

# Duality

How do we get an upper bound to a maximization LP?

$$\begin{aligned} \max \quad & 13a + 23b \\ \text{s.t.} \quad & 5a + 15b \leq 480 \\ & 4a + 4b \leq 160 \\ & 35a + 20b \leq 1190 \\ & a, b \geq 0 \end{aligned}$$

Note that a lower bound is easy to derive. Every choice of  $a, b \geq 0$  gives us a lower bound (e.g.  $a = 12, b = 28$  gives us a lower bound of 800).

If you take a conic combination of the rows (multiply the  $i$ -th row with  $y_i \geq 0$ ) such that  $\sum_i y_i a_{ij} \geq c_j$  then  $\sum_i y_i b_i$  will be an upper bound.

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## Definition 2

Let  $z = \max\{c^T x \mid Ax \leq b, x \geq 0\}$  be a linear program  $P$  (called the primal linear program).

The linear program  $D$  defined by

$$w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$$

is called the **dual problem**.

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## Lemma 3

*The dual of the dual problem is the primal problem.*

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# Weak Duality

Let  $z = \max\{c^T x \mid Ax \leq b, x \geq 0\}$  and  
 $w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$  be a primal dual pair.

$x$  is primal feasible iff  $x \in \{x \mid Ax \leq b, x \geq 0\}$

$y$  is dual feasible, iff  $y \in \{y \mid A^T y \geq c, y \geq 0\}$ .

## Theorem 4 (Weak Duality)

*Let  $\hat{x}$  be primal feasible and let  $\hat{y}$  be dual feasible. Then*

$$c^T \hat{x} \leq z \leq w \leq b^T \hat{y} .$$

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# Weak Duality

$$A^T \hat{y} \geq c \Rightarrow \hat{x}^T A^T \hat{y} \geq \hat{x}^T c \quad (\hat{x} \geq 0)$$

$$A \hat{x} \leq b \Rightarrow y^T A \hat{x} \leq y^T b \quad (y \geq 0)$$

This gives

$$c^T \hat{x} \leq y^T A \hat{x} \leq b^T y .$$

Since, there exists primal feasible  $\hat{x}$  with  $c^T \hat{x} = z$ , and dual feasible  $\hat{y}$  with  $b^T \hat{y} = w$  we get  $z \leq w$ .

If  $P$  is unbounded then  $D$  is infeasible.

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