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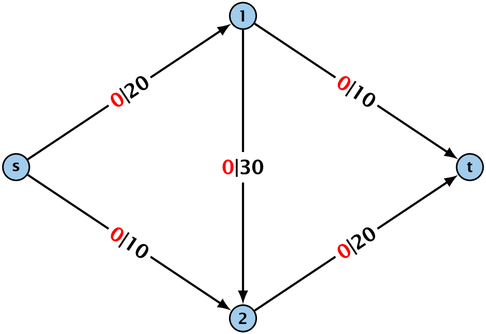

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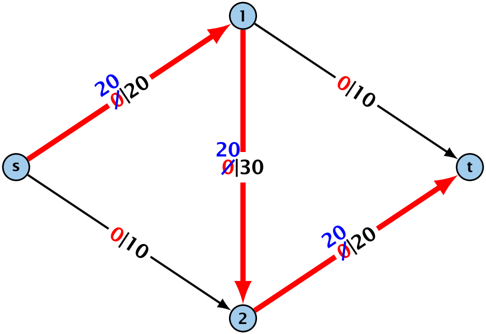

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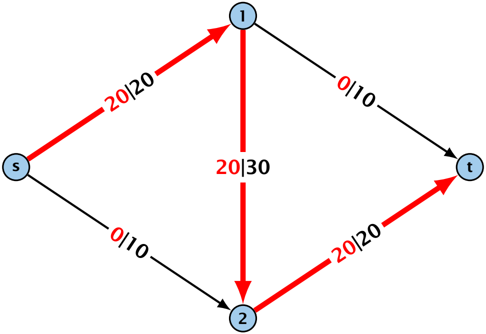

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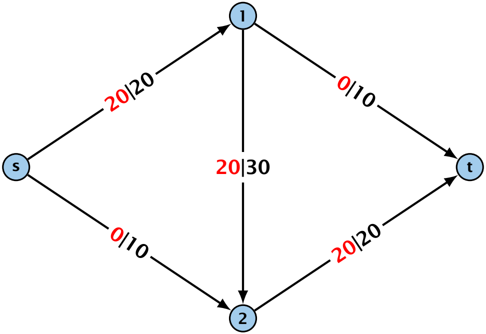

# The Residual Graph

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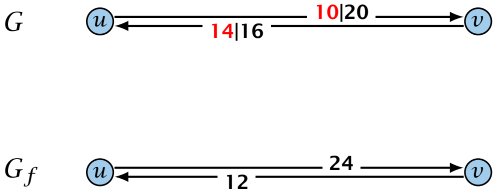

# Augmenting Path Algorithm

**Definition 1**

An augmenting path with respect to flow $f$, is a path from $s$ to $t$ in the auxiliary graph $G_f$ that contains only edges with non-zero capacity.

**Algorithm 1** FordFulkerson($G = (V, E, c)$)

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Initialize f(e) ← 0 for all edges.
while ∃ augmenting path p in G_f do
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1: Initialize $f(e) \leftarrow 0$ for all edges.
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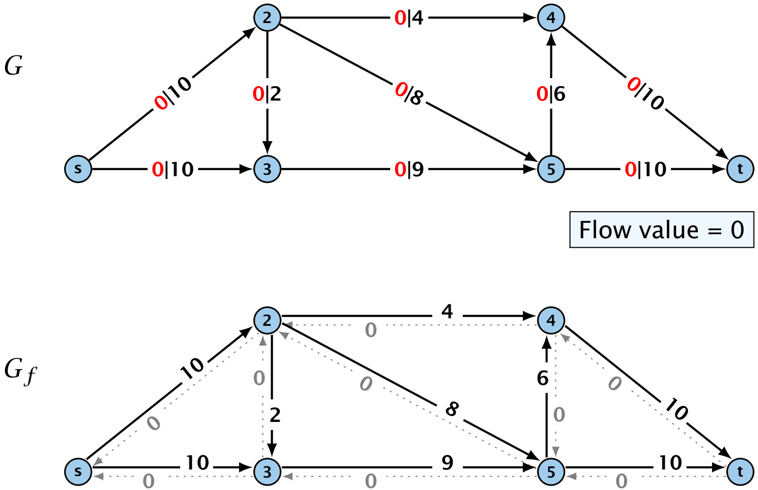

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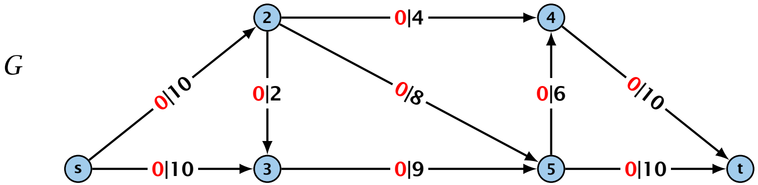
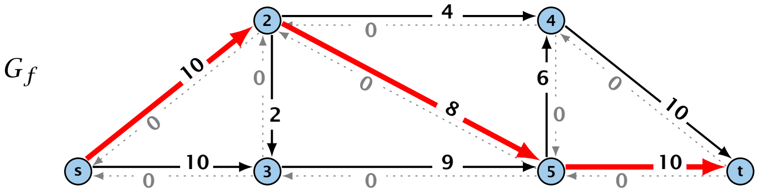



