

Repetition: Primal Dual for Set Cover

Primal Relaxation:

$$\begin{array}{ll} \min & \sum_{i=1}^k w_i x_i \\ \text{s.t.} & \forall u \in U \quad \sum_{i:u \in S_i} x_i \geq 1 \\ & \forall i \in \{1, \dots, k\} \quad x_i \geq 0 \end{array}$$

Dual Formulation:

$$\begin{array}{ll} \max & \sum_{u \in U} y_u \\ \text{s.t.} & \forall i \in \{1, \dots, k\} \quad \sum_{u:u \in S_i} y_u \leq w_i \\ & y_u \geq 0 \end{array}$$

Repetition: Primal Dual for Set Cover

Algorithm:

- ▶ Start with $y = 0$ (feasible dual solution).
Start with $x = 0$ (integral primal solution that may be infeasible).
- ▶ While x not feasible
 - ▶ Identify an element e that is not covered in current primal integral solution.
 - ▶ Increase dual variable y_e until a dual constraint becomes tight (maybe increase by 0!).
 - ▶ If this is the constraint for set S_j set $x_j = 1$ (add this set to your solution).

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Analysis:

- ▶ For every set S_j with $x_j = 1$ we have

$$\sum_{e \in S_j} y_e = w_j$$

- ▶ Hence our cost is

$$\begin{aligned} \sum_j w_j x_j &= \sum_j \sum_{e \in S_j} y_e = \sum_e |\{j : e \in S_j\}| \cdot y_e \\ &\leq f \cdot \sum_e y_e \leq f \cdot \text{OPT} \end{aligned}$$

Note that the constructed pair of primal and dual solution fulfills **primal slackness conditions**.

This means

$$x_j > 0 \Rightarrow \sum_{e \in S_j} y_e = w_j$$

If we would also fulfill **dual slackness conditions**

$$y_e > 0 \Rightarrow \sum_{j: e \in S_j} x_j = 1$$

then the solution would be **optimal!!!**

We don't fulfill these constraint but we fulfill an approximate version:

$$y_e > 0 \Rightarrow 1 \leq \sum_{j:e \in S_j} x_j \leq f$$

This is sufficient to show that the solution is an f -approximation.

Suppose we have a primal/dual pair

$\begin{aligned} \min \quad & \sum_j c_j x_j \\ \text{s.t.} \quad & \forall i \sum_j a_{ij} x_j \geq b_i \\ & \forall j \quad x_j \geq 0 \end{aligned}$	$\begin{aligned} \max \quad & \sum_i b_i y_i \\ \text{s.t.} \quad & \forall j \sum_i a_{ij} y_i \leq c_j \\ & \forall i \quad y_i \geq 0 \end{aligned}$
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and solutions that fulfill approximate slackness conditions:

$$x_j > 0 \Rightarrow \sum_i a_{ij} y_i \geq \frac{1}{\alpha} c_j$$

$$y_i > 0 \Rightarrow \sum_j a_{ij} x_j \leq \beta b_i$$

Then

$$\begin{aligned} & \boxed{\sum_j c_j x_j} \leq \alpha \sum_j \left(\sum_i a_{ij} y_i \right) x_j \\ & \quad \uparrow \text{primal cost} \\ & \alpha \sum_i \left(\sum_j a_{ij} x_j \right) y_i \\ & \leq \alpha \beta \cdot \boxed{\sum_i b_i y_i} \\ & \quad \uparrow \text{dual objective} \end{aligned}$$

right hand side of j -th dual constraint

Feedback Vertex Set for Undirected Graphs

- ▶ Given a graph $G = (V, E)$ and non-negative weights $w_v \geq 0$ for vertex $v \in V$.
- ▶ Choose a minimum cost subset of vertices s.t. every cycle contains at least one vertex.

We can encode this as an instance of Set Cover

- ▶ Each vertex can be viewed as a set that contains some cycles.
- ▶ However, this encoding gives a Set Cover instance of non-polynomial size.
- ▶ The $O(\log n)$ -approximation for Set Cover does not help us to get a good solution.

