7.4 Augmenting Data Structures

Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

1. We choose a red-black tree as the underlying data-structure.
2. We store in each node v the size of the sub-tree rooted at v.
3. We need to be able to update the size-field in each node without asymptotically affecting the running time of insert, delete, and search. We come back to this step later...

7.4 Augmenting Data Structures

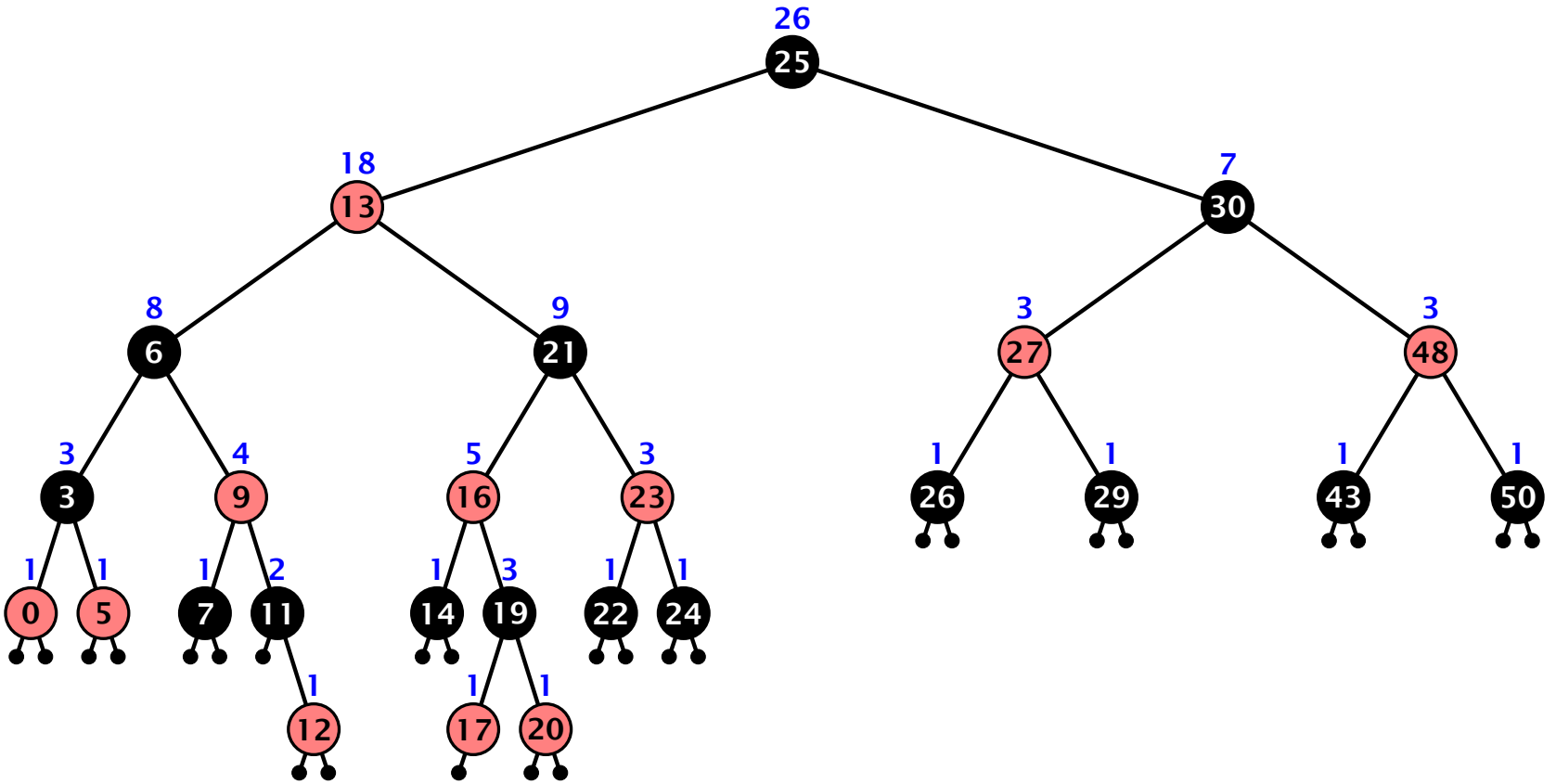
Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

4. How does find-by-rank work?
   Find-by-rank(k) := Select(root,k) with

Algorithm 11 Select(x, i)

```
1: if x = null then return error
2: if left[x] ≠ null then r ← left[x].size +1 else r ← 1
3: if i = r then return x
4: if i < r then
5:         return Select(left[x], i)
6: else
7:         return Select(right[x], i − r)
```

Select(x, i)

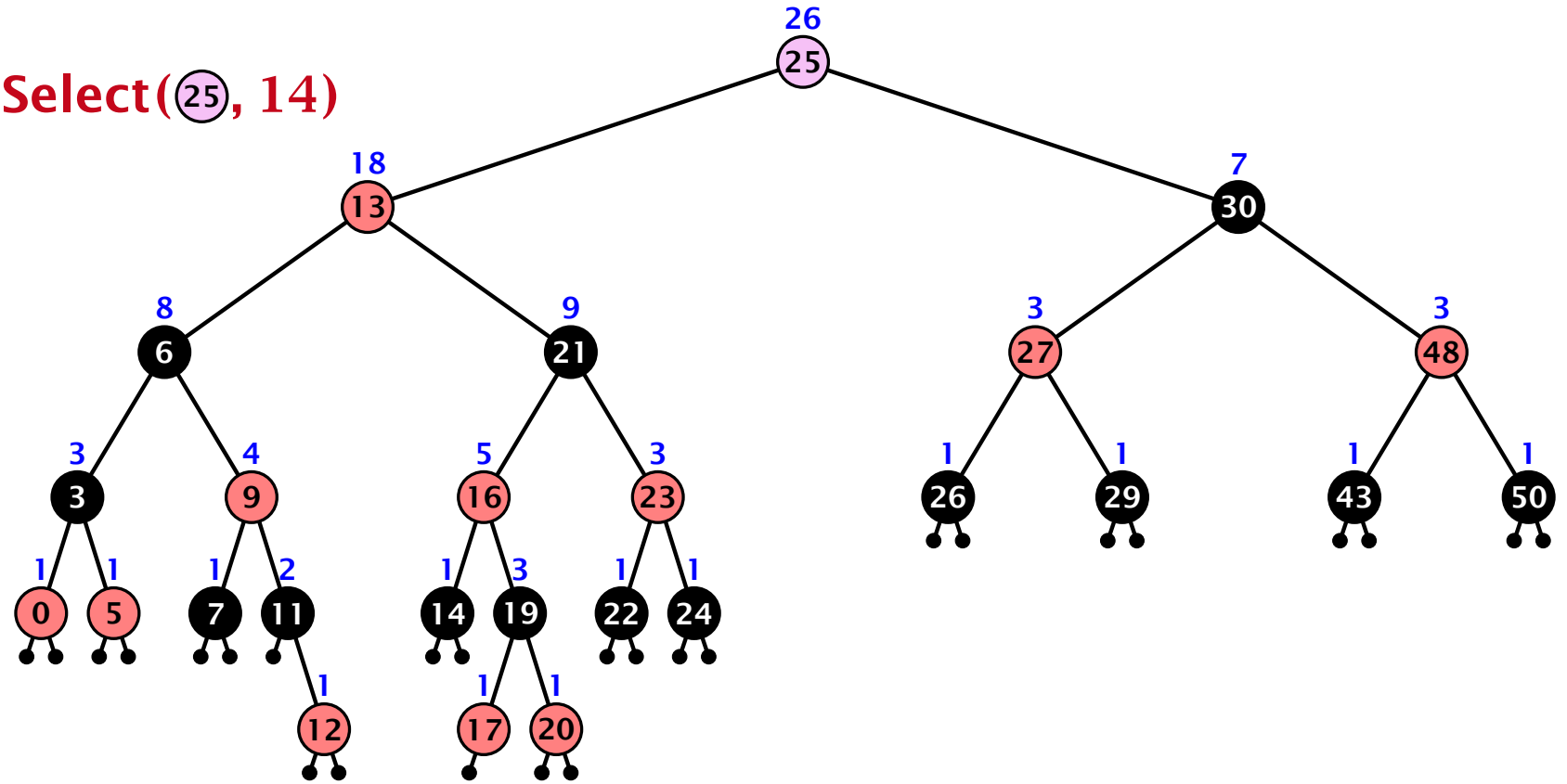


Find-by-rank:

- decide whether you have to proceed into the left or right sub-tree
- adjust the rank that you are searching for if you go right

Select(x, i)

Select(25, 14)



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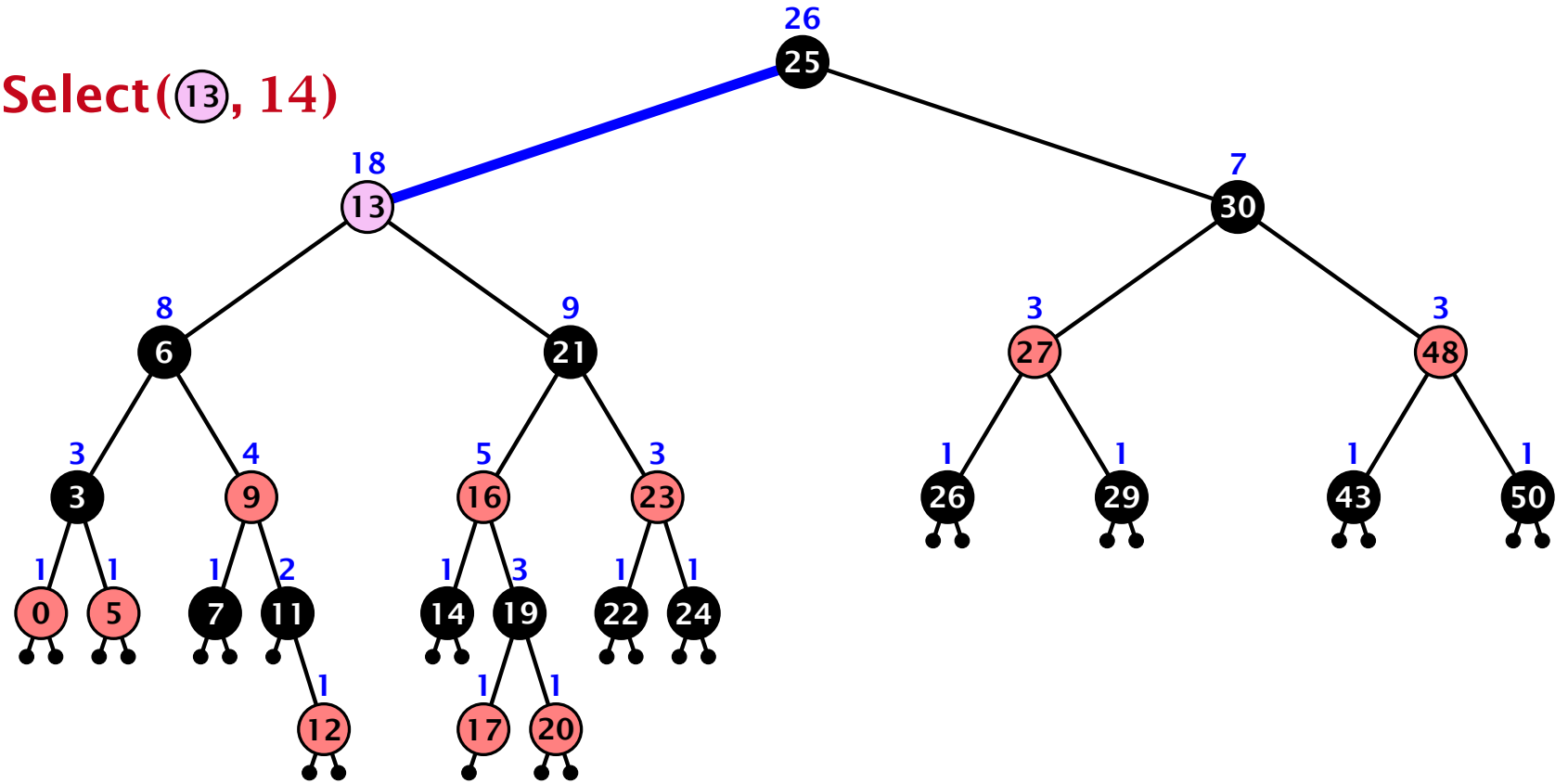
Select(x, i)

Select(13, 14)



Find-by-rank:

- decide whether you have to proceed into the left or right sub-tree
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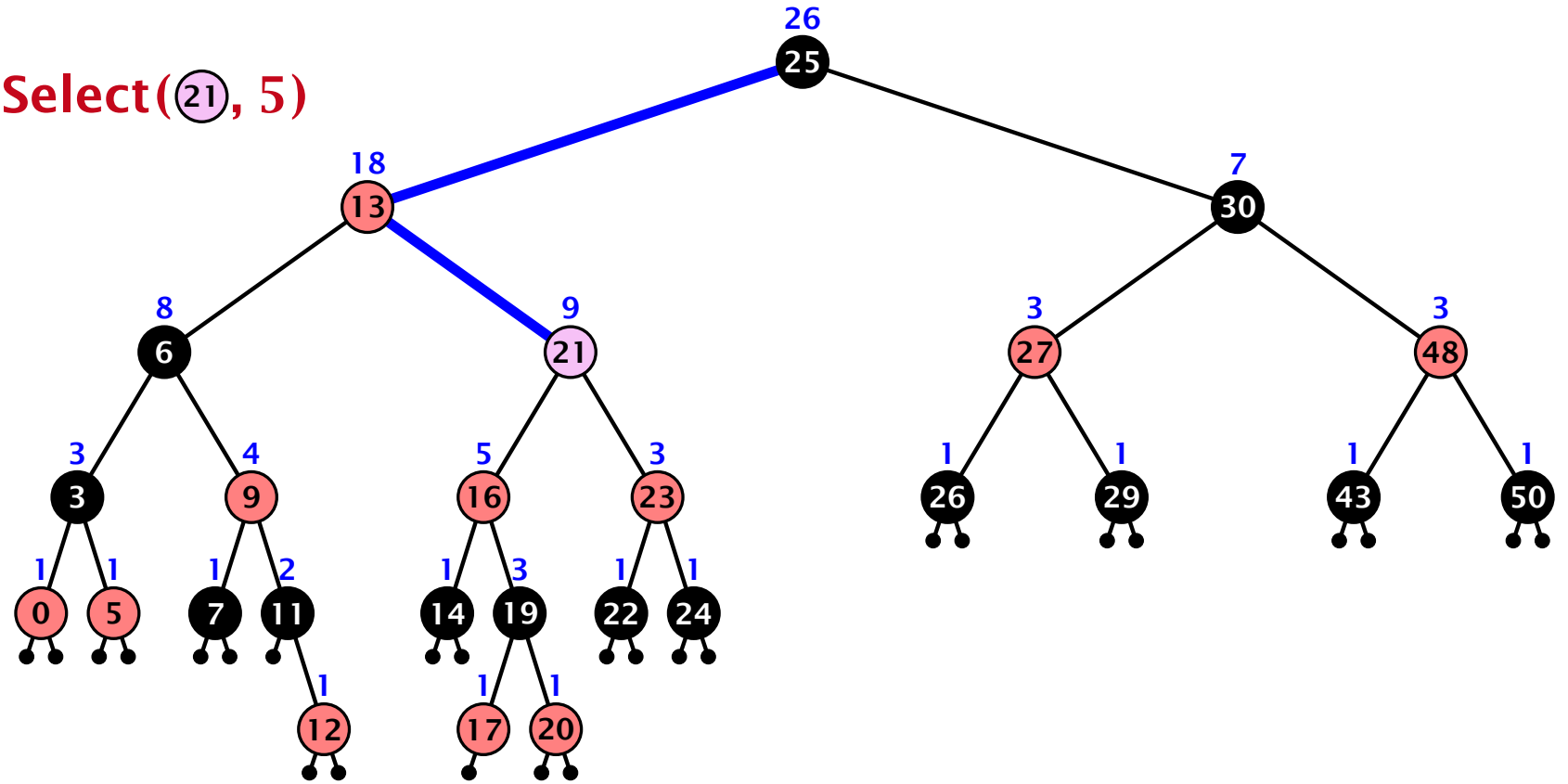
Select(x, i)