Flow value = 0



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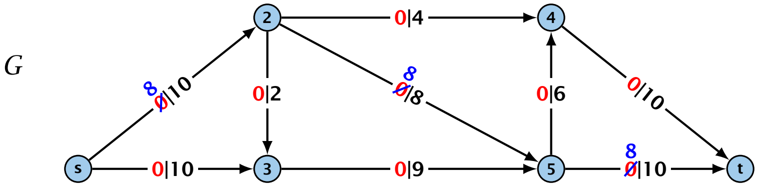
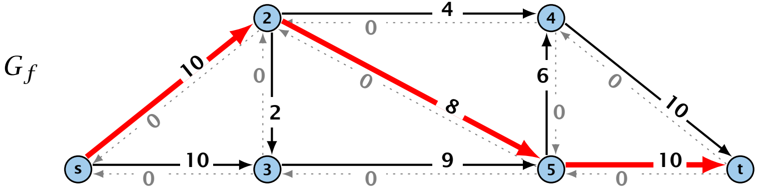



Flow value = 8



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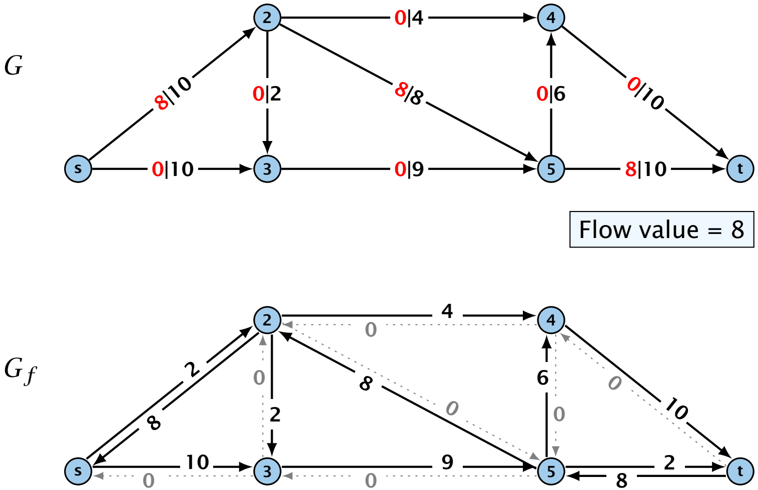

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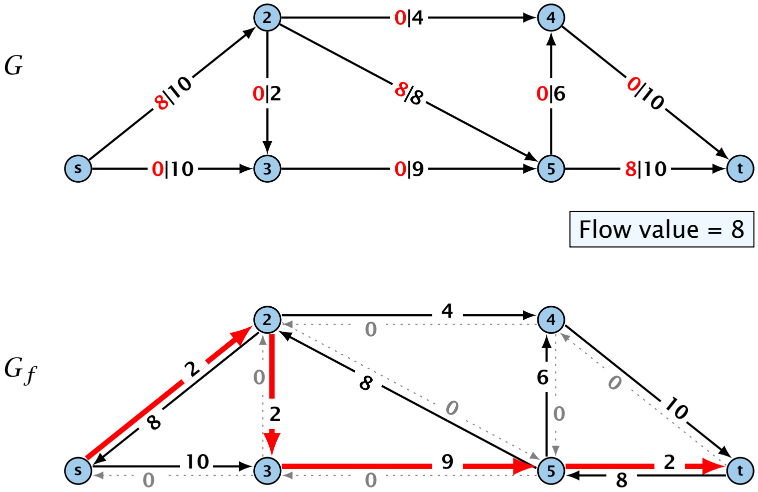

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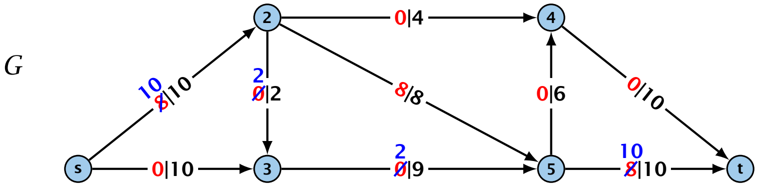
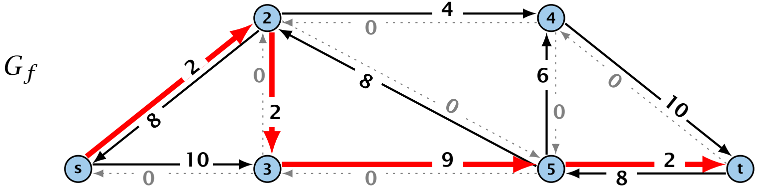



