

## 16 Rounding Data + Dynamic Programming

### Knapsack:

Given a set of items  $\{1, \dots, n\}$ , where the  $i$ -th item has weight  $w_i \in \mathbb{N}$  and profit  $p_i \in \mathbb{N}$ , and given a threshold  $W$ . Find a subset  $I \subseteq \{1, \dots, n\}$  of items of total weight at most  $W$  such that the profit is maximized (we can assume each  $w_i \leq W$ ).

$$\begin{array}{ll} \max & \sum_{i=1}^n p_i x_i \\ \text{s.t.} & \sum_{i=1}^n w_i x_i \leq W \\ & \forall i \in \{1, \dots, n\} \quad x_i \in \{0, 1\} \end{array}$$

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### Algorithm 1 Knapsack

```
1:  $A(1) \leftarrow [(0, 0), (p_1, w_1)]$ 
2: for  $j \leftarrow 2$  to  $n$  do
3:    $A(j) \leftarrow A(j - 1)$ 
4:   for each  $(p, w) \in A(j - 1)$  do
5:     if  $w + w_j \leq W$  then
6:       add  $(p + p_j, w + w_j)$  to  $A(j)$ 
7:       remove dominated pairs from  $A(j)$ 
8: return  $\max_{(p, w) \in A(n)} p$ 
```

The running time is  $\mathcal{O}(n \cdot \min\{W, P\})$ , where  $P = \sum_i p_i$  is the total profit of all items. This is only **pseudo-polynomial**.

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## Definition 2

An algorithm is said to have pseudo-polynomial running time if the running time is polynomial when the numerical part of the input is encoded in unary.

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# Scheduling Revisited

The previous analysis of the scheduling algorithm gave a makespan of

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Together with the observation that if each  $p_i \geq \frac{1}{3} C_{\max}^*$  then LPT is optimal this gave a  $4/3$ -approximation.

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**Idea:**

1. Find the optimum Makespan for the long jobs by brute force.

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**Idea:**

1. Find the optimum Makespan for the long jobs by brute force.
2. Then use the list scheduling algorithm for the short jobs, always assigning the next job to the least loaded machine.

We still have a cost of

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If  $\ell$  is a short job its length is at most

$$p_\ell \leq \sum_j p_j / (mk)$$

which is at most  $C_{\max}^* / k$ .

Hence we get a schedule of length at most

$$\left(1 + \frac{1}{k}\right) C_{\max}^*$$

There are at most  $km$  long jobs. Hence, the number of possibilities of scheduling these jobs on  $m$  machines is at most  $m^{km}$ , which is constant if  $m$  is constant. Hence, it is easy to implement the algorithm in polynomial time.

### Theorem 3

*The above algorithm gives a polynomial time approximation scheme (PTAS) for the problem of scheduling  $n$  jobs on  $m$  identical machines if  $m$  is constant.*

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## How to get rid of the requirement that $m$ is constant?

We first design an algorithm that works as follows:

On input of  $T$  it either finds a schedule of length  $(1 + \frac{1}{k})T$  or certifies that no schedule of length at most  $T$  exists (assume  $T \geq \frac{1}{m} \sum_j p_j$ ).

We partition the jobs into **long** jobs and **short** jobs:

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- ▶ We round all **long jobs** down to multiples of  $T/k^2$ .
- ▶ For these rounded sizes we first find an optimal schedule.
- ▶ If this schedule does not have length at most  $T$  we conclude that also the original sizes don't allow such a schedule.
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After the first phase the rounded sizes of the long jobs assigned to a machine add up to at most  $T$ .

There can be at most  $k$  (long) jobs assigned to a machine as otherwise their rounded sizes would add up to more than  $T$  (note that the rounded size of a long job is at least  $T/k$ ).

Since, jobs had been rounded to multiples of  $T/k^2$  going from rounded sizes to original sizes gives that the Makespan is at most

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Assigning the current (short) job to such a machine gives that the new load is at most

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**Running Time for scheduling large jobs:** There should not be a job with rounded size more than  $T$  as otw. the problem becomes trivial.

