Enumerating all basic feasible solutions (BFS), in order to find the optimum is slow.

Simplex Algorithm [George Dantzig 1947]
Move from BFS to adjacent BFS, without decreasing objective function.

Two BFSs are called adjacent if the bases just differ in one variable.

4 Simplex Algorithm

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4 Simplex Algorithm

$$\begin{array}{llcl} \max & 13a + 23b & & \\ \text{s.t.} & 5a + 15b + s_c & = & 480 \\ & 4a + 4b + s_h & = & 160 \\ & 35a + 20b + s_m & = & 1190 \\ & a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m & \geq & 0 \end{array}$$

$$\begin{array}{lcl} \max Z & & \\ 13a + 23b - Z & = & 0 \\ 5a + 15b + s_c & = & 480 \\ 4a + 4b + s_h & = & 160 \\ 35a + 20b + s_m & = & 1190 \\ a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m & \geq & 0 \end{array}$$

basis = $\{s_c, s_h, s_m\}$
$A = B = 0$
$Z = 0$

$s_c = 480$
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Pivoting Step

$$\begin{array}{lrrrrrrr} \max Z & & & & & & & \\ & 13a + 23b & & & & -Z & = & 0 \\ & 5a + 15b & + s_c & & & & = & 480 \\ & 4a + 4b & & + s_h & & & = & 160 \\ & 35a + 20b & & & + s_m & & = & 1190 \\ & a\ ,\ b & ,\ s_c & ,\ s_h & ,\ s_m & & \geq & 0 \end{array}$$

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4 Simplex Algorithm

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Pivoting Step

$$
\begin{array}{l}
\text{max } Z \\
\begin{array}{rcrcrcrcrcrcl}
13a & + & 23b & & & & & & & - & Z & = & 0 \\
5a & + & 15b & + & s_c & & & & & & & = & 480 \\
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- choose variable to bring into the basis
- chosen variable should have positive coefficient in objective function
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4 Simplex Algorithm

$$
\begin{array}{llcrcrcrcrcl}
\text{max} & 13a + 23b \\
\text{s.t.} & 5a & + & 15b & + & s_c & & & & & = & 480 \\
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\begin{array}{rcrcrcrcrcrcl}
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$$
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Pivoting Step

$$
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\max\ Z \\
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Pivoting Step

$$\begin{array}{l} \max Z \\ 13a + 23b \qquad\qquad - Z = 0 \\ 5a + 15b + s_c \qquad\qquad = 480 \\ 4a + 4b \quad + s_h \qquad = 160 \\ 35a + 20b \qquad + s_m \quad = 1190 \\ a, b, s_c, s_h, s_m \geq 0 \end{array}$$

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Pivoting Step

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$$\begin{array}{l} \max Z \\ 13a + 23\boldsymbol{b} - Z = 0 \\ 5a + 15\boldsymbol{b} + s_c = 480 \\ 4a + 4\boldsymbol{b} + s_h = 160 \\ 35a + 20\boldsymbol{b} + s_m = 1190 \\ a, \boldsymbol{b}, s_c, s_h, s_m \geq 0 \end{array}$$

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$Z = 0$

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- Choose variable with coefficient > 0 as entering variable.

Pivoting Step

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$$\begin{array}{llllll} \max Z & & & & & \\ 13a + 23b & & & & -Z & = 0 \\ 5a + 15b & + s_c & & & & = 480 \\ 4a + 4b & & + s_h & & & = 160 \\ 35a + 20b & & & + s_m & & = 1190 \\ a \;,\; b & ,\; s_c & ,\; s_h & ,\; s_m & & \geq 0 \end{array}$$

basis $= \{s_c, s_h, s_m\}$

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$Z = 0$

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$s_h = 160$

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- Choose variable with coefficient > 0 as entering variable.
- If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.

Pivoting Step

$$\begin{array}{llllll} \max Z & & & & & \\ 13a + 23b & & & & -Z & = 0 \\ 5a + 15b & + s_c & & & & = 480 \\ 4a + 4b & & + s_h & & & = 160 \\ 35a + 20b & & & + s_m & & = 1190 \\ a \;,\; b & ,\; s_c & ,\; s_h & ,\; s_m & & \geq 0 \end{array}$$

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$Z = 0$

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\begin{array}{l}
\max Z \\
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$$
\begin{array}{l}
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Z = 0 \\
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s_h = 160 \\
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\end{array}
$$

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- If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.
- For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.

Pivoting Step

$$
\begin{array}{l}
\max Z \\
13a + 23b \qquad\qquad - Z = 0 \\
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max Z

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- For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.
- Choosing $\theta = \min\{480/15, 160/4, 1190/20\}$ ensures that in the new solution one current basic variable becomes 0, and no variable goes negative.

Pivoting Step

max Z

$$\begin{aligned} 13a + 23b \quad\quad\quad\quad\quad\quad - Z &= 0 \\ 5a + 15b + s_c \quad\quad\quad &= 480 \\ 4a + 4b \quad + s_h \quad\quad &= 160 \\ 35a + 20b \quad\quad + s_m &= 1190 \\ a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m &\geq 0 \end{aligned}$$

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- The basic variable in the row that gives $\min\{480/15, 160/4, 1190/20\}$ becomes the leaving variable.

Pivoting Step

max Z

$$\begin{array}{rcl} 13a + 23b & - Z & = 0 \\ 5a + 15b + s_c & & = 480 \\ 4a + 4b \quad + s_h & & = 160 \\ 35a + 20b \quad\quad + s_m & & = 1190 \\ a, \; b, \; s_c, \; s_h, \; s_m & & \geq 0 \end{array}$$

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- The basic variable in the row that gives $\min\{480/15, 160/4, 1190/20\}$ becomes the leaving variable.

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$s_h = 160$
$s_m = 1190$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

$$\begin{array}{lr} \max Z & \\ 13a + 23b \qquad\qquad - Z & = 0 \\ 5a + 15b + s_c & = 480 \\ 4a + 4b + s_h & = 160 \\ 35a + 20b + s_m & = 1190 \\ a, b, s_c, s_h, s_m & \geq 0 \end{array}$$

basis $= \{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$
$s_c = 480$
$s_h = 160$
$s_m = 1190$

- Choose variable with coefficient > 0 as entering variable.
- If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.
- For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.
- Choosing $\theta = \min\{480/15, 160/4, 1190/20\}$ ensures that in the new solution one current basic variable becomes 0, and no variable goes negative.
- The basic variable in the row that gives $\min\{480/15, 160/4, 1190/20\}$ becomes the leaving variable.

$$\begin{array}{l} \max Z \\ 13a + 23b \qquad\qquad - Z = 0 \\ 5a + 15b + s_c \qquad = 480 \\ 4a + 4b + s_h \qquad = 160 \\ 35a + 20b + s_m = 1190 \\ a, b, s_c, s_h, s_m \geq 0 \end{array}$$

basis = $\{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$
$s_c = 480$
$s_h = 160$
$s_m = 1190$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

$$\begin{array}{l} \max Z \\ \frac{16}{3}a - \frac{23}{15}s_c \qquad - Z = -736 \\ \frac{1}{3}a + b + \frac{1}{15}s_c \qquad = 32 \\ \frac{8}{3}a - \frac{4}{15}s_c + s_h \qquad = 32 \\ \frac{85}{3}a - \frac{4}{3}s_c + s_m = 550 \\ a, b, s_c, s_h, s_m \geq 0 \end{array}$$

basis = $\{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$
$b = 32$
$s_h = 32$
$s_m = 550$

$$\begin{array}{l} \max Z \\ 13a + 23b \qquad\qquad - Z = 0 \\ 5a + 15b + s_c \qquad = 480 \\ 4a + 4b + s_h \qquad = 160 \\ 35a + 20b + s_m = 1190 \\ a, b, s_c, s_h, s_m \geq 0 \end{array}$$

basis = $\{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$
$s_c = 480$
$s_h = 160$
$s_m = 1190$

- Choose variable with coefficient > 0 as entering variable.
- If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.
- For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.
- Choosing $\theta = \min\{480/15, 160/4, 1190/20\}$ ensures that in the new solution one current basic variable becomes 0, and no variable goes negative.
- The basic variable in the row that gives $\min\{480/15, 160/4, 1190/20\}$ becomes the leaving variable.

max Z

$$\begin{aligned}
\tfrac{16}{3}a \quad & - \tfrac{23}{15}s_c & - Z &= -736 \\
\tfrac{1}{3}a + b + & \tfrac{1}{15}s_c & &= 32 \\
\tfrac{8}{3}a \quad & - \tfrac{4}{15}s_c + s_h & &= 32 \\
\tfrac{85}{3}a \quad & - \tfrac{4}{3}s_c \quad + s_m & &= 550 \\
a\ ,\ b\ ,\ & s_c\ ,\ s_h\ ,\ s_m & &\geq 0
\end{aligned}$$

basis $= \{b, s_h, s_m\}$

$a = s_c = 0$

$Z = 736$

$b = 32$

$s_h = 32$

$s_m = 550$

max Z

$$\begin{aligned}
13a + 23b \quad & & - Z &= 0 \\
5a + 15b + s_c & & &= 480 \\
4a + 4b \quad & + s_h & &= 160 \\
35a + 20b \quad & \quad + s_m & &= 1190 \\
a\ ,\ b\ ,\ s_c\ ,\ & s_h\ ,\ s_m & &\geq 0
\end{aligned}$$

basis $= \{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

max Z

$$\begin{aligned}
\tfrac{16}{3}a \quad & - \tfrac{23}{15}s_c & - Z &= -736 \\
\tfrac{1}{3}a + b + & \tfrac{1}{15}s_c & &= 32 \\
\tfrac{8}{3}a \quad & - \tfrac{4}{15}s_c + s_h & &= 32 \\
\tfrac{85}{3}a \quad & - \tfrac{4}{3}s_c \quad + s_m & &= 550 \\
a\ ,\ b\ ,\ & s_c\ ,\ s_h\ ,\ s_m & &\geq 0
\end{aligned}$$

basis $= \{b, s_h, s_m\}$

$a = s_c = 0$

$Z = 736$

$b = 32$

$s_h = 32$

$s_m = 550$

$$\begin{array}{l}\max Z\\ \frac{16}{3}\boldsymbol{a} - \frac{23}{15}s_c - Z = -736\\ \frac{1}{3}\boldsymbol{a} + b + \frac{1}{15}s_c = 32\\ \frac{8}{3}\boldsymbol{a} - \frac{4}{15}s_c + s_h = 32\\ \frac{85}{3}\boldsymbol{a} - \frac{4}{3}s_c + s_m = 550\\ \boldsymbol{a}, b, s_c, s_h, s_m \geq 0\end{array}$$

basis $= \{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$
$b = 32$
$s_h = 32$
$s_m = 550$