Flow value = 10



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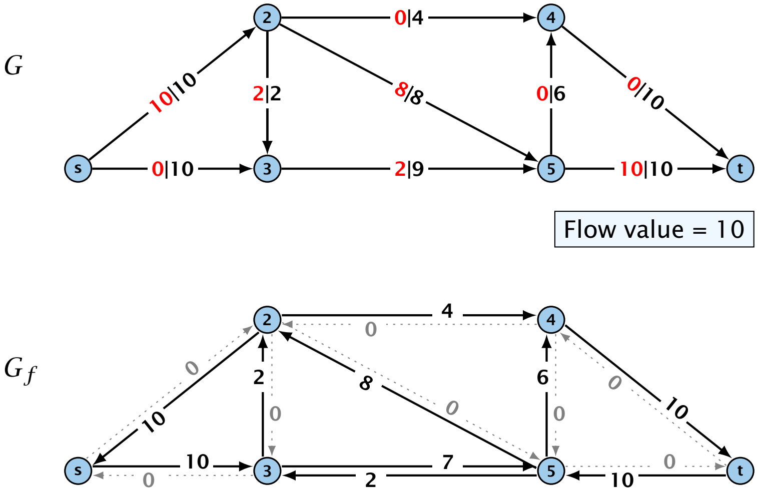

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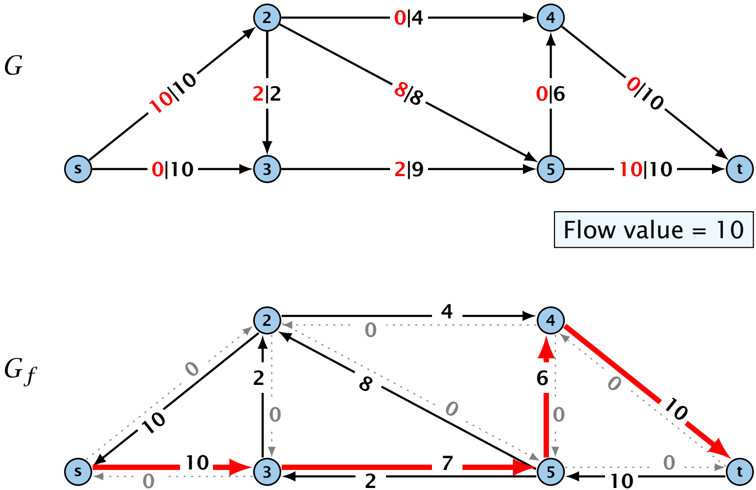

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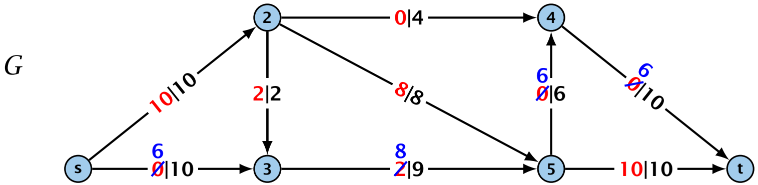
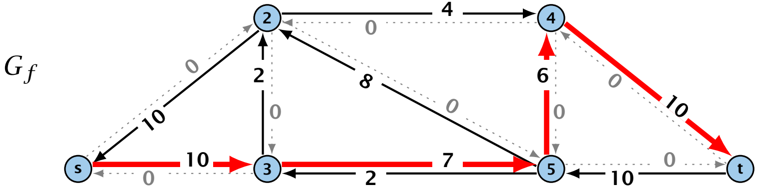



Flow value = 16



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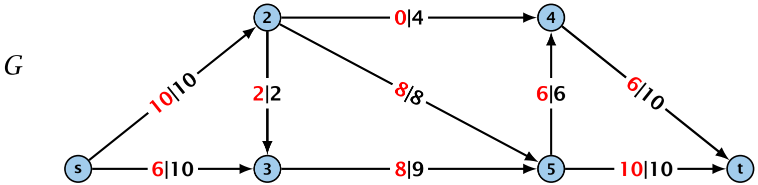


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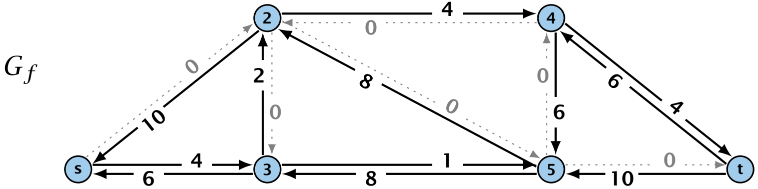