Hence, any large job has rounded size of  $\frac{i}{k^2}T$  for  $i \in \{k, \dots, k^2\}$ . Therefore the number of different inputs is at most  $n^{k^2}$  (described by a vector of length  $k^2$  where, the  $i$ -th entry describes the number of jobs of size  $\frac{i}{k^2}T$ ). This is polynomial.

The schedule/configuration of a particular machine  $x$  can be described by a vector of length  $k^2$  where the  $i$ -th entry describes the number of jobs of rounded size  $\frac{i}{k^2}T$  assigned to  $x$ . There are only  $(k+1)^{k^2}$  different vectors.

This means there are a constant number of different machine configurations.

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Let  $\text{OPT}(n_1, \dots, n_{k^2})$  be the **number of machines** that are required to schedule input vector  $(n_1, \dots, n_{k^2})$  with Makespan at most  $T$ .

If  $\text{OPT}(n_1, \dots, n_{k^2}) \leq m$  we can schedule the input.

We have

$$\text{OPT}(n_1, \dots, n_{k^2}) = \begin{cases} 0 & (n_1, \dots, n_{k^2}) = 0 \\ 1 + \min_{(s_1, \dots, s_{k^2}) \in C} \text{OPT}(n_1 - s_1, \dots, n_{k^2} - s_{k^2}) & (n_1, \dots, n_{k^2}) \geq 0 \\ \infty & \text{otw.} \end{cases}$$

where  $C$  is the set of all configurations.

Hence, the running time is roughly  $(k+1)^{k^2} n^{k^2} \approx (nk)^{k^2}$ .

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We can turn this into a PTAS by choosing  $k = \lceil 1/\epsilon \rceil$  and using binary search. This gives a running time that is exponential in  $1/\epsilon$ .

Can we do better?

Scheduling on identical machines with the goal of minimizing Makespan is a **strongly NP-complete** problem.

**Theorem 4**

*There is no FPTAS for problems that are strongly NP-hard.*

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- ▶ Suppose we have an instance with polynomially bounded processing times  $p_i \leq q(n)$

- ▶ We set  $k := \lceil 2nq(n) \rceil \geq 2 \text{OPT}$

- ▶ Then

$$\text{ALG} \leq \left(1 + \frac{1}{k}\right) \text{OPT} \leq \text{OPT} + \frac{1}{2}$$

- ▶ But this means that the algorithm computes the optimal solution as the optimum is integral.
- ▶ This means we can solve problem instances if processing times are polynomially bounded
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## More General

Let  $\text{OPT}(n_1, \dots, n_A)$  be the number of machines that are required to schedule input vector  $(n_1, \dots, n_A)$  with Makespan at most  $T$  ( $A$ : number of different sizes).

If  $\text{OPT}(n_1, \dots, n_A) \leq m$  we can schedule the input.

$$\text{OPT}(n_1, \dots, n_A) = \begin{cases} 0 & (n_1, \dots, n_A) = 0 \\ 1 + \min_{(s_1, \dots, s_A) \in C} \text{OPT}(n_1 - s_1, \dots, n_A - s_A) & (n_1, \dots, n_A) \neq 0 \\ \infty & \text{otw.} \end{cases}$$

where  $C$  is the set of all configurations.

$|C| \leq (B + 1)^A$ , where  $B$  is the number of jobs that possibly can fit on the same machine.

The running time is then  $O((B + 1)^A n^A)$  because the dynamic programming table has just  $n^A$  entries.

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# Bin Packing

Given  $n$  items with sizes  $s_1, \dots, s_n$  where

$$1 > s_1 \geq \dots \geq s_n > 0 .$$

Pack items into a minimum number of bins where each bin can hold items of total size at most 1.

