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Let X be random variable taking non-negative values.
Then

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Trivial!

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That's awfully weak :(

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For $0 \leq \delta \leq 1$ we have that

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Reason:
As long as derivative of left side is smaller than derivative of right side the inequality holds.

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$$f(\delta) := -\ln(1+\delta) + 2\delta/3 \le 0$$

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$$f(\delta) := -\ln(1+\delta) + 2\delta/3 \le 0$$

A convex function ($f''(\delta) \ge 0$) on an interval takes maximum at the boundaries.

Show:

$$\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} \le e^{-U\delta^2/3}$$

Take logarithms:

$$U(\delta - (1+\delta)\ln(1+\delta)) \le -U\delta^2/3$$

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Reason:
As long as derivative of left side is smaller than derivative of right side the inequality holds.

For $\delta \geq 1$ we show

$$\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} \leq e^{-U\delta/3}$$

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Take logarithms:

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$$\ln(1-\delta) \leq -\delta$$

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Integer Multicommodity Flows

- Given s_i-t_i pairs in a graph.
- Connect each pair by a path such that not too many path use any given edge.

$$
\begin{array}{llrcl}
\min & & & W & \\
\text{s.t.} & \forall i & \sum_{p\in\mathcal{P}_i} x_p & = & 1 \\
& & \sum_{p:e\in p} x_p & \le & W \\
& & x_p & \in & \{0,1\}
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$$

$$\ln(1-\delta) \le -\delta$$

True for $\delta = 0$. Take derivatives:

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Integer Multicommodity Flows

Randomized Rounding:
For each i choose one path from the set $\mathcal{P}_i$ at random according to the probability distribution given by the Linear Programming solution.

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Theorem 5
If $W^* \geq c \ln n$ for some constant c, then with probability at least $n^{-c/3}$ the total number of paths using any edge is at most $W^* + \sqrt{cW^* \ln n}$.

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With probability at least $n^{-c/3}$ the total number of paths using any edge is at most $W^* + c \ln n$.

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Integer Multicommodity Flows

Let X_e^i be a random variable that indicates whether the path for s_i-t_i uses edge e.

Then the number of paths using edge e is $Y_e = \sum_i X_e^i$.

$$E[Y_e] = \sum_i \sum_{p \in \mathcal{P}_i : e \in p} x_p^* = \sum_{p : e \in P} x_p^* \le W^*$$

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Choose $\delta = \sqrt{(c \ln n)/W^*}$.

Then

$$\Pr[Y_e \geq (1+\delta)W^*] < e^{-W^*\delta^2/3} = \frac{1}{n^{c/3}}$$

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17.3 MAXSAT

Problem definition:

- n Boolean variables
- m clauses $C_1,\dots,C_m$. For example

$$C_7 = x_3 \vee \bar{x}_5 \vee \bar{x}_9$$

- Non-negative weight w_j for each clause C_j.
- Find an assignment of true/false to the variables sucht that the total weight of clauses that are satisfied is maximum.

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- Hence, each clause consists of a set of literals (i.e., no duplications: $x_i \vee x_i \vee \bar{x}_j$ is **not** a clause).
- We assume a clause does not contain x_i and $\bar{x}_i$ for any i.
- x_i is called a positive literal while the negation $\bar{x}_i$ is called a negative literal.
- For a given clause C_j the number of its literals is called its length or size and denoted with ℓ_j.
- Clauses of length one are called unit clauses.

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MAXSAT: Flipping Coins

Set each x_i independently to true with probability $\frac{1}{2}$ (and, hence, to false with probability $\frac{1}{2}$, as well).

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- Clauses of length one are called unit clauses.

Define random variable X_j with

$$X_j = \begin{cases} 1 & \text{if } C_j \text{ satisfied} \\ 0 & \text{otw.} \end{cases}$$

Then the total weight W of satisfied clauses is given by

$$W = \sum_j w_j X_j$$

MAXSAT: Flipping Coins

Set each x_i independently to true with probability $\frac{1}{2}$ (and, hence, to false with probability $\frac{1}{2}$, as well).

