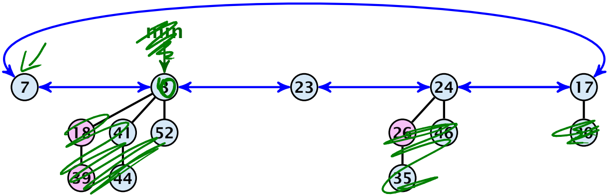
8.3 Fibonacci Heaps

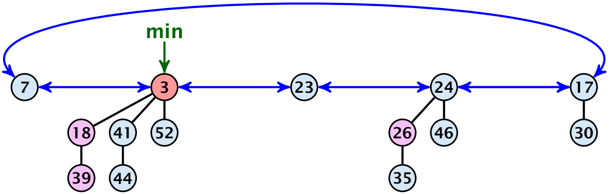
S. delete-min(x)



8.3 Fibonacci Heaps

S. delete-min(x)

- Delete minimum; add child-trees to heap; time: $D(\min) \cdot \mathcal{O}(1)$.



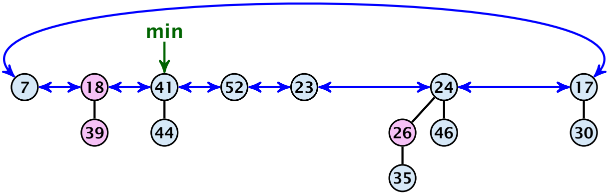
8.3 Fibonacci Heaps

S. delete-min(x)

- Delete minimum; add child-trees to heap; time: $D(\min) \cdot \mathcal{O}(1)$.

-1

- Update min-pointer; time: $(t + D(\min)) \cdot \mathcal{O}(1)$.



8.3 Fibonacci Heaps

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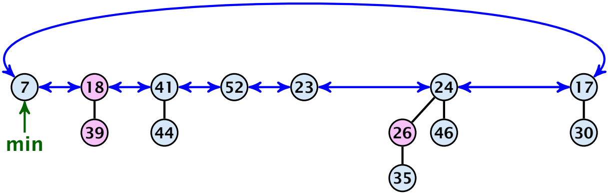

8.3 Fibonacci Heaps

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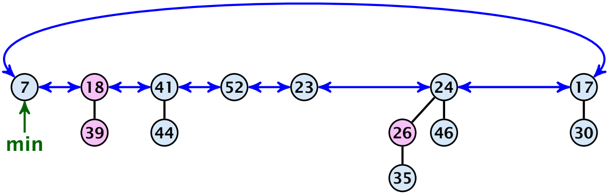
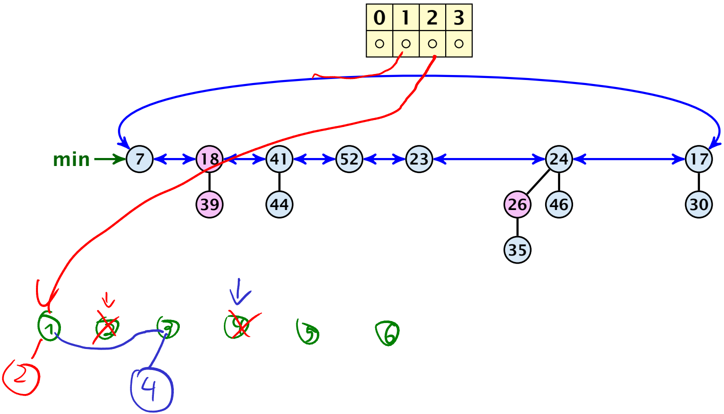


- Consolidate root-list so that no roots have the same degree. Time $t \cdot \mathcal{O}(1)$ (see next slide).