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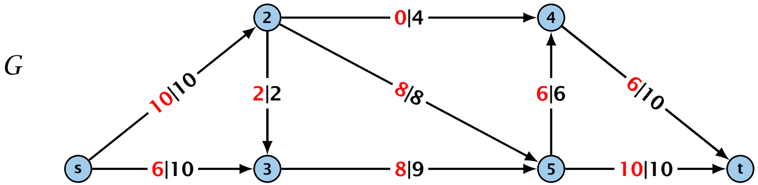


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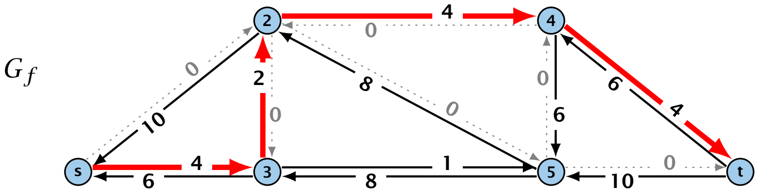

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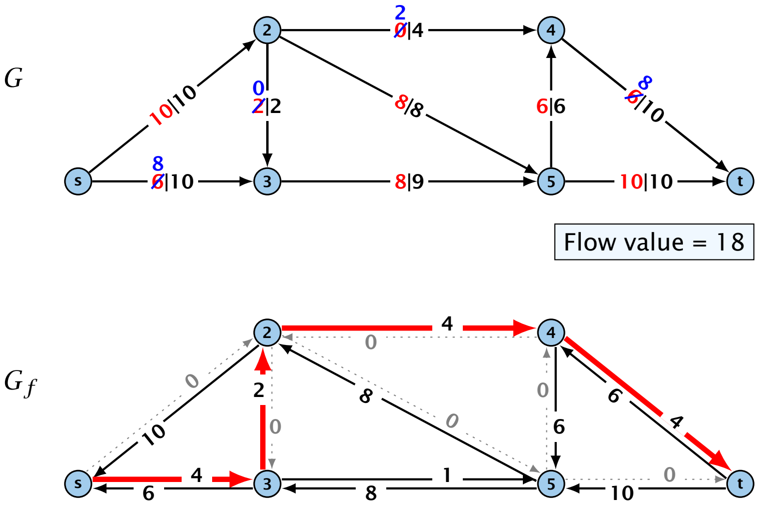

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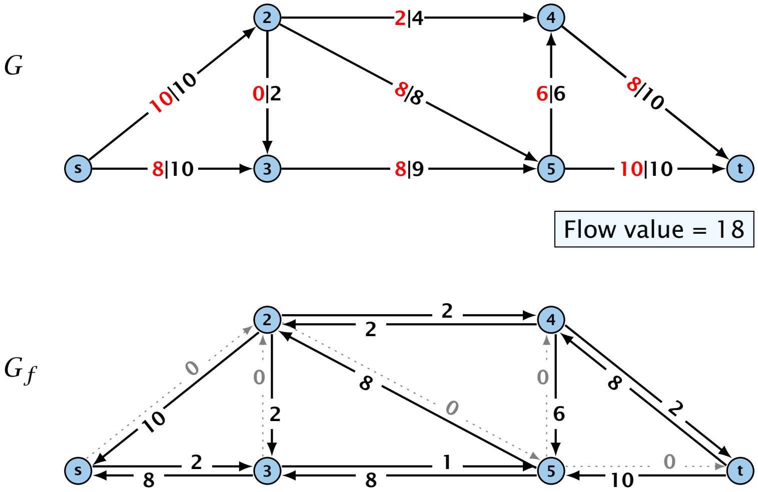

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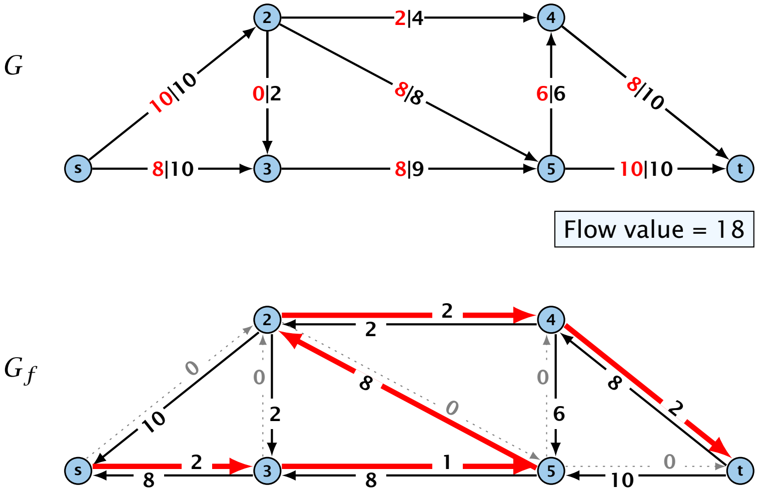

