5.2 Simplex and Duality

The following linear programs form a primal dual pair:

$$
\begin{aligned}
z &= \max\{c^T x \mid Ax = b, x \geq 0\} \\
w &= \min\{b^T y \mid A^T y \geq c\}
\end{aligned}
$$

This means for computing the dual of a standard form LP, we do not have non-negativity constraints for the dual variables.

Proof

Primal:

$$\max\{c^T x \mid Ax = b, x \geq 0\}$$

5.2 Simplex and Duality

The following linear programs form a primal dual pair:

$$z = \max\{c^T x \mid Ax = b, x \geq 0\}$$
$$w = \min\{b^T y \mid A^T y \geq c\}$$

This means for computing the dual of a standard form LP, we do not have non-negativity constraints for the dual variables.

Proof

Primal:

$$\begin{aligned}&\max\{c^Tx \mid Ax = b, x \geq 0\}\\&= \max\{c^Tx \mid Ax \leq b, -Ax \leq -b, x \geq 0\}\end{aligned}$$

5.2 Simplex and Duality

The following linear programs form a primal dual pair:

$$\begin{aligned}z &= \max\{c^Tx \mid Ax = b, x \geq 0\}\\w &= \min\{b^Ty \mid A^Ty \geq c\}\end{aligned}$$

This means for computing the dual of a standard form LP, we do not have non-negativity constraints for the dual variables.

Proof

Primal:

$$\begin{aligned}
&\max\{c^T x \mid Ax = b, x \geq 0\} \\
&\quad = \max\{c^T x \mid Ax \leq b, -Ax \leq -b, x \geq 0\} \\
&\quad = \max\left\{c^T x \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix}, x \geq 0\right\}
\end{aligned}$$

5.2 Simplex and Duality

The following linear programs form a primal dual pair:

$$\begin{aligned}
z &= \max\{c^T x \mid Ax = b, x \geq 0\} \\
w &= \min\{b^T y \mid A^T y \geq c\}
\end{aligned}$$

This means for computing the dual of a standard form LP, we do not have non-negativity constraints for the dual variables.

Proof

Primal:

$$\begin{aligned}\max\{c^Tx \mid Ax = b, x \geq 0\} \\ &= \max\{c^Tx \mid Ax \leq b, -Ax \leq -b, x \geq 0\} \\ &= \max\{c^Tx \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix}, x \geq 0\}\end{aligned}$$

Dual:

$$\min\{[b^T \; -b^T]y \mid [A^T \; -A^T]y \geq c, y \geq 0\}$$

5.2 Simplex and Duality

The following linear programs form a primal dual pair:

$$\begin{aligned} z &= \max\{c^Tx \mid Ax = b, x \geq 0\} \\ w &= \min\{b^Ty \mid A^Ty \geq c\} \end{aligned}$$

This means for computing the dual of a standard form LP, we do not have non-negativity constraints for the dual variables.

Proof

Primal:

$$
\begin{aligned}
&\max\{c^T x \mid Ax = b, x \geq 0\} \\
&\quad = \max\{c^T x \mid Ax \leq b, -Ax \leq -b, x \geq 0\} \\
&\quad = \max\left\{c^T x \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix}, x \geq 0\right\}
\end{aligned}
$$

Dual:

$$
\begin{aligned}
&\min\{[b^T \ -b^T] y \mid [A^T \ -A^T] y \geq c, y \geq 0\} \\
&\quad = \min\left\{[b^T \ -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \,\middle|\, [A^T \ -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \geq c, y^- \geq 0, y^+ \geq 0\right\}
\end{aligned}
$$

5.2 Simplex and Duality

The following linear programs form a primal dual pair:

$$
\begin{aligned}
z &= \max\{c^T x \mid Ax = b, x \geq 0\} \\
w &= \min\{b^T y \mid A^T y \geq c\}
\end{aligned}
$$

This means for computing the dual of a standard form LP, we do not have non-negativity constraints for the dual variables.