Choose variable a to bring into basis.

$$\begin{array}{l}\max Z\\ 13a + 23\boldsymbol{b} - Z = 0\\ 5a + 15\boldsymbol{b} + s_c = 480\\ 4a + 4\boldsymbol{b} + s_h = 160\\ 35a + 20\boldsymbol{b} + s_m = 1190\\ a, \boldsymbol{b}, s_c, s_h, s_m \geq 0\end{array}$$

basis $= \{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$
$s_c = 480$
$s_h = 160$
$s_m = 1190$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

$$\begin{array}{l}\max Z\\ \frac{16}{3}a - \frac{23}{15}s_c - Z = -736\\ \frac{1}{3}a + b + \frac{1}{15}s_c = 32\\ \frac{8}{3}a - \frac{4}{15}s_c + s_h = 32\\ \frac{85}{3}a - \frac{4}{3}s_c + s_m = 550\\ a, b, s_c, s_h, s_m \geq 0\end{array}$$

basis $= \{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$
$b = 32$
$s_h = 32$
$s_m = 550$

$$\begin{aligned}
&\max Z\\
&\tfrac{16}{3}a - \tfrac{23}{15}s_c - Z = -736\\
&\tfrac{1}{3}a + b + \tfrac{1}{15}s_c = 32\\
&\tfrac{8}{3}a - \tfrac{4}{15}s_c + s_h = 32\\
&\tfrac{85}{3}a - \tfrac{4}{3}s_c + s_m = 550\\
&a, b, s_c, s_h, s_m \ge 0
\end{aligned}$$

basis $= \{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$

$b = 32$
$s_h = 32$
$s_m = 550$

Choose variable a to bring into basis.
Computing $\min\{3 \cdot 32, {}^{3\cdot 32}/8, {}^{3\cdot 550}/85\}$ means pivot on line 2.

$$\begin{aligned}
&\max Z\\
&13a + 23b - Z = 0\\
&5a + 15b + s_c = 480\\
&4a + 4b + s_h = 160\\
&35a + 20b + s_m = 1190\\
&a, b, s_c, s_h, s_m \ge 0
\end{aligned}$$

basis $= \{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$

$s_c = 480$
$s_h = 160$
$s_m = 1190$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

$$\begin{aligned}
&\max Z\\
&\tfrac{16}{3}a - \tfrac{23}{15}s_c - Z = -736\\
&\tfrac{1}{3}a + b + \tfrac{1}{15}s_c = 32\\
&\tfrac{8}{3}a - \tfrac{4}{15}s_c + s_h = 32\\
&\tfrac{85}{3}a - \tfrac{4}{3}s_c + s_m = 550\\
&a, b, s_c, s_h, s_m \ge 0
\end{aligned}$$

basis $= \{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$

$b = 32$
$s_h = 32$
$s_m = 550$

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}\mathbf{a} & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{b, s_h, s_m\} \\
a = s_c = 0 \\
Z = 736 \\
b = 32 \\
s_h = 32 \\
s_m = 550
\end{array}
$$

Choose variable a to bring into basis.

Computing $\min\{3 \cdot 32, {}^{3\cdot 32}/_{8}, {}^{3\cdot 550}/_{85}\}$ means pivot on line 2.

Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
13a & + & 23\mathbf{b} & & & & & & & - & Z & = & 0 \\
5a & + & 15\mathbf{b} & + & s_c & & & & & & & = & 480 \\
4a & + & 4\mathbf{b} & & & + & s_h & & & & & = & 160 \\
35a & + & 20\mathbf{b} & & & & & + & s_m & & & = & 1190 \\
a & , & \mathbf{b} & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{s_c, s_h, s_m\} \\
a = b = 0 \\
Z = 0 \\
s_c = 480 \\
s_h = 160 \\
s_m = 1190
\end{array}
$$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}a & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{b, s_h, s_m\} \\
a = s_c = 0 \\
Z = 736 \\
b = 32 \\
s_h = 32 \\
s_m = 550
\end{array}
$$

$$\begin{array}{l} \max Z \\ \begin{array}{rcrcrcrcrcrcl} \frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\ \frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\ \frac{8}{3}\mathbf{a} & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\ \frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\ a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0 \end{array} \end{array}$$

basis $= \{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$
$b = 32$
$s_h = 32$
$s_m = 550$

Choose variable a to bring into basis.
Computing $\min\{3 \cdot 32, {}^{3\cdot 32}/_{8}, {}^{3\cdot 550}/_{85}\}$ means pivot on line 2.
Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$\begin{array}{l} \max Z \\ \begin{array}{rcrcrcrcrcl} & & & - & s_c & - & 2s_h & & & - Z & = -800 \\ & & b & + & \frac{1}{10}s_c & - & \frac{1}{8}s_h & & & & = 28 \\ a & & & - & \frac{1}{10}s_c & + & \frac{3}{8}s_h & & & & = 12 \\ & & & & \frac{3}{2}s_c & - & \frac{85}{8}s_h & + & s_m & & = 210 \\ a & , & b & , & s_c & , & s_h & , & s_m & & \geq 0 \end{array} \end{array}$$

basis $= \{a, b, s_m\}$
$s_c = s_h = 0$
$Z = 800$
$b = 28$
$a = 12$
$s_m = 210$

$$\begin{array}{l} \max Z \\ \begin{array}{rcrcrcrcrcl} 13a & + & 23\mathbf{b} & & & & & & & - Z & = 0 \\ 5a & + & 15\mathbf{b} & + & s_c & & & & & & = 480 \\ 4a & + & 4\mathbf{b} & & & + & s_h & & & & = 160 \\ 35a & + & 20\mathbf{b} & & & & & + & s_m & & = 1190 \\ a & , & \mathbf{b} & , & s_c & , & s_h & , & s_m & & \geq 0 \end{array} \end{array}$$

basis $= \{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$
$s_c = 480$
$s_h = 160$
$s_m = 1190$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

$$\begin{array}{l} \max Z \\ \begin{array}{rcrcrcrcrcrcl} \frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\ \frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\ \frac{8}{3}a & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\ \frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\ a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0 \end{array} \end{array}$$

basis $= \{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$
$b = 32$
$s_h = 32$
$s_m = 550$

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}a & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

basis $= \{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$

$b = 32$
$s_h = 32$
$s_m = 550$

Choose variable a to bring into basis.
Computing $\min\{3 \cdot 32, {}^{3\cdot 32}/_{8}, {}^{3\cdot 550}/_{85}\}$ means pivot on line 2.
Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
 & & & - & s_c & - & 2s_h & & & - & Z & = & -800 \\
 & & b & + & \frac{1}{10}s_c & - & \frac{1}{8}s_h & & & & & = & 28 \\
a & & & - & \frac{1}{10}s_c & + & \frac{3}{8}s_h & & & & & = & 12 \\
 & & & & \frac{3}{2}s_c & - & \frac{85}{8}s_h & + & s_m & & & = & 210 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

basis $= \{a, b, s_m\}$
$s_c = s_h = 0$
$Z = 800$

$b = 28$
$a = 12$
$s_m = 210$

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- any feasible solution satisfies all equations in the tableaux
- in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0$, $s_h \geq 0$
- hence optimum solution value is at most 800
- the current solution has value 800

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}a & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{b, s_h, s_m\} \\
a = s_c = 0 \\
Z = 736 \\
b = 32 \\
s_h = 32 \\
s_m = 550
\end{array}
$$

Choose variable a to bring into basis.
Computing $\min\{3 \cdot 32, {}^{3\cdot 32}/_{8}, {}^{3\cdot 550}/_{85}\}$ means pivot on line 2.
Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
 & & & - & s_c & - & 2s_h & & & - & Z & = & -800 \\
 & & b & + & \frac{1}{10}s_c & - & \frac{1}{8}s_h & & & & & = & 28 \\
a & & & - & \frac{1}{10}s_c & + & \frac{3}{8}s_h & & & & & = & 12 \\
 & & & & \frac{3}{2}s_c & - & \frac{85}{8}s_h & + & s_m & & & = & 210 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{a, b, s_m\} \\
s_c = s_h = 0 \\
Z = 800 \\
b = 28 \\
a = 12 \\
s_m = 210
\end{array}
$$

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- any feasible solution satisfies all equations in the tableaux
- in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0$, $s_h \geq 0$
- hence optimum solution value is at most 800
- the current solution has value 800

$$\begin{array}{l} \max Z \\ \begin{array}{rcrcrcrcrcrcl} \frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\ \frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\ \frac{8}{3}a & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\ \frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\ a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0 \end{array} \end{array}$$

basis = $\{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$
$b = 32$
$s_h = 32$
$s_m = 550$

Choose variable a to bring into basis.
Computing $\min\{3 \cdot 32, {}^{3\cdot 32}/_{8}, {}^{3\cdot 550}/_{85}\}$ means pivot on line 2.
Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$\begin{array}{l} \max Z \\ \begin{array}{rcrcrcrcrcrcl} & & & - & s_c & - & 2s_h & & & - & Z & = & -800 \\ & & b & + & \frac{1}{10}s_c & - & \frac{1}{8}s_h & & & & & = & 28 \\ a & & & - & \frac{1}{10}s_c & + & \frac{3}{8}s_h & & & & & = & 12 \\ & & & & \frac{3}{2}s_c & - & \frac{85}{8}s_h & + & s_m & & & = & 210 \\ a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0 \end{array} \end{array}$$

basis = $\{a, b, s_m\}$
$s_c = s_h = 0$
$Z = 800$
$b = 28$
$a = 12$
$s_m = 210$

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- any feasible solution satisfies all equations in the tableaux
- in particular: $Z = 800 - s_c - 2s_h$, $s_c \ge 0, s_h \ge 0$
- hence optimum solution value is at most 800
- the current solution has value 800

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}a & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \ge & 0
\end{array}
\end{array}
$$

$$
\begin{array}{lcl}
\text{basis} & = & \{b, s_h, s_m\} \\
a & = & s_c = 0 \\
Z & = & 736 \\
b & = & 32 \\
s_h & = & 32 \\
s_m & = & 550
\end{array}
$$

Choose variable a to bring into basis.
Computing $\min\{3 \cdot 32, 3\cdot32/8, 3\cdot550/85\}$ means pivot on line 2.
Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
 & & & - & s_c & - & 2s_h & & & - & Z & = & -800 \\
 & & b & + & \frac{1}{10}s_c & - & \frac{1}{8}s_h & & & & & = & 28 \\
a & & & - & \frac{1}{10}s_c & + & \frac{3}{8}s_h & & & & & = & 12 \\
 & & & & \frac{3}{2}s_c & - & \frac{85}{8}s_h & + & s_m & & & = & 210 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \ge & 0
\end{array}
\end{array}
$$

$$
\begin{array}{lcl}
\text{basis} & = & \{a, b, s_m\} \\
s_c & = & s_h = 0 \\
Z & = & 800 \\
b & = & 28 \\
a & = & 12 \\
s_m & = & 210
\end{array}
$$

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- any feasible solution satisfies all equations in the tableaux
- in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0, s_h \geq 0$
- hence optimum solution value is at most 800
- the current solution has value 800

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}\mathbf{a} & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{b, s_h, s_m\} \\
a = s_c = 0 \\
Z = 736 \\
b = 32 \\
s_h = 32 \\
s_m = 550
\end{array}
$$

Choose variable a to bring into basis.