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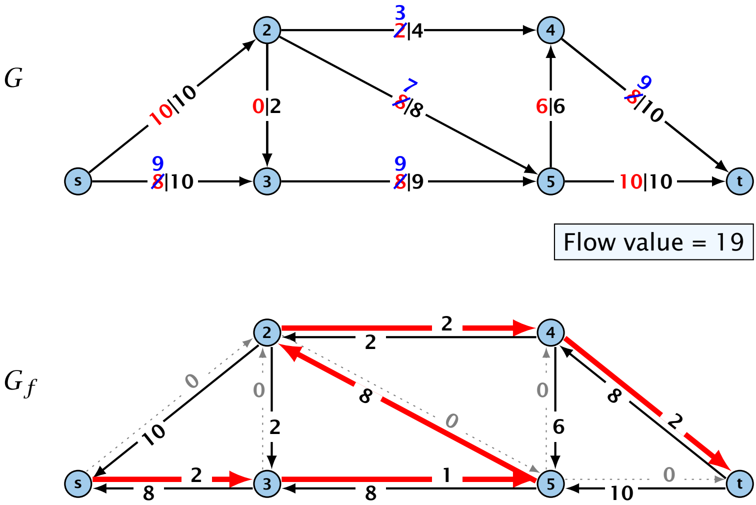

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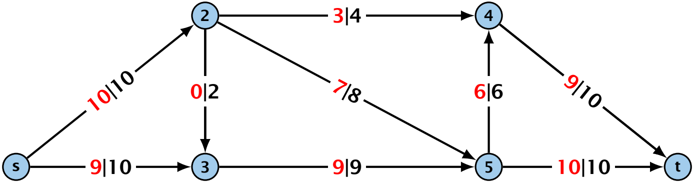
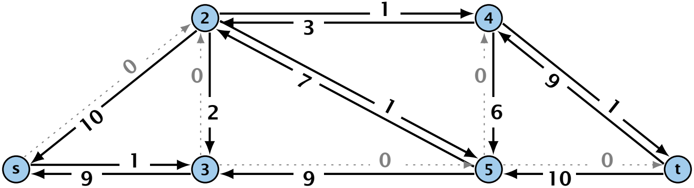
$G$




Flow value = 19

$G_f$



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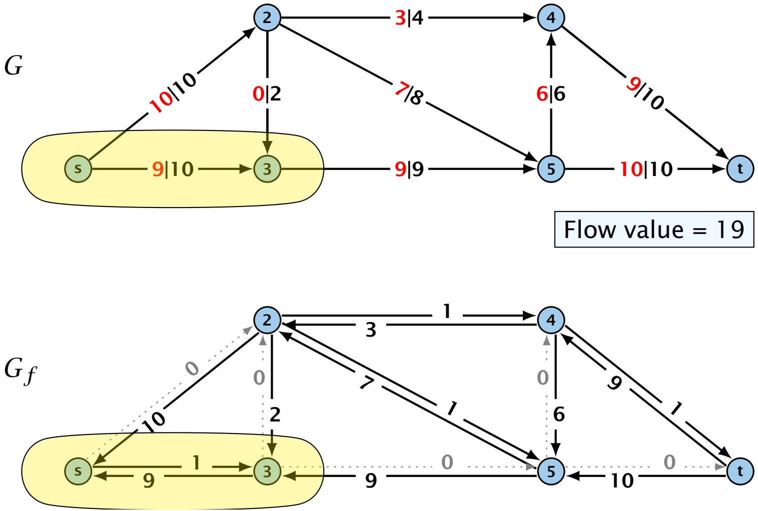

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**Theorem 2**
*A flow $f$ is a maximum flow **iff** there are no augmenting paths.*

**Theorem 3**
*The value of a maximum flow is equal to the value of a minimum cut.*

**Proof.**
Let $f$ be a flow. The following are equivalent:

1. There exists a cut $A$ such that $\mathrm{val}(f) = \mathrm{cap}(A, V \setminus A)$.
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This we already showed.

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This finishes the proof.

Here the first equality uses the flow value lemma, and the second exploits the fact that the flow along incoming edges must be $0$ as the residual graph does not have edges leaving $A$.

# Analysis

Assumption:
All capacities are integers between $1$ and $C$.

Invariant:
Every flow value $f(e)$ and every residual capacity $c_f(e)$ remains integral troughout the algorithm.

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*The algorithm terminates in at most* $\mathrm{val}(f^*) \le nC$ *iterations, where* $f^*$ *denotes the maximum flow. Each iteration can be implemented in time* $\mathcal{O}(m)$*. This gives a total running time of* $\mathcal{O}(nmC)$*.*

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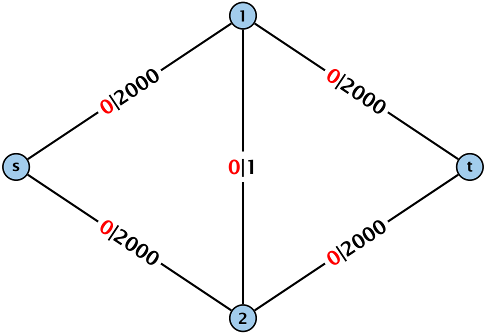
# A Bad Input

Problem: The running time may not be polynomial.