Select(21, 5)



Find-by-rank:

- decide whether you have to proceed into the left or right sub-tree
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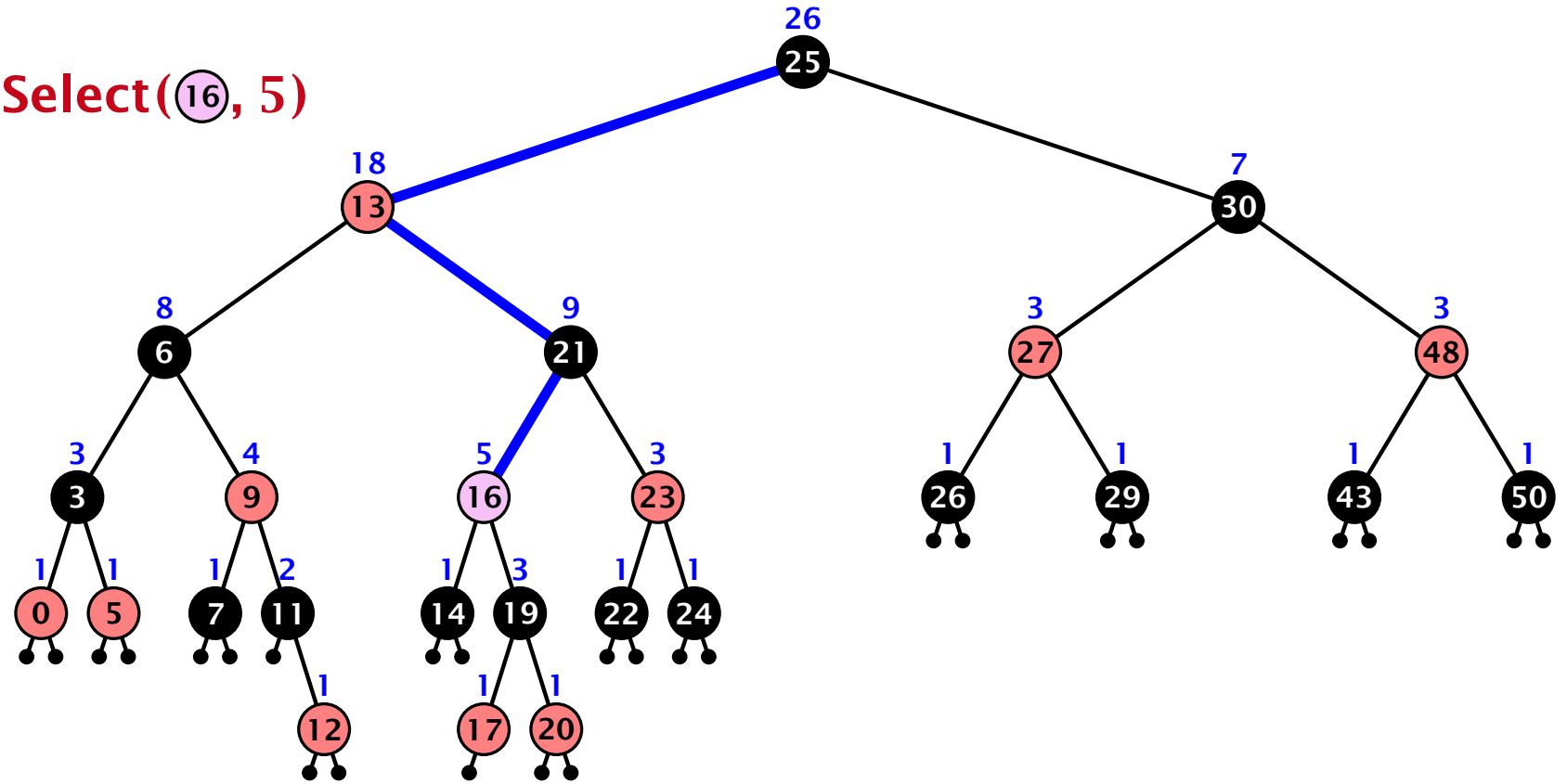
Select(x, i)

Select(16, 5)

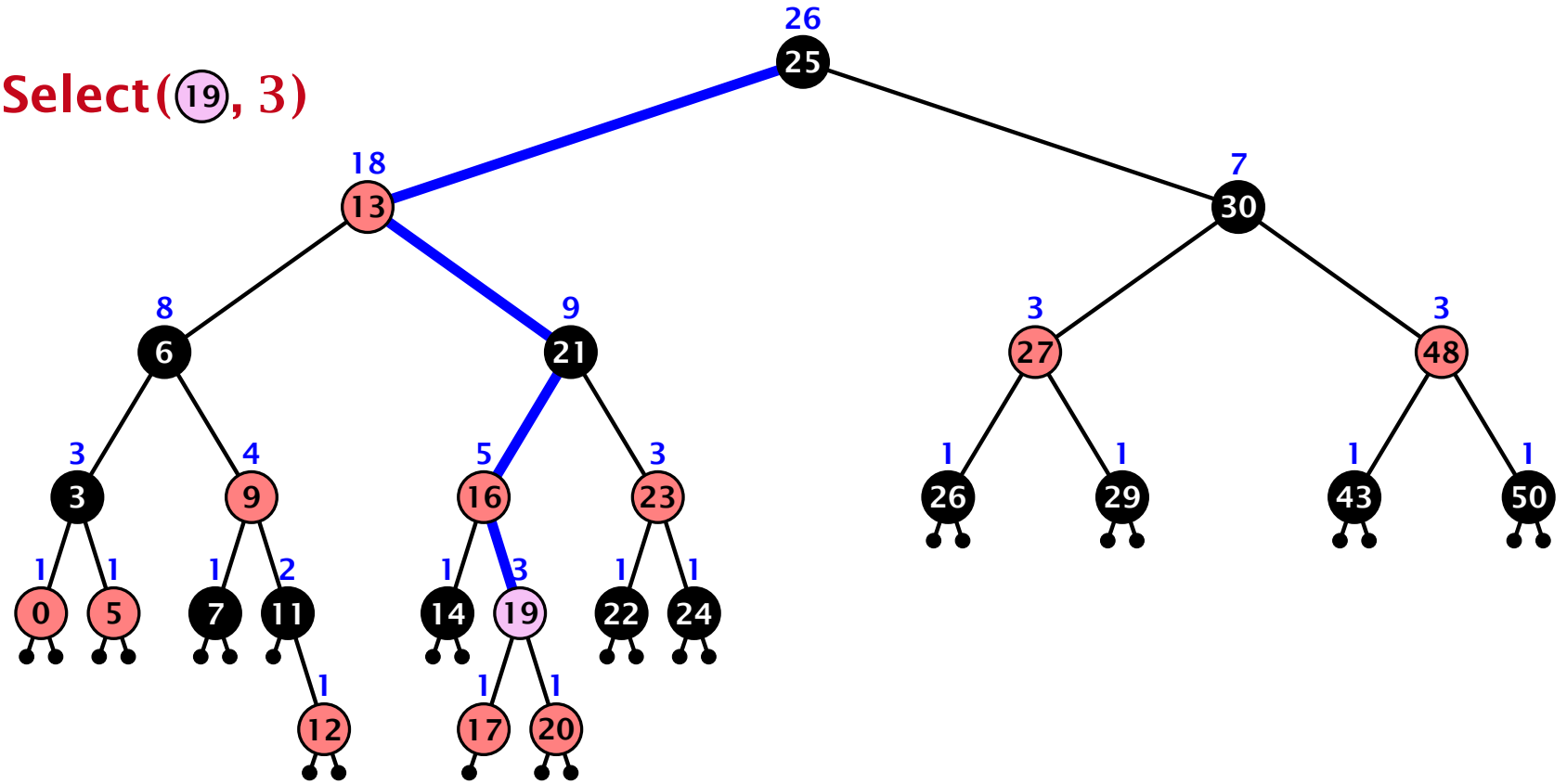


Find-by-rank:

- decide whether you have to proceed into the left or right sub-tree
- adjust the rank that you are searching for if you go right

Select(x, i)

Select(19, 3)



Find-by-rank:

- decide whether you have to proceed into the left or right sub-tree
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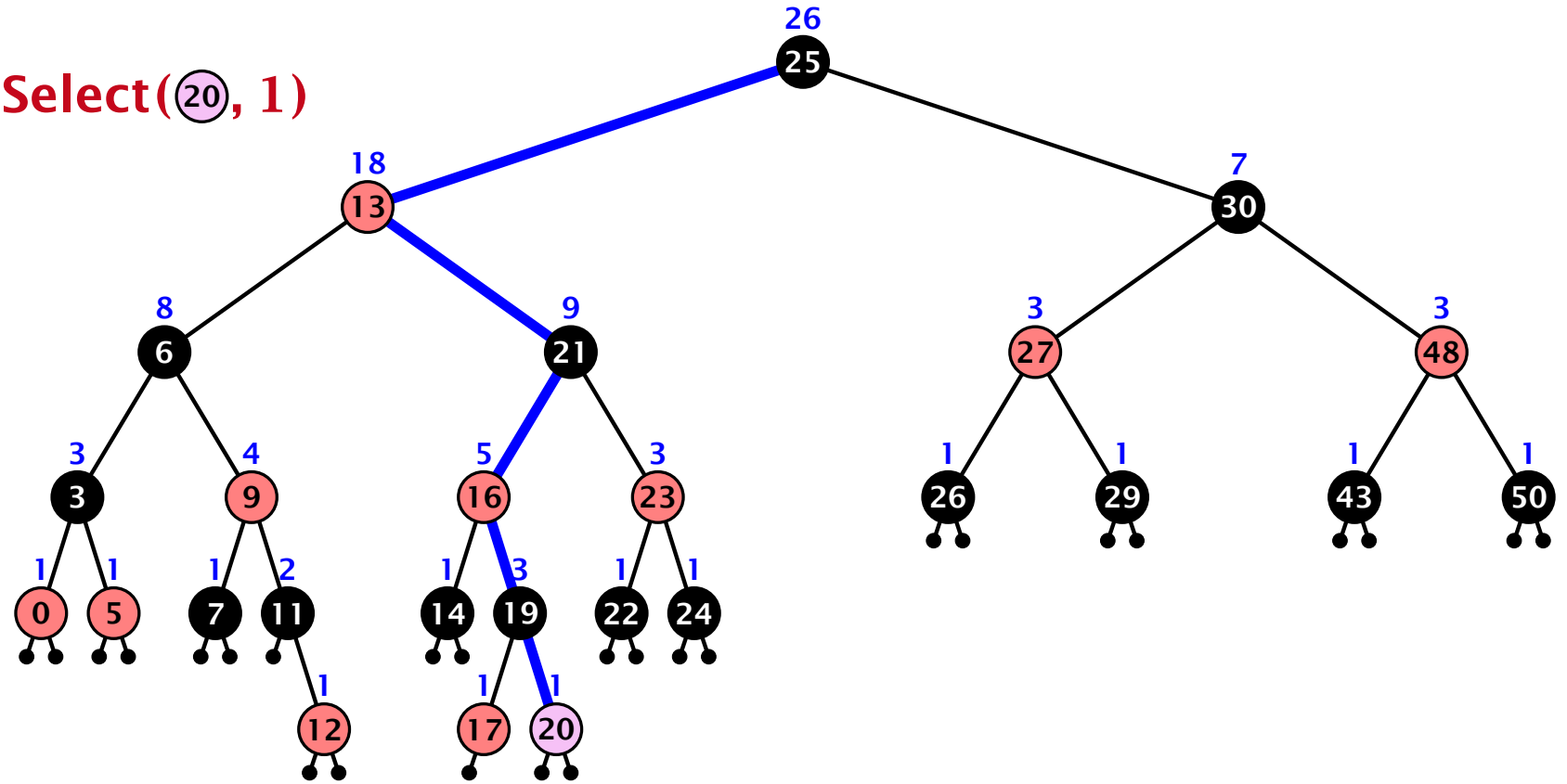
Select(x, i)

Select(20, 1)



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7.4 Augmenting Data Structures

Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

3. How do we maintain information?

Search(k): Nothing to do.

Insert(x): When going down the search path increase the size field for each visited node. Maintain the size field during rotations.

Delete(x): Directly after splicing out a node traverse the path from the spliced out node upwards, and decrease the size counter on every node on this path. Maintain the size field during rotations.

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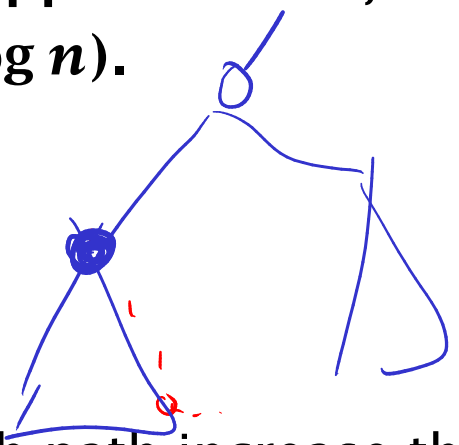

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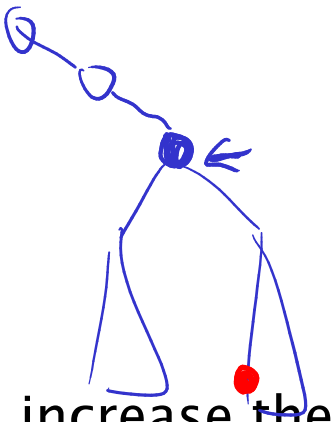

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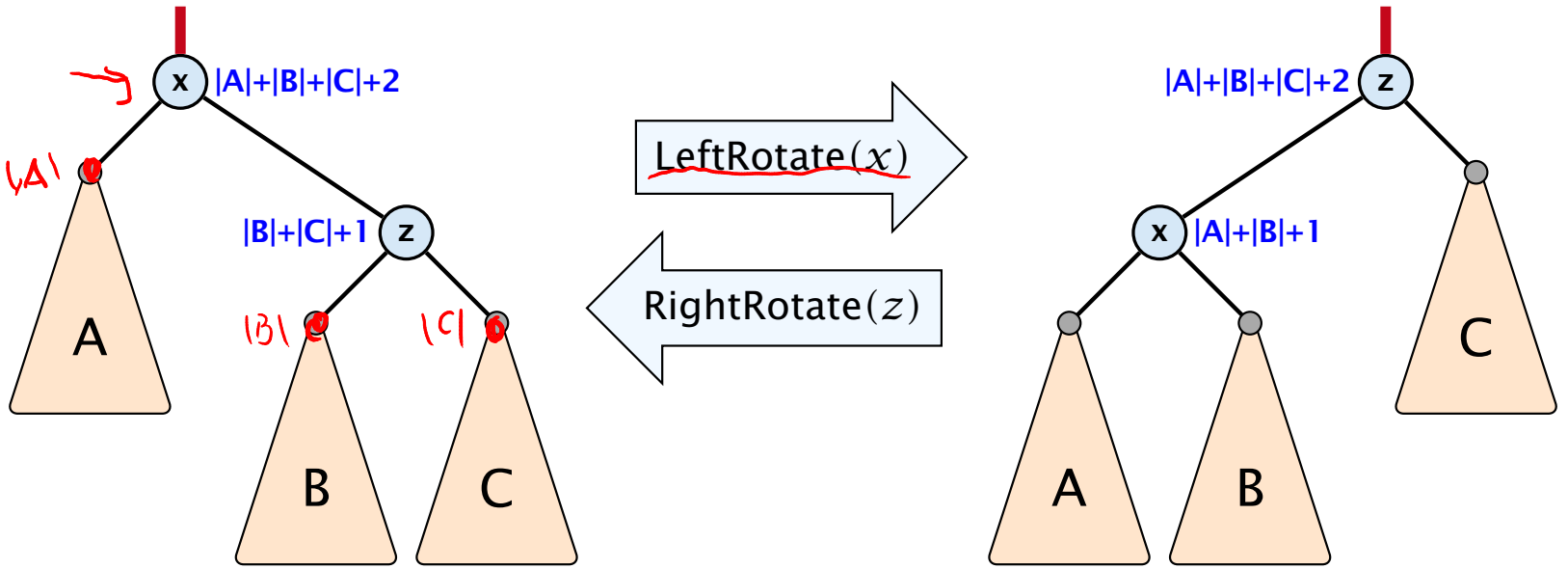
Rotations

The only operation during the fix-up procedure that alters the tree and requires an update of the size-field:



The nodes x and z are the only nodes changing their size-fields.