Let \mathcal{C} denote the set of all cycles (where a cycle is identified by its set of vertices)

Primal Relaxation:

$$\begin{array}{ll} \min & \sum_v w_v x_v \\ \text{s.t.} & \forall C \in \mathcal{C} \quad \sum_{v \in C} x_v \geq 1 \\ & \forall v \quad x_v \geq 0 \end{array}$$

Dual Formulation:

$$\begin{array}{ll} \max & \sum_{C \in \mathcal{C}} y_C \\ \text{s.t.} & \forall v \in V \quad \sum_{C: v \in C} y_C \leq w_v \\ & \forall C \quad y_C \geq 0 \end{array}$$

If we perform the previous dual technique for Set Cover we get the following:

- ▶ Start with $x = 0$ and $y = 0$
- ▶ While there is a cycle C that is not covered (does not contain a chosen vertex).
 - ▶ Increase y_C until dual constraint for some vertex v becomes tight.
 - ▶ set $x_v = 1$.

Then

$$\begin{aligned} \sum_v w_v x_v &= \sum_v \sum_{C: v \in C} y_C x_v \\ &= \sum_{v \in S} \sum_{C: v \in C} y_C \\ &= \sum_C |S \cap C| \cdot y_C \end{aligned}$$

where S is the set of vertices we choose.

If every cycle is short we get a good approximation ratio, but this is unrealistic.

Algorithm 1 FeedbackVertexSet

```
1:  $y \leftarrow 0$ 
2:  $x \leftarrow 0$ 
3: while exists cycle  $C$  in  $G$  do
4:   increase  $y_C$  until there is  $v \in C$  s.t.  $\sum_{C:v \in C} y_C = w_v$ 
5:    $x_v = 1$ 
6:   remove  $v$  from  $G$ 
7:   repeatedly remove vertices of degree 1 from  $G$ 
```

Idea:

Always choose a short cycle that is not covered. If we always find a cycle of length at most α we get an α -approximation.

Observation:

For any path P of vertices of degree 2 in G the algorithm chooses at most one vertex from P .

Observation:

If we always choose a cycle for which the number of vertices of degree at least 3 is at most α we get a 2α -approximation.

Theorem 2

In any graph with no vertices of degree 1, there always exists a cycle that has at most $\mathcal{O}(\log n)$ vertices of degree 3 or more. We can find such a cycle in linear time.

This means we have

$$y_C > 0 \Rightarrow |S \cap C| \leq \mathcal{O}(\log n) .$$

Primal Dual for Shortest Path

Given a graph $G = (V, E)$ with two nodes $s, t \in V$ and edge-weights $c : E \rightarrow \mathbb{R}^+$ find a shortest path between s and t w.r.t. edge-weights c .

$$\begin{array}{ll} \min & \sum_e c(e)x_e \\ \text{s.t.} & \forall S \in \mathcal{S} \quad \sum_{e: \delta(S)} x_e \geq 1 \\ & \forall e \in E \quad x_e \in \{0, 1\} \end{array}$$

Here $\delta(S)$ denotes the set of edges with exactly one end-point in S , and $\mathcal{S} = \{S \subseteq V : s \in S, t \notin S\}$.

Primal Dual for Shortest Path

The Dual:

$$\begin{array}{ll} \max & \sum_S \gamma_S \\ \text{s.t.} & \forall e \in E \quad \sum_{S: e \in \delta(S)} \gamma_S \leq c(e) \\ & \forall S \in \mathcal{S} \quad \gamma_S \geq 0 \end{array}$$

Here $\delta(S)$ denotes the set of edges with exactly one end-point in S , and $\mathcal{S} = \{S \subseteq V : s \in S, t \notin S\}$.

Primal Dual for Shortest Path

We can interpret the value γ_S as the width of a moat surrounding the set S .

Each set can have its own moat but all moats must be disjoint.

An edge cannot be shorter than all the moats that it has to cross.

Algorithm 1 PrimalDualShortestPath

```
1:  $\gamma \leftarrow 0$ 
2:  $F \leftarrow \emptyset$ 
3: while there is no  $s$ - $t$  path in  $(V, F)$  do
4:   Let  $C$  be the connected component of  $(V, F)$  containing  $s$ 
5:   Increase  $\gamma_C$  until there is an edge  $e' \in \delta(C)$  such that  $\sum_{S: e' \in \delta(S)} \gamma_S = c(e')$ .
6:    $F \leftarrow F \cup \{e'\}$ 
7: Let  $P$  be an  $s$ - $t$  path in  $(V, F)$ 
8: return  $P$ 
```

Lemma 3

At each point in time the set F forms a tree.