## Theorem 5

*There is no  $\rho$ -approximation for Bin Packing with  $\rho < 3/2$  unless  $P = NP$ .*

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## Proof

- ▶ In the partition problem we are given positive integers  $b_1, \dots, b_n$  with  $B = \sum_i b_i$  even. Can we partition the integers into two sets  $S$  and  $T$  s.t.

$$\sum_{i \in S} b_i = \sum_{i \in T} b_i \quad ?$$

- ▶ We can solve this problem by setting  $s_i := 2b_i/B$  and asking whether we can pack the resulting items into 2 bins or not.
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## Definition 6

An asymptotic polynomial-time approximation scheme (APTAS) is a family of algorithms  $\{A_\epsilon\}$  along with a constant  $c$  such that  $A_\epsilon$  returns a solution of value at most  $(1 + \epsilon)\text{OPT} + c$  for minimization problems.

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# Bin Packing

Again we can differentiate between small and large items.

## Lemma 7

Any packing of items into  $\ell$  bins can be extended with items of size at most  $\gamma$  s.t. we use only  $\max\{\ell, \frac{1}{1-\gamma}\text{SIZE}(I) + 1\}$  bins, where  $\text{SIZE}(I) = \sum_i s_i$  is the sum of all item sizes.

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- ▶ If after Greedy we use more than  $\ell$  bins, all bins (apart from the last) must be full to at least  $1 - \gamma$ .
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Choose  $\gamma = \epsilon/2$ . Then we either use  $\ell$  bins or at most

$$\frac{1}{1 - \epsilon/2} \cdot \text{OPT} + 1 \leq (1 + \epsilon) \cdot \text{OPT} + 1$$

bins.

It remains to find an algorithm for the large items.

## Linear Grouping:

Generate an instance  $I'$  (for large items) as follows.

