

## 13.3 Highest Label

### Algorithm 6 highest-label( $G, s, t$ )

- 1: initialize preflow
- 2: **foreach**  $u \in V \setminus \{s, t\}$  **do**
- 3:      $u.current\text{-neighbour} \leftarrow u.neighbour\text{-list-head}$
- 4: **while**  $\exists$  active node  $u$  **do**
- 5:     select active node  $u$  with highest label
- 6:     discharge( $u$ )

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### Lemma 1

*When using highest label the number of non-saturating pushes is only  $\mathcal{O}(n^3)$ .*

A push from a node on level  $\ell$  can only “activate” nodes on levels strictly less than  $\ell$ .

This means, after a non-saturating push from  $u$  a relabel is required to make  $u$  active again.

Hence, after  $n$  non-saturating pushes without an intermediate relabel there are no active nodes left.

Therefore, the number of non-saturating pushes is at most  $n(\#relabels + 1) = \mathcal{O}(n^3)$ .

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Since a discharge-operation is terminated by a non-saturating push this gives an upper bound of  $\mathcal{O}(n^3)$  on the number of discharge-operations.

The cost for relabels and saturating pushes can be estimated in exactly the same way as in the case of the generic push-relabel algorithm.

### **Question:**

How do we find the next node for a discharge operation?

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Maintain lists  $L_i$ ,  $i \in \{0, \dots, 2n\}$ , where list  $L_i$  contains active nodes with label  $i$  (maintaining these lists induces only constant additional cost for every push-operation and for every relabel-operation).

After a discharge operation terminated for a node  $u$  with label  $k$ , traverse the lists  $L_k, L_{k-1}, \dots, L_0$ , (in that order) until you find a non-empty list.

Unless the last (non-saturating) push was to  $s$  or  $t$  the list  $k - 1$  must be non-empty (i.e., the search takes constant time).

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Hence, the total time required for searching for active nodes is at most

$$\mathcal{O}(n^3) + n(\#non-saturating-pushes-to-s-or-t)$$

### Lemma 2

*The number of non-saturating pushes to  $s$  or  $t$  is at most  $\mathcal{O}(n^2)$ .*

With this lemma we get

### Theorem 3

*The push-relabel algorithm with the rule highest-label takes time  $\mathcal{O}(n^3)$ .*

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### Proof of the Lemma.

- ▶ We only show that the number of pushes to the source is at most  $\mathcal{O}(n^2)$ . A similar argument holds for the target.
- ▶ After a node  $v$  (which must have  $\ell(v) = n + 1$ ) made a non-saturating push to the source there needs to be another node whose label is increased from  $\leq n + 1$  to  $n + 2$  before  $v$  can become active again.
- ▶ This happens for every push that  $v$  makes to the source. Since, every node can pass the threshold  $n + 2$  at most once,  $v$  can make at most  $n$  pushes to the source.
- ▶ As this holds for every node the total number of pushes to the source is at most  $\mathcal{O}(n^2)$ .