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$E[W]$

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$$E[W] = \sum_j w_j E[X_j]$$

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$$\begin{aligned} E[W] &= \sum_j w_j E[X_j] \\ &= \sum_j w_j \Pr[C_j \text{ is satisified}] \end{aligned}$$

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E[W] &= \sum_j w_j E[X_j] \\
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\end{aligned}
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MAXSAT: LP formulation

- Let for a clause C_j, P_j be the set of positive literals and N_j the set of negative literals.

$$C_j = \bigvee_{j\in P_j} x_i \vee \bigvee_{j\in N_j} \bar{x}_i$$

$$\begin{array}{llrcl} \max & & \multicolumn{3}{c}{\sum_j w_j z_j} \\ \text{s.t.} & \forall j & \sum_{i\in P_j} y_i + \sum_{i\in N_j}(1-y_i) & \geq & z_j \\ & \forall i & y_i & \in & \{0,1\} \\ & \forall j & z_j & \leq & 1 \end{array}$$

$$\begin{aligned} E[W] &= \sum_j w_j E[X_j] \\ &= \sum_j w_j \Pr[C_j \text{ is satisified}] \\ &= \sum_j w_j \Big(1 - \Big(\frac{1}{2}\Big)^{\ell_j}\Big) \\ &\geq \frac{1}{2}\sum_j w_j \\ &\geq \frac{1}{2}\mathrm{OPT} \end{aligned}$$

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MAXSAT: Randomized Rounding

Set each x_i independently to true with probability y_i (and, hence, to false with probability $(1 - y_i)$).

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Lemma 7 (Geometric Mean $\leq$ Arithmetic Mean)

For any nonnegative $a_1, \dots, a_k$

$$\left(\prod_{i=1}^{k} a_i\right)^{1/k} \leq \frac{1}{k}\sum_{i=1}^{k} a_i$$

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Definition 8

A function f on an interval I is concave if for any two points s and r from I and any $\lambda \in [0,1]$ we have

$$f(\lambda s + (1-\lambda)r) \geq \lambda f(s) + (1-\lambda)f(r)$$

Lemma 9

Let f be a concave function on the interval $[0,1]$, with $f(0) = a$ and $f(1) = a + b$. Then

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$\Pr[C_j \text{ not satisfied}]$

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$$\le \left[\frac{1}{\ell_j}\left(\sum_{i\in P_j}(1-y_i) + \sum_{i\in N_j} y_i\right)\right]^{\ell_j}$$

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&\le \left[\frac{1}{\ell_j}\left(\sum_{i\in P_j}(1-y_i)+\sum_{i\in N_j} y_i\right)\right]^{\ell_j} \\
&= \left[1-\frac{1}{\ell_j}\left(\sum_{i\in P_j} y_i+\sum_{i\in N_j}(1-y_i)\right)\right]^{\ell_j}
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Let f be a concave function on the interval $[0,1]$, with $f(0)=a$ and $f(1)=a+b$. Then

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\begin{aligned}
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&= \left[1-\frac{1}{\ell_j}\left(\sum_{i\in P_j} y_i+\sum_{i\in N_j}(1-y_i)\right)\right]^{\ell_j} \\
&\le \left(1-\frac{z_j}{\ell_j}\right)^{\ell_j} .
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The function $f(z) = 1 - (1 - \frac{z}{\ell})^{\ell}$ is concave. Hence,

$\Pr[C_j \text{ satisfied}]$

$$
\begin{aligned}
\Pr[C_j \text{ not satisfied}] &= \prod_{i \in P_j} (1 - y_i) \prod_{i \in N_j} y_i \\
&\le \left[\frac{1}{\ell_j} \left(\sum_{i \in P_j} (1 - y_i) + \sum_{i \in N_j} y_i \right) \right]^{\ell_j} \\
&= \left[1 - \frac{1}{\ell_j} \left(\sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \right) \right]^{\ell_j} \\
&\le \left(1 - \frac{z_j}{\ell_j} \right)^{\ell_j} .
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The function $f(z) = 1 - (1 - \frac{z}{\ell})^{\ell}$ is concave. Hence,

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&\leq \left[\frac{1}{\ell_j} \left(\sum_{i \in P_j} (1 - y_i) + \sum_{i \in N_j} y_i \right) \right]^{\ell_j} \\
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\end{aligned}$$