Computing $\min\{3 \cdot 32, {}^{3\cdot 32}/_{8}, {}^{3\cdot 550}/_{85}\}$ means pivot on line 2.

Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
 & & & - & s_c & - & 2s_h & & & - & Z & = & -800 \\
 & & b & + & \frac{1}{10}s_c & - & \frac{1}{8}s_h & & & & & = & 28 \\
a & & & - & \frac{1}{10}s_c & + & \frac{3}{8}s_h & & & & & = & 12 \\
 & & & & \frac{3}{2}s_c & - & \frac{85}{8}s_h & + & s_m & & & = & 210 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{a, b, s_m\} \\
s_c = s_h = 0 \\
Z = 800 \\
b = 28 \\
a = 12 \\
s_m = 210
\end{array}
$$

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- any feasible solution satisfies all equations in the tableaux
- in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0, s_h \geq 0$
- hence optimum solution value is at most 800
- the current solution has value 800

$$\begin{array}{l} \max Z \\ \frac{16}{3}a - \frac{23}{15}s_c - Z = -736 \\ \frac{1}{3}a + b + \frac{1}{15}s_c = 32 \\ \frac{8}{3}a - \frac{4}{15}s_c + s_h = 32 \\ \frac{85}{3}a - \frac{4}{3}s_c + s_m = 550 \\ a, b, s_c, s_h, s_m \geq 0 \end{array}$$

basis = $\{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$
$b = 32$
$s_h = 32$
$s_m = 550$

Choose variable a to bring into basis.
Computing $\min\{3 \cdot 32, 3\cdot 32/8, 3\cdot 550/85\}$ means pivot on line 2.
Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$\begin{array}{l} \max Z \\ - s_c - 2s_h - Z = -800 \\ b + \frac{1}{10}s_c - \frac{1}{8}s_h = 28 \\ a - \frac{1}{10}s_c + \frac{3}{8}s_h = 12 \\ \frac{3}{2}s_c - \frac{85}{8}s_h + s_m = 210 \\ a, b, s_c, s_h, s_m \geq 0 \end{array}$$

basis = $\{a, b, s_m\}$
$s_c = s_h = 0$
$Z = 800$
$b = 28$
$a = 12$
$s_m = 210$

Matrix View

Let our linear program be

$$
\begin{array}{rcrcl}
c_B^T x_B & + & c_N^T x_N & = & Z \\
A_B x_B & + & A_N x_N & = & b \\
x_B & , & x_N & \geq & 0
\end{array}
$$

The simplex tableaux for basis B is

$$
\begin{array}{rcrcl}
 & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\
I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\
x_B & , & x_N & \geq & 0
\end{array}
$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- any feasible solution satisfies all equations in the tableaux
- in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0, s_h \geq 0$
- hence optimum solution value is at most 800
- the current solution has value 800

Matrix View

Let our linear program be

$$\begin{array}{rcrcl} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

$$\begin{array}{rcrcl} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

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Matrix View

Let our linear program be

$$\begin{array}{rcrcl} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

$$\begin{array}{rcrcl} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

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- the current solution has value 800

Matrix View

Let our linear program be

$$\begin{array}{rcrcl} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

$$\begin{array}{rcrcl} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

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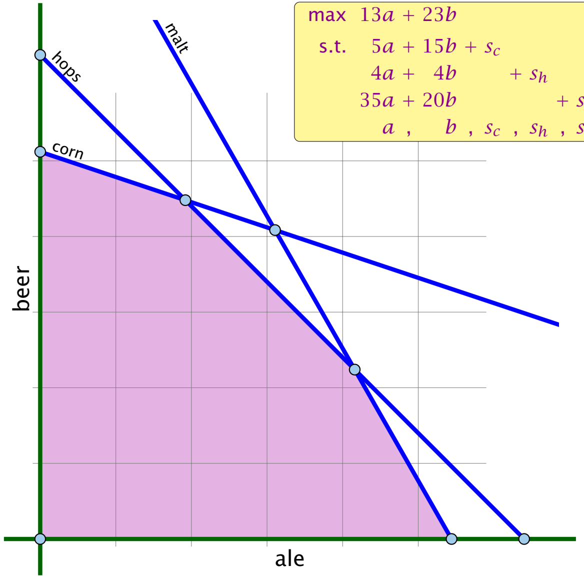
- in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0, s_h \geq 0$
- hence optimum solution value is at most 800
- the current solution has value 800

Geometric View of Pivoting



$$\begin{array}{llllllll} \max & 13a & + & 23b & & & & \\ \text{s.t.} & 5a & + & 15b & + s_c & & & = 480 \\ & 4a & + & 4b & & + s_h & & = 160 \\ & 35a & + & 20b & & & + s_m & = 1190 \\ & a & , & b & , s_c & , s_h & , s_m & \geq 0 \end{array}$$

Matrix View

Let our linear program be

$$\begin{array}{ccccc} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

$$\begin{array}{ccccc} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

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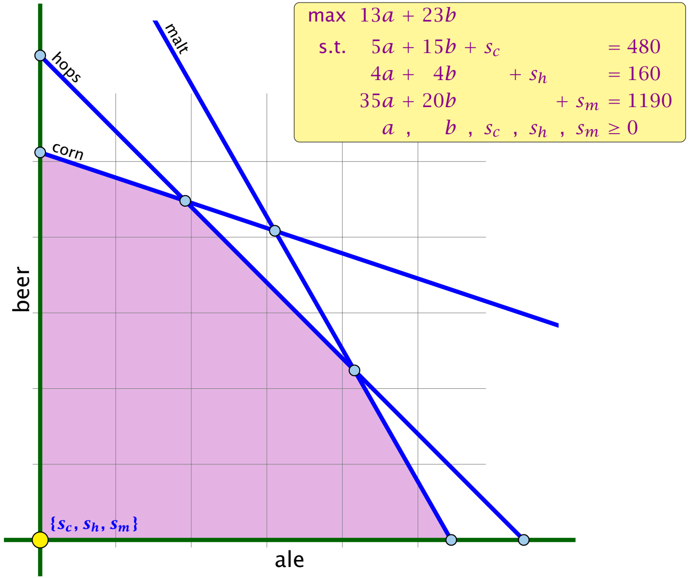
Geometric View of Pivoting



Matrix View

Let our linear program be

$$\begin{array}{rcrcl} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

$$\begin{array}{rcrcl} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

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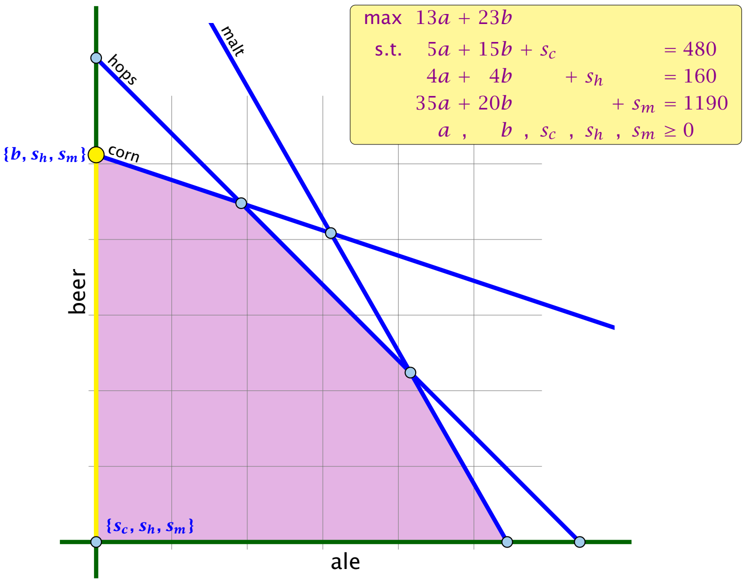

Matrix View

Let our linear program be

$$\begin{array}{ccccc} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

$$\begin{array}{ccccc} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

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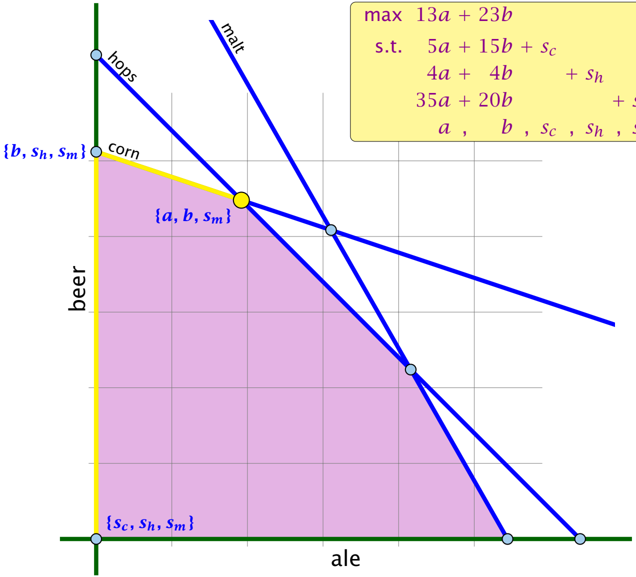
Geometric View of Pivoting



$$\begin{array}{lrcl} \max & 13a + 23b & & \\ \text{s.t.} & 5a + 15b + s_c & = & 480 \\ & 4a + 4b \quad + s_h & = & 160 \\ & 35a + 20b \quad + s_m & = & 1190 \\ & a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m & \geq & 0 \end{array}$$

