

## 6.2 Master Theorem

### Lemma 1

Let  $a \geq 1$ ,  $b \geq 1$  and  $\epsilon > 0$  denote constants. Consider the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) .$$

#### Case 1.

If  $f(n) = \mathcal{O}(n^{\log_b(a)-\epsilon})$  then  $T(n) = \Theta(n^{\log_b a})$ .

#### Case 2.

If  $f(n) = \Theta(n^{\log_b(a)} \log^k n)$  then  $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$ ,  
 $k \geq 0$ .

#### Case 3.

If  $f(n) = \Omega(n^{\log_b(a)+\epsilon})$  and for sufficiently large  $n$   
 $af\left(\frac{n}{b}\right) \leq cf(n)$  for some constant  $c < 1$  then  $T(n) = \Theta(f(n))$ .

## 6.2 Master Theorem

We prove the Master Theorem for the case that  $n$  is of the form  $b^{\ell}$ , and we assume that the non-recursive case occurs for problem size  $1$  and incurs cost  $1$ .

# The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

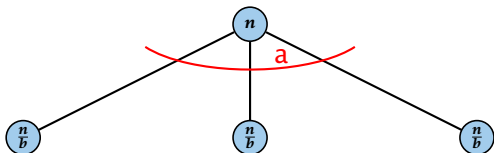
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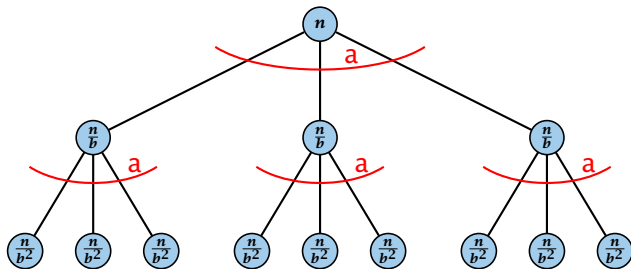
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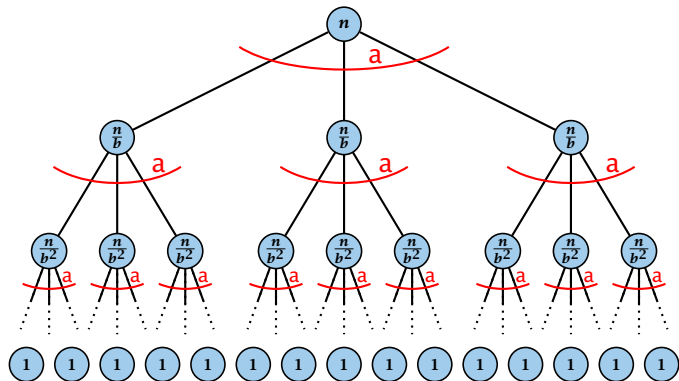
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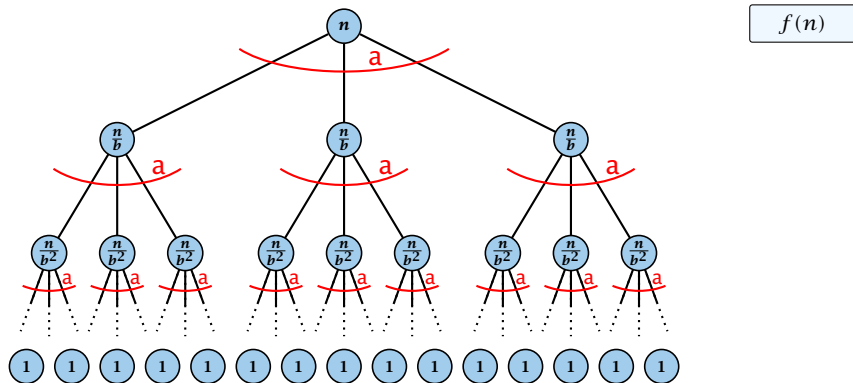
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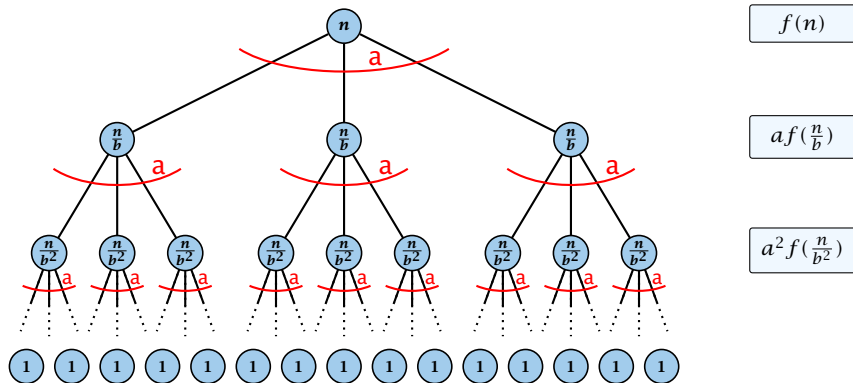