The function $f(z) = 1 - (1 - \frac{z}{\ell})^{\ell}$ is concave. Hence,

$$\begin{aligned}\Pr[C_j \text{ satisfied}] &\geq 1 - \left(1 - \frac{z_j}{\ell_j}\right)^{\ell_j} \\ &\geq \left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right] \cdot z_j \ .\end{aligned}$$

$$\begin{aligned}\Pr[C_j \text{ not satisfied}] &= \prod_{i \in P_j}(1 - y_i) \prod_{i \in N_j} y_i \\ &\leq \left[\frac{1}{\ell_j}\left(\sum_{i \in P_j}(1 - y_i) + \sum_{i \in N_j} y_i\right)\right]^{\ell_j} \\ &= \left[1 - \frac{1}{\ell_j}\left(\sum_{i \in P_j} y_i + \sum_{i \in N_j}(1 - y_i)\right)\right]^{\ell_j} \\ &\leq \left(1 - \frac{z_j}{\ell_j}\right)^{\ell_j} \ .\end{aligned}$$

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$f''(z) = -\frac{\ell-1}{\ell}\left[1 - \frac{z}{\ell}\right]^{\ell-2} \leq 0$ for $z \in [0,1]$. Therefore, f is concave.

$$\begin{aligned}\Pr[C_j \text{ not satisfied}] &= \prod_{i\in P_j}(1-y_i)\prod_{i\in N_j} y_i \\ &\leq \left[\frac{1}{\ell_j}\left(\sum_{i\in P_j}(1-y_i) + \sum_{i\in N_j} y_i\right)\right]^{\ell_j} \\ &= \left[1 - \frac{1}{\ell_j}\left(\sum_{i\in P_j} y_i + \sum_{i\in N_j}(1-y_i)\right)\right]^{\ell_j} \\ &\leq \left(1 - \frac{z_j}{\ell_j}\right)^{\ell_j} \ .\end{aligned}$$

$$E[W]$$

The function $f(z) = 1 - (1 - \frac{z}{\ell})^{\ell}$ is concave. Hence,

$$\Pr[C_j \text{ satisfied}] \geq 1 - \left(1 - \frac{z_j}{\ell_j}\right)^{\ell_j}$$
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\begin{aligned}
E[W] &= \sum_j w_j \Pr[C_j \text{ is satisfied}] \\
&\geq \sum_j w_j z_j \left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right] \\
&\geq \left(1 - \frac{1}{e}\right) \mathrm{OPT} \ .
\end{aligned}
$$

The function $f(z) = 1 - (1 - \frac{z}{\ell})^{\ell}$ is concave. Hence,

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\Pr[C_j \text{ satisfied}] &\geq 1 - \left(1 - \frac{z_j}{\ell_j}\right)^{\ell_j} \\
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MAXSAT: The better of two

Theorem 10
Choosing the better of the two solutions given by randomized rounding and coin flipping yields a $\frac{3}{4}$-approximation.

$$
\begin{aligned}
E[W] &= \sum_j w_j \Pr[C_j \text{ is satisfied}] \\
&\geq \sum_j w_j z_j \left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right] \\
&\geq \left(1 - \frac{1}{e}\right) \mathrm{OPT} \ .
\end{aligned}
$$

Let W_1 be the value of randomized rounding and W_2 the value obtained by coin flipping.

$$E[\max\{W_1, W_2\}]$$

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Choosing the better of the two solutions given by randomized rounding and coin flipping yields a $\frac{3}{4}$*-approximation.*

Let W_1 be the value of randomized rounding and W_2 the value obtained by coin flipping.