Matrix View

Let our linear program be

$$\begin{array}{ccccc} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

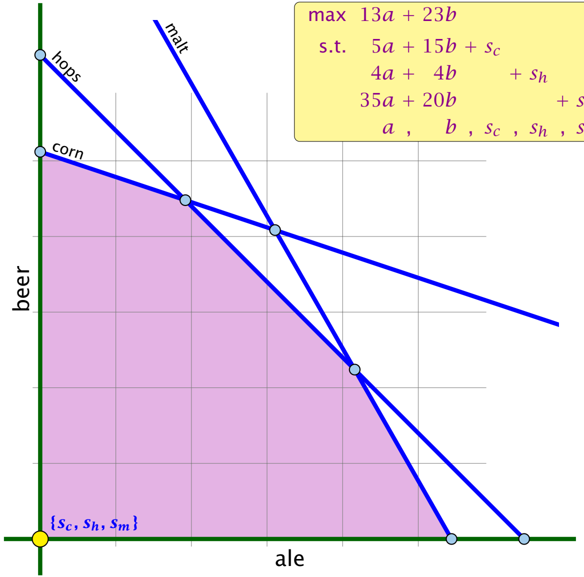
$$\begin{array}{ccccc} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

Geometric View of Pivoting

$$
\begin{array}{lrcrcrcrcrcl}
\max & 13a & + & 23b & & & & & & & & \\
\text{s.t.} & 5a & + & 15b & + & s_c & & & & & = & 480 \\
& 4a & + & 4b & & & + & s_h & & & = & 160 \\
& 35a & + & 20b & & & & & + & s_m & = & 1190 \\
& a & , & b & , & s_c & , & s_h & , & s_m & \geq & 0
\end{array}
$$



Matrix View

Let our linear program be

$$
\begin{array}{rcrcl}
c_B^T x_B & + & c_N^T x_N & = & Z \\
A_B x_B & + & A_N x_N & = & b \\
x_B & , & x_N & \geq & 0
\end{array}
$$

The simplex tableaux for basis B is

$$
\begin{array}{rcrcl}
 & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\
I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\
x_B & , & x_N & \geq & 0
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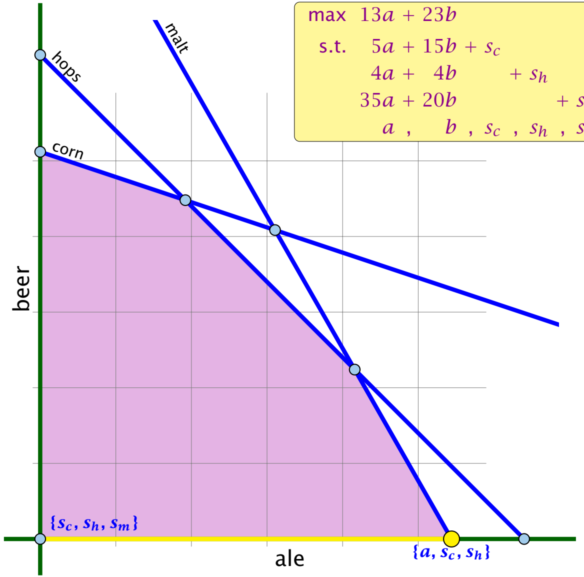
Geometric View of Pivoting



$$
\begin{array}{lrcl}
\max & 13a + 23b & & \\
\text{s.t.} & 5a + 15b + s_c & = & 480 \\
 & 4a + 4b \quad + s_h & = & 160 \\
 & 35a + 20b \qquad + s_m & = & 1190 \\
 & a\;,\; b\;,\; s_c\;,\; s_h\;,\; s_m & \geq & 0
\end{array}
$$

Matrix View

Let our linear program be

$$
\begin{array}{ccccc}
c_B^T x_B & + & c_N^T x_N & = & Z \\
A_B x_B & + & A_N x_N & = & b \\
x_B & , & x_N & \geq & 0
\end{array}
$$

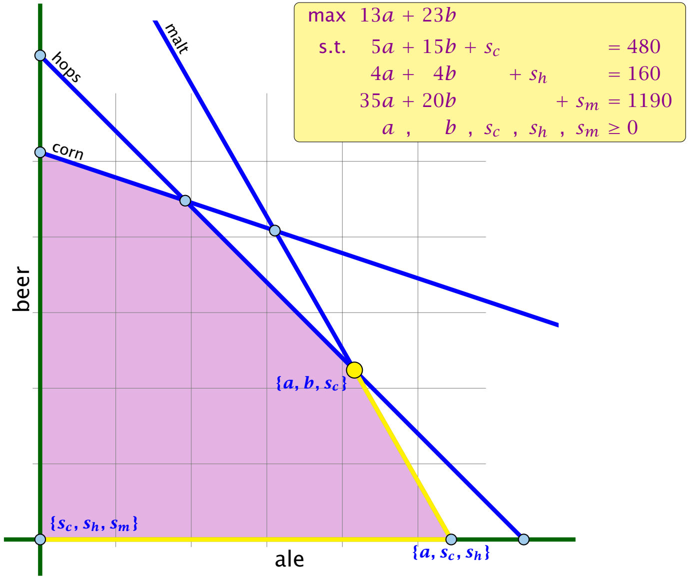
The simplex tableaux for basis B is

$$
\begin{array}{ccccc}
 & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\
I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\
x_B & , & x_N & \geq & 0
\end{array}
$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

Geometric View of Pivoting



Matrix View

Let our linear program be

$$\begin{array}{rcrcl} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

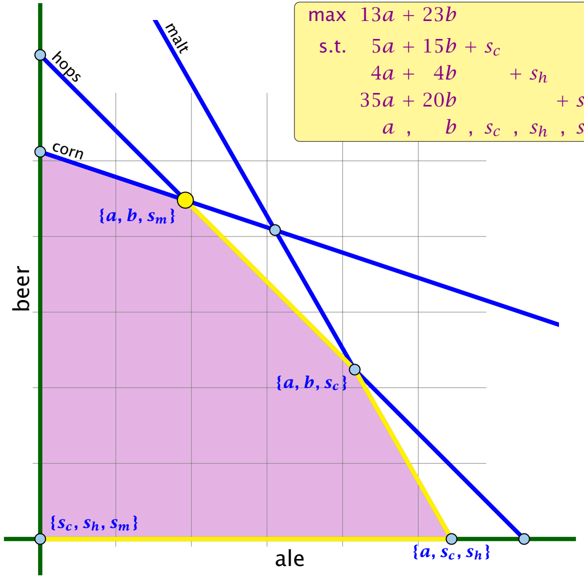
The simplex tableaux for basis B is

$$\begin{array}{rcrcl} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

Geometric View of Pivoting



$$
\begin{array}{llllllll}
\max & 13a + 23b & & & & & & \\
\text{s.t.} & 5a + 15b & + s_c & & & = & 480 \\
 & 4a + 4b & & + s_h & & = & 160 \\
 & 35a + 20b & & & + s_m & = & 1190 \\
 & a \;,\; b & ,\; s_c & ,\; s_h & ,\; s_m & \geq & 0
\end{array}
$$

Matrix View

Let our linear program be

$$
\begin{array}{ccccc}
c_B^T x_B & + & c_N^T x_N & = & Z \\
A_B x_B & + & A_N x_N & = & b \\
x_B & , & x_N & \geq & 0
\end{array}
$$

The simplex tableaux for basis B is

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\begin{array}{ccccc}
 & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\
I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\
x_B & , & x_N & \geq & 0
\end{array}
$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

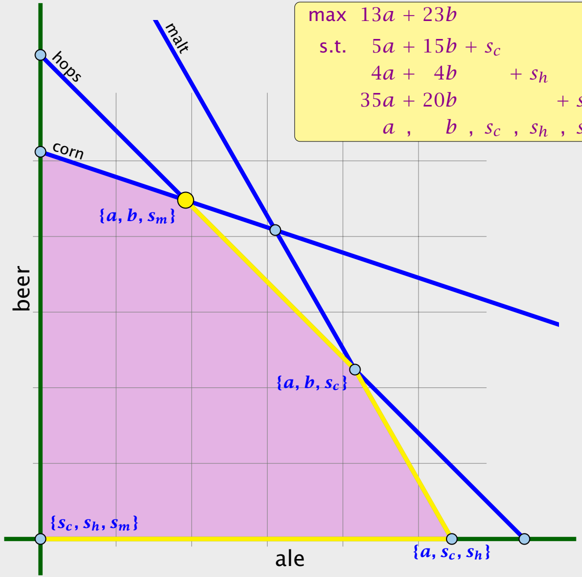
Algebraic Definition of Pivoting

- Given basis B with BFS x^*.
- Choose index $j \notin B$ in order to increase x_j^* from 0 to $\theta > 0$.
- Go from x^* to $x^* + \theta \cdot d$.

Requirements for d:

Geometric View of Pivoting

$$\begin{aligned}\max\ & 13a + 23b \\ \text{s.t. } & 5a + 15b + s_c = 480 \\ & 4a + 4b + s_h = 160 \\ & 35a + 20b + s_m = 1190 \\ & a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m \geq 0\end{aligned}$$

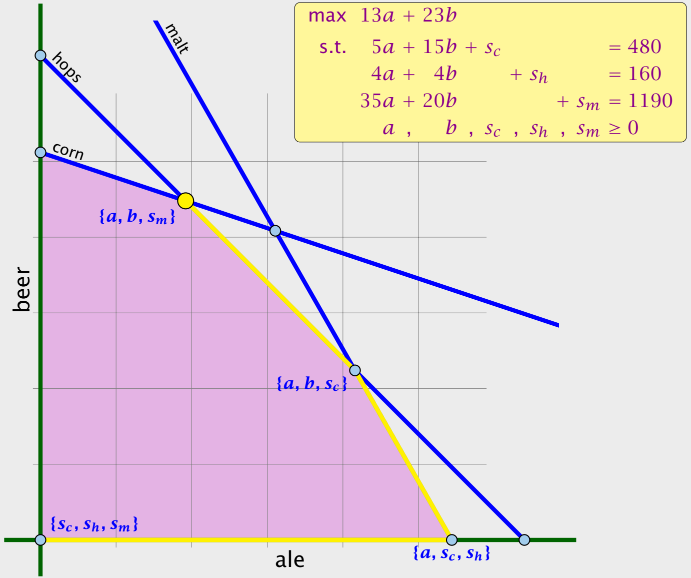


Algebraic Definition of Pivoting

- Given basis B with BFS x^*.
- Choose index $j \notin B$ in order to increase x_j^* from 0 to $\theta > 0$.
  - Other non-basis variables should stay at 0.
  - Basis variables change to maintain feasibility.
- Go from x^* to $x^* + \theta \cdot d$.