The new size-fields can be computed locally from the size-fields of the children.

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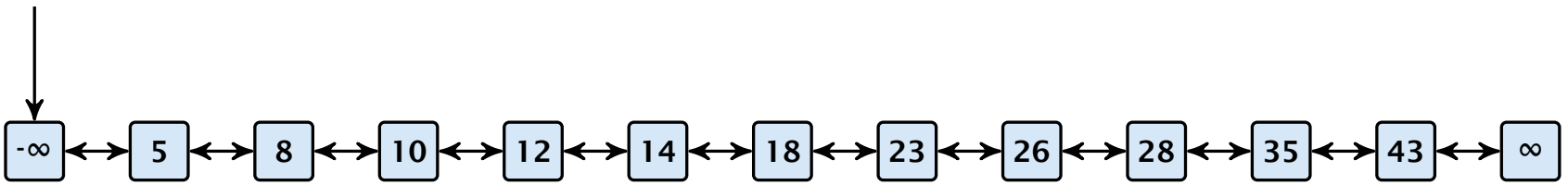

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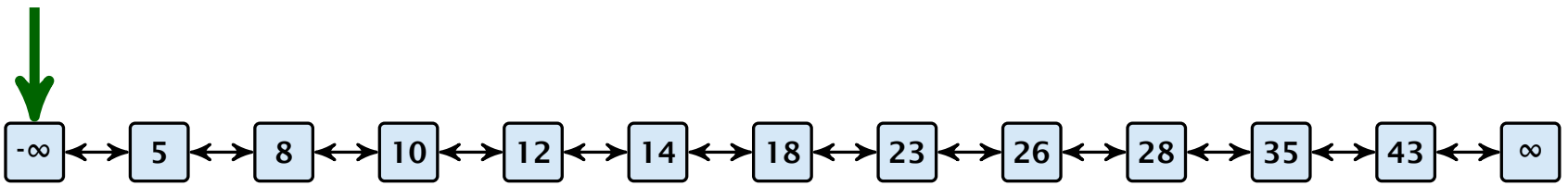

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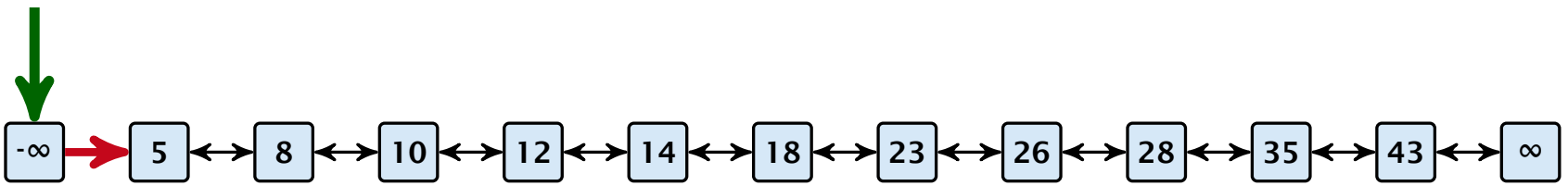

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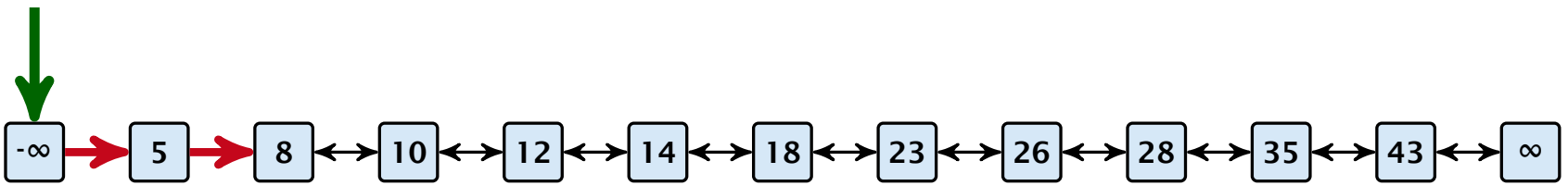

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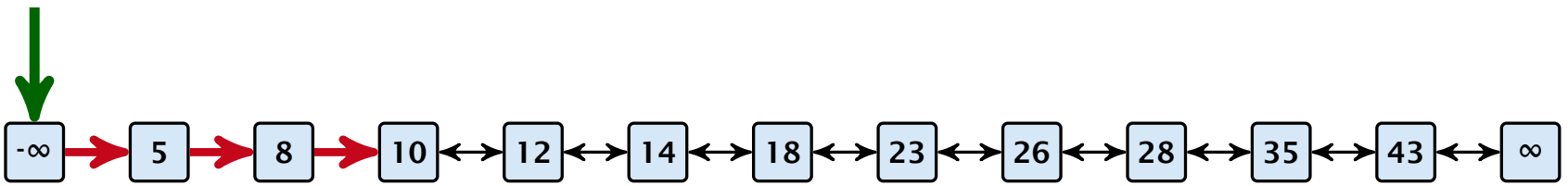

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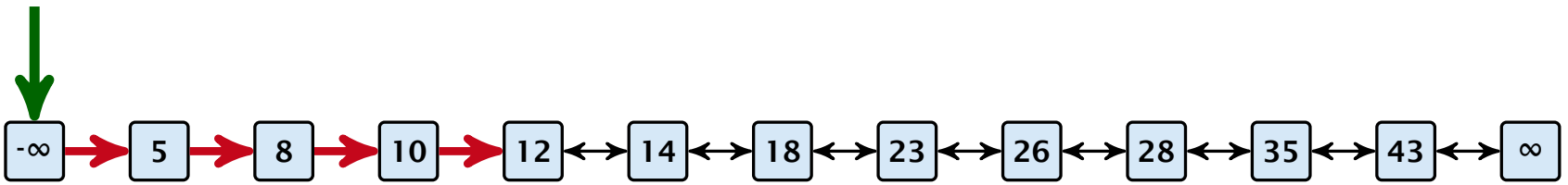

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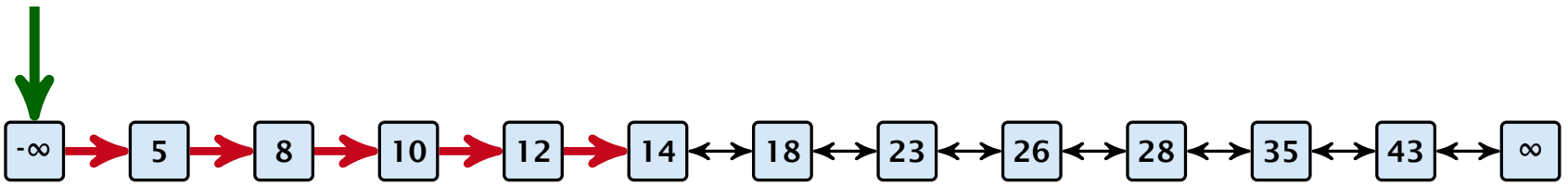

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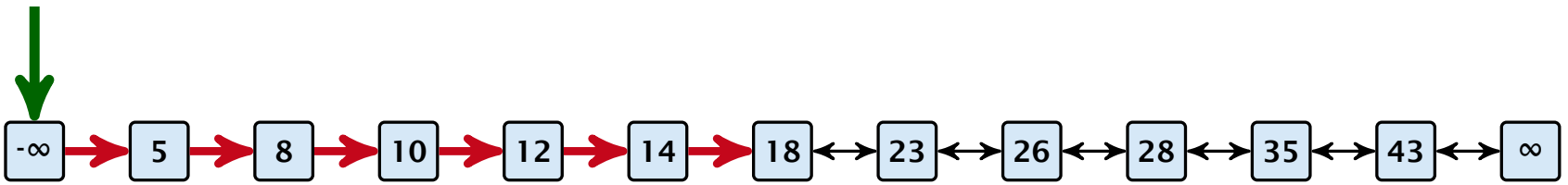

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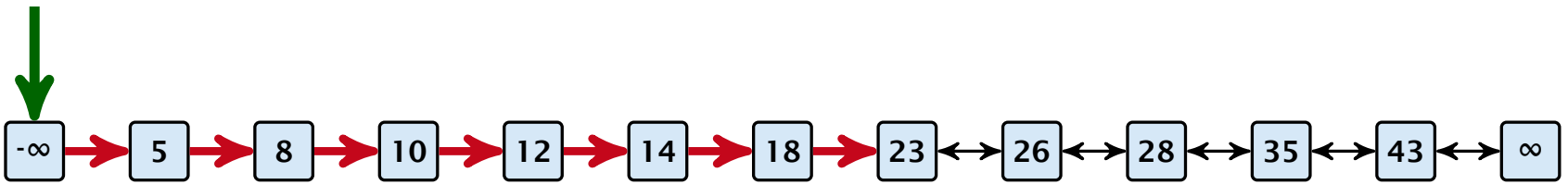

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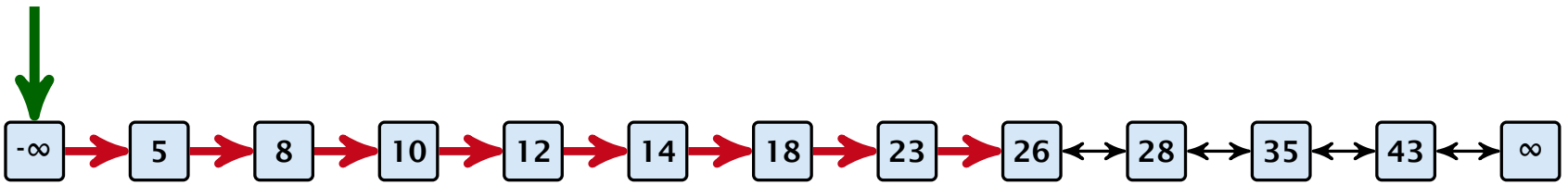

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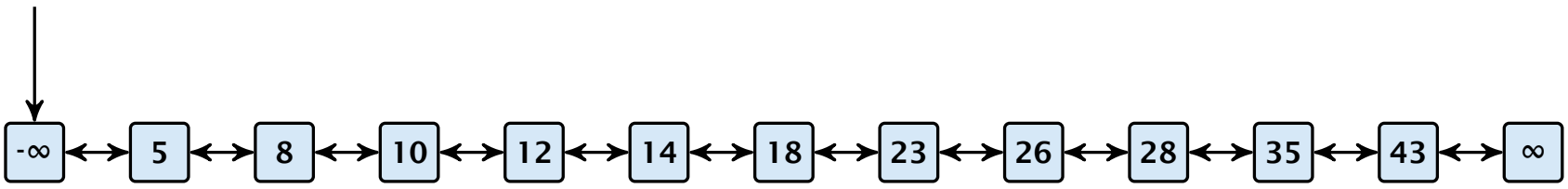

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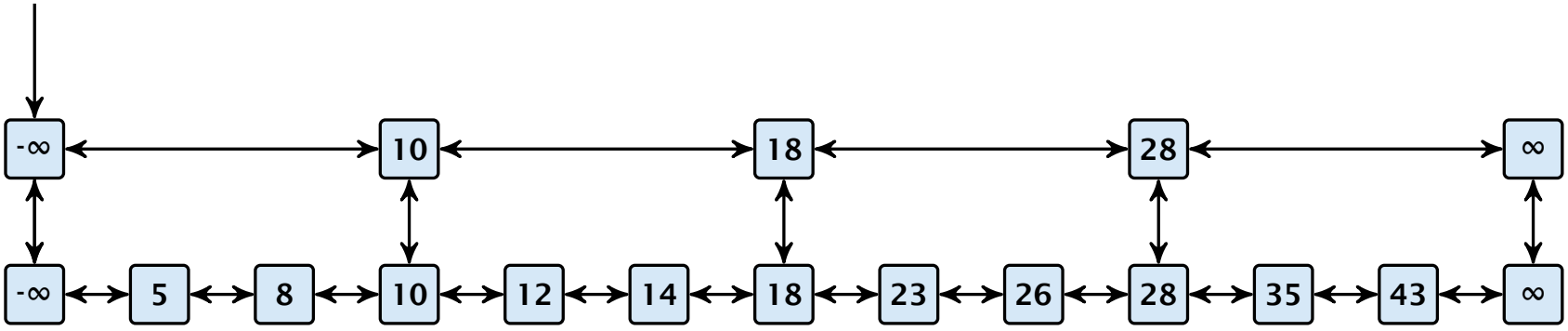

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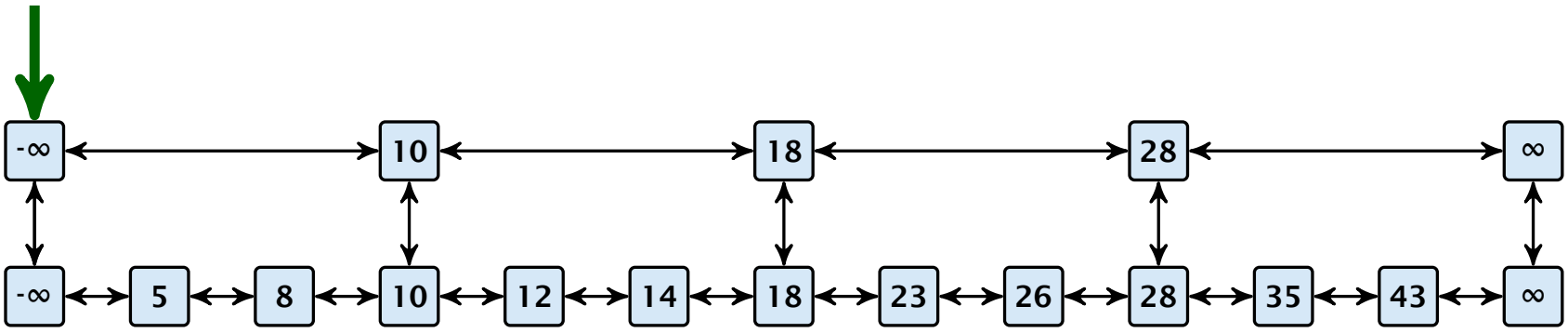

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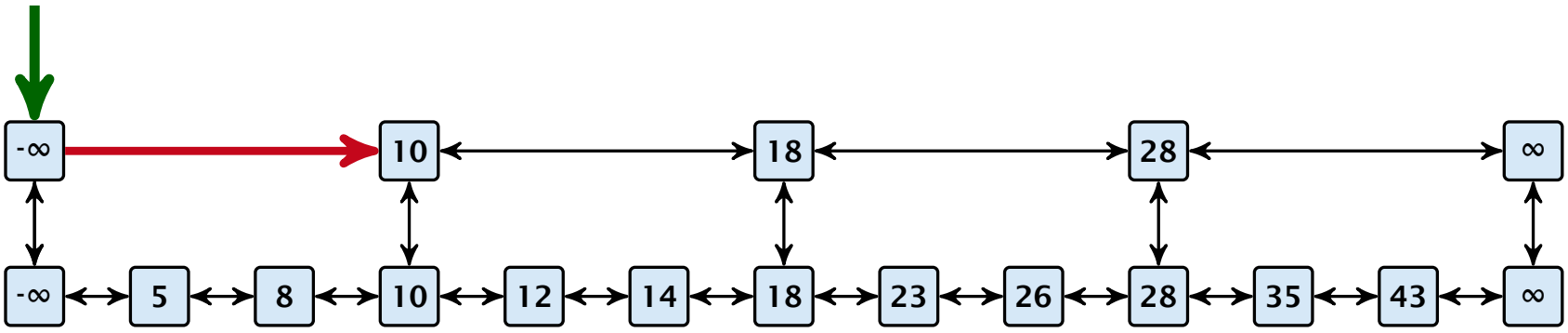