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E[\max\{W_1, W_2\}]& \\
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&\geq E[\tfrac{1}{2}W_1 + \tfrac{1}{2}W_2] \\
&\geq \frac{1}{2}\sum_j w_j z_j \left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right] + \frac{1}{2}\sum_j w_j \left(1 - \left(\frac{1}{2}\right)^{\ell_j}\right)
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&\geq \sum_j w_j z_j \Bigg[\underbrace{\frac{1}{2}\left(1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right) + \frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^{\ell_j}\right)}_{\geq \frac{3}{4} \text{ for all integers}}\Bigg]
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&\quad \geq \sum_j w_j z_j \Bigg[\underbrace{\frac{1}{2}\left(1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right) + \frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^{\ell_j}\right)}_{\geq \frac{3}{4} \text{for all integers}}\Bigg] \\
&\quad \geq \frac{3}{4}\mathrm{OPT}
\end{aligned}
$$

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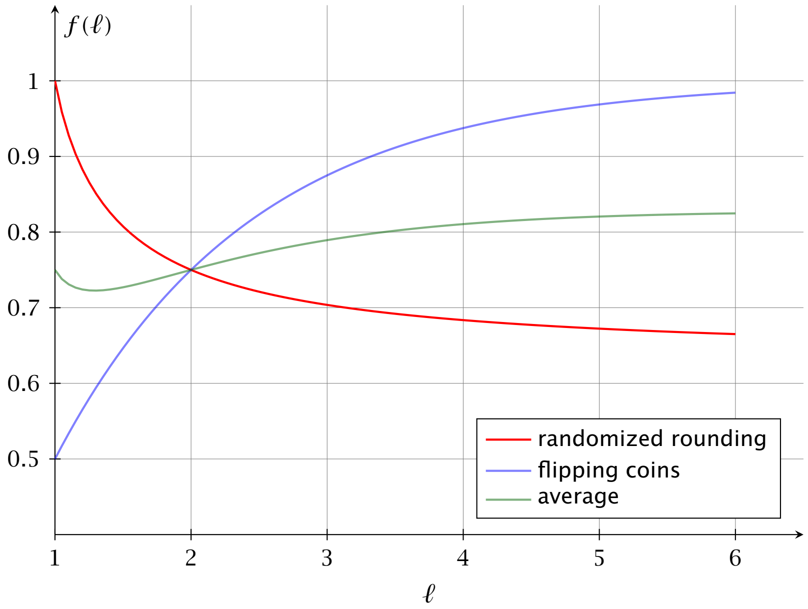

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&\qquad \geq E[\tfrac{1}{2}W_1 + \tfrac{1}{2}W_2] \\
&\qquad \geq \frac{1}{2}\sum_j w_j z_j \left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right] + \frac{1}{2}\sum_j w_j \left(1 - \left(\frac{1}{2}\right)^{\ell_j}\right) \\
&\qquad \geq \sum_j w_j z_j \underbrace{\left[\frac{1}{2}\left(1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right) + \frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^{\ell_j}\right)\right]}_{\geq \frac{3}{4} \text{ for all integers}} \\
&\qquad \geq \frac{3}{4}\text{OPT}
\end{aligned}
$$

MAXSAT: Nonlinear Randomized Rounding

So far we used linear randomized rounding, i.e., the probability that a variable is set to 1/true was exactly the value of the corresponding variable in the linear program.

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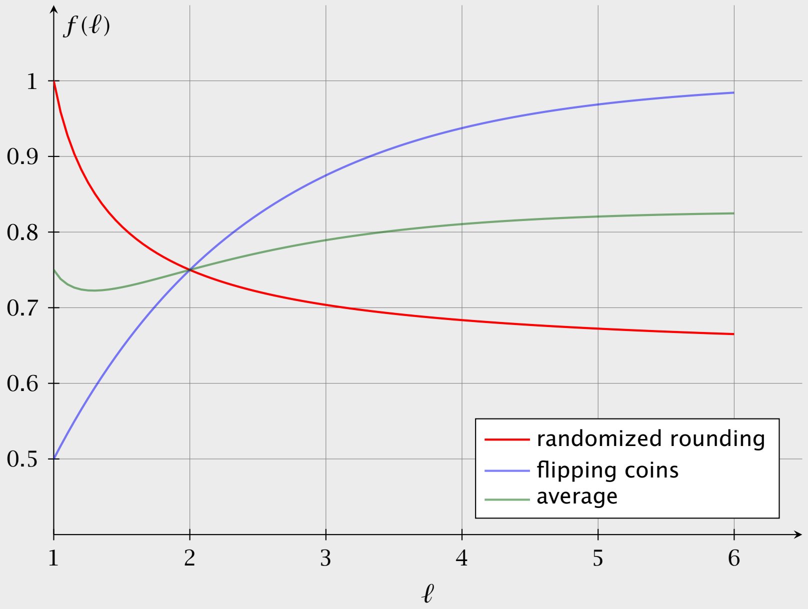