Requirements for d:

Geometric View of Pivoting



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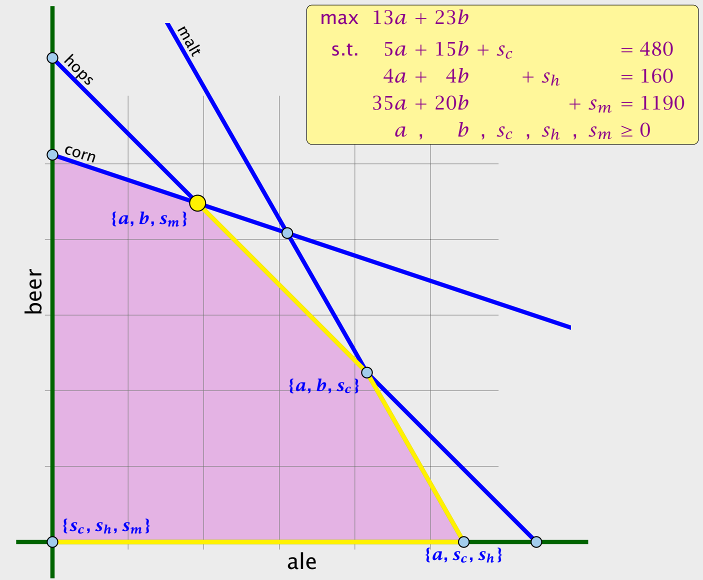

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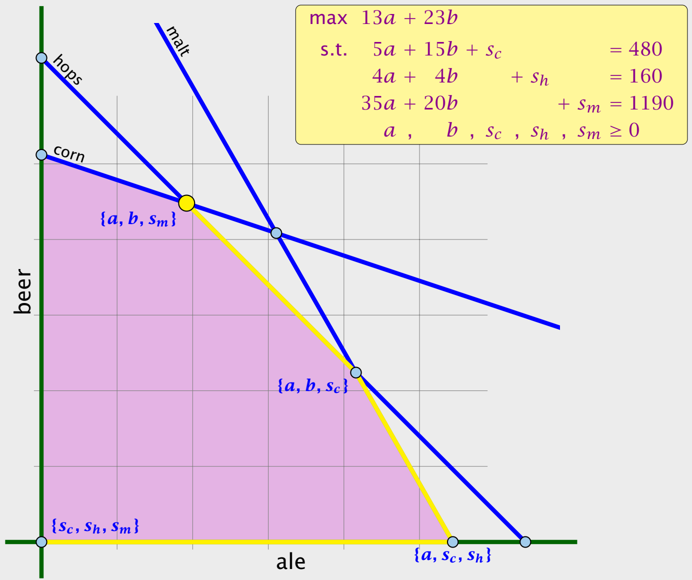
Requirements for d:

Geometric View of Pivoting



Algebraic Definition of Pivoting

- Given basis B with BFS x^*.
- Choose index $j \notin B$ in order to increase x_j^* from 0 to $\theta > 0$.
  - Other non-basis variables should stay at 0.
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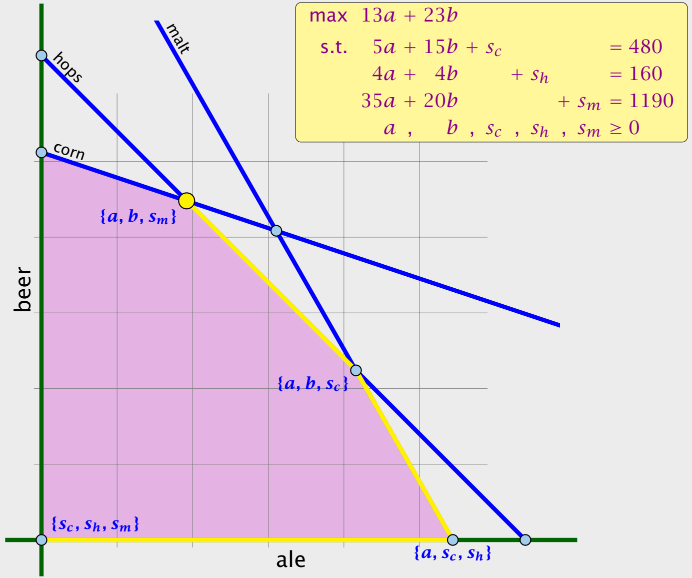
- Go from x^* to $x^* + \theta \cdot d$.

Requirements for d:

Geometric View of Pivoting



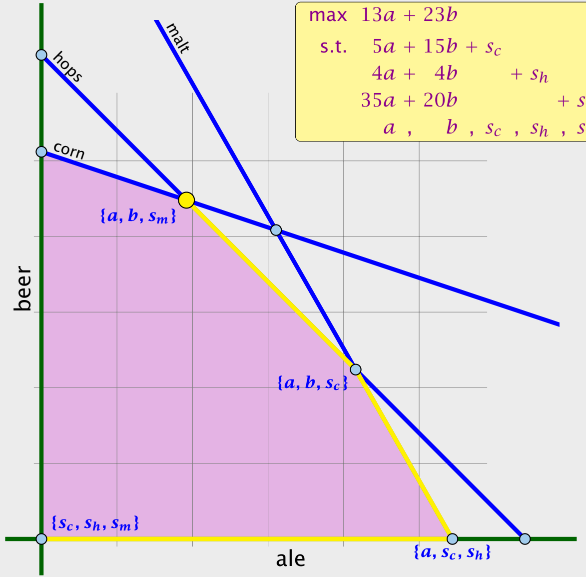
Algebraic Definition of Pivoting

- Given basis B with BFS x^*.
- Choose index $j \notin B$ in order to increase x_j^* from 0 to $\theta > 0$.
  - Other non-basis variables should stay at 0.
  - Basis variables change to maintain feasibility.
- Go from x^* to $x^* + \theta \cdot d$.

Requirements for d:

- $d_j = 1$ (normalization)
- $d_\ell = 0,\ \ell \notin B,\ \ell \neq j$
- $A(x^* + \theta d) = b$ must hold. Hence $Ad = 0$.
- Altogether: $A_B d_B + A_{*j} = Ad = 0$, which gives $d_B = -A_B^{-1} A_{*j}$.

Geometric View of Pivoting



$$
\begin{array}{lrcl}
\max & 13a + 23b & & \\
\text{s.t.} & 5a + 15b + s_c & = & 480 \\
 & 4a + 4b + s_h & = & 160 \\
 & 35a + 20b + s_m & = & 1190 \\
 & a,\ b,\ s_c,\ s_h,\ s_m & \geq & 0
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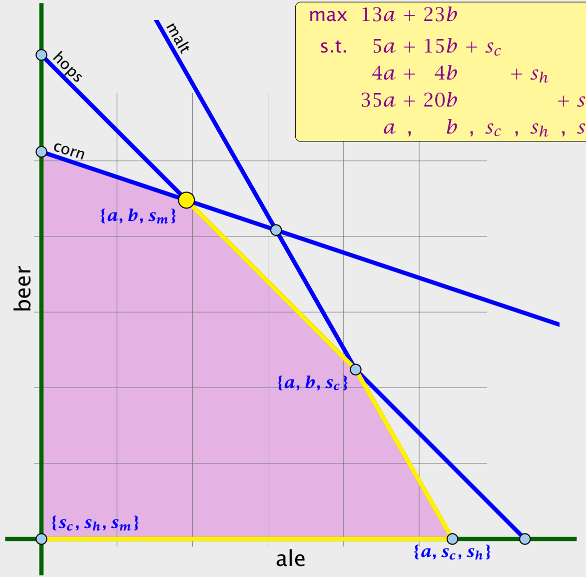
Geometric View of Pivoting

$$\begin{array}{lrcl} \max & 13a + 23b & & \\ \text{s.t.} & 5a + 15b + s_c & = & 480 \\ & 4a + 4b + s_h & = & 160 \\ & 35a + 20b + s_m & = & 1190 \\ & a, b, s_c, s_h, s_m & \geq & 0 \end{array}$$



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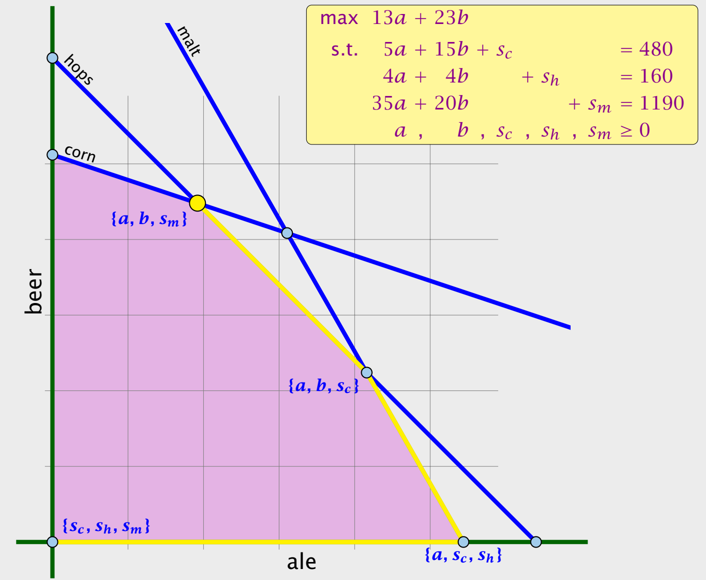