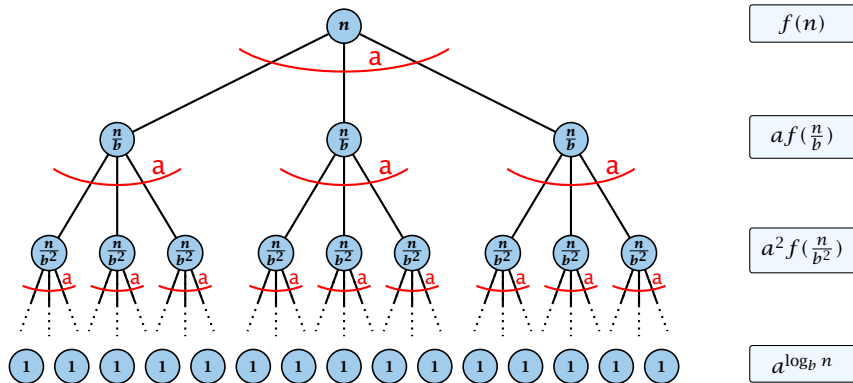
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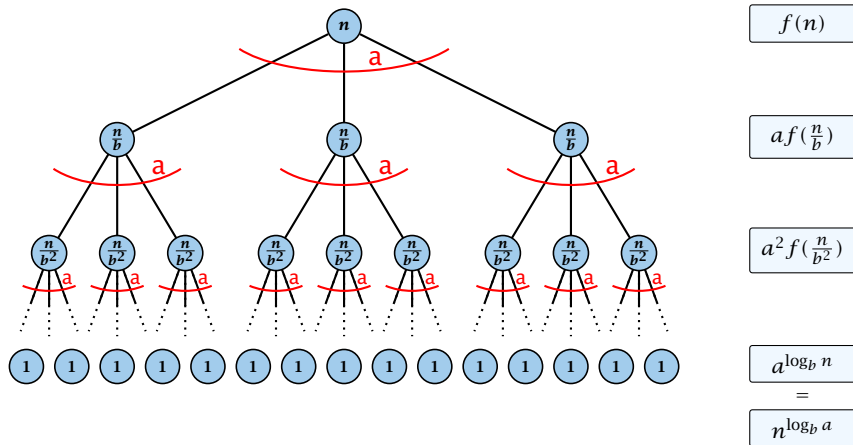
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## 6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) .$$

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Hence,

$$T(n) \leq \left( \frac{c}{b^{\epsilon} - 1} + 1 \right) n^{\log_b(a)}$$



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Hence,

$$T(n) \leq \left( \frac{c}{b^{\epsilon} - 1} + 1 \right) n^{\log_b(a)} \quad \Rightarrow T(n) = \mathcal{O}(n^{\log_b a}).$$

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$n = b^\ell \Rightarrow \ell = \log_b n$	$= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$
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$$= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \approx \frac{1}{k} \ell^{k+1}$$

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$$\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n).$$

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From this we get  $a^i f(n/b^i) \leq c^i f(n)$ , where we assume that  $n/b^{i-1} \geq n_0$  is still sufficiently large.

**Case 3.** Now suppose that  $f(n) \geq dn^{\log_b a + \epsilon}$ , and that for sufficiently large  $n$ :  $af(n/b) \leq cf(n)$ , for  $c < 1$ .

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$



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$$q < 1 : \sum_{i=0}^n q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$$

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Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\ &\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

$$q < 1 : \sum_{i=0}^n q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

$$\Rightarrow T(n) = \Theta(f(n)).$$

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

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$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline \end{array}$$

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<hr/>									



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<hr/>									
								0	

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1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
							1		
								0	

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1	0	0	0	1	0	0	1	1	$B$
							1	1	
							0	0	

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<hr/>									
						1	1		
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1	0	0	0	1	0	0	1	1	$B$
<hr/>						0	0	0	

The diagram illustrates the addition of two 9-bit integers, A and B. The bits of A are 1 1 0 1 1 0 1 0 1 and the bits of B are 1 0 0 0 1 0 0 1 1. A horizontal line is drawn under the bits of B. The result of the addition is shown below the line, with the bits 0 0 0. The carry bits are indicated by small '1's below the bits of B. The bit '1' under the 7th bit of B is highlighted with a light blue box.