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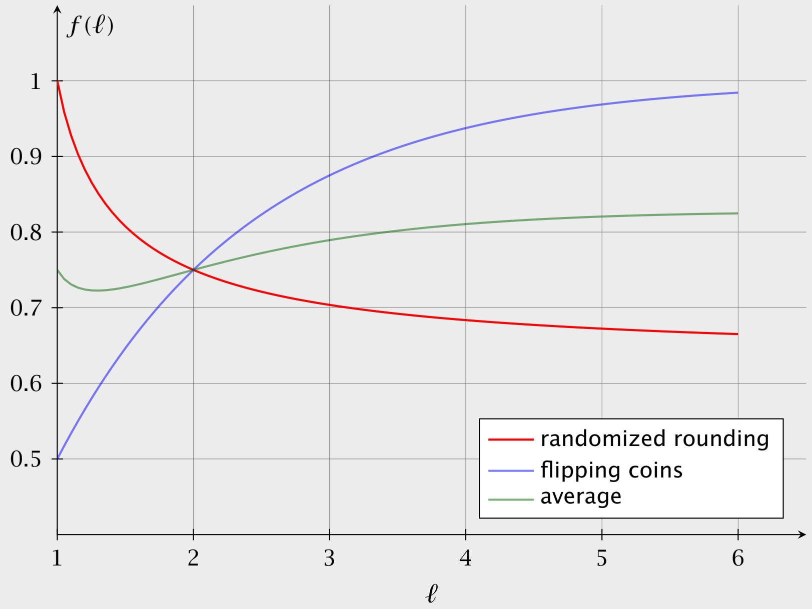

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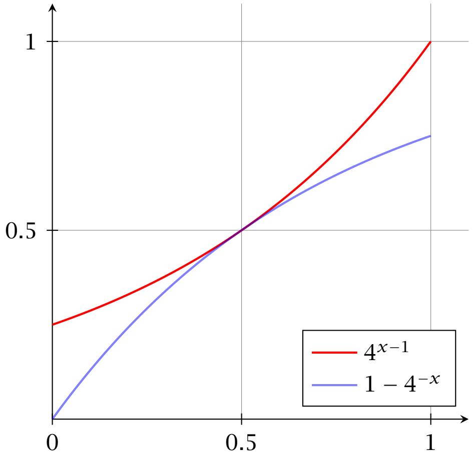

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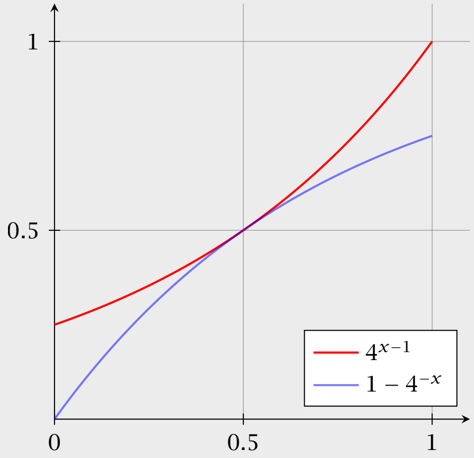
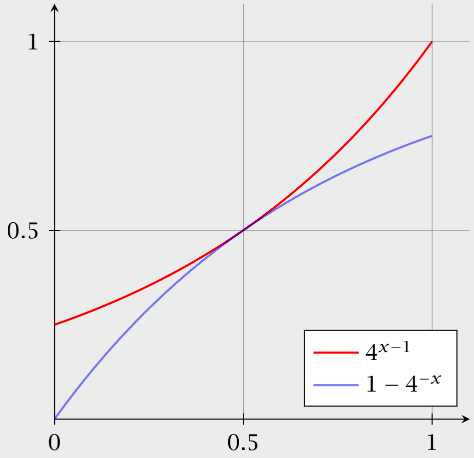
$\Pr[C_j \text{ not satisfied}]$



$$\Pr[C_j \text{ not satisfied}] = \prod_{i\in P_j}(1-f(y_i)) \prod_{i\in N_j} f(y_i)$$



