element x to the root in its tree. **Not constant.**

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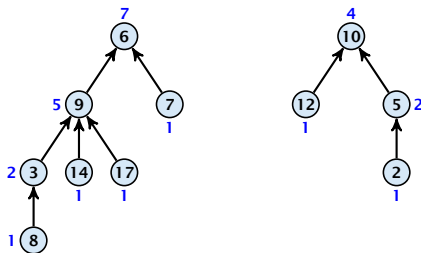
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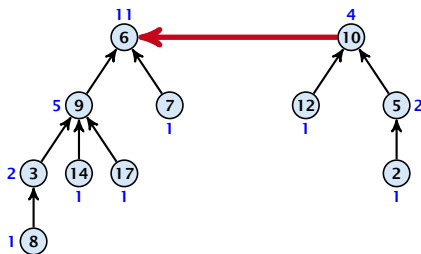


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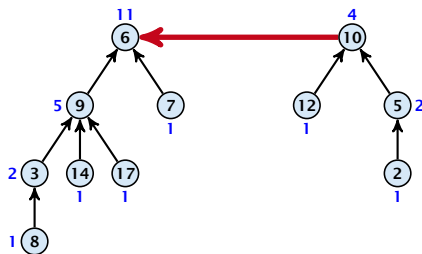


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- ▶ Time: constant for $\text{link}(a, b)$ plus two find-operations.

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- ▶ Hence, at any point in time a tree fulfills $\text{size}(p) \geq 2 \text{size}(c)$, for any pair of nodes (p, c) , where p is a parent of c .

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- ▶ When we attach a tree with root c to become a child of a tree with root p , then $\text{size}(p) \geq 2 \text{size}(c)$, where size denotes the value of the size-field right after the operation.
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- ▶ Hence, at any point in time a tree fulfills $\text{size}(p) \geq 2 \text{size}(c)$, for any pair of nodes (p, c) , where p is a parent of c .



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find(x):

- ▶ Go upward until you find the root.

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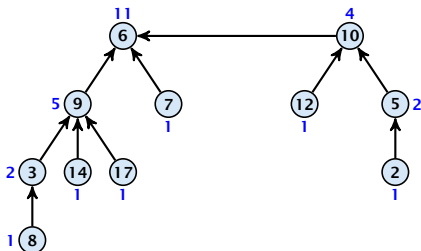
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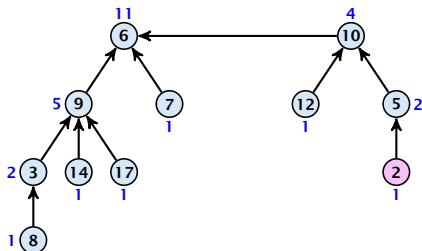
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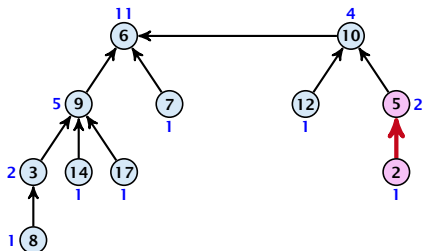
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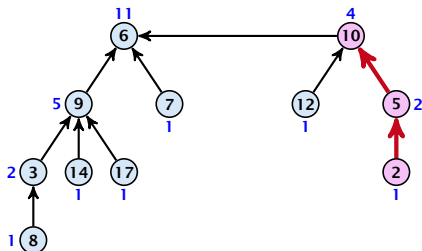
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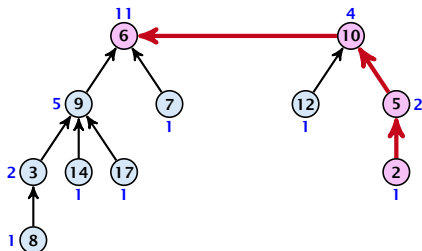
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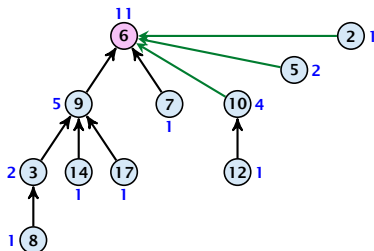
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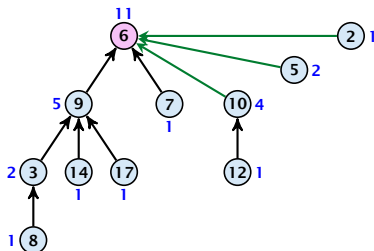
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- ▶ Note that the size-fields now only give an upper bound on the size of a sub-tree.

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However, for a worst-case analysis there is no improvement on the running time. It can still happen that a find-operation takes time $\mathcal{O}(\log n)$.

Amortized Analysis

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- ▶ $\text{size}(v)$:= the number of nodes that were in the sub-tree rooted at v when v became the child of another node (or the number of nodes if v is the root).

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Lemma 26

The rank of a parent must be strictly larger than the rank of a child.

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- ▶ A node v sees at most one node of rank s during the running time of the algorithm.
- ▶ This holds because the rank-sequence of the roots of the different trees that contain v during the running time of the algorithm is a strictly increasing sequence.
- ▶ Hence, every node sees at most one rank s node, but every rank s node is seen by at least 2^s different nodes. □

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Theorem 28

Union find with path compression fulfills the following amortized running times:

- ▶ $\text{makeset}(x) : \mathcal{O}(\log^*(n))$
- ▶ $\text{find}(x) : \mathcal{O}(\log^*(n))$
- ▶ $\text{union}(x, y) : \mathcal{O}(\log^*(n))$

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- ▶ The maximum non-empty rank group is $\log^*(\lfloor \log n \rfloor) \leq \log^*(n) - 1$ (which holds for $n \geq 2$).
- ▶ Hence, the total number of rank-groups is at most $\log^* n$.

Amortized Analysis



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- ▶ Otherwise we charge the cost to the find-account.

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- ▶ After some charges to v the parent will be in a larger rank-group. $\Rightarrow v$ will **never** be charged again.
- ▶ The total charge made to a node in rank-group g is at most $\text{tow}(g) - \text{tow}(g - 1) - 1 \leq \text{tow}(g)$.

Amortized Analysis

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- ▶ The total charge is at most

$$\sum_g n(g) \cdot \text{tow}(g) ,$$

where $n(g)$ is the number of nodes in group g .

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Hence,

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This means if we inflate the cost of **makeset** to $\log^* n$ and add this to the node account of v then the balances of all node accounts will sum up to a positive value (this is sufficient to obtain an amortized bound).

Amortized Analysis



Amortized Analysis

The analysis is not tight. In fact it has been shown that the amortized time for the union-find data structure with path compression is $\mathcal{O}(\alpha(m, n))$, where $\alpha(m, n)$ is the inverse Ackermann function which grows a lot lot slower than $\log^* n$. (Here, we consider the average running time of m operations on at most n elements).

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There is also a lower bound of $\Omega(\alpha(m, n))$.

Amortized Analysis

$$A(x, y) = \begin{cases} y + 1 & \text{if } x = 0 \\ A(x - 1, 1) & \text{if } y = 0 \\ A(x - 1, A(x, y - 1)) & \text{otw.} \end{cases}$$

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- ▶ $A(0, y) = y + 1$
- ▶ $A(1, y) = y + 2$
- ▶ $A(2, y) = 2y + 3$
- ▶ $A(3, y) = 2^{y+3} - 3$
- ▶ $A(4, y) = \underbrace{2^{2^{2^2}}}_{y+3 \text{ times}} - 3$