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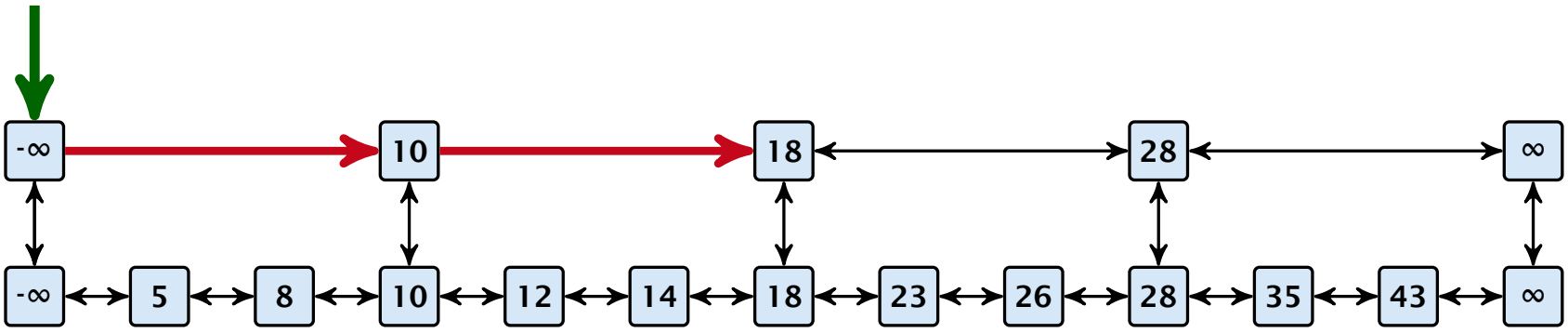

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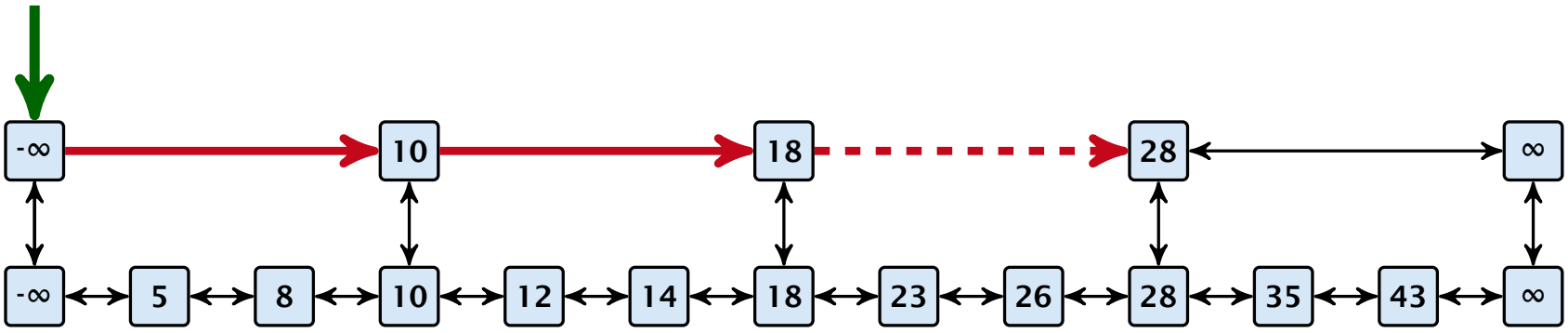

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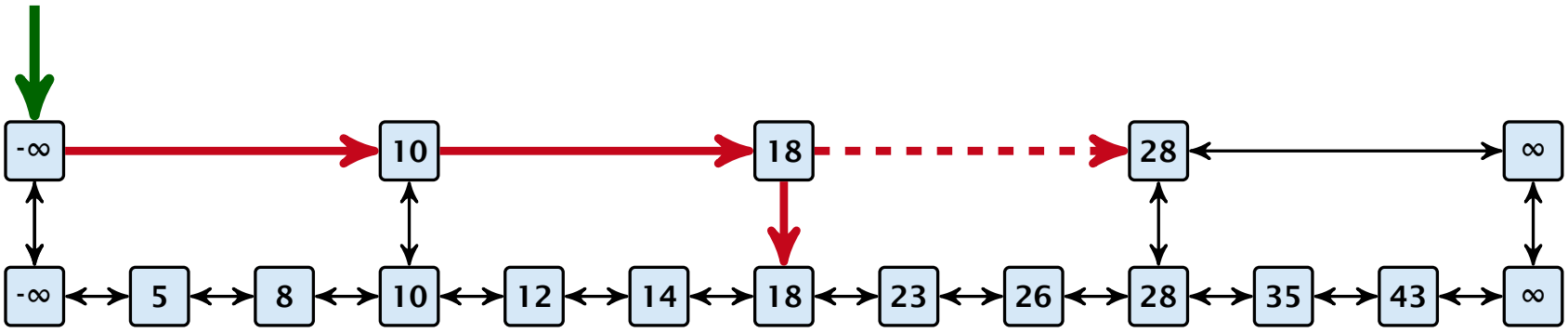

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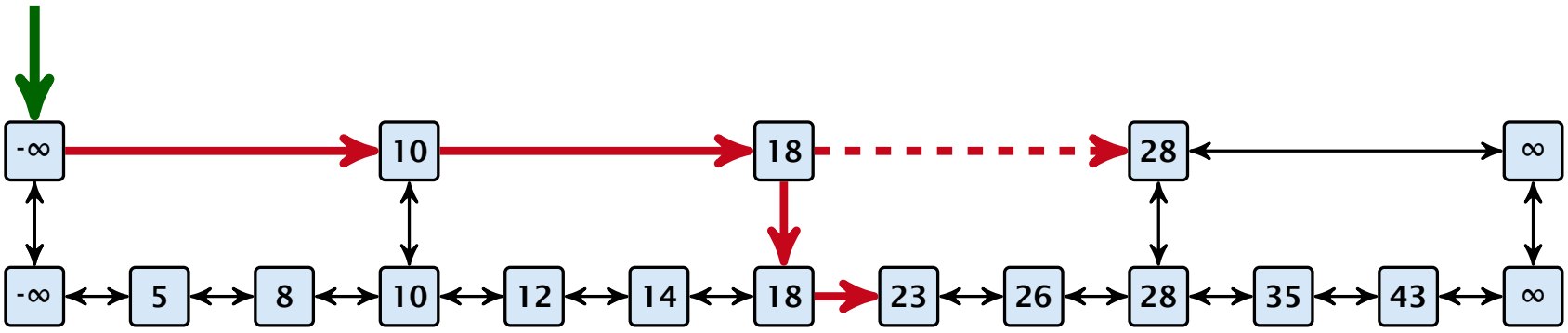

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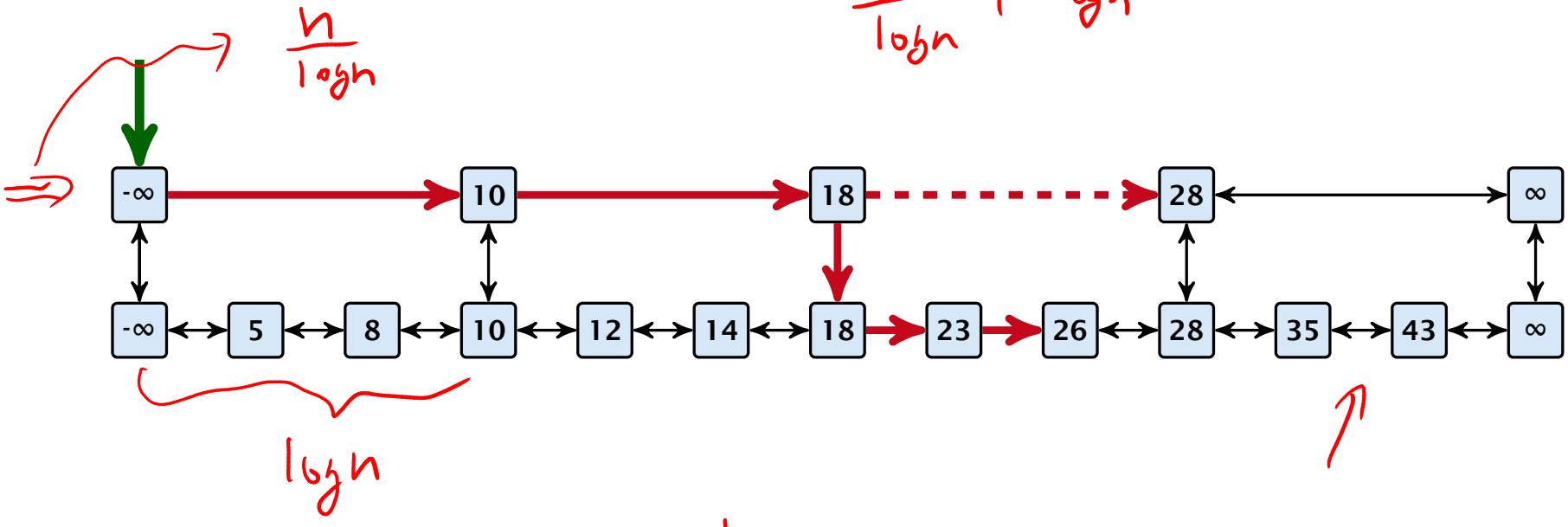

$\frac{n}{\log n}$

$\frac{n}{\log n} + \log n$

$\log n$

$\frac{n}{s} + s \leq 2\sqrt{n}$

$s = \sqrt{n}$

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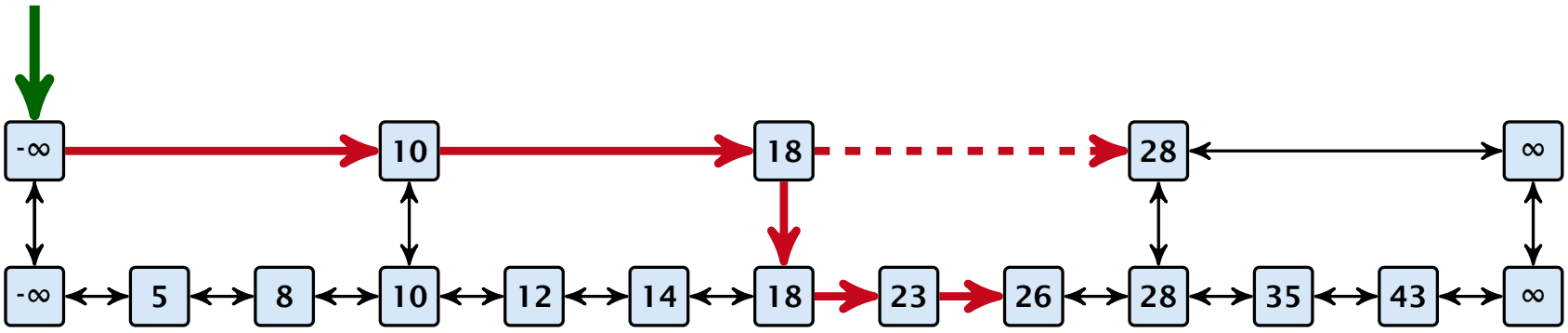

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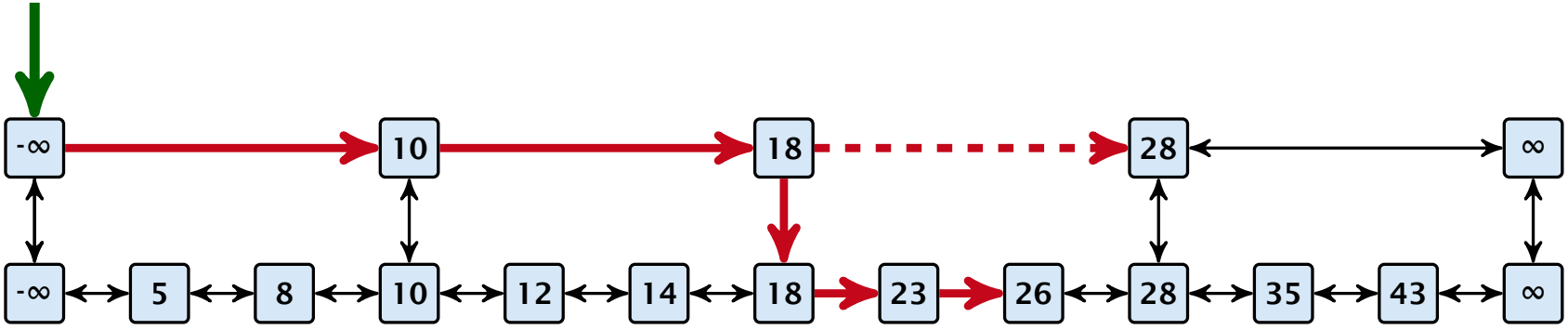

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Worst case search time: $|L_1| + \frac{|L_0|}{|L_1|}$ (ignoring additive constants)

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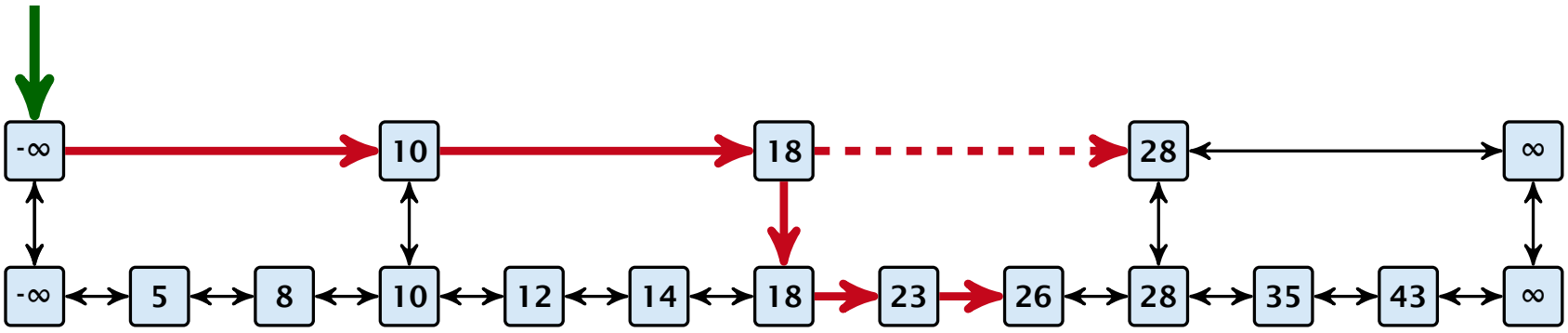

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Choose $|L_1| = \sqrt{n}$. Then search time $\Theta(\sqrt{n})$.

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- Find the largest item in list L_k that is smaller than x. At most $|L_k| + 2$ steps.

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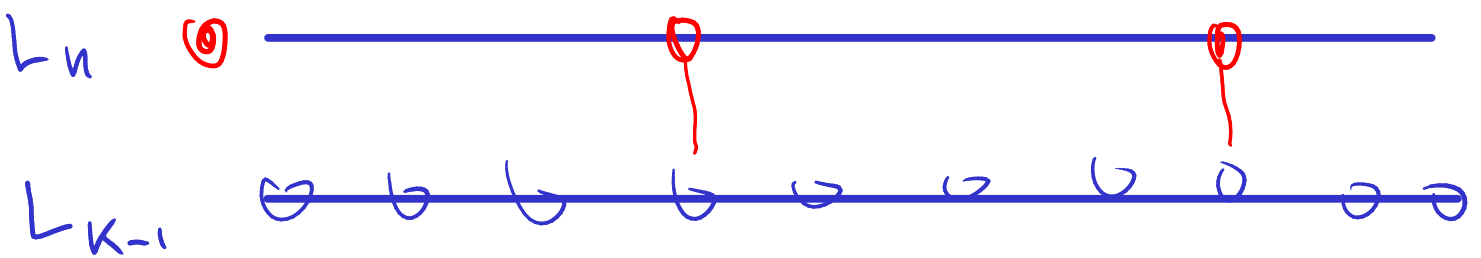

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- Find the largest item in list L_{k-2} that is smaller than x. At most $\lceil \frac{|L_{k-2}|}{|L_{k-1}|+1} \rceil + 2$ steps.
- ...
- At most $|L_k| + \sum_{i=1}^{k} \frac{L_{i-1}}{L_i} + 3(k+1)$ steps.

