16 Rounding Data + Dynamic Programming

Knapsack:
Given a set of items $\{1, \ldots, n\}$, where the i-th item has weight $w_i \in \mathbb{N}$ and profit $p_i \in \mathbb{N}$, and given a threshold W. Find a subset $I \subseteq \{1, \ldots, n\}$ of items of total weight at most W such that the profit is maximized (we can assume each $w_i \leq W$).

$$\begin{array}{lrcl} \max & \sum_{i=1}^{n} p_i x_i & & \\ \text{s.t.} & \sum_{i=1}^{n} w_i x_i & \leq & W \\ \forall i \in \{1, \ldots, n\} & x_i & \in & \{0, 1\} \end{array}$$

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Algorithm 1 Knapsack

$A(1) \leftarrow [(0,0),(p_1,w_1)]$
for $j \leftarrow 2$ to n **do**
  $A(j) \leftarrow A(j-1)$
  for each $(p,w) \in A(j-1)$ **do**
    if $w + w_j \leq W$ **then**
      add $(p+p_j, w+w_j)$ to $A(j)$
    remove dominated pairs from $A(j)$
return $\max_{(p,w)\in A(n)} p$

The running time is $\mathcal{O}(n \cdot \min\{W,P\})$, where $P = \sum_i p_i$ is the total profit of all items. This is only pseudo-polynomial.

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Definition 2
An algorithm is said to have pseudo-polynomial running time if the running time is polynomial when the numerical part of the input is encoded in unary.

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- Set $\mu := \epsilon M/n$.

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Let S be the set of items returned by the algorithm, and let O be an optimum set of items.

$$\sum_{i\in S} p_i$$

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Scheduling Revisited

The previous analysis of the scheduling algorithm gave a makespan of

$$\frac{1}{m}\sum_{j\neq\ell} p_j + p_\ell$$

where ℓ is the last job to complete.

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Partition the input into long jobs and short jobs.

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We still have a cost of

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where ℓ is the last job (this only requires that all machines are busy before time S_ℓ).

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If ℓ is a long job, then the schedule must be optimal, as it consists of an optimal schedule of long jobs plus a schedule for short jobs.

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If ℓ is a short job its length is at most

$$p_\ell \le \sum_j p_j/(mk)$$

which is at most $C^*_{\max}/k$.

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Hence we get a schedule of length at most

$$\left(1 + \frac{1}{k}\right) C^*_{\max}$$

There are at most km long jobs. Hence, the number of possibilities of scheduling these jobs on m machines is at most m^{km}, which is constant if m is constant. Hence, it is easy to implement the algorithm in polynomial time.

Theorem 3

The above algorithm gives a polynomial time approximation scheme (PTAS) for the problem of scheduling n jobs on m identical machines if m is constant.

We choose $k = \lceil \frac{1}{\epsilon} \rceil$.

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How to get rid of the requirement that m is constant?

We first design an algorithm that works as follows:
On input of T it either finds a schedule of length $(1+\frac{1}{k})T$ or certifies that no schedule of length at most T exists (assume $T \geq \frac{1}{m}\sum_j p_j$).

We partition the jobs into long jobs and short jobs:

- A job is long if its size is larger than T/k.
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We choose $k = \lceil \frac{1}{\epsilon} \rceil$.

- We round all long jobs down to multiples of T/k^2.
- For these rounded sizes we first find an optimal schedule.
- If this schedule does not have length at most T we conclude that also the original sizes don't allow such a schedule.
- If we have a good schedule we extend it by adding the short jobs according to the LPT rule.

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On input of T it either finds a schedule of length $(1+\frac{1}{k})T$ or certifies that no schedule of length at most T exists (assume $T \geq \frac{1}{m}\sum_j p_j$).

We partition the jobs into long jobs and short jobs:

- A job is long if its size is larger than T/k.
- Otw. it is a short job.

- We round all long jobs down to multiples of T/k^2.
- For these rounded sizes we first find an optimal schedule.
- If this schedule does not have length at most T we conclude that also the original sizes don't allow such a schedule.
- If we have a good schedule we extend it by adding the short jobs according to the LPT rule.

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After the first phase the rounded sizes of the long jobs assigned to a machine add up to at most T.

There can be at most k (long) jobs assigned to a machine as otw. their rounded sizes would add up to more than T (note that the rounded size of a long job is at least T/k).