Proof

Primal:

$$\begin{aligned}&\max\{c^Tx \mid Ax = b, x \geq 0\}\\ &\quad = \max\{c^Tx \mid Ax \leq b, -Ax \leq -b, x \geq 0\}\\ &\quad = \max\left\{c^Tx \;\middle|\; \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix}, x \geq 0\right\}\end{aligned}$$

Dual:

$$\begin{aligned}&\min\{[b^T\ -b^T]y \mid [A^T\ -A^T]y \geq c, y \geq 0\}\\ &\quad = \min\left\{[b^T\ -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \;\middle|\; [A^T\ -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \geq c, y^- \geq 0, y^+ \geq 0\right\}\\ &\quad = \min\left\{b^T \cdot (y^+ - y^-) \;\middle|\; A^T \cdot (y^+ - y^-) \geq c, y^- \geq 0, y^+ \geq 0\right\}\end{aligned}$$

5.2 Simplex and Duality

The following linear programs form a primal dual pair:

$$\begin{aligned} z &= \max\{c^Tx \mid Ax = b, x \geq 0\}\\ w &= \min\{b^Ty \mid A^Ty \geq c\}\end{aligned}$$

This means for computing the dual of a standard form LP, we do not have non-negativity constraints for the dual variables.

Proof

Primal:

$$\begin{aligned}&\max\{c^Tx \mid Ax = b, x \ge 0\}\\ &\quad = \max\{c^Tx \mid Ax \le b, -Ax \le -b, x \ge 0\}\\ &\quad = \max\{c^Tx \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \le \begin{bmatrix} b \\ -b \end{bmatrix}, x \ge 0\}\end{aligned}$$

Dual:

$$\begin{aligned}&\min\{[b^T \; -b^T]y \mid [A^T \; -A^T]y \ge c, y \ge 0\}\\ &\quad = \min\left\{[b^T \; -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \;\middle|\; [A^T \; -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \ge c, y^- \ge 0, y^+ \ge 0\right\}\\ &\quad = \min\left\{b^T \cdot (y^+ - y^-) \;\middle|\; A^T \cdot (y^+ - y^-) \ge c, y^- \ge 0, y^+ \ge 0\right\}\\ &\quad = \min\left\{b^T y' \;\middle|\; A^T y' \ge c\right\}\end{aligned}$$

5.2 Simplex and Duality

The following linear programs form a primal dual pair:

$$\begin{aligned} z &= \max\{c^Tx \mid Ax = b, x \ge 0\}\\ w &= \min\{b^Ty \mid A^Ty \ge c\}\end{aligned}$$

This means for computing the dual of a standard form LP, we do not have non-negativity constraints for the dual variables.

Proof of Optimality Criterion for Simplex

Suppose that we have a basic feasible solution with reduced cost

$$\tilde{c} = c^T - c_B^T A_B^{-1} A \leq 0$$

This is equivalent to $A^T (A_B^{-1})^T c_B \geq c$

$y^* = (A_B^{-1})^T c_B$ is solution to the dual $\min\{b^T y | A^T y \geq c\}$.

$$\begin{aligned} b^T y^* &= (Ax^*)^T y^* = (A_B x_B^*)^T y^* \\ &= (A_B x_B^*)^T (A_B^{-1})^T c_B = (x_B^*)^T A_B^T (A_B^{-1})^T c_B \\ &= c^T x^* \end{aligned}$$

Hence, the solution is optimal.

Proof

Primal:

$$\begin{aligned} &\max\{c^T x \mid Ax = b, x \geq 0\} \\ &\quad = \max\{c^T x \mid Ax \leq b, -Ax \leq -b, x \geq 0\} \\ &\quad = \max\{c^T x \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix}, x \geq 0\} \end{aligned}$$

Dual:

$$\begin{aligned} &\min\{[b^T \; -b^T] y \mid [A^T \; -A^T] y \geq c, y \geq 0\} \\ &\quad = \min\left\{[b^T \; -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \,\middle|\, [A^T \; -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \geq c, y^- \geq 0, y^+ \geq 0\right\} \\ &\quad = \min\left\{b^T \cdot (y^+ - y^-) \,\middle|\, A^T \cdot (y^+ - y^-) \geq c, y^- \geq 0, y^+ \geq 0\right\} \\ &\quad = \min\left\{b^T y' \,\middle|\, A^T y' \geq c\right\} \end{aligned}$$

Proof of Optimality Criterion for Simplex

Suppose that we have a basic feasible solution with reduced cost

$$\tilde{c} = c^T - c_B^T A_B^{-1} A \leq 0$$

This is equivalent to $A^T (A_B^{-1})^T c_B \geq c$

$y^* = (A_B^{-1})^T c_B$ is solution to the dual $\min\{b^T y | A^T y \geq c\}$.

Hence, the solution is optimal.

