Part II

Foundations

3 Goals

- Gain knowledge about efficient algorithms for important problems, i.e., learn how to solve certain types of problems efficiently.
- Learn how to analyze and judge the efficiency of algorithms.
- Learn how to design efficient algorithms.

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What do you measure?

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- Running time
- Number of comparisons
- Number of multiplications
- Number of hard-disc accesses
- Program size
- Power consumption
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How do you measure?

- Implementing and testing on representative inputs
  - How do you choose your inputs?
  - May be very time-consuming.
  - Very reliable results if done correctly.
  - Results only hold for a specific machine and for a specific set of inputs.

- Theoretical analysis in a specific model of computation.
  - Gives asymptotic bounds like "this algorithm always runs in time $\mathcal{O}(n^2)$".
  - Typically focuses on the worst case.
  - Can give lower bounds like "any comparison-based sorting algorithm needs at least $\Omega(n \log n)$ comparisons in the worst case".

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Input length

The theoretical bounds are usually given by a function $f : \mathbb{N} \to \mathbb{N}$ that maps the input length to the running time (or storage space, comparisons, multiplications, program size etc.).

The input length may e.g. be

- the size of the input (number of bits)
- the number of arguments

Suppose n numbers from the interval $\{1, \ldots, N\}$ have to be sorted. In this case we usually say that the input length is n instead of e.g. $n \log N$, which would be the number of bits required to encode the input.

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Model of Computation

How to measure performance

- Calculate running time and storage space etc. on a simplified, idealized model of computation, e.g. Random Access Machine (RAM), Turing Machine (TM), …
- Calculate number of certain basic operations: comparisons, multiplications, harddisc accesses, …

Version is often easier, but focusing on one type of operation makes it more difficult to obtain meaningful results.

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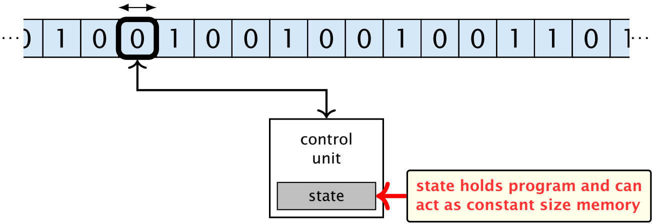
Turing Machine

- Very simple model of computation.
- Only the “current” memory location can be altered.
- Very good model for discussing computabiliy, or polynomial vs. exponential time.
- Some simple problems like recognizing whether input is of the form xx, where x is a string, have quadratic lower bound.

$\Rightarrow$ **Not a good model for developing efficient algorithms.**



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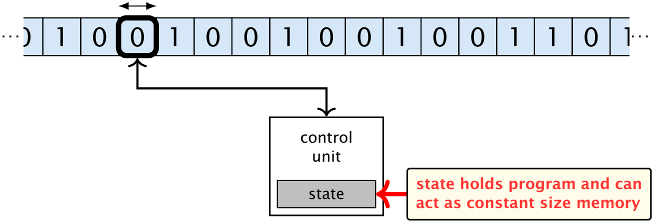

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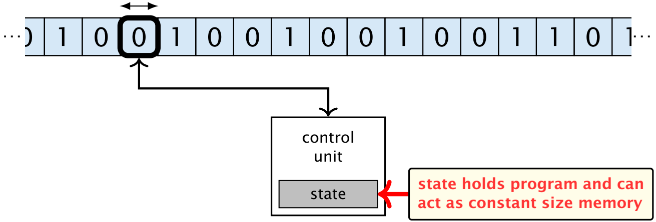

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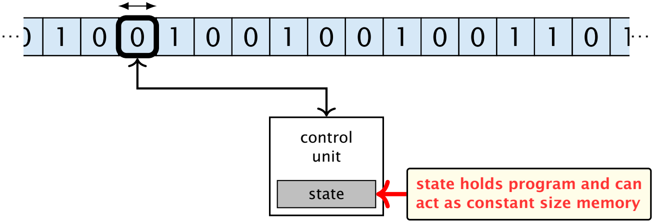

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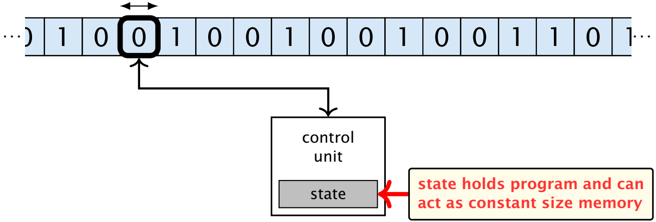

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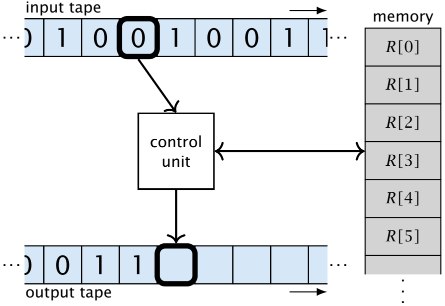
Random Access Machine (RAM)

- Input tape and output tape (sequences of zeros and ones; unbounded length).
- Memory unit: infinite but countable number of registers $R[0], R[1], R[2], \dots$.
- Registers hold integers.
- Indirect addressing.



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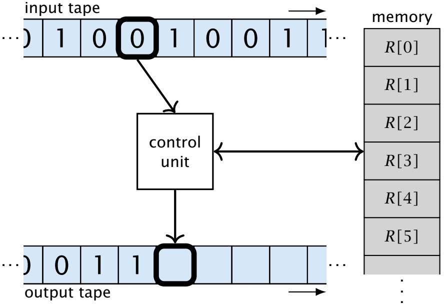

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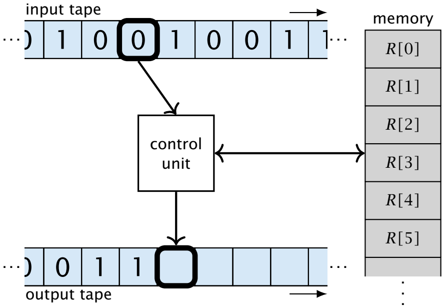

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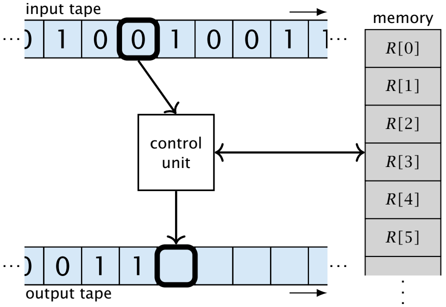

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Random Access Machine (RAM)

Operations

- input operations (input tape → $R[i]$)
  - READ i
- output operations ($R[i]$ → output tape)
  - WRITE i
- register-register transfers
  - $R[j] := R[i]$
  - $R[j] := 4$
- **indirect** addressing
  - $R[j] := R[R[i]]$
    loads the content of the $R[i]$-th register into the j-th register
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Operations

- branching (including loops) based on comparisons
  - jump x
    jumps to position x in the program;
    sets instruction counter to x;
    reads the next operation to perform from register $R[x]$
  - jumpz x $R[i]$
    jump to x if $R[i] = 0$
    if not the instruction counter is increased by 1;
  - jumpi i
    jump to $R[i]$ (indirect jump);
- arithmetic instructions: $+$, $-$, $\times$, $/$
  - $R[i] := R[j] + R[k];$
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The jump-directives are very close to the jump-instructions contained in the assembler language of real machines.