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Choose ratios between list-lengths evenly, i.e., $\frac{|L_{i-1}|}{|L_i|} = r$, and, hence, $L_k \approx r^{-k} n$.

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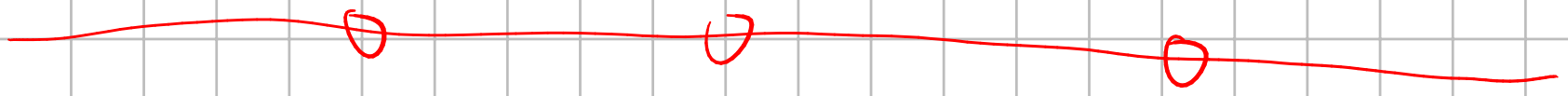
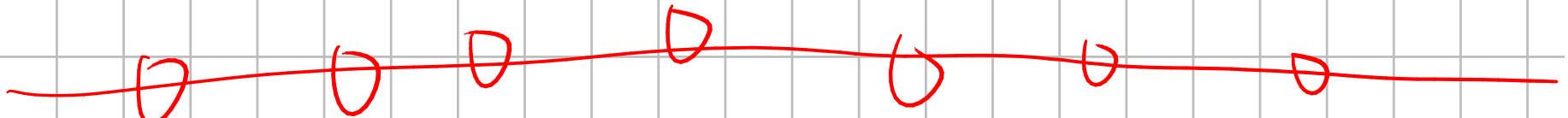
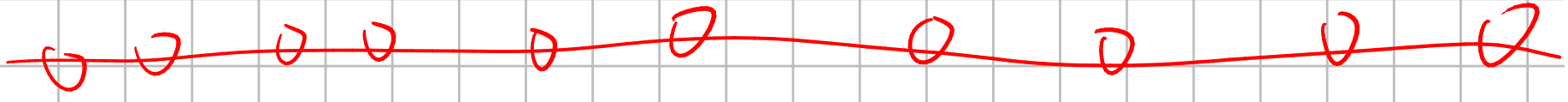
$$n^{\frac{1}{\log_2 n}} = \left[2^{\log_2 n}\right]^{\frac{1}{\log_2 n}}$$

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Choosing $k = \Theta(\log n)$ gives a logarithmic running time.

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How to do insert and delete?

- If we want that in L_i we always skip over roughly the same number of elements in L_{i-1}, an insert or delete may require a lot of re-organisation.

Use randomization instead!

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Insert:

- A search operation gives you the insert position for element x in every list.
- Flip a coin until it shows head, and record the number $t \in \{1, 2, \ldots\}$ of trials needed.
- Insert x into lists $L_0, \ldots, L_{t-1}$.

Delete:

- You get all predecessors via backward pointers.
- Delete x in all lists it actually appears in.

The time for both operations is dominated by the search time.

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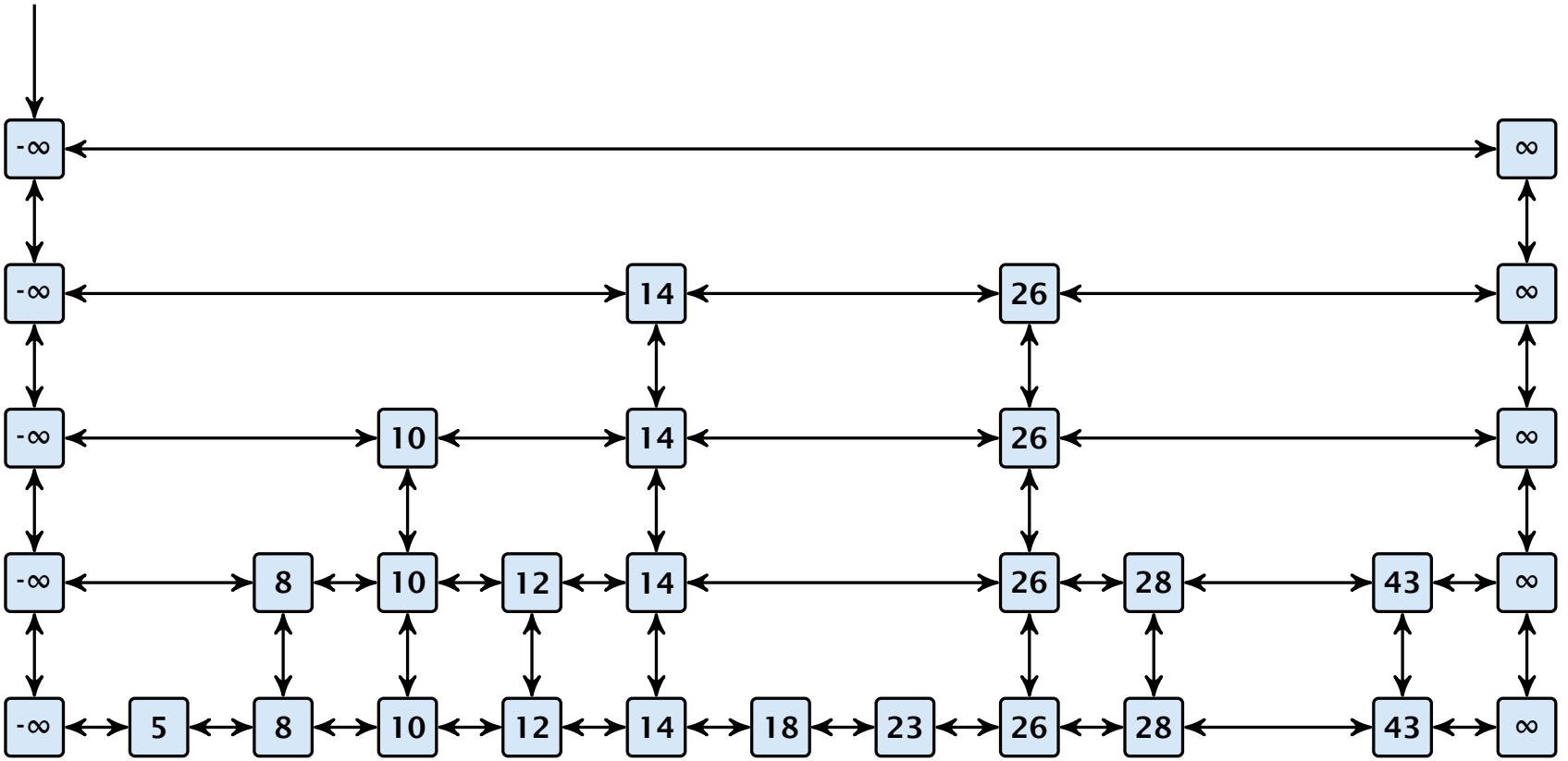

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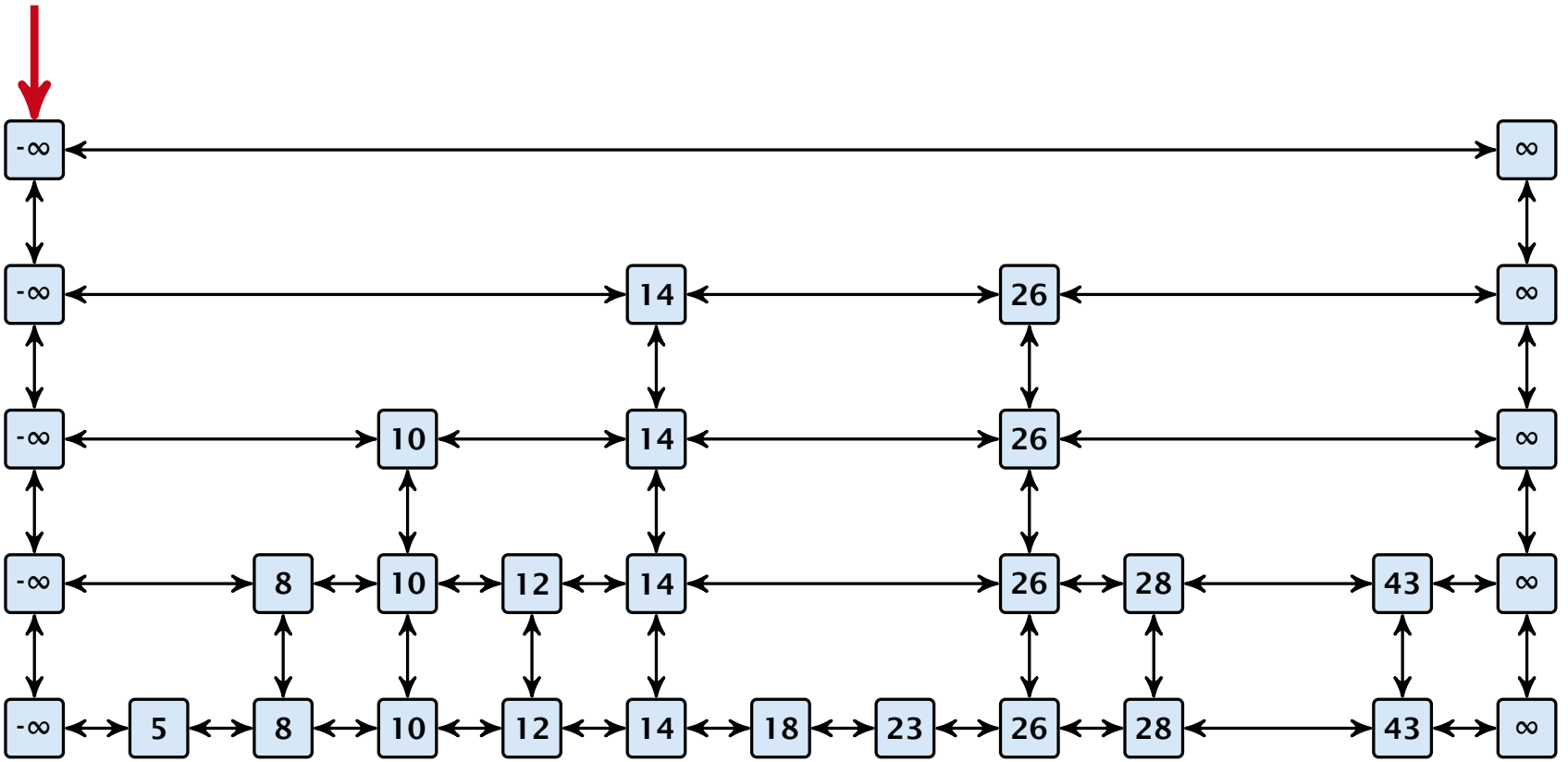

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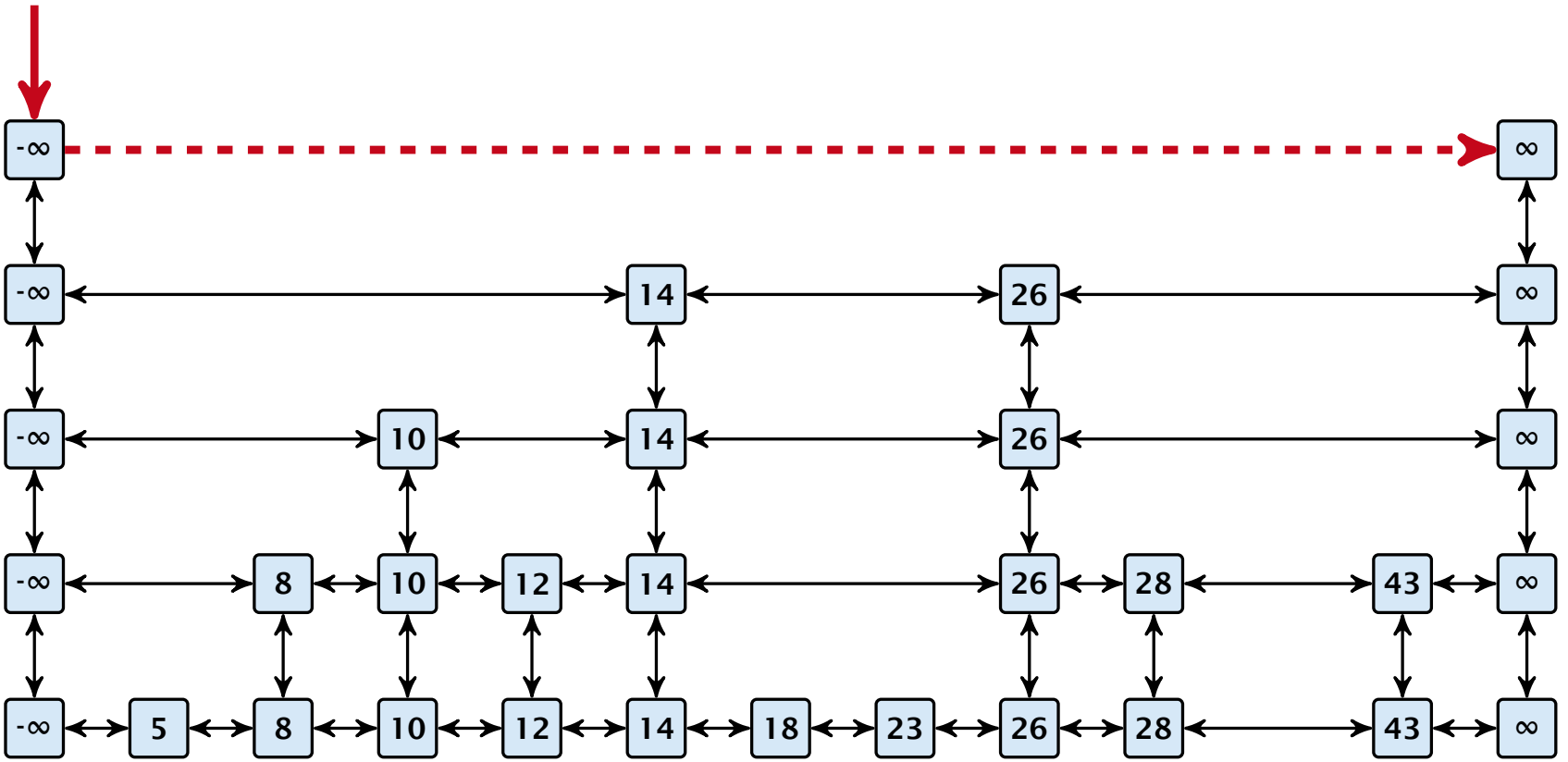

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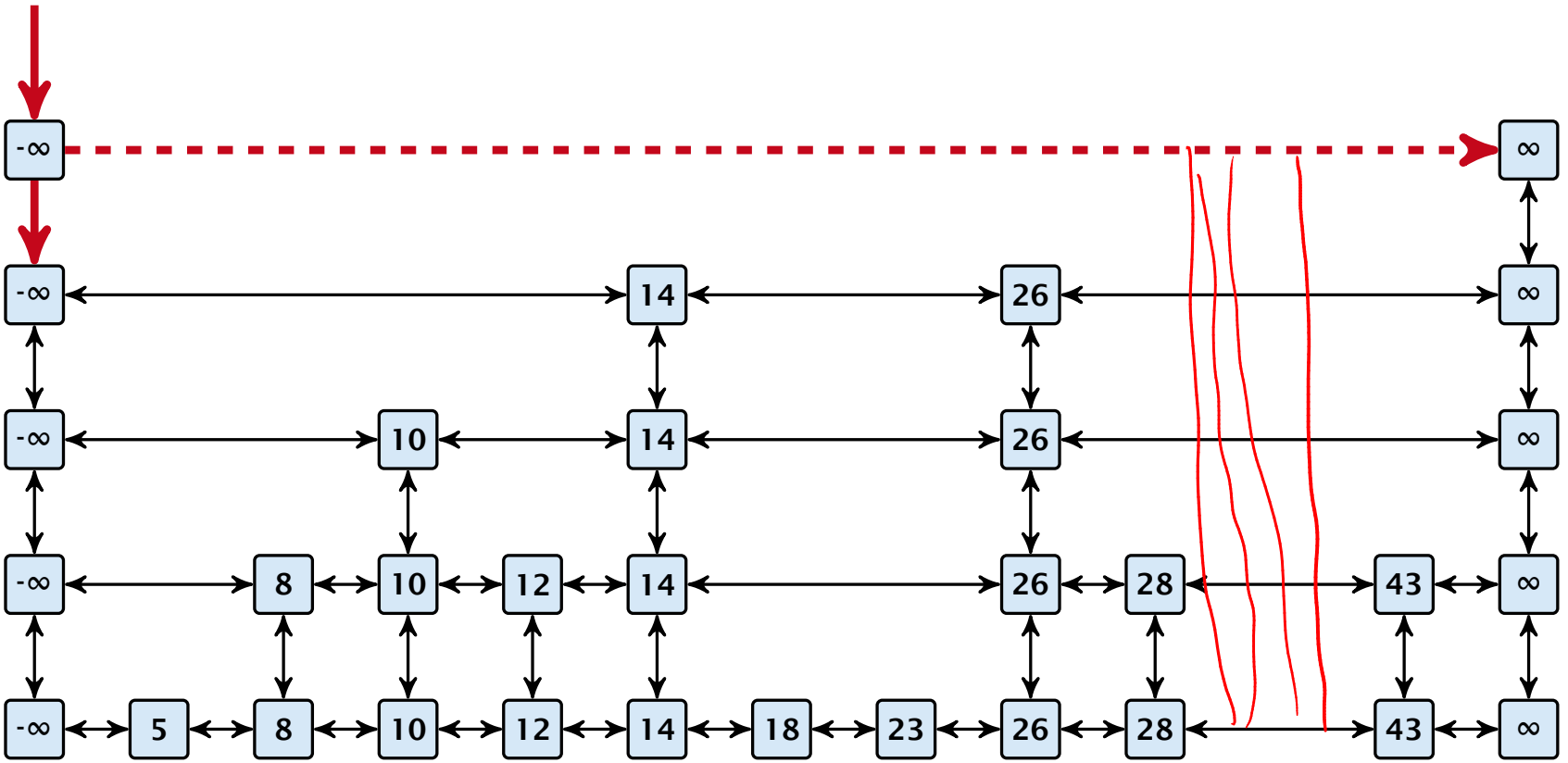