Proof

Primal:

$$\begin{aligned}
&\max\{c^T x \mid Ax = b, x \geq 0\}\\
&\quad = \max\{c^T x \mid Ax \leq b, -Ax \leq -b, x \geq 0\}\\
&\quad = \max\{c^T x \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix}, x \geq 0\}
\end{aligned}$$

Dual:

$$\begin{aligned}
&\min\{[b^T \; -b^T] y \mid [A^T \; -A^T] y \geq c, y \geq 0\}\\
&\quad = \min\left\{ [b^T \; -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \,\middle|\, [A^T \; -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \geq c, y^- \geq 0, y^+ \geq 0 \right\}\\
&\quad = \min\left\{ b^T \cdot (y^+ - y^-) \,\middle|\, A^T \cdot (y^+ - y^-) \geq c, y^- \geq 0, y^+ \geq 0 \right\}\\
&\quad = \min\left\{ b^T y' \,\middle|\, A^T y' \geq c \right\}
\end{aligned}$$

Proof of Optimality Criterion for Simplex

Suppose that we have a basic feasible solution with reduced cost

$$\tilde{c} = c^T - c_B^T A_B^{-1} A \le 0$$

This is equivalent to $A^T (A_B^{-1})^T c_B \ge c$

$y^* = (A_B^{-1})^T c_B$ is solution to the dual $\min\{b^T y | A^T y \ge c\}$.

$$\begin{aligned} b^T y^* &= (Ax^*)^T y^* = (A_B x_B^*)^T y^* \\ &= (A_B x_B^*)^T (A_B^{-1})^T c_B = (x_B^*)^T A_B^T (A_B^{-1})^T c_B \\ &= c^T x^* \end{aligned}$$

Hence, the solution is optimal.

Proof

Primal:

$$\begin{aligned} &\max\{c^T x \mid Ax = b, x \ge 0\} \\ &\quad = \max\{c^T x \mid Ax \le b, -Ax \le -b, x \ge 0\} \\ &\quad = \max\{c^T x \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \le \begin{bmatrix} b \\ -b \end{bmatrix}, x \ge 0\} \end{aligned}$$

Dual:

$$\begin{aligned} &\min\{[b^T \; -b^T] y \mid [A^T \; -A^T] y \ge c, y \ge 0\} \\ &\quad = \min\left\{[b^T \; -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \;\middle|\; [A^T \; -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \ge c, y^- \ge 0, y^+ \ge 0\right\} \\ &\quad = \min\left\{b^T \cdot (y^+ - y^-) \;\middle|\; A^T \cdot (y^+ - y^-) \ge c, y^- \ge 0, y^+ \ge 0\right\} \\ &\quad = \min\left\{b^T y' \;\middle|\; A^T y' \ge c\right\} \end{aligned}$$

Proof of Optimality Criterion for Simplex

Suppose that we have a basic feasible solution with reduced cost

$$\tilde{c} = c^T - c_B^T A_B^{-1} A \leq 0$$

This is equivalent to $A^T (A_B^{-1})^T c_B \geq c$

$y^* = (A_B^{-1})^T c_B$ is solution to the dual $\min\{b^T y | A^T y \geq c\}$.

$$\begin{aligned} b^T y^* &= (Ax^*)^T y^* = (A_B x_B^*)^T y^* \\ &= (A_B x_B^*)^T (A_B^{-1})^T c_B = (x_B^*)^T A_B^T (A_B^{-1})^T c_B \\ &= c^T x^* \end{aligned}$$

Hence, the solution is optimal.

Proof

Primal:

$$\begin{aligned} &\max\{c^T x \mid Ax = b, x \geq 0\} \\ &\quad = \max\{c^T x \mid Ax \leq b, -Ax \leq -b, x \geq 0\} \\ &\quad = \max\{c^T x \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix}, x \geq 0\} \end{aligned}$$

Dual:

$$\begin{aligned} &\min\{[b^T \; -b^T] y \mid [A^T \; -A^T] y \geq c, y \geq 0\} \\ &\quad = \min\left\{[b^T \; -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \;\middle|\; [A^T \; -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \geq c, y^- \geq 0, y^+ \geq 0\right\} \\ &\quad = \min\left\{b^T \cdot (y^+ - y^-) \;\middle|\; A^T \cdot (y^+ - y^-) \geq c, y^- \geq 0, y^+ \geq 0\right\} \\ &\quad = \min\left\{b^T y' \;\middle|\; A^T y' \geq c\right\} \end{aligned}$$