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Model of Computation

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  The cost depends on the content of memory cells:
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Bounded word RAM model: cost is uniform but the largest value stored in a register may not exceed 2^w, where usually $w = \log_2 n$.

The latter model is quite realistic as the word-size of a standard computer that handles a problem of size n must be at least $\log_2 n$ as otherwise the computer could either not store the problem instance or not address all its memory.

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4 Modelling Issues

Example 2

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Algorithm 1 RepeatedSquaring($n$)
$r \leftarrow 2$;
for $i = 1 \rightarrow n$ do
    $r \leftarrow r^2$
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running time:

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- best-case complexity:

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Formal Definition

Let f denote functions from $\mathbb{N}$ to $\mathbb{R}^+$.

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There is an equivalent definition using limes notation (assuming that the respective limes exists). f and g are functions from $\mathbb{N}_0$ to $\mathbb{R}_0^+$.

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- Note that for the version of the Landau notation defined here, we assume that f and g are positive functions.
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Abuse of notation

1. People write $f = \mathcal{O}(g)$, when they mean $f \in \mathcal{O}(g)$. This is **not** an equality (how could a function be equal to a set of functions).
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How do we interpret an expression like:

$$2n^2 + 3n + 1 = 2n^2 + \Theta(n)$$

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How do we interpret an expression like:

$$\sum_{i=1}^{n} \Theta(i) = \Theta(n^2)$$

The $\Theta(i)$-symbol on the left represents one anonymous function $f : \mathbb{N} \rightarrow \mathbb{R}^+$, and then $\sum_i f(i)$ is computed.

Careful!

“It is understood” that every occurence of an $\mathcal{O}$-symbol (or $\Theta, \Omega, o, \omega$) on the left represents one anonymous function.

Hence, the left side is **not** equal to

$$\Theta(1) + \Theta(2) + \cdots + \Theta(n-1) + \Theta(n)$$

$\Theta(1)+\Theta(2)+\cdots+\Theta(n-1)+\Theta(n)$ does not really have a reasonable interpretation.

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Asymptotic Notation in Equations

We can view an expression containing asymptotic notation as generating a set:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n)$$

represents

$$\Big\{ f : \mathbb{N} \to \mathbb{R}^+ \mid f(n) = n^2 \cdot g(n) + h(n) \text{ with } g(n) \in \mathcal{O}(n) \text{ and } h(n) \in \mathcal{O}(\log n) \Big\}$$

Recall that according to the previous slide e.g. the expressions $\sum_{i=1}^{n} \mathcal{O}(i)$ and $\sum_{i=1}^{n/2} \mathcal{O}(i) + \sum_{i=n/2+1}^{n} \mathcal{O}(i)$ generate different sets.

Asymptotic Notation in Equations

Then an asymptotic equation can be interpreted as containement btw. two sets:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) = \Theta(n^2)$$

represents

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) \subseteq \Theta(n^2)$$

Note that the equation does not hold.

Asymptotic Notation

Lemma 3

Let f, g be functions with the property $\exists n_0 > 0\, \forall n \geq n_0 : f(n) > 0$ (the same for g). Then

- $c \cdot f(n) \in \Theta(f(n))$ *for any constant* c
- $\mathcal{O}(f(n)) + \mathcal{O}(g(n)) = \mathcal{O}(f(n) + g(n))$
- $\mathcal{O}(f(n)) \cdot \mathcal{O}(g(n)) = \mathcal{O}(f(n) \cdot g(n))$
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The expressions also hold for Ω. Note that this means that $f(n) + g(n) \in \Theta(\max\{f(n), g(n)\})$.

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Comments

- Do not use asymptotic notation within induction proofs.
- For any constants a, b we have $\log_a n = \Theta(\log_b n)$. Therefore, we will usually ignore the base of a logarithm within asymptotic notation.
- In general $\log n = \log_2 n$, i.e., we use 2 as the default base for the logarithm.

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Asymptotic Notation

In general asymptotic classification of running times is a good measure for comparing algorithms:

- If the running time analysis is tight and actually occurs in practise (i.e., the asymptotic bound is not a purely theoretical worst-case bound), then the algorithm that has better asymptotic running time will always outperform a weaker algorithm for large enough values of n.
- However, suppose that I have two algorithms:
  - Algorithm A. Running time $f(n) = 1000\log n = \mathcal{O}(\log n)$.
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  Clearly $f = o(g)$. However, as long as $\log n \le 1000$ Algorithm B will be more efficient.