8.3 Fibonacci Heaps

Consolidate:



8.3 Fibonacci Heaps

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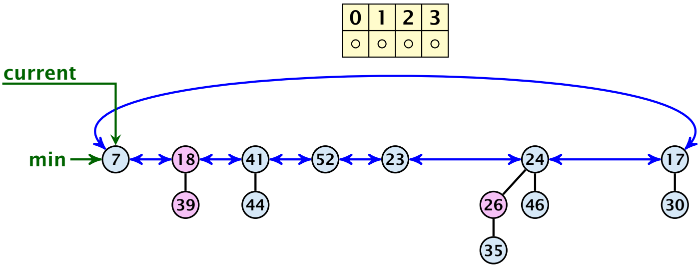

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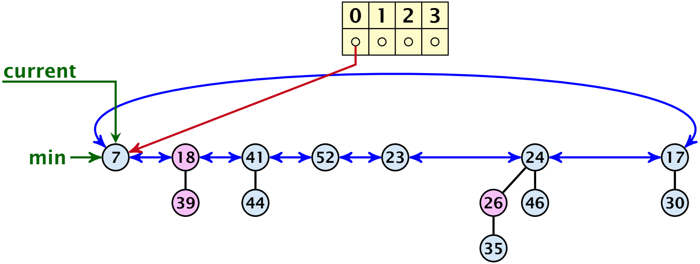

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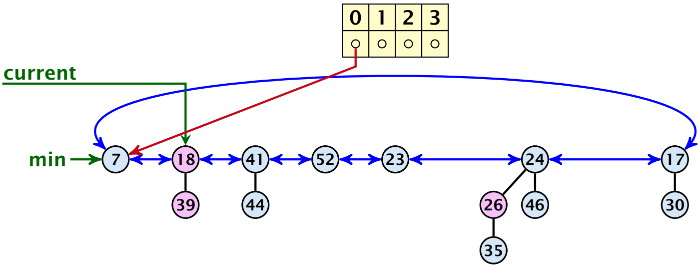

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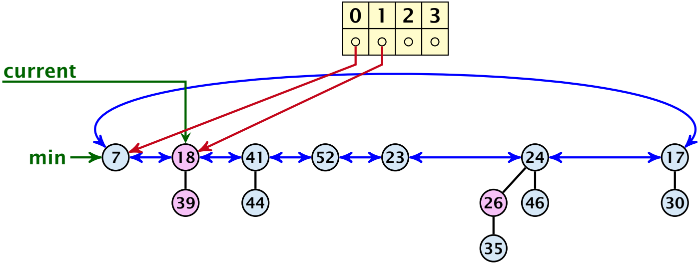

8.3 Fibonacci Heaps

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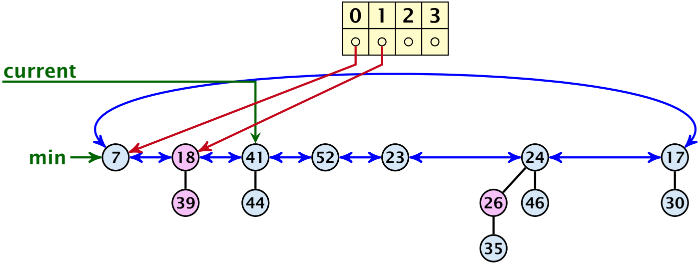

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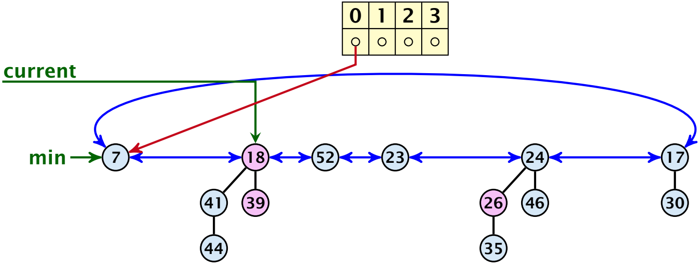