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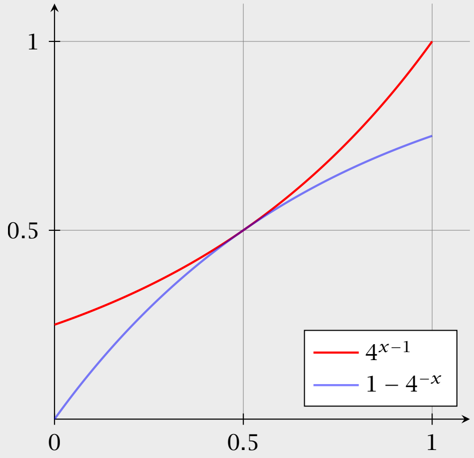

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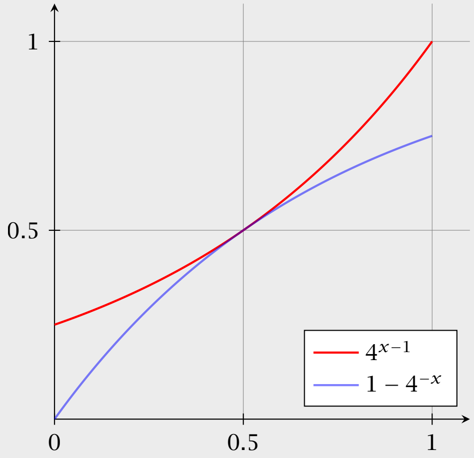
$$



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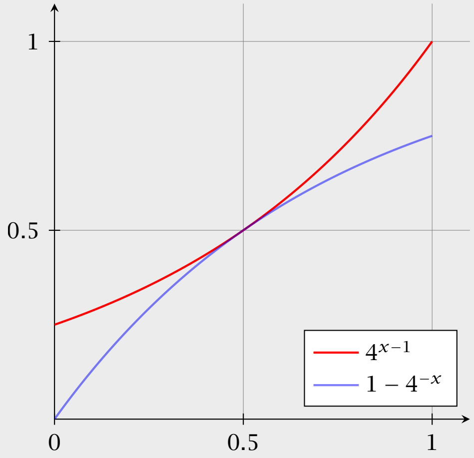

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Can we do better?

Not if we compare ourselves to the value of an optimum LP-solution.

Definition 12 (Integrality Gap)

The integrality gap for an ILP is the worst-case ratio over all instances of the problem of the value of an optimal IP-solution to the value of an optimal solution to its linear programming relaxation.

Note that the integrality is less than one for maximization problems and larger than one for minimization problems (of course, equality is possible).

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Our ILP-formulation for the MAXSAT problem has integrality gap at most $\frac{3}{4}$.

$$\begin{array}{llrcl} \max & & \sum_j w_j z_j & & \\ \text{s.t.} & \forall j & \sum_{i\in P_j} y_i + \sum_{i\in N_j}(1-y_i) & \geq & z_j \\ & \forall i & y_i & \in & \{0,1\} \\ & \forall j & z_j & \leq & 1 \end{array}$$

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MaxCut

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Given a weighted graph $G = (V, E, w)$, $w(v) \geq 0$, partition the vertices into two parts. Maximize the weight of edges between the parts.

Trivial 2-approximation

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Semidefinite Programming

$$\begin{array}{llrcl} \text{max / min} & & \sum_{i,j} c_{ij}x_{ij} & & \\ \text{s.t.} & \forall k & \sum_{i,j,k} a_{ijk}x_{ij} & = & b_k \\ & \forall i,j & x_{ij} & = & x_{ji} \\ & & X = (x_{ij}) \text{ is psd.} & & \end{array}$$

- linear objective, linear contraints
- we can constrain a square matrix of variables to be symmetric positive definite

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- variables are vectors in n-dimensional space
- objective functions and contraints are linear in inner products of the vectors

This is equivalent!