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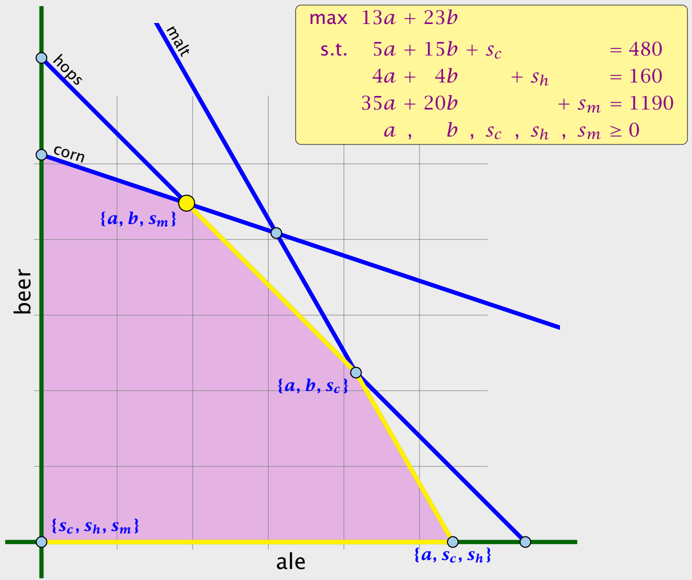

Algebraic Definition of Pivoting

Definition 2 (j-th basis direction)
Let B be a basis, and let $j \notin B$. The vector d with $d_j = 1$ and $d_\ell = 0, \ell \notin B, \ell \neq j$ and $d_B = -A_B^{-1}A_{*j}$ is called the j-th basis direction for B.

Going from x^* to $x^* + \theta \cdot d$ the objective function changes by

$$\theta \cdot c^T d = \theta(c_j - c_B^T A_B^{-1} A_{*j})$$

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Let our linear program be

$$\begin{array}{rcrcl} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

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What does it mean that the ratio b_i/A_{ie} (and hence A_{ie}) is negative for a constraint?

This means that the corresponding basic variable will increase if we increase b. Hence, there is no danger of this basic variable becoming negative

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The objective function may not increase!

Because a variable x_ℓ with $\ell \in B$ is already 0.

The set of inequalities is degenerate (also the basis is degenerate).

Definition 4 (Degeneracy)
A BFS x^* is called degenerate if the set $J = \{j \mid x_j^* > 0\}$ fulfills $|J| < m$.

It is possible that the algorithm cycles, i.e., it cycles through a sequence of different bases without ever terminating. Happens, very rarely in practise.

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Does it always increase?

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The objective function may not increase!

Because a variable x_ℓ with $\ell \in B$ is already 0.

The set of inequalities is degenerate (also the basis is degenerate).

Definition 4 (Degeneracy)

A BFS x^* is called degenerate if the set $J = \{j \mid x_j^* > 0\}$ fulfills $|J| < m$.

It is possible that the algorithm cycles, i.e., it cycles through a sequence of different bases without ever terminating. Happens, very rarely in practise.

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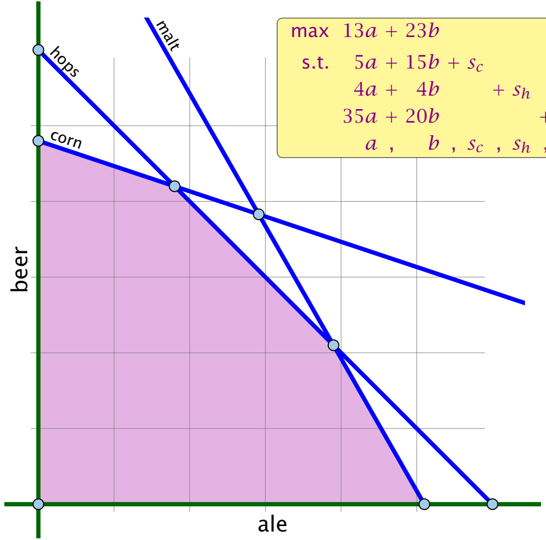

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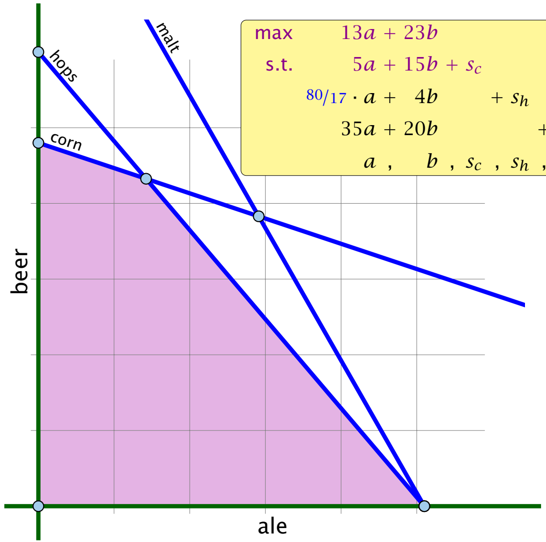

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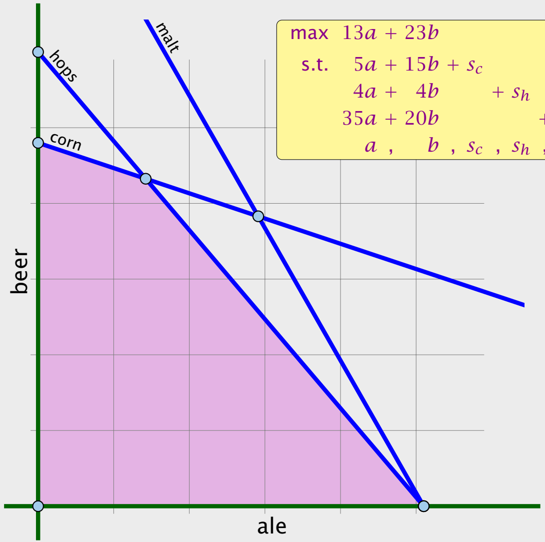

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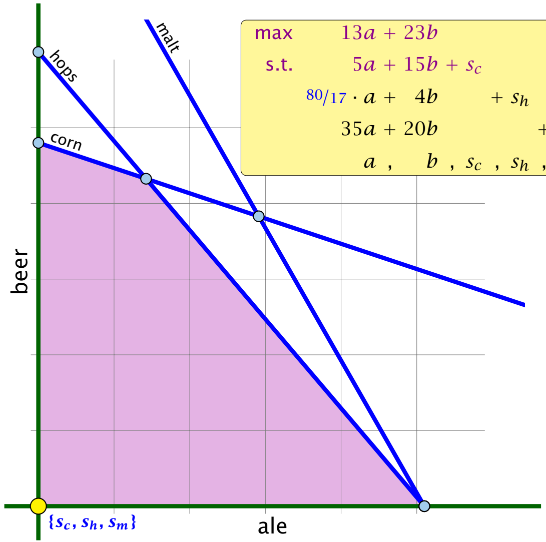

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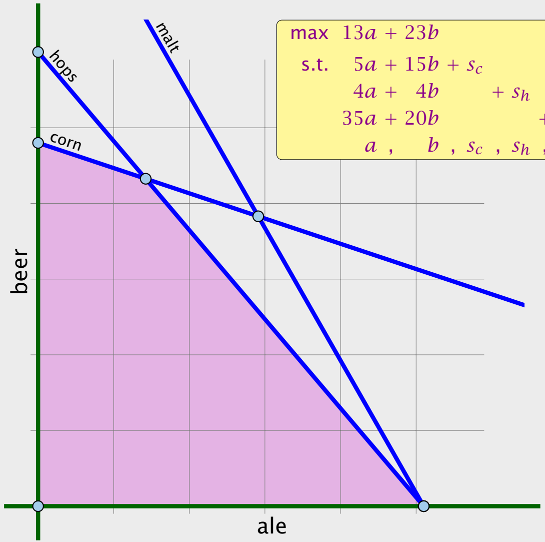

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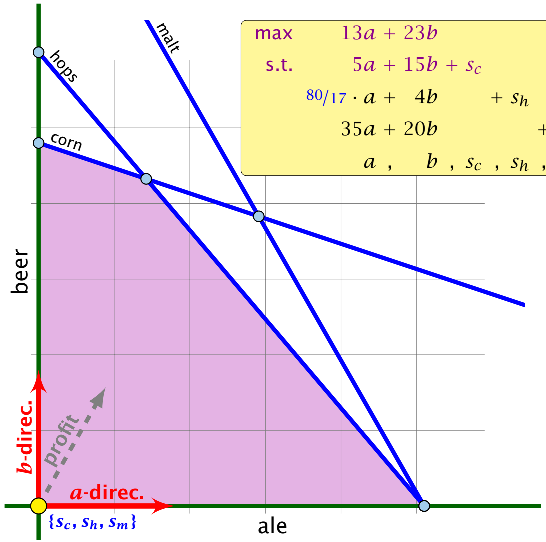

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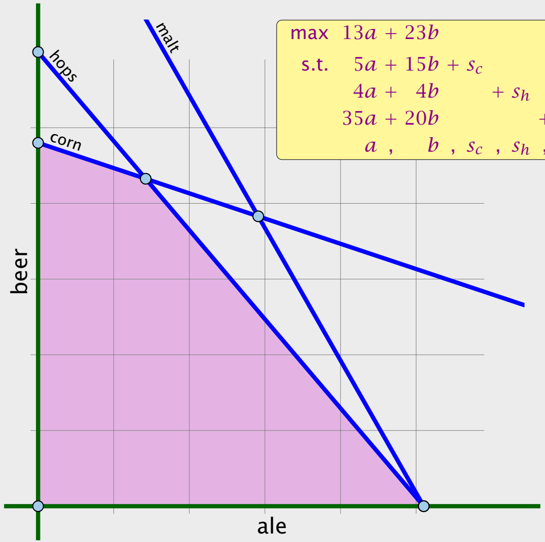