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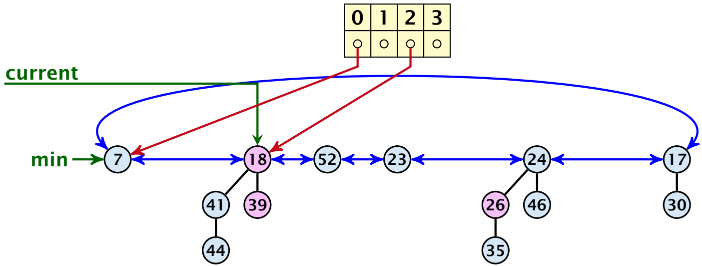

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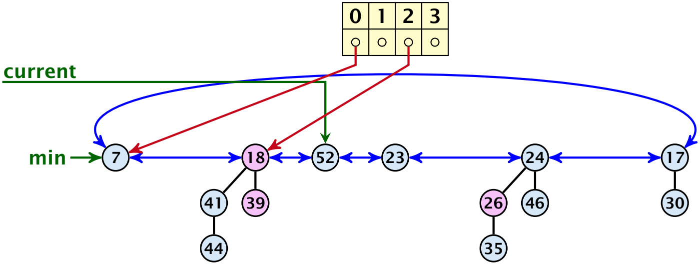

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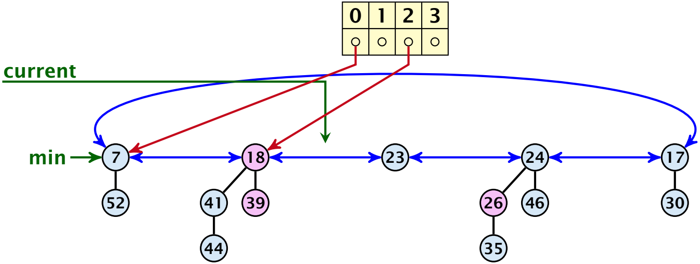

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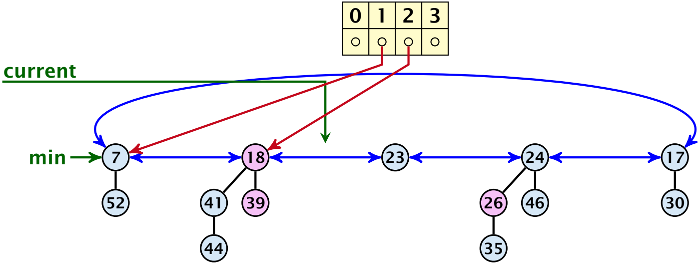
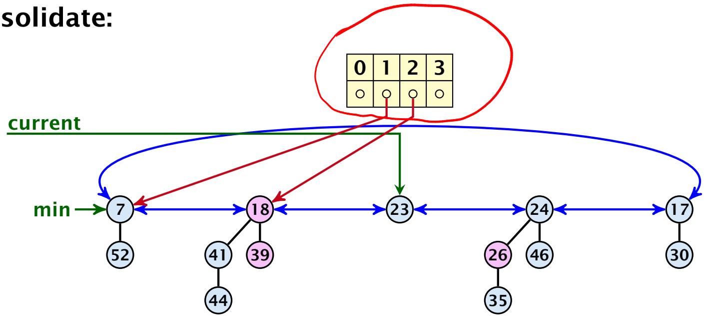


8.3 Fibonacci Heaps

max-degree

Consolidate:

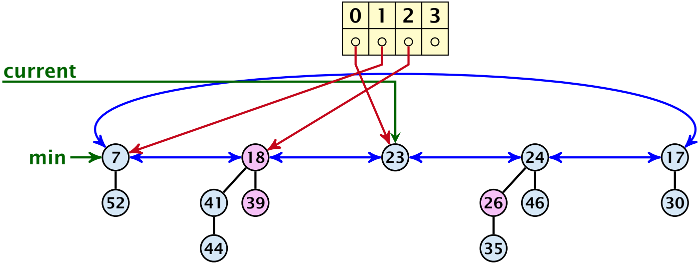
t



store max-degree

8.3 Fibonacci Heaps

Consolidate:



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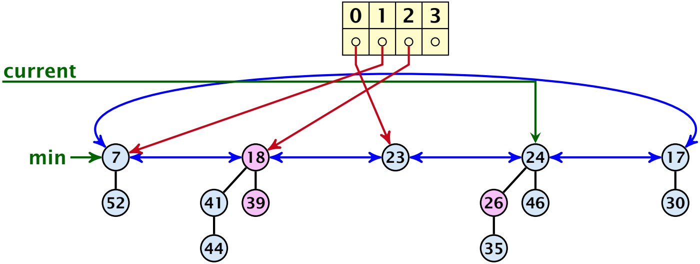

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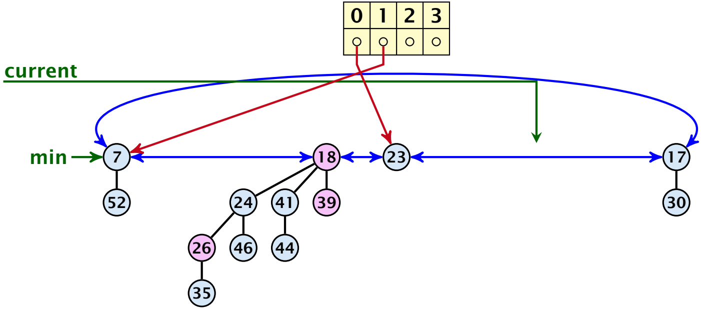

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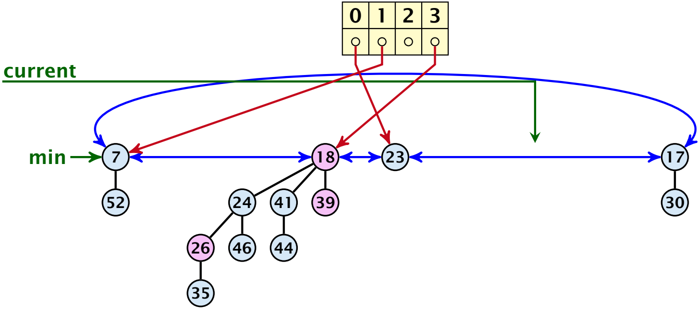

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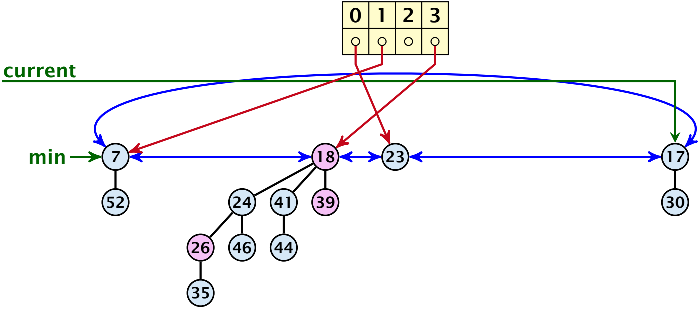

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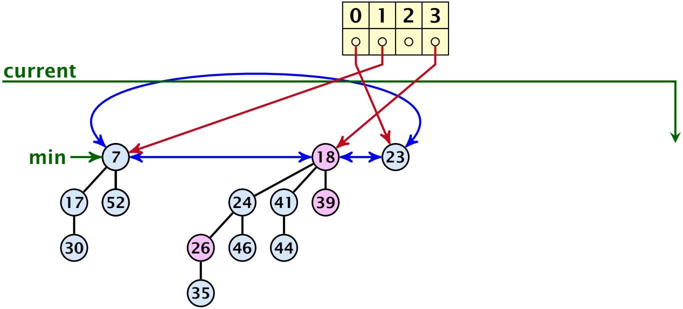