Proof:

- ▶ In each iteration we take the current connected component from (V, F) that contains s (call this component C) and add some edge from $\delta(C)$ to F .
- ▶ Since, at most one end-point of the new edge is in C the edge cannot close a cycle.

$$\begin{aligned} \sum_{e \in P} c(e) &= \sum_{e \in P} \sum_{S: e \in \delta(S)} y_S \\ &= \sum_{S: s \in S, t \notin S} |P \cap \delta(S)| \cdot y_S. \end{aligned}$$

If we can show that $y_S > 0$ implies $|P \cap \delta(S)| = 1$ gives

$$\sum_{e \in P} c(e) = \sum_S y_S \leq \text{OPT}$$

by weak duality.

Hence, we find a shortest path.

If S contains two edges from P then there must exist a subpath P' of P that starts and ends with a vertex from S (and all interior vertices are not in S).

When we increased y_S , S was a connected component of the set of edges F' that we had chosen till this point.

$F' \cup P'$ contains a cycle. Hence, also the final set of edges contains a cycle.

This is a contradiction.

Steiner Forest Problem:

Given a graph $G = (V, E)$, together with source-target pairs s_i, t_i , $i = 1, \dots, k$, and a cost function $c: E \rightarrow \mathbb{R}^+$ on the edges. Find a subset $F \subseteq E$ of the edges such that for every $i \in \{1, \dots, k\}$ there is a path between s_i and t_i only using edges in F .

$$\begin{array}{ll} \min & \sum_e c(e) x_e \\ \text{s.t.} & \forall S \subseteq V: S \in S_i \text{ for some } i \quad \sum_{e \in \delta(S)} x_e \geq 1 \\ & \forall e \in E \quad x_e \in \{0, 1\} \end{array}$$

Here S_i contains all sets S such that $s_i \in S$ and $t_i \notin S$.

$$\begin{array}{ll} \max & \sum_{S: \exists i \text{ s.t. } S \in S_i} y_S \\ \text{s.t.} & \forall e \in E \quad \sum_{S: e \in \delta(S)} y_S \leq c(e) \\ & y_S \geq 0 \end{array}$$

The difference to the dual of the shortest path problem is that we have many more variables (sets for which we can generate a moat of non-zero width).

Algorithm 1 FirstTry

```
1:  $\gamma \leftarrow 0$ 
2:  $F \leftarrow \emptyset$ 
3: while not all  $s_i$ - $t_i$  pairs connected in  $F$  do
4:   Let  $C$  be some connected component of  $(V, F)$ 
     such that  $|C \cap \{s_i, t_i\}| = 1$  for some  $i$ .
5:   Increase  $\gamma_C$  until there is an edge  $e' \in \delta(C)$  s.t.
      $\sum_{S \in \mathcal{S}; e' \in \delta(S)} \gamma_S = c_{e'}$ 
6:    $F \leftarrow F \cup \{e'\}$ 
7: return  $\bigcup_i P_i$ 
```

$$\sum_{e \in F} c(e) = \sum_{e \in F} \sum_{S: e \in \delta(S)} \gamma_S = \sum_S |\delta(S) \cap F| \cdot \gamma_S.$$

If we show that $\gamma_S > 0$ implies that $|\delta(S) \cap F| \leq \alpha$ we are in good shape.

However, this is not true:

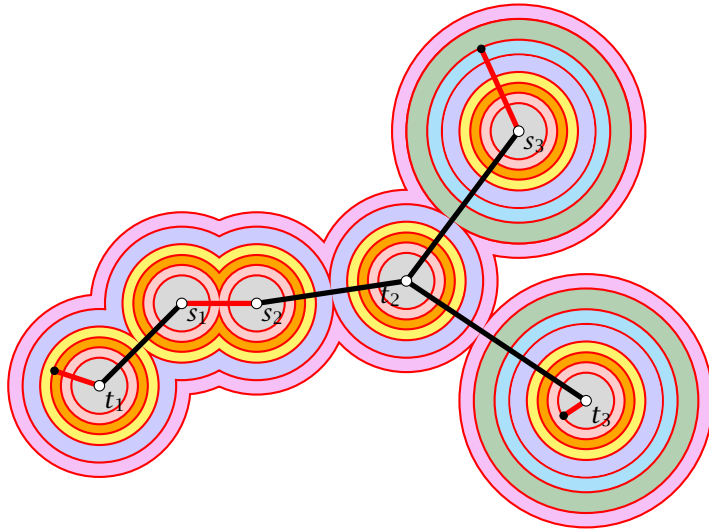
- ▶ Take a complete graph on $k + 1$ vertices v_0, v_1, \dots, v_k .
- ▶ The i -th pair is v_0 - v_i .
- ▶ The first component C could be $\{v_0\}$.
- ▶ We only set $\gamma_{\{v_0\}} = 1$. All other dual variables stay 0.
- ▶ The final set F contains all edges $\{v_0, v_i\}$, $i = 1, \dots, k$.
- ▶ $\gamma_{\{v_0\}} > 0$ but $|\delta(\{v_0\}) \cap F| = k$.