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6 Recurrences

Algorithm 2 mergesort(list L)

```
$n \leftarrow \text{size}(L)$
if $n \le 1$ return $L$
$L_1 \leftarrow L[1 \cdots \lfloor \frac{n}{2} \rfloor]$
$L_2 \leftarrow L[\lfloor \frac{n}{2} \rfloor + 1 \cdots n]$
$\text{mergesort}(L_1)$
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$L \leftarrow \text{merge}(L_1, L_2)$
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```

This algorithm requires

$$T(n) = T\left(\left\lceil \frac{n}{2} \right\rceil\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \mathcal{O}(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + \mathcal{O}(n)$$

comparisons when $n > 1$ and 0 comparisons when $n \leq 1$.

Recurrences

How do we bring the expression for the number of comparisons ($\approx$ running time) into a closed form?

For this we need to solve the recurrence.

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Methods for Solving Recurrences

1. **Guessing+Induction**
   Guess the right solution and prove that it is correct via induction. It needs experience to make the right guess.
2. **Master Theorem**
   For a lot of recurrences that appear in the analysis of algorithms this theorem can be used to obtain tight asymptotic bounds. It does not provide exact solutions.
3. **Characteristic Polynomial**
   Linear homogenous recurrences can be solved via this method.

Methods for Solving Recurrences

4. **Generating Functions**
   A more general technique that allows to solve certain types of linear inhomogenous relations and also sometimes non-linear recurrence relations.
5. **Transformation of the Recurrence**
   Sometimes one can transform the given recurrence relations so that it e.g. becomes linear and can therefore be solved with one of the other techniques.

6.1 Guessing+Induction

First we need to get rid of the $\mathcal{O}$-notation in our recurrence:

$$T(n) \le \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \ge 2 \\ 0 & \text{otherwise} \end{cases}$$

Informal way:

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Informal way:

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One way of solving such a recurrence is to guess a solution, and check that it is correct by plugging it in.

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if we choose $d \geq c$.

Formally, this is not correct if n is not a power of 2. Also even in this case one would need to do an induction proof.

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How do we get a result for all values of n?

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We consider the following recurrence instead of the original one:

$$T(n) \leq \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 16 \\ b & \text{otherwise} \end{cases}$$

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$$T(n) \leq \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 16 \\ b & \text{otherwise} \end{cases}$$

Note that we can do this as for constant-sized inputs the running time is always some constant (b in the above case).

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We also make a guess of $T(n) \le dn\log n$ and get

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\boxed{\log n \leq \tfrac{n}{4}} \quad &\leq dn\log n + (\log 9 - 3.5)dn + cn
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\boxed{\log \tfrac{9}{16}n = \log n + (\log 9 - 4)} \quad & \quad = dn\log n + (\log 9 - 4)dn + 2d\log n + cn \\
\boxed{\log n \le \tfrac{n}{4}} \quad & \quad \le dn\log n + (\log 9 - 3.5)dn + cn \\
 & \quad \le dn\log n - 0.33dn + cn \\
 & \quad \le dn\log n
\end{aligned}$$

for a suitable choice of d.

6.2 Master Theorem

Note that the cases do not cover all possibilities.

Lemma 4

Let $a \geq 1, b \geq 1$ *and* $\epsilon > 0$ *denote constants. Consider the recurrence*

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) \ .$$

Case 1.
If $f(n) = \mathcal{O}(n^{\log_b(a)-\epsilon})$ *then* $T(n) = \Theta(n^{\log_b a})$.

Case 2.
If $f(n) = \Theta(n^{\log_b(a)} \log^k n)$ *then* $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$, $k \geq 0$.

Case 3.
If $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ *and for sufficiently large* n $af(\frac{n}{b}) \leq cf(n)$ *for some constant* $c < 1$ *then* $T(n) = \Theta(f(n))$.

6.2 Master Theorem

We prove the Master Theorem for the case that n is of the form b^{ℓ}, and we assume that the non-recursive case occurs for problem size 1 and incurs cost 1.

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

The Recursion Tree

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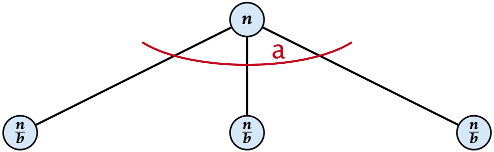

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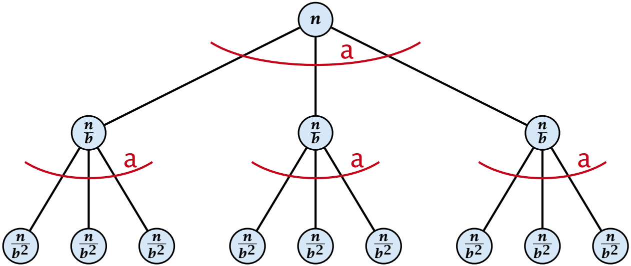
The running time of a recursive algorithm can be visualized by a recursion tree:



The Recursion Tree

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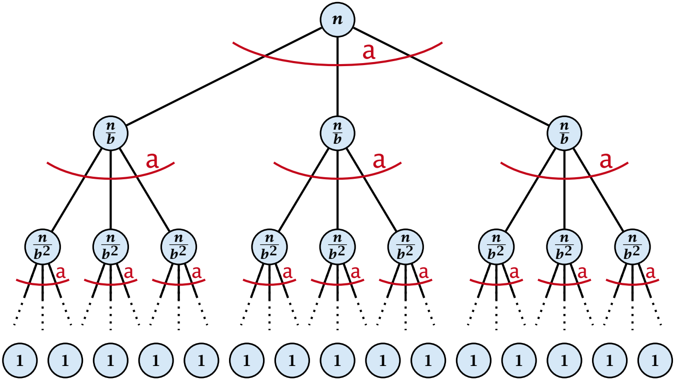

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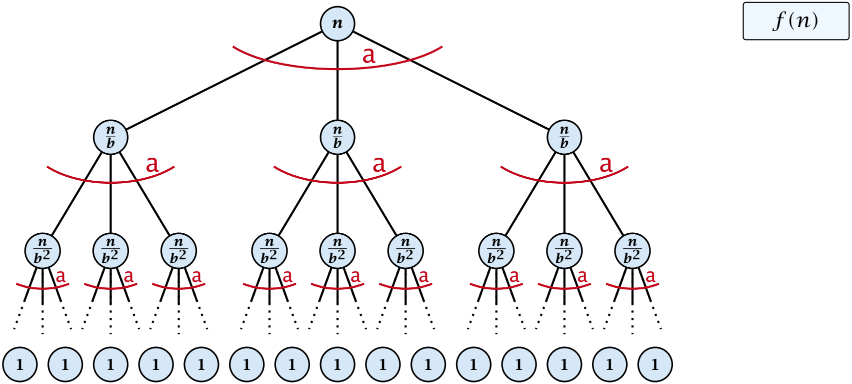

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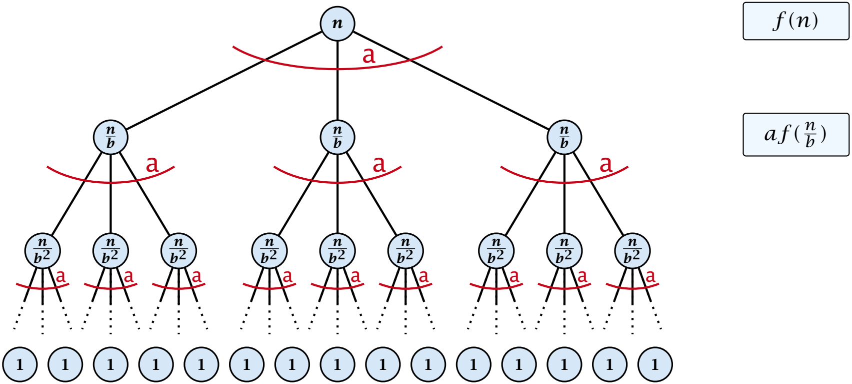
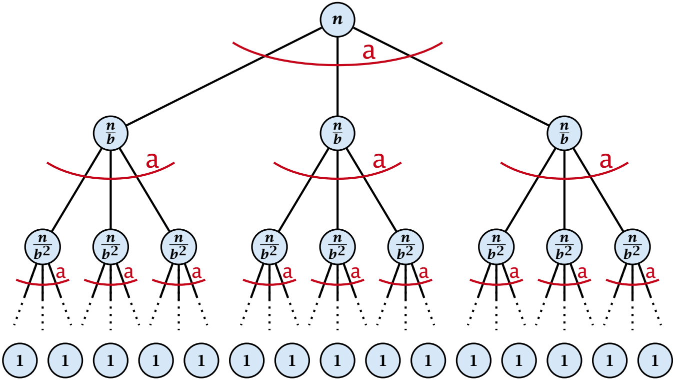


The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



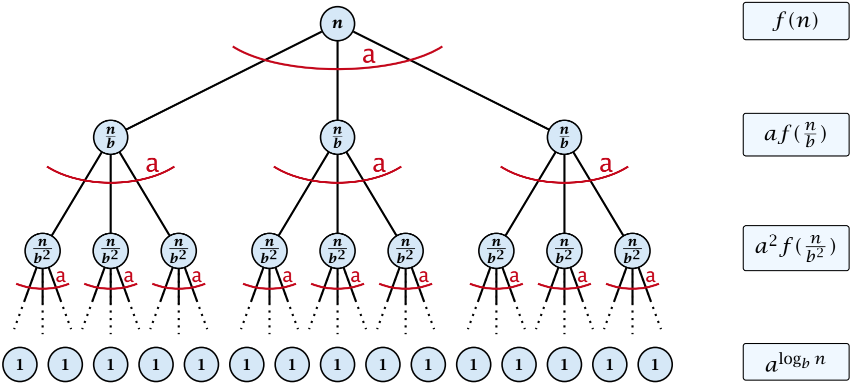
$f(n)$

$af(\frac{n}{b})$

$a^2f(\frac{n}{b^2})$

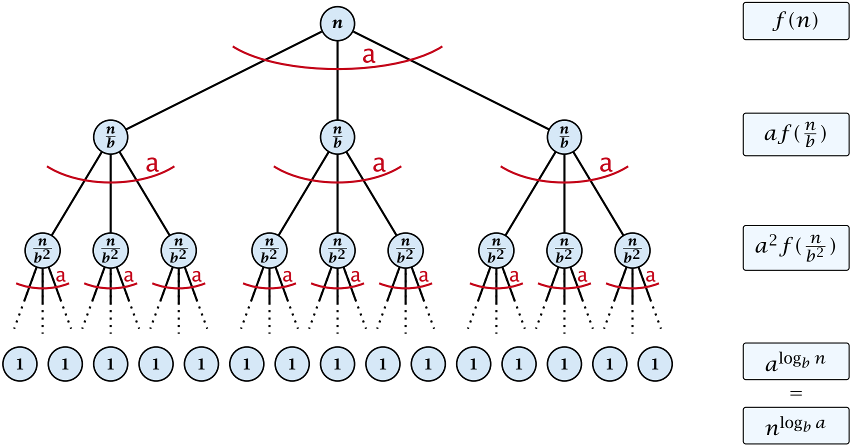
The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \ .$$

Case 1. Now suppose that $f(n) \le cn^{\log_b a - \epsilon}$.

Case 1. Now suppose that $f(n) \leq cn^{\log_b a-\epsilon}$.

$$T(n) - n^{\log_b a}$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a-\epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a-\epsilon}$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a-\epsilon} \end{aligned}$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a-\epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a-\epsilon}$$

$$\boxed{b^{-i(\log_b a-\epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i}a^{-i}}$$