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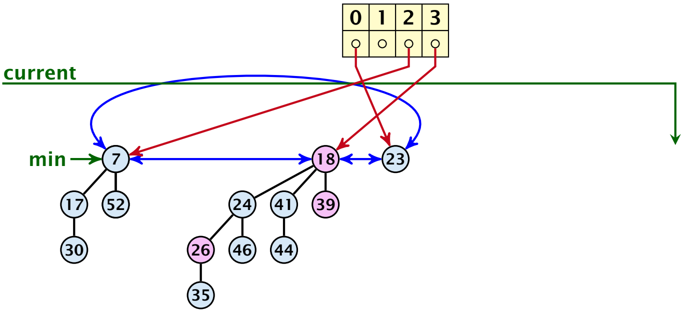

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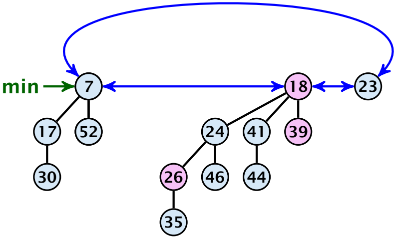

8.3 Fibonacci Heaps

Actual cost for delete-min()

- At most $D_n + t$ elements in root-list before consolidate.

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Amortized cost for delete-min()

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number of elements in list before delete-min

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$$
\begin{aligned}
c_1 \cdot (D_n + t) &- c \cdot (t - D_n - 1) \\
&\leq (c_1 + c)D_n + (c_1 - c)t + c
\end{aligned}
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& \qquad \le (c_1 + c)D_n + (c_1 - c)t + c \le 2c(D_n + 1) \le \mathcal{O}(D_n)
\end{aligned}
$$

for $c \ge c_1$.

8.3 Fibonacci Heaps

If the input trees of the consolidation procedure are binomial trees (for example only singleton vertices) then the output will be a set of distinct binomial trees, and, hence, the Fibonacci heap will be (more or less) a Binomial heap right after the consolidation.

If we do not have delete or decrease-key operations then $D_n \leq \log n$.

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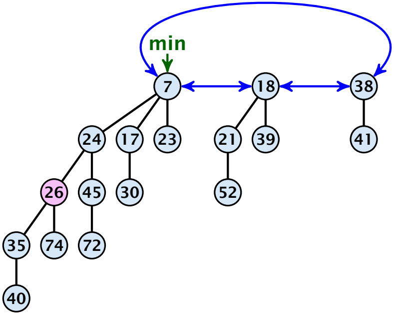
Fibonacci Heaps: decrease-key(handle h, v)



Case 1: decrease-key does not violate heap-property

- Just decrease the key-value of element referenced by h. Nothing else to do.

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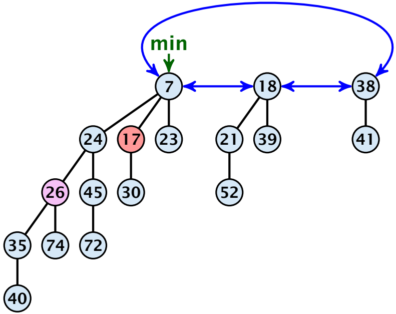

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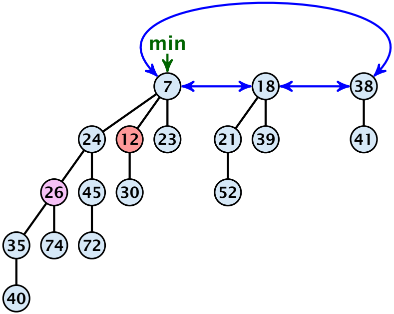

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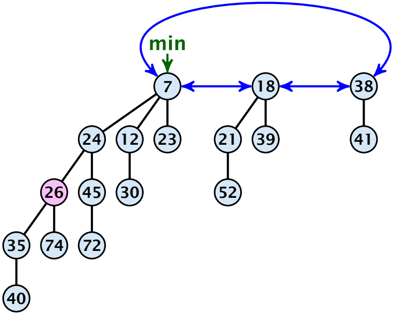