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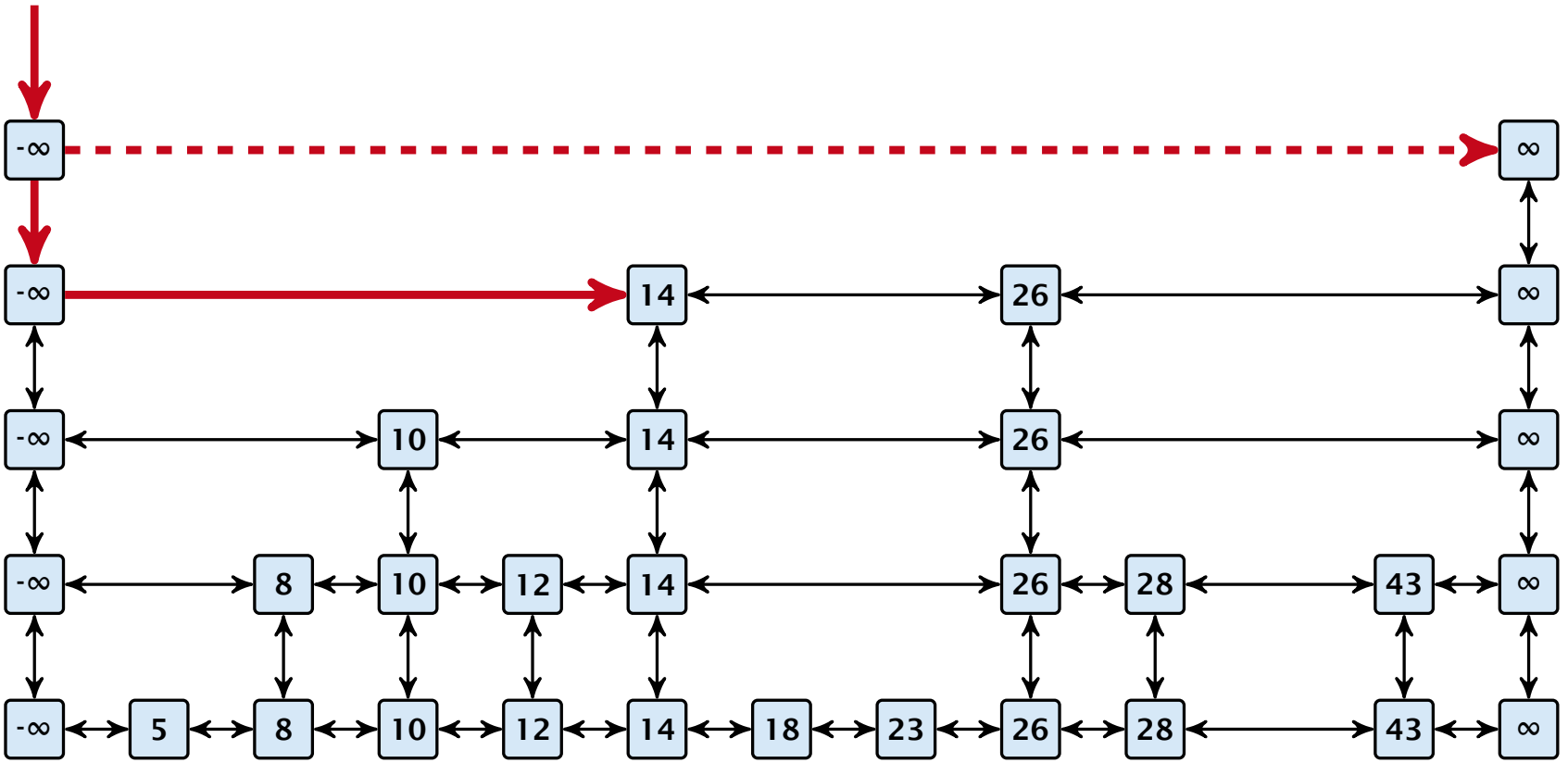

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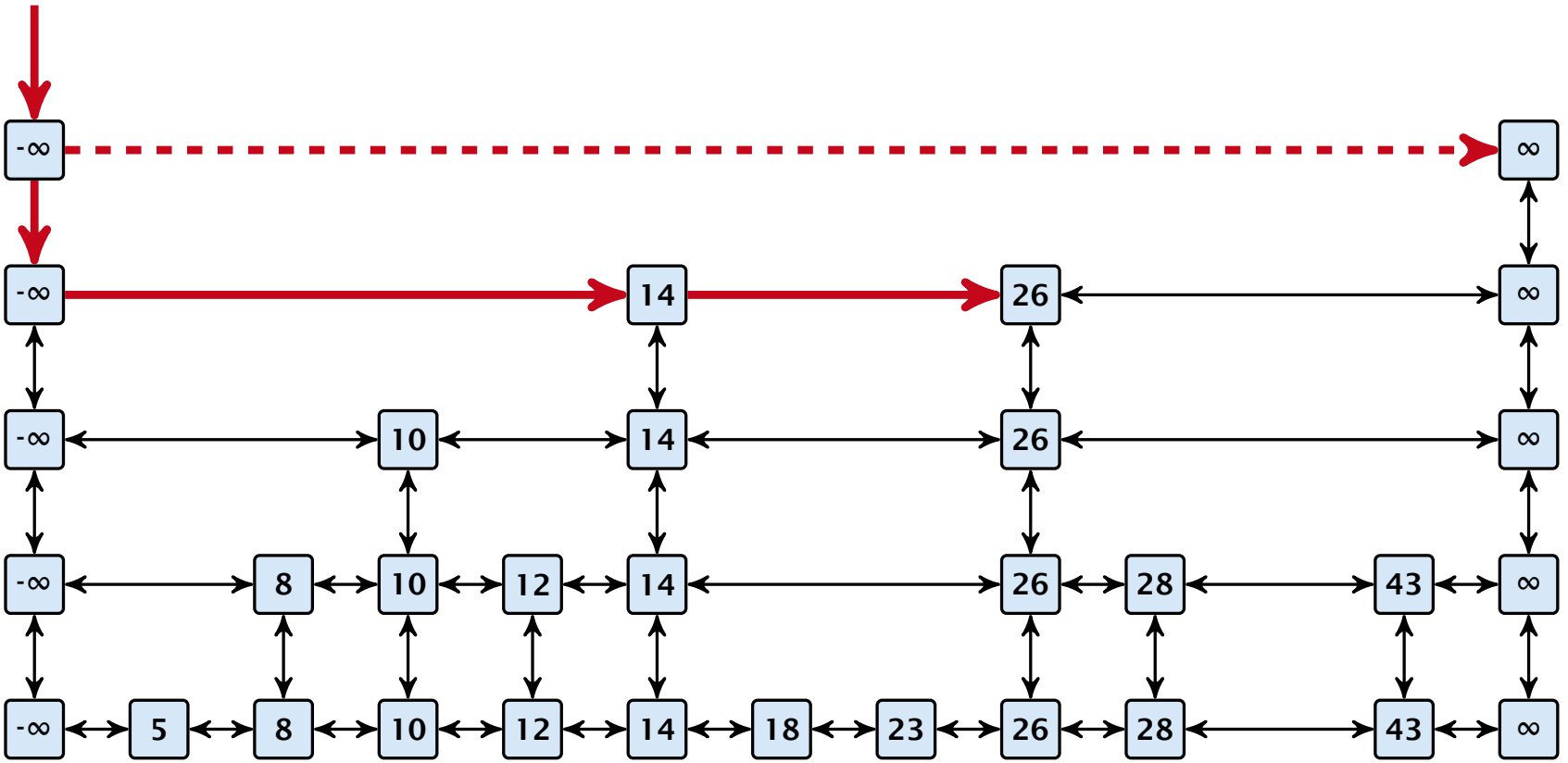

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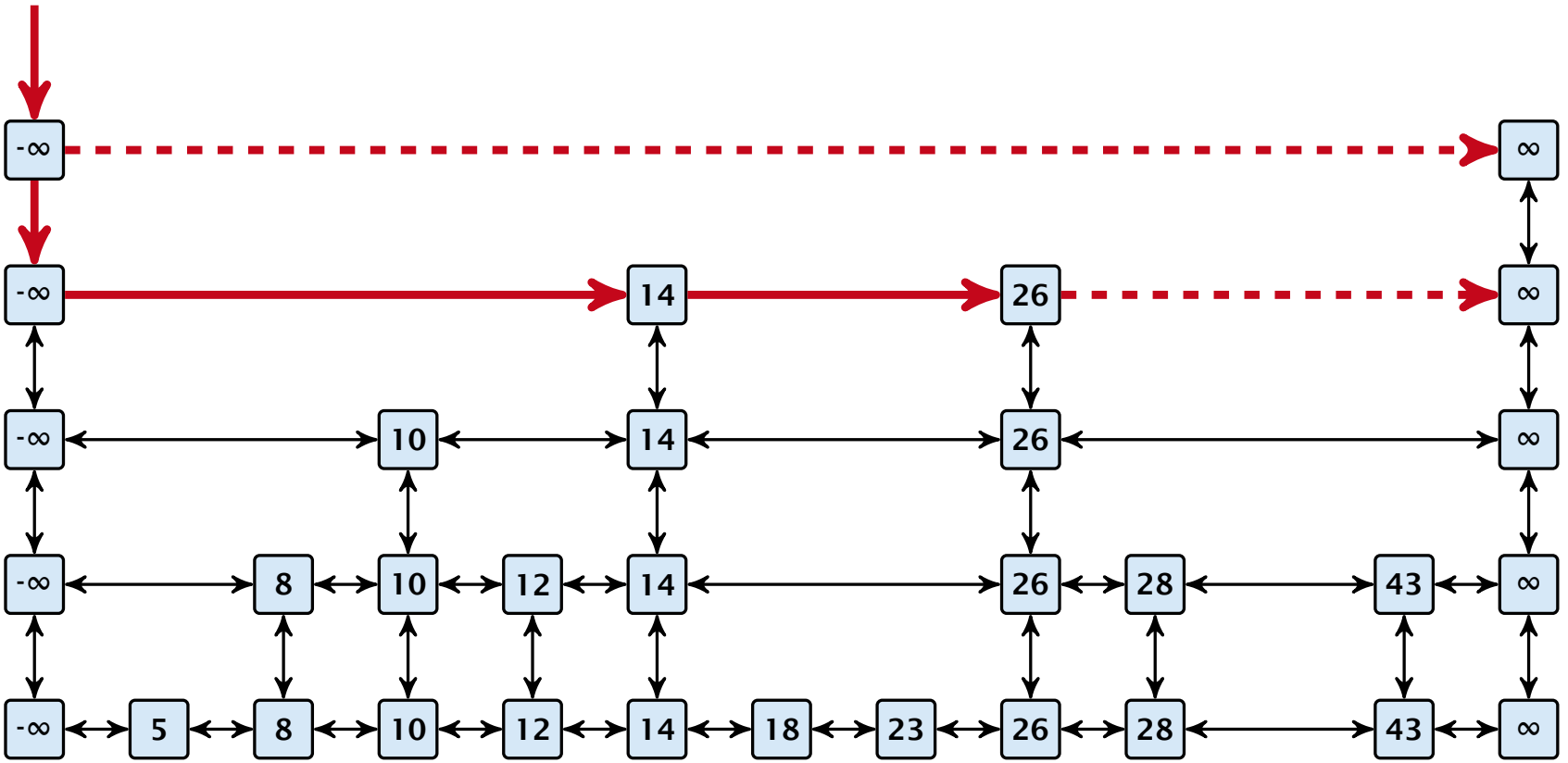

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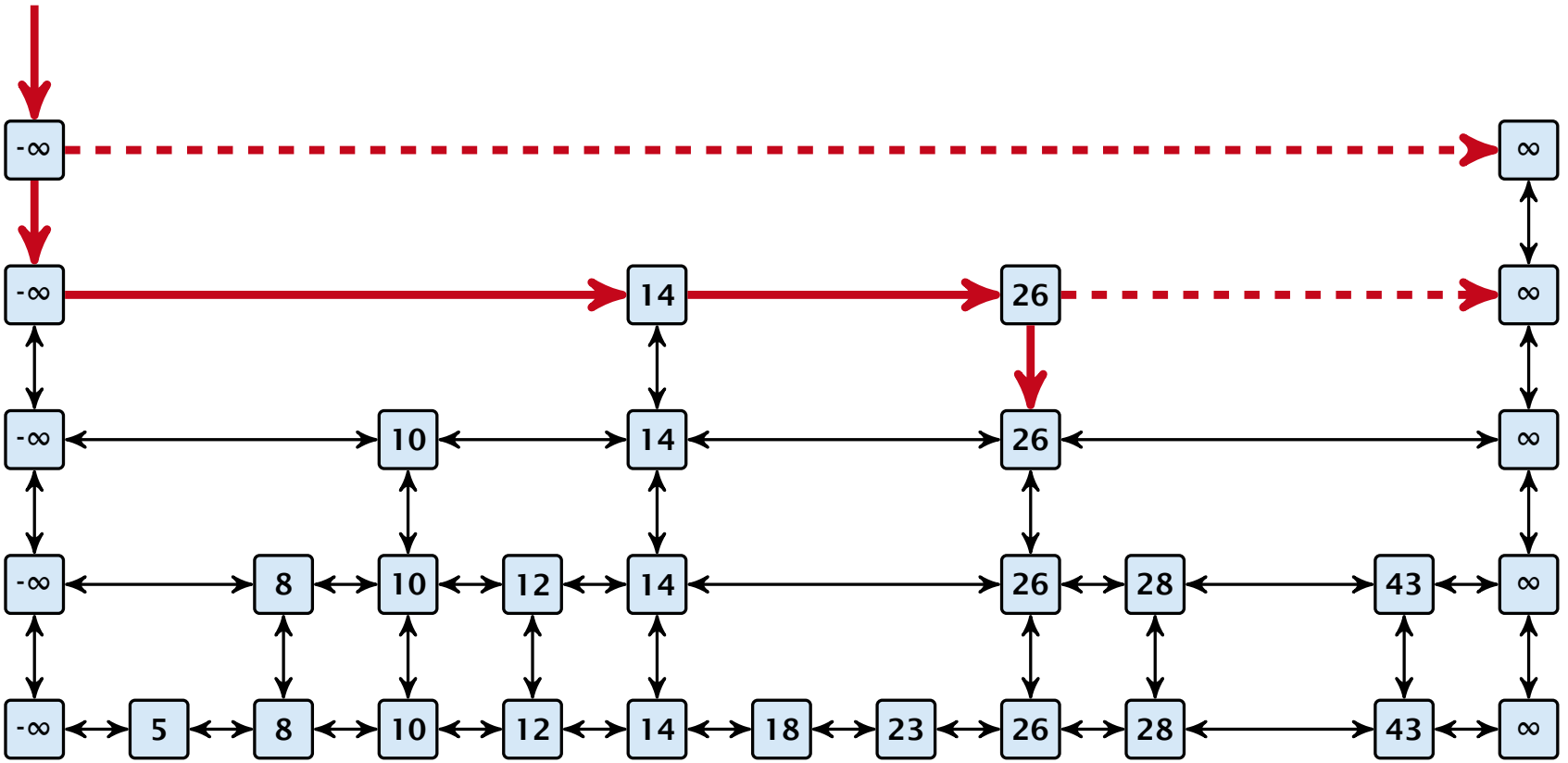