Case 1. Now suppose that $f(n) \le cn^{\log_b a-\epsilon}$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\ &\le c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a-\epsilon} \\ &= cn^{\log_b a-\epsilon} \sum_{i=0}^{\log_b n-1} (b^\epsilon)^i \end{aligned}$$

$$\boxed{b^{-i(\log_b a-\epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i}a^{-i}}$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a-\epsilon}$.

$$\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a-\epsilon} \\
&= cn^{\log_b a-\epsilon} \sum_{i=0}^{\log_b n-1} \left(b^\epsilon\right)^i
\end{aligned}$$

$$\boxed{b^{-i(\log_b a-\epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i}a^{-i}}$$

$$\boxed{\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}}$$

Case 1. Now suppose that $f(n) \le cn^{\log_b a-\epsilon}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)\\
&\le c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a-\epsilon}\\
\boxed{b^{-i(\log_b a-\epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i}a^{-i}} \quad &= cn^{\log_b a-\epsilon} \sum_{i=0}^{\log_b n-1} (b^{\epsilon})^i\\
\boxed{\textstyle\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}} \quad &= cn^{\log_b a-\epsilon}(b^{\epsilon\log_b n} - 1)/(b^{\epsilon} - 1)
\end{aligned}
$$

Case 1. Now suppose that $f(n) \le cn^{\log_b a - \epsilon}$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \\ \boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} \quad &= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^{\epsilon})^i \\ \boxed{\textstyle\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}} \quad &= cn^{\log_b a - \epsilon}(b^{\epsilon \log_b n} - 1)/(b^{\epsilon} - 1) \\ &= cn^{\log_b a - \epsilon}(n^{\epsilon} - 1)/(b^{\epsilon} - 1) \end{aligned}$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a-\epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a-\epsilon}$$

$$\boxed{b^{-i(\log_b a-\epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i}a^{-i}}$$

$$= cn^{\log_b a-\epsilon} \sum_{i=0}^{\log_b n-1} (b^\epsilon)^i$$

$$\boxed{\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}}$$

$$= cn^{\log_b a-\epsilon}(b^{\epsilon \log_b n} - 1)/(b^\epsilon - 1)$$

$$= cn^{\log_b a-\epsilon}(n^\epsilon - 1)/(b^\epsilon - 1)$$

$$= \frac{c}{b^\epsilon - 1} n^{\log_b a}(n^\epsilon - 1)/(n^\epsilon)$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \\
\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} \quad &= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^\epsilon)^i \\
\boxed{\textstyle\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}} \quad &= cn^{\log_b a - \epsilon}(b^{\epsilon \log_b n} - 1)/(b^\epsilon - 1) \\
&= cn^{\log_b a - \epsilon}(n^\epsilon - 1)/(b^\epsilon - 1) \\
&= \frac{c}{b^\epsilon - 1} n^{\log_b a}(n^\epsilon - 1)/(n^\epsilon)
\end{aligned}$$

Hence,

$$T(n) \leq \left(\frac{c}{b^\epsilon - 1} + 1\right) n^{\log_b (a)}$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a-\epsilon}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a-\epsilon} \\
\boxed{b^{-i(\log_b a-\epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i}a^{-i}} \quad &= cn^{\log_b a-\epsilon} \sum_{i=0}^{\log_b n-1} (b^\epsilon)^i \\
\boxed{\textstyle\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}} \quad &= cn^{\log_b a-\epsilon}(b^{\epsilon\log_b n} - 1)/(b^\epsilon - 1) \\
&= cn^{\log_b a-\epsilon}(n^\epsilon - 1)/(b^\epsilon - 1) \\
&= \frac{c}{b^\epsilon - 1} n^{\log_b a}(n^\epsilon - 1)/(n^\epsilon)
\end{aligned}
$$

Hence,

$$
T(n) \leq \left(\frac{c}{b^\epsilon - 1} + 1\right) n^{\log_b (a)} \qquad \boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a}).}
$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}$.

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\ &= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\ &= cn^{\log_b a} \log_b n \end{aligned}$$

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Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n)$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\ &= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\ &= cn^{\log_b a} \log_b n \end{aligned}$$

Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n)$$

$\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log n)$.

Case 2. Now suppose that $f(n) \geq cn^{\log_b a}$.

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$$T(n) - n^{\log_b a}$$

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Case 2. Now suppose that $f(n) \geq cn^{\log_b a}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\geq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}
\end{aligned}
$$

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Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n)$$

Case 2. Now suppose that $f(n) \geq cn^{\log_b a}$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\geq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\ &= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\ &= cn^{\log_b a} \log_b n \end{aligned}$$

Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n)$$

$$\boxed{\Rightarrow T(n) = \Omega(n^{\log_b a} \log n).}$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}(\log_b(n))^k$.

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$$T(n) - n^{\log_b a}$$

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

Case 2. Now suppose that $f(n) \le c n^{\log_b a} (\log_b(n))^k$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \end{aligned}$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}(\log_b(n))^k$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \end{aligned}$$

$$\boxed{n = b^\ell \Rightarrow \ell = \log_b n}$$

Case 2. Now suppose that $f(n) \le c n^{\log_b a} (\log_b(n))^k$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k$$

$\boxed{n = b^\ell \Rightarrow \ell = \log_b n}$

$$= c n^{\log_b a} \sum_{i=0}^{\ell - 1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$$

Case 2. Now suppose that $f(n) \le cn^{\log_b a}(\log_b(n))^k$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
&\le c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \\
&= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\
&= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k
\end{aligned}
$$

$\boxed{n = b^\ell \Rightarrow \ell = \log_b n}$

Case 2. Now suppose that $f(n) \le c n^{\log_b a} (\log_b(n))^k$.

$$
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\boxed{n = b^\ell \Rightarrow \ell = \log_b n} \qquad &= c n^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\
&= c n^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \\
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\end{aligned}
$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}(\log_b(n))^k$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \\ &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\ &= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \\ &= cn^{\log_b a} \boxed{\sum_{i=1}^{\ell} i^k} \approx \tfrac{1}{k}\ell^{k+1} \end{aligned}$$

$$\boxed{n = b^\ell \Rightarrow \ell = \log_b n}$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}(\log_b(n))^k$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \\
\boxed{n = b^\ell \Rightarrow \ell = \log_b n} \quad &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\
&= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \\
&= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \\
&\approx \frac{c}{k} n^{\log_b a} \ell^{k+1}
\end{aligned}
$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}(\log_b(n))^k$.

$$\boxed{n = b^\ell \Rightarrow \ell = \log_b n}$$

$$\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \\
&= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\
&= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \\
&= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \\
&\approx \frac{c}{k} n^{\log_b a} \ell^{k+1}
\end{aligned}$$

$$\boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n).}$$

Case 3. Now suppose that $f(n) \geq dn^{\log_b a+\epsilon}$, and that for sufficiently large n: $af(n/b) \leq cf(n)$, for $c < 1$.

Where did we use $f(n) \geq \Omega(n^{\log_b a+\epsilon})$?

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$$\leq \sum_{i=0}^{\log_b n-1} c^i f(n) + \mathcal{O}(n^{\log_b a})$$

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$$\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a})$$

Hence,

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$$\boxed{\Rightarrow T(n) = \Theta(f(n)).}$$

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Example: Multiplying Two Integers

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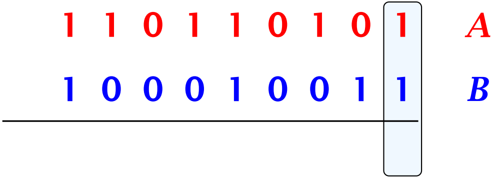

$$\begin{array}{ccccccccc|c} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & \boxed{1} & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & \boxed{1} & B \\ \hline \end{array}$$

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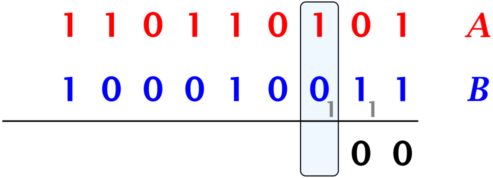

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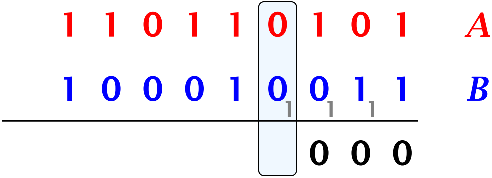

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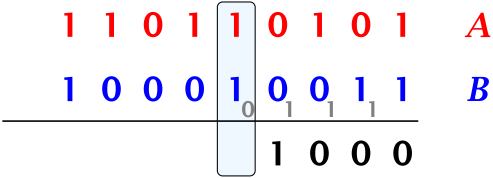

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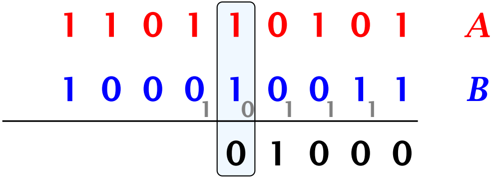

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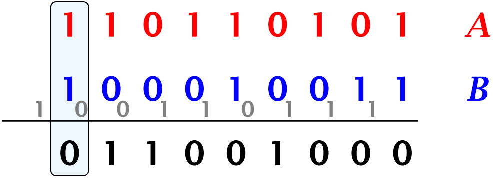

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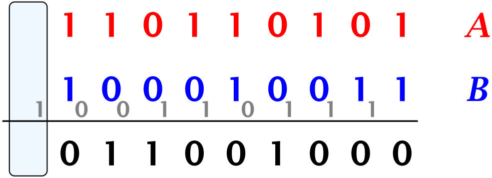

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\hline
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\end{array}$$

This gives that two n-bit integers can be added in time $\mathcal{O}(n)$.

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

- This is also nown as the "school method" for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

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\end{array}$$

- This is also nown as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$\begin{array}{r}
1\;0\;0\;0\;1 \times 1\;0\;1\;1 \\
\hline
1\;0\;0\;0\;1 \\
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- Computing intermediate results: $\mathcal{O}(nm)$.

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- Note that the intermediate numbers that are generated can have at most $m + n \le 2n$ bits.

Time requirement:

- Computing intermediate results: $\mathcal{O}(nm)$.
- Adding m numbers of length $\le 2n$: $\mathcal{O}((m+n)m) = \mathcal{O}(nm)$.

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Suppose that integers A and B are of length $n = 2^k$, for some k.

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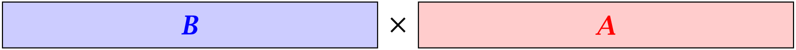

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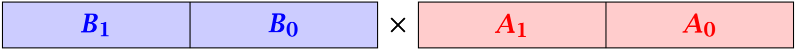

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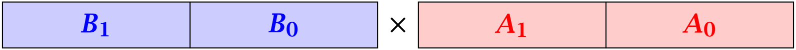

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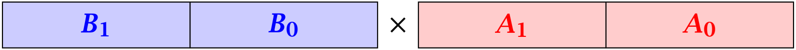

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Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 B_0$$

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if $|A| = |B| = 1$ then
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$\mathcal{O}(1)$

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We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) \ .$$

Example: Multiplying Two Integers

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- Case 1: $f(n) = \mathcal{O}(n^{\log_b a-\epsilon})$ $T(n) = \Theta(n^{\log_b a})$
- Case 2: $f(n) = \Theta(n^{\log_b a}\log^k n)$ $T(n) = \Theta(n^{\log_b a}\log^{k+1} n)$
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In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a-\epsilon})$.

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We get a running time of $\mathcal{O}(n^2)$ for our algorithm.

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$\Longrightarrow$ Not better then the “school method”.

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We can use the following identity to compute Z_1:

A more precise (correct) analysis would say that computing Z_1 needs time $T(\frac{n}{2}+1)+\mathcal{O}(n)$.