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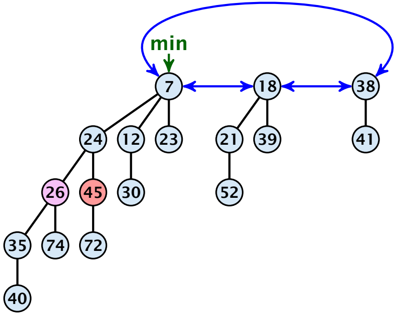

Case 2: heap-property is violated, but parent is not marked

- Decrease key-value of element x reference by h.
- If the heap-property is violated, cut the parent edge of x, and make x into a root.
- Adjust min-pointers, if necessary.
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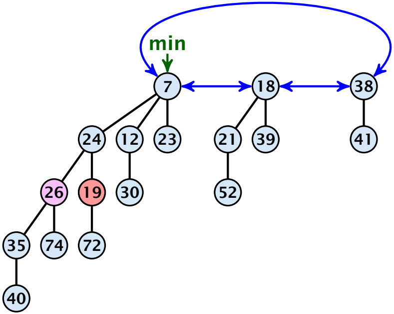

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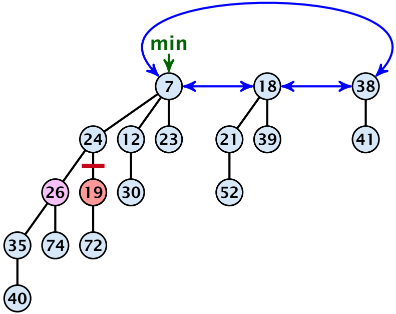

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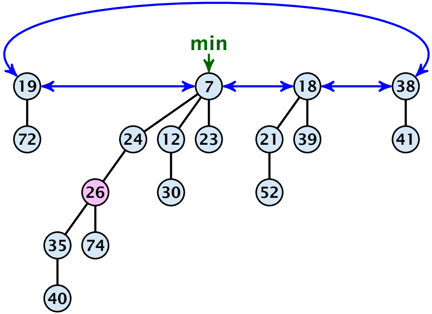

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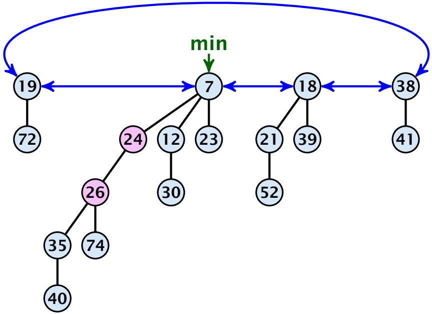

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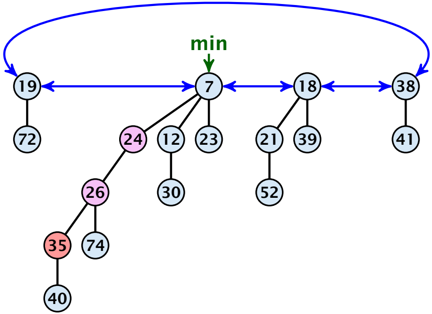

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- Decrease key-value of element x reference by h.
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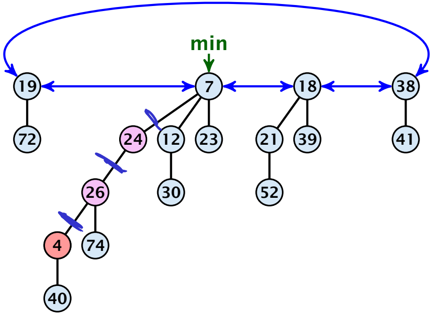
Fibonacci Heaps: decrease-key(handle h, v)



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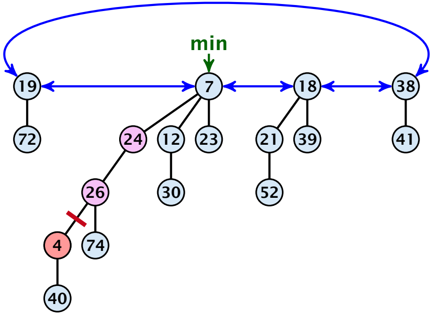