Algorithm 1 SecondTry

```
1:  $\gamma \leftarrow 0$ ;  $F \leftarrow \emptyset$ ;  $\ell \leftarrow 0$ 
2: while not all  $s_i$ - $t_i$  pairs connected in  $F$  do
3:    $\ell \leftarrow \ell + 1$ 
4:   Let  $\mathcal{C}$  be set of all connected components  $C$  of  $(V, F)$ 
     such that  $|C \cap \{s_i, t_i\}| = 1$  for some  $i$ .
5:   Increase  $\gamma_C$  for all  $C \in \mathcal{C}$  uniformly until for some edge
      $e_\ell \in \delta(C')$ ,  $C' \in \mathcal{C}$  s.t.  $\sum_{S: e_\ell \in \delta(S)} \gamma_S = c_{e_\ell}$ 
6:    $F \leftarrow F \cup \{e_\ell\}$ 
7:    $F' \leftarrow F$ 
8:   for  $k \leftarrow \ell$  downto 1 do // reverse deletion
9:     if  $F' - e_k$  is feasible solution then
10:       remove  $e_k$  from  $F'$ 
11: return  $F'$ 
```

The reverse deletion step is not strictly necessary this way. It would also be sufficient to simply delete all unnecessary edges in any order.

Example



Lemma 4

For any \mathcal{C} in any iteration of the algorithm

$$\sum_{C \in \mathcal{C}} |\delta(C) \cap F'| \leq 2|\mathcal{C}|$$

This means that the number of times a moat from \mathcal{C} is crossed in the final solution is at most twice the number of moats.

Proof: later...

$$\sum_{e \in F'} c_e = \sum_{e \in F'} \sum_{S: e \in \delta(S)} \gamma_S = \sum_S |F' \cap \delta(S)| \cdot \gamma_S.$$

We want to show that

$$\sum_S |F' \cap \delta(S)| \cdot \gamma_S \leq 2 \sum_S \gamma_S$$

- ▶ In the i -th iteration the increase of the left-hand side is

$$\epsilon \sum_{C \in \mathcal{C}} |F' \cap \delta(C)|$$

and the increase of the right hand side is $2\epsilon|\mathcal{C}|$.

- ▶ Hence, by the previous lemma the inequality holds after the iteration if it holds in the beginning of the iteration.

Lemma 5

For any set of connected components \mathcal{C} in any iteration of the algorithm

$$\sum_{C \in \mathcal{C}} |\delta(C) \cap F'| \leq 2|\mathcal{C}|$$

Proof:

- ▶ At any point during the algorithm the set of edges forms a forest (why?).
- ▶ Fix iteration i . Let F_i be the set of edges in F at the beginning of the iteration.
- ▶ Let $H = F' - F_i$.
- ▶ All edges in H are necessary for the solution.

- ▶ Contract all edges in F_i into single vertices V' .
- ▶ We can consider the forest H on the set of vertices V' .
- ▶ Let $\deg(v)$ be the degree of a vertex $v \in V'$ within this forest.
- ▶ Color a vertex $v \in V'$ **red** if it corresponds to a component from \mathcal{C} (an active component). Otw. color it blue. (Let B the set of blue vertices (with non-zero degree) and R the set of red vertices)
- ▶ We have

$$\sum_{v \in R} \deg(v) \geq \sum_{C \in \mathcal{C}} |\delta(C) \cap F'| \stackrel{?}{\leq} 2|\mathcal{C}| = 2|R|$$

- ▶ Suppose that no node in B has degree one.
- ▶ Then

$$\begin{aligned} \sum_{v \in R} \deg(v) &= \sum_{v \in R \cup B} \deg(v) - \sum_{v \in B} \deg(v) \\ &\leq 2(|R| + |B|) - 2|B| = 2|R| \end{aligned}$$

- ▶ Every blue vertex with non-zero degree must have degree at least two.
 - ▶ Suppose not. The single edge connecting $b \in B$ comes from H , and, hence, is necessary.
 - ▶ But this means that the cluster corresponding to b must separate a source-target pair.
 - ▶ But then it must be a red node.