- ▶ Order large items according to size.
- ▶ Let the first  $k$  items belong to group 1; the following  $k$  items belong to group 2; etc.
- ▶ Delete items in the first group;
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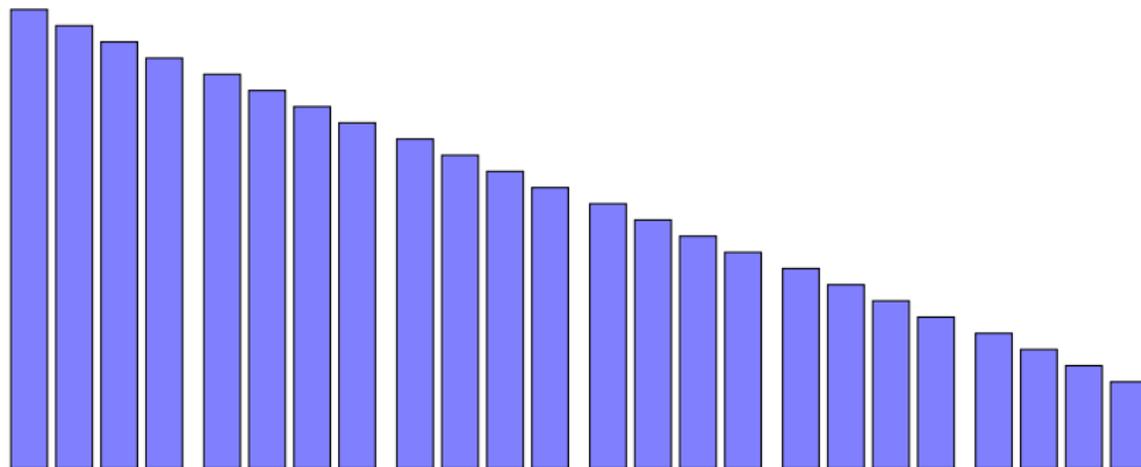
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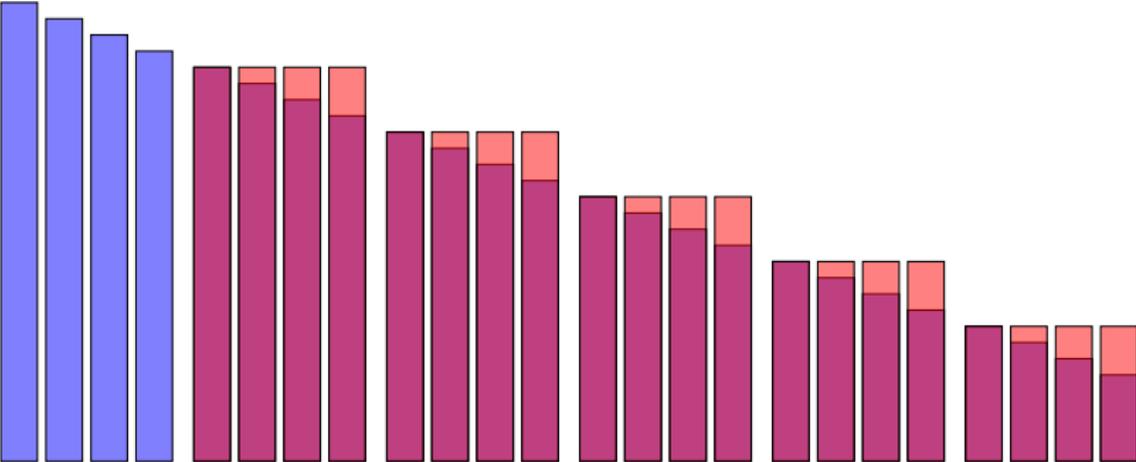
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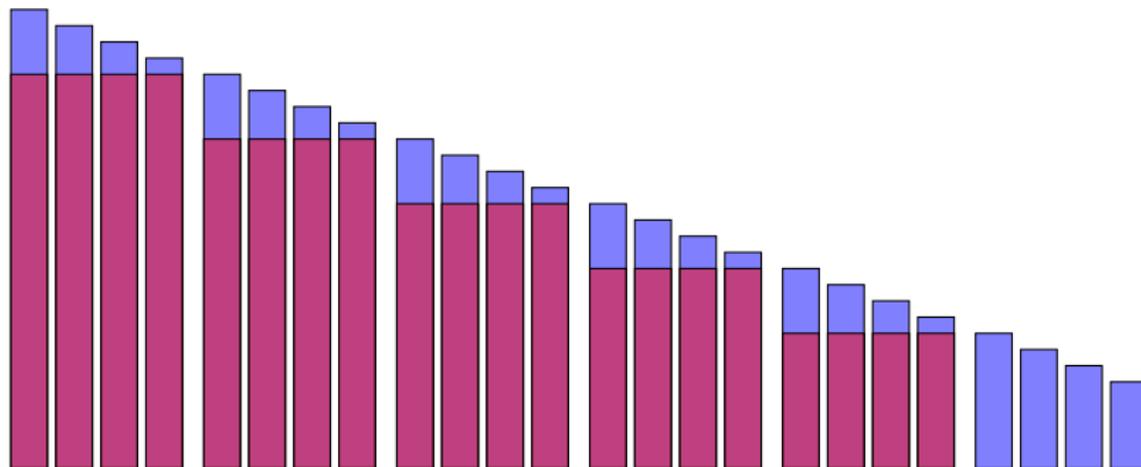
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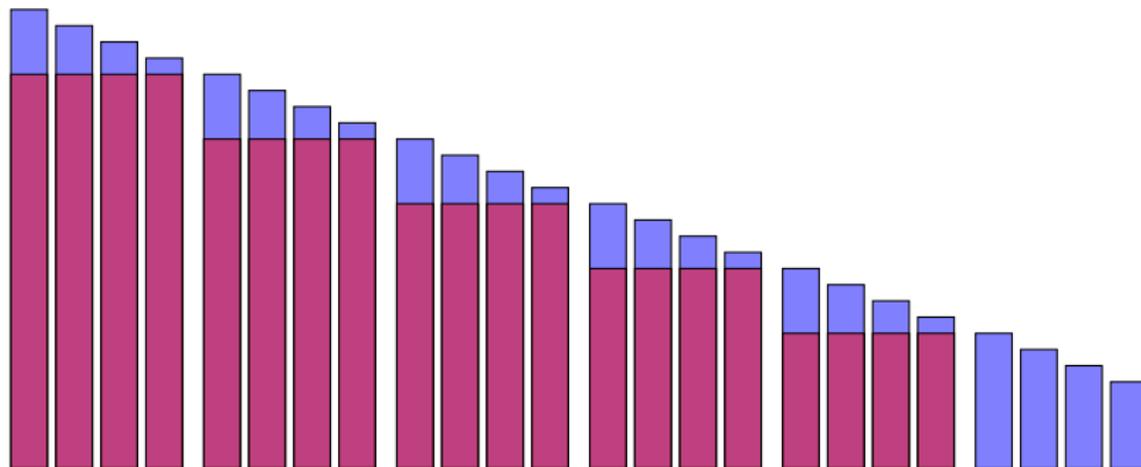
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## Lemma 8