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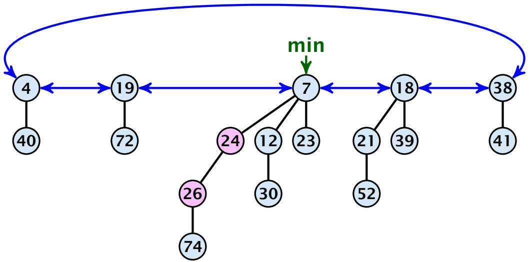

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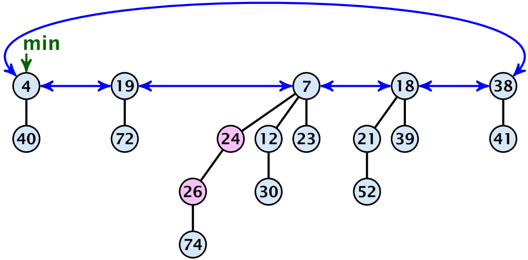

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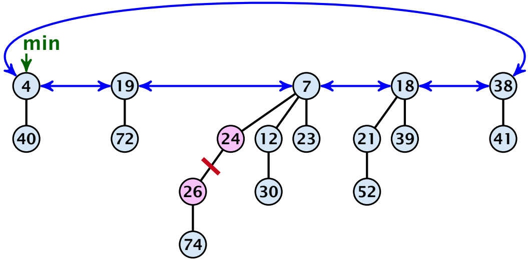
Fibonacci Heaps: decrease-key(handle h, v)



Case 3: heap-property is violated, and parent is marked

- Decrease key-value of element x reference by h.
- Cut the parent edge of x, and make x into a root.
- Adjust min-pointers, if necessary.
- Continue cutting the parent until you arrive at an unmarked node.

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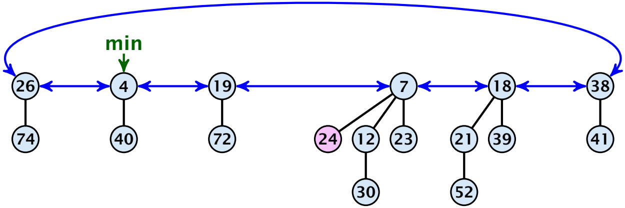

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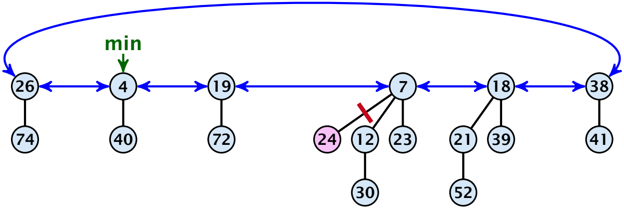

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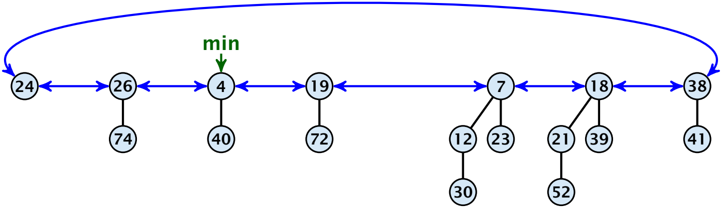

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- Cut the parent edge of x, and make x into a root.
- Adjust min-pointers, if necessary.
- Execute the following:

  $p \leftarrow \mathrm{parent}[x]$;
  while (p is marked)
      $pp \leftarrow \mathrm{parent}[p]$;
      cut of p; make it into a root; unmark it;
      $p \leftarrow pp$;
  if p is unmarked and not a root mark it;

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

- Constant cost for decreasing the value.
- Constant cost for each of ℓ cuts.
- Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2.