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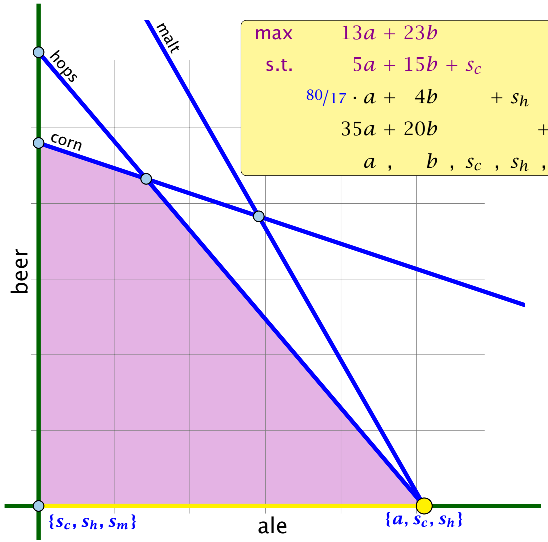

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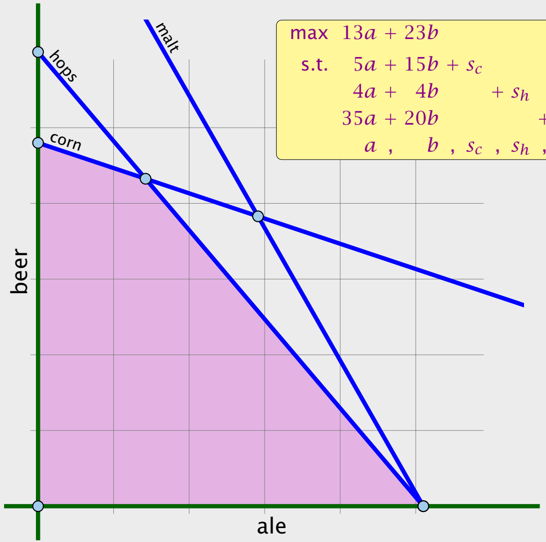

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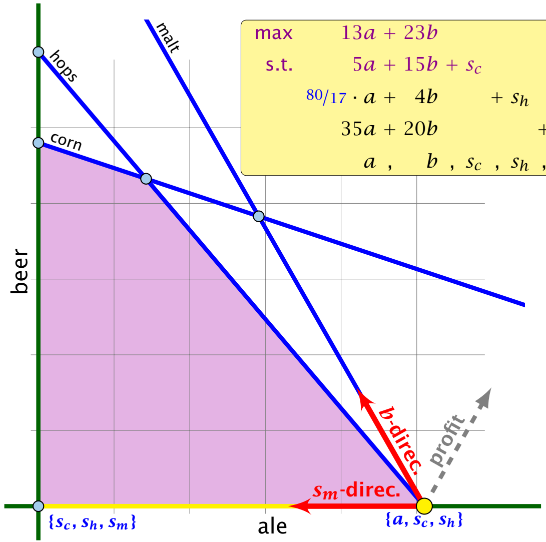

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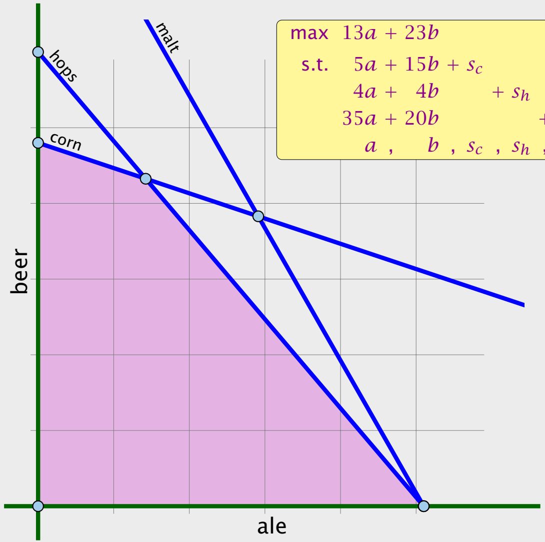

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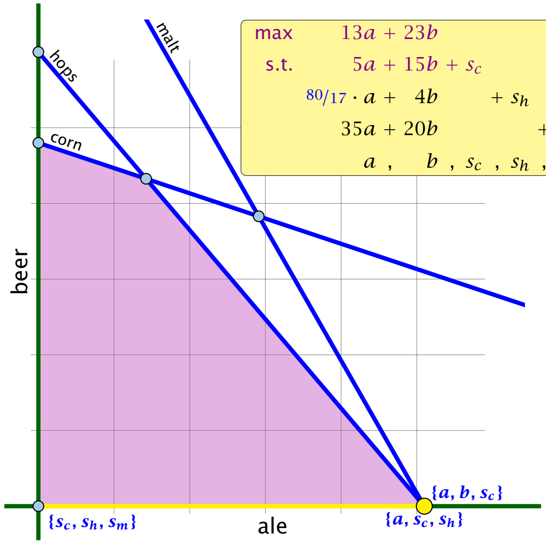

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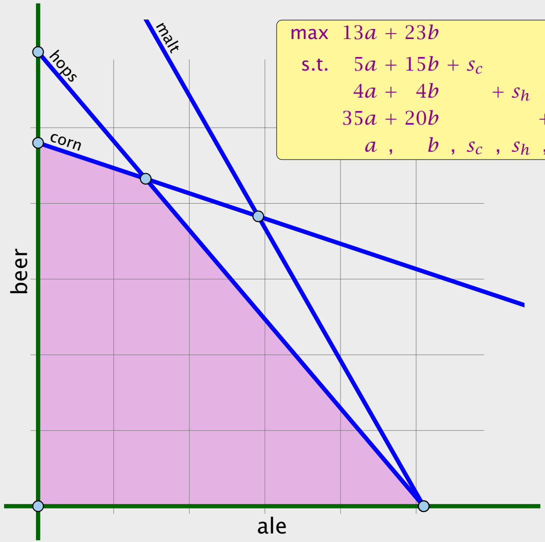

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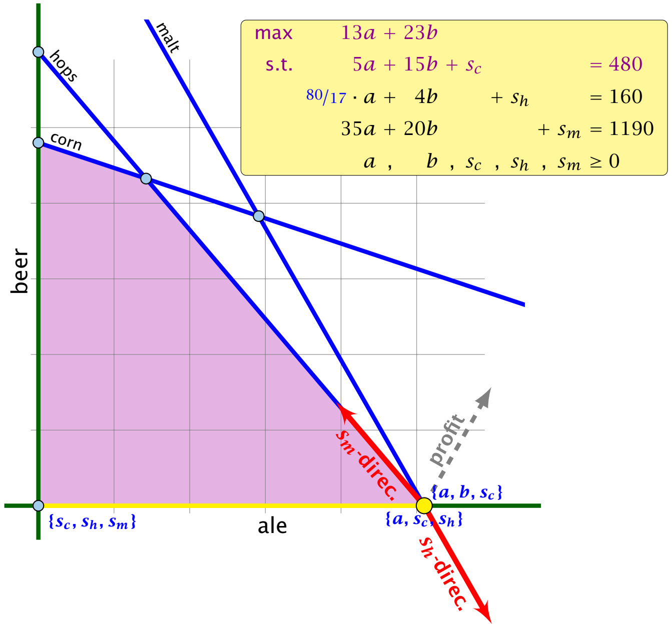

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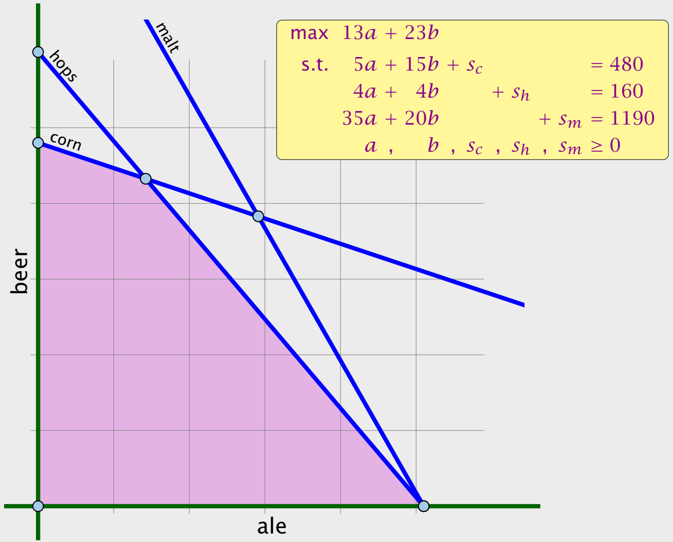

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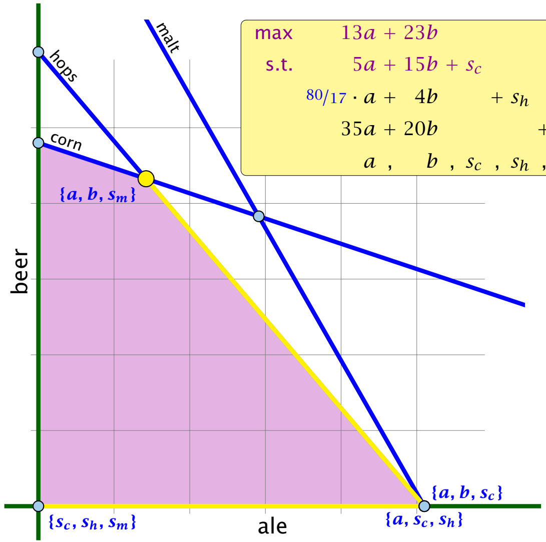

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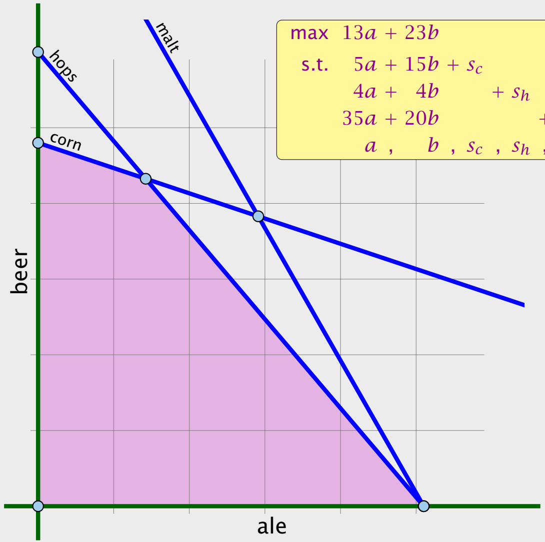