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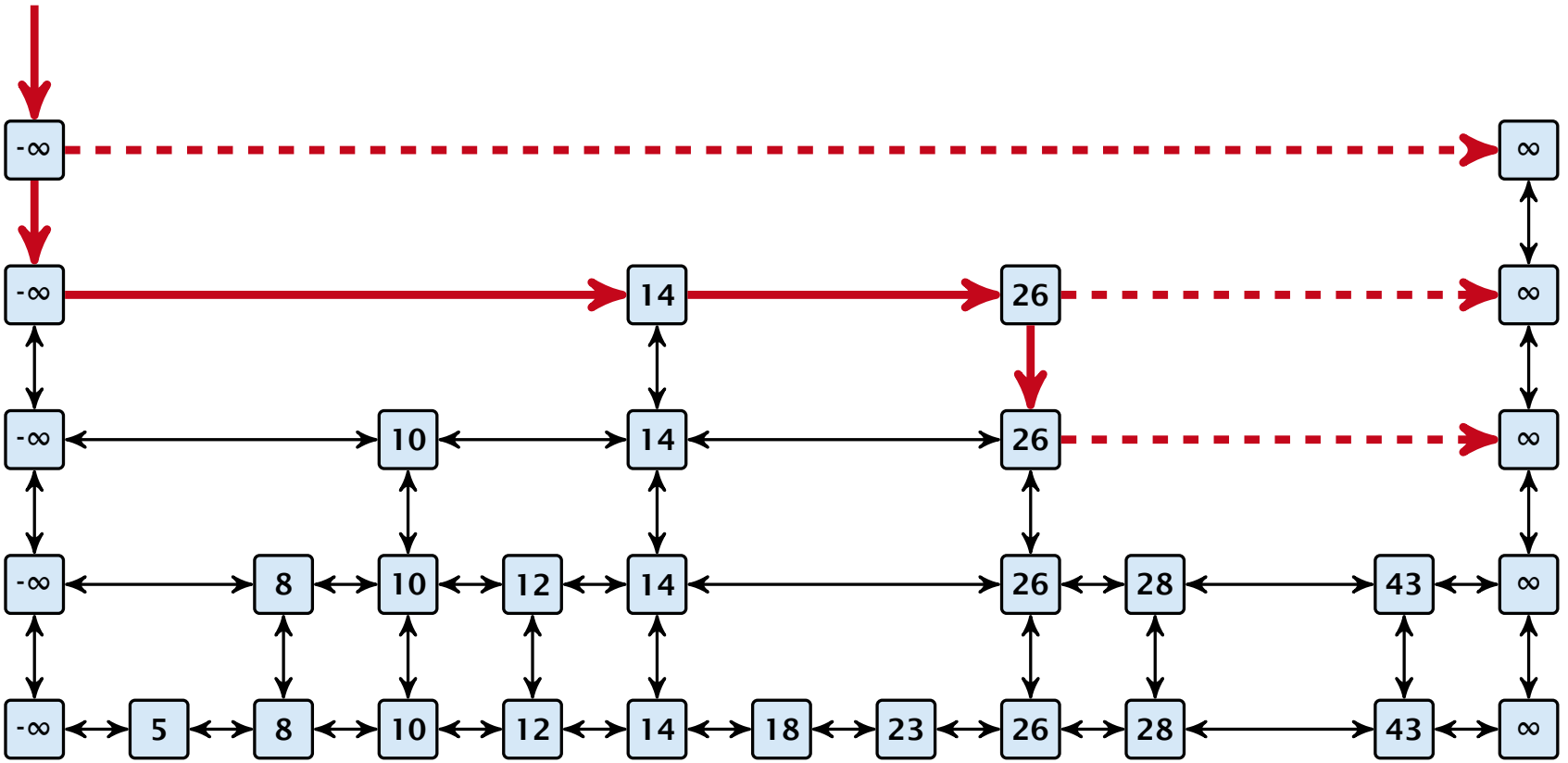

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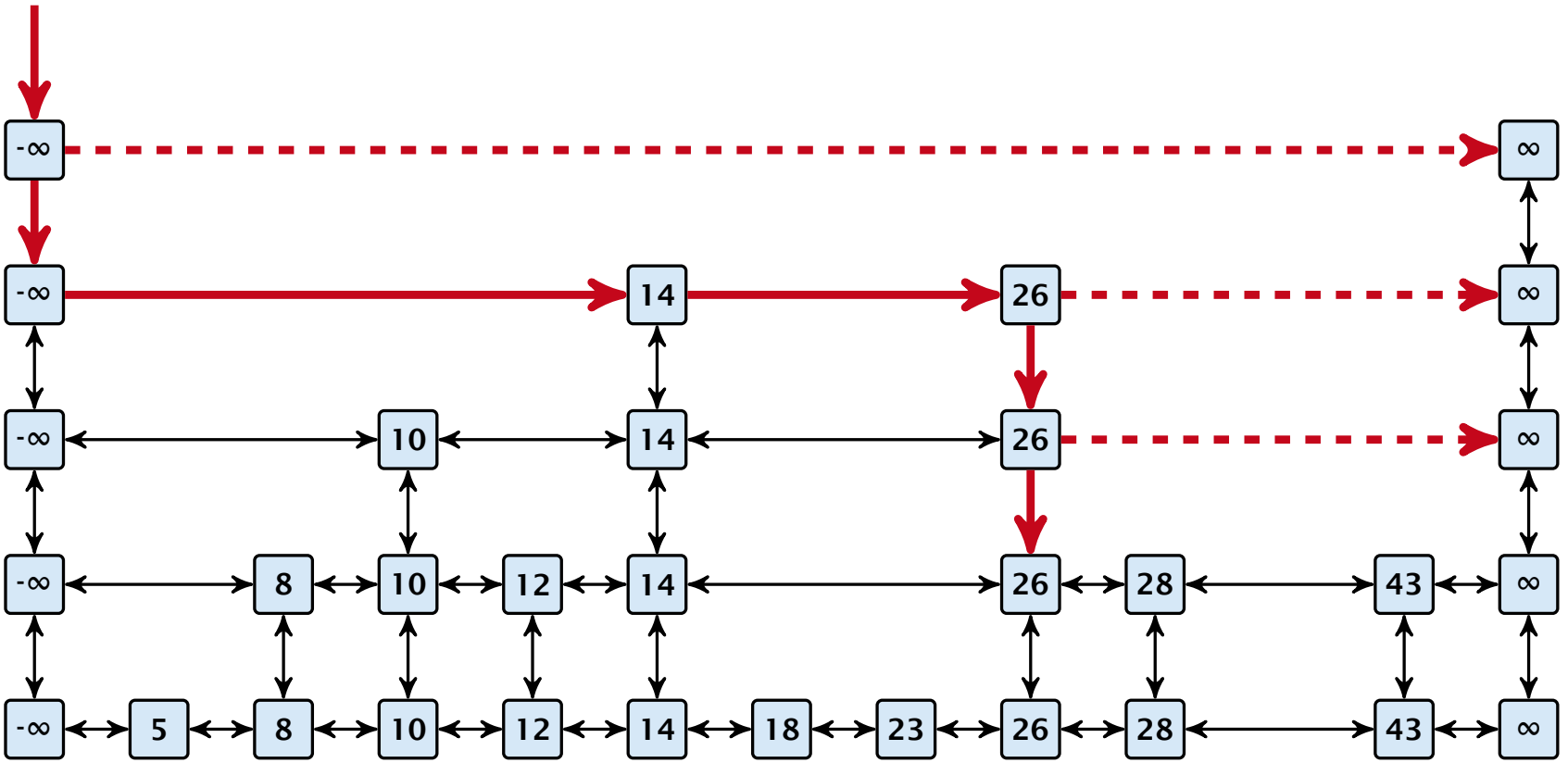

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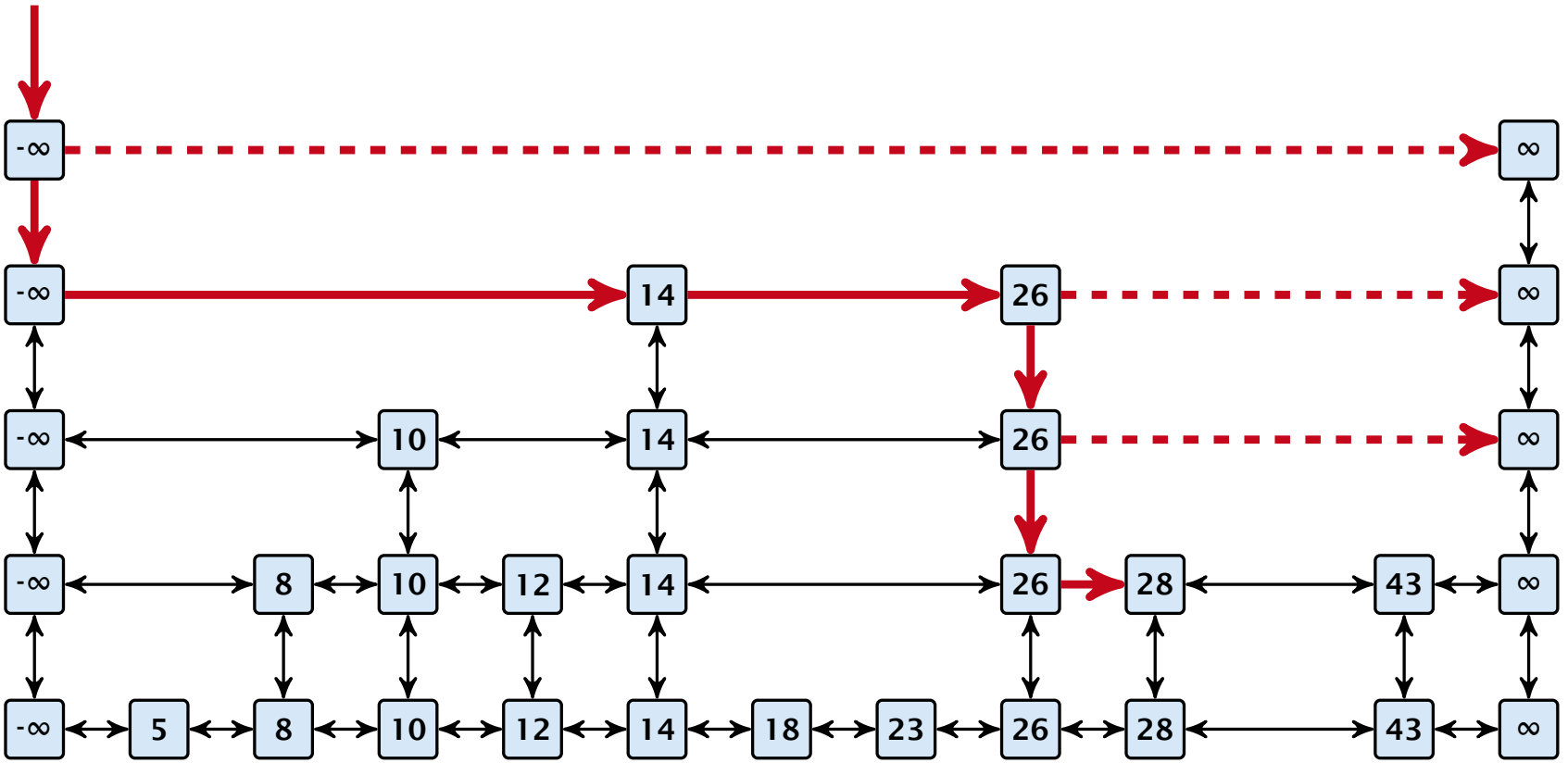

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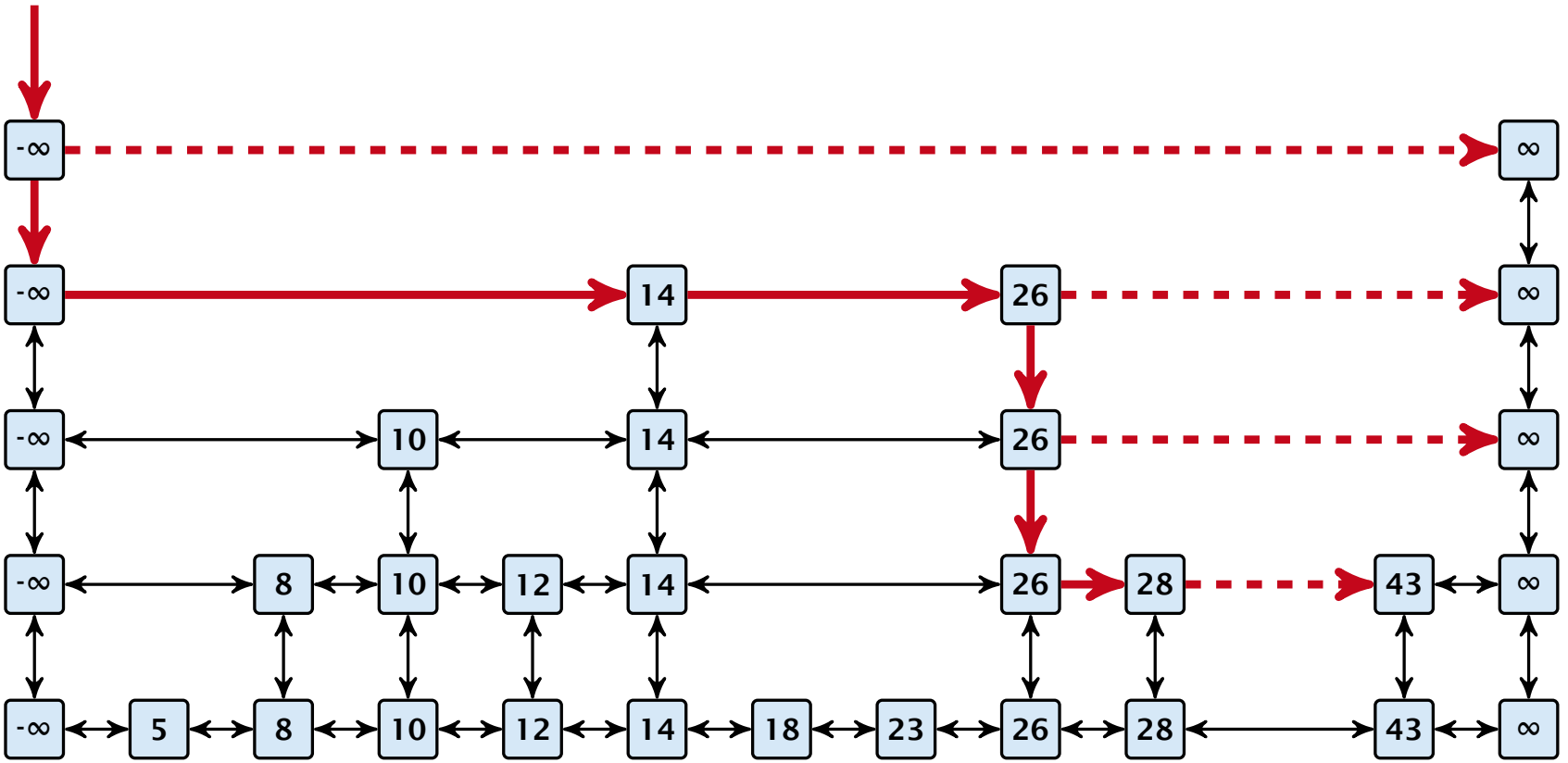