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A more precise (correct) analysis would say that computing Z_1 needs time $T(\frac{n}{2} + 1) + \mathcal{O}(n)$.

Example: Multiplying Two Integers

We can use the following identity to compute Z_1:

$$\begin{aligned} Z_1 &= A_1B_0 + A_0B_1 \\ &= (A_0 + A_1) \cdot (B_0 + B_1) - \overbrace{A_1B_1}^{=Z_2} - \overbrace{A_0B_0}^{=Z_0} \end{aligned}$$

Hence,

Algorithm 4 mult(A, B)

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if $|A| = |B| = 1$ then                                  $\mathcal{O}(1)$
        return $a_0 \cdot b_0$                           $\mathcal{O}(1)$
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Example: Multiplying Two Integers

We get the following recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + \mathcal{O}(n) \ .$$

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $\quad T(n) = \Theta(n^{\log_b a})$
- Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $\quad T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
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Again we are in Case 1. We get a running time of $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$.

A huge improvement over the “school method”.

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6.3 The Characteristic Polynomial

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \cdots + c_kT(n-k) = f(n)$$

This is the general form of a linear recurrence relation of order k with constant coefficients ($c_0, c_k \neq 0$).

- $T(n)$ only depends on the k preceding values. This means the recurrence relation is of order k.
- The recurrence is linear as there are no products of $T[n]$'s.
- If $f(n) = 0$ then the recurrence relation becomes a linear, homogeneous recurrence relation of order k.

Note that we ignore boundary conditions for the moment.

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Observations:

- The solution $T[1], T[2], T[3], \ldots$ is completely determined by a set of boundary conditions that specify values for $T[1], \ldots, T[k]$.
- In fact, any k consecutive values completely determine the solution.
- k non-concecutive values might not be an appropriate set of boundary conditions (depends on the problem).

Approach:

- First determine all solutions that satisfy recurrence relation.
- Then pick the right one by analyzing boundary conditions.
- First consider the homogenous case.

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The Homogenous Case

The solution space

$$S = \Big\{ \mathcal{T} = T[1], T[2], T[3], \ldots \ \Big|\ \mathcal{T} \text{ fulfills recurrence relation} \Big\}$$

is a vector space. This means that if $\mathcal{T}_1, \mathcal{T}_2 \in S$, then also $\alpha\mathcal{T}_1 + \beta\mathcal{T}_2 \in S$, for arbitrary constants α, β.

How do we find a non-trivial solution?

We guess that the solution is of the form λ^n, $\lambda \neq 0$, and see what happens. In order for this guess to fulfill the recurrence we need

$$c_0\lambda^n + c_1\lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \cdots + c_k \cdot \lambda^{n-k} = 0$$

for all $n \geq k$.

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Dividing by λ^{n-k} gives that all these constraints are identical to

$$c_0\lambda^k + c_1\lambda^{k-1} + c_2 \cdot \lambda^{k-2} + \cdots + c_k = 0$$

This means that if λ_i is a root (Nullstelle) of $P[\lambda]$ then $T[n] = \lambda_i^n$ is a solution to the recurrence relation.

Let $\lambda_1, \dots, \lambda_k$ be the k (complex) roots of $P[\lambda]$. Then, because of the vector space property

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Lemma 5

Assume that the characteristic polynomial has k distinct roots $\lambda_1, \dots, \lambda_k$. Then all solutions to the recurrence relation are of the form

$$\alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \cdots + \alpha_k\lambda_k^n \ .$$

Proof.

There is one solution for every possible choice of boundary conditions for $T[1], \dots, T[k]$.

We show that the above set of solutions contains one solution for every choice of boundary conditions.

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Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i's$ such that these conditions are met:

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Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i's$ such that these conditions are met:

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_k^2 \\ & & \vdots & \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_k^k \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix} = \begin{pmatrix} T[1] \\ T[2] \\ \vdots \\ T[k] \end{pmatrix}$$

We show that the column vectors are linearly independent. Then the above equation has a solution.

Computing the Determinant

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} =$$

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$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^{k} \lambda_i \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_{k-1}^{k-1} & \lambda_k^{k-1} \end{vmatrix}$$

$$= \prod_{i=1}^{k} \lambda_i \cdot \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{k-2} & \lambda_1^{k-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{k-2} & \lambda_2^{k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-2} & \lambda_k^{k-1} \end{vmatrix}$$

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$$\begin{vmatrix} 1 & \lambda_1 - \lambda_1 \cdot 1 & \cdots & \lambda_1^{k-2} - \lambda_1 \cdot \lambda_1^{k-3} & \lambda_1^{k-1} - \lambda_1 \cdot \lambda_1^{k-2} \\ 1 & \lambda_2 - \lambda_1 \cdot 1 & \cdots & \lambda_2^{k-2} - \lambda_1 \cdot \lambda_2^{k-3} & \lambda_2^{k-1} - \lambda_1 \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k - \lambda_1 \cdot 1 & \cdots & \lambda_k^{k-2} - \lambda_1 \cdot \lambda_k^{k-3} & \lambda_k^{k-1} - \lambda_1 \cdot \lambda_k^{k-2} \end{vmatrix}$$

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$$\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2} \end{vmatrix}$$