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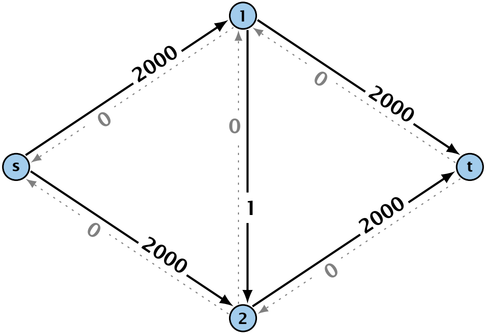


Question:

Can we tweak the algorithm so that the running time is polynomial in the input length?

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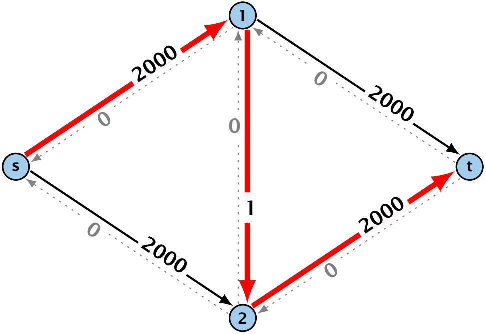


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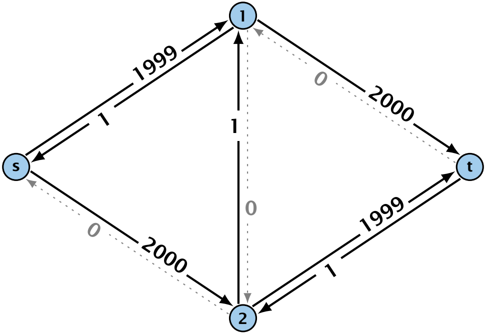


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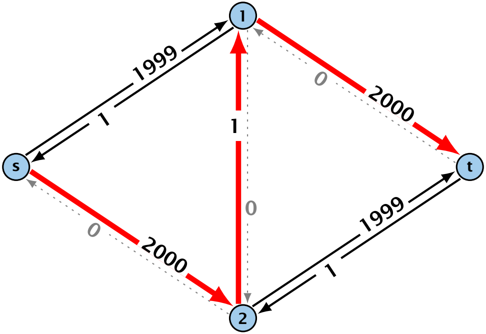


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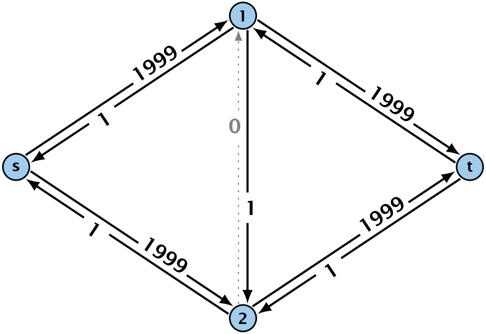


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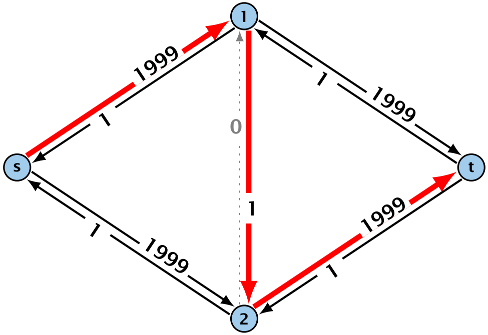


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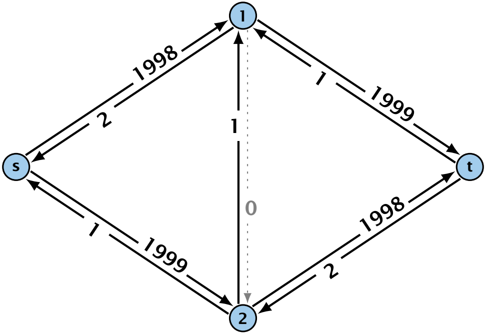


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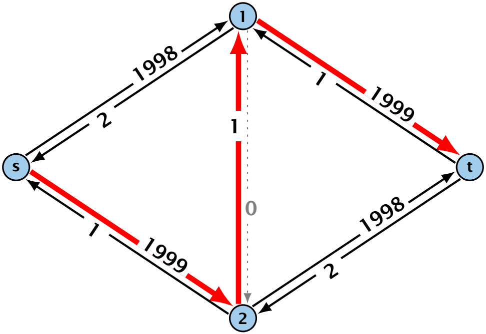