Proof of Optimality Criterion for Simplex

Suppose that we have a basic feasible solution with reduced cost

$$\tilde{c} = c^T - c_B^T A_B^{-1} A \le 0$$

This is equivalent to $A^T (A_B^{-1})^T c_B \ge c$

$y^* = (A_B^{-1})^T c_B$ is solution to the dual $\min\{b^T y | A^T y \ge c\}$.

$$\begin{aligned} b^T y^* &= (Ax^*)^T y^* = (A_B x_B^*)^T y^* \\ &= (A_B x_B^*)^T (A_B^{-1})^T c_B = (x_B^*)^T A_B^T (A_B^{-1})^T c_B \\ &= c^T x^* \end{aligned}$$

Hence, the solution is optimal.

Proof

Primal:

$$\begin{aligned} &\max\{c^T x \mid Ax = b, x \ge 0\} \\ &= \max\{c^T x \mid Ax \le b, -Ax \le -b, x \ge 0\} \\ &= \max\{c^T x \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \le \begin{bmatrix} b \\ -b \end{bmatrix}, x \ge 0\} \end{aligned}$$

Dual:

$$\begin{aligned} &\min\{[b^T \; -b^T] y \mid [A^T \; -A^T] y \ge c, y \ge 0\} \\ &= \min\left\{ [b^T \; -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \;\middle|\; [A^T \; -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \ge c, y^- \ge 0, y^+ \ge 0 \right\} \\ &= \min\left\{ b^T \cdot (y^+ - y^-) \;\middle|\; A^T \cdot (y^+ - y^-) \ge c, y^- \ge 0, y^+ \ge 0 \right\} \\ &= \min\left\{ b^T y' \;\middle|\; A^T y' \ge c \right\} \end{aligned}$$

Proof of Optimality Criterion for Simplex

Suppose that we have a basic feasible solution with reduced cost

$$\tilde{c} = c^T - c_B^T A_B^{-1} A \le 0$$

This is equivalent to $A^T (A_B^{-1})^T c_B \ge c$

$y^* = (A_B^{-1})^T c_B$ is solution to the dual $\min\{b^T y | A^T y \ge c\}$.

$$\begin{aligned} b^T y^* &= (Ax^*)^T y^* = (A_B x_B^*)^T y^* \\ &= (A_B x_B^*)^T (A_B^{-1})^T c_B = (x_B^*)^T A_B^T (A_B^{-1})^T c_B \\ &= c^T x^* \end{aligned}$$

Hence, the solution is optimal.

Proof

Primal:

$$\begin{aligned} &\max\{c^T x \mid Ax = b, x \ge 0\} \\ &= \max\{c^T x \mid Ax \le b, -Ax \le -b, x \ge 0\} \\ &= \max\{c^T x \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \le \begin{bmatrix} b \\ -b \end{bmatrix}, x \ge 0\} \end{aligned}$$

Dual:

$$\begin{aligned} &\min\{[b^T \ -b^T] y \mid [A^T \ -A^T] y \ge c, y \ge 0\} \\ &= \min\left\{ [b^T \ -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \,\middle|\, [A^T \ -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \ge c, y^- \ge 0, y^+ \ge 0 \right\} \\ &= \min\left\{ b^T \cdot (y^+ - y^-) \,\middle|\, A^T \cdot (y^+ - y^-) \ge c, y^- \ge 0, y^+ \ge 0 \right\} \\ &= \min\left\{ b^T y' \,\middle|\, A^T y' \ge c \right\} \end{aligned}$$

Proof of Optimality Criterion for Simplex

Suppose that we have a basic feasible solution with reduced cost

$$\tilde{c} = c^T - c_B^T A_B^{-1} A \le 0$$

This is equivalent to $A^T (A_B^{-1})^T c_B \ge c$

$y^* = (A_B^{-1})^T c_B$ is solution to the dual $\min\{b^T y | A^T y \ge c\}$.

$$\begin{aligned} b^T y^* &= (Ax^*)^T y^* = (A_B x_B^*)^T y^* \\ &= (A_B x_B^*)^T (A_B^{-1})^T c_B = (x_B^*)^T A_B^T (A_B^{-1})^T c_B \\ &= c^T x^* \end{aligned}$$

Hence, the solution is optimal.