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$$\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & (\lambda_2 - \lambda_1)\cdot \color{red}{1} & \cdots & (\lambda_2 - \lambda_1)\cdot \color{blue}{\lambda_2^{k-3}} & (\lambda_2 - \lambda_1)\cdot \color{olive}{\lambda_2^{k-2}} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & (\lambda_k - \lambda_1)\cdot \color{red}{1} & \cdots & (\lambda_k - \lambda_1)\cdot \color{blue}{\lambda_k^{k-3}} & (\lambda_k - \lambda_1)\cdot \color{olive}{\lambda_k^{k-2}} \end{vmatrix} =$$

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$$\prod_{i=2}^{k}(\lambda_i-\lambda_1)\cdot\begin{vmatrix} 1 & \lambda_2 & \cdots & \lambda_2^{k-3} & \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-3} & \lambda_k^{k-2} \end{vmatrix}$$

Computing the Determinant

Repeating the above steps gives:

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^{k} \lambda_i \cdot \prod_{i>\ell} (\lambda_i - \lambda_\ell)$$

Hence, if all λ_i's are different, then the determinant is non-zero.

The Homogeneous Case

What happens if the roots are not all distinct?

Suppose we have a root λ_i with multiplicity (Vielfachheit) at least 2. Then not only is λ_i^n a solution to the recurrence but also $n\lambda_i^n$.

To see this consider the polynomial

$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_k\lambda^{n-k}$$

Since λ_i is a root we can write this as $Q[\lambda] \cdot (\lambda - \lambda_i)^2$. Calculating the derivative gives a polynomial that still has root λ_i.

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This means

$$c_0 n\lambda_i^{n-1} + c_1(n-1)\lambda_i^{n-2} + \cdots + c_k(n-k)\lambda_i^{n-k-1} = 0$$

Hence,

$$c_0\underbrace{n\lambda_i^{n}}_{T[n]} + c_1\underbrace{(n-1)\lambda_i^{n-1}}_{T[n-1]} + \cdots + c_k\underbrace{(n-k)\lambda_i^{n-k}}_{T[n-k]} = 0$$

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Suppose λ_i has multiplicity j. We know that

$$c_0 n\lambda_i^n + c_1(n-1)\lambda_i^{n-1} + \cdots + c_k(n-k)\lambda_i^{n-k} = 0$$

(after taking the derivative; multiplying with λ; plugging in λ_i)

Doing this again gives

$$c_0 n^2\lambda_i^n + c_1(n-1)^2\lambda_i^{n-1} + \cdots + c_k(n-k)^2\lambda_i^{n-k} = 0$$

We can continue $j-1$ times.

Hence, $n^\ell\lambda_i^n$ is a solution for $\ell \in 0,\dots,j-1$.

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The Homogeneous Case

Lemma 6
Let $P[\lambda]$ denote the characteristic polynomial to the recurrence

$$c_0T[n] + c_1T[n-1] + \cdots + c_kT[n-k] = 0$$

Let λ_i, $i = 1, \dots, m$ be the (complex) roots of $P[\lambda]$ with multiplicities ℓ_i. Then the general solution to the recurrence is given by

$$T[n] = \sum_{i=1}^{m} \sum_{j=0}^{\ell_i - 1} \alpha_{ij} \cdot (n^j \lambda_i^n) \ .$$

The full proof is omitted. We have only shown that any choice of α_{ij}'s is a solution to the recurrence.

Example: Fibonacci Sequence

$$\begin{aligned} T[0] &= 0 \\ T[1] &= 1 \\ T[n] &= T[n-1] + T[n-2] \text{ for } n \geq 2 \end{aligned}$$

The characteristic polynomial is

$$\lambda^2 - \lambda - 1$$

Finding the roots, gives

$$\lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$$

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Hence, the solution is of the form

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$$\alpha\left(\frac{1+\sqrt{5}}{2}\right)+\beta\left(\frac{1-\sqrt{5}}{2}\right)=1\Longrightarrow\alpha-\beta=\frac{2}{\sqrt{5}}$$

Example: Fibonacci Sequence

Hence, the solution is

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

The Inhomogeneous Case

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \cdots + c_kT(n-k) = f(n)$$

with $f(n) \neq 0$.

While we have a fairly general technique for solving homogeneous, linear recurrence relations the inhomogeneous case is different.

The Inhomogeneous Case

The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n) \ ,$$

where T_h is any solution to the homogeneous equation, and T_p is one particular solution to the inhomogeneous equation.

There is no general method to find a particular solution.

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The Inhomogeneous Case

Example:

$$T[n] = T[n-1] + 1 \qquad T[0] = 1$$

Then,

$$T[n-1] = T[n-2] + 1 \qquad (n \geq 2)$$

Subtracting the first from the second equation gives,

$$T[n] - T[n-1] = T[n-1] - T[n-2] \qquad (n \geq 2)$$

or

$$T[n] = 2T[n-1] - T[n-2] \qquad (n \geq 2)$$

I get a completely determined recurrence if I add $T[0] = 1$ and $T[1] = 2$.

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$T[1] = 2$ gives $1 + \beta = 2 \Longrightarrow \beta = 1$.

The Inhomogeneous Case

If $f(n)$ is a polynomial of degree r this method can be applied $r+1$ times to obtain a homogeneous equation:

$$T[n] = T[n-1] + n^2$$

Shift:

$$T[n-1] = T[n-2] + (n-1)^2 = T[n-2] + n^2 - 2n + 1$$

Difference:

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$$T[n-1] = T[n-2] + (n-1)^2 = T[n-2] + n^2 - 2n + 1$$

Difference:

$$T[n] - T[n-1] = T[n-1] - T[n-2] + 2n - 1$$

$$T[n] = 2T[n-1] - T[n-2] + 2n - 1$$

The Inhomogeneous Case

If $f(n)$ is a polynomial of degree r this method can be applied $r+1$ times to obtain a homogeneous equation:

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$$T[n-1] = 2T[n-2] - T[n-3] + 2(n-1) - 1$$

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Shift:

$$\begin{aligned} T[n-1] &= 2T[n-2] - T[n-3] + 2(n-1) - 1 \\ &= 2T[n-2] - T[n-3] + 2n - 3 \end{aligned}$$

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and so on...

6.4 Generating Functions

Definition 7 (Generating Function)

Let $(a_n)_{n\geq 0}$ be a sequence. The corresponding

- generating function (Erzeugendenfunktion) is

$$F(z) := \sum_{n\geq 0} a_n z^n;$$

- exponential generating function (exponentielle Erzeugendenfunktion) is

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1. The generating function of the sequence $(1, 0, 0, \ldots)$ is
$$F(z) = 1\,.$$

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6.4 Generating Functions

There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

Let $f = \sum_{n\ge 0} a_n z^n$ and $g = \sum_{n\ge 0} b_n z^n$.