$$\text{OPT}(I') \leq \text{OPT}(I) \leq \text{OPT}(I') + k$$

Proof 1:

Any bin packing for  $I'$  gives a bin packing for  $I$  as follows:

1. Pack the items of group 1 into the bins of the packing for  $I'$ .  
2. If items for group 2 have been packed,

3. pack the items of group 3, where in the packing for  $I'$  the items for group 2 have been packed.

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Assume that our instance does not contain pieces smaller than  $\epsilon/2$ . Then  $\text{SIZE}(I) \geq \epsilon n/2$ .

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We can find an optimal packing for such instances by the previous Dynamic Programming approach.

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$$\text{OPT}(I') + k \leq \text{OPT}(I) + \epsilon \text{SIZE}(I) \leq (1 + \epsilon) \text{OPT}(I)$$

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## Change of Notation:

- ▶ Group pieces of identical size.
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- ▶  $s_2$  is second largest size and  $b_2$  number of pieces of size  $s_2$ ;
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- ▶  $s_m$  smallest size and  $b_m$  number of pieces of size  $s_m$ .

# Configuration LP

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A possible packing of a bin can be described by an  $m$ -tuple  $(t_1, \dots, t_m)$ , where  $t_i$  describes the number of pieces of size  $s_i$ .  
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**How to solve this LP?**

later...

We can assume that each item has size at least  $1/\text{SIZE}(I)$ .

# Harmonic Grouping

- ▶ Sort items according to size (monotonically decreasing).
- ▶ Process items in this order; close the current group if size of items in the group is at least 2 (or larger). Then open new group.
- ▶ I.e.,  $G_1$  is the smallest cardinality set of largest items s.t. total size sums up to at least 2. Similarly, for  $G_2, \dots, G_{r-1}$ .
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- ▶ Round all items in a group to the size of the largest group member.
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## Lemma 10

*The number of different sizes in  $I'$  is at most  $\text{SIZE}(I)/2$ .*

Let  $I'$  be the set of items that are not packed in the first bin. Let  $S$  be the set of sizes of items in  $I'$ . Let  $n_s$  be the number of items of size  $s$  in  $I'$ . Let  $n$  be the number of items in  $I'$ . Let  $k$  be the number of items in  $I'$  that are of size  $s$ .

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- ▶ Each group that survives (recall that  $G_1$  and  $G_r$  are deleted) has total size at least 2.
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- ▶ The total size of items in  $G_1$  and  $G_r$  is at most 6 as a group has total size at most 3.
- ▶ Consider a group  $G_i$  that has strictly more items than  $G_{i-1}$ .
- ▶ It discards  $n_i - n_{i-1}$  pieces of total size at most

$$3 \frac{n_i - n_{i-1}}{n_i} \leq \sum_{j=n_{i-1}+1}^{n_i} \frac{3}{j}$$

since the smallest piece has size at most  $3/n_i$ .