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Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
<hr/>									
						0	0	0	

The diagram illustrates the addition of two 9-bit integers, A and B. A vertical box highlights the carry propagation from the 6th bit to the 7th bit. The carry bits are shown as small '1's below the digits.

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1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
					0	1	1	1	
					1	0	0	0	

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1	0	0	0	1	0	0	1	1	$B$
					0	1	1	1	
					1	0	0	0	



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1	1	0	1	1	0	1	0	1	$A$
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				1	0	1	1	1	
-----				0	1	0	0	0	

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For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & 0 & 0 & 1 & 0 & 0 & 0 & \end{array}$$

The diagram illustrates the addition of two 9-bit integers, A and B, using a carry propagation method. The numbers are aligned by their least significant bits. A horizontal line is drawn under the second row. The result of the addition is shown in the third row. The carry propagation is indicated by small '1's below the digits in the second row, and the carry bit '0' is shown in the third row under the fourth column.

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Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
<hr/>									
			0	0	1	0	0	0	

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1	0	0	0	1	0	0	1	1		$B$
		1	0	0	1	0	0	0		

Carry bits: 0, 1, 1, 0, 1, 1, 1

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<hr/>									
		1	0	0	1	0	0	0	

*Note: In the original image, a vertical box highlights the first two bits of A and B (1 and 1), and the carry bit 0 is shown below the first bit of B.*

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<hr/>									
	1	1	0	0	1	0	0	0	

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	1	0	0	0	1	0	0	1	1		$B$
	<hr/>										
	0	1	1	0	0	1	0	0	0		

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	<hr/>									
1	0	1	1	0	0	1	0	0	0	

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For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0 \end{array}$$

This gives that two  $n$ -bit integers can be added in time  $\mathcal{O}(n)$ .

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

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$$\begin{array}{r} 10001 \times 1011 \\ \hline \end{array}$$

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Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ \boxed{1} \\ \hline \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1 \end{array}$$



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$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1 \\ \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1\ 0 \\ \phantom{1\ 0\ 0\ 0\ 1} 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1 \end{array}$$

**Time requirement:**

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1 \end{array}$$

**Time requirement:**

- ▶ Computing intermediate results:  $\mathcal{O}(nm)$ .

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$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1 \\ \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1\ 0 \\ \phantom{1\ 0\ 0\ 0\ 1} 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1 \end{array}$$

**Time requirement:**

- ▶ Computing intermediate results:  $\mathcal{O}(nm)$ .
- ▶ Adding  $m$  numbers of length  $\leq 2n$ :  
 $\mathcal{O}((m+n)m) = \mathcal{O}(nm)$ .

## Example: Multiplying Two Integers

**A recursive approach:**

Suppose that integers  $A$  and  $B$  are of length  $n = 2^k$ , for some  $k$ .

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Suppose that integers  $A$  and  $B$  are of length  $n = 2^k$ , for some  $k$ .

$$\boxed{b_{n-1} \quad \dots \quad b_0} \times \boxed{a_{n-1} \quad \dots \quad a_0}$$

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## Example: Multiplying Two Integers

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Suppose that integers  $A$  and  $B$  are of length  $n = 2^k$ , for some  $k$ .

$$\begin{array}{|c|c|} \hline B_1 & B_0 \\ \hline \end{array} \times \begin{array}{|c|c|} \hline A_1 & A_0 \\ \hline \end{array}$$

## Example: Multiplying Two Integers

**A recursive approach:**

Suppose that integers  $A$  and  $B$  are of length  $n = 2^k$ , for some  $k$ .



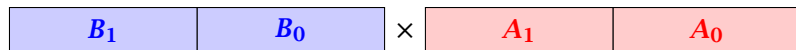
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$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

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Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 B_0$$

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We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) .$$

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**Master Theorem:** Recurrence:  $T[n] = aT(\frac{n}{b}) + f(n)$ .

- ▶ Case 1:  $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$        $T(n) = \Theta(n^{\log_b a})$
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⇒ Not better than the “school method”.

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8: <b>return</b> $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	$\mathcal{O}(n)$

## Example: Multiplying Two Integers

We get the following recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + \mathcal{O}(n) .$$

Master Theorem: Recurrence:  $T[n] = aT\left(\frac{n}{b}\right) + f(n)$ .

- ▶ Case 1:  $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$        $T(n) = \Theta(n^{\log_b a})$
- ▶ Case 2:  $f(n) = \Theta(n^{\log_b a} \log^k n)$        $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
- ▶ Case 3:  $f(n) = \Omega(n^{\log_b a + \epsilon})$        $T(n) = \Theta(f(n))$

Again we are in Case 1. We get a running time of  $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$ .

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