Amortized cost:

- $t' = t + \ell$, as every cut creates one new root.
- $m' \leq m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.
- $\Delta\Phi \leq \ell + 2(-\ell + 2) = 4 - \ell$
- Amortized cost is at most

if $c \geq c_2$.

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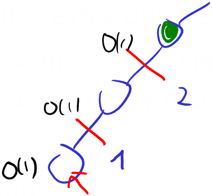

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  $c_2(\ell + 1) + c(4 - \ell) \le (c_2 - c)\ell + 4c + c_2 = \mathcal{O}(1)$,

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Delete node

$H.\mathbf{delete}(x)$:

- decrease value of x to $-\infty$.
- delete-min.

Amortized cost: $\mathcal{O}(D_n)$

- $\mathcal{O}(1)$ for decrease-key.
- $\mathcal{O}(D_n)$ for delete-min.

Lemma 32

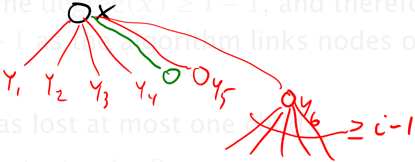
Let x be a node with degree k and let $y_1, \dots, y_k$ denote the children of x in the order that they were linked to x. Then

$$\mathrm{degree}(y_i) \geq \begin{cases} 0 & \textit{if } i = 1 \\ i - 2 & \textit{if } i > 1 \end{cases}$$

8.3 Fibonacci Heaps

Proof

- When y_i was linked to x, at least $y_1, \ldots, y_{i-1}$ were already linked to x.
- Hence, at this time $\text{degree}(x) \geq i - 1$, and therefore also $\text{degree}(y_i) \geq i - 1$ as the algorithm links nodes of equal degree only.



- Since, then y_i has lost at most one child.
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8.3 Fibonacci Heaps

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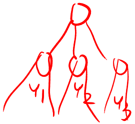
- s_k monotonically increases with k
- $s_0 = 1$ and $s_1 = 2$.

Let x be a degree k node of size s_k and let $y_1, \ldots, y_k$ be its children.

$$s_k = 2 + \sum_{i=2}^{k} \mathrm{size}(y_i)$$



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$$\begin{aligned} s_k &= 2 + \sum_{i=2}^{k} \mathrm{size}(y_i) \\ &\geq 2 + \sum_{i=2}^{k} s_{i-2} \end{aligned}$$

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$$\begin{aligned} s_k &= 2 + \sum_{i=2}^{k} \mathrm{size}(y_i) \\ &\geq 2 + \sum_{i=2}^{k} s_{i-2} \\ &= 2 + \sum_{i=0}^{k-2} s_i \end{aligned}$$

8.3 Fibonacci Heaps

Definition 33

Consider the following non-standard Fibonacci type sequence:

$$F_k = \begin{cases} 1 & \text{if } k = 0 \\ 2 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

Facts:

1. $F_k \geq \phi^k$.
2. For $k \geq 2$: $F_k = 2 + \sum_{i=0}^{k-2} F_i$.

The above facts can be easily proved by induction. From this it follows that $s_k \geq F_k \geq \phi^k$, which gives that the maximum degree in a Fibonacci heap is logarithmic.

$\phi = 1{,}61\ldots$

$1 + \phi = \phi^2$

k=0: $1 = F_0 \geq \Phi^0 = 1$ ✓

k=1: $2 = F_1 \geq \Phi^1 \approx 1.61$ ✓

k-2,k-1 → k: $F_k = F_{k-1} + F_{k-2} \geq \Phi^{k-1} + \Phi^{k-2} = \Phi^{k-2}\overbrace{(\Phi + 1)}^{\Phi^2} = \Phi^k$

k=2: $3 = F_2 = 2 + 1 = 2 + F_0$ ✓

k-1 → k: $F_k = F_{k-1} + F_{k-2} = 2 + \sum_{i=0}^{k-3} F_i + F_{k-2} = 2 + \sum_{i=0}^{k-2} F_i$