- **Equality:** f and g are equal if $a_n = b_n$ for all n.
- **Addition:** $f + g := \sum_{n\ge 0}(a_n + b_n)z^n$.
- **Multiplication:** $f \cdot g := \sum_{n\ge 0} c_n z^n$ with $c_n = \sum_{p=0}^{n} a_p b_{n-p}$.

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The arithmetic view:

We view a power series as a function $f : \mathbb{C} \to \mathbb{C}$.

Then, it is important to think about convergence/convergence radius etc.

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What does $\sum_{n\ge 0} z^n = \frac{1}{1-z}$ mean in the algebraic view?

It means that the power series $1-z$ and the power series $\sum_{n\ge 0} z^n$ are invers, i.e.,

$$\left(1-z\right)\cdot\left(\sum_{n\ge 0}^{\infty} z^n\right) = 1\ .$$

This is well-defined.

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$$\underbrace{\sum_{n\geq 1} nz^{n-1}}_{\sum_{n\geq 0}(n+1)z^n} = \frac{1}{(1-z)^2}$$

6.4 Generating Functions

Formally the derivative of a formal power series $\sum_{n\geq 0} a_n z^n$ is defined as $\sum_{n\geq 0} n a_n z^{n-1}$.

The known rules for differentiation work for this definition. In particular, e.g. the derivative of $\frac{1}{1-z}$ is $\frac{1}{(1-z)^2}$.

Note that this requires a proof if we consider power series as algebraic objects. However, we did not prove this in the lecture.

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Hence, the generating function of the sequence $a_n = n + 1$ is $1/(1-z)^2$.

6.4 Generating Functions

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Hence, the generating function of the sequence $a_n = (n+1)(n+2)$ is $\frac{2}{(1-z)^3}$.

6.4 Generating Functions

Computing the k-th derivative of $\sum z^n$.

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Hence:

$$\sum_{n\geq 0}\binom{n+k}{k}z^n = \frac{1}{(1-z)^{k+1}} \ .$$

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Hence:

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The generating function of the sequence $a_n = \binom{n+k}{k}$ is $\frac{1}{(1-z)^{k+1}}$.

6.4 Generating Functions

$$\sum_{n\geq 0} nz^n = \sum_{n\geq 0} (n+1)z^n - \sum_{n\geq 0} z^n$$

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The generating function of the sequence $a_n = n$ is $\frac{z}{(1-z)^2}$.

6.4 Generating Functions

We know

$$\sum_{n\ge 0} y^n = \frac{1}{1-y}$$

Hence,

$$\sum_{n\ge 0} a^n z^n = \frac{1}{1-az}$$

The generating function of the sequence $f_n = a^n$ is $\frac{1}{1-az}$.

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Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

$$A(z)$$

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&= zA(z) + \sum_{n\geq 0} z^n \\
&= zA(z) + \frac{1}{1-z}
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Hence, $a_n = n + 1$.

Some Generating Functions

n-th sequence element	*generating function*

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$\frac{1}{n!}$	e^z

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$\sum_{i=0}^{n} f_i$	$\frac{F(z)}{1-z}$

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$\sum_{i=0}^{n} f_i g_{n-i}$	$F \cdot G$
$f_{n-k} \quad (n \geq k); \quad 0$ otw.	$z^k F$
$\sum_{i=0}^{n} f_i$	$\frac{F(z)}{1-z}$
nf_n	$z\frac{\mathrm{d}F(z)}{\mathrm{d}z}$

Some Generating Functions

***n*-th sequence element**	***generating function***
cf_n	cF
$f_n + g_n$	$F + G$
$\sum_{i=0}^{n} f_i g_{n-i}$	$F \cdot G$
$f_{n-k} \quad (n \geq k); \quad 0$ otw.	$z^k F$
$\sum_{i=0}^{n} f_i$	$\dfrac{F(z)}{1-z}$
nf_n	$z\dfrac{\mathrm{d}F(z)}{\mathrm{d}z}$
$c^n f_n$	$F(cz)$

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6. The coefficients of the resulting power series are the a_n.

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

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&= 1 + 3z \sum_{n\geq 0} a_n z^n + \sum_{n\geq 0} n z^n \\
&= 1 + 3zA(z) + \frac{z}{(1-z)^2}
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$$\begin{aligned} z^2 - z + 1 &= A(1-z)^2 + B(1-3z)(1-z) + C(1-3z) \\ &= A(1-2z+z^2) + B(1-4z+3z^2) + C(1-3z) \end{aligned}$$

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This leads to the following conditions:

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This leads to the following conditions:

$$\begin{aligned} A + B + C &= 1 \\ 2A + 4B + 3C &= 1 \\ A + 3B &= 1 \end{aligned}$$

which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

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&= \sum_{n\geq 0} \Big(\frac{7}{4} \cdot 3^n - \frac{1}{4} - \frac{1}{2}(n+1)\Big) z^n \\
&= \sum_{n\geq 0} \Big(\frac{7}{4} \cdot 3^n - \frac{1}{2} n - \frac{3}{4}\Big) z^n
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6. This means $a_n = \frac{7}{4}3^n - \frac{1}{2}n - \frac{3}{4}$.

6.5 Transformation of the Recurrence

Example 9

$$
\begin{aligned}
f_0 &= 1 \\
f_1 &= 2 \\
f_n &= f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 \; .
\end{aligned}
$$

6.5 Transformation of the Recurrence

Example 9

$$\begin{aligned} f_0 &= 1 \\ f_1 &= 2 \\ f_n &= f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 \,. \end{aligned}$$

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Then:

$$\begin{aligned} g_0 &= 1 \\ g_k &= 3g_{k-1} + 2^k,\ k \geq 1 \end{aligned}$$

6 Recurrences

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6 Recurrences

We get

$$\begin{aligned} g_k &= 3\left[g_{k-1}\right] + 2^k \\ &= 3\left[3g_{k-2} + 2^{k-1}\right] + 2^k \\ &= 3^2\left[g_{k-2}\right] + 32^{k-1} + 2^k \\ &= 3^2\left[3g_{k-3} + 2^{k-2}\right] + 32^{k-1} + 2^k \\ &= 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k \\ &= 2^k \cdot \sum_{i=0}^{k} \left(\frac{3}{2}\right)^i \end{aligned}$$

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&= 2^k \cdot \sum_{i=0}^{k} \left(\frac{3}{2}\right)^i \\
&= 2^k \cdot \frac{(\frac{3}{2})^{k+1} - 1}{1/2} = 3^{k+1} - 2^{k+1}
\end{aligned}
$$

6 Recurrences

Let $n = 2^k$:

$$\begin{aligned} g_k &= 3^{k+1} - 2^{k+1}, \text{ hence} \\ f_n &= 3 \cdot 3^k - 2 \cdot 2^k \end{aligned}$$

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