Since, jobs had been rounded to multiples of T/k^2 going from rounded sizes to original sizes gives that the Makespan is at most

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During the second phase there always must exist a machine with load at most T, since T is larger than the average load.

Assigning the current (short) job to such a machine gives that the new load is at most

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Running Time for scheduling large jobs: There should not be a job with rounded size more than T as otw. the problem becomes trivial.

Hence, any large job has rounded size of $\frac{i}{k^2}T$ for $i \in \{k, \dots, k^2\}$. Therefore the number of different inputs is at most n^{k^2} (described by a vector of length k^2 where, the i-th entry describes the number of jobs of size $\frac{i}{k^2}T$). This is polynomial.

The schedule/configuration of a particular machine x can be described by a vector of length k^2 where the i-th entry describes the number of jobs of rounded size $\frac{i}{k^2}T$ assigned to x. There are only $(k+1)^{k^2}$ different vectors.

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Let $\mathrm{OPT}(n_1,\dots,n_{k^2})$ be the number of machines that are required to schedule input vector $(n_1,\dots,n_{k^2})$ with Makespan at most T.

If $\mathrm{OPT}(n_1,\dots,n_{k^2}) \leq m$ we can schedule the input.

We have

$$\mathrm{OPT}(n_1,\dots,n_{k^2}) = \begin{cases} 0 & (n_1,\dots,n_{k^2}) = 0 \\ 1 + \min\limits_{(s_1,\dots,s_{k^2})\in C} \mathrm{OPT}(n_1 - s_1,\dots,n_{k^2} - s_{k^2}) & (n_1,\dots,n_{k^2}) \geq 0 \\ \infty & \text{otw.} \end{cases}$$

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We can turn this into a PTAS by choosing $k = \lceil 1/\epsilon \rceil$ and using binary search. This gives a running time that is exponential in $1/\epsilon$.

Can we do better?
Scheduling on identical machines with the goal of minimizing Makespan is a strongly NP-complete problem.

Theorem 4
There is no FPTAS for problems that are strongly NP-hard.

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- Suppose we have an instance with polynomially bounded processing times $p_i \leq q(n)$
- We set $k := \lceil 2nq(n) \rceil \geq 2\,\mathrm{OPT}$
- Then

$$\mathrm{ALG} \leq \left(1 + \frac{1}{k}\right)\mathrm{OPT} \leq \mathrm{OPT} + \frac{1}{2}$$

- But this means that the algorithm computes the optimal solution as the optimum is integral.
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More General

Let $\mathrm{OPT}(n_1, \dots, n_A)$ be the number of machines that are required to schedule input vector $(n_1, \dots, n_A)$ with Makespan at most T (A: number of different sizes).

If $\mathrm{OPT}(n_1, \dots, n_A) \le m$ we can schedule the input.

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where C is the set of all configurations.

$|C| \le (B+1)^A$, where B is the number of jobs that possibly can fit on the same machine.

The running time is then $O((B+1)^A n^A)$ because the dynamic programming table has just n^A entries.

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If $\mathrm{OPT}(n_1, \dots, n_A) \le m$ we can schedule the input.

$$\mathrm{OPT}(n_1, \dots, n_A) = \begin{cases} 0 & (n_1, \dots, n_A) = 0 \\ 1 + \min\limits_{(s_1, \dots, s_A) \in C} \mathrm{OPT}(n_1 - s_1, \dots, n_A - s_A) & (n_1, \dots, n_A) \gneq 0 \\ \infty & \text{otw.} \end{cases}$$

where C is the set of all configurations.

$|C| \le (B+1)^A$, where B is the number of jobs that possibly can fit on the same machine.

The running time is then $O((B+1)^A n^A)$ because the dynamic programming table has just n^A entries.

- Suppose we have an instance with polynomially bounded processing times $p_i \le q(n)$
- We set $k := \lceil 2nq(n) \rceil \ge 2\,\mathrm{OPT}$
- Then

$$\mathrm{ALG} \le \left(1 + \frac{1}{k}\right)\mathrm{OPT} \le \mathrm{OPT} + \frac{1}{2}$$

- But this means that the algorithm computes the optimal solution as the optimum is integral.
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Bin Packing

Given n items with sizes $s_1, \dots, s_n$ where

$$1 > s_1 \geq \cdots \geq s_n > 0 \ .$$

Pack items into a minimum number of bins where each bin can hold items of total size at most 1.