Proof

Primal:

$$\begin{aligned} &\max\{c^T x \mid Ax = b, x \ge 0\} \\ &\quad = \max\{c^T x \mid Ax \le b, -Ax \le -b, x \ge 0\} \\ &\quad = \max\{c^T x \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \le \begin{bmatrix} b \\ -b \end{bmatrix}, x \ge 0\} \end{aligned}$$

Dual:

$$\begin{aligned} &\min\{[b^T \ -b^T] y \mid [A^T \ -A^T] y \ge c, y \ge 0\} \\ &\quad = \min\left\{[b^T \ -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \;\middle|\; [A^T \ -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \ge c, y^- \ge 0, y^+ \ge 0\right\} \\ &\quad = \min\left\{b^T \cdot (y^+ - y^-) \;\middle|\; A^T \cdot (y^+ - y^-) \ge c, y^- \ge 0, y^+ \ge 0\right\} \\ &\quad = \min\left\{b^T y' \;\middle|\; A^T y' \ge c\right\} \end{aligned}$$

Proof of Optimality Criterion for Simplex

Suppose that we have a basic feasible solution with reduced cost

$$\tilde{c} = c^T - c_B^T A_B^{-1} A \le 0$$

This is equivalent to $A^T (A_B^{-1})^T c_B \ge c$

$y^* = (A_B^{-1})^T c_B$ is solution to the dual $\min\{b^T y | A^T y \ge c\}$.

$$\begin{aligned} b^T y^* &= (Ax^*)^T y^* = (A_B x_B^*)^T y^* \\ &= (A_B x_B^*)^T (A_B^{-1})^T c_B = (x_B^*)^T A_B^T (A_B^{-1})^T c_B \\ &= c^T x^* \end{aligned}$$

Hence, the solution is optimal.

Proof

Primal:

$$\begin{aligned} &\max\{c^T x \mid Ax = b, x \ge 0\} \\ &\quad = \max\{c^T x \mid Ax \le b, -Ax \le -b, x \ge 0\} \\ &\quad = \max\{c^T x \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \le \begin{bmatrix} b \\ -b \end{bmatrix}, x \ge 0\} \end{aligned}$$

Dual:

$$\begin{aligned} &\min\{[b^T \; -b^T] y \mid [A^T \; -A^T] y \ge c, y \ge 0\} \\ &\quad = \min\left\{[b^T \; -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \;\middle|\; [A^T \; -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \ge c, y^- \ge 0, y^+ \ge 0\right\} \\ &\quad = \min\left\{b^T \cdot (y^+ - y^-) \;\middle|\; A^T \cdot (y^+ - y^-) \ge c, y^- \ge 0, y^+ \ge 0\right\} \\ &\quad = \min\left\{b^T y' \;\middle|\; A^T y' \ge c\right\} \end{aligned}$$

Proof of Optimality Criterion for Simplex

Suppose that we have a basic feasible solution with reduced cost

$$\tilde{c} = c^T - c_B^T A_B^{-1} A \leq 0$$

This is equivalent to $A^T (A_B^{-1})^T c_B \geq c$

$y^* = (A_B^{-1})^T c_B$ is solution to the dual $\min\{b^T y | A^T y \geq c\}$.

$$\begin{aligned} b^T y^* &= (Ax^*)^T y^* = (A_B x_B^*)^T y^* \\ &= (A_B x_B^*)^T (A_B^{-1})^T c_B = (x_B^*)^T A_B^T (A_B^{-1})^T c_B \\ &= c^T x^* \end{aligned}$$

Hence, the solution is optimal.

Proof

Primal:

$$\begin{aligned} &\max\{c^T x \mid Ax = b, x \geq 0\} \\ &\quad = \max\{c^T x \mid Ax \leq b, -Ax \leq -b, x \geq 0\} \\ &\quad = \max\{c^T x \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix}, x \geq 0\} \end{aligned}$$

Dual:

$$\begin{aligned} &\min\{[b^T \; -b^T] y \mid [A^T \; -A^T] y \geq c, y \geq 0\} \\ &\quad = \min\left\{[b^T \; -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \;\middle|\; [A^T \; -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \geq c, y^- \geq 0, y^+ \geq 0\right\} \\ &\quad = \min\left\{b^T \cdot (y^+ - y^-) \;\middle|\; A^T \cdot (y^+ - y^-) \geq c, y^- \geq 0, y^+ \geq 0\right\} \\ &\quad = \min\left\{b^T y' \;\middle|\; A^T y' \geq c\right\} \end{aligned}$$