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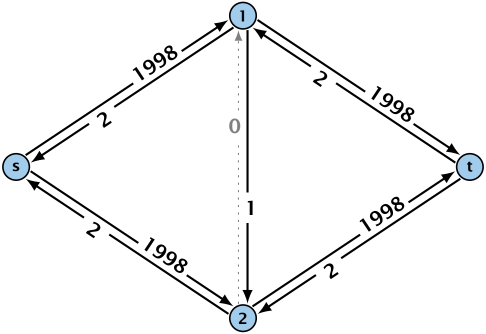


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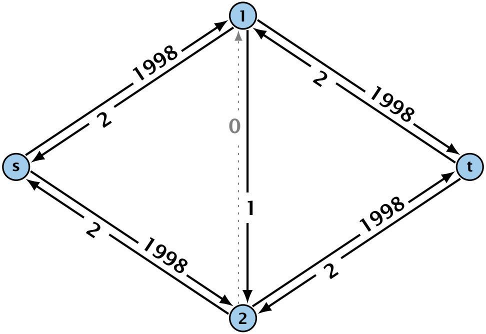
# A Bad Input

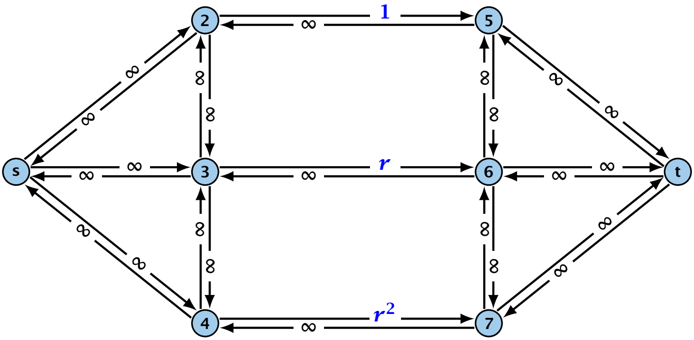
Problem: The running time may not be polynomial.




Question:
Can we tweak the algorithm so that the running time is polynomial in the input length?

# A Pathological Input

Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.



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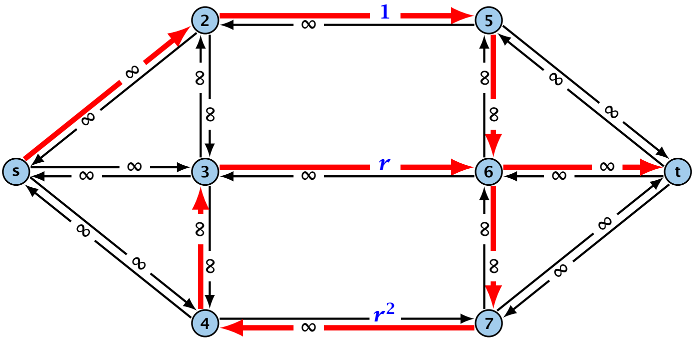

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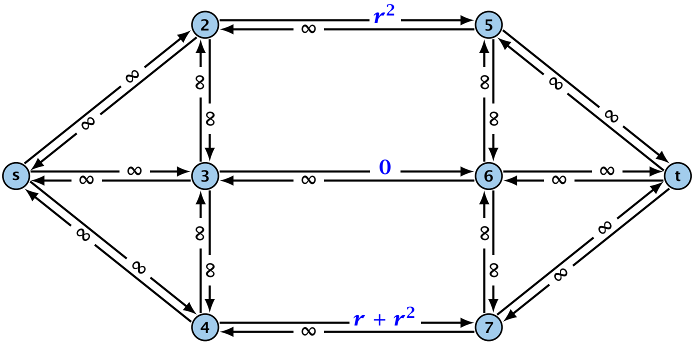

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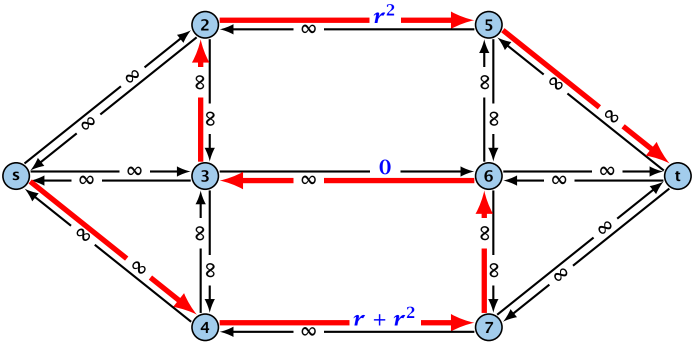

