Flows

Definition 2

An (s,t)-flow in a (complete) directed graph $G = (V, V \times V, c)$ is a function $f : V \times V \mapsto \mathbb{R}_0^+$ that satisfies

1. For each edge (x, y)

$$0 \leq f_{xy} \leq c_{xy} \ .$$

(capacity constraints)

2. For each $v \in V \setminus \{s, t\}$

$$\sum_x f_{vx} = \sum_x f_{xv} \ .$$

(flow conservation constraints)

Flows

Definition 3

The value of an (s,t)-flow f is defined as

$$\mathrm{val}(f) = \sum_x f_{sx} - \sum_x f_{xs} \ .$$

Maximum Flow Problem:

Find an (s,t)-flow with maximum value.

LP-Formulation of Maxflow

$$
\begin{array}{rllll}
\max & \sum_z f_{sz} - \sum_z f_{zs} & & & \\
\text{s.t.} \quad \forall (z, w) \in V \times V & f_{zw} & \leq & c_{zw} & \ell_{zw} \\
\forall w \neq s, t \quad \sum_z f_{zw} - \sum_z f_{wz} & = & 0 & p_w \\
& f_{zw} & \geq & 0 &
\end{array}
$$

$$
\begin{array}{rllll}
\min & \sum_{(xy)} c_{xy} \ell_{xy} & & \\
\text{s.t.} \quad f_{xy} \ (x, y \neq s, t): & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
f_{sy} \ (y \neq s, t): & 1\ell_{sy} \qquad + 1p_y & \geq & 1 \\
f_{xs} \ (x \neq s, t): & 1\ell_{xs} - 1p_x & \geq & -1 \\
f_{ty} \ (y \neq s, t): & 1\ell_{ty} \qquad + 1p_y & \geq & 0 \\
f_{xt} \ (x \neq s, t): & 1\ell_{xt} - 1p_x & \geq & 0 \\
f_{st}: & 1\ell_{st} & \geq & 1 \\
f_{ts}: & 1\ell_{ts} & \geq & -1 \\
& \ell_{xy} & \geq & 0
\end{array}
$$

LP-Formulation of Maxflow

$$
\begin{array}{rllll}
\min & \sum_{(xy)} c_{xy} \ell_{xy} & & \\
\text{s.t.} \quad f_{xy} \ (x, y \neq s, t): & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
f_{sy} \ (y \neq s, t): & 1\ell_{sy} - \quad 1 + 1p_y & \geq & 0 \\
f_{xs} \ (x \neq s, t): & 1\ell_{xs} - 1p_x + \quad 1 & \geq & 0 \\
f_{ty} \ (y \neq s, t): & 1\ell_{ty} - \quad 0 + 1p_y & \geq & 0 \\
f_{xt} \ (x \neq s, t): & 1\ell_{xt} - 1p_x + \quad 0 & \geq & 0 \\
f_{st}: & 1\ell_{st} - \quad 1 + \quad 0 & \geq & 0 \\
f_{ts}: & 1\ell_{ts} - \quad 0 + \quad 1 & \geq & 0 \\
& \ell_{xy} & \geq & 0
\end{array}
$$

LP-Formulation of Maxflow

$$
\begin{array}{llrcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \neq s,t): & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
& f_{sy}\ (y \neq s,t): & 1\ell_{sy} - p_s + 1p_y & \geq & 0 \\
& f_{xs}\ (x \neq s,t): & 1\ell_{xs} - 1p_x + p_s & \geq & 0 \\
& f_{ty}\ (y \neq s,t): & 1\ell_{ty} - p_t + 1p_y & \geq & 0 \\
& f_{xt}\ (x \neq s,t): & 1\ell_{xt} - 1p_x + p_t & \geq & 0 \\
& f_{st}: & 1\ell_{st} - p_s + p_t & \geq & 0 \\
& f_{ts}: & 1\ell_{ts} - p_t + p_s & \geq & 0 \\
& & \ell_{xy} & \geq & 0 \\
\end{array}
$$

with $p_t = 0$ and $p_s = 1$.

$$
\begin{array}{llrcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}: & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
& & \ell_{xy} & \geq & 0 \\
& & p_s & = & 1 \\
& & p_t & = & 0 \\
\end{array}
$$

We can interpret the ℓ_{xy} value as assigning a length to every edge.

The value p_x for a variable, then can be seen as the distance of x to t (where the distance from s to t is required to be 1 since $p_s = 1$).

The constraint $p_x \leq \ell_{xy} + p_y$ then simply follows from triangle inequality ($d(x,t) \leq d(x,y) + d(y,t) \Rightarrow d(x,t) \leq \ell_{xy} + d(y,t)$).

One can show that there is an optimum LP-solution for the dual problem that gives an integral assignment of variables.

This means $p_x = 1$ or $p_x = 0$ for our case. This gives rise to a cut in the graph with vertices having value 1 on one side and the other vertices on the other side. The objective function then evaluates the capacity of this cut.

This shows that the Maxflow/Mincut theorem follows from linear programming duality.