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Fact [without proof]

We (essentially) can solve Semidefinite Programs in polynomial time...

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Quadratic Program for MaxCut:

$$\boxed{\begin{array}{llcl} \max & \frac{1}{2}\sum_{i,j} w_{ij}(1-y_iy_j) & & \\ \forall i & y_i & \in & \{-1,1\} \end{array}}$$

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- this is clearly a relaxation
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Rounding the SDP-Solution

- Choose a random vector r such that $r/\|r\|$ is uniformly distributed on the unit sphere.
- If $r^t v_i > 0$ set $y_i = 1$ else set $y_i = -1$

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Rounding the SDP-Solution

Choose the i-th coordinate r_i as a Gaussian with mean 0 and variance 1, i.e., $r_i \sim \mathcal{N}(0,1)$.

Density function:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{x^2/2}$$

Then

$$\begin{aligned}\Pr[r = (x_1,\dots,x_n)] &= \frac{1}{(\sqrt{2\pi})^n} e^{x_1^2/2} \cdot e^{x_2^2/2} \cdot \ldots \cdot e^{x_n^2/2}\, dx_1 \cdot \ldots \cdot dx_n \\ &= \frac{1}{(\sqrt{2\pi})^n} e^{\frac{1}{2}(x_1^2+\ldots+x_n^2)}\, dx_1 \cdot \ldots \cdot dx_n\end{aligned}$$

Hence the probability for a point only depends on its distance to the origin.

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&\qquad = \frac{1}{(\sqrt{2\pi})^n} e^{\frac{1}{2}(x_1^2+\ldots+x_n^2)} \, \mathrm{d}x_1 \cdot \ldots \cdot \mathrm{d}x_n
\end{aligned}$$

Hence the probability for a point only depends on its distance to the origin.

Rounding the SDP-Solution

- Choose a random vector r such that $r/\|r\|$ is uniformly distributed on the unit sphere.
- If $r^t v_i > 0$ set $y_i = 1$ else set $y_i = -1$

Rounding the SDP-Solution

Fact
The projection of r onto two unit vectors e_1 and e_2 are independent and are normally distributed with mean 0 and variance 1 iff e_1 and e_2 are orthogonal.

Note that this is clear if e_1 and e_2 are standard basis vectors.

Rounding the SDP-Solution

Choose the i-th coordinate r_i as a Gaussian with mean 0 and variance 1, i.e., $r_i \sim \mathcal{N}(0,1)$.

Density function:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{x^2/2}$$

Then

$$\begin{aligned}
&\Pr[r = (x_1, \dots, x_n)] \\
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Hence the probability for a point only depends on its distance to the origin.

Rounding the SDP-Solution

Corollary
If we project r onto a hyperplane its normalized projection $(r'/\|r'\|)$ is uniformly distributed on the unit circle within the hyperplane.

Rounding the SDP-Solution

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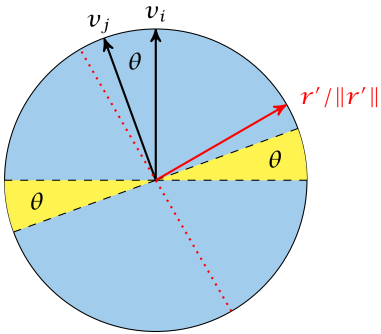
Rounding the SDP-Solution



- if the normalized projection falls into the shaded region, v_i and v_j are rounded to different values
- this happens with probability θ/π

Rounding the SDP-Solution

Corollary

If we project r onto a hyperplane its normalized projection ($r'/\|r'\|$) is uniformly distributed on the unit circle within the hyperplane.