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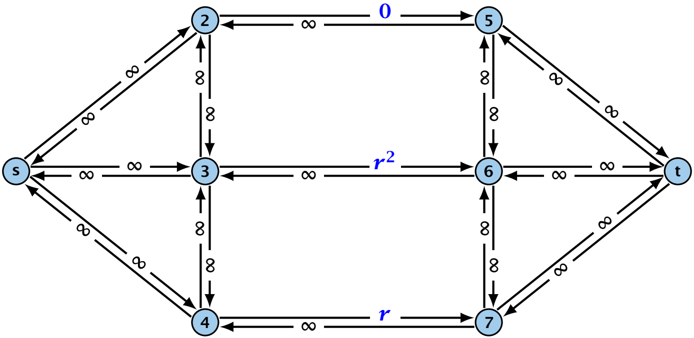

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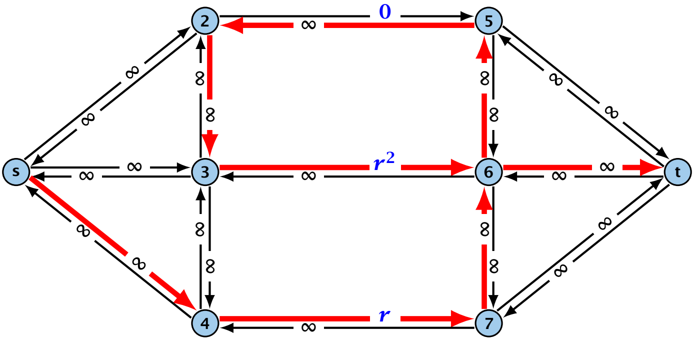

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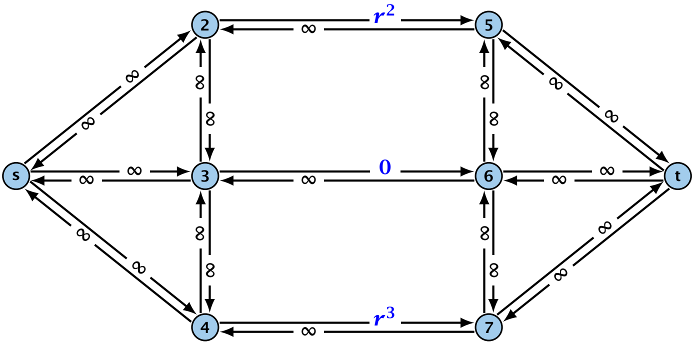

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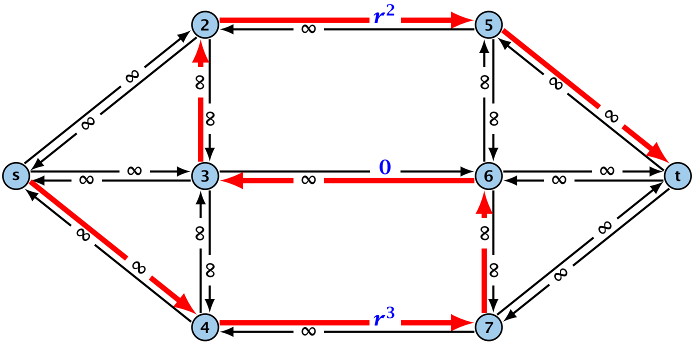

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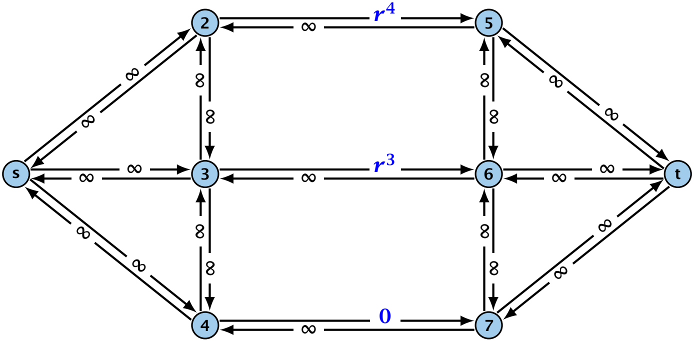

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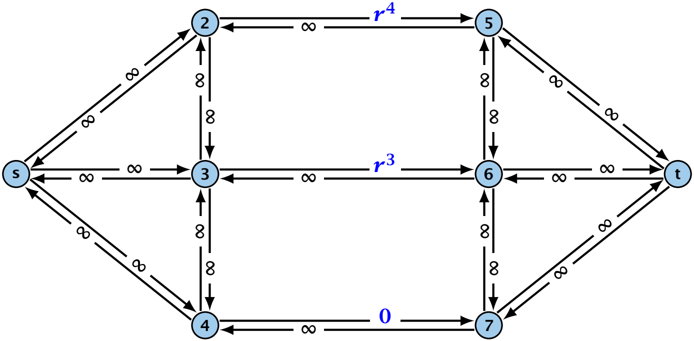


Running time may be infinite!!!

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- We need to find paths efficiently.
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**Several possibilities:**

- Choose path with maximum bottleneck capacity.
- Choose path with sufficiently large bottleneck capacity.
- Choose the shortest augmenting path.