Theorem 5

There is no ρ-approximation for Bin Packing with $\rho < 3/2$ unless $P = NP$.

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- In the partition problem we are given positive integers $b_1, \ldots, b_n$ with $B = \sum_i b_i$ even. Can we partition the integers into two sets S and T s.t.

$$\sum_{i \in S} b_i = \sum_{i \in T} b_i \quad ?$$

- We can solve this problem by setting $s_i := 2b_i/B$ and asking whether we can pack the resulting items into 2 bins or not.
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Definition 6

An asymptotic polynomial-time approximation scheme (APTAS) is a family of algorithms $\{A_\epsilon\}$ along with a constant c such that A_ϵ returns a solution of value at most $(1+\epsilon)\mathrm{OPT}+c$ for minimization problems.

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Again we can differentiate between small and large items.

Lemma 7

Any packing of items into ℓ bins can be extended with items of size at most γ s.t. we use only $\max\{\ell, \frac{1}{1-\gamma}\text{SIZE}(I) + 1\}$ bins, where $\text{SIZE}(I) = \sum_i s_i$ is the sum of all item sizes.

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- If after Greedy we use more than ℓ bins, all bins (apart from the last) must be full to at least $1 - \gamma$.
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Choose $\gamma = \epsilon/2$. Then we either use ℓ bins or at most

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Linear Grouping:

Generate an instance I' (for large items) as follows.

- Order large items according to size.
- Let the first k items belong to group 1; the following k items belong to group 2; etc.
- Delete items in the first group;
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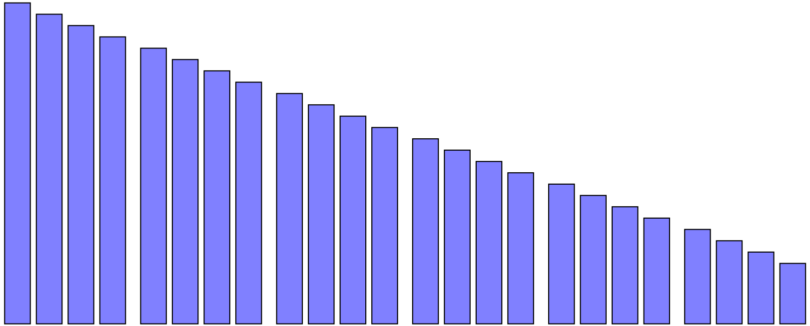

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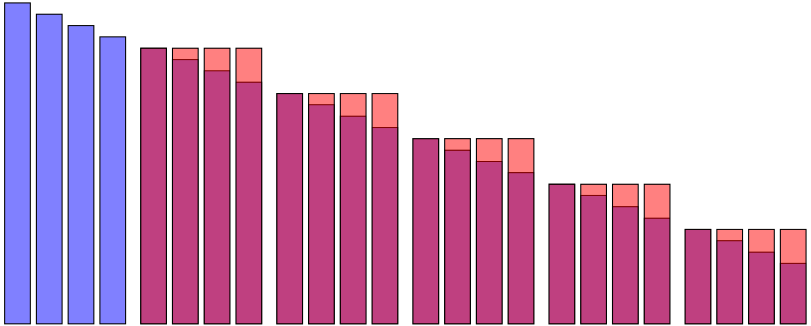

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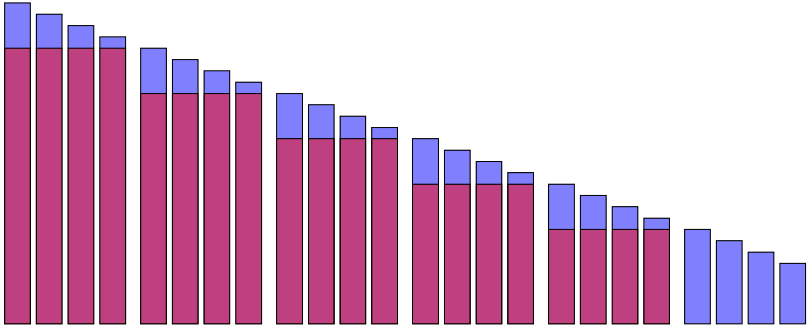