Rounding the SDP-Solution

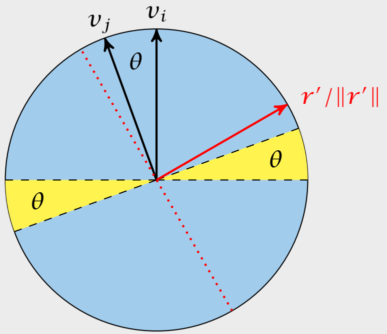
- contribution of edge (i,j) to the SDP-relaxation:

$$\frac{1}{2}w_{ij}\left(1 - v_i^t v_j\right)$$

- (expected) contribution of edge (i,j) to the rounded instance $w_{ij}\arccos(v_i^t v_j)/\pi$
- ratio is at most

$$\min_{x\in[-1,1]} \frac{2\arccos(x)}{\pi(1-x)} \geq 0.878$$

Rounding the SDP-Solution



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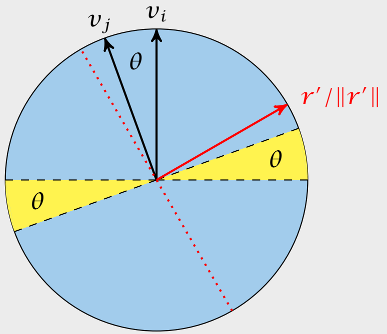

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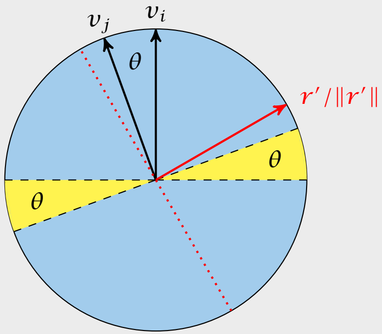

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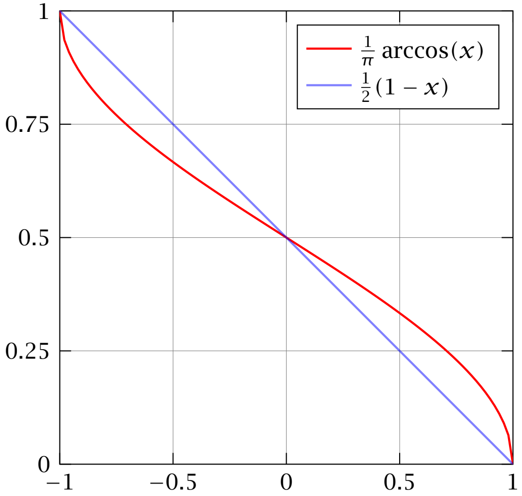

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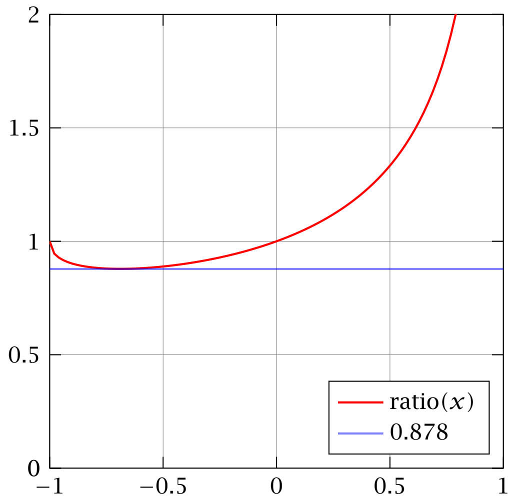

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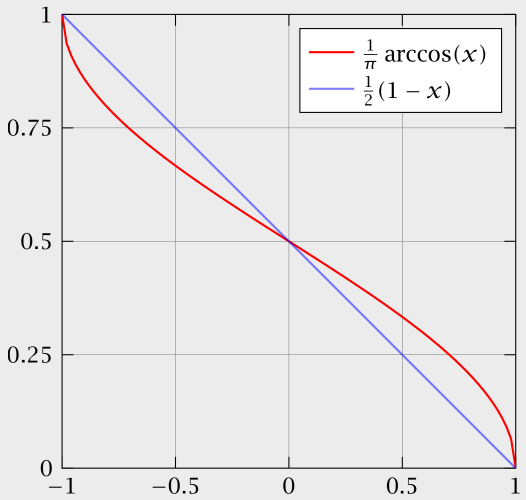

Rounding the SDP-Solution

Theorem 14
Given the unique games conjecture, there is no α-approximation for the maximum cut problem with constant

$$\alpha > \min_{x\in[-1,1]} \frac{2\arccos(x)}{\pi(1-x)}$$

unless $\mathrm{P} = \mathrm{NP}$.

Rounding the SDP-Solution