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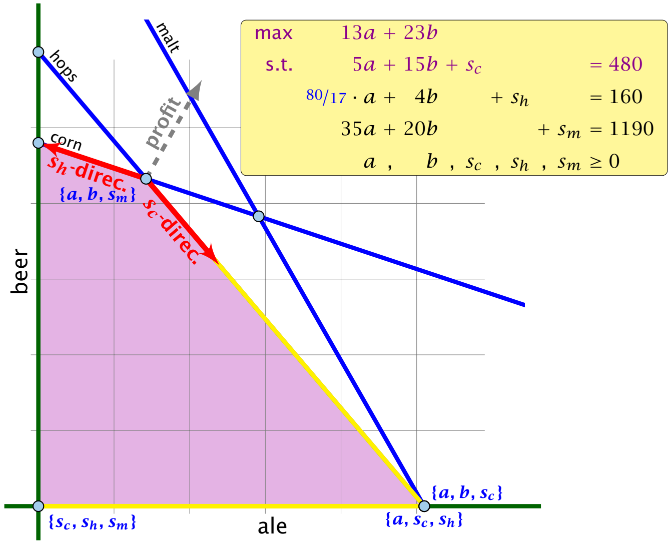

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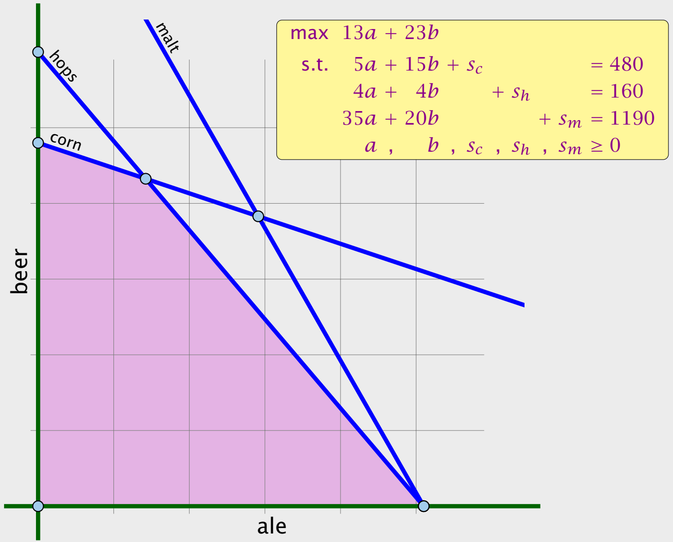

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- We can choose a column e as an entering variable if $\tilde{c}_e > 0$ ($\tilde{c}_e$ is reduced cost for x_e).
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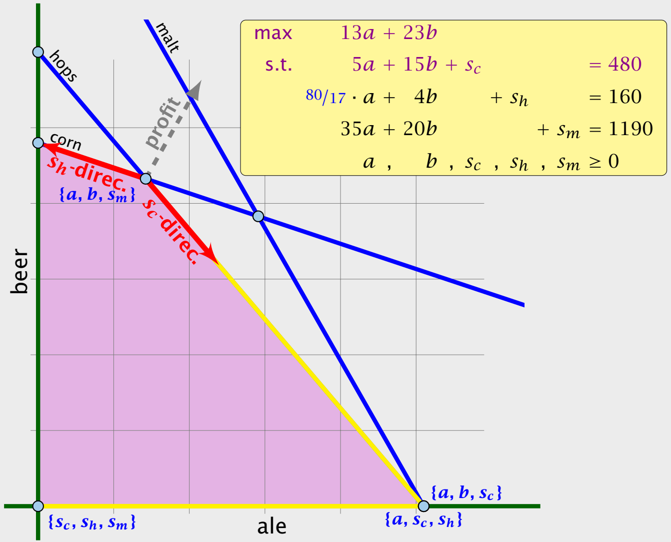

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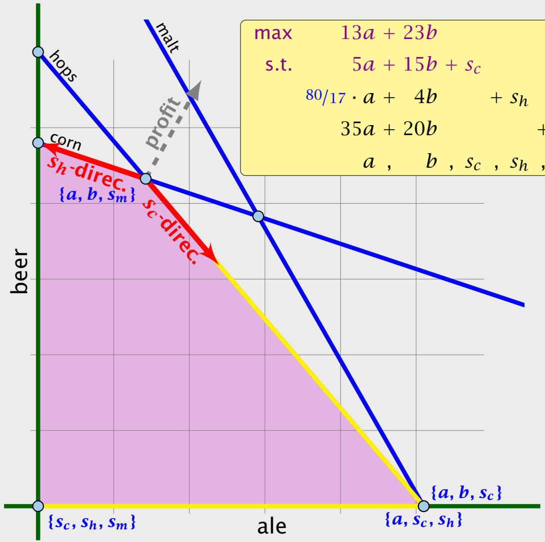

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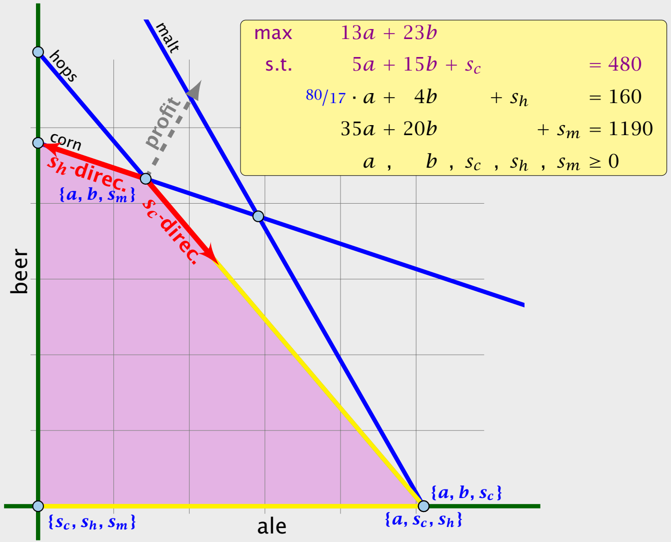

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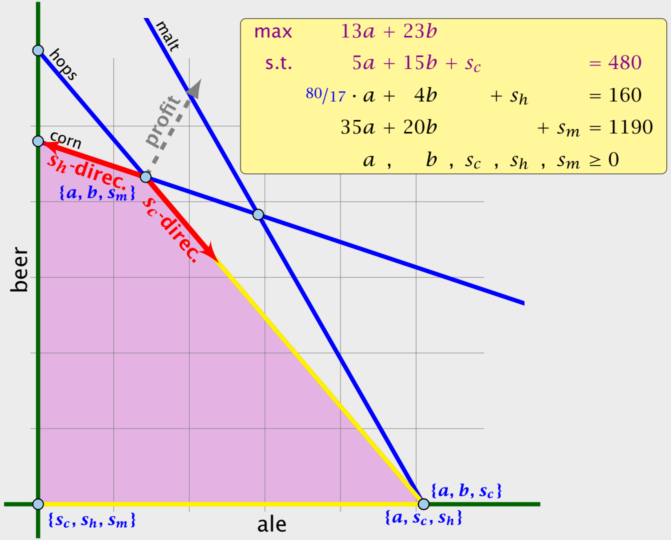

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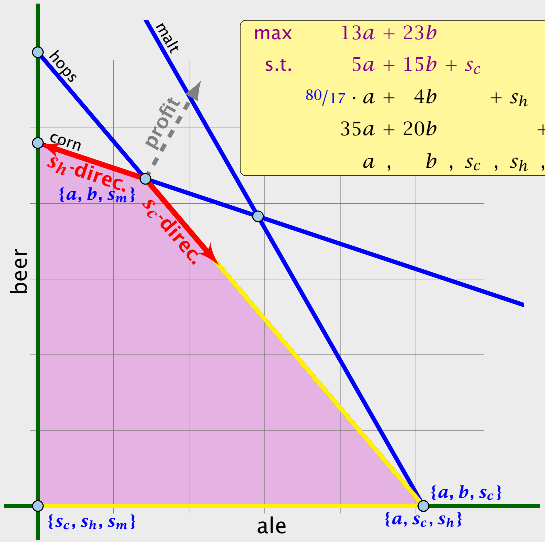

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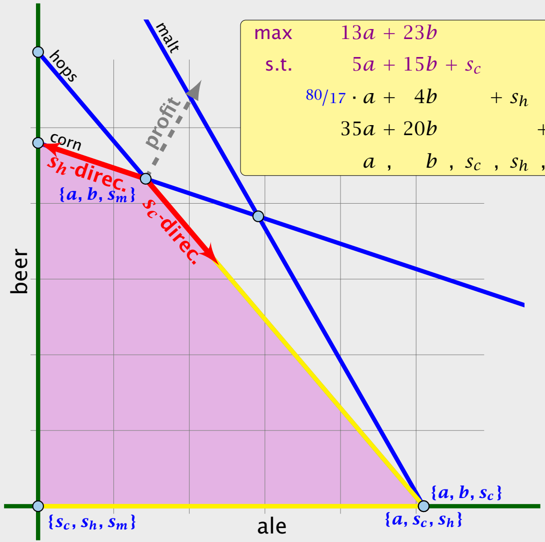

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How do we come up with an initial solution?

- $Ax \leq b, x \geq 0$, and $\boldsymbol{b} \geq \mathbf{0}$.
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How do we come up with an initial solution?

- $Ax \leq b, x \geq 0$, and $\boldsymbol{b} \geq \boldsymbol{0}$.
- The standard slack from for this problem is $Ax + Is = b, x \geq 0, s \geq 0$, where s denotes the vector of slack variables.
- Then $s = b$, $x = 0$ is a basic feasible solution (how?).
- We directly can start the simplex algorithm.

How do we find an initial basic feasible solution for an arbitrary problem?

Termination

What do we have so far?
Suppose we are given an initial feasible solution to an LP. If the LP is non-degenerate then Simplex will terminate.

Note that we either terminate because the min-ratio test fails and we can conclude that the LP is unbounded, or we terminate because the vector of reduced cost is non-positive. In the latter case we have an optimum solution.

Two phase algorithm

Suppose we want to maximize $c^T x$ s.t. $Ax = b, x \geq 0$.

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1. Multiply all rows with $b_i < 0$ by -1.
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3. If $\sum_i v_i > 0$ then the original problem is infeasible.
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5. From this you can get basic feasible solution.
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Optimality

Lemma 5
Let B be a basis and x^* a BFS corresponding to basis B. $\tilde{c} \le 0$ implies that x^* is an optimum solution to the LP.

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