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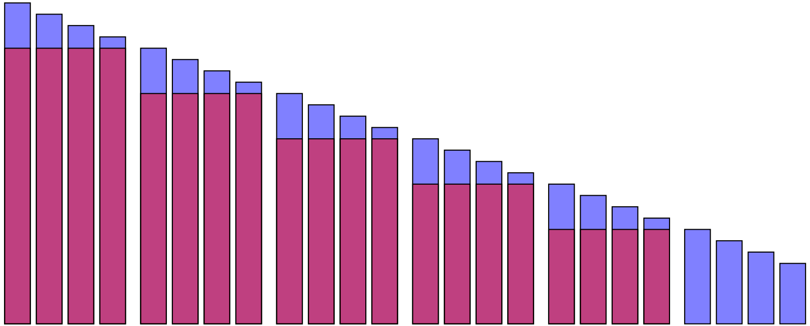

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$\mathrm{OPT}(I') \le \mathrm{OPT}(I) \le \mathrm{OPT}(I') + k$

Proof 1:

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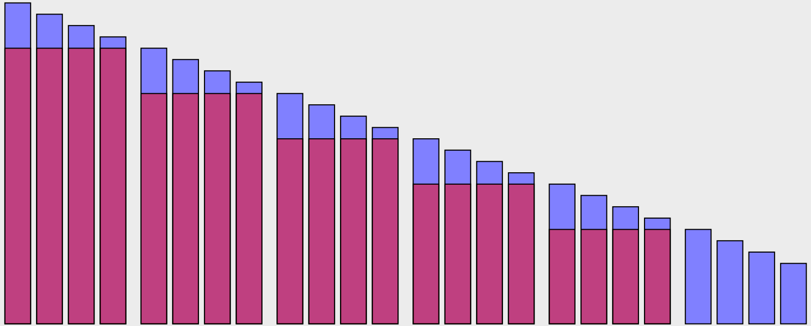

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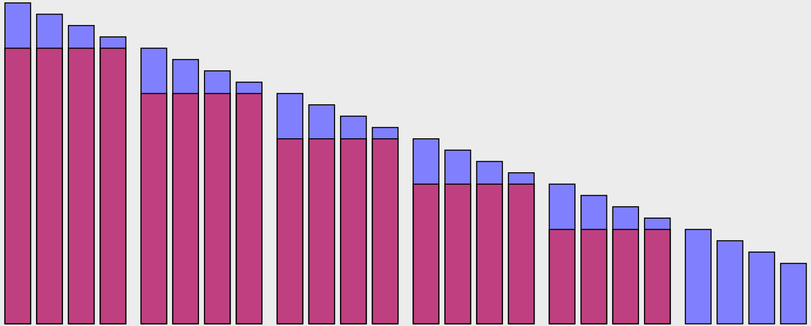

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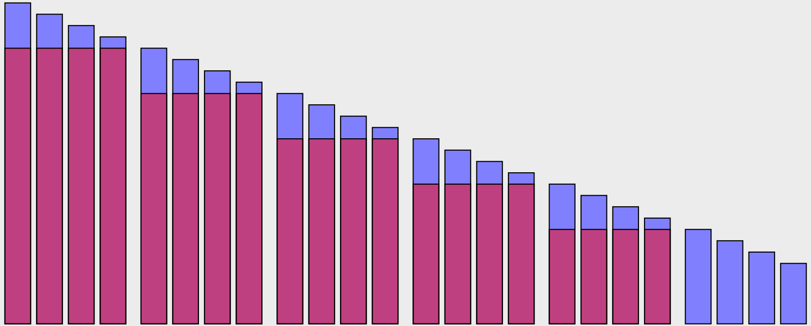

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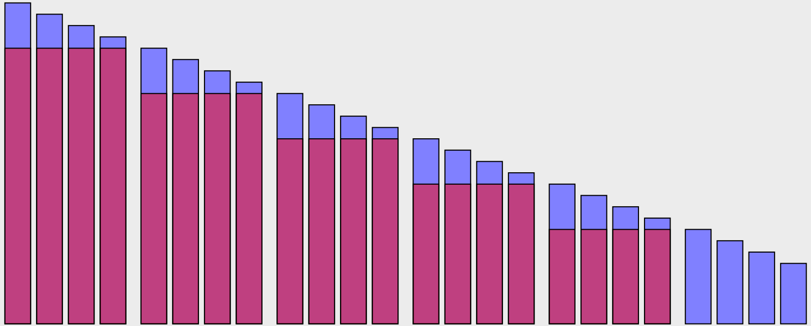

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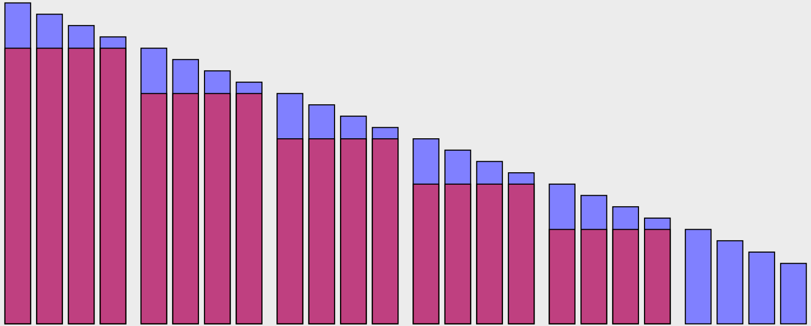

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- ...

Lemma 8

$\mathrm{OPT}(I') \leq \mathrm{OPT}(I) \leq \mathrm{OPT}(I') + k$

Proof 1:

- Any bin packing for I gives a bin packing for I' as follows.
- Pack the items of group 2, where in the packing for I the items for group 1 have been packed;
- Pack the items of groups 3, where in the packing for I the items for group 2 have been packed;
- ...

Assume that our instance does not contain pieces smaller than $\epsilon/2$. Then $\mathrm{SIZE}(I) \geq \epsilon n/2$.

We set $k = \lfloor \epsilon \mathrm{SIZE}(I) \rfloor$.

Then $n/k \leq n/\lfloor \epsilon^2 n/2 \rfloor \leq 4/\epsilon^2$ (here we used $\lfloor \alpha \rfloor \geq \alpha/2$ for $\alpha \geq 1$).

Hence, after grouping we have a constant number of piece sizes ($4/\epsilon^2$) and at most a constant number ($2/\epsilon$) can fit into any bin.

We can find an optimal packing for such instances by the previous Dynamic Programming approach.

- cost (for large items) at most

$$\mathrm{OPT}(I') + k \leq \mathrm{OPT}(I) + \epsilon \mathrm{SIZE}(I) \leq (1+\epsilon)\mathrm{OPT}(I)$$

- running time $\mathcal{O}((\frac{2}{\epsilon} n)^{4/\epsilon^2})$.

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Can we do better?

In the following we show how to obtain a solution where the number of bins is only

$$\mathrm{OPT}(I) + \mathcal{O}(\log^2(\mathrm{SIZE}(I))) \enspace .$$

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Change of Notation:

- Group pieces of identical size.
- Let s_1 denote the largest size, and let b_1 denote the number of pieces of size s_1.
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Configuration LP

A possible packing of a bin can be described by an m-tuple $(t_1, \dots, t_m)$, where t_i describes the number of pieces of size s_i. Clearly,

$$\sum_i t_i \cdot s_i \le 1 \ .$$

We call a vector that fulfills the above constraint a configuration.

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Configuration LP

Let N be the number of configurations (exponential).

Let $T_1, \ldots, T_N$ be the sequence of all possible configurations (a configuration T_j has T_{ji} pieces of size s_i).

$$\begin{array}{llrcl} \min & & \sum_{j=1}^N x_j & & \\ \text{s.t.} & \forall i \in \{1 \ldots m\} & \sum_{j=1}^N T_{ji} x_j & \geq & b_i \\ & \forall j \in \{1, \ldots, N\} & x_j & \geq & 0 \\ & \forall j \in \{1, \ldots, N\} & x_j & \text{integral} & \end{array}$$

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We can assume that each item has size at least $1/\text{SIZE}(I)$.

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Harmonic Grouping

- Sort items according to size (monotonically decreasing).
- Process items in this order; close the current group if size of items in the group is at least 2 (or larger). Then open new group.
- I.e., G_1 is the smallest cardinality set of largest items s.t. total size sums up to at least 2. Similarly, for $G_2, \dots, G_{r-1}$.
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From the grouping we obtain instance I' as follows:

- Round all items in a group to the size of the largest group member.
- Delete all items from group G_1 and G_r.
- For groups $G_2, \ldots, G_{r-1}$ delete $n_i - n_{i-1}$ items.
- Observe that $n_i \geq n_{i-1}$.

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Lemma 10

The number of different sizes in I' is at most $\mathrm{SIZE}(I)/2$.

- Each group that survives (recall that G_1 and G_r are deleted) has total size at least 2.
- Hence, the number of surviving groups is at most $\mathrm{SIZE}(I)/2$.
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- Delete all items from group G_1 and G_r.
- For groups $G_2, \dots, G_{r-1}$ delete $n_i - n_{i-1}$ items.
- Observe that $n_i \geq n_{i-1}$.

Lemma 11
The total size of deleted items is at most $\mathcal{O}(\log(\mathrm{SIZE}(I)))$.

Lemma 10
The number of different sizes in I' *is at most* $\mathrm{SIZE}(I)/2$.

- Each group that survives (recall that G_1 and G_r are deleted) has total size at least 2.
- Hence, the number of surviving groups is at most $\mathrm{SIZE}(I)/2$.
- All items in a group have the same size in I'.

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The total size of deleted items is at most $\mathcal{O}(\log(\text{SIZE}(I)))$.

- The total size of items in G_1 and G_r is at most 6 as a group has total size at most 3.
- Consider a group G_i that has strictly more items than G_{i-1}.
- It discards $n_i - n_{i-1}$ pieces of total size at most

$$3\frac{n_i - n_{i-1}}{n_i} \le \sum_{j=n_{i-1}+1}^{n_i} \frac{3}{j}$$

since the smallest piece has size at most $3/n_i$.
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Algorithm 1 BinPack

```
if SIZE(I) < 10 then
    pack remaining items greedily
Apply harmonic grouping to create instance I'; pack discarded items in at most O(log(SIZE(I))) bins.
Let x be optimal solution to configuration LP
Pack ⌊x_j⌋ bins in configuration T_j for all j; call the packed instance I_1.
Let I_2 be remaining pieces from I'
Pack I_2 via BinPack(I_2)
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$$\mathrm{OPT}_{\mathrm{LP}}(I_1) + \mathrm{OPT}_{\mathrm{LP}}(I_2) \le \mathrm{OPT}_{\mathrm{LP}}(I') \le \mathrm{OPT}_{\mathrm{LP}}(I)$$

Proof:

- Each piece surviving in I' can be mapped to a piece in I of no lesser size. Hence, $\mathrm{OPT}_{\mathrm{LP}}(I') \le \mathrm{OPT}_{\mathrm{LP}}(I)$
- $\lfloor x_j \rfloor$ is feasible solution for I_1 (even integral).
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EADS II
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16.4 Advanced Rounding for Bin Packing

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Each level of the recursion partitions pieces into three types

1. Pieces discarded at this level.
2. Pieces scheduled because they are in I_1.
3. Pieces in I_2 are handed down to the next level.

Pieces of type 2 summed over all recursion levels are packed into at most $\mathrm{OPT}_{\mathrm{LP}}$ many bins.

Pieces of type 1 are packed into at most

$$\mathcal{O}(\log(\mathrm{SIZE}(I))) \cdot L$$

many bins where L is the number of recursion levels.

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Let $T_1, \dots, T_N$ be the sequence of all possible configurations (a configuration T_j has T_{ji} pieces of size s_i).

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Primal

$$\begin{array}{llrcl} \min & & \sum_{j=1}^{N} x_j & & \\ \text{s.t.} & \forall i \in \{1 \dots m\} & \sum_{j=1}^{N} T_{ji} x_j & \geq & b_i \\ & \forall j \in \{1, \dots, N\} & x_j & \geq & 0 \end{array}$$

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$$\begin{array}{llrcl} \max & & \sum_{i=1}^{m} y_i b_i & & \\ \text{s.t.} & \forall j \in \{1, \dots, N\} & \sum_{i=1}^{m} T_{ji} y_i & \leq & 1 \\ & \forall i \in \{1, \dots, m\} & y_i & \geq & 0 \end{array}$$

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Suppose that I am given variable assignment y for the dual.

How do I find a violated constraint?

I have to find a configuration $T_j = (T_{j1}, \dots, T_{jm})$ that

is feasible, i.e.,

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But this is the Knapsack problem.

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$$\begin{array}{llrcl} \min & & \sum_{j=1}^{N} x_j & & \\ \text{s.t.} & \forall i \in \{1 \dots m\} & \sum_{j=1}^{N} T_{ji} x_j & \ge & b_i \\ & \forall j \in \{1, \dots, N\} & x_j & \ge & 0 \end{array}$$

Dual

$$\begin{array}{llrcl} \max & & \sum_{i=1}^{m} y_i b_i & & \\ \text{s.t.} & \forall j \in \{1, \dots, N\} & \sum_{i=1}^{m} T_{ji} y_i & \le & 1 \\ & \forall i \in \{1, \dots, m\} & y_i & \ge & 0 \end{array}$$

Separation Oracle

We have FPTAS for Knapsack. This means if a constraint is violated with $1+\epsilon' = 1+\frac{\epsilon}{1-\epsilon}$ we find it, since we can obtain at least $(1-\epsilon)$ of the optimal profit.

The solution we get is feasible for:

Dual′

$$\begin{array}{llll} \max & & \sum_{i=1}^{m} y_i b_i & \\ \text{s.t.} & \forall j \in \{1,\dots,N\} & \sum_{i=1}^{m} T_{ji} y_i & \le 1+\epsilon' \\ & \forall i \in \{1,\dots,m\} & y_i & \ge 0 \end{array}$$

Primal′

$$\begin{array}{llll} \min & & (1+\epsilon')\sum_{j=1}^{N} x_j & \\ \text{s.t.} & \forall i \in \{1\dots m\} & \sum_{j=1}^{N} T_{ji} x_j & \ge b_i \\ & \forall j \in \{1,\dots,N\} & x_j & \ge 0 \end{array}$$

Separation Oracle

Suppose that I am given variable assignment y for the dual.

How do I find a violated constraint?

I have to find a configuration $T_j = (T_{j1},\dots,T_{jm})$ that

- is feasible, i.e.,

$$\sum_{i=1}^{m} T_{ji} \cdot s_i \le 1 \ ,$$

- and has a large profit

$$\sum_{i=1}^{m} T_{ji} y_i > 1$$

But this is the Knapsack problem.

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Dual'

$$\begin{array}{llrcl} \max & & \sum_{i=1}^m y_i b_i & & \\ \text{s.t.} & \forall j\in\{1,\dots,N\} & \sum_{i=1}^m T_{ji} y_i & \le & 1+\epsilon' \\ & \forall i\in\{1,\dots,m\} & y_i & \ge & 0 \end{array}$$

Primal'

$$\begin{array}{llrcl} \min & & (1+\epsilon')\sum_{j=1}^N x_j & & \\ \text{s.t.} & \forall i\in\{1\dots m\} & \sum_{j=1}^N T_{ji} x_j & \ge & b_i \\ & \forall j\in\{1,\dots,N\} & x_j & \ge & 0 \end{array}$$

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Primal′

$$\begin{array}{llrcl} \min & & (1+\epsilon') \sum_{j=1}^{N} x_j & & \\ \text{s.t.} & \forall i \in \{1 \dots m\} & \sum_{j=1}^{N} T_{ji} x_j & \geq & b_i \\ & \forall j \in \{1,\dots,N\} & x_j & \geq & 0 \end{array}$$

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If the value of the computed dual solution (which may be infeasible) is z then

$$\mathrm{OPT} \le z \le (1+\epsilon')\mathrm{OPT}$$

How do we get good primal solution (not just the value)?

- The constraints used when computing z certify that the solution is feasible for DUAL$'$.
- Suppose that we drop all unused constraints in DUAL. We will compute the same solution feasible for DUAL$'$.
- Let DUAL$''$ be DUAL without unused constraints.
- The dual to DUAL$''$ is PRIMAL where we ignore variables for which the corresponding dual constraint has not been used.
- The optimum value for PRIMAL$''$ is at most $(1+\epsilon')\mathrm{OPT}$.
- We can compute the corresponding solution in polytime.

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\end{array}$$

Primal$'$

$$\begin{array}{llrcl}
\min & & (1+\epsilon')\sum_{j=1}^N x_j & & \\
\text{s.t.} & \forall i \in \{1\dots m\} & \sum_{j=1}^N T_{ji} x_j & \ge & b_i \\
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This gives that overall we need at most

$$(1+\epsilon')\mathrm{OPT}_{\mathrm{LP}}(I) + \mathcal{O}(\log^2(\mathrm{SIZE}(I)))$$

bins.

We can choose $\epsilon' = \frac{1}{\mathrm{OPT}}$ as $\mathrm{OPT} \leq \#\text{items}$ and since we have a fully polynomial time approximation scheme (FPTAS) for knapsack.

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