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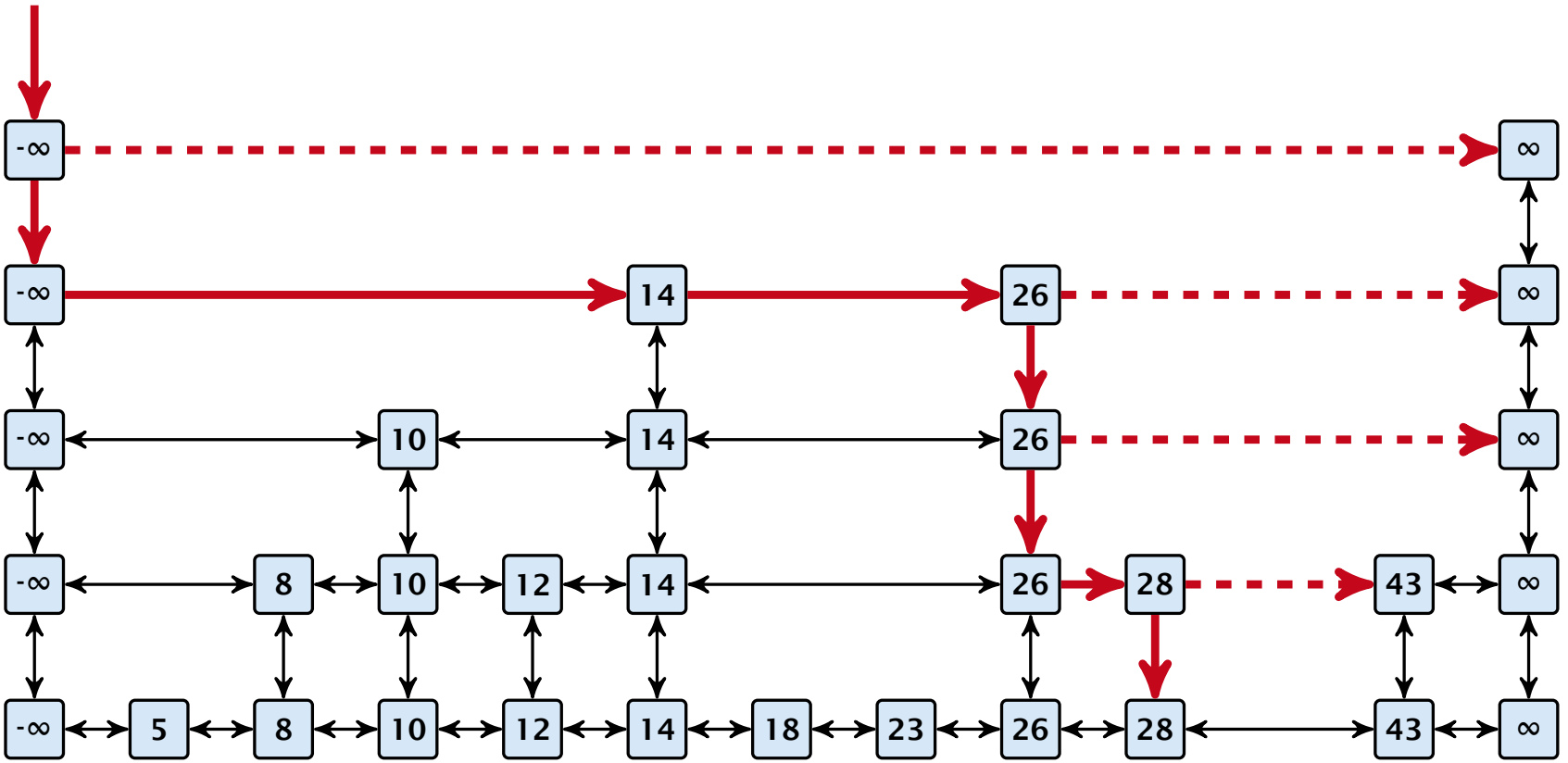

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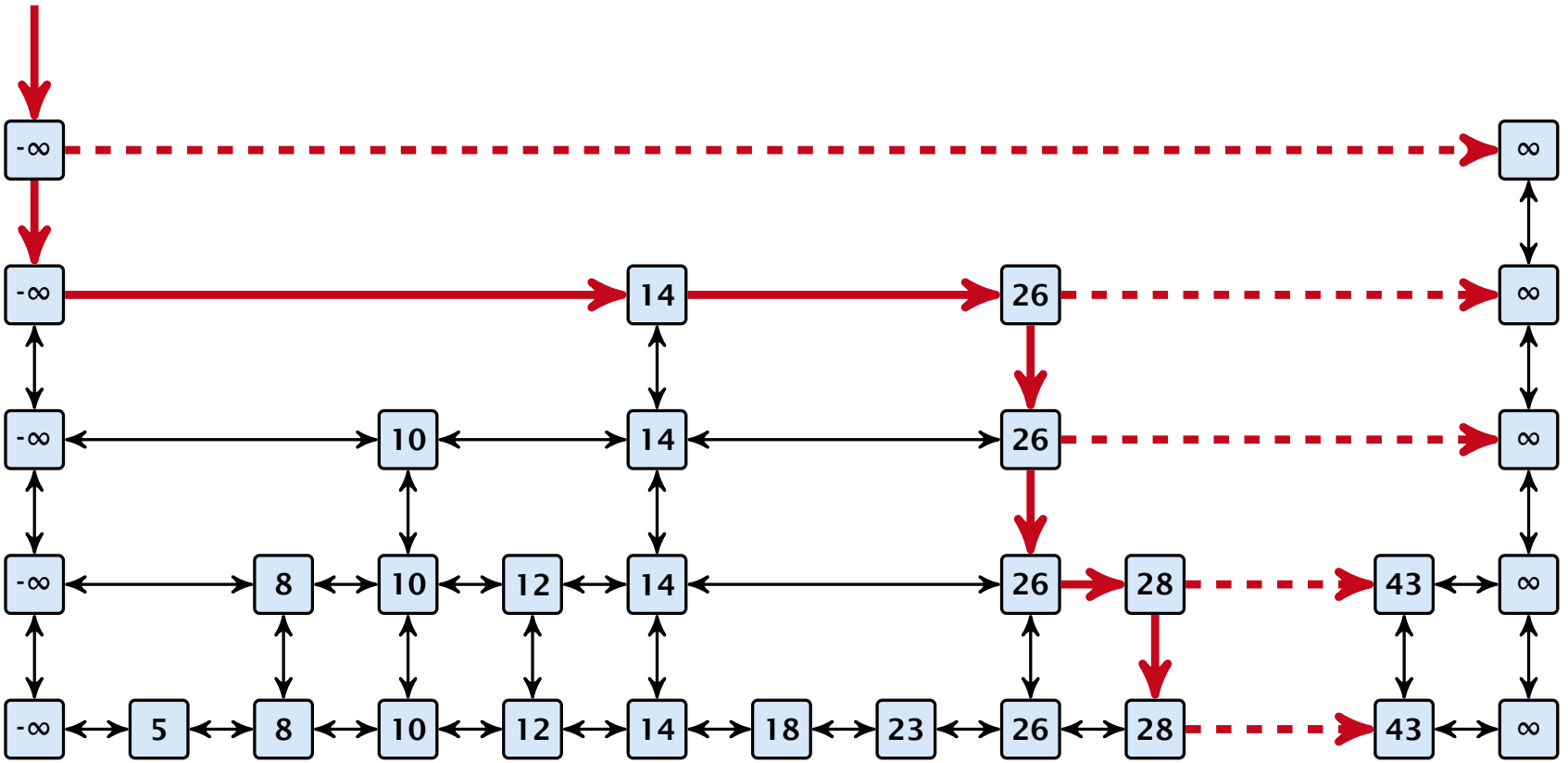

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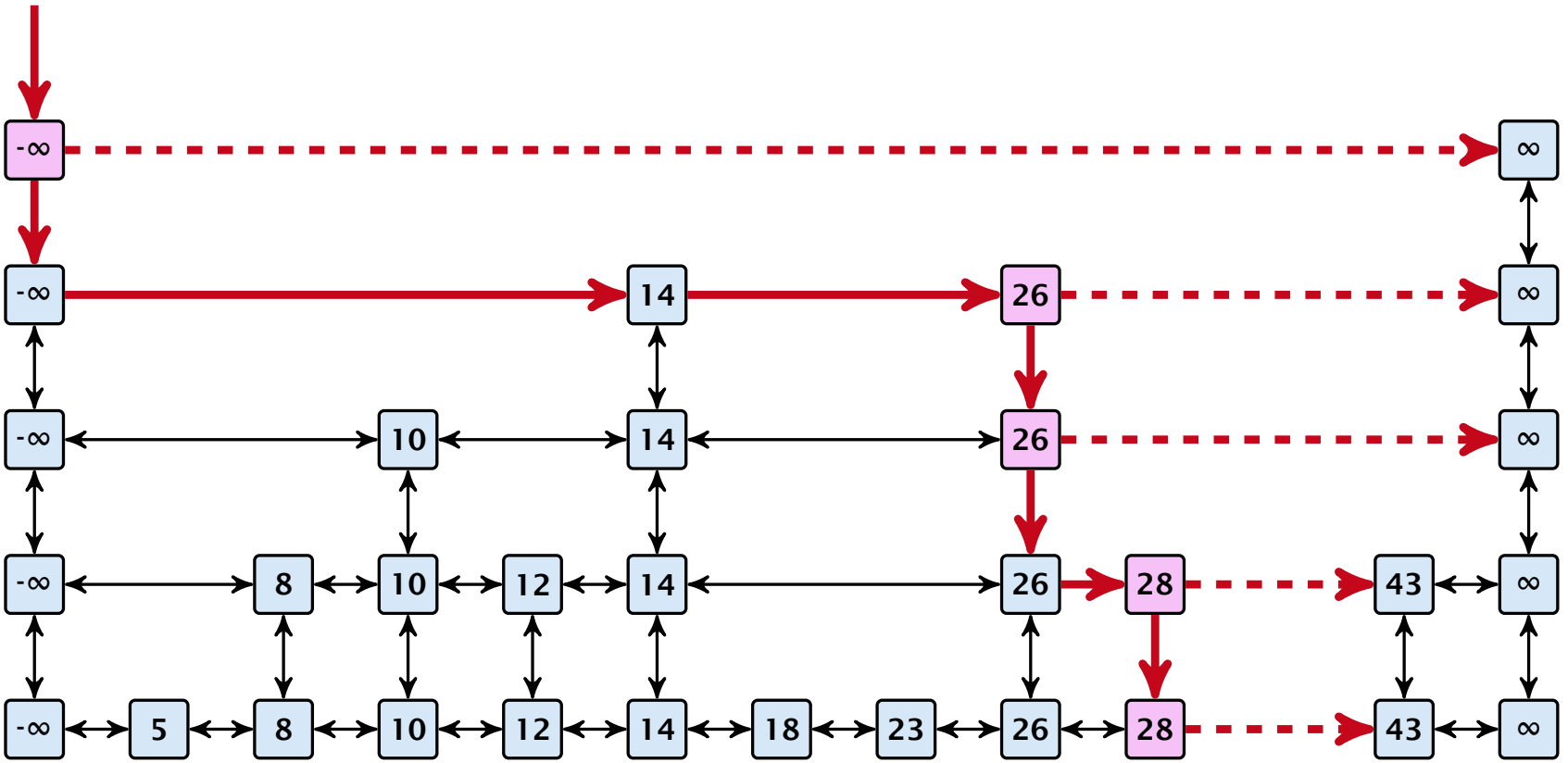

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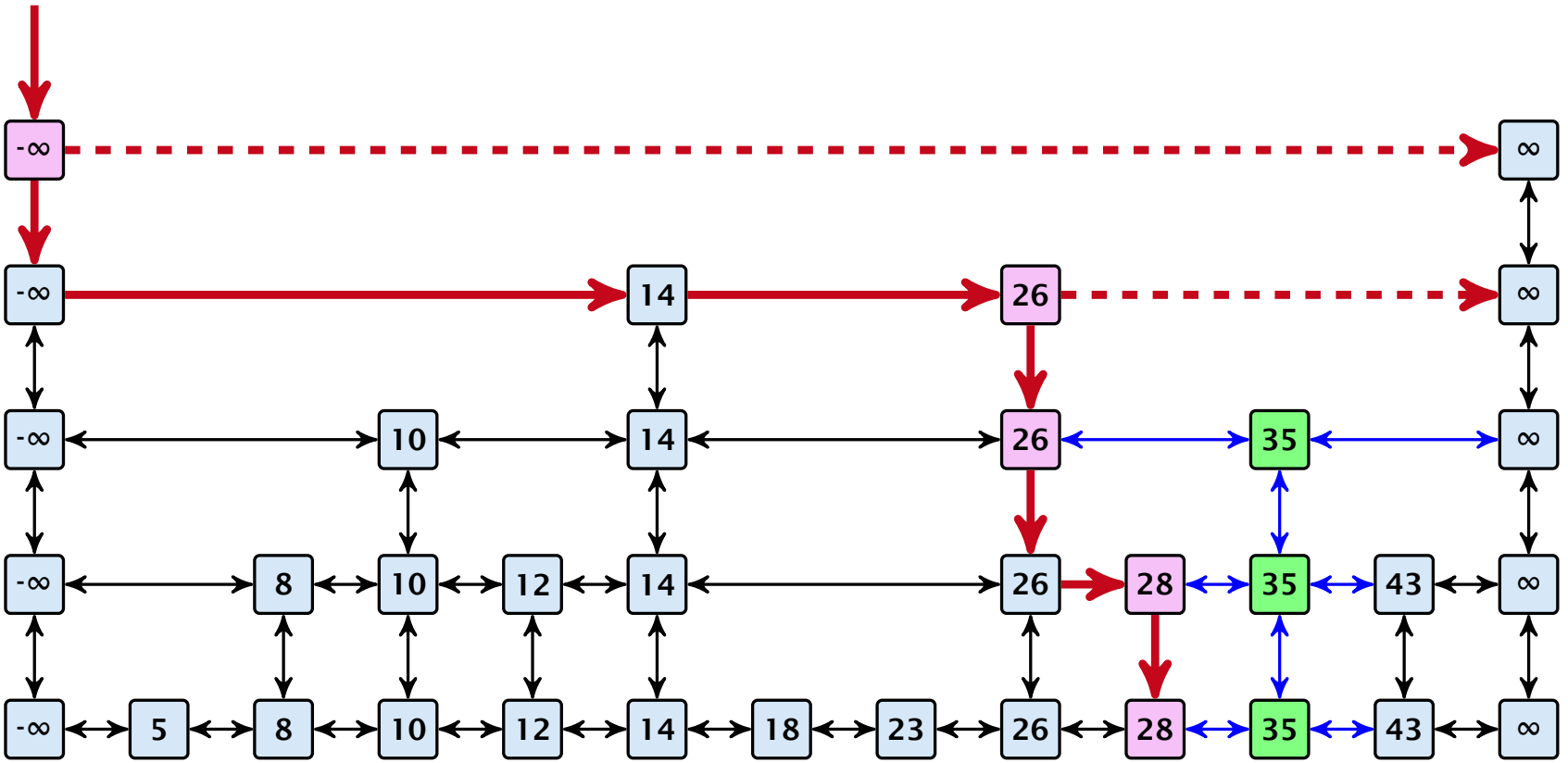

High Probability

$\alpha^2 \cdot \log n + \log\log n = \Theta(\log n)$

Definition 18 (High Probability)

We say a **randomized** algorithm has running time $\mathcal{O}(\log n)$ with high probability if for any constant α the running time is at most $\mathcal{O}(\log n)$ with probability at least $1 - \frac{1}{n^\alpha}$.

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Suppose there are polynomially many events $E_1, E_2, \ldots, E_\ell$, $\ell = n^c$ each holding with high probability (e.g. E_i may be the event that the i-th search in a skip list takes time at most $\mathcal{O}(\log n)$).

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This means $\Pr[E_1 \wedge \cdots \wedge E_\ell]$ holds with high probability.