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### Algorithm 1 BinPack

- 1: **if**  $\text{SIZE}(I) < 10$  **then**
- 2:     pack remaining items greedily
- 3: Apply harmonic grouping to create instance  $I'$ ; pack discarded items in at most  $\mathcal{O}(\log(\text{SIZE}(I)))$  bins.
- 4: Let  $x$  be optimal solution to configuration LP
- 5: Pack  $\lfloor x_j \rfloor$  bins in configuration  $T_j$  for all  $j$ ; call the packed instance  $I_1$ .
- 6: Let  $I_2$  be remaining pieces from  $I'$
- 7: Pack  $I_2$  via  $\text{BinPack}(I_2)$

$$\text{OPT}_{\text{LP}}(I_1) + \text{OPT}_{\text{LP}}(I_2) \leq \text{OPT}_{\text{LP}}(I') \leq \text{OPT}_{\text{LP}}(I)$$

Proof:

Each LP feasible solution for  $I'$  can be mapped to a feasible solution for  $I$  by an item-wise move. Hence,  $\text{OPT}_{\text{LP}}(I') \leq \text{OPT}_{\text{LP}}(I)$ .

Each LP feasible solution for  $I$  is also feasible for  $I'$ .

Hence,  $\text{OPT}_{\text{LP}}(I) \leq \text{OPT}_{\text{LP}}(I')$ .

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- ▶ Each piece surviving in  $I'$  can be mapped to a piece in  $I$  of no lesser size. Hence,  $\text{OPT}_{\text{LP}}(I') \leq \text{OPT}_{\text{LP}}(I)$
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# Analysis

Each level of the recursion partitions pieces into three types

1. Pieces discarded at this level.
2. Pieces scheduled because they are in  $I_1$ .
3. Pieces in  $I_2$  are handed down to the next level.

Pieces of type 2 summed over all recursion levels are packed into at most  $\text{OPT}_{1P}$  many bins.

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Let  $T_1, \dots, T_N$  be the sequence of all possible configurations (a configuration  $T_j$  has  $T_{ji}$  pieces of size  $s_i$ ).

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### Dual

$$\begin{array}{ll} \max & \sum_{i=1}^m \gamma_i b_i \\ \text{s.t.} & \forall j \in \{1, \dots, N\} \quad \sum_{i=1}^m T_{ji} \gamma_i \leq 1 \\ & \forall i \in \{1, \dots, m\} \quad \gamma_i \geq 0 \end{array}$$

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## Separation Oracle

Suppose that I am given variable assignment  $y$  for the dual.

**How do I find a violated constraint?**

I have to find a configuration  $T_j = (T_{j1}, \dots, T_{jm})$  that

is feasible for  $P_j$ .

and has a large profit.

But this is the Knapsack problem.

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# Separation Oracle

We have FPTAS for Knapsack. This means if a constraint is violated with  $1 + \epsilon' = 1 + \frac{\epsilon}{1-\epsilon}$  we find it, since we can obtain at least  $(1 - \epsilon)$  of the optimal profit.

The solution we get is feasible for:

Dual'

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## Separation Oracle

We have FPTAS for Knapsack. This means if a constraint is violated with  $1 + \epsilon' = 1 + \frac{\epsilon}{1-\epsilon}$  we find it, since we can obtain at least  $(1 - \epsilon)$  of the optimal profit.

The solution we get is feasible for:

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If the value of the computed dual solution (which may be infeasible) is  $z$  then

$$\text{OPT} \leq z \leq (1 + \epsilon')\text{OPT}$$

How do we get good primal solution (not just the value)?

The constraints used when computing  $z$  imply that the solution is feasible for

the LP that we drop all unused constraints in  $z$ . We will assume the remaining constraints are

feasible. The LP with all unused constraints

is dual to the LP that we drop all unused constraints for  $z$ . The corresponding dual constraint has not been used.

The primal value for this LP is  $z - \epsilon' \text{OPT}$ .

The primal value for the original LP is  $\text{OPT}$ .

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- ▶ The constraints used when computing  $z$  **certify** that the solution is feasible for **DUAL'**.
- ▶ Suppose that we drop all unused constraints in **DUAL**. We will compute the same solution feasible for **DUAL'**.
- ▶ Let **DUAL''** be **DUAL** without unused constraints.
